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Upper Bounds of Zagreb Connection Indices of Tensor and Strong Product on Graphs

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Abstract. A topological index (TI) is a function from $\sum$ to the set of real numbers, where $\sum$ is the set of finite simple graphs. In fact, it is a final outcome of a logical, systematical and mathematical process that transforms feature encoded in a molecular graph to a fixed real number. Gutman and Trinajstic (1972) first time defined degree based TI named as first Zagreb index to compute the total π-electron energy of a molecular graph. They also exposed another TI that is renamed as modified first Zagreb connection index in $[Ali\ and\ Trinajstic,\ Mol.\ Inform.\ 37(2018),\ 1-7]$. In this paper, we compute the upper bounds for the Zagreb connection indices i.e. first Zagreb connection index, second Zagreb connection index and modified first Zagreb connection index of the resultant graphs which are obtained by applying the tensor and strong product of two graphs.

AMS (MOS) Subject Classification Codes: 05C78.
Key Words: Zagreb indices; Connection number; Tensor product and Strong product.

1. INTRODUCTION

A topological index (TI) is a numerical value that correlates a chemical structure with its several physical properties, chemical reactions and biological experiments. Especially, it plays an important role in the study of quantitative structures activity relationships (QSAR) and quantitative structures property relationships (QSPR). For more detail, see [31]. A molecular graph is a simple graph that represents relation between vetices and edges as equal to atoms and bonds respectively. Recently, molecular graph has become an essential part in the study of chemical graph theory. Different TI's have been presented in the field of graph theory but degree based TI's are more studied than others, see [26, 12]. These are used to find motor octane number, total surface area, heat of formation, boiling point, critical temperature, connectivity and solubility, see [30, 21, 22]. In addition, medical behaviours of the crystallin materials, nano-materials and drugs which are very important for pharmaceutical and chemical industries are also discovered with the help of TI's, see [15, 10].

In 1972, Gutman and Trinajstic [13] first time studied degree based TI named as first Zagreb index to examine the dependence of the total π-electron energy and checked its behaviour on a molecular structure. Later on, Gutman and Ruscic [14] defined another TI named as second Zagreb index. Up till now, many results have been derived on these TI's. For more detail, see [5, 4, 33]. Gutman and Trinajstic [13] also exposed another TI but this descriptor could not gain more attention of the researchers till 2017. Recently, Ali and Trinajstic [2] reinvestigated it and called as modified first Zagreb connection index. They also checked different physico chemical properties of octane isomers and reported that it has more precise correlation coefficient results of octane isomers. For more study, we refer to [23, 8, 27, 9].

The graph operations especially graph products play an important role in the studies of pure and applied mathematics as well as computer science. In particularly, it is well known fact that many chemical graph structures are established from simple graphs via product-related operations on graphs. First of all, Graovac and Pisanski [11] computed different results of Wiener index using product operations on graphs. Moreover, several chemical graphs can be formed using graph operations such as linking computer model is constructed by cartesian product of Hamiltonian paths and cycles in the network, see [38]. The fence and closed fence graph is lexicographic product of P_n & P_2 and C_n & P_2 respectively. The closed fence is also designed by strong product of C_n & K_2. For basic properties and formulation of structures of product on graphs and its applications, see [18, 34, 37, 29, 6, 7, 19, 20, 1, 3, 39, 24, 17, 16, 25].

In this paper, we obtain general results of upper bounds for the Zagreb connection indices i.e. first Zagreb connection index, second Zagreb connection index and modified first Zagreb connection index on the resultant graphs which are obtained by operating tensor and strong product on two simple graphs. The rest of the paper is settled as: Section II represents the preliminary definitions and results. Section III covers the general results of product graphs and Section IV includes the applications & conclusion.

2. Notations and Preliminaries

Let Q be a simple and connected graph. The vertex and edge set of Q is denoted by $V(Q)$ and $E(Q)$ respectively. The degree of vertex b in Q is the number of edges incident on it. In view of Todesehini and Consonni [33], it is defined that $d_Q(b)$ is a simple degree (number of vertices at distance one) and $\tau_Q(b)$ is a connection number (number of vertices at distance two). Now, at the end of this paper, we assume that Q_1 and Q_2 are two connected graphs such that $|V(Q_1)| = n_1$, $|V(Q_2)| = n_2$, $|E(Q_1)| = e_1$ and $|E(Q_2)| = e_2$.

Definition 2.1. For a graph Q, the first Zagreb index $(M_1(Q))$ and second Zagreb index $(M_2(Q))$ are defined as

$M_1(Q) = \sum_{b \in V(Q)} [d_Q(b)]^2 = \sum_{ab \in E(Q)} [d_Q(a) + d_Q(b)]$ and
$M_2(Q) = \sum_{ab \in E(Q)} [d_Q(a) \times d_Q(b)]$.
These degree-based indices are defined by Gutman, Trinajstic, and Ruscic, see [13, 14]. These are frequently used to predict better outcomes in molecular structures such as absolute value of correlation coefficient, acentric factor, molar volume and standard enthalpy of formation. Corresponding to these degree-based TI's, the connection-based TI's are defined in Definition 2.2. For more studies related to connection-based TI's, see [2, 23, 9, 24, 28].

Definition 2.2. For a graph Q, the first Zagreb connection index $(ZC_1(Q))$, second Zagreb connection index $(ZC_2(Q))$ and modified first Zagreb connection index $(ZC_1^*(Q))$ are defined as $ZC_1(Q) = \sum_{b \in V(Q)} [\tau_Q(b)]^2$, $ZC_2(Q) = \sum_{ab \in E(Q)} [\tau_Q(a) \times \tau_Q(b)]$ and $ZC_1^*(Q) = \sum_{b \in V(Q)} d_Q(b)\tau_Q(b) = \sum_{ab \in E(Q)} [\tau_Q(a) + \tau_Q(b)]$.

Definition 2.3. The tensor product or kronecker product $(Q_1 \otimes Q_2)$ of two graphs Q_1 and Q_2 is obtained by taking vertex set and edge set as: $V(Q_1 \otimes Q_2) = V(Q_1) \times V(Q_2)$ and $E(Q_1 \otimes Q_2) = \{(a_1, b_1)(a_2, b_2)$; where $(a_1, b_1), (a_2, b_2) \in V(Q_1) \times V(Q_2)\}$ with conditions,
• $[a_1a_2 \in E(Q_1)] \wedge [b_1b_2 \in E(Q_2)]$. The connection number of a vertex (a, b) of $Q_1 \otimes Q_2$ is defined by $\tau_{Q_1 \otimes Q_2}(a, b) \leq \tau_{Q_1}(a) + d_{Q_1}(a)d_{Q_2}(b) + d_{Q_1}(a)$. For more detail, see Figure 1.

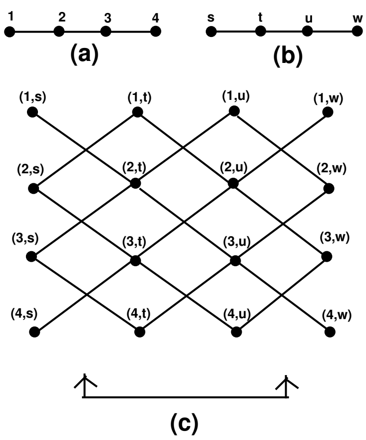

FIGURE 1. (**a**) $Q_1 \cong P_4$ (**b**) $Q_2 \cong P_4$ and (**c**) Tensor Product $(P_4 \otimes P_4)$.

Definition 2.4. The strong product or normal product $(Q_1 \boxtimes Q_2)$ of two graphs Q_1 and Q_2 is obtained by taking vertex set and edge set as: $V(Q_1 \boxtimes Q_2) = V(Q_1) \times V(Q_2)$ and $E(Q_1 \boxtimes Q_2) = \{(a_1, b_1)(a_2, b_2)$; where $(a_1, b_1), (a_2, b_2) \in V(Q_1) \times V(Q_2)\}$ with conditions,
• either $[a_1 = a_2 \in V(G_1) \wedge b_1b_2 \in E(Q_2)]$ or $[b_1 = b_2 \in V(Q_2) \wedge a_1a_2 \in E(Q_1)]$ or $[a_1a_2 \in E(Q_1)] \wedge [b_1b_2 \in E(Q_2)]$. The connection number of a vertex (a, b) of $Q_1 \boxtimes Q_2$ is defined by $\tau_{Q_1 \boxtimes Q_2}(a, b) \leq n_2\tau_{Q_1}(a) + d_{Q_1}(a)d_{Q_2}(b) - \tau_{Q_2}(b) + 1$. For more detail, see Figure 2.

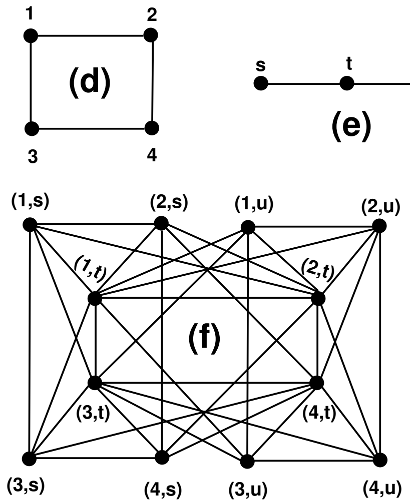

FIGURE 2. **(d)** $Q_1 \cong C_4$ **(e)** $Q_2 \cong P_3$ and **(f)** Strong Product $(C_4 \boxtimes P_3)$.

Lemma 2.1.[35] Let Q be a connected graph. Then,
$\sum\limits_{b\in V(Q)} d_Q(b) = 2e$, where $|E(Q)| = e$.

Lemma 2.2.[36] Let Q be a connected graph with n vertices and e edges. Then $\tau_Q(a) + d_Q(a) \leq \sum\limits_{b\in N_Q(a)} (d_Q(b)$, where equality holds if and only if Q is a $\{C_3, C_4\}-$ free graph.

Lemma 2.3. Let Q be a connected and $\{C_3, C_4\}-$ free graph with n vertices and e edges. Then, $\sum\limits_{a\in V(Q)} \tau_Q(a) = M_1(Q) - 2e$.

Proof. Proof is obvious using Lemma 2.1 and Lemma 2.2.

3. Main Results

This section consists on the main results.

Theorem 3.1. Let Q_1 and Q_2 be two connected graphs. Then, the first Zagreb connection indices of their tensor and strong product are

(a) $ZC_1(Q_1 \otimes Q_2) \leq 2(n_2+2e_2)ZC_1^*(Q_1)+n_2ZC_1(G_1)+(n_2+4e_2)M_1(Q_1)$
$+M_1(Q_1)M_1(Q_2)$,

(b) $ZC_1(Q_1 \boxtimes Q_2) \leq n_2^3 ZC_1(Q_1)+M_1(Q_1)M_1(Q_2)+n_1n_2+n_1ZC_1(Q_2)-2n_1$
$[M_1(Q_2)-2e_2]+4e_2n_2ZC_1^*(Q_1)+8e_1e_2-4e_1ZC_1^*(Q_2)+2n_2^2[M_1(Q_1)-2e_1]$
$-2n_2[M_1(Q_1)-2e_1][M_1(Q_2)-2e_2]$.

Proof (a).

$$ZC_1(Q_1 \otimes Q_2) = \sum_{(a,b)\in V(Q_1\otimes Q_2)} [\tau_{Q_1\otimes Q_2}(a,b)]^2$$

$$\leq \sum_{a\in V(Q_1)} \sum_{b\in V(Q_2)} [\tau_{Q_1}(a)+d_{Q_1}(a)d_{Q_2}(b)+d_{Q_1}(a)]^2$$

$$= \sum_{a\in V(Q_1)} \sum_{b\in V(Q_2)} [\tau^2_{Q_1}(a)+d^2_{Q_1}(a)d^2_{Q_2}(b)+d^2_{Q_1}(a)+2\tau_{Q_1}(a)d_{Q_1}(a)d_{Q_2}(b)$$
$$+2d_{Q_1}(a)d_{Q_2}(b)d_{Q_1}(a)+2d_{Q_1}(a)\tau_{Q_1}(a)]$$
$$= n_2ZC_1(Q_1)+M_1(Q_1)M_1(Q_2)+n_2M_1(Q_1)+4e_2ZC^*_1(Q_1)$$
$$+4e_2M_1(Q_1)+2n_2ZC^*_1(Q_1)$$
$$ZC_1(Q_1\otimes Q_2) \leq 2(n_2+2e_2)ZC^*_1(Q_1)+n_2ZC_1(Q_1)+(n_2+4e_2)M_1(Q_1)$$
$$+M_1(Q_1)M_1(Q_2).$$

(b).

$$ZC_1(Q_1\boxtimes Q_2) = \sum_{(a,b)\in V(Q_1\boxtimes Q_2)} [\tau_{Q_1\boxtimes Q_2}(a,b)]^2$$
$$\leq \sum_{a\in V(Q_1)} \sum_{b\in V(Q_2)} [n_2\tau_{Q_1}(a)+d_{Q_1}(a)d_{Q_2}(b)+\{1-\tau_{Q_2}(b)\}]^2$$
$$= \sum_{a\in V(Q_1)} \sum_{b\in V(Q_2)} [n^2_2\tau^2_{Q_1}(a)+d^2_{Q_1}(a)d^2_{Q_2}(b)+\{1-\tau_{Q_2}(b)\}^2+2n_2\tau_{Q_1}(a)$$
$$d_{Q_1}(a)d_{Q_2}(b)+2d_{Q_1}(a)d_{Q_2}(b)\{1-\tau_{Q_2}(b)\}+2\{1-\tau_{Q_2}(b)\}n_2\tau_{Q_1}(a)]$$
$$= \sum_{a\in V(G_1)} \sum_{b\in V(Q_2)} [n^2_2\tau^2_{Q_1}(a)+d^2_{Q_1}(a)d^2_{Q_2}(b)+1+\tau^2_{Q_2}(b)-2\tau_{Q_2}(b)+2n_2d_{Q_1}(a)$$
$$\tau_{Q_1}(a)d_{Q_2}(b)+2d_{Q_1}(a)d_{Q_2}(b)-2d_{Q_1}(a)d_{Q_2}(b)+2n_2\tau_{Q_1}(a)-2n_2\tau_{Q_1}(a)\tau_{Q_2}(b)]$$
$$ZC_1(Q_1\boxtimes Q_2) \leq n^3_2ZC_1(Q_1)+M_1(Q_1)M_1(Q_2)+n_1n_2+n_1ZC_1(Q_2)-2n_1$$
$$[M_1(Q_2)-2e_2]+4e_2n_2ZC^*_1(Q_1)+8e_1e_2-4e_1ZC^*_1(Q_2)+2n^2_2[M_1(Q_1)-2e_1]$$
$$-2n_2[M_1(Q_1)-2e_1][M_1(Q_2)-2e_2].$$

Theorem 3.2. Let Q_1 and Q_2 be two connected graphs. Then, the second Zagreb connection indices of their tensor and strong product are

(a) $$ZC_2(Q_1\otimes Q_2) \leq 2e_2ZC_2(Q_1)+2\sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)} [d_{Q_1}(a_1)\tau_{Q_1}(a_2)d_{Q_2}(b_1)$$
$$+d_{Q_1}(a_2)\tau_{Q_1}(a_1)d_{Q_2}(b_2)]+2e_2\sum_{a_1a_2\in E(Q_1)} [d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)]$$
$$+2M_2(Q_1)M_2(Q_2)+2M_2(Q_1)M_1(Q_2)+2e_2M_2(Q_1),$$

(b) $$ZC_2(Q_1\boxtimes Q_2) \leq n_2(n_2+2e_2)ZC^*_1(Q_1)+ZC^*_1(Q_2)[e_1(2n_2-1)-n_1-(n_2+1)M_1(Q_1)]$$
$$+n^2_2(n_2+e_2)ZC_2(Q_1)+(n_1+e_1)ZC_2(Q_2)+n^2_2e_2ZC_1(Q_1)+e_1ZC_1(Q_2)+2n_2e_2[M_1(Q_1)-2e_1]$$
$$-2e_1[M_1(Q_2)-2e_2]+M_2(Q_2)[M_1(Q_1)+M_2(Q_1)]+2[e_1M_1(Q_2)+e_2M_1(Q_1)]+M_1(Q_2)M_2(Q_1)$$
$$+n_1e_2+e_1(n_2+e_2)-2e_1\sum_{b_1b_2\in E(Q_2)} [d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)]+2n_2e_2\sum_{a_1a_2\in E(G_1)}$$
$$[d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)]+n_2\sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)} [d_{Q_1}(a_1)\tau_{Q_1}(a_2)d_{Q_2}(b_1)$$
$$+d_{Q_1}(a_2)\tau_{Q_1}(a_1)d_{Q_2}(b_2)]-n_2\sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)} [\tau_{Q_1}(a_1)\tau_{Q_2}(b_2)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_1)]$$
$$-\sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)} [d_{Q_1}(a_1)d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)\tau_{Q_2}(b_1)]$$

$$+\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)].$$

Proof (a).

$$ZC_2(Q_1\otimes Q_2)=\sum_{(a_1,b_1)(a_2,b_2)\in E(Q_1\otimes Q_2)}[\tau_{Q_1\otimes Q_2}(a_1,b_1)\times\tau_{Q_1\otimes Q_2}(a_2,b_2)]$$
$$=\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\otimes Q_2}(a_1,b_1)\times\tau_{Q_1\otimes Q_2}(a_2,b_2)]$$
$$\leq 2\sum_{a_1a_2\in E(G_1)}\sum_{b_1b_2\in E(Q_2)}[\{\tau_{Q_1}(a_1)+d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_1)\}$$
$$\times\{\tau_{Q_1}(a_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)+d_{Q_1}(a_2)\}]$$
$$=2\sum_{a_1a_2\in E(G_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)+\tau_{Q_1}(a_1)d_{Q_1}(a_2)d_{Q_2}(b_2)+\tau_{Q_1}(a_1)d_{Q_1}(a_2)$$
$$+d_{Q_1}(a_1)d_{Q_2}(b_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_1)d_{Q_2}(b_1)d_{Q_1}(a_2)d_{Q_2}(b_2)+d_{Q_1}(a_1)d_{Q_2}(b_1)d_{Q_1}(a_2)$$
$$+d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_1)d_{Q_1}(a_2)d_{Q_2}(b_2)+d_{Q_1}(a_1)d_{Q_1}(a_2)]$$
$$=2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)+\{d_{Q_1}(a_1)\tau_{Q_1}(a_2)d_{Q_2}(b_1)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)d_{Q_2}(b_2)\}$$
$$+\{d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)\}+\{d_{Q_1}(a_1)d_{Q_1}(a_2)\}\{d_{Q_2}(b_1)d_{Q_2}(b_2)\}+d_{Q_1}(a_1)d_{Q_1}(a_2)$$
$$\{d_{Q_2}(b_1)+d_{Q_2}(b_2)\}+d_{Q_1}(a_1)d_{G_1}(a_2)]$$
$$ZC_2(Q_1\otimes Q_2)\leq 2e_2ZC_2(Q_1)+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}\{d_{Q_1}(a_1)\tau_{Q_1}(a_2)d_{Q_2}(b_1)+d_{Q_1}(a_2)$$
$$\tau_{Q_1}(a_1)d_{Q_2}(b_2)\}+2e_2\sum_{a_1a_2\in E(Q_1)}\{d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)\}+2M_2(Q_1)M_2(Q_2)$$
$$+2M_2(Q_1)M_1(Q_2)+2e_2M_2(Q_1).$$

(b).

$$ZC_2(Q_1\boxtimes Q_2)=\sum_{(a_1,b_1)(a_2,b_2)\in E(Q_1\boxtimes Q_2)}[\tau_{Q_1\boxtimes Q_2}(a_1,b_1)\times\tau_{Q_1\boxtimes Q_2}(a_2,b_2)]$$
$$=\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a,b_1)\times\tau_{Q_1\boxtimes Q_2}(a,b_2)]+\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\boxtimes Q_2}(a_1,b)$$
$$\times\tau_{Q_1\boxtimes Q_2}(a_2,b)]+\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a_1,b_1)\times\tau_{Q_1\boxtimes Q_2}(a_2,b_2)].$$

Taking

$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a,b_1)\times\tau_{Q_1\boxtimes Q_2}(a,b_2)]$$
$$\leq\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{n_2\tau_{Q_1}(a)+d_{Q_1}(a)d_{Q_2}(b_1)-\tau_{Q_2}(b_1)+1\}$$
$$\times\{n_2\tau_{Q_1}(a)+d_{Q_1}(a)d_{Q_2}(b_2)-\tau_{Q_2}(b_2)+1\}$$
$$=\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[n_2^2\tau_{Q_1}^2(a)+n_2\tau_{Q_1}(a)d_{Q_1}(a)d_{Q_2}(b_2)-n_2\tau_{Q_1}(a)\tau_{Q_2}(b_2)+n_2\tau_{Q_1}(a)+n_2d_{Q_1}(a)$$
$$d_{Q_2}(b_1)\tau_{Q_1}(a)+d_{Q_1}^2(a)d_{Q_2}(b_1)d_{Q_2}(b_2)-d_{Q_1}(a)d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_1}(a)d_{Q_2}(b_1)-n_2\tau_{Q_2}(b_1)\tau_{Q_1}(a)$$

$-\tau_{Q_2}(b_1)d_{Q_1}(a)d_{Q_2}(b_2)+\tau_{Q_2}(b_1)\tau_{Q_2}(b_2)-\tau_{Q_2}(b_1)+n_2\tau_{Q_1}(a)+d_{Q_1}(a)d_{Q_2}(b_2)-\tau_{Q_2}(b_2)+1]$

$= \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[n_2^2\tau_{Q_1}^2(a)+n_2d_{Q_1}(a)\tau_{Q_1}(a)\{d_{Q_2}(b_1)+d_{Q_2}(b_2)\}-n_2\tau_{Q_1}(a)\{\tau_{Q_2}(b_1)$

$+\tau_{G_2}(b_2)\}+2n_2\tau_{Q_1}(a)+d_{Q_1}^2(a)d_{Q_2}(b_1)d_{Q_2}(b_2)-d_{Q_1}(u)\{d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)\}$

$+d_{Q_1}(a)\{d_{Q_2}(b_1)+d_{Q_2}(b_2)\}+\tau_{Q_2}(b_1)\tau_{Q_2}(b_2)-\{\tau_{Q_2}(b_1)+\tau_{Q_2}(b_2)\}+1]$

$= n_2^2e_2ZC_1(Q_1)+n_2M_1(Q_2)ZC_1^*(Q_1)-n_2[M_1(Q_1)-2e_1]ZC_1^*(Q_2)+2n_2e_2[M_1(Q_1)-2e_1]$

$+M_1(Q_1)M_2(Q_2)-2e_1\sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)]+2e_1M_1(Q_2)$

$+n_1ZC_2(Q_2)-n_1ZC_1^*(Q_2)+n_1e_2.$

Also taking

$\sum_{v\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\boxtimes Q_2}(a_1,b)\times\tau_{Q_1\boxtimes Q_2}(a_2,b)]$

$\leq \sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\{n_2\tau_{Q_1}(a_1)+d_{Q_1}(a_1)d_{Q_2}(b)-\tau_{Q_2}(b)+1\}$

$\times\{n_2\tau_{Q_1}(a_2)+d_{Q_1}(a_2)d_{Q_2}(b)-\tau_{Q_2}(b)+1\}]$

$= \sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[n_2^2\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)+n_2\tau_{Q_1}(a_1)d_{Q_1}(a_2)d_{Q_2}(b)-n_2\tau_{Q_1}(a_1)\tau_{Q_2}(b)$

$+n_2\tau_{Q_1}(a_1)+n_2d_{Q_1}(a_1)d_{Q_2}(b)\tau_{Q_1}(a_2)+d_{Q_1}(a_1)d_{Q_1}(a_2)d_{Q_2}^2(b)-d_{Q_1}(a_1)d_{Q_2}(b)\tau_{Q_2}(b)$

$+d_{Q_1}(a_1)d_{Q_2}(b)-n_2\tau_{Q_2}(b)\tau_{Q_1}(a_2)-\tau_{Q_2}(b)d_{Q_1}(a_2)d_{Q_2}(b)+\tau_{Q_2}^2(b)-\tau_{Q_2}(b)+n_2\tau_{Q_1}(a_2)$

$+d_{Q_1}(a_2)d_{Q_2}(b)-\tau_{Q_2}(b)+1]$

$= \sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[n_2^2\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)+n_2d_{Q_2}(b)\{d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)\}$

$-n_2\tau_{Q_2}(b)\{\tau_{Q_1}(a_1)+\tau_{Q_1}(a_2)\}+n_2\{\tau_{Q_1}(a_1)+\tau_{Q_1}(a_2)\}+d_{Q_1}(a_1)d_{Q_1}(a_2)d_{Q_2}^2(b)$

$-d_{Q_2}(b)\tau_{Q_2}(b)\{d_{Q_1}(a_1)+d_{Q_1}(a_2)\}+d_{Q_2}(b)\{d_{Q_1}(a_1)+d_{Q_1}(a_2)\}+\tau_{Q_2}^2(b)-2\tau_{Q_2}(b)+1]$

$= n_2^3ZC_2(Q_1)+2n_2e_2\sum_{a_1a_2\in E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)]-n_2[M_1(Q_2)-2e_2]$

$ZC_1^*(Q_1)+n_2^2ZC_1^*(Q_1)+2e_2M_1(Q_1)+e_1ZC_1(Q_2)-2e_1[M_1(Q_2)-2e_2]+n_2e_1.$

Again taking

$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a_1,b_1)\times\tau_{Q_1\boxtimes Q_2}(a_2,b_2)]$

$\leq 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{n_2\tau_{Q_1}(a_1)+d_{Q_1}(a_1)d_{Q_2}(b_1)-\tau_{Q_2}(b_1)+1\}$

$\times\{n_2\tau_{Q_1}(a_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)-\tau_{Q_2}(b_2)+1\}]$

$= 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[n_2^2\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)+n_2\tau_{Q_1}(a_1)d_{Q_1}(a_2)d_{Q_2}(b_2)-n_2\tau_{Q_1}(a_1)\tau_{Q_2}(b_2)$

$+n_2\tau_{Q_1}(a_1) + n_2d_{Q_1}(a_1)d_{Q_2}(b_1)\tau_{Q_1}(a_2) + d_{Q_1}(a_1)d_{Q_1}(a_2)d_{Q_2}(b_1)d_{Q_2}(b_2) - d_{Q_1}(a_1)$

$d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_1}(a_1)d_{Q_2}(b_1)-n_2\tau_{Q_2}(b_1)\tau_{Q_1}(a_2)-\tau_{Q_2}(b_1)d_{Q_1}(a_2)d_{Q_2}(b_2)+\tau_{Q_2}(b_1)$

$\tau_{Q_2}(b_2)-\tau_{Q_2}(b_1)+n_2\tau_{Q_1}(a_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)-\tau_{Q_2}(b_2)+1]$

$= 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[n_2^2\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)+n_2\{d_{Q_1}(a_1)\tau_{Q_1}(a_2)d_{Q_2}(b_1)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)$

$d_{Q_2}(b_2)\}-n_2\{\tau_{Q_1}(a_1)\tau_{Q_2}(b_2)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_1)\}+n_2\{\tau_{Q_1}(a_1)+\tau_{Q_1}(a_2)\}+\{d_{Q_1}(a_1)d_{Q_1}(a_2)\}$
$\{d_{Q_2}(b_1)d_{Q_2}(b_2)\}-\{d_{Q_1}(a_1)d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)\tau_{Q_2}(b_1)\}+\{d_{Q_1}(a_1)d_{Q_2}(b_1)$
$+d_{Q_1}(a_2)d_{Q_2}(b_2)\}+\tau_{Q_2}(b_1)\tau_{Q_2}(b_2)-\{\tau_{Q_2}(b_1)+\tau_{Q_2}(b_2)\}+1]$
$= 2n_2^2e_2ZC_2(Q_1)+2n_2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)d_{Q_2}(b_1)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)d_{Q_2}(b_2)]$
$-2n_2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_2)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_1)]+2n_2e_2ZC_1^*(Q_1)+2M_2(Q_1)M_2(Q_2)$
$-2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)\tau_{Q_2}(b_1)]+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}$
$[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)]+2e_1ZC_2(Q_2)-2e_1ZC_1^*(Q_2)+2e_1e_2.$

Consequently,

$ZC_2(Q_1\boxtimes Q_2) \leq n_2(n_2+4e_2)ZC_1^*(Q_1)+ZC_1^*(Q_2)[e_1(2n_2-2)-n_1-(n_2+1)M_1(Q_1)]$
$+n_2^2(n_2+2e_2)ZC_2(Q_1)+(n_1+2e_1)ZC_2(Q_2)+n_2^2e_2ZC_1(Q_1)+e_1ZC_1(Q_2)+2n_2e_2[M_1(Q_1)-2e_1]$
$-2e_1[M_1(Q_2)-2e_2]+M_2(Q_2)[M_1(Q_1)+2M_2(Q_1)]+2[e_1M_1(Q_2)+e_2M_1(Q_1)]+M_1(Q_2)M_2(Q_1)$
$+n_1e_2+e_1(n_2+2e_2)-2e_1\sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)]+2n_2e_2\sum_{a_1a_2\in E(Q_1)}$
$[d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)]+2n_2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}\{d_{Q_1}(a_1)\tau_{Q_1}(a_2)d_{Q_2}(b_1)$
$+d_{Q_1}(a_2)\tau_{Q_1}(a_1)d_{Q_2}(b_2)\}-2n_2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_2)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_1)]$
$-2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)\tau_{Q_2}(b_1)]$
$+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)].$

Theorem 3.3. Let Q_1 and Q_2 be two connected graphs. Then, the modified first Zagreb connection indices of their tensor and strong product are

(a)$ZC_1^*(Q_1\otimes Q_2) \leq 2e_2ZC_1^*(Q_1)+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)]$
$+2e_2M_1(Q_1),$

(b)$ZC_1^*(Q_1\boxtimes Q_2) \leq n_2(n_2+e_2)ZC_1^*(Q_1)-(n_1+e_1)ZC_1^*(Q_2)+2[e_2M_1(Q_1)+e_1M_1(Q_2)]$
$+2n_2e_2[M_1(Q_1)-2e_1]-2e_1[M_1(Q_2)-2e_2]+2n_1e_2+2n_2e_1+2e_1e_2$
$+\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)].$

Proof (a).

$$ZC_1^*(Q_1\otimes Q_2) = \sum_{(a_1,b_1)(a_2,b_2)\in E(G_1\otimes Q_2)}[\tau_{Q_1\otimes Q_2}(a_1,b_1)+\tau_{Q_1\otimes Q_2}(a_2,b_2)]$$
$$= \sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\otimes Q_2}(a_1,b_1)+\tau_{Q_1\otimes Q_2}(a_2,b_2)]$$

$$\leq 2\sum_{a_1a_2\in E(G_1)}\sum_{b_1b_2\in E(Q_2)}[\{\tau_{Q_1}(a_1)+d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_1)\}$$
$$+\{\tau_{Q_1}(a_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)+d_{Q_1}(a_2)\}]$$
$$=2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{\tau_{Q_1}(a_1)+\tau_{Q_1}(a_2)\}+\{d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)\}$$
$$+\{d_{Q_1}(a_1)+d_{Q_1}(a_2)\}]$$
$$ZC_1^*(Q_1\otimes Q_2)\leq 2e_2ZC_1^*(Q_1)+2\sum_{u_1u_2\in E(Q_1)}\sum_{v_1v_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)]$$
$$+2e_2M_1(Q_1).$$

(b).

$$ZC_1^*(Q_1\boxtimes Q_2)=\sum_{(a_1,b_1)(a_2,b_2)\in E(Q_1\boxtimes Q_2)}[\tau_{Q_1\boxtimes Q_2}(a_1,b_1)+\tau_{Q_1\boxtimes Q_2}(a_2,b_2)]$$
$$=\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a,b_1)+\tau_{Q_1\boxtimes Q_2}(a,b_2)]+\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\boxtimes Q_2}(a_1,b)$$
$$+\tau_{Q_1\boxtimes Q_2}(a_2,b)]+\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a_1,b_1)+\tau_{Q_1\boxtimes Q_2}(a_2,b_2)].$$

Taking

$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a,b_1)+\tau_{Q_1\boxtimes Q_2}(a,b_2)]$$
$$\leq\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{n_2\tau_{Q_1}(a)+d_{Q_1}(a)d_{Q_2}(b_1)-\tau_{Q_2}(b_1)+1\}$$
$$+\{n_2\tau_{Q_1}(a)+d_{Q_1}(a)d_{Q_2}(b_2)-\tau_{Q_2}(b_2)+1\}$$
$$=\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[2n_2\tau_{Q_1}(a)+d_{Q_1}(a)\{d_{Q_2}(b_1)+d_{Q_2}(b_2)\}-\{\tau_{Q_2}(b_1)+\tau_{Q_2}(b_2)\}+2]$$
$$=2n_2e_2[M_1(Q_1)-2e_1]+2e_1M_1(Q_2)-n_1ZC_1^*(Q_2)+2e_2n_1.$$

Also taking

$$\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\boxtimes Q_2}(a_1,b)+\tau_{Q_1\boxtimes Q_2}(a_2,b)]$$
$$\leq\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\{n_2\tau_{Q_1}(a_1)+d_{Q_1}(a_1)d_{Q_2}(b)-\tau_{Q_2}(b)+1\}$$
$$+\{n_2\tau_{Q_1}(a_2)+d_{Q_1}(a_2)d_{Q_2}(b)-\tau_{Q_2}(b)+1\}]$$
$$=\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[n_2\{\tau_{Q_1}(a_1)+\tau_{Q_1}(a_2)\}+d_{Q_2}(b)\{d_{Q_1}(a_1)+d_{Q_1}(a_2)\}-2\tau_{Q_2}(b)+2]$$
$$=n_2^2ZC_1^*(Q_1)+2e_2M_1(Q_1)-2e_1[M_1(Q_2)-2e_2]+2n_2e_1.$$

Again taking

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\boxtimes Q_2}(a_1,b_1)+\tau_{Q_1\boxtimes Q_2}(a_2,b_2)]$$
$$\leq 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{n_2\tau_{Q_1}(a_1)+d_{Q_1}(a_1)d_{Q_2}(b_1)-\tau_{Q_2}(b_1)+1\}$$

$+\{n_2\tau_{Q_1}(a_2)+d_{Q_1}(a_2)d_{Q_2}(b_2)-\tau_{Q_2}(b_2)+1\}]$

$= 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[n_2\{\tau_{Q_1}(a_1)+\tau_{Q_1}(a_2)\}+\{d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)\}$

$-\{\tau_{Q_2}(v_1)+\tau_{Q_2}(v_2)\}+2]$

$= 2n_2e_2ZC_1^*(Q_1)+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)]$

$-2e_1ZC_1^*(Q_2)+4e_1e_2.$

Consequently,

$ZC_1^*(Q_1\boxtimes Q_2) \leq n_2(n_2+2e_2)ZC_1^*(Q_1)-(n_1+2e_1)ZC_1^*(Q_2)+2[e_2M_1(Q_1)+e_1M_1(Q_2)]$

$+2n_2e_2[M_1(Q_1)-2e_1]-2e_1[M_1(Q_2)-2e_2]+2n_1e_2+2n_2e_1+4e_1e_2$

$+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_1)+d_{Q_1}(a_2)d_{Q_2}(b_2)].$

4. Applications and Conclusion

We can compute relation between exact value and computed value for the application of these product related operations. Assume that $Q_1 \cong P_3$ & $Q_2 \cong P_3$ be two particular alkanes called by paths. Then, verified main results of the Zagreb connection indices of tensor and strong product on graphs as

1. Tensor Product:

- Exact value of $ZC_1(P_3 \otimes P_3) = 40$,
- Computed value of upper bound $ZC_1(P_3 \otimes P_3) \leq 136$,

- Exact value of $ZC_2(P_3 \otimes P_3) = 4$,
- Computed value of upper bound $ZC_2(P_3 \otimes P_3) \leq 128$,

- Exact value of $ZC_1^*(P_3 \otimes P_3) = 20$,
- Computed value of upper bound $ZC_1^*(P_3 \otimes P_3) \leq 52$.

2. Strong Product:

- Exact value of $ZC_1(P_3 \boxtimes P_3) = 136$,
- Computed value of upper bound $ZC_1(P_3 \boxtimes P_3) \leq 169$,

- Exact value of $ZC_2(P_3 \boxtimes P_3) = 156$,
- Computed value of upper bound $ZC_2(P_3 \boxtimes P_3) \leq 264$,

- Exact value of $ZC_1^*(P_3 \boxtimes P_3) = 112$,
- Computed value of upper bound $ZC_1^*(P_3 \boxtimes P_3) \leq 152$.

In this paper, we have computed the general results in shape of upper bounds related to first Zagreb connection index, second Zagreb connection index and modified first Zagreb connection index of the resultant graphs which are obtained by applying two graphs product i.e. tensor product and strong product on simple graphs. For different resultant graphs obtained by various graphs operations, the problem is still open to compute Zagreb connection indices in their general forms.

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