

A Generalized System of Nonlinear Variational Inequalities in Hilbert Spaces

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Abstract. In this paper, we consider convergence of iterative-projection method for solutions of a generalized system for three different nonlinear relaxed co-coercive mappings in the framework of Hilbert spaces. Strong convergence theorems are established. Our results improve and extend the recent ones announced by many others.

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1. INTRODUCTION AND PRELIMINARIES

Variational inequalities are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inequality problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve the variational inequalities have been developed by many authors. There exists a vast literature [1]-[28] on the approximation solvability of nonlinear variational inequalities as well as nonlinear variational inclusions using projection type methods, resolvent operator type methods or averaging techniques. It is well known that variational inequalities are equivalent to fixed point problems. This alternative equivalent formulation is very important

from the numerical analysis point of view and has played a significant part in several numerical methods for solving variational inequalities and complementarity problems. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. For the convergence of the projection method, one may require that the operator is strongly monotone and Lipschitz continuous. Gabay [10] has shown that the convergence of a projection method can be proved for co-coercive operators. Note that co-coercivity is a weaker condition than strong monotonicity. Recently, Verma [22]-[25] introduced a new system of nonlinear strongly monotone variational inequalities and studied the approximate solvability of this system based on a system of projection methods. Projection methods have been applied widely to problems arising especially from complementarity, convex quadratic programming, and variational problems.

In this paper, we consider, based on the projection method, the approximate solvability of a system of nonlinear variational inequalities with different co-coercive mappings in the framework of Hilbert spaces. Solutions of the system of nonlinear relaxed co-coercive variational inequalities are also fixed points of asymptotically nonexpansive mappings. Our results obtained in this paper generalize the results announced by Chang et al [3], Verma [22]-[24] and some others.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let C be a closed convex subset of H and let $T : C \rightarrow H$ be a nonlinear mapping. Let P_C be the projection of H onto the convex subset C . The classical variational inequality, denoted by $VI(C, T)$, is to find $u \in C$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.1)$$

We see that the point $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda T)$, where $\lambda > 0$ is a constant. That is,

$$u = P_C(u - \lambda Tu). \quad (1.2)$$

One can easily see variational inequalities and fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the study of the variational inequalities and related optimization problems.

Let $T : C \rightarrow H$ be a mapping. Recall the following definitions.

(1) T is said to be monotone if

$$\langle Tu - Tv, u - v \rangle \geq 0, \quad \forall u, v \in C.$$

(2) T is said to be δ -strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in C.$$

This implies that

$$\|Tx - Ty\| \geq \delta \|x - y\|, \quad \forall x, y \in C,$$

that is, T is δ -expansive and, when $\delta = 1$, it is expansive.

(3) T is said to be γ -cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \gamma \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(4) T is said to be relaxed γ -cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\gamma) \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(5) T is said to be relaxed (γ, δ) -cocoercive if there exist two constants $\gamma, \delta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\gamma)\|Tx - Ty\|^2 + \delta\|x - y\|^2, \quad \forall x, y \in C.$$

Let $S : C \rightarrow C$ be a mapping. Recall the following definitions.

(6) S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(7) S is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|S^n x - S^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, n \geq 0.$$

Next, we denote the fixed point of S by $F(S)$. If $x^* \in F(S) \cap VI(C, T)$, then we have the following

$$x^* = S^n x^* = P_C[x^* - \rho T x^*] = S^n P_C[x^* - \rho T x^*], \quad \forall n \geq 0.$$

This formulation is used to suggest the following iterative methods for finding a common element of the set of fixed points of asymptotically nonexpansive mappings and of the set of solutions to variational inequalities. Recently, three-step iterative method was studied by many authors to approximate solutions of variational inequalities and nonlinear operator equations. It has been shown that three-step schemes are numerically better than two-step and one step methods; see, for example, [15-17,27,28] and the reference therein.

Let $T_1, T_2, T_3 : C \times C \times C \rightarrow H$ be nonlinear mappings. Consider a system of nonlinear variational inequality problems (SNVI) as follows:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT_1(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.3)$$

$$\langle tT_2(z^*, x^*, y^*) + y^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.4)$$

$$\langle rT_3(x^*, y^*, z^*) + z^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, r > 0. \quad (1.5)$$

One can easily see the SNVI problem (1.3)-(1.5) is equivalent to the following projection formulas:

$$x^* = P_C[y^* - sT_1(y^*, z^*, x^*)], \quad s > 0,$$

$$y^* = P_C[z^* - tT_2(z^*, x^*, y^*)], \quad t > 0,$$

$$z^* = P_C[x^* - rT_3(x^*, y^*, z^*)], \quad r > 0,$$

respectively, where P_C is the projection of H onto C .

Next, we consider some special classes of the SNVI problems (1.3)-(1.5) as follows:

(I) If C is a closed convex cone of H , then the SNVI problem (1.3)-(1.5) is equivalent to the following system (SNC) of nonlinear complementarity problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$T_1(y^*, z^*, x^*) \in C^*, \quad T_2(z^*, x^*, y^*) \in C^*, \quad T_3(x^*, y^*, z^*) \in C^*$$

$$\langle sT_1(y^*, z^*, x^*) + x^* - y^*, x^* \rangle = 0, \quad s > 0, \quad (1.6)$$

$$\langle tT_2(z^*, x^*, y^*) + y^* - z^*, x^* \rangle = 0, \quad t > 0, \quad (1.7)$$

$$\langle rT_3(x^*, y^*, z^*) + z^* - x^*, x^* \rangle = 0, \quad r > 0, \quad (1.8)$$

where C^* is the polar cone to C defined by

$$C^* = \{f \in H : \langle f, x \rangle \geq 0, \quad \forall x \in C\}. \quad (1.9)$$

(II) If $T_1 = T_2 = T_3$, then then the SVNI problem (1.3)-(1.5) reduces to the following SNVI problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.10)$$

$$\langle tT(z^*, x^*, y^*) + y^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.11)$$

$$\langle rT(x^*, y^*, z^*) + z^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, r > 0. \quad (1.12)$$

(III) If $T_1, T_2, T_3 : C \rightarrow H$ are univariate mappings, then the SVNI problem (1.3)-(1.5) reduces to the following SNVI problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT_1(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.13)$$

$$\langle tT_2(z^*) + y^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.14)$$

$$\langle rT_3(x^*) + z^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, r > 0. \quad (1.15)$$

(IV) If $T_1 = T_2 = T_3 = T : C \rightarrow H$ is a univariate mapping, then the SVNI problem (1.3)-(1.5) reduces to the following SNVI problems:

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\langle sT(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.16)$$

$$\langle tT(z^*) + y^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.17)$$

$$\langle rT(x^*) + z^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, r > 0. \quad (1.18)$$

One can easily get the SNVI problem (1.16)-(1.18) is equivalent to the following projection formulas:

$$x^* = P_C[y^* - sT(y^*)], \quad s > 0, \quad (1.19)$$

$$y^* = P_C[z^* - tT(z^*)], \quad t > 0, \quad (1.20)$$

$$z^* = P_C[x^* - rT(x^*)], \quad r > 0. \quad (1.21)$$

Next, we introduce the following iterative methods for the above SNVI problems.

Algorithm 1. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = S^n P_C[x_n - r_n T_3(x_n, y_n, z_n)], \\ y_n = S^n P_C[z_n - t_n T_2(z_n, x_n, y_n)], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n, z_n, x_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and S is an asymptotically nonexpansive mapping.

If $T_1 = T_2 = T_3 = T$ and $S = I$, the identity mapping, then Algorithm 1 is reduced to the following:

Algorithm 2. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = P_C[x_n - r_n T(x_n, y_n, z_n)], \\ y_n = P_C[z_n - t_n T(z_n, x_n, y_n)], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C[y_n - s_n T(y_n, z_n, x_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ for all $n \geq 0$.

If $T_1, T_2, T_3 : C \rightarrow H$ are univariate mappings, then the Algorithm 1 is reduced to the following:

Algorithm 3. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = S^n P_C[x_n - r_n T_3 x_n], \\ y_n = S^n P_C[z_n - t_n T_2 z_n], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and S is an asymptotically nonexpansive mapping.

If $T_1 = T_2 = T_3 = T : C \rightarrow H$ is a univariate mapping and $S = I$, the identity mapping, then we have the following:

Algorithm 4. For any $x_0, y_0, z_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} z_n = P_C[x_n - r_n T x_n], \\ y_n = P_C[z_n - t_n T z_n], \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C[y_n - s_n T(y_n)], \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 1. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 2. A mapping $T : C \times C \times C \rightarrow H$ is said to be relaxed $((\gamma, \delta))$ -cocoercive if there exist constants $(\gamma, \delta) > 0$ such that, for all $x, x' \in C$

$$\begin{aligned} & \langle T(x, y, z) - T(x', y', z'), x - x' \rangle \\ & \geq (-\gamma) \|T(x, y, z) - T(x', y', z')\|^2 + \delta \|x - x'\|^2, \quad \forall y, y', z, z' \in C. \end{aligned}$$

Definition 3. A mapping $T : C \times C \times C \rightarrow H$ is said to be μ -Lipschitz continuous in the first variable if there exists a constant $\mu > 0$ such that, for all $x, x' \in C$,

$$\|T(x, y, z) - T(x', y', z')\| \leq \mu \|x - x'\|, \quad \forall y, y', z, z' \in C.$$

2. MAIN RESULTS

Theorem 4. Let C be a closed convex subset of a real Hilbert space H . Let $T_i : C \times C \times C \rightarrow H$ be a relaxed (γ_i, δ_i) -cocoercive and μ_i -Lipschitz continuous mapping in the first variable for each $i = 1, 2, 3$ and $S : C \rightarrow C$ an asymptotically nonexpansive mapping with a fixed point. Suppose that $x^*, y^*, z^* \in F(S)$ and $(x^*, y^*, z^*) \in \Omega_1$, where Ω_1 denotes the set of solutions to the SNVI problems (1.3)-(1.5). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by Algorithm 1 and let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Assume that the following conditions are satisfied.

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $k_n^3 \theta_{1n} \theta_{2n} \theta_{3n} < 1$, where

$$\theta_{1n} = \sqrt{1 + s_n^2 \mu_1^2 - 2s_n \delta_1 + 2s_n \gamma_1 \mu_1^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2 \mu_2^2 - 2t_n \delta_2 + 2t_n \gamma_2 \mu_2^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2 \mu_3^2 - 2r_n \delta_3 + 2r_n \gamma_3 \mu_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^* , y^* and z^* , respectively.

Proof. From the assumption, we have

$$\begin{cases} x^* = (1 - \alpha_n)x^* + \alpha_n S^n P_C[y^* - s_n T_1(y^*, z^*, x^*)], \\ y^* = S^n P_C[z^* - t_n T_2(z^*, x^*, y^*)], \\ z^* = S^n P_C[x^* - r_n T_3(x^*, y^*, z^*)]. \end{cases}$$

It follows from Algorithm 1 that

$$\begin{aligned} & \|x_{n+1} - x^*\| \\ &= \|(1 - \alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n, z_n, x_n)] - x^*\| \\ &= \|(1 - \alpha_n)x_n + \alpha_n S^n P_C[y_n - s_n T_1(y_n, z_n, x_n)] \\ &\quad - (1 - \alpha_n)x^* + \alpha_n S^n P_C[y^* - s_n T_1(y^*, z^*, x^*)]\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n k_n \|y_n - y^* - s_n [T(y_n, z_n, x_n) - T(y^*, z^*, x^*)]\|. \end{aligned} \quad (2.1)$$

By the assumption that T_1 is relaxed (γ_1, δ_1) -cocoercive and μ_1 -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} & \|y_n - y^* - s_n [T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|^2 \\ &= \|y_n - y^*\|^2 - 2s_n \langle y_n - y^*, T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*) \rangle \\ &\quad + s_n^2 \|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\ &\leq \|y_n - y^*\|^2 - 2s_n [-\gamma_1 \|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 + \delta_1 \|y_n - y^*\|^2] \\ &\quad + s_n^2 \mu_1^2 \|y_n - y^*\|^2 \\ &\leq \|y_n - y^*\|^2 + 2s_n \gamma_1 \mu_1^2 \|y_n - y^*\|^2 - 2s_n \delta_1 \|y_n - y^*\|^2 + s_n^2 \mu_1^2 \|y_n - y^*\|^2 \\ &= \theta_{1n}^2 \|y_n - y^*\|^2, \end{aligned} \quad (2.2)$$

where $\theta_{1n} = \sqrt{1 + s_n^2 \mu_1^2 - 2s_n \delta_1 + 2s_n \gamma_1 \mu_1^2}$. Now, we estimate

$$\begin{aligned} \|y_n - y^*\| &= \|S^n P_C[z_n - t_n T_2(z_n, x_n, y_n)] - y^*\| \\ &= \|S^n P_C[z_n - t_n T_2(z_n, x_n, y_n)] - S^n P_C[z^* - t_n T_2(z^*, x^*, y^*)]\| \\ &\leq k_n \|z_n - z^* - t_n [T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\|. \end{aligned} \quad (2.3)$$

By the assumption that T_2 is relaxed (γ_2, δ_2) -cocoercive and μ_2 -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned}
& \|z_n - z^* - t_n[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\|^2 \\
&= \|z_n - z^*\|^2 - 2t_n \langle z_n - z^*, T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*) \rangle \\
&\quad + t_n^2 \|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\
&\leq \|z_n - z^*\|^2 - 2t_n [-\gamma_2 \|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 + \delta_2 \|z_n - z^*\|^2] \quad (2.4) \\
&\quad + t_n^2 \mu_2^2 \|z_n - z^*\|^2 \\
&\leq \|z_n - z^*\|^2 + 2t_n \gamma_2 \mu_2^2 \|z_n - z^*\|^2 - 2t_n \delta_2 \|z_n - z^*\|^2 + t_n^2 \mu_2^2 \|z_n - z^*\|^2 \\
&\leq \theta_{2n}^2 \|z_n - z^*\|^2,
\end{aligned}$$

where $\theta_{2n} = \sqrt{1 + t_n^2 \mu_2^2 - 2t_n \delta_2 + 2t_n \gamma_2 \mu_2^2}$. On the other hand, we have

$$\begin{aligned}
\|z_n - z^*\| &= \|S^n P_C[x_n - r_n T_3(x_n, y_n, z_n)] - z^*\| \\
&= \|S^n P_C[x_n - r_n T_3(x_n, y_n, z_n)] - S^n P_C[x^* - r_n T_3(x^*, y^*, z^*)]\| \quad (2.5) \\
&= k_n \|P_C[x_n - r_n T_3(x_n, y_n, z_n)] - P_C[x^* - r_n T_3(x^*, y^*, z^*)]\| \\
&\leq k_n \|x_n - x^* - r_n [T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)]\|.
\end{aligned}$$

By the assumption that T_3 is relaxed (γ_3, δ_3) -cocoercive and μ_3 -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned}
& \|x_n - x^* - r_n [T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)]\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n \langle x_n - x^*, T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*) \rangle \\
&\quad + r_n^2 \|T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2r_n [-\gamma_3 \|T_3(x_n, y_n, z_n) - T_3(x^*, y^*, z^*)\|^2 + \delta_3 \|x_n - x^*\|^2] \quad (2.6) \\
&\quad + r_n^2 \mu_3^2 \|x_n - x^*\|^2 \\
&\leq \theta_{3n}^2 \|x_n - x^*\|^2,
\end{aligned}$$

where $\theta_{3n} = \sqrt{1 + r_n^2 \mu_3^2 - 2r_n \delta_3 + 2r_n \gamma_3 \mu_3^2}$. Combining (2.2), (2.3), (2.4), (2.5) and (2.6), we see

$$\|y_n - y^* - s_n [T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| \leq k_n^2 \theta_{1n} \theta_{2n} \theta_{3n} \|x_n - x^*\|. \quad (2.7)$$

Substitute (2.7) into (2.1) yields that

$$\|x_{n+1} - x^*\| \leq [1 - \alpha_n (1 - k_n^3 \theta_{1n} \theta_{2n} \theta_{3n})] \|x_n - x^*\|. \quad (2.8)$$

Applying Lemma 1 to (2.8), we can get the desired conclusion easily. This completes the proof. \square

Remark 5. Theorem 4 mainly improves the corresponding results of Chang, Lee and Chan [3] and also extends the results of Huang and Noor [11] to some extent.

As applications of Theorem 4, we have the following results immediately.

Corollary 6. *Let C be a closed convex subset of a real Hilbert space H . Let $T : C \times C \times C \rightarrow H$ be a relaxed (γ, δ) -cocoercive and μ -Lipschitz continuous mapping in the first variable. Suppose that $(x^*, y^*, z^*) \in \Omega_2$, where Ω_2 denotes the set of solutions to the SNVI problems (1.10)-(1.12). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by*

Algorithm 2 and let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Assume that the following conditions are satisfied.

- (a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
 (b) $\theta_{1n}\theta_{2n}\theta_{3n} < 1$, where

$$\theta_{1n} = \sqrt{1 + s_n^2\mu^2 - 2s_n\delta + 2s_n\gamma\mu^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2\mu^2 - 2t_n\delta + 2t_n\gamma\mu^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2\mu^2 - 2r_n\delta + 2r_n\gamma\mu^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^* , y^* and z^* , respectively.

Corollary 7. Let C be a closed convex subset of a real Hilbert space H . Let $T_i : C \rightarrow H$ be a relaxed (γ_i, δ_i) -cocoercive and μ_i -Lipschitz continuous mapping for each $i = 1, 2, 3$ and $S : C \rightarrow C$ an asymptotically nonexpansive mapping with a fixed point. Suppose that $x^*, y^*, z^* \in F(S)$ and $(x^*, y^*, z^*) \in \Omega_3$, where Ω_3 denotes the set of solutions to the SNVI problems (1.13)-(1.15). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by Algorithm 3 and let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Assume that the following conditions are satisfied.

- (a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
 (b) $k_n^3\theta_{1n}\theta_{2n}\theta_{3n} < 1$, where

$$\theta_{1n} = \sqrt{1 + s_n^2\mu_1^2 - 2s_n\delta_1 + 2s_n\gamma_1\mu_1^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2\mu_2^2 - 2t_n\delta_2 + 2t_n\gamma_2\mu_2^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2\mu_3^2 - 2r_n\delta_3 + 2r_n\gamma_3\mu_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^* , y^* and z^* , respectively.

Corollary 8. Let C be a closed convex subset of a real Hilbert space H . Let $T : C \rightarrow H$ be a relaxed (γ, δ) -cocoercive and μ -Lipschitz continuous mapping. Suppose that $(x^*, y^*, z^*) \in \Omega_4$, where Ω_4 denotes the set of solutions to the SNVI problems (1.16)-(1.18). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by Algorithm 4 and let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Assume that the following conditions are satisfied.

- (a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
 (b) $\theta_{1n}\theta_{2n}\theta_{3n} < 1$, where

$$\theta_{1n} = \sqrt{1 + s_n^2\mu^2 - 2s_n\delta + 2s_n\gamma\mu^2}, \quad \theta_{2n} = \sqrt{1 + t_n^2\mu^2 - 2t_n\delta + 2t_n\gamma\mu^2}$$

and

$$\theta_{3n} = \sqrt{1 + r_n^2\mu^2 - 2r_n\delta + 2r_n\gamma\mu^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^* , y^* and z^* , respectively.

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