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A semilocal convergence analysis of an inexact Newton method using recurrent relations

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Abstract. We extend the applicability of an inexact Newton method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. The recurrent relations method is used to prove the existence-convergence theorem. Our error bounds are tighter and the information on the location of the solution at least as precise under the same information as before. Our results compare favorably with earlier studies in [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20]. A numerical example involving a nonlinear integral equation of a Chandrasekhar type is also presented in this study.

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1. Introduction

Let X and Y be Banach spaces, let D be a nonempty open convex subset of X and let $F\colon \bar{D} \longmapsto Y$ be a continuous function that is Fréchet differentiable on D. In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0. \tag{1.1}$$

Computational sciences have received substantial and significant interest of researchers in recent years in several areas such as engineering sciences, economic equilibrium theory and mathematics. These sciences can solve various problems by passing first through mathematical modelling and then later looking for the solution iteratively [4, 5]. For example, finding a local minimum of function is connected to solving a set of nonlinear equations

(1.1). So, numerical methods are crucial and necessary for solving of these nonlinear equations. Note that similar equations than (1.1) are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving control and optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. Finally, note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

In [2, 3, 4], Argyros introduced the inexact Newton method INM as

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = y_n - z_n, \quad n = 0, 1, \ldots \end{cases} \tag{1.2}$$

in order to approximate the solution x^* of (1.1). Here $x_0 \in D$ and z_n is a sequence in X depending on x_n, y_n and earlier iterates. If $z_n = 0$, we obtain Newton's method and if $z_n = F'(y_n)^{-1}F(y_n)$ we obtain the standard two-step Newton method. Several other choices of z_n and analyses of the resulting iterative schemes can be found in [2, 3, 4].

In a previous paper [6], Argyros and Hilout used a Kantorovich-type approach to prove the semilocal convergence of INM. In the present paper we will use recurrent relations to provide the semilocal convergence analysis of INM under different Kantorovich-type convergence conditions. The results obtained here can be extended to a version of INM involving outer or generalized inverses along the lines of Chen and Nashed [12] and Argyros and Hilout [7]. Other Newton-type iterative methods for equation (1.1) can be found in [13, 14, 15, 16, 17, 18, 19, 20].

This paper is organized as follows. In Section 2 we provide the semilocal convergence analysis of INM and present comparisons of our results with earlier studies in the literature. Section 3 contains special cases of our main results. Finally, we present a numerical example involving a nonlinear integral equation of a Chandrasekhar type appearing in the study of radiative heat transfer.

2. Semilocal Convergence analysis of INM

Let $U(x_0, \tau)$ and $\bar{U}(x_0, \tau)$ be, respectively, the open and closed ball in X with center x_0 and radius $\tau > 0$. Let $L(Y, X)$ be the space of bounded linear operators from Y into X. Our main result in this section is the following semilocal convergence result for INM.

Theorem 1. *Let $F\colon D \subset X \mapsto Y$ be a Fréchet differentiable operator on a nonempty open convex subset of a Banach space X with values in a Banach space Y. Suppose that $J_0 = F'(x_0)^{-1} \in L(Y, X)$ for some $x_0 \in D$ and that the following conditions hold:*

$$\|J_0\| \leq \beta, \tag{2.1}$$

$$\|F(x_0)\| \leq \eta, \tag{2.2}$$

$$\|F'(x) - F'(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in D, \tag{2.3}$$

$$\|F'(x)\| \leq c \quad \text{for all } x \in D. \tag{2.4}$$

Suppose further that the sequence z_n is chosen so that

$$\|z_n\| \leq \frac{b}{2}\|J_n\|\|y_n - x_n\|^2 \tag{2.5}$$

for some $b \geq 0$ and $J_n = F'(x_n)^{-1}$. Let

$$\gamma = \max\{b, L\beta, \frac{1}{2}(L + bc\beta), \frac{b}{2}\sqrt{\frac{L}{L + b\beta c}}\}. \tag{2.6}$$

Let $f(t) = \frac{2}{2-2t-t^2}$ and $g(t) = t(1+t^2)$ and let $\tau_2 \approx 0.334375061169603$ be the smallest positive zero of the scalar equation $f^2(t)g(t) - 1 = 0$. Suppose further that

$$a_0 = \beta\gamma\eta\beta < \tau_2, \tag{2.7}$$

$$U(x_0, \tau) \subseteq D \tag{2.8}$$

where $\tau = \frac{\eta(1+a_0/2)}{1-f(a_0)g(a_0)}$. Then, the sequence INM (1.2) is well defined, remains in $U(x_0, \tau)$ and converges to a solution x^* of equation (1.1) in the ball $\bar{U}(x_0, \tau)$. This solution is unique in $U(x_0, \frac{2}{L\beta} - \tau) \cap D$ if $\tau < 2/(L\beta)$.

Proof. It is easy to verify that $\tau_1 = [(143 + 6\sqrt{906})^{1/3} - 23(143 + 6\sqrt{906})^{-1/3}) - 1]/6 \approx 0.4192238370$ is the smallest positive zero of the scalar equation $f(t)g(t) - 1 = 0$. Since $a_0 < \tau_2 < \tau_1 < 1/2$, we must have $f(a_0)g(a_0) < 1$ and $f^2(a_0)g(a_0) < 1$ and $a_0(1 + a_0) = \frac{1}{2}(a_0 + 1)^2 - \frac{1}{2} < 1$. It follows from (2.5), (2.6), (2.8) and (2.7) that

$$\begin{aligned}\|x_1 - y_0\| = \|z_0\| &\leq \frac{b}{2}\|J_0\|\|y_0 - x_0\|^2 \leq \frac{b}{2}\beta\eta\|y_0 - x_0\| \\ &\leq \frac{\gamma}{2}\beta\eta\|y_0 - x_0\| = \frac{a_0}{2}\|y_0 - x_0\|\end{aligned}$$

and

$$\begin{aligned}\|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq (1 + a_0/2)\|y_0 - x_0\| \\ &\leq (1 + a_0/2)\eta = \tau[1 - f(a_0)g(a_0)] < \tau.\end{aligned}$$

Using (2.6) and (2.8) we see that

$$\begin{aligned}\|J_0\|\|F'(x_1) - F'(x_0)]\| &\leq L\beta\|x_1 - x_0\| \leq \beta L(1 + a_0/2)\|y_0 - x_0\| \\ &\leq \gamma\eta(1 + a_0/2) = a_0(1 + a_0/2) < 1\end{aligned}$$

It follows from the Banach Lemma on invertible operators [18] that J_1 exists and $\|J_1\| \leq \frac{\|J_0\|}{1-a_0(1+a_0/2)} = f(a_0)\|J_0\|$. Therefore $y_1 \in D$ is well defined and

$$\begin{aligned}F(x_1) &= [F(x_1) - F(y_0) - F'(y_0)(x_1 - y_0)] + [F(y_0) + F'(y_0)(x_1 - y_0)] \\ &= \int_0^1 [F'(y_0 + \theta(x_1 - y_0)) - F'(y_0)]\, d\theta(x_1 - y_0) + [F(y_0) + F'(y_0)(x_1 - y_0)] \\ &= \int_0^1 [F'(y_0 + \theta(x_1 - y_0)) - F'(y_0)]\, d\theta(x_1 - y_0) \\ &+ \int_0^1 [F'(x_0 + \theta(y_0 - x_0)) - F'(x_0)]\, d\theta(y_0 - x_0) + F'(y_0)(x_1 - y_0).\end{aligned}$$

Using (2.3)-(2.6) and the above expressions, we see that

$$\begin{aligned}\|F(x_1)\| &\le \frac{L}{2}\|x_1-y_0\|^2+\frac{L}{2}\|y_0-x_0\|^2+c\|x_1-y_0\|\\ &\le \frac{L}{2}(\frac{b}{2}\|J_0\|\|y_0-x_0\|^2)^2+\frac{L}{2}\|y_0-x_0\|^2+\frac{cb\|J_0\|}{2}\|y_0-x_0\|^2\\ &\le \left[\frac{Lb^2\beta^2\eta^2}{8}+\frac{L+cb\beta}{2}\right]\|y_0-x_0\|^2\\ &= \frac{1}{2}(L+c\beta b)[(\eta\beta\frac{b}{2}\sqrt{\frac{L}{L+b\beta c}})^2+1]\|y_0-x_0\|^2\\ &\le \gamma[(\eta\beta\gamma)^2+1]\|y_0-x_0\|^2\le\gamma\eta(1+a_0^2)\|y_0-x_0\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|y_1-x_1\| &\le \|J_1\|\|F(x_1\|\\ &\le \|J_0\|f(a_0)\gamma\eta(1+a_0^2)\|y_0-x_0\|\\ &\le \beta\gamma\eta(1+a_0^2)f(a_0)\|y_0-x_0\|\\ &= a_0(1+a_0^2)f(a_0)\|y_0-x_0\|=f(a_0)g(a_0)\|y_0-x_0\|,\end{aligned}$$

$$\begin{aligned}L\|J_1\|\|y_1-x_1\| &\le \beta Lf^2(a_0)g(a_0)\|y_0-x_0\|,\\ &\le \beta\gamma\eta f^2(a_0)g(a_0)=a_0f^2(a_0)g(a_0),\end{aligned}$$

$$\begin{aligned}\|x_2-y_1\| &\le \frac{b}{2}\|J_1\|\|y_1-x_1\|^2\\ &\le \frac{b}{2}\|J_0\|f^2(a_0)g(a_0)\|y_0-x_0\|\|y_1-x_1\|\\ &\le \frac{a_0}{2}f^2(a_0)g(a_0)\|y_1-x_1\|,\end{aligned}$$

$$\begin{aligned}\|x_2-x_1\| &\le \|x_2-y_1\|+\|y_1-x_1\|\\ &\le \left[1+\frac{a_0}{2}f^2(a_0)g(a_0)\right]\|y_1-x_1\|\\ &\le \eta f(a_0)g(a_0)\left[1+\frac{a_0}{2}f^2(a_0)g(a_0)\right]\\ &\le \eta f(a_0)g(a_0)(1+a_0/2),\end{aligned}$$

$$\begin{aligned}\|x_2-x_0\| &\le \|x_2-x_1\|+\|x_1-x_0\|\\ &\le (1+a_0/2)(1+f(a_0)g(a_0))\eta=\tau[1-f(a_0)g(a_0)][1+f(a_0)g(a_0)]\\ &\le \tau.\end{aligned}$$

We conclude that $x_2\in D$. Moreover, since $\|J_1\|\|F'(x_2)-F'(x_1)\|\le a_0f^2(a_0)g(a_0)[1+\frac{a_0}{2}f^2(a_0)g(a_0)]<1$, we see that J_2 exists, by the Banach Lemma on invertible operators and $\|J_2\|\le f(a_0f(a_0)^2g(a_0))\|J_1\|$. Therefore we can deduce that y_2, x_2, $x_3\in D$ in an analogous way.

If we set $a_1=a_0f^2(a_0)g(a_0)$ then a straightforward induction argument shows that the sequence

$$a_{n+1}=a_nf^2(a_n)g(a_n) \tag{2.9}$$

is increasing and satisfies the condition $a_n(a_n + a_n/2) < 1$ for all $n \geq 0$. Furthermore, if $y_n, z_n, x_{n+1} \in D$, another induction argument shows that $J_n = F'(x_n)^{-1}$ exists and

$$\begin{aligned}
\|J_n\| &\leq f(a_{n-1})\|J_{n-1}\| \\
\|y_n - x_n\| &\leq f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \\
&\leq (f(a_0)g(a_0))^n\|y_0 - x_0\| < \eta, \\
L\|J_n\|\|y_n - x_n\| &\leq a_n, \\
\|x_{n+1} - y_n\| &\leq \frac{a_n}{2}\|y_n - x_n\|, \\
\|x_{n+1} - x_n\| &\leq (1 + \frac{a_n}{2})\|y_n - x_n\|, \\
\|x_{n+1} - x_0\| &\leq (1 + \frac{a_n}{2})\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - f(a_0)g(a_0)}\|y_0 - x_0\| \\
&< \frac{(1 + a_0/2)\eta}{1 - f(a_0)g(a_0)} = \tau
\end{aligned}$$

These recurrence relations show that the defining sequence (1.2) of INM is well defined. It follows that

$$\begin{aligned}
\|y_n - x_n\| &\leq f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \\
&\leq \cdots \leq (\prod_{i=0}^{n-1} f(a_i)g(a_i))\|y_0 - x_0\| \\
&\leq (f(a_0)g(a_0))^n\|y_0 - x_0\|, \\
\|x_{n+1} - x_n\| &\leq \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \\
&\leq \sum_{j=n}^{n+m-1} (1 + \frac{a_j}{2})\|x_j - x_{j-1}\| \\
&\leq (1 + \frac{a_n}{2}) \sum_{j=n}^{n+m-1} (\prod_{i=0}^{j-1} f(a_i)g(a_i))\|y_0 - x_0\| \\
&\leq (1 + \frac{a_0}{2})(f(a_0)g(a_0))^n \frac{1 - (f(a_0)g(a_0))^m}{1 - f(a_0)g(a_0)}\eta. \qquad (2.10)
\end{aligned}$$

By letting $n = 0$ in (2.10), we get

$$\|x_m - x_0\| \leq (1 + \frac{a_0}{2})\frac{1 - (f(a_0)g(a_0))^m}{1 - f(a_0)g(a_0)}\eta \leq \tau.$$

Therefore $x_m \in U(x_0, \tau)$ for all $m \geq 1$. Similarly, y_m, $z_m \in U(x_0, \tau)$ for all $m \geq 0$. It follows that x_m, y_m, $z_m \in D$ for $m \geq 1$.

Since x_n is a Cauchy sequence – because of (2.10) – it converges to some $x^* \in \bar{U}(x_0, \tau)$. Moreover, since $\|J_nF(x_n)\| \mapsto 0$ and $\|F(x_n)\| \leq \|F'(x_n)\|\|J_nF(x_n)\| \leq c\|J_nF(x_n)\| \mapsto 0$, we conclude that $F(x^*) = 0$.

Finally, we suppose that there exists another solution y^* of (1.1) in $U(x_0, \frac{2}{L\beta} - \tau) \cap D$. Then

$$0 = J_0[F(y^*) - F(x^*)] = \int_0^1 J_0F'(x^* + t(y^* - x^*))\, dt(y^* - x^*).$$

But since

$$\begin{aligned}\|I-T\| &\le \|J_0\| \int_0^1 \|F'(x^*+t(y^*-x^*))-F'(x_0)\|\,dt\\ &\le L\beta \int_0^1 [t\|y^*-x_0\|+(1-t)\|x^*-x_0\|]\,dt\\ &= \frac{1}{2}L\beta[\|y^*-x_0\|+\|x^*-x_0\|] < 1,\end{aligned}$$

where $T = J_0 \int_0^1 F'(x^*+t(y^*-x^*))\,dt$, we see that T is invertible and hence that $y^* = x^*$. That completes the proof of the theorem. □

Remark 2. In view of the Lipschitz condition in (2.3), there exists $L_0 \le L$ such that

$$\|F'(x)-F'(x_0)\| \le L_0\|x-x_0\|$$

for all $x \in D$. It follows from the arguments in the proof of the uniqueness assertion in Theorem 1 that if $\tau < 2/L_0\beta$ the solution x^* of equation (1.1) is unique in the ball $U(x_0, \frac{2}{L\beta}-\tau)\cap D$. If $L = L_0$, the uniqueness balls coincide, but when $L_0 < L$, the ball $U(x_0, \frac{2}{L\beta}-\tau)\cap D$ is larger than the corresponding ball in Theorem 1.

Remark 3. Condition (2.4) can be dropped provided that given L, b, η, β, the inequality

$$L\tau + \|F'(x_0)\| \le t \tag{2.11}$$

has a positive solution denoted by c. Note that in this case, τ is a function of t only when t replaces c in the definition of τ given in (2.8). Indeed, in this case, it would follow from (2.3) that:

$$\begin{aligned}\|F'(x)\| &= \|[F'(x)-F'(x)]+F'(x_0)\|\\ &\le \|F'(x)-F'(x)\|+\|F'(x_0)\|\\ &\le L\|x-x_0\|+\|F'(x_0)\| = c\end{aligned}$$

for all $x \in \bar{U}(x_0,\tau)$.

Remark 4. According to the proof of Theorem 1, the sequence z_n does not have to be included in D or $\bar{U}(x_0,\tau)$. An interesting choice of z_n seems to be

$$z_n = \varepsilon(y_n - x_n), \quad \varepsilon \ge 0.$$

3. Special cases and applications

We provide numerical examples and special cases.

Example 5. Case $z_n \neq 0$.

Let $X = Y = \mathcal{C}[0,1]$, $D = U(1,1)$ and define operator $\mathcal{P}$ on D by

$$\mathcal{P}(x)(s) = \lambda\, x(s) \int_0^1 \mathcal{K}(s,t)\,x(t)\,dt - x(s) + y(s). \tag{3.1}$$

Note that every zero of $\mathcal{P}$ satisfies the equation

$$x(s) = y(s) + \lambda\, x(s) \int_0^1 \mathcal{K}(s,t)\,x(t)\,dt. \tag{3.2}$$

Nonlinear integral equations of the form (3.2) are considered Chandrasekhar–type equations [3, 5, 8, 10] *and they arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gases* [3, 5]. *Here λ is a real number called the "albedo" for scattering and the kernel $\mathcal{K}(s,t)$ is a continuous function in two variables s, t such that*

(i) $0 < \mathcal{K}(s,t) < 1$,
(ii) $\mathcal{K}(s,t) + \mathcal{K}(t,s) = 1$

for all $(s,t) \in [0,1]^2$.

The space X *is equipped with the max-norm:*

$$\|x\| = \max_{0\le s\le 1} |x(s)|.$$

Let us assume for simplicity that

$$\mathcal{K}(s,t) = \frac{s}{s+t} \quad \text{for all} \quad (s,t) \in [0,1]^2. \tag{3.3}$$

Note that the function $\mathcal{K}$ *satisfies conditions (i) and (ii).*

Choose $x_0(s) = y(s) = 1$ *for all* $s \in [0,1]$, $\lambda = .25$, *and*

$$z_n = \frac{1}{100} F''(x_n)\,(y_n - x_n)^2, \tag{3.4}$$

where F'' *is the second Fréchet derivative of operator*F [3]. *Then, using (2.1)–(2.6), we see that*

$$L_0 = L = 2\,|\lambda| \max_{0\le s\le 1} \left| \int_0^1 \frac{s}{s+t}\, dt \right| = 0.346573589$$

$$\|\mathcal{P}'(x_0(s))^{-1}\| \le 1.530394215 = \beta,$$

$$\|\mathcal{P}'(x_0(s))^{-1}\mathcal{P}(x_0(s))\| \le \beta\,|\lambda|\,\ln 2 = 0.265197107 = \eta$$

and

$$b = \frac{4|\lambda|\ln 2}{100} = 0.00693147.$$

Choose $c = 1.4$ *and* $D = \bar{U}(x_0, \tau)$. *Then a computation shows that*

$$\beta L = 0.811712235, \quad \frac{L + c\beta b}{2} = 0.1807123116,$$

$$\frac{b}{2}\sqrt{\frac{L}{L + c\beta b}} = .001539163574, \quad \gamma = \beta L = 0.811712235,$$

$$a_0 = 0.3294383769 < \tau_2 = 0.334375061169603,$$

$$g(a_0) = 0.3651922067, \quad f(a_0) = 1.622594825,$$

$$\tau = 0.7580978534 \quad \text{and} \quad \bar{\tau} = 2/(L\beta) - \tau = 3.012682334.$$

Since condition(2.7) holds, it follows from Theorem 1 that (NM) converges to a unique solution of problem (1.1) in $\bar{U}(x_0, \bar{\tau})$.

CONCLUSION

Using our new concept of recurrent functions and Lipschitz condition, we provided new convergence results for INM. Our semilocal convergence and our error bounds are more precise than earlier ones [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Special cases and numerical example are also provided in this study.

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