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Maps in Tangent Complex of Order Three

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Abstract.: Previously we have extended the notion of tangent complex of first order to second order and proposed various morphisms in order to connect the tangent complex to well known Grassmannian complex. Now we are motivated to find similar constructions and maps for order greater than 2. Therefore in this paper we present the maps and other ingredients for the tangent complex of order three in dialogarithmic settings. This work will play a key role in the generalization of these constructions.

AMS (MOS) Subject Classification Codes: 18G; 14L; 16G
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1. Introduction

Siddiqui initiated the formation of tangent complex for the first order using geometric configurations(see [12], [13]). He introduced a group $T\mathcal{B}_2(F)$ called tangent group of first order and constructed cross ratio, Seigel's identity for cross ratio and its associated determinant of 2×2. On the basis of these ingredients he proposed the maps $\tau_{0,\varepsilon}$, $\tau_{1,\varepsilon}$ and ∂_ε to relate the tangent complex and Suslin's Grassmannian complex (see [12] , [13]). We extended these constructions by introducing a similar group $T\mathcal{B}_2^2(F)$ but of second order [9], [11]. Using the map ∂_{ε^2} we established second ordered tangent complex and presented the maps π_{0,ε^2} and π_{1,ε^2} between this newly establish complex and Suslin's Grassmannian complex. The commutativity of resulting figure is also proved in the same work.

Naturally we are motivated to extend these constructions up to a general order. To do this the study of only first two orders are not enough as they do not reflect any specific pattern. So the study of next order is essential after which we can extract the required goal.

In the first section we define the group $T\mathcal{B}_2^3(F)$ called third ordered tangent group and determinant $\Delta(v_i^*, v_j^*)$ associated to it. This group also satisfies functional equations of dialogarithmic which are mentioned in ($2.2). After the construction of cross ratio and its identity we give a map ∂_{ε^3} in equation (3) to construct dialogarithmic tangential complex . Our proposed maps π^2_{0,ε^3} and π^2_{1,ε^3} enable us to the formation of commutative diagram (F). In the last section we demonstrate proof of the commutativity of (F)(see theorem (3.2))

2. Notations and Preliminaries

2.1. **Tangent Group of order 3 in weight 2.** Let $F[\varepsilon]_4$ be a truncated polynomial ring over an arbitrary field F then we call the $\mathbb{Z}$-module $T\mathcal{B}_2^3(F)$ a tangent group of order 3 if it is generated by $\langle s; s', s'', s'''] \in \mathbb{Z}[F[\varepsilon]_4]$ and quotient by the expression [1].

$$\begin{aligned}&\langle s; s', s'', s'''] - \langle t; t', t'', t'''] + \left\langle \frac{t}{s}; \left(\frac{t}{s}\right)', \left(\frac{t}{s}\right)'', \left(\frac{t}{s}\right)''' \right] \\ &- \left\langle \frac{1-t}{1-s}; \left(\frac{1-t}{1-s}\right)', \left(\frac{1-t}{1-s}\right)'', \left(\frac{1-t}{1-s}\right)''' \right] \\ &+ \left\langle \frac{s(1-t)}{t(1-s)}; \left(\frac{s(1-t)}{t(1-s)}\right)', \left(\frac{s(1-t)}{t(1-s)}\right)'', \left(\frac{s(1-t)}{t(1-s)}\right)''' \right], \quad s, t \neq 0, 1, s \neq t \qquad (2.\ 1)\end{aligned}$$

where $\langle s; s', s'', s'''] = [s + s'\varepsilon + s''\varepsilon^2 + s'''\varepsilon^3] - [s]$ and $s, s', s'', s''' \in F$.

The expressions $\left(\frac{t}{s}\right)'$, $\left(\frac{t}{s}\right)''$, $\left(\frac{1-t}{1-s}\right)'$, $\left(\frac{1-t}{1-s}\right)''$, $\left(\frac{s(1-t)}{t(1-s)}\right)'$ and $\left(\frac{s(1-t)}{t(1-s)}\right)''$ are defined in [12] and [9].which are given as

$$\begin{aligned}&\left(\frac{t}{s}\right)' = \frac{st' - s't}{s^2}; \quad \left(\frac{1-t}{1-s}\right)' = \frac{(1-t)s' - (1-s)t'}{(1-s)^2}; \\ &\left(\frac{s(1-t)}{t(1-s)}\right)' = \frac{t(1-t)s' - s(1-s)t'}{t(1-s)^2} \qquad (2.\ 2)\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{t}{s}\right)'' = \frac{s^2t'' - sts'' - ss't' + t(s')^2}{s^3}; \quad \left(\frac{1-t}{1-s}\right)'' = \frac{A}{(1-s)^3}; \\ &\left(\frac{s(1-t)}{t(1-s)}\right)'' = \frac{B}{s^3(1-t)^3} \qquad (2.\ 3)\end{aligned}$$

where

$$A = (1-s)(1-t)s'' - (1-s)^2t'' - (1-s)s't' + (1-t)(s')^2$$

and

$$\begin{aligned}B =&(t')^2s^3 - tt''s^3 + 2tt''s^2 - 2(t')^2s^2 - tt''s + (t')^2s + tst's' - ts't' \\ &+t^3ss'' - t^3(s')^2 - t^3s'' - t^2ss'' + t^2(s')^2 + t^2s''\end{aligned}$$

Other terms are given as under

$$\left(\frac{t}{s}\right)''' = \frac{t'''}{s} - \frac{s'}{s}\left(\frac{t}{s}\right)'' - \frac{s''}{s}\left(\frac{t}{s}\right)' - \frac{s'''}{s}\left(\frac{t}{s}\right)$$

$$\left(\frac{1-t}{1-s}\right)''' = \frac{s'''}{(1-s)}\left(\frac{1-t}{1-s}\right) + \frac{s''}{(1-s)}\left(\frac{1-t}{1-s}\right)' + \frac{s'}{(1-s)}\left(\frac{1-t}{1-s}\right)'' - \frac{s'''}{(1-s)}$$

and

$$\left(\frac{s(1-t)}{t(1-s)}\right)'' = \frac{C}{s^3(1-t)^3}$$

where

$$C = (t')^2s^3 - tt''s^3 + 2tt''s^2 - 2(t')^2s^2 - tt''s + (t')^2s + tst's' - ts't' + t^3ss'' - t^3(s')^2 - t^3s'' - t^2ss'' + t^2(s')^2 + t^2s''$$

2.2. **Relations in $T\mathcal{B}_2^3(F)$.** Functional equation for an algebraic structure is the useful relations satisfied by the elements of the structure [8], [10] . Only three such relations are known for the group $T\mathcal{B}_2^3(F)$ known as inversion, two term and five term functional equations of $T\mathcal{B}_2^3(F)$ which are given as under (see [2], [14], [14], [12] and [11]).

(1) $\langle s; t_1, t_2, t_3]_2^3 = -\langle 1-s; -t_1, -t_2, -t_3]_2^3$

(2) $\langle s; t_1, t_2, t_3]_2^3 = \left\langle \frac{1}{s}; -\frac{t_1}{s^2} - \frac{st_2-(t_1)^2}{s^3}\right]_2^3$

Equation (2. 1) represents the five term relation of $T\mathcal{B}_2^3(F)$.

2.3. **Cross-Ratio.** In [9] cross ratio has been constructed for second order tangential settings so we use here the similar technique to extend it to third order. For more details see [3], [4], [5] and [6].

Let $\mathbb{A}^3_{F[\varepsilon]_4}$ represents an affine space over the turncated ring of polynomials $F[\varepsilon]_4$ and $(v_0^*, \ldots, v_3^*)$ belongs to $C_4(\mathbb{A}^3_{F[\varepsilon]_4})$ then we obtain

$$\mathbf{r}(v_0^*, \ldots, v_3^*) = \frac{\{\sum_{i=0}^3 \Delta(v_0^*, v_3^*)_{\varepsilon^i}\varepsilon^i\}\{\sum_{i=0}^3 \Delta(v_1^*, v_2^*)_{\varepsilon^i}\varepsilon^i\}}{\{\sum_{i=0}^3 \Delta(v_0^*, v_2^*)_{\varepsilon^i}\varepsilon^i\}\{\sum_{i=0}^3 \Delta(v_1^*, v_3^*)_{\varepsilon^i}\varepsilon^i\}}$$

where $\Delta(v_i^*, v_j^*)$ is 2×2 determinant (See [9]).

From now we use a short hand $\mathbb{V}_{0n}$ to denote the tuple $(v_0^*, \ldots, v_n^*)$ of $C_n(\mathbb{A}^n_{F[\varepsilon]_{n+1}})$ and $(v_i^*v_j^*)$ to denote $\Delta(v_i^*, v_j^*)$. Now we reform the above ratio into simpler form $G + H\varepsilon + I\varepsilon^2 + J\varepsilon^3$ where the unknowns G, H and I represents $r(\mathbb{V}_{03})$, $r_\varepsilon(\mathbb{V}_{03})$ and $r_{\varepsilon^2}(\mathbb{V}_{03})$ respectively. Values of first two of these coefficients are calculated in [9] and [13] respectively. In the same way we can evaluate "J" or $r_{\varepsilon^3}(\mathbb{V}_{03})$ as follows

$$r_{\varepsilon^3}(\mathbb{V}_{03}) = \frac{\{(v_0^*v_3^*)(v_1^*v_2^*)\}_{\varepsilon^3}}{(v_0v_2)(v_1v_3)} - r_\varepsilon(\mathbb{V}_{03})\frac{\{(v_0^*v_2^*)(v_1^*v_3^*)\}_{\varepsilon^2}}{(v_0v_2)(v_1v_3)} - r_{\varepsilon^2}(\mathbb{V}_{03})\frac{\{(v_0^*v_2^*)(v_1^*v_3^*)\}_{\varepsilon}}{(v_0v_2)(v_1v_3)} - r(\mathbb{V}_{03})\frac{\{(v_0^*v_2^*)(v_1^*v_3^*)\}_{\varepsilon^3}}{(v_0v_2)(v_1v_3)} \quad (2.\ 4)$$

where $\{(v_j^*v_k^*)(v_m^*v_n^*)\}_{\varepsilon^3} = \sum_{i=0}^3 (v_j^*v_k^*)_{\varepsilon^{3-i}}(v_m^*v_n^*)_{\varepsilon^i}$

Lemma 2.4. *If $\mathbb{V}_{03}$ be an element of $C_4(\mathbb{A}^3_{F[\varepsilon]_4})$ then*

$$\{(v_0^*v_1^*)(v_2^*v_3^*)\}_{\varepsilon^3} = \{(v_0^*v_2^*)(v_1^*v_3^*)\}_{\varepsilon^3} - \{(v_0^*v_3^*)(v_1^*v_2^*)\}_{\varepsilon^3}$$

with $v_i^* = \sum_{k=0}^3 v_{i,\varepsilon^k}\varepsilon^k \quad ; \quad v_{i,\varepsilon^0} = v_i \quad (v_i^*v_j^*) = \sum_{k=0}^3 (v_i^*v_j^*)_{\varepsilon^k}\varepsilon^k$
and $(v_i^*v_j^*)_{\varepsilon^3} = (v_i, v_{j,\varepsilon^3}) + (v_{i,\varepsilon}, v_{j,\varepsilon^2}) + (v_{i,\varepsilon^2}, v_{j,\varepsilon}) + (v_{i,\varepsilon^3}, v_j)$

Proof. We can deduce the proof directly from the result of Lemma 4.1.1 of ([12]) by setting $n = 3$. □

3. Results

3.1. **Dilogarathmic Tangential Complexes.** We use the map (3) to establish the dialogarithmic tangent complex of order 3 as follows

$$T\mathcal{B}_2^3(F) \xrightarrow{\partial_{\varepsilon^3}} F \otimes F^\times \oplus \bigwedge^2 F$$

where $T\mathcal{B}_2^3(F)$ is defined earlier to be tangent group. Now for a field of characteristic zero the $m-$dimensional tuples $(v_0^*, \ldots, v_{(m-1)}^*$ generates a free commutative group $C_m(\mathbb{A}^2_{F[\varepsilon]_4})$ over the affine space $\mathbb{A}^2_{F[\varepsilon]_4}$. Furthermore, we can form the analogue of Grassmannian complex as

$$\ldots \to^d C_5(\mathbb{A}^2_{F[\varepsilon]_4}) \to^d C_4(\mathbb{A}^2_{F[\varepsilon]_4}) \to^d C_3(\mathbb{A}^2_{F[\varepsilon]_4})$$

where

$$d : (v_0^*, \ldots, v_m^*) \mapsto \sum_{i=0}^m (-1)^i (v_0^*, \ldots, \hat{v}_i^*, \ldots, v_m^*),$$

Our aim is to connect this analogue of Grassmannian sub- complex and the third order complex in tanential settings. The result of connection of both complexes gives the diagram below.

$$\begin{array}{ccccc} C_5(\mathbb{A}^2_{F[\varepsilon]_4}) & \xrightarrow{d} & C_4(\mathbb{A}^2_{F[\varepsilon]_4}) & \xrightarrow{d} & C_3(\mathbb{A}^2_{F[\varepsilon]_4}) \\ & & \downarrow{\scriptstyle \pi^2_{0,\varepsilon^3}} & & \downarrow{\scriptstyle \pi^2_{1,\varepsilon^3}} \\ & & T\mathcal{B}_2^3(F) & \xrightarrow{\partial_{\varepsilon^3}} & F \otimes F^\times \oplus \bigwedge^2 F \end{array} \tag{F}$$

here ∂_{ε^3} is a map which behaves like

$$\begin{aligned} \partial_{\epsilon^3}\left(\langle s; t_1, t_2, t_3]_2^3\right) = & \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\} \otimes (1-s) \\ & + \left\{\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)\right\} \otimes s \\ & + \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{t^3}\right)\right\} \wedge \left\{\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)\right\} \end{aligned} \tag{3. 5}$$

where $\quad \langle s; t_1, t_2, t_3]_2^3 \in T\mathcal{B}_2^3(F) \; ; \quad s, t_1, t_2, t_3 \in F; a \neq 0, 1$

To minimize the complication we express the map π^2_{0,ϵ^2} as $\pi^2_{0,\epsilon^2} = \pi^1 + \pi^2$

$$\pi^1(\mathbb{V}^*_{02}) = \sum_{i=0}^{2}(-1)^i\left\{\left(3\left(\frac{(v_i^*v_{i+1}^*)_{\epsilon^3}}{(v_iv_{i+1})} - \frac{(v_i^*v_{i+1}^*)_{\epsilon^2}(v_i^*v_{i+1}^*)_{\epsilon}}{(v_iv_{i+1})^2}\right) + \frac{(v_i^*v_{i+1}^*)^3{}_{\epsilon}}{(v_iv_{i+1})^3}\right)\otimes\frac{(v_iv_{i+2})}{(v_{i+1}v_{i+2})}\right\} \quad i \mod 3 \tag{3. 6}$$

$$\pi^2(\mathbb{V}^*_{02}) = \sum_{i=0}^{2}(-1)^i\left\{3\frac{(v_i^*v_{i+1}^*)_{\epsilon^3}}{(v_iv_{i+1})} - 3\frac{(v_i^*v_{i+1}^*)_{\epsilon^2}(v_i^*v_{i+1}^*)_{\epsilon}}{(v_iv_{i+1})^2} + \frac{(v_i^*v_{i+1}^*)^3{}_{\epsilon}}{(v_iv_{i+1})^3}\right.$$
$$\left.\wedge\left(3\frac{(v_i^*v_{i+2}^*)_{\epsilon^3}}{(v_iv_{i+2})} - 3\frac{(v_i^*v_{i+2}^*)_{\epsilon^2}(v_i^*v_{i+2}^*)_{\epsilon}}{(v_iv_{i+2})^2} + \frac{(v_i^*v_{i+2}^*)^3{}_{\epsilon}}{(v_iv_{i+2})^3}\right)\right\}; \qquad i \mod 3 \tag{3. 7}$$

$$\pi^2_{1,\epsilon^3}(\mathbb{V}^*_{03}) = \langle r(\mathbb{V}_{03}); r_{\epsilon}(\mathbb{V}^*_{03}), r_{\epsilon^2}(\mathbb{V}^*_{03}), r_{\epsilon^3}(\mathbb{V}^*_{03})] \tag{3. 8}$$

The maps π^2_{0,ϵ^3} and π^2_{1,ϵ^3} are to be checked whether these are well defined or not. The second map is actually a cross ratio which we have defined earlier and hence it is well defined. We only investigate for τ^2_{0,ϵ^3} in the following lemma .

Lemma 3.2. *The map π^2_{0,ϵ^3} defined above is free of vector's length.*

Proof. From the equations(3. 6) and (3. 7) we have

$$\pi^2_{0,\epsilon^3}(\mathbb{V}^*_{02}) = \sum_{i=0}^{2}(-1)^i\left\{\left(3\frac{(v_i^*v_{i+1}^*)_{\epsilon^3}}{(v_iv_{i+1})} + 3\frac{(v_i^*,v_{i+1}^*)_{\epsilon^2}(v_i^*v_{i+1}^*)_{\epsilon}}{(v_iv_{i+1})^2} + \frac{(v_i^*v_{i+1}^*)^3{}_{\epsilon}}{(v_iv_{i+1})^3}\right)\otimes\frac{(v_iv_{i+2})}{(v_{i+1}v_{i+2})}\right\}$$
$$+\left\{3\frac{(v_i^*v_{i+1}^*)_{\epsilon^3}}{(v_iv_{i+1})} - 3\frac{(v_i^*v_{i+1}^*)_{\epsilon^2}(v_i^*v_{i+1}^*)_{\epsilon}}{(v_iv_{i+1})^2} + \frac{(v_i^*v_{i+1}^*)^3{}_{\epsilon}}{(v_iv_{i+1})^3}\right)$$
$$\wedge\left(3\frac{(v_i^*v_{i+2}^*)_{\epsilon^3}}{(v_iv_{i+2})} - 3\frac{(v_i^*v_{i+2}^*)_{\epsilon^2}(v_i^*v_{i+2}^*)_{\epsilon}}{(v_iv_{i+2})^2} - \frac{(v_i^*v_{i+2}^*)^3{}_{\epsilon}}{(v_iv_{i+2})^3}\right)\right\} \tag{3. 9}$$

The expansion of π^2_{0,ϵ^3} shows that it contain three types of expressions $\frac{(u)_{\varepsilon^3}}{u}$, $\frac{(u)^3_{\varepsilon}}{u^3}$ and $\frac{(u)_{\varepsilon^2}(v)_{\varepsilon}}{uv}$. Here we chose $\rho \in F^{\times}$ then we may have

$$\frac{(\rho u)_{\varepsilon^3}}{\rho u} = \frac{\rho(u)_{\varepsilon^3}}{\rho u} = \frac{(u)_{\varepsilon^3}}{u}$$

Similarly

$$\frac{(\rho u)^3_{\varepsilon}}{(\rho u)^3} = \frac{(u)^3_{\varepsilon}}{u^3} \qquad \text{and} \qquad \frac{(\rho u)_{\varepsilon^2}(\rho v)_{\varepsilon}}{(\rho u)(\rho v)} = \frac{(u)_{\varepsilon^2}(v)_{\varepsilon}}{uv}$$

This shows if we change the length of a vector u to ρu the value of these expressions remain unchanged. So we can conclude that the map π^2_{0,ϵ^3} is well-defined.

□

Theorem 3.3. *Commutation holds for the diagram (F) of complexes. i.e.*

$$\pi^2_{0,\epsilon^3} \circ d = \partial^2_{\epsilon^3} \circ \pi^2_{1,\epsilon^3}$$

Proof. In (3) we have already described $\partial^2_{\epsilon^3}$ which seems to be lengthy enough and may create complication in our calculations . To avoid such situation we divide $\partial^2_{\epsilon^3}$ into two simpler maps σ_1 and σ_2

$$\partial^2_{\epsilon^3} = \sigma_1 + \sigma_2. \tag{3. 10}$$

This implies

$$\partial^2_{\epsilon^3} \circ \pi^2_{1,\epsilon^3} = \sigma_1 \circ \pi^2_{1,\epsilon^3} + \sigma_2 \circ \pi^2_{1,\epsilon^3} \tag{3. 11}$$

where

$$\begin{aligned}&\sigma_1(\langle s; t_1, t_2, t_3]^3_2)\\ &= \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\} \otimes (1-s) + \left\{\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)\right\} \otimes s)\end{aligned} \tag{3. 12}$$

and

$$\sigma_2(\langle s; t_1, t_2, t_3]^3_2) = \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\} \wedge \left\{\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)\right\} \tag{3. 13}$$

If we put $s = r(v_0, \ldots, v_3)$; $t_1 = r_\epsilon(v_0^*, \ldots, v_3^*)$; $t_2 = r_{\epsilon^2}(v_0^*, \ldots, v_3^*)$
and $t_3 = r_{\epsilon^3}(v_0^*, \ldots, v_3^*)$. Then we get

$$\begin{aligned}&\sigma_1 \circ \pi^2_{1,\epsilon^3}(\mathbb{V}^*_{03})\\ &= \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\} \otimes (1-s) + \left\{\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)\right\} \otimes s\end{aligned} \tag{3. 14}$$

Here we calculate the value of $\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)$ and $\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)$.

$$\begin{aligned}\frac{t_1^3}{s^3} &= \left(\frac{r_\epsilon(\mathbb{V}^*_{03})}{r(\mathbb{V}_{03})}\right)^3\\
&= \frac{(v_o^*v_3^*)^3_\epsilon}{(v_0v_3)^3} + \frac{(v_1^*v_2^*)^3_\epsilon}{(v_1v_2)^3} - \frac{(v_0^*v_2^*)^3_\epsilon}{(v_0v_2)^3} - \frac{(v_1^*v_3^*)^3_\epsilon}{(v_1v_3)^3} + 3\frac{(v_o^*v_3^*)^2_\epsilon}{(v_0v_3)^2}\frac{(v_1^*v_2^*)_\epsilon}{(v_1v_2)} - 3\frac{(v_0^*v_2^*)^2_\epsilon}{(v_0v_2)^2}\frac{(v_1^*v_3^*)_\epsilon}{(v_1v_3)}\\
&+ 3\frac{(v_o^*v_3^*)_\epsilon}{(v_0v_3)}\frac{(v_1^*v_2^*)^2_\epsilon}{(v_1v_2)^2} - 3\frac{(v_0^*v_2^*)_\epsilon}{(v_0v_2)}\frac{(v_1^*v_3^*)^2_\epsilon}{(v_1v_3)^2} - 3\frac{(v_o^*v_3^*)^2_\epsilon}{(v_0v_3)^2}\frac{(v_0^*v_2^*)_\epsilon}{(v_0v_2)} + 3\frac{(v_o^*v_3^*)_\epsilon}{(v_0v_3)}\frac{(v_0^*v_2^*)^2_\epsilon}{(v_0v_2)^2}\\
&- 3\frac{(v_o^*v_3^*)^2_\epsilon}{(v_0v_3)^2}\frac{(v_1^*v_3^*)_\epsilon}{(v_1v_3)} + 3\frac{(v_o^*v_3^*)_\epsilon}{(v_0v_3)}\frac{(v_1^*v_3^*)^2_\epsilon}{(v_1v_3)^2} - 3\frac{(v_o^*v_2^*)_\epsilon}{(v_0v_2)}\frac{(v_1^*v_2^*)^2_\epsilon}{(v_1v_2)^2} + 3\frac{(v_o^*v_2^*)^2_\epsilon}{(v_0v_2)^2}\frac{(v_1^*v_2^*)_\epsilon}{(v_1v_2)}\\
&- 3\frac{(v_1^*v_3^*)_\epsilon}{(v_1v_3)}\frac{(v_1^*v_2^*)^2_\epsilon}{(v_1v_2)^2} + 3\frac{(v_1^*v_3^*)^2_\epsilon}{(v_1v_3)^2}\frac{(v_1^*v_2^*)_\epsilon}{(v_1v_2)} - 6\frac{(v_o^*v_3^*)_\epsilon}{(v_0v_3)}\frac{(v_1^*v_2^*)_\epsilon}{(v_1v_2)}\frac{(v_0^*v_2^*)_\epsilon}{(v_0v_2)}\\
&- 6\frac{(v_o^*v_3^*)_\epsilon}{(v_0v_3)}\frac{(v_1^*v_2^*)_\epsilon}{(v_1v_2)}\frac{(v_1^*v_3^*)_\epsilon}{(v_1v_3)} + 6\frac{(v_o^*v_3^*)_\epsilon}{(v_0v_3)}\frac{(v_1^*v_3^*)_\epsilon}{(v_1v_3)}\frac{(v_0^*v_2^*)_\epsilon}{(v_0v_2)} + 6\frac{(v_1^*v_3^*)_\epsilon}{(v_1v_3)}\frac{(v_1^*v_2^*)_\epsilon}{(v_1v_2)}\frac{(v_0^*v_2^*)_\epsilon}{(v_0v_2)}\end{aligned} \tag{3. 15}$$

$$
\begin{aligned}
\frac{t_1t_2}{s^2} &= \left(\frac{r_\epsilon(\mathbb{V}^*_{03})r_{\epsilon^2}(\mathbb{V}^*_{03})}{r(\mathbb{V}_{03})}\right)^2\\
&= \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_ov^*_3)_{\epsilon^2}}{(v_0v_3)} + \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_2)_{\epsilon^2}}{(v_1v_2)} + \frac{(v^*_ov^*_3)_{\epsilon^2}}{(v_0v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)} - \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_ov^*_2)_{\epsilon^2}}{(v_0v_2)}\\
&- \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)} - \frac{(v^*_ov^*_3)_{\epsilon^2}}{(v_0v_3)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} - \frac{(v^*_ov^*_3)_{\epsilon^2}}{(v_0v_3)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} + \frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_1v^*_2)_{\epsilon^2}}{(v_1v_2)}\\
&+ \frac{(v^*_ov^*_2)_\epsilon}{(v_0v_2)}\frac{(v^*_ov^*_2)_{\epsilon^2}}{(v_0v_2)} + \frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)} - \frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_0v^*_2)_{\epsilon^2}}{(v_0v_2)} - \frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)}\\
&- \frac{(v^*_1v^*_2)_{\epsilon^2}}{(v_1v_2)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} - \frac{(v^*_1v^*_2)_{\epsilon^2}}{(v_1v_2)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} + \frac{(v^*_ov^*_2)_\epsilon}{(v_0v_2)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)} + \frac{(v^*_ov^*_2)_{\epsilon^2}}{(v_0v_2)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\\
&+ \frac{(v^*_ov^*_3)^2_\epsilon}{(v_0v_3)^2}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)} - 2\frac{(v^*_0v^*_2)^2_\epsilon}{(v_0v_2)^2}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} + \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_2)^2_\epsilon}{(v_1v_2)^2} - 2\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)}\frac{(v^*_1v^*_3)^2_\epsilon}{(v_1v_3)^2}\\
&- \frac{(v^*_ov^*_3)^2_\epsilon}{(v_0v_3)^2}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} + 2\frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_0v^*_2)^2_\epsilon}{(v_0v_2)^2} - \frac{(v^*_ov^*_3)^2_\epsilon}{(v_0v_3)^2}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} + 2\frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_3)^2_\epsilon}{(v_1v_3)^2}\\
&- \frac{(v^*_ov^*_2)_\epsilon}{(v_0v_2)}\frac{(v^*_1v^*_2)^2_\epsilon}{(v_1v_2)^2} + 2\frac{(v^*_ov^*_2)^2_\epsilon}{(v_0v_2)^2}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)} - \frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\frac{(v^*_1v^*_2)^2_\epsilon}{(v_1v_2)^2} + 2\frac{(v^*_1v^*_3)^2_\epsilon}{(v_1v_3)^2}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\\
&- \frac{(v^*_0v^*_2)^3_\epsilon}{(v_0v_2)^3} - \frac{(v^*_1v^*_3)^3_\epsilon}{(v_1v_3)^3} - 3\frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} - 3\frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\\
&+ 3\frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} + 3\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)}
\end{aligned}
\tag{3. 16}
$$

$$
\begin{aligned}
\frac{t_3}{s} &= \frac{(v^*_ov^*_3)_{\epsilon^3}}{(v_0v_3)} + \frac{(v^*_1v^*_2)_{\epsilon^3}}{(v_1v_2)} - \frac{(v^*_0v^*_2)_{\epsilon^3}}{(v_0v_2)} - \frac{(v^*_1v^*_3)_{\epsilon^3}}{(v_1v_3)} + 2\frac{(v^*_ov^*_2)_\epsilon}{(v_0v_2)}\frac{(v^*_ov^*_2)_{\epsilon^2}}{(v_0v_2)} + 2\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)}\\
&+ \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_2)_{\epsilon^2}}{(v_1v_2)} + \frac{(v^*_ov^*_3)_{\epsilon^2}}{(v_0v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)} - \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_ov^*_2)_{\epsilon^2}}{(v_0v_2)} - \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)}\\
&- \frac{(v^*_ov^*_3)_{\epsilon^2}}{(v_0v_3)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} - \frac{(v^*_ov^*_3)_{\epsilon^2}}{(v_0v_3)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} - \frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_0v^*_2)_{\epsilon^2}}{(v_0v_2)} - \frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)}\\
&- \frac{(v^*_1v^*_2)_{\epsilon^2}}{(v_1v_2)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} - \frac{(v^*_1v^*_2)_{\epsilon^2}}{(v_1v_2)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} + \frac{(v^*_ov^*_2)_\epsilon}{(v_0v_2)}\frac{(v^*_1v^*_3)_{\epsilon^2}}{(v_1v_3)} + \frac{(v^*_ov^*_2)_{\epsilon^2}}{(v_0v_2)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\\
&- \frac{(v^*_0v^*_2)^2_\epsilon}{(v_0v_2)^2}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} - \frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)}\frac{(v^*_1v^*_3)^2_\epsilon}{(v_1v_3)^2} + \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_0v^*_2)^2_\epsilon}{(v_0v_2)^2} + \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_3)^2_\epsilon}{(v_1v_3)^2}\\
&+ \frac{(v^*_ov^*_2)^2_\epsilon}{(v_0v_2)^2}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)} + \frac{(v^*_1v^*_3)^2_\epsilon}{(v_1v_3)^2}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)} - \frac{(v^*_0v^*_2)^3_\epsilon}{(v_0v_2)^3} - \frac{(v^*_1v^*_3)^3_\epsilon}{(v_1v_3)^3} - \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)}\\
&- \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)} + \frac{(v^*_ov^*_3)_\epsilon}{(v_0v_3)}\frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)} + \frac{(v^*_1v^*_3)_\epsilon}{(v_1v_3)}\frac{(v^*_1v^*_2)_\epsilon}{(v_1v_2)}\frac{(v^*_0v^*_2)_\epsilon}{(v_0v_2)}
\end{aligned}
\tag{3. 17}
$$

From equations (3. 15) , (3. 16) and (3. 17) we get

$$\frac{3t_3}{s}-\left(\frac{3t_1t_2}{s^2}-\frac{t_1^3}{s^3}\right)$$
$$=3\left(\frac{(v_o^*v_3^*)_{\epsilon^3}}{(v_0v_3)}+\frac{(v_1^*v_2^*)_{\epsilon^3}}{(v_1v_2)}-\frac{(v_0^*v_2^*)_{\epsilon^3}}{(v_0v_2)}-\frac{(v_1^*v_3^*)_{\epsilon^3}}{(v_1v_3)}+\frac{(v_o^*v_2^*)_{\epsilon}}{(v_0v_2)}\frac{(v_o^*v_2^*)_{\epsilon^2}}{(v_0v_2)}+\frac{(v_1^*v_3^*)_{\epsilon}}{(v_1v_3)}\frac{(v_1^*v_3^*)_{\epsilon^2}}{(v_1v_3)}\right.$$
$$\left.-\frac{(v_o^*v_3^*)_{\epsilon}}{(v_0v_3)}\frac{(v_o^*v_3^*)_{\epsilon^2}}{(v_0v_3)}-\frac{(v_1^*v_2^*)_{\epsilon}}{(v_1v_2)}\frac{(v_1^*v_2^*)_{\epsilon^2}}{(v_1v_2)}\right)+\frac{(v_o^*v_3^*)^3_{\epsilon}}{(v_0v_3)^3}+\frac{(v_1^*v_2^*)^3_{\epsilon}}{(v_1v_2)^3}-\frac{(v_0^*v_2^*)^3_{\epsilon}}{(v_0v_2)^3}-\frac{(v_1^*v_3^*)^3_{\epsilon}}{(v_1v_3)^3} \quad (3.\ 18)$$

similarly

$$\frac{3t_3}{1-s}-\left(\frac{3t_1t_2}{(1-s)^2}-\frac{t_1^3}{(1-s)^3}\right)$$
$$=3\left(\frac{(v_0^*v_2^*)_{\varepsilon^3}}{(v_0v_2)}+\frac{(v_1^*v_3^*)_{\varepsilon^3}}{(v_1v_3)}-\frac{(v_0^*v_1^*)_{\varepsilon^3}}{(v_0v_1)}-\frac{(v_2^*v_3^*)_{\varepsilon^3}}{(v_2v_3)}-\frac{(v_0^*v_2^*)_{\varepsilon^2}}{(v_0v_2)}\frac{(v_0^*v_2^*)_{\varepsilon}}{(v_0v_2)}-\frac{(v_1^*v_3^*)_{\varepsilon^2}}{(v_1v_3)}\frac{(v_1^*v_3^*)_{\varepsilon}}{(v_1v_3)}\right.$$
$$\left.+\frac{(v_0^*v_1^*)_{\varepsilon^2}}{(v_0v_1)}\frac{(v_0^*v_1^*)_{\varepsilon}}{(v_0v_1)}+\frac{(v_2^*v_3^*)_{\varepsilon^2}}{(v_2v_3)}\frac{(v_2^*v_3^*)_{\varepsilon}}{(v_2v_3)}\right)+\frac{(v_0^*v_2^*)^3_{\varepsilon}}{(v_0v_2)^3}+\frac{(v_1^*v_3^*)^3_{\varepsilon}}{(v_1v_3)^3}-\frac{(v_0^*v_1^*)^3_{\varepsilon}}{v_0v_1)^3}-\frac{(v_2^*v_3^*)^3_{\varepsilon}}{(v_2v_3)^3} \quad (3.\ 19)$$

Substitute the value of (16) in (12) we obtain the result of $\sigma_1\circ\pi^2_{1,\epsilon^3}(\mathbb{V}^*_{03})$. Similarly we evaluate the value of $\sigma_2\circ\pi^2_{1,\epsilon^3}(\mathbb{V}^*_{03})$. And thus addition of both these we acquire the following conclusion for the map $\partial^2_{\epsilon^3}\circ\pi^2_{1,\epsilon^3}(\mathbb{V}^*_{03})$.

$$\partial^2_{\epsilon^3}\circ\pi^2_{1,\epsilon^3}(\mathbb{V}^*_{03})$$
$$=\left\{3\left(\frac{(v_o^*v_3^*)_{\epsilon^3}}{(v_0v_3)}+\frac{(v_1^*v_2^*)_{\epsilon^3}}{(v_1v_2)}-\frac{(v_0^*v_2^*)_{\epsilon^3}}{(v_0v_2)}-\frac{(v_1^*v_3^*)_{\epsilon^3}}{(v_1v_3)}+\frac{(v_o^*v_2^*)_{\epsilon}}{(v_0v_2)}\frac{(v_o^*v_2^*)_{\epsilon^2}}{(v_0v_2)}+\frac{(v_1^*v_3^*)_{\epsilon}}{(v_1v_3)}\frac{(v_1^*v_3^*)_{\epsilon^2}}{(v_1v_3)}\right.\right.$$
$$\left.\left.-\frac{(v_o^*v_3^*)_{\epsilon}}{(v_0v_3)}\frac{(v_o^*v_3^*)_{\epsilon^2}}{(v_0v_3)}-\frac{(v_1^*v_2^*)_{\epsilon}}{(v_1v_2)}\frac{(v_1^*v_2^*)_{\epsilon^2}}{(v_1v_2)}\right)+\frac{(v_o^*v_3^*)^3_{\epsilon}}{(v_0v_3)^3}+\frac{(v_1^*v_2^*)^3_{\epsilon}}{(v_1v_2)^3}-\frac{(v_0^*v_2^*)^3_{\epsilon}}{(v_0v_2)^3}-\frac{(v_1^*v_3^*)^3_{\epsilon}}{(v_1v_3)^3}\right\}$$
$$\otimes\frac{(v_0,v_1)(v_2,v_3)}{(v_0,v_2)(v_1,v_3)}+\left\{3\left(\frac{(v_0^*v_2^*)_{\varepsilon^3}}{(v_0v_2)}+\frac{(v_1^*v_3^*)_{\varepsilon^3}}{(v_1v_3)}-\frac{(v_0^*v_1^*)_{\varepsilon^3}}{(v_0v_1)}-\frac{(v_2^*v_3^*)_{\varepsilon^3}}{(v_2v_3)}-\frac{(v_0^*v_2^*)_{\varepsilon^2}}{(v_0v_2)}\frac{(v_0^*v_2^*)_{\varepsilon}}{(v_0v_2)}\right.\right.$$
$$\left.-\frac{(v_1^*v_3^*)_{\varepsilon^2}}{(v_1v_3)}\frac{(v_1^*v_3^*)_{\varepsilon}}{(v_1v_3)}+\frac{(v_0^*v_1^*)_{\varepsilon^2}}{(v_0v_1)}\frac{(v_0^*v_1^*)_{\varepsilon}}{(v_0v_1)}+\frac{(v_2^*v_3^*)_{\varepsilon^2}}{(v_2v_3)}\frac{(v_2^*v_3^*)_{\varepsilon}}{(v_2v_3)}\right)+\frac{(v_0^*v_2^*)^3_{\varepsilon}}{(v_0v_2)^3}+\frac{(v_1^*v_3^*)^3_{\varepsilon}}{(v_1v_3)^3}-\frac{(v_0^*v_1^*)^3_{\varepsilon}}{v_0v_1)^3}$$
$$\left.-\frac{(v_2^*v_3^*)^3_{\varepsilon}}{(v_2v_3)^3}\right\}\otimes\frac{(v_0,v_3)(v_1,v_2)}{(v_0,v_2)(v_1,v_3)} \quad (3.\ 20)$$

Next we move to evaluate the other side $\pi^2_{0,\epsilon^3}\circ d(\mathbb{V}^*_{03})$. Since we have

$$\pi^2_{0,\epsilon^3}\circ d(\mathbb{V}^*_{03})=\pi^1\circ d(\mathbb{V}^*_{03})+\pi^2\circ d(\mathbb{V}^*_{03}) \quad (3.\ 21)$$

Applying the definitions of π^1, π^2 and "d" we obtain

$$\pi^1 \circ d(\mathbb{V}^*_{03}) = \widetilde{\text{Alt}}_{(0123)}\left\{ \sum_{i=0}^{2}(-1)^i \left\{ \left(3\frac{(v^*_i, v^*_{i+1})_{\epsilon^3}}{(v_i, v_{i+1})} - 3\frac{(v^*_i, v^*_{i+1})_{\epsilon^2}(v^*_i, v^*_{i+1})_{\epsilon}}{(v_i, v_{i+1})^2} + \frac{{(v^*_i, v^*_{i+1})^3}_{\epsilon}}{(v_i, v_{i+1})^3} \right) \otimes \frac{(v_i, v_{i+2})}{(v_{i+1}, v_{i+2})} \right\} \right\}; \quad i \quad mod \quad 3 \tag{3. 22}$$

Furthermore we use the facts $p \otimes \frac{q}{r} = p \otimes q - p \otimes r$ in the expansion of inner sum. This gives us total 18 terms which can further be classified into the terms like $\frac{(u)_{\varepsilon^3}}{u} \otimes v$, $\frac{(u)_{\varepsilon^2}(u)_{\varepsilon}}{u^2} \otimes v$ and $\frac{(u)^3_{\varepsilon}}{u^3} \otimes v$. Now if we expand through the sum of alternation then total number of terms will be raised up to ninety. After cancellations and simplifications we acquire an expression identical with (18).

□

The above result allows us to conclude the following.

Corollary 3.4. *For the map d' the below chains.*

$$C_4(\mathbb{A}^3_{F[\varepsilon]_4}) \xrightarrow{d'} C_3(\mathbb{A}^2_{F[\varepsilon]_4}) \xrightarrow{\pi^2_{0,\varepsilon^3}} F \otimes F^\times \oplus \bigwedge^2 F$$

and

$$C_5(\mathbb{A}^3_{F[\varepsilon]_4}) \xrightarrow{d'} C_4(\mathbb{A}^2_{F[\varepsilon]_4}) \xrightarrow{\pi^2_{1,\varepsilon^3}} T\mathcal{B}_2(F)$$

are complexes where $d(u^*_0, \ldots, u^*_m) = \sum_{i=0}^{m}(-1)^i(u^*_0, \ldots, \hat{u^*_i}, \ldots, u^*_m)$

Proof. We only have to exhibit $\pi^2_{0,\varepsilon^3} \circ d' = \pi^2_{1,\varepsilon^3} \circ d' = 0$. □

4. Conclusion

Basically our aim was to determine whether there exist maps between Grassmannian complex and third order tangent complex of weight 2 or not and from the work above we can conclude that there exist maps π^2_{0,ε^3} and π^2_{1,ε^3} which connects commutatively both the complexes mentioned earlier. Also this result allow us to do such constructions for higher order tangent complexes and finally we can generalize this notion.

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