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On Certain Subclasses of Analytic Functions of Complex Order Defined by Generalized Hypergeometric Functions

Halit Orhan
Department of Mathematics
Faculty of Science
Ataturk University
25240, Erzurum Turkey
Email: horhan@atauni.edu.tr

Dorina Răducanu
Faculty of Mathematics and Computer Science
Transilvania University of Braşov
50091, Iuliu Maniu, 50,Braşov, Romania
Email: dorinaraducanu@yahoo.com

Abstract. By making use of the generalized hypergeometric functions, in this paper we introduce and investigate certain new subclasses of analytic functions of complex order defined in the open unit disk. Coefficient inequalities, radii of close-to-convexity, starlikeness and convexity, closure theorems, integral means inequalities and several relations associated with (n, δ)-neighborhood for these classes are obtained.

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1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, \ldots\}) \tag{1. 1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ consisting of functions f of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0\,; n \in \mathbb{N}). \tag{1. 2}$$

Let $f \in \mathcal{A}(n)$ given by (1. 1) and let $g \in \mathcal{A}(n)$ given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k.$$

The Hadamard product(or convolution) of f and g is defined by

$$(f*g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k = (g*f)(z) \ (z \in \mathbb{U}).$$

For complex parameters $\alpha_i, \beta_j \in \mathbb{C} - \{0, -1, -2, \ldots\}$ $(i = 1, 2, \ldots, l \, ; j = 1, 2, \ldots, m)$ the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$${}_lF_m(z) \equiv {}_l F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!}$$

$$(l \le m+1 \ , \ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \ , z \in \mathbb{U})$$

where $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1, & k = 0 \\ \lambda(\lambda+1)\ldots(\lambda+k-1), & k \in \mathbb{N}. \end{cases}$$

Let $H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A}(n) \to \mathcal{A}(n)$ be the linear operator defined by

$$\begin{aligned} H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) &:= z {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{k=n+1}^{\infty} \Gamma_k a_k z^k \qquad (1.\ 3) \end{aligned}$$

where

$$\Gamma_k = \frac{(\alpha_1)_{k-1} \ldots (\alpha_l)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_m)_{k-1}} \frac{1}{(k-1)!} \quad (k \ge n+1). \qquad (1.\ 4)$$

If $f \in \mathcal{T}(n)$ is given by (1. 2) then

$$H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := z - \sum_{k=n+1}^{\infty} |\Gamma_k| a_k z^k. \qquad (1.\ 5)$$

For simplicity, in the sequel, we shall write $H_{l,m}(\alpha_1, \beta_1) f(z)$ instead of $H_{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z)$.

The linear operator $H_{l,m}(\alpha_1, \beta_1) f(z)$ is called the Dziok-Srivastava operator [5] and it contains, amongst its special cases, various other operators introduced and studied by Hohlov [7], Carlson-Shaffer [3], Bernardi [2], Libera [10], Livingston [12], Ruscheweyh [18], Srivastava-Owa ([15], [22]).

We say that a function $f \in \mathcal{A}(n)$ is in the class $S_{l,m}(\lambda, \beta, \gamma)$ if

$$\left| \frac{\dfrac{z\nu'(z)}{\nu(z)} - 1}{2\beta\gamma + \dfrac{z\nu'(z)}{\nu(z)} - 1} \right| < 1 \qquad (1.\ 6)$$

$$(z \in \mathbb{U} \ \ , 0 < \beta \le 1 \ \ , \gamma \in \mathbb{C} - \{0\})$$

where

$$\frac{z\nu'(z)}{\nu(z)} = \frac{z(H_{l,m}(\alpha_1, \beta_1) f(z))' + \lambda z^2 (H_{l,m}(\alpha_1, \beta_1) f(z))''}{(1-\lambda) H_{l,m}(\alpha_1, \beta_1) f(z) + \lambda z (H_{l,m}(\alpha_1, \beta_1) f(z))'} \qquad (1.\ 7)$$

$$(z \in \mathbb{U} \ , 0 \le \lambda \le 1 \ , \alpha_i, \beta_j \in \mathbb{C} - \{0, -1, -2, \ldots\} \ , (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m)).$$

A function f in the class $\mathcal{T}(n)$ is said to be in the class $TS_{l,m}(\lambda, \beta, \gamma)$ if it satisfies the following inequality:

$$\left| \frac{1}{\gamma} \left(\frac{z\nu'(z)}{\nu(z)} - 1 \right) \right| < \beta \qquad (1.\ 8)$$

$$(z \in \mathbb{U}\ , \gamma \in \mathbb{C} - \{0\}\ \ , 0 \leq \lambda \leq 1\ , 0 < \beta \leq 1).$$

Finally, we say that a function $f \in \mathcal{T}(n)$ is in the class $TR_{l,m}(\lambda, \beta, \gamma)$ if

$$\left| \frac{1}{\gamma} \left(\nu'(z) - 1 \right) \right| < \beta \tag{1. 9}$$

$$(z \in \mathbb{U}\ , \gamma \in \mathbb{C} - \{0\}\ \ , 0 \leq \lambda \leq 1\ , 0 < \beta \leq 1).$$

We note that there are some known subclasses of our classes of functions $S_{l,m}(\lambda, \beta, \gamma)$, $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$.

Example1. If $l = 2$ and $m = 1$ with $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$ then

$$TS_{2,1}(\lambda, \beta, \gamma) \equiv S_n(\gamma, \lambda, \beta).$$

Example2. If $l = 2$ and $m = 1$ with $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$ then

$$TR_{2,1}(\lambda, \beta, \gamma) \equiv R_n(\gamma, \lambda, \beta).$$

The classes $S_n(\gamma, \lambda, \beta)$ and $R_n(\gamma, \lambda, \beta)$ were investigated in [1].

Example3. If $l = 2$ and $m = 1$ with $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$, $\lambda = 0$, $\beta = |b|$, $\gamma = 1$ then

$$TS_{2,1}(0, |b|, 1) \equiv S_1^*(b),$$

where $b \in \mathbb{C} - \{0\}$.The class $S_1^*(b)$ was studied in [9].

In the present paper we obtain a sufficient condition, in terms of coefficient bounds, for a function to be in the class $S_{l,m}(\lambda, \beta, \gamma)$. We also determine an upper bound for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the class $S_{l,m}(\lambda, \beta, \gamma)$.

Furthermore, coefficient inequalities, radii of starlikeness and convexity, closure theorems, integral means inequalities and several inclusion relations associated with (n, δ)-neighborhoods for the classes $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$ are obtained.

2. Coefficient estimates for the class $S_{l,m}(\lambda, \beta, \gamma)$

In this section we obtain a sufficient condition for a function $f \in \mathcal{A}(n)$ to be in the class $S_{l,m}(\lambda, \beta, \gamma)$ and we also determine an upper bound for the functional $|a_3 - \mu a_2^2|$.

Theorem 1. *Let $f \in \mathcal{A}(n)$ given by (1. 1). If*

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k||a_k| \leq \beta|\gamma| \ \ (z \in \mathbb{U}) \tag{2. 1}$$

where Γ_k is given by (1. 4), then the function f is in the class $S_{l,m}(\lambda, \beta, \gamma)$.

Proof. Suppose that the inequality (2. 1) holds. We have for $z \in \mathbb{U}$,

$$|z\nu'(z) - \nu(z)| - |\nu(z)(2\beta\gamma - 1) + z\nu'(z)| = \left| \sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k-1)\Gamma_k a_k z^k \right| -$$

$$\left| 2\beta\gamma z + \sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + 2\beta\gamma - 1)\Gamma_k a_k z^k \right|$$

$\leq \sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k-1)|\Gamma_k||a_k||z|^k - 2\beta|\gamma||z| + \sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k +$

$2\beta|\gamma| - 1)|\Gamma_k||a_k||z|^k$

$\leq 2 \left(\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k||a_k| - \beta|\gamma| \right) \leq 0.$

which shows that f belongs to $S_{l,m}(\lambda,\beta,\gamma)$. Thus, the proof of the theorem is completed. $\square$

In order to prove our next theorem we need the following lemma.

Lemma 2 ([8]). *Let $w(z)$ of the form*

$$w(z)=\sum_{k=1}^{\infty}c_kz^k \tag{2. 2}$$

be an analytic function in $\mathbb{U}$ such that $|w(z)|<1$ $(z\in\mathbb{U})$. If μ is a complex number then

$$|c_2-\mu c_1^2|\le\max\{1,|\mu|\}\,. \tag{2. 3}$$

Theorem 3. *Let μ be a complex number. If the function $f\in\mathcal{A}(2)$ given by (1. 1) is in the class $S_{l,m}(\lambda,\beta,\gamma)$ then*

$$|a_3-\mu a_2^2|\le\frac{\beta|\gamma|\prod_{j=1}^{m}|\beta_j||\beta_j+1|}{(2\lambda+1)\prod_{i=1}^{l}|\alpha_i||\alpha_i+1|}\max\{1,|d|\}$$

where

$$d:=\frac{4\beta\gamma\mu(2\lambda+1)\prod_{i=1}^{l}(\alpha_i+1)\prod_{j=1}^{m}\beta_j}{(\lambda+1)^2\prod_{i=1}^{l}\alpha_i\prod_{j=1}^{m}(\beta_j+1)}-(2\beta\gamma+1).$$

Proof. Let

$$w(z):=\frac{z\nu'(z)-\nu(z)}{(2\beta\gamma-1)\nu(z)+z\nu'(z)}.$$

It follows that w is an analytic function in $\mathbb{U}$, $w(0)=0$ and $|w(z)|<1$ $(z\in\mathbb{U})$. Consider $w(z)=\sum_{k=1}^{\infty}c_kz^k$. Then

$$(c_1z+c_2z^2+\ldots)=\frac{\frac{1}{2\beta\gamma}\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]\Gamma_ka_kz^{k-1}}{1+\frac{1}{2\beta\gamma}\sum_{k=2}^{\infty}[1+\lambda(k-1)](2\beta\gamma+k-1)\Gamma_ka_kz^{k-1}}$$

which implies

$$(c_1z+c_2z^2+\ldots)\left(1+\frac{1}{2\beta\gamma}\sum_{k=2}^{\infty}[1+\lambda(k-1)](2\beta\gamma+k-1)\Gamma_ka_kz^{k-1}\right)$$
$$=\frac{1}{2\beta\gamma}\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]\Gamma_ka_kz^{k-1}. \tag{2. 4}$$

Equating the coefficients of z and z^2 in both sides of (2. 4) we obtain

$$a_2 = \frac{2\beta\gamma \prod_{j=1}^{m} \beta_j}{(\lambda+1) \prod_{i=1}^{l} \alpha_i} c_1$$

and

$$a_3 = \frac{\beta\gamma \prod_{j=1}^{m} \beta_j(\beta_j+1)}{(2\lambda+1) \prod_{i=1}^{l} \alpha_i(\alpha_i+1)} [c_2 + (2\beta\gamma+1)c_1^2].$$

Hence

$$a_3 - \mu a_2^2 = \frac{\beta\gamma \prod_{j=1}^{m} \beta_j(\beta_j+1)}{(2\lambda+1) \prod_{i=1}^{l} \alpha_i(\alpha_i+1)} (c_2 - dc_1^2) \tag{2. 5}$$

where

$$d := \frac{4\beta\gamma\mu(2\lambda+1) \prod_{i=1}^{l}(\alpha_i+1) \prod_{j=1}^{m} \beta_j}{(\lambda+1)^2 \prod_{i=1}^{l} \alpha_i \prod_{j=1}^{m}(\beta_j+1)} - (2\beta\gamma+1).$$

It follows from (2. 5) that

$$|a_3 - \mu a_2^2| \leq \frac{\beta|\gamma| \prod_{j=1}^{m} |\beta_j||\beta_j+1|}{(2\lambda+1) \prod_{i=1}^{l} |\alpha_i||\alpha_i+1|} |c_2 - dc_1^2|.$$

In virtue of Lemma 2 we obtain

$$|a_3 - \mu a_2^2| \leq \frac{\beta|\gamma| \prod_{j=1}^{m} |\beta_j||\beta_j+1|}{(2\lambda+1) \prod_{i=1}^{l} |\alpha_i||\alpha_i+1|} \max\{1, |d|\}.$$

Thus, we completed the proof of the theorem. □

In the next sections we establish certain properties of the classes $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$.

3. Coefficient inequalities

In this section we obtain necessary and sufficient conditions for a function to be in the class $TS_{l,m}(\lambda,\beta,\gamma)$ and $TR_{l,m}(\lambda,\beta,\gamma)$, respectively.

Theorem 4. *Let $f \in \mathcal{T}(n)$ given by (1. 2). Then f belongs to the class $TS_{l,m}(\lambda,\beta,\gamma)$ if and only if*

$$\sum_{k=n+1}^{\infty}[1+\lambda(k-1)](k+\beta|\gamma|-1)|\Gamma_k|a_k \leq \beta|\gamma| \tag{3. 1}$$

$$(\gamma \in \mathbb{C}-\{0\} \quad ,0 \leq \lambda \leq 1 \quad ,0<\beta \leq 1).$$

Proof. Suppose $f \in TS_{l,m}(\lambda,\beta,\gamma)$. By making use of (1. 8) we easily obtain

$$\mathrm{Re}\left(\frac{z\nu'(z)}{\nu(z)}-1\right) > -\beta|\gamma| \quad (z \in \mathbb{U}) \tag{3. 2}$$

which, in view of (1. 7), gives

$$\mathrm{Re}\left\{\frac{-\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)](k-1)|\Gamma_k|a_k z^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k z^{k-1}}\right\} > -\beta|\gamma| \tag{3. 3}$$

Setting $z=r$ $(0 \leq r < 1)$in (3. 3), we observe that the expression in the denominator on the left hand side of (3. 3) is positive for $r=0$ and also for all $r \in (0,1)$. Thus by letting $r \to 1^-$ through real values, (3. 3) leads us to the desired condition (3. 1) of the theorem.

Conversely, by applying the hypothesis (3. 1) and setting $|z|=1$, we find by using (1. 2) that

$$\left|\frac{z\nu'(z)}{\nu(z)}-1\right| = \left|\frac{\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)](k-1)|\Gamma_k|a_k z^k}{z-\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k z^k}\right|$$

$$\leq \frac{\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)](k-1)|\Gamma_k|a_k|z|^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k|z|^{k-1}}$$

$$\leq \frac{\beta|\gamma|\left(1-\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k\right)}{1-\sum\limits_{k=n+1}^{\infty}[1+\lambda(k-1)]|\Gamma_k|a_k} = \beta|\gamma|.$$

Hence, by the maximum modulus principle, we have $f \in TS_{l,m}(\lambda,\beta,\gamma)$, which evidently completes the proof of the theorem. □

Corollary 5. *If $f \in TS_{l,m}(\lambda, \beta, \gamma)$ then*

$$a_k \leq \frac{\beta|\gamma|}{\Phi(\lambda, \beta, \gamma, k)} \quad (k \geq n+1)$$

where

$$\Phi(\lambda, \beta, \gamma, k) = [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| \tag{3. 4}$$
$$(0 \leq \lambda \leq 1 \ , \ \gamma \in \mathbb{C} - \{0\} \ , \ 0 < \beta \leq 1 \ , \ k \geq n+1).$$

Equality holds for the functions

$$f_k(z) = z - \frac{\beta|\gamma|}{\Phi(\lambda, \beta, \gamma, k)} z^k \quad (z \in \mathbb{U} \ , \ k \geq n+1).$$

By using the same arguments as in the proof of Theorem 4 we can establish the next theorem.

Theorem 6. *Let $f \in \mathcal{T}(n)$ given by (1. 2). Then f is in the class $TR_{l,m}(\lambda, \beta, \gamma)$ if and only if*

$$\sum_{k=n+1}^{\infty} k[1 + \lambda(k-1)]|\Gamma_k| a_k \leq \beta|\gamma| \tag{3. 5}$$
$$(\gamma \in \mathbb{C} - \{0\} \quad , 0 \leq \lambda \leq 1 \quad , 0 < \beta \leq 1)$$

Corollary 7. *If $f \in TR_{l,m}(\lambda, \beta, \gamma)$ then*

$$a_k \leq \frac{\beta|\gamma|}{\Psi(\lambda, \beta, \gamma, k)} \quad (k \geq n+1)$$

where

$$\Psi(\lambda, \beta, \gamma, k) = k[1 + \lambda(k-1)]|\Gamma_k| \tag{3. 6}$$
$$(0 \leq \lambda \leq 1 \ , \ \gamma \in \mathbb{C} - \{0\} \ , \ 0 < \beta \leq 1 \ , \ k \geq n+1).$$

Equality holds for the functions

$$f_k(z) = z - \frac{\beta|\gamma|}{\Psi(\lambda, \beta, \gamma, k)} z^k \quad (z \in \mathbb{U} \ , \ k \geq n+1).$$

4. Radii of close-to-convexity, starlikeness and convexity

We begin this section with the following theorem.

Theorem 8. *Let the function f defined by (1. 2) be in the class $TS_{l,m}(\lambda, \beta, \gamma)$. Then f is close-to-convex of order δ $(0 \leq \delta < 1)$ in $|z| < r_1(\lambda, \beta, \gamma, \delta)$, where*

$$r_1(\lambda, \beta, \gamma, \delta) = \inf_{k \geq n+1} \left[\frac{(1-\delta)[1 + \lambda(k-1)](k + \beta|\gamma| - 1)}{k\beta|\gamma|} |\Gamma_k| \right]^{\frac{1}{k-1}}.$$

Proof. It is sufficient to prove that $|f'(z) - 1| \leq 1 - \delta$ $(0 \leq \delta < 1)$ for $z \in \mathbb{U}$ such that $|z| < r_1(\lambda, \beta, \gamma, \delta)$. We have

$$|f'(z) - 1| = \left| \sum_{k=n+1}^{\infty} k a_k z^{k-1} \right| \leq \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \delta$ if

$$\sum_{k=n+1}^{\infty} \left(\frac{k}{1-\delta} \right) a_k |z|^{k-1} \leq 1. \tag{4. 1}$$

By making use of (3. 1) we obtain

$$\sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k|a_k \leq 1.$$

Then the inequality (4. 1) will be true if

$$\left(\frac{k}{1-\delta}\right)|z|^{k-1} \leq \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k| \ (k \geq n+1)$$

or equivalently

$$|z| \leq \left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}} \quad (k \geq n+1) \tag{4. 2}$$

The theorem follows easily from (4. 2). □

Theorem 9. *Let the function f defined by (1. 2) be in the class $TS_{l,m}(\lambda,\beta,\gamma)$. Then f is starlike of order δ $(0 \leq \delta < 1)$ in $|z| < r_2(\lambda,\beta,\gamma,\delta)$, where*

$$r_2(\lambda,\beta,\gamma,\delta) = \inf_{k\geq n+1}\left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

Proof. We must prove that $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta$ $(0 \leq \delta < 1)$ for $z \in \mathbb{U}$ such that $|z| < r_2(\lambda,\beta,\gamma,\delta)$. We have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{-\sum\limits_{k=n+1}^{\infty}(k-1)a_k z^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}a_k z^{k-1}}\right| \leq \frac{\sum\limits_{k=n+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta$ if

$$\sum_{k=n+1}^{\infty}\left(\frac{k-\delta}{1-\delta}\right)a_k|z|^{k-1} \leq 1. \tag{4. 3}$$

In virtue of (3. 1) we have

$$\sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k|a_k \leq 1.$$

Hence, the inequality (4.3) will be true if

$$\left(\frac{k-\delta}{1-\delta}\right)|z|^{k-1} \leq \frac{[1+\lambda(k-1)](k+\beta|\gamma|-1)}{\beta|\gamma|}|\Gamma_k| \ (k \geq n+1)$$

or if

$$|z| \leq \left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}} \quad (k \geq n+1).$$

Thus, the proof of our theorem is completed. □

Corollary 10. *Let the function f defined by (1. 2) be in the class $TS_{l,m}(\lambda,\beta,\gamma)$. Then f is convex of order δ $(0 \le \delta < 1)$ in $|z| < r_3(\lambda,\beta,\gamma,\delta)$, where*

$$r_3(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{(1-\delta)[1+\lambda(k-1)](k+\beta|\gamma|-1)}{k(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

By using the same arguments as in the proofs of Theorems 8 and 9 we can obtain the radii of close-to-convexity, starlikeness and convexity for the class $TR_{l,m}(\lambda,\beta,\gamma)$.

Theorem 11. *Let f given by (1. 2) be in the class $TR_{l,m}(\lambda,\beta,\gamma)$. Then the function f is close-to-convex of order δ $(0 \le \delta < 1)$ in $|z| < \rho_1(\lambda,\beta,\gamma,\delta)$, where*

$$\rho_1(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{(1-\delta)[1+\lambda(k-1)]}{\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

Theorem 12. *Let f given by (1. 2) be in the class $TR_{l,m}(\lambda,\beta,\gamma)$. Then the function f is starlike of order δ $(0 \le \delta < 1)$ in $|z| < \rho_2(\lambda,\beta,\gamma,\delta)$, where*

$$\rho_2(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{k(1-\delta)[1+\lambda(k-1)]}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

Corollary 13. *Let f given by (1. 2) be in the class $TR_{l,m}(\lambda,\beta,\gamma)$. Then the function f is convex of order δ $(0 \le \delta < 1)$ in $|z| < \rho_3(\lambda,\beta,\gamma,\delta)$, where*

$$\rho_3(\lambda,\beta,\gamma,\delta) = \inf_{k\ge n+1}\left[\frac{(1-\delta)[1+\lambda(k-1)]}{(k-\delta)\beta|\gamma|}|\Gamma_k|\right]^{\frac{1}{k-1}}.$$

5. Closure theorems

Let the functions $f_j \in \mathcal{T}(n)$ $(j = 1,2,\ldots,p)$ defined by

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \quad (z \in \mathbb{U}). \tag{5. 1}$$

We obtain the following results for the closure of functions in the classes $TS_{l,m}(\lambda,\beta,\gamma)$ and $TR_{l,m}(\lambda,\beta,\gamma)$.

Theorem 14. *Let the functions f_j $(j = 1,2,\ldots,p)$ given by (5. 1) be in the class $TS_{l,m}(\lambda,\beta,\gamma)$ and let $c_j \ge 0$ $(j = 1,2,\ldots,p)$ such that $\sum_{j=1}^{p} c_j = 1$. Then the function h defined by*

$$h(z) = \sum_{j=1}^{p} c_j f_j$$

is also in the class $TS_{l,m}(\lambda,\beta,\gamma)$.

Proof. In virtue of the definition of h, we can write

$$h(z) = \sum_{j=1}^{p} c_j\left[z - \sum_{k=n+1}^{\infty} a_{k,j} z^k\right]$$

$$= \left(\sum_{j=1}^{p} c_j\right) z - \sum_{k=n+1}^{\infty}\left(\sum_{j=1}^{p} c_j a_{k,j}\right) z^k$$

$$= z - \sum_{k=n+1}^{\infty} \left(\sum_{j=1}^{p} c_j a_{k,j} \right) z^k.$$

Since the functions f_j are in $TS_{l,m}(\lambda, \beta, \gamma)$, for every $j = 1, 2, \ldots, p$ we have

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| a_{k,j} \leq \beta|\gamma|.$$

Hence we get

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| \left(\sum_{j=1}^{p} c_j a_{k,j} \right)$$

$$= \sum_{j=1}^{p} c_j \left(\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| a_{k,j} \right) \leq \sum_{j=1}^{p} c_j (\beta|\gamma|) = \beta|\gamma|$$

which implies that h is in the class $TS_{l,m}(\lambda, \beta, \gamma)$. Thus, the proof of the theorem is completed. □

Corollary 15. *The class $TS_{l,m}(\lambda, \beta, \gamma)$ is closed under convex linear combination.*

Proof. Assume that the functions f_j $(j = 1, 2)$ given by (5. 1) are in the class $TS_{l,m}(\lambda, \beta, \gamma)$. It is sufficient to show that the function h defined by

$$h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \ \ (0 \leq \mu \leq 1)$$

is in the class $TS_{l,m}(\lambda, \beta, \gamma)$.

By taking $p = 2$, $c_1 = \mu$ and $c_2 = 1 - \mu$ in Theorem 14 we obtain the corollary. □

Making use of the same arguments as in the proofs of Theorem 14 and Corollary 15, closure properties for the class $TR_{l,m}(\lambda, \beta, \gamma)$ can also be obtained.

6. Convolution and integral properties

In this section we shall prove that the classes $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$ are closed under convolution and integral operator.

Theorem 16. *Let $g(z)$ of the form*

$$g(z) = z - \sum_{k=n+1}^{\infty} c_k z^k \ \ (0 \leq c_k \leq 1 \ , \ k \geq n+1)$$

be analytic in $\mathbb{U}$. If the function f belongs to the class $TS_{l,m}(\lambda, \beta, \gamma)$ then the function $f * g$ is also in the class $TS_{l,m}(\lambda, \beta, \gamma)$.

Proof. Since$f \in TS_{l,m}(\lambda, \beta, \gamma)$ by (3. 1) we have

$$\sum_{k=n+1}^{\infty} [1 + \lambda(k-1)](k + \beta|\gamma| - 1)|\Gamma_k| a_k \leq \beta|\gamma|.$$

By making use of the last inequality and the fact that

$$(f * g)(z) = z - \sum_{k=n+1}^{\infty} a_k c_k z^k$$

we obtain

$$\sum_{k=n+1}^{\infty}[1+\lambda(k-1)](k+\beta|\gamma|-1)|\Gamma_k|a_kc_k$$
$$\leq \sum_{k=n+1}^{\infty}[1+\lambda(k-1)](k+\beta|\gamma|-1)|\Gamma_k|a_k \leq \beta|\gamma|$$

and hence, in virtue of Theorem 4, the result follows. □

Let $I_c : \mathcal{T}(n) \to \mathcal{T}(n)$ be the integral operator defined by

$$F(z) = I_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (c > -1\ ,\ z \in \mathbb{U}) \tag{6. 1}$$

We note that if $f \in \mathcal{T}(n)$ is given by (1.2) then

$$F(z) = z - \sum_{k=n+1}^{\infty}\frac{c+1}{c+k}a_kz^k. \tag{6. 2}$$

Theorem 17. *If the function f is in the class $TS_{l,m}(\lambda,\beta,\gamma)$ then the function F given by (6. 1) is also in $TS_{l,m}(\lambda,\beta,\gamma)$.*

Proof. From (6. 2) it results that $F(z) = (f * g)(z)\ (z \in \mathbb{U})$, where

$$g(z) = z - \sum_{k=n+1}^{\infty}\frac{c+1}{c+k}z^k \ \text{ and }\ 0 \leq \frac{c+1}{c+k} \leq 1.$$

By Theorem 16, the proof is trivial. □

The proofs for the convolution and integral properties for the class $TR_{l,m}(\lambda,\beta,\gamma)$ are similar.

7. Integral means inequalities

In order to prove the results regarding integral means inequalities we need the concept of subordination between analytic functions and also a subordination theorem due to Littlewood [11].

Let f and g be two analytic functions in $\mathbb{U}$. The function g is said to be subordinate to f,denoted by $g \prec f$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1\ (z \in \mathbb{U})$ such that $g(z) = f(w(z))\ (z \in \mathbb{U})$.

Lemma 18 ([11]). *If f and g are two analytic functions in $\mathbb{U}$ such that $g \prec f$ then*

$$\int_0^{2\pi}|g(re^{i\theta})|^{\mu}d\theta \leq \int_0^{2\pi}|f(re^{i\theta})|^{\mu}d\theta \quad (\mu > 0\ ,\ 0 < r < 1).$$

Theorem 19. *Suppose $f \in TS_{l,m}(\lambda,\beta,\gamma)$ and let the function $f_2(z)$ defined by*

$$f_2(z) = z - \frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)}z^2 \quad (z \in \mathbb{U})$$

where $\Phi(\lambda,\beta,\gamma,k)$ is defined by (3 . 4).If $\{\Phi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, then

$$\int_0^{2\pi}|f(z)|^{\mu}d\theta \leq \int_0^{2\pi}|f_2(z)|^{\mu}d\theta \quad (z = re^{i\theta}\ ,\ 0 < r < 1\ ,\ \mu > 0).$$

Proof. Let $f(z)=z-\sum_{k=2}^{\infty}a_kz^k$. For $z=re^{i\theta}$ $(0<r<1)$ and $\mu>0$ we must show that

$$\int_0^{2\pi}|1-\sum_{k=2}^{\infty}a_kz^{k-1}|d\theta\leq\int_0^{2\pi}|1-\frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)}z|d\theta.$$

By applying Lemma 18 it would suffice to prove that

$$1-\sum_{k=2}^{\infty}a_kz^{k-1}\prec 1-\frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)}z \tag{7. 1}$$

Setting

$$1-\sum_{k=2}^{\infty}a_kz^{k-1}=1-\frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)}w(z)\ \ (z\in\mathbb{U})$$

we find that

$$w(z)=\sum_{k=2}^{\infty}\frac{\Phi(\lambda,\beta,\gamma,2)}{\beta|\gamma|}a_kz^{k-1}$$

which readily yields $w(0)=0$. Since $\{\Phi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, we have

$$\Phi(\lambda,\beta,\gamma,2)\leq\Phi(\lambda,\beta,\gamma,k)\ \ (k\geq 2).$$

In virtue of (3. 1) we obtain

$$|w(z)|=|\sum_{k=2}^{\infty}\frac{\Phi(\lambda,\beta,\gamma,2)}{\beta|\gamma|}a_kz^{k-1}|$$

$$\leq|z|\sum_{k=2}^{\infty}\frac{\Phi(\lambda,\beta,\gamma,k)}{\beta|\gamma|}a_k\leq|z|<1.$$

The last inequality shows that we have the subordination (7. 1), which evidently proves our theorem. □

The proof of the next theorem is the same with the proof of the Theorem (19).

Theorem 20. *Suppose $f\in TS_{l,m}(\lambda,\beta,\gamma)$ and let the function $f_2(z)$ defined by*

$$f_2(z)=z-\frac{\beta|\gamma|}{\Phi(\lambda,\beta,\gamma,2)}z^2\ \ (z\in\mathbb{U})$$

where $\Phi(\lambda,\beta,\gamma,k)$ is defined by (3. 4). If $\{\Phi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, then

$$\int_0^{2\pi}|f'(z)|^{\mu}d\theta\leq\int_0^{2\pi}|f_2'(z)|^{\mu}d\theta\ \ (z=re^{i\theta}\ ,\ 0<r<1\ ,\ \mu>0).$$

In the same way we can obtain integral means inequalities for the class $TR_{l,m}(\lambda,\beta,\gamma)$.

Theorem 21. *Suppose $f\in TR_{l,m}(\lambda,\beta,\gamma)$ and let the function $f_2(z)$ defined by*

$$f_2(z)=z-\frac{\beta|\gamma|}{\Psi(\lambda,\beta,\gamma,2)}z^2\ \ (z\in\mathbb{U})$$

where $\Psi(\lambda,\beta,\gamma,k)$ is defined by (3 . 6).If $\{\Psi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ is a non-decreasing sequence, then

$$\int_0^{2\pi}|f(z)|^{\mu}d\theta\leq\int_0^{2\pi}|f_2(z)|^{\mu}d\theta\ \ (z=re^{i\theta}\ ,\ 0<r<1\ ,\ \mu>0)$$

and

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |f_2'(z)|^\mu d\theta \ \ (z = re^{i\theta}\ ,\ 0 < r < 1\ ,\ \mu > 0).$$

8. Inclusion relationships involving the (n, δ)-neighborhoods

In this section we establish some inclusion relationships involving the (n, δ)-neighborhoods for each of the classes $TS_{l,m}(\lambda, \beta, \gamma)$ and $TR_{l,m}(\lambda, \beta, \gamma)$.

Following the earlier investigations by Goodman [6], Ruscheweyh [19], Silverman [20]and others ([14], [23]) we define the (n, δ)-neighborhood of a function $f \in \mathcal{T}(n)$ by

$$N_{n,\delta}(f) = \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\}. \quad (8.\ 1)$$

In particular, for identity function $e(z) = z$ $(z \in \mathbb{U})$ we immediately have

$$N_{n,\delta}(e) = \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}. \quad (8.\ 2)$$

Theorem 22. *If* $\{\Phi(\lambda, \beta, \gamma, k)\}_{k=2}^{\infty}$ *is a non-decreasing sequence and*

$$\delta := \frac{(n+1)\beta|\gamma|}{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}|} \quad (8.\ 3)$$

then

$$TS_{l,m}(\lambda, \beta, \gamma) \subset N_{n,\delta}(e).$$

Proof. Let $f \in TS_{l,m}(\lambda, \beta, \gamma)$. Then in view of the assertion (3. 1) of Theorem 4 and the given condition

$$\Phi(\lambda, \beta, \gamma, n+1) \leq \Phi(\lambda, \beta, \gamma, k) \ \ (k \geq n+1)$$

we get

$$\Phi(\lambda, \beta, \gamma, n+1) \sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|$$

or

$$(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}| \sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|$$

which implies that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}|}. \quad (8.\ 4)$$

Applying the assertion (3. 1) of Theorem 4 again, in conjuction with (8. 4) we find

$$(\lambda n + 1)|\Gamma_{n+1}| \sum_{k=n+1}^{\infty} k a_k \leq \beta|\gamma| + (1 - \beta|\gamma|)(\lambda n + 1)|\Gamma_{n+1}| \sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta|\gamma| + (1 - \beta|\gamma|)(\lambda n + 1)|\Gamma_{n+1}| \frac{\beta|\gamma|}{(\lambda n + 1)(n + \beta|\gamma|)|\Gamma_{n+1}|}$$

$$= \frac{(n+1)\beta|\gamma|}{n + \beta|\gamma|}.$$

Hence

$$\sum_{k=n+1}^{\infty} ka_k \le \frac{(n+1)\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|} =: \delta$$

which in virtue of (8. 2), proves our theorem. □

Similarly, by applying the assertion (3. 5) of Theorem 6 instead of the assertion (3. 1) in Theorem 4 we can prove the following theorem.

Theorem 23. *If* $\{\Psi(\lambda,\beta,\gamma,k)\}_{k=2}^{\infty}$ *is a non-decreasing sequence and*

$$\delta := \frac{\beta|\gamma|}{(\lambda n+1)|\Gamma_{n+1}|}$$

then

$$TR_{l,m}(\lambda,\beta,\gamma) \subset N_{n,\delta}(e).$$

9. Neighborhoods for the classes $TS_{l,m}(\lambda,\beta,\gamma)$, $TR_{l,m}(\lambda,\beta,\gamma)$

In the sequence, we shall determine the neighborhood properties for each of the classes of functions

$$TS_{l,m}^{(\alpha)}(\lambda,\beta,\gamma) \text{ and } TR_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$$

which are defined as follows.

A function $f \in \mathcal{T}(n)$ is said to be in the class $TS_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$ if there exists a function $g \in TS_{l,m}(\lambda,\beta,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha \quad (z \in \mathbb{U}\ ,\ 0 \le \alpha < 1) \tag{9. 1}$$

Analogously, a function $f \in \mathcal{T}(n)$ is said to be in the class $TR_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$ if there exists a function $g \in TR_{l,m}(\lambda,\beta,\gamma)$ such that the inequality (9. 1) holds true.

Theorem 24. *If* $g \in TS_{l,m}(\lambda,\beta,\gamma)$ *and*

$$\alpha := 1 - \frac{\delta(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|}{(n+1)[(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}| - \beta|\gamma|]} \quad (\beta|\gamma| \ge 1) \tag{9. 2}$$

then $N_{n,\delta}(g) \subset TS_{l,m}^{(\alpha)}(\lambda,\beta,\gamma)$.

Proof. Suppose that $f \in N_{n,\delta}(g)$. Then, from definition (8. 1) we find that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \le \delta$$

which readily implies the inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \le \frac{\delta}{n+1} \quad (n \in \mathbb{N}).$$

Since $g \in TS_{l,m}(\lambda,\beta,\gamma)$, from (8. 4) we have

$$\sum_{k=n+1}^{\infty} b_k \le \frac{\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|}.$$

It follows that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum\limits_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=n+1}^{\infty}b_k}$$

$$\leq \frac{\delta}{n+1}\cdot\frac{(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|}{(\lambda n+1)(n+\beta|\gamma|)|\Gamma_{n+1}|-\beta|\gamma|}=1-\alpha$$

provided that α is given by (9. 2). Thus, the proof of our theorem is completed. □

The proof of Theorem 25 below is much akin to that of the Theorem 24.

Theorem 25. *If* $g \in TR_{l,m}(\lambda,\beta,\gamma)$ *and*

$$\alpha := 1-\frac{\delta(\lambda n+1)|\Gamma_{n+1}|}{(n+1)(\lambda n+1)|\Gamma_{n+1}|-\beta|\gamma|} \quad (\beta|\gamma|\geq 1)$$

then $N_{n,\delta}(g) \subset TR^{(\alpha)}_{l,m}(\lambda,\beta,\gamma)$.

Remark 26. By taking $\Gamma_{n+1}=1$ in Theorem 24 and also in Theorem 25 we obtain the inclusion relation of Altintaş et al. [1].

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