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On the Convergence of a Modified Newton Method for Solving Equations

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Abstract. We approximate a solution of a nonlinear operator equation in a Banach space setting, where the differentiability of the operator involved is not assumed using a modified Newton method considered also in [1], [2], [6], [11]–[15]. We provide new sufficient convergence conditions, which are weaker than before [1], [2], [6], [11]–[15], under the same computational cost. Numerical examples where our results apply, where earlier ones fail are also provided in this study.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where, F is a continuous operator defined on a subset $\mathcal{D}_F$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We use the modified Newton method:

$$x_{n+1} = x_n - G'(x_n)^{-1} \ F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \tag{1.2}$$

to generate a sequence approximating $x^\star$. Operator $G'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, denote the Fréchet–derivative of operator G [2], [5], [11]. If $G'(x) = F'(x)$ $(x \in \mathcal{D})$, we obtain Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} \ F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \tag{1.3}$$

[3]–[5], [8]–[11].
The semilocal convergence of the modified Newton method (1.2) has been given by several authors under Lipschitz–type conditions (see [1]–[15], and the references there).
Here, motivated by optimization considerations, we introduce the needed center–Lipschitz conditions (see (2.14)) to find upper bounds on the norms $\parallel G'(x)^{-1} \ G'(x_0) \parallel$ instead of the less precise Lipschitz conditions (see (2.4)). In turn out that this way, we obtain a new semilocal convergence analysis with the following advantages over the corresponding ones in [1], [2], [6], [11]–[15]: larger convergence domain, finer estimates on the distances $\parallel x_{n+1} - x_n \parallel$, $\parallel x_n - x^\star \parallel$ $(n \geq 0)$, and an at least as precise information on the location of the solution $x^\star$.
Numerical examples where our results apply to solve nonlinear equations (1.1) are provided, in cases earlier ones cannot [6]–[15].

2. Semilocal convergence using increasing majorizing sequences

We state a semilocal convergence result for the modified Newton method (1.2).

Theorem 1. *Let $F : \mathcal{D}_F \subset \mathcal{X} \longrightarrow \mathcal{Y}$ be continuous, and $G : D_G \subset \mathcal{X} \longrightarrow Y$, be satisfying the Fréchet differentiability on a disk $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$.*
Assume there exist $x_0 \in \mathcal{D}$, and constants $\eta \geq 0$, $M \geq 0$, $K > 0$, such that for $x, y \in \mathcal{D}_0$:

$$G'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}); \tag{2.1}$$

$$\parallel G'(x_0)^{-1} \ F(x_0) \parallel \leq \eta; \tag{2.2}$$

$$\parallel G'(x_0)^{-1} \ ((F - G)(x) - (F - G)(y)) \parallel \leq M \ \parallel x - y \parallel; \tag{2.3}$$

$$\parallel G'(x_0)^{-1} \ (G'(x) - G'(y)) \parallel \leq K \ \parallel x - y \parallel; \tag{2.4}$$

$$2 \ K \ \eta \leq (1 - M)^2; \tag{2.5}$$

and

$$\overline{U}(x_0, r_0^\star) = \{x \in \mathcal{X} \ : \parallel x - x_0 \parallel \leq r_0^\star\} \subseteq \mathcal{D}, \tag{2.6}$$

where

$$r_0^\star \geq r^\star = \frac{1 - M}{K} \left(1 - \sqrt{1 - \frac{2 \ K \ \eta}{(1 - M)^2}}\right). \tag{2.7}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$, generated by modified Newton method (1.2) is well defined for all $n \geq 0$, remains in $\overline{U}(x_0, r_0^\star)$, and converges to a solution $x^\star$ of equation $F(x) = 0$. Moreover, the following estimates hold for all $n \geq 0$:

$$\parallel x_{n+1} - x_n \parallel \leq r_{n+1} - r_n \tag{2.8}$$

and

$$\parallel x_n - x^\star \parallel \leq r^\star - r_n, \tag{2.9}$$

where, scalar sequence $\{r_n\}$ is given by:

$$r_0 = 0,$$

$$r_{n+1} = r_n - \frac{\Big(K\ (r_n - r_{n-1}) + 2\ M\Big)\ (r_n - r_{n-1})}{2\ (1 - K\ r_n)} = r_n - \frac{g(r_n)}{1 - K\ r_n} \quad (n \geq 1) \tag{2.10}$$

and

$$g(r) = \frac{K}{2}\ r^2 - (1 - M)\ r + \eta. \tag{2.11}$$

Proof. Proof was provided in non–affine invariant form in ([12], p. 189). Here, we state that the same proof can be provided in affine invariant form, if we use $G'(x_0)^{-1}\ G(x)$, $G'(x_0)^{-1}\ F(x)$ instead of $G(x)$, $F(x)$ respectively used in [12].
That completes the proof of Theorem 1. □

Remark 2. Theorem 1 improves corresponding Theorem 5.3 in [12], since our results are provided in affine invariant form. The advantages of affine versus non–affine convergence results have been explained in detail in [8] (see also [5]).
It turns out that the results in Theorem 1 can be improved even further. Indeed, let us define scalar sequences $\{q_n\}$, $\{s_n\}$ $(n \geq 0)$ for some $L > 0$ by:

$$q_0 = 0, \qquad q_1 = \eta,$$

$$q_{n+2} = q_{n+1} - \frac{\Big(K\ (q_n - q_{n-1}) + 2\ M\Big)\ (q_n - q_{n-1})}{2\ (1 - L\ q_n)} \quad (n \geq 1), \tag{2.12}$$

and

$$s_0 = 0,$$

$$s_{n+1} = s_n - \frac{g(s_n)}{1 - L\ s_n} \quad (n \geq 0). \tag{2.13}$$

In view of (2.4), there exists $L > 0$ such that

$$\| G'(x_0)^{-1}\ (G'(x) - G'(x_0)) \| \leq L\ \ \| x - x_0 \| \qquad \text{for all} \quad x \in \mathcal{D}. \tag{2.14}$$

Note that in general

$$L \leq K \tag{2.15}$$

holds, and $\dfrac{K}{L}$ can be arbitrarily large [3]–[5].

We can show the semilocal convergence theorem for the modified Newton's method (1.2).

Theorem 3. (1) *If hypothesis (2.5) holds, then following hold for all $n \geq 0$:*

$$0 \leq q_n \leq s_n \leq r_n, \tag{2.16}$$

$$0 \leq q_{n+1} - q_n \leq s_{n+1} - s_n \leq r_{n+1} - r_n, \tag{2.17}$$

$$0 \leq q^\star - q_n \leq s^\star - s_n \leq r^\star - r_n, \tag{2.18}$$

and

$$q^\star \leq s^\star \leq r^\star, \tag{2.19}$$

where

$$q^\star = \lim_{n\to\infty} q_n, \qquad s^\star = \lim_{n\to\infty} s_n, \tag{2.20}$$

and $r^\star$ is given in (2.7).

(2) *Under the hypotheses of Theorem 1, sequence $\{x_n\}$ ($n \geq 0$), generated by modified Newton method (1.2) is well defined, remains in $\overline{U}(x_0, q^\star)$ for all $n \geq 0$, and converges to a solution $x^\star \in \overline{U}(x_0, q^\star)$ of equation $F(x) = 0$.*
Moreover, the following estimates hold for all $n \geq 0$:

$$\| x_{n+1} - x_n \| \leq q_{n+1} - q_n \tag{2.21}$$

and

$$\| x_n - x^\star \| \leq q^\star - q_n. \tag{2.22}$$

Proof. (1) The proof of this part follows using induction on n, (2.10)–(2.13), (2.15), and standard majorization techniques [2], [5], [11].

(2) Let $x \in \overline{U}(x_0, q^\star)$. Using (2.7), (2.14), and (2.19), we obtain

$$\| G'(x_0)^{-1}\ (G'(x) - G'(x_0)) \| \leq L\ \| x - x_0 \| \leq L\ r^\star \leq K\ t^\star < 1. \tag{2.23}$$

It follows from (2.23), and the Banach lemma on invertible operators [5], [11], that $G'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\| G'(x)^{-1}\ G'(x_0) \| \leq \frac{1}{1 - L\ \| x - x_0 \|}. \tag{2.24}$$

Using (1.2) for $n = 0$, (2.2), and (2.6), we get $\| x_1 - x_0 \| \leq \eta \leq r^\star$. That is $x_1 \in \overline{U}(x_0, r^\star)$, and (2.21) holds for $n = 0$. Let us assume that $x_k \in \overline{U}(x_0, r^\star)$ for all $k \leq n$. Then, using (1.2), (2.3), (2.4), (2.10), (2.12), (2.15), (2.24), and the identity

$$\frac{\frac{K}{2}\ r_k^2 - (1 - M)\ r_k + \eta}{1 - K\ r_k} = \frac{\frac{K}{2}\ (r_k - r_{k-1})^2 + M\ (r_k - r_{k-1})}{1 - K\ r_k} \tag{2.25}$$

we obtain in turn

$$\begin{aligned}
&\| x_{k+1} - x_k \| = \| -(G'(x_k)^{-1}\ G'(x_0))\ (G'(x_0)^{-1}\ F(x_k)) \| \\
&\leq \frac{1}{1 - L\ q_k} \Bigg(\| G'(x_0)^{-1}\ (G(x_k) - G(x_{k-1}) - G'(x)\ (x_k - x_{k-1})) \| + \\
&\| G'(x_0)^{-1}\ (F(x_k) - G(x_k) - (F(x_{k-1}) - G(x_{k-1}))) \| \\
&\leq \frac{1}{1 - L\ q_k} \left(\frac{K}{2}\ \| x_k - x_{k-1} \| + M \right)\ \| x_k - x_{k-1} \| \\
&\leq \frac{1}{1 - L\ q_k} \left(\frac{K}{2}\ (q_k - q_{k-1}) + M \right) (q_k - q_{k-1}) = q_{k+1} - q_k,
\end{aligned} \tag{2.26}$$

which shows (2.21) for all $n \geq 0$.

Using part 1. of Theorem 3, and (2.26), we deduce sequence $\{x_n\}$ is Cauchy in a Banach space $\mathcal{X}$, and such it converges to some $x^\star \in \overline{U}(x_0, q^\star)$ (since $\overline{U}(x_0, q^\star)$ is a closed set). By letting $k \longrightarrow \infty$ in (2.26), we obtain $F(x^\star) = 0$. Estimate (2.22) is obtained from (2.21) by using standard majorization techniques [5].

That completes the proof of Theorem 3. □

Remark 4. It follows from Theorem 3 that finer majorizing sequences than $\{r_n\}$ can be obtained, and under the same computational cost, since in practice, the evaluation of Lipschitz constant K requires the evaluation of center–Lipschitz constant L. One is then wondering if the Newton–Kantorovich–type hypothesis (2.5) can be weakened, since finer sequence $\{q_n\}$ may be converging under weaker hypotheses. In [3]–[5], we provided sufficient convergence conditions for more general majorizing sequences that $\{q_n\}$. One such condition which can be weaker than (2.5) is given in [4], [5]:

$$\left(\frac{K}{2}+\frac{2\ L}{2-\delta}\right)\ \delta\ \eta+2\ M\leq\delta\qquad\text{for some}\ \ \delta\in[0,2).\tag{2.27}$$

In the next section, we provide sufficient convergence conditions other that (2.5), and (2.27) using decreasing instead of increasing majorizing sequences.

3. Semilocal convergence using decreasing majorizing sequences

We need the following result on majorizing sequences for modified Newton method (1.2)

Lemma 5. *Let $\eta\geq 0$, $K>0$, $M\geq 0$, and $L>0$ be given constants. Set $t_0=\dfrac{1}{L}$. Define functions Δ, A, C on $[0,+\infty)^2$, and B on $[0,+\infty)$ by*

$$\Delta(t,\gamma)=(K\ t+M)^2-(K-2\ \gamma\ L)\ (K\ t+2\ M)\ t,$$

$$A(t,\gamma)=2\ \left(K\ t+M+\sqrt{\Delta(t,\gamma)}\right),$$

$$B(t)=2\ (K\ t+2\ M),$$

and

$$C(t,\gamma)=\frac{B(t)}{A(t,\gamma)}.$$

Assume any of the following hold:

function

$$f_t(x)=1-x-C(t,x)\tag{3.1}$$

has a non–negative zero $\gamma_0=\gamma(t_0)$ at $t=t_0$, such that:

$$\beta=2\ L\ \eta\leq\gamma_0\leq\frac{K}{2\ L};\tag{3.2}$$

or

$$K+M-L\geq 0,\tag{3.3}$$

$$f_{t_0}(\beta)\geq 0,\tag{3.4}$$

and hypothesis (3.2) holds;
or
function f_t has a non–negative zero γ_0^1 at $t=t_1$, such that:

$$\gamma_0^1\leq\frac{K}{2\ L}\tag{3.5}$$

and

$$\beta<\beta_0,\tag{3.6}$$

where

$$\beta_0 = 4 \left\{ M + 2 + \sqrt{(M+2)^2 + 4\ (\frac{K}{2\ L} - 1)} \right\}^{-1}; \tag{3.7}$$

or

$$f_{t_1}(\beta_1) \leq 0, \qquad \beta_1 = \min\{\beta_0,\ \frac{K}{2\ L}\}, \tag{3.8}$$

$$f_{t_1}(\beta) \geq 0, \tag{3.9}$$

(3.5), and (3.6) hold.
Note that the existence of γ_0 (or γ_0^1) follows from the intermediate value theorem applied to function f_{t_0} (or f_{t_1}) on the interval $\left[\beta, \frac{K}{2\ L}\right]$ (or $[\beta, \beta_1]$) respectively.
Then, scalar sequence $\{t_n\}$ $(n \geq 0)$ generated by

$$t_1 = t_0 - \eta,$$

$$t_{n+1} = t_n - \frac{\Big(K\ (t_{n-1} - t_n) + 2\ M\Big)\ (t_{n-1} - t_n)}{2\ L\ t_n} \quad (n \geq 1), \tag{3.10}$$

is well defined, decreasing and converges to some $t^\star \in [0, t_0]$.

Proof. If $\eta = 0$, then $t_n = t_0 = t^\star \quad (n \geq 0)$. Let us assume $\eta \neq 0$. Function Δ is a quadratic polynomial with leading coefficient $2\ K\ \gamma_0\ L$ (for $\gamma = \gamma_0$), and whose sign of the discriminant is the same with: $2\ \gamma_0\ L\ (2\ \gamma_0\ L - K)$. It then follows by (2.2) that functions A and C are well defined on $[0, \infty) \times [0, \gamma_0]$. It also follows by definition of γ_0 that $\gamma_0 \in (0, 1)$. Set:

$$\frac{t_{n+1}}{t_n} = 1 - \gamma_n,$$

where,

$$\gamma_n = \gamma(t_n) = \frac{\Big(K\ (t_{n-1} - t_n) + 2\ M\Big)\ (t_{n-1} - t_n)}{2\ L\ t_n^2} \quad (n \geq 1).$$

We shall show: $t_k \geq (1 - \gamma_0)\ t_{k-1}$, which if $t_{k-1} > 0$, implies $0 < t_k < t_{k-1}$. But, $t_k \geq (1 - \gamma_0)\ t_{k-1}$ holds if $1 - \gamma_k \geq 1 - \gamma_0$ or $\gamma_k \leq \gamma_0$ or

$$(K - 2\ L\ \gamma_0)\ t_k^2 - 2\ (K\ t_{k-1} + M)\ t_k + (K\ t_{k-1} + 2\ M)\ t_{k-1} \leq 0,$$

or $t_k \geq C(t_{k-1}, \gamma_0)\ t_{k-1}$. Using (3.10), and the definition of t_0, we get $t_1 \geq (1 - \gamma_0)\ t_0$.
That is we have:
$t_1 \geq (1 - \gamma_0)\ t_0 \Longrightarrow t_1 \geq C(t_0, \gamma_0)\ t_0 \Longleftrightarrow t_2 \geq (1 - \gamma_0)\ t_1$.
Similarly, we show this implication holds in general, i.e.,
$t_{k-1} \geq (1 - \gamma_0)\ t_{k-2} \Longrightarrow t_{k-1} \geq C(t_0, \gamma_0)\ t_{k-2} \Longleftrightarrow t_k \geq (1 - \gamma_0)\ t_{k-1}$ $(k \geq 1)$.
If the alternative conditions (3.5)–(3.9) hold, then $t_2 > 0$, and $t_3 \geq (1 - \gamma_0^1)\ t_2$, and the induction follows by analogy.
The induction is completed. Hence, sequence $\{t_n\}$ $(n \geq 0)$ is decreasing positive, and as such it converges to some $t^\star \in [0, t_0]$.
That completes the proof of Lemma 5. □

We can show the following semilocal convergence theorem for modified Newton method (1.2):

Theorem 6. *Let $F : \mathcal{D}_F \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be continuous, and $G : \mathcal{D}_G \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be satisfying the Fréchet differentiability on a disk $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$. Assume there exist $x_0 \in \mathcal{D}_0 \subseteq \mathcal{D}$, and constants $\eta \geq 0$, $K > 0$, $M \geq 0$, $L > 0$, such that for $x, y \in \mathcal{D}$:*

$$G'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X});$$

$$\| G'(x_0)^{-1} \; F(x_0) \| \leq \eta;$$

$$\| G'(x_0)^{-1} \; ((F-G)(x) - (F-G)(y)) \| \leq M \;\; \| x - x \|;$$

$$\| G'(x_0)^{-1} \; (G'(x) - G'(y)) \| \leq K \;\; \| x - y \|;$$

$$\| G'(x_0)^{-1} \; (G'(x) - G'(x_0)) \| \leq L \;\; \| x - x_0 \|;$$

$$\overline{U}(x_0, t_0 - t^\star) \subseteq \mathcal{D} \qquad (\text{or} \quad \overline{U}(x_0, \frac{1}{L}) \subseteq \mathcal{D}),$$

and hypotheses of Lemma 5 hold.
Then, sequence $\{x_n\}$ $(n \geq 0)$, generated by modified Newton method (1.2) is well defined, remains in $\overline{U}(x_0, t_0 - t^\star)$ for all $n \geq 0$, and converges to a solution $x^\star \in \overline{U}(x_0, t_0 - t^\star)$ of equation $F(x) = 0$.
Moreover, the following estimates hold for all $n \geq 0$:

$$\| x_{n+1} - x_n \| \leq t_n - t_{n+1} \tag{3.11}$$

and

$$\| x_n - x^\star \| \leq t_n - t^\star. \tag{3.12}$$

Furthemore, if

$$K \; (t_0 - t^\star) + M < L \; t^\star, \tag{3.13}$$

or

$$\frac{K}{2 \; L} + M < 1, \tag{3.14}$$

$x^\star$ is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, t_0 - t^\star)$.

Proof. As in (2.26) for $x_k \in \overline{U}(x_0, t^\star)$, we arrive at:

$$\| x_{k+1} - x_k \| \leq \frac{1}{1 - L \; (t_0 - t_k)} \left(\frac{K}{2} \; (t_{k-1} - t_k) + M \right) (t_{k-1} - t_k) = t_k - t_{k+1}, \tag{3.15}$$

which implies (3.11). Estimate (3.12) follows from (3.11) by using standard majorizing techniques [5], [11].
It is show uniqueness part. Let $y^\star \in \overline{U}(x_0, t_0 - t^\star)$ be a solution of equation $F(x) = 0$. Using (1.2), we obtain the identity

$$\begin{aligned} x_{k+1} - y^\star &= x_k - G'(x_k)^{-1} \; F(x_k) - y^\star \\ &= -(G'(x_k)^{-1} \; G'(x_0)) \Big((G'(x_k) \; (x_k - y^\star) - (G(x_k) - G(y^\star))) + \\ & \quad (F(x_k) - G(x_k) - (F(y^\star) - G(y^\star))) \Big), \end{aligned} \tag{3.16}$$

which as in (2.26) leads to:

$$\begin{aligned}
&\| x_{k+1} - y^\star \| \\
&\leq \frac{1}{1 - L \ \| x_k - y^\star \|} \left(\frac{K}{2} \ \| x_k - y^\star \| + M \right) \ \| x_k - y^\star \| \\
&\leq \frac{1}{1 - L \ (t_0 - t^\star)} \left(\frac{K}{2} \ (t_0 - t^\star) + M \right) \ \| x_k - y^\star \| \\
&\leq \frac{1}{L \ t^\star} \left(\frac{K}{2} \ (t_0 - t^\star) + M \right) \ \| x_k - y^\star \| \\
&\leq \| x_k - y^\star \| \qquad (\text{by } (3.13) \text{ or } (3.14)),
\end{aligned}$$

which shows $\lim_{k \to \infty} x_k = y^\star$. But, we already showed $\lim_{k \to \infty} x_k = x^\star$. Hence, we conlude $x^\star = y^\star$.
That completes the proof of Theorem 6. □

4. Special cases and applications

A direct comparison between Theorems 1 and 6 is not possible, since the former uses an increasing majorizing sequence and the latter a decreasing one. However, we can compare the sufficient convergence condition (2.5) with the corresponding ones in Lemma 5, at least in some intersting special cases.

(1) Case $F(x) = G(x)$, $(x \in \mathcal{D})$. (Newton's method). Condition (2.5) reduces to the famous Newton–Kantorovich hypothesis [5], [8], [11]:

$$h_K = 2 \ K \ \eta \leq 1, \tag{4.1}$$

since $M = 0$, where as condition (3.2) becomes:

$$h_A = 2 \ \overline{K} \ \eta \leq 1, \tag{4.2}$$

where

$$\overline{K} = \frac{1}{8} \left(K + 4 \ L + \sqrt{K^2 + 8 \ L \ K} \right), \tag{4.3}$$

since,

$$\gamma_0 = \frac{\sqrt{K^2 + 8 \ L \ K} - K}{\sqrt{K^2 + 8 \ L \ K} + K}. \tag{4.4}$$

Note that

$$K \leq \overline{K} \tag{4.5}$$

hold in general, and $\dfrac{\overline{K}}{K}$ can be arbitrarily large [3]–[5]. In case $L < K$, then strict inequality holds (4.5).
It follows from (4.1), (4.2), and (4.5) that

$$h_K \leq 1 \Longrightarrow h_A \leq 1, \tag{4.6}$$

but not necessarily vice verca unles, if $K = L$ (see also Example1).

(2) Case $F(x) \neq G(x)$ $(x \in \mathcal{D})$. Then we can only compare Theorem 1 with Theorem 6 using numerical examples.

Example1. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $\mathcal{D} = [\alpha, 2-\alpha]$, $\alpha \in [0, \frac{1}{2})$, and define function F and G on $\mathcal{D}$ by

$$F(x) = G(x) + \epsilon\ |x-1| \qquad \text{and} \qquad G(x) = x^3 - \alpha, \tag{4.7}$$

where, ϵ is a given real number.
Using (2.1)–(2.4), we obtain:

$$\eta = \frac{1}{3}\ (1-\alpha), \quad L = 3-\alpha, \quad K = 2\ (2-\alpha), \quad \text{and} \quad M = |\epsilon|.$$

Note that function F is continuous but not differentiable on D, since a $F'(1)$ does not exist. Hypothesis (4.1) is violated, since

$$4\ (2-\alpha)\ \left(\frac{1}{3}\ (1-\alpha) + |\epsilon|\right) > 1 \qquad \text{for all} \quad \epsilon, \quad \text{and} \quad \alpha \in [0, \frac{1}{2}). \tag{4.8}$$

That is there is no guarantee that sequence $\{x_n\}$, starting at $x_0 = 1$ converges under the hypotheses of Theorem 1.
However, our Theorem 6 can apply to solve equation $F(x) = 0$.

Let us consider two cases:

(1) Case $\epsilon = 0$. Then condition (4.2) holds for $\alpha \in [.450339002, \frac{1}{2}]$, which is the same range, given by us in [4] using a different approach.
(2) Case $\epsilon \neq 0$. Choose e.g.: $\epsilon = .1$, and $\alpha = .49$. Then, we get:

$$\eta = .17, \quad L = 2.51, \quad K = 3.02, \quad M = .1, \quad t_0 = .398406374,$$
$$t_1 = .22840637, \quad \gamma_0 = .410812, \quad \gamma_0^1 = .369936,$$

$$\beta = .8534, \qquad \frac{K}{2\ L} = .601593625, \qquad \text{and} \quad \beta_0 = 1.058703597.$$

Hence hypotheses (3.5) and (3.6) of Lemma 5 are satisfied. That is the conclusions of Theorem 6 apply to solve equation $F(x) = 0$.

We complete this study with an example involving a nonlinear integral equation of Chandrasekhar–type [1], [2], [5], [7], [11]. For simplicity, we choose $F(x) = G(x)$ $(x \in \mathcal{D})$.
Example2. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ be the space of real–valued continuous functions defined on the interval $[0,1]$ with norm

$$\| x \| = \max_{0 \le s \le 1} |x(s)|.$$

Let $\theta \in [0,1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\, dt + y(s) - \theta. \tag{4.9}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1]\times[0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (4.9) arise in the theory of radiative transfer, neutron transport, and the kinetic theory of gasses [1], [2], [4], [7].
For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \dfrac{s}{s+t}$, for all $s \in [0,1]$, and

$t \in [0,1]$, with $s+t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$, $g = 0$ and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\, dt + y(s) - \theta, \tag{4.10}$$

for all $s \in [0,1]$, then every zero of F satisfies equation (4.9). We have the estimates

$$\max_{0 \leq s \leq 1} \mid \int \frac{s}{s+t}\, dt \mid = \ln 2.$$

Therfore, if we set $b_0 = \parallel F'(u_0)^{-1} \parallel$, then it follows from (2.1)–(2.4) that:

$$\eta = q\ (|\lambda|\ \ln 2 + 1 - \theta), \quad K = 2\ q\ (|\lambda|\ \ln 2 + 3\ (2 - \theta)),$$
$$L = q\ (2\ |\lambda|\ \ln 2 + 3\ (3 - \theta)).$$

It follows from Theorem 6 that if condition (4.2) holds, then problem (4.9) has a unique solution near u_0. This condition is weaker than the conditions given before using the Newton–Kantorovich hypothesis (4.1).
Note also that $L < K$ for all $\theta \in [0,1]$.

Conclusion

We provide a semilocal convergence analysis for a modified Newton method considered also in [4], [5], [12], [13], [15], in order to approximate a locally unique solution of an equation in a Banach space.
Using a combination of Lipschitz and center–Lipschitz conditions, insted of only Lipschitz conditions used in the works above, we provide an analysis with the following advantages: larger convergence domain and weaker sufficient convergence conditions. Note that these advantages are obtained under the same computational cost, since in practice the computation of the Lipschitz constant K requires the computation of L.
Numerical examples further validating the results are also provided.

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