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Computation of the Values for the Riemann-Liouville Fractional Derivative of the Generalized Polylogarithm Functions

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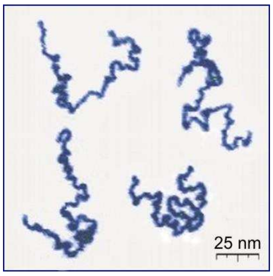
Abstract. In this article, we compute tables of values for the Riemann-Liouville fractional derivative of the generalized polylogarithm functions considering parameter values $\mu = 3; 4; 5$ and $s = \frac{1}{2}; \frac{3}{2}; \frac{-1}{2}; \frac{-3}{2}$. Several authors investigated such functions and their analytic properties, but no work can be found in the literature for the computation of their values. We perform numerical computations to evaluate Riemann-Liouville fractional derivative of the generalized polylogarithm functions for different values of the involved parameters. We validate the data obtained by using our new mathematical model (given in the form of a difference equation) and the known classical integral representations for $\mu = 3; 4; 5$ and $s = \frac{1}{2}; \frac{3}{2}$. It is worth mentioning that for the positive values of parameter $s = \frac{1}{2}; \frac{3}{2}$, our calculations are consistent with the directly computed results by using their integral representation and 100% accuracy is achieved. Furthermore, it is obvious that the involved integrals $\int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-ze^{-t})^3}; \int_0^\infty \frac{t^{s-1}e^{-4t}}{(1-ze^{-t})^4}; \int_0^\infty \frac{t^{s-1}e^{-5t}}{(1-ze^{-t})^5}$; are not convergent for the negative values of parameter s and in this investigation we evaluate these integrals for the negative values of s.

AMS (MOS) Subject Classification Codes: 33E20; 65Axx; 65Qxx
Key Words: fractional derivative; analytic number theory; polylogarithm functions; recurrence relations; integral representations; Mathematica; generalized HurwitzLerch zeta functions.

1. Introduction

1.1. *Motivation*

The study of Polymers plays a fundamental role in modern Sciences like Polymer Nanotechnology; Polymer Physics; Polymer Chemistry and Biology. More specifically, the natural length scales of polymer chains that lie in the nanometer domain, make polymers perfect building blocks for nanotechnology. A number of recent developments in the use of polymers for the fabrication of nanostructures by self-assembling strategies are discussed in [18].



Therefore, Polylogarithm functions were first known to C. Truesdell when Mr. H. Jacobson informed him that these function play an important role in his researches on the theory of structure of polymers [7]. Then Truesdell [35] studied its different properties and representations to compute its values for $s = -1/2; 1/2; 3/2$. Recently, it is a well-known mathematical function that can be obtained as a special case of Hurwitz-Lerch zeta functions. It is related with the important functions of Analytic Number Theory and Quantum Physics. By taking motivation from all these facts we compute the values of Riemann-Liouville fractional derivative of the generalized polylogarithm functions by using computational software Mathematica. We consider the same values of the parameter z as were considered by Truesdell. To achieve the purpose, we use the following recently obtained difference equations involving the generalized Hurwitz-Lerch zeta functions [29].

$$2!z^2\Phi_3^*(z,s,a+2) = \Phi(z,s-2,a) - (2a+1)\Phi(z,s-1,a) + a(a+1)\Phi(z,s,a) \tag{1. 1}$$

$$\begin{aligned} 3!z^3\Phi_4^*(z,s,a+3) = \Phi(z,s-3,a) &- 3(a+1)\Phi(z,s-2,a) \\ &+ (3a^2+6a+2)\Phi(z,s-1,a) \\ &+ a(a+1)(a+2)\Phi(z,s,a) \end{aligned} \tag{1. 2}$$

$$\begin{aligned} 4!z^4\Phi_5^*(z,s,a+4) = \Phi(z,s-4,a) &- 2(2a+3)\Phi(z,s-3,a) \\ &+ (6a^2+18a+11)\Phi(z,s-2,a) \\ &- (4a^3+18a^2+22a+6)(a+2)\Phi(z,s-1,a) \\ &+ a(a+1)(a+2)(a+3)\Phi(x,s,a) \end{aligned} \tag{1. 3}$$

therefore, before going on our research results, we review the literature to present some preliminaries and basic definitions that are necessary to understand the details of this research.

1.2. *Preliminaries and basic definitions*

The polylogarithm function [35] is defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}; s \in \mathbb{C} \qquad (|z| < 1; \Re(s) > 1, |z| = 1) \tag{1. 4}$$

It generalizes the Riemann zeta function [4, p.32] as we have

$$Li_s(1) = \Phi(1,s) = \zeta(s) \qquad (\Re(s) > 1) \tag{1. 5}$$

and can also be represented as an integral

$$Li_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{5-1}}{e^t - z}\, dt \qquad (s \in \mathbb{C} \text{ when } |z| < 1 \text{ and when } |z| = 1) \tag{1. 6}$$

Here, it is important to mention that the polylogarithm function is also related with the important functions of Quantum Statistics namely, Bose-Einstein $B_{s-1}(x)$ and the Fermi-Dirac functions $F_{s-1}(x)$. This relation is given by [22, Equation (1.14-1.15)].

$$Li_s(e^x) = B_{s-1}; -Li_s(-e^x) = F_{s-1}(x) \tag{1. 7}$$

One important representation of this function namely Lindelfs representation is given by [35, p. 149(13)],

$$\begin{gathered} Li_s(z,s) = \Gamma(1-s)(\log z)^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n) \frac{\log z)^n}{n!}, \\ (|\log z| < 2\pi, s \neq 1, 2, 3, ...). \end{gathered} \tag{1. 8}$$

Hurwitz-Lerch zeta function [4, p. 27] as a generalization of the polylogarithm is given by

$$\begin{gathered} \Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \\ (a \in \mathbb{C} \setminus \mathbb{Z}^0; s \in c \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1) \end{gathered} \tag{1. 9}$$

It has a meromorPhic extension to the whole complex s-plane while it has a simple singularity at s = 1 of residue 1. It is also represented by [4, p. 27(1.6)(3)]

$$\begin{gathered} \Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1-ze^-t}\,dt \\ (|z| < 1 \Longrightarrow \Re(s) > 0; \Re(a) > 0; z = 1 \Longrightarrow \Re(s) > 1) \end{gathered} \tag{1. 10}$$

Apart from other applications, the Hurwitz Lerch zeta function or the generalized polylogarithm function is the most general function in the original zeta family. For example, for different values of involved parameters in equations (9)-(10) yield the following relationships with the polylogarithm, Hurwitz and Riemann functions respectively:

$$\begin{aligned} Li_s(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(s,z,1), \\ \zeta(s,a) &= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(s,1,a), \\ \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{(n+a)^s} = \Phi(s,1,1) = \zeta(s,1) \end{aligned} \tag{1. 11}$$

Similar to as for the polylogarithm function, the Hurwitz-Lerch zeta function also has a series representation [4, pp.28–29]

$$\begin{gathered} \Phi(s,z,\alpha) = \frac{\Gamma(1-s)}{z^{\alpha}} (\log \frac{1}{2})^{\alpha} + z^{-\alpha} \sum_{n=0}^{\infty} \zeta(s-n,\alpha) \frac{(\log z)^n}{n!}, \\ (|\log z| < 2\pi, s \neq 1, 2, 3, ..., \alpha \neq 0, -1, -2, ...,), \end{gathered} \tag{1. 12}$$

which generalizes Lindelfs representation (1. 8). As shown already by Lin and Srivastava, [11] the following generalized definition is simply the Riemann-Liouville fractional derivative of the Hurwitz-Lerch zeta function itself. This is of special interest for this investigation and is given by

$$\Phi_{\mu}^{*}(z,s,a)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{at}}{(1-ze^{-t})^{\mu}}dt \tag{1. 13}$$
$$(\Re(a)>0,\Re(s)>0 \text{ when } |z|\leq 1 (z\neq 1); \Re(s-\mu)>0 \text{ when } z=1)$$

and its series representation is given by

$$\Phi_{\mu}^{*}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(a+n)^s}\frac{z^n}{n!} \tag{1. 14}$$

For $\mu=1$, equations (1. 13) and (1. 14) reduce to the original Hurwitz-Lerch zeta function (1. 9)-(1. 10).

More recently Srivastava et. al [19] have used Riemann-Liouville fractional derivative to establish some new fractional-calculus connections between MittagLeffler functions. Fractional calculus has become vital to model the real-world problems. The use of fractional derivatives and integrals have reshaped the scientific research [13, 28, 37]. Several different definitions of the fractional deriv-atives have been used and developed to study different mathematical problems and applications [6, 9, 16, 36]. Ji-Huan He [6] beautifully describes fractal calculus and its geometrical explanation by introducing new space variables. Of our interest here is the one of the basic definitions namely the Riemann-Liouville fractional derivative operator D_z^{μ} defined by (see, for example, [10, p. 181], [17, p. 70] and [11])

$$D_z^{\mu}\{f(z)\}=\begin{cases}\frac{1}{\Gamma(-\mu)}\int_0^z (z-t)^{\mu-1}f(t)\,dt & \Re(\mu)>0\\ \frac{d^m}{dz^m}\{D_z^{\mu-m}f(z)\} & (m-1\leq\Re(\mu)<m(m\in N))\end{cases} \tag{1. 15}$$

Therefore, it is significant to study the generalized polylogarithm function in the light of the following interesting (and useful) relation

$$\Phi_{\mu}^{*}(z,s,a)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-at}}{(1-ze^{-t})^{\mu}}\,dt=\frac{1}{\Gamma(\mu)}D_z^{\mu-1}\{z^{\mu-1}\Phi(z,s,a)\};\Re(\mu)>0 \tag{1. 16}$$

which (as previously observed by Lin and Srivastava [11, p.730]) reveals that the function $\Phi_{\mu}^{*}(z,s,a)$ is basically a Riemann-Liouville fractional derivative of the generalized polylogarithm function $\Phi(z,s,a)$ (some other closely-related researches by Garg et al. [5] and Lin et al. [12] can be studied for further details).

Bayad and Chikhi [1], Srivastava and Choi [20], Choi and Srivastava [3], Srivastava [2] Nakamura [15] and Kanemitsu et al. [8] studied different properties and applications of the generalized polylogarithm functions (Hurwitz-Lerch zeta function). Srivastava [23], has established some formulas for the Bernoulli and Euler polynomials at rational arguments in view of their relationship with the generalized polylogarithm functions. Srivastava et. al [24] have also studied integral and computational representations of the extended Hurwitz-Lerch Zeta function. Several different generalizations of the generalized Hurwitz-Lerch zeta function (14) can be found in the literature, for example, Srivastava discusses almost all these generalizations and works in his article [25]. Choi and Parmer [2] further generalized these functions by introducing one more parameter. For more such discussions the interested reader is referred to author work [29] and [32].

1.3. *Significance and Objectives*

From the above discussion, we can notice that several authors presented and studied worthwhile generalizations of the HurwitzLerch zeta functions. Various analytic formulae, integral, and series representations of these functions are known in the literature. However, as we deeply study the basic zeta and polylogarithm functions, we know their values, their graphs, and several other basic aspects. To the best of our knowledge no such work has been reported for the computation of the values of the Riemann-Liouville fractional derivative of the generalized polylogarithm functions. Computation of values performed in this investigation are worthwhile to evaluate these functions accurately that are consistent with the results obtained by known integral representation. The objective of our present investigation is achieved by focusing on the above equation (1. 16). Our results are presented in the next Section 2 as follows.

- We compute Table 2 of the values for the Riemann-Liouville fractional derivative of the generalized polylogarithm functions by considering parameter values $\mu = 3$ and $s = \frac{1}{2}; \frac{3}{2}; \frac{-1}{2}; \frac{-3}{2}$ in Subsection 2.1.
- We compute Table 3 of values for the Riemann-Liouville fractional derivative of the generalized polylogarithm functions by considering parameter values $\mu = 4$ and $s = \frac{1}{2}; \frac{3}{2}; \frac{-1}{2}; \frac{-3}{2}$ in Subsection 2.2.
- We compute Table 4 of values for the Riemann-Liouville fractional derivative of the generalized polylogarithm functions by considering parameter values $\mu = 4$ and $s = \frac{1}{2}; \frac{3}{2}; \frac{-1}{2}; \frac{-3}{2}$ in Subsection 2.3.
- In these Tables namely Table 2; Table 3 and Table 4, the values obtained for $s = \frac{-1}{2}$ are negative due to the factor $\Gamma(\frac{-1}{2})$ occurring in the expression used to compute the values. In each case the magnitude of the computed values is increasing in relation with the increasing value of the parameter z. In the limiting case $z \to 1$,this increase in the values of these functions becomes suddenly more prominent.
- The obtained data is confirmed consistent by using our new mathematical model and the known classical integral representations for $\mu = 3; 4; 5$ and $s = \frac{1}{2}; \frac{3}{2}$.
- It is remarkable that for the positive values of parameter $s = \frac{1}{2}; \frac{3}{2}$, our calculations are consistent with the directly computed results by using their integral representation and 100% accuracy is achieved.
- Moreover, from equation (1. 16), it is obvious that the involved integrals $\int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-ze^{-t})^3}\,dt$; $\int_0^\infty \frac{t^{s-1}e^{-4t}}{(1-ze^{-t})^4}\,dt$ and $\int_0^\infty \frac{t^{s-1}e^{-5t}}{(1-ze^{-t})^5}\,dt$ are not convergent for the negative values of parameter s and in this research, we evaluate these integrals for these negative values.

2. RESULTS

2.1. *Computation of the values of Riemann-Liouville fractional derivative of the generalized polylogarithm functions; $\mu = 3$*

Taking $a = 1$ in eqution (1), we get the following mathematical model to compute the values of Riemann-Liouville fractional derivative of the generalized polylogarithm functions

$$\int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-ze^{-t})^3}\,dt = \frac{\Gamma(s)}{1.2.z^3}\ \ Li_{s-2}(z) - 3Li_{s-1}(z) + 2Li_s(z) \tag{2. 17}$$

Mathematical Language :(Gamma[s] (PolyLog[-2 + s, z] - 3 PolyLog[-1 + s, z] + 2 PolyLog[s, z]))/(2 z ˆ3)=Integrate[$t\hat{}(s-1)(1/(E\hat{}(3t)(1-z/E\hat{}t)\hat{}3))$, t, 0, ∞]

TABLE 1. Computation of $\int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-ze^{-t})^3}\,dt$

s	Direct Evaluation		Using difference Equation (2. 17)			
z	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{-1}{2}$	$s=\frac{-3}{2}$
0.05	1.16912	0.18844	1.16912	0.18844	-7.33414	15.5601
0.10	1.34515	0.209228	1.34515	0.209228	-8.84585	19.9634
0.15	1.55988	0.233579	1.55988	0.233579	-10.7850	25.9708
0.20	1.82488	0.262356	1.82488	0.262356	-13.3094	34.3159
0.25	2.1562	0.296701	2.1562	0.296701	-16.6509	46.1467
0.30	2.57661	0.338149	2.57661	0.338149	-21.1580	63.3107
0.35	3.11914	0.388809	3.11914	0.388809	-27.3702	88.8786
0.40	3.83304	0.451641	3.83304	0.451641	-36.1484	128.145
0.45	4.79405	0.530908	4.79405	0.530908	-48.9194	190.640
0.50	6.12302	0.632935	6.12302	0.632935	-68.1515	294.395
0.55	8.02127	0.767454	8.02127	0.767454	-98.3405	475.638
0.60	10.8423	0.950096	10.8423	0.950096	-148.203	812.620
0.65	15.2484	1.20734	15.2484	1.20734	-235.997	1490.26
0.70	22.587	1.58713	22.587	1.58713	-403.907	2998.64
0.75	35.9122	2.18438	35.9122	2.18438	-762.886	6849.01
0.80	63.2609	3.21085	63.2609	3.21085	-1662.15	18797.5
0.85	131.017	5.2308	131.017	5.2308	-4539	68974.6
0.90	364.484	10.2551	364.484	10.2551	-18718.1	429985
0.95	2083.23	31.3818	2083.23	31.3818	-211264	$9.78208*10^6$
0.96	2083.23	44.6709	2083.23	44.6709	-461105	$2.67297*10^7$
0.97	7503.87	70.1447	7503.87	70.1447	$-1.26146*10^6$	$9.76527*10^7$
0.98	20725.9	131.66	20725.9	131.66	$-5.21168*10^6$	$6.06124*10^8$
0.99	117521	381.439	117521	381.439	$-5.89342*10^7$	$1.37297*10^{10}$
0.995	665609	1093.54	665609	1093.54	$-6.66599*10^8$	$3.10835*10^{11}$
0.996	$1.16306*10^6$	1532.69	$1.16306*10^6$	1532.69	$-1.45555*10^9$	$8.48539*10^{11}$
0.997	$2.38811*10^6$	2366.76	$2.38811*10^6$	2366.76	$-3.98375*10^9$	$3.09701*10^{12}$
0.998	$6.58249*10^6$	4361.41	$6.58249*10^6$	4361.41	$-1.64661*10^{10}$	$1.92044*10^{13}$
0.9999	$1.17807*10^{10}$	392556	$1.17807*10^{10}$	392556	$-1.86283*10^{11}$	$1.37443*10^{19}$

2.2. *Computation of the values of Riemann-Liouville fractional derivative of the generalized polylogarithm functions;* $\mu=4$

Taking $a=1$ in eqution (1), we get the following mathematical model to compute the values of Riemann-Liouville fractional derivative of the generalized polylogarithm functions

$$\int_0^\infty \frac{t^{s-1}e^{-4t}}{(1-ze^{-t})^4}\,dt = \frac{\Gamma(s)}{1.2.3.z^4}\ \ Li_{s-3}(z) - 6Li_{s-2}(z) + 11Li_{s-1} + 6Li_s(z) \tag{2. 18}$$

Mathematica Language :(Gamma[s] (PolyLog[-3 + s, z] -6PolyLog[-2 + s, z] +11 PolyLog[-1 + s, z] -6 PolyLog[s, z]))/(3! z ˆ4) = Integrate[t ˆ(s - 1) (1/(E ˆ(4 t) (1 - z/E ˆ t)ˆ 4)), t, 0, ∞]

TABLE 2. Computation of $\int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-ze^{-t})^4}\,dt$

s	Direct Evaluation		Using difference Equation (2. 18)			
z	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{-1}{2}$	$s=\frac{-3}{2}$
0.05	1.06467	0.128268	1.06467	0.128268	-8.91807	25.1813
0.10	1.29163	0.149630	1.29163	0.149630	-11.3585	34.0541
0.15	1.58415	0.176002	1.58415	0.176002	-14.6693	46.8421
0.20	1.96679	0.208949	1.96679	0.208949	-19.2426	65.6712
0.25	2.47578	0.250665	2.47578	0.250665	-25.6898	94.0708
0.30	3.16576	0.304288	3.16576	0.304288	-34.9911	138.091
0.35	4.12159	0.374422	4.12159	0.374422	-48.7689	208.491
0.40	5.47925	0.467996	5.47925	0.467996	-69.8105	325.220
0.45	7.46487	0.595779	7.46487	0.595779	-103.112	527.115
0.50	10.4711	0.775137	10.4711	0.775137	-158.092	894.227
0.55	15.2160	1.03532..	15.2160	1.03532..	-253.597	1603.21
0.60	23.0971	1.42826	23.0971	1.42826	-430.177	3077.49
0.65	37.0530	2.05242	37.0530	2.05242	-783.285	6441.89
0.70	63.9012	3.11053	63.9012	3.11053	-1564.87	15103.4
0.75	121.647	5.06797	121.647	5.06797	-3548.83	41344.5
0.80	267.205	9.1658	267.205	9.1658	-9670.74	141665
0.85	735.863	19.5357	735.863	19.5357	-35233.3	692244
0.90	3061.34	56.1024	3061.34	56.1024	-218083	$6.46527*10^6$
0.95	34871	332.445	34871	332.445	$-4.92608*10^6$	$2.93813*10^8$
0.96	76253.7	586.669	76253.7	586.669	$-1.34414*10^7$	$1.00332*10^9$
0.97	209009	1217.01	209009	1217.01	$-4.9036*10^7$	$4.88613*10^9$
0.98	865191	3390.47	865191	3390.47	$-3.03929*10^8$	$4.54810*10^10$
0.99	$9.80288*10^6$	19399.5	$9.80288*10^6$	19399.5	$-6.87468*10^9$	$2.05995*10^{12}$
0.995	110989000	110393	110989000	110393	$-1.55529*10^{11}$	$9.32617*10^{13}$
0.996	$2.42399*10^8$	193082	$2.42399*10^8$	193082	$-4.24512*10^{11}$	$3.18232*10^{14}$
0.997	$6.63561*10^8$	396842	$6.63561*10^8$	396842	$-1.54917*10^{12}$	$1.54862*10^{15}*$
0.998	$2.74325*10^9$	$1.09491*10^6$	$2.74325*10^9$	$1.09491*10^6$	$-9.26312*10^{12}$	$1.4404*10^{16}$
0.9999	$9.81733*10^{13}$	$1.96325*10^9$	$9.81733*10^{13}$	$1.96325*10^9$	$-6.87226*10^{18}$	$2.06165*10^{23}$

2.3. *Computation of the values of Riemann-Liouville fractional derivative of the generalized polylogarithm functions;* $\mu = 5$

Taking $a = 1$ in eqution (1), we get the following mathematical model to compute the values of Riemann-Liouville fractional derivative of the generalized polylogarithm functions

$$\int_0^\infty \frac{t^{s-1}e^{-5t}}{(1-ze^{-t})^5}\,dt = \frac{\Gamma(s)}{1.2.3.4.z^5}\;\; Li_{s-4}(z) - 10Li_{s-3}(z) + 35Li_{s-2}(z) - 50Li_{s-1} + 24Li_s(z) \tag{2. 19}$$

Mathematica Language :(Gamma[s] (PolyLog[-4 + s, z] -10PolyLog[-3 + s, z] +35PolyLog[-2 + s, z] -50 PolyLog[-1 + s, z] +24 PolyLog[s, z]))/(4! z ˆ 5)=Integrate[t ˆ (s - 1) (1/(Eˆ (5 t) (1 - z/E ˆ t)ˆ 5)), t, 0, ∞]

TABLE 3. Computation of $\int_0^\infty \frac{t^{s-1}e^{-5t}}{(1-ze^{-t})^5}\, dt$

s	Direct Evaluation		Using difference Equation (2. 19)			
z	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{1}{2}$	$s=\frac{3}{2}$	$s=\frac{-1}{2}$	$s=\frac{-3}{2}$
0.05	1.00171	0.0963229	1.00171	0.0963229	-10.4982	37.0118
0.10	1.28186	0.118236	1.28186	0.118236	-14.1175	52.7883
0.15	1.66344	0.146773	1.66344	0.146773	-19.3101	76.8167
0.20	2.19266	0.184499	2.19266	0.184499	-26.9206	114.328
0.25	2.9418	0.23523	2.9418	0.23523	-38.3469	174.540
0.30	4.02708	0.304772	4.02708	0.304772	-55.9772	274.287
0.35	5.6415	0.40222	5.6415	0.40222	-84.0436	445.605
0.40	8.11765	0.542277	8.11765	0.542277	-130.368	752.389
0.45	12.0537	0.74962	12.0537	0.74962	-210.124	1329.24
0.50	18.5808	1.06751	18.5808	1.06751	-354.486	2478.47
0.55	29.9702	1.57579	29.9702	1.57579	-632.011	4933.24
0.60	51.1251	2.43147	51.1251	2.43147	-1206.47	10644.9
0.65	93.6271	3.96801	93.6271	3.96801	-2511.42	25445.2
0.70	188.154	6.96732	188.154	6.96732	-5855.57	69545.6
0.75	429.275	13.5172	429.275	13.5172	-15940.5.	$2.28272*10^5$
0.80	1177.05	30.2935	1177.05	30.2935	-54317.1	$9.76943*10^5$
0.85	4315.64	85.2319	4315.64	85.2319	-263951	$6.36015*10^6$
0.90	26887.8	362.85	26887.8	362.85	$-2.45155*10^6$	$8.90332*10^7$
0.95	611462	4238.35	611462	4238.35	$-1.10794*10^8$	$8.08601*10^9$
0.96	$1.67075*10^6$	9317.75	$1.67075*10^6$	9317.75	$-3.77920*10^8$	$3.45101*10^{10}$
0.97	$6.1036*10^6$	25679.5	$6.1036*10^6$	25679.5	$-1.83842*10^9$	$2.24050*10^{11}$
0.98	$3.78836*10^7$	106896	$3.78836*10^7$	106896	-1.7093310^{10}	$3.12777*10^{12}$
0.99	$3.78836*10^7$	$1.21814*10^6$	$3.78836*10^7$	$1.21814*10^6$	$-7.73341*10^{11}$	$2.83286*10^{14}$
0.995	$1.94272*10^{10}$	$1.38324*10^7$	$1.94272*10^{10}$	$1.38324*10^7$	$-3.49926*10^{13}$	$2.56489*10^{16}$
0.996	$5.30337*10^{10}$	$3.02277*10^7$	$5.30337*10^{10}$	$3.02277*10^7$	$-1.19390*10^{14}$	$1.09399*10^{17}$
0.997	$1.93563*10^{11}$	$8.27966*10^7$	$1.93563*10^{11}$	$8.27966*10^7$	$-5.80925*10^{14}$	$7.09815*10^{17}$
0.998	$1.20027*10^{12}$	$3.42496*10^8$	$1.20027*10^{12}$	$3.42496*10^8$	$-5.40268*10^15$	$9.90302*10^{18}$
0.9999	$8.5902*10^{17}$	$1.22709*10^{13}$	$8.5902*10^{17}$	$1.22709*10^{13}$	$-7.73128*10^{22}$	$2.83478*10^{27}$

3. FUTURE DIRECTIONS

Approximation of the values of special functions have always been remained an important aspect for the analysis of special functions by using different representations. In this study, we performed computational analysis for the Riemann-Liouville fractional derivative of the generalized polylogarithm functions by using newly established difference equations. This analysis proved valuable to compute the values of the these functions. The outcomes were also confirmed by using two different approaches for the positive values of s. By following the method, we can obtain significant new results by considering the further specific values of the involved parameters. This is useful for the further analysis of these functions by plotting the graphs and deriving different series and asymptotic representations, etc. This work is in progress and would be a part of some future research. This practice to acquire the data of values for these important functions by making use of our new mathematical model explores the required simplicity that is always needed. It is mentionable that the method established in this research is in fact noteworthy for the analysis and study of these higher transcendental functions. Furthermore, this research will have future effects due to the following important facts:

- The Riemann hypothesis is a well-known unsolved problem in analytic number theory [34]. It states that all the non-trivial zeros of the zeta function exist on the real line $s = \frac{1}{2}$. These zeros seem to be complex conjugates and hence symmetric on this line. The integrals of the zeta function and its generalizations are vital in the study of Riemann hypothesis and for the investigation of zeta functions themselves.
- The study of distributions in statistical inference and reliability theory [14, 26] also involves such integrals. By following the approach developed in this paper, we can initiate a deeper analysis of these functions that will enhance their applications in the mentioned studies.
- These generalized functions have simple relations with the BoseEinstein and FermiDirac functions [27,30, 31, 33]. These functions are of basic importance in quantum statistics that contracts by means of two specific categories of spin symmetry, that is, fermions and bosons. Fitting together these functions here with the generalized polylogarithm functions yields accurate values for these functions. While approximation of these integrals have always remained a challenge in Quantum Physics.

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