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On The Total Differential of Almost Quasiconformal Mappings

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Abstract. We give some sufficient conditions of existence of total differential at a point (which may be a boundary point)for almost quasiconformal mappings.

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1. Auxiliary Concepts

1.1. Let D be a domain in $\mathbb{R}^n$ and $f : D \to \mathbb{R}^m$ be a vector function. The vector function $f : D \to \mathbb{R}^m$ has at a point $a \in D$ a *total differential*, if there exists a constant matrix

$$C = \{C_{ij}\}_{\substack{1\leq i\leq n \\ 1\leq j\leq m}}$$

such that

$$f(x) - f(a) = C \cdot (x - a) + o(|x - a|) \quad (x \to a, \quad x \in D)\,. \tag{1. 1}$$

It is known, that a function f has a total differential at a point $a \in D$, if in a neighborhood of a there exist partial derivatives $\partial f_i/\,\partial x_j$ $(i = 1,\dots,n, j = 1,\dots,m)$, which are continuous at a. There are examples which show that the continuity of partial derivatives at a is not a necessary condition for the existence a total differential at a.

1.2. Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of the class $W^{1,n}_{\rm loc}(D)$. We put

$$f'(x) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}\cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdots & \cdots \\ \dfrac{\partial f_m}{\partial x_1}\cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix} and, next, \|f'(x)\| = \left(\sum_{i=1}^m\sum_{j=1}^n\left(\frac{\partial f_i}{\partial x_j}(x)\right)^2\right)^{1/2}.$$

We shall say, that a mapping $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ belongs to a class $\mathcal{W}^{1,n}_{\rm loc}(D)$, if for an arbitrary subdomain $D' \subset\subset D$ there exists a constant $p > n$ which, in general, depends on D', such that $f \in W^{1,p}(D')$. A continuous mapping $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is *almost quasiconformal* in D with a measurable function $K(x) \geq 0$ and locally integrable function

$\delta(x) : D \to \mathbb{R}$, if $f \in \mathcal{W}^{1,n}_{\rm loc}(D)$ and almost everywhere on D the following property holds

$$\|f'(x)\|^n \le K(x)\ |J(x,f)| + \delta(x)\,, \qquad (1.\ 2)$$

where

$$J(x,f) = \det\ (f'(x))\ .$$

The concept of almost quasiconformal maps belongs to Callender [6], however we note that the condition (1. 2) has in [6] a different form. Namely, in [6] it is assumed that $K(x) \equiv \text{const}$, and instead of $|J(x,f)|$ it is written $J(x,f)$. Thus the class of maps considered here is essentially wider than the class considered by Callender in [6]. In particular, our definition permits to consider degenerate quasiconformal maps.

Under condition of preservation of the Jacobian sign and the assumptions

$$K \equiv \text{const} > 0, \quad \delta \equiv 0,$$

the supposition (1. 2) means that the mapping f is quasiregular [23, §3 Ch. I], [25, Sect. 14.1]. It should be noted that in the case of quasiregular maps it is assumed only that the vector-function f is continuous and belongs to $W^{1,n}_{\rm loc}(D)$, and the supposition $\mathcal{W}^{1,n}_{\rm loc}(D)$ holds automatically.

The assumption (1. 2) does not require that the sign of $\det\ (f'(x))$ is constant. Thus, almost quasiconformal maps can change their orientation.

The following simple statement [12, Sect. 8.1] shows that the class of considered maps is very wide.

Proposition 1. *Let $f : D \to \mathbb{R}^n$ be a mapping and moreover*

$$f \in \text{ACL}(D) \quad \textit{and} \quad \operatorname{ess\,sup}_{x\in D}\|f'(x)\| \le q < \infty\,.$$

Then f is almost quasiconformal with $K = \epsilon\, n^{n/2}$ and $\delta = (1+\epsilon)\, q^n$, where $\epsilon = \text{const} > 0$ is arbitrary.

1.3. In the case, if $D \subset \mathbb{R}^2$ and $a \in \partial D$ is a multiple point of a boundary, the relation (1. 1) can depend on a direction of the approach to the point a from D and, consequently, the definition of the total differential must be more precise.

We define ends of a domain D using analogy with the Carathéodory theory of prime ends (see, for example, [13, §3]).

For an arbitrary set $U \subset D$ we put $[U] = \overline{U} \setminus \partial D$, where $\overline{U}$ is the closure with respect to $\mathbb{R}^n$. Let $\{U_k\}$, $k = 1, 2, \ldots$ be a family of subdomains $U_k \subset D$ with properties:

(i) for every $k = 1, 2, \ldots \quad [U_{k+1}] \subset U_k$,

(ii) $\bigcap\limits_{k=1}^{\infty} [U_k] = \emptyset$.

An arbitrary sequence $\{U_k\}$ with these properties is called a *chain* in D.

Let $\{U'_k\}$, $\{U''_k\}$ be two chains of subdomains of D. We say, that U'_k is *contained* in $\{U''_k\}$, if for every $m \ge 1$ there is a number $k(m)$ such that for all $k > k(m)$ the following property holds $U'_k \subset U''_m$. Two chains are called *equivalent*, if each of them is contained in the other one. The classes of equivalence ξ of chains are called *ends* of D.

To define an end ξ it is sufficient to set even one representative of the class of equivalence. If an end ξ is defined with a chain $\{U_k\}$, then we write $\xi \asymp \{U_k\}$.

A *body* of a end $\xi \asymp \{U_k\}$ is the set

$$|\xi| = \cap_{i=1}^{\infty} \overline{U_i}\,.$$

It is clear, that this set does not depend on the choice of a chain $\{U_k\}$.

Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of points $x_m \in D$ which does not have condensation point in D. Such sequences are called *nonconvergent* in D.

Let $a_\xi \in |\xi|$ be an arbitrary point. A nonconvergent in D sequence of points $x_k \in D$ *converges to a point* a_ξ *with respect to the topology of* ξ, if $x_k \to a_\xi$ (with respect to the topology $\mathbb{R}^n$) and for some chain $\{U_k\} \in \xi$ the following property holds: for every $k = 1, 2, \dots$ there is a number $m(k)$ such that $x_m \in U_k$ for arbitrary $m > m(k)$.

Let D be a domain in $\mathbb{R}^n$, ξ be an end of D, $a_\xi \in |\xi|$ be a point. We shall say, that a subdomain D' of D *adjoins* at a_ξ, if $a_\xi \in \partial D'$ and any sequence of points $x_k \in D'$, converging to a_ξ with respect to the topology $\mathbb{R}^n$, converges to this point with respect to the topology of ξ. We shall say, that a vector-function $f : D \to \mathbb{R}^m$ satisfies to the property

$$\lim_{x \to a_\xi} f(x) = A\,, \quad A = (A_1, \dots, A_m)$$

if

$$f(x_k) \to A \quad \text{as} \quad x_k \to a_\xi$$

along every sequence of point $x_k \in D$, which converges to a_ξ with respect to the topology of ξ. The vector A is denoted by $f(a_\xi)$.

Suppose that a vector-function $f : D \to \mathbb{R}^m$ and a point a_ξ are such that $f(a_\xi)$ exists. We say, that f has a *total differential at a boundary point* a_ξ, if there exists a constant matrix $C = \{C_{ij}\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}$, for which

$$f(x) - f(a_\xi) = C \cdot (x - a_\xi) + o(|x - a_\xi|) \quad (x \to a_\xi, \quad x \in D). \tag{1. 3}$$

As in the case of an inner point, we shall say that

$$df(a_\xi) = C \cdot (x - a_\xi)$$

is a differential of f at a_ξ.

The differential of the vector-function at the boundary point need not be unique (see corresponding examples in [9]).

2. **The weighted modulus**

2.1. Recall the definition of the class $\mathrm{ACL}^{\mathrm{p}}_{\sigma}$. Let $D \subset \mathbb{R}^n$ be an open set. Fix i, $1 \le i \le n$, and denote by D_i^* an orthogonal projection of D onto the hyperplane $x_i = 0$. For an arbitrary locally summable in D function f we put

$$f_i^*(x_i', t, x_i'') \equiv f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\,,$$

$$x_i' = (x_1, \dots, x_{i-1}), \quad x_i'' = (x_{i+1}, \dots, x_n)\,.$$

Next, let

$$D_i(x_i', x_i'') \equiv \{(x_i', t, x_i'') \in \mathbb{R}^n : (x_i', 0, x_i'') \in D_i^*\}\,.$$

A continuous function $f : D \to \mathbb{R}$ is called *absolutely continuous on lines* (or shortly, ACL), if for every $i = 1, \dots, n$, the coordinate functions $f_i^*(x_i', t, x_i'')$ are absolutely continuous (with respect to the variable t) inside the union of linear intervals $D \cap D_i(x_i', x_i'')$ for $\mathcal{H}^{n-1}$-almost all points $(x_i', 0, x_i'') \in D_i^*$. (Here and below the symbol $d\mathcal{H}^p$ means an element of p-dimensional Hausdorff measure.)

Every ACL-function $f : D \to \mathbb{R}$ has partial derivatives $\partial f / \partial x_i$ $(i = 1, \dots, n)$ almost everywhere in D. By the symbol $f' \equiv (\partial f_i / \partial x_j)$ we denote a formal derivative of f at points, where all partial derivatives exist. At points, in which the matrix f' is not defined, let us agree to take all $\partial f_i/\partial x_j = +\infty$ $(i = 1, \dots, n; j = 1, \dots, m)$.

Let $\sigma : D \to \mathbb{R}^1$ be a nonnegative measurable function, which is defined almost everywhere in D, and let $p \geq 1$ be a constant. The class $\mathrm{ACL}^p_\sigma(D)$ is the set of ACL-functions in D, for which

$$\int\limits_D \|f'(x)\|^p \, \sigma(x) \, d\mathcal{H}^n < \infty \, .$$

In the case, if the weight function $\sigma \equiv 1$, we have the well known class $\mathrm{ACL}^p(D)$, which coincides with the set of continuous $W^1_p(D)$-functions [17, Theorems 5.6.2-3].

2.2. Let D be a domain in $\mathbb{R}^m$, $m > 1$, let $U \subset D$ be a countably $(\mathcal{H}^k, k)$-rectifiable set, $1 \leq k \leq m$, and let $\sigma : U \to \mathbb{R}^1$ be a nonnegative $\mathcal{H}^k$-measurable function. Fix a constant $p > 1$ and for an arbitrary family Γ of locally rectifiable arcs $\gamma \subset U$, we define a (p, σ)-modulus

$$\mathrm{mod}_{p,\sigma}(\Gamma; U) = \inf_\rho \frac{\int\limits_U \rho^p \, \sigma \, d\mathcal{H}^k}{\left(\inf\limits_{\gamma \in \Gamma} \int\limits_\gamma \rho \, d\mathcal{H}^1 \right)^p} \, , \tag{2. 4}$$

where the infimum is taken over all nonnegative, Borel measurable functions ρ in U. If $\Gamma = \emptyset$, then we put $\mathrm{mod}_{p,\sigma}(\Gamma; U) = \infty$.

In the case $U = D$ we have a standard definition of the weighted (p, σ)-modulus of the family Γ in $\mathbb{R}^n$ (see, for example, [18, Sect. 3.2]).

2.3. Let y and a be a pair of points such that $y \in D$ and either a is an interior point of D, or $a = a_\xi \in |\xi|$, where ξ is an end of the domain $D \subset \mathbb{R}^n$. We say that a simple Jordan arc γ, defined by a parametrization $x(\tau) : [0, 1) \to D$, *leads from* y *to* a, if $x(0) = y$ and

$$\lim_{\tau \to 1} x(\tau) = a \quad \text{as} \quad a \in D$$

and there is a sequence $\tau_k \to 1$, along which

$$\lim_{\tau_k \to 1} x(\tau_k) = a_\xi \quad \text{as} \quad a \in |\xi| \, .$$

We consider a family Γ of all locally rectifiable, simple Jordan arcs $\gamma \subset D$, leading from y to a. We put

$$\mathrm{mod}_{p,\sigma} \Gamma(y, a; D) = \mathrm{mod}_{p,\sigma}(\Gamma; D) \, . \tag{2. 5}$$

2.4. Let $D \subset \mathbb{R}^n$ be a domain and $a = a_\xi$ be its interior or boundary point. Fix a continuous vector-function $\nu : \overline{D} \to \mathbb{R}^k$, $1 \leq k < \infty$ and put $B^\nu(a, r) = \{x \in D : |\nu(x) - \nu(a)| < r\}$. By $B^\nu_D(a, r)$ we denote a connected component of $B^\nu(a, r)$, containing a if a is an interior point of D, and adjoining at a if $a \in |\xi|$.

By $S^\nu_D(a, r)$ we denote the relative boundary

$$S^\nu_D(a, r) = \partial B^\nu_D(a, r) \setminus \partial D \, .$$

In the case $\nu(x) \equiv x$ we shall use notations $B^n(a, r)$, $B^n_D(a, r)$ and $S^\nu_D(a, r)$, respectively.

Suppose that $\nu(x)$ is locally Lipschitz. Let $h(x) = |\nu(x) - \nu(a)|$ and let

$$0 < \mathrm{ess} \inf_{x \in D'} |\nabla h(x)| \leq \mathrm{ess} \sup_{x \in D'} |\nabla h(x)| < \infty \tag{2. 6}$$

on every subset $D' \Subset D$.

By Theorem 3.2.15 [16] (see also [18, Theorem 1.6.1]) for a.e. $t \in \mathbb{R}^1$ the sets $S_D^\nu(a,t)$ are countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable.

Fix a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set $S_D^\nu(a,t)$ and a measurable function $\sigma : S_D^\nu(a,t) \to \mathbb{R}^1$. Let U be a connected component of $S_D^\nu(a,t)$. For a pair of points $a_1,\ a_2 \in U$ let $\Gamma = \Gamma(a_1, a_2)$ stands for the family of all locally rectifiable arcs $\gamma \subset U$, joining points a_1 and a_2. We define a weighted modulus

$$\operatorname{mod}(a_1, a_2; \sigma) = \operatorname{mod}_{n,\sigma}\Gamma(a_1, a_2)\,. \tag{2. 7}$$

Next, let

$$\kappa(S_D^\nu(a,t), \sigma) = \inf_U \inf_{a_1, a_2 \in U} \operatorname{mod}(a_1, a_2; \sigma), \tag{2. 8}$$

where the first of infimums is taken over the collection $\{U\}$ of all connected components U of $S_D^\nu(a,t)$.

We put

$$\kappa^\nu(a,t) = \kappa(S_D^\nu(a,t), \sigma^*)\,, \quad \sigma^* = \frac{\sigma}{|\nabla h|},$$

where $\sigma : D \to \mathbb{R}^1$ is a nonnegative measurable function.

2.5. We shall need the following multidimensional version of known "Length and Area Principle" (see, for example, [19], [13]).

Lemma 2. [7] *Let D be a domain in $\mathbb{R}^n$, let $a = a_\xi \in \overline{D}$, let a vector-function $\nu : D \to \mathbb{R}^k$ satisfy* (2. 6) *and $\sigma(x) : D \to \mathbb{R}^1$ is a nonnegative, measurable function. Let $f : D \to \mathbb{R}^m$ be a vector-function of the class $\mathrm{ACL}^n_\sigma(D)$. Then for arbitrary $t',\ t'' \in h(D)$, $t' < t''$, the following inequality holds*

$$\int\limits_{t'}^{t''} \Omega^n\left(f, S_D^\nu(a,t)\right) \kappa^\nu(a,t)\, dt \le \int\limits_{D(t',t'')} \|f'(x)\|^n\, \sigma(x)\, d\mathcal{H}^n(x)\,. \tag{2. 9}$$

Here

$$D(t', t'') = \{x \in D : t' < |\nu(x) - \nu(a)| < t''\},$$

$$\Omega\left(f, S_D^\nu(a,t)\right) = \sup_U \operatorname{osc}\left(f, U\right)$$

and the infimum is taken over all connected components U of $S_D^\nu(a,t)$.

3. Main results

3.1. Let $D \subset \mathbb{R}^n$ be an open set. We say that a function $f : D \to \mathbb{R}^m$, $m \ge 1$, is *monotone*, if for every subdomain $U \subset D$ the following property holds

$$\operatorname{osc}\,(f, U) \le \operatorname{osc}\,(f, \partial' U)\,, \quad \partial' U = \partial U \setminus D\,.$$

Here and below by the symbol

$$\operatorname{osc}\,(\phi, E) = \sup_{x,y \in E} |\phi(x) - \phi(y)|$$

we denote the oscillation of a function ϕ on E.

Let $h(t) : [0,\infty) \to [0,\infty)$ be an upper semicontinuity function. We shall say, that $f : D \to \mathbb{R}^m$, $m \ge 1$, is h-*monotone*, if for every subdomain $U \subset D$ we have

$$h\left(\operatorname{osc}\,(f,U)\right) \le \operatorname{osc}\,(f,\partial' U)\,,$$

and is α-*monotone*, $0 < \alpha \equiv \text{const} < \infty$, if f is h-monotone with $h(t) = t^\alpha$.

Some examples of α-monotone functions were given in [8].

Fix a continuous vector-function $\nu : \overline{D} \to \mathbb{R}^k$. We say, that a vector-function $f : D \to \mathbb{R}^m$, $m \ge 1$, is *weakly* (h,ν)-*monotone close to a point* $a = a_\xi$ (interior or boundary), if

$$\limsup_{r\to 0} \frac{h\left(\operatorname{osc}\left(f, B^\nu_D(a,r)\right)\right)}{\operatorname{osc}\left(f, S^\nu_D(a,r)\right)} < \infty\,, \tag{3. 10}$$

and *weakly* (α,ν)-*monotone close to a point* a, if f is weakly (h,ν)-monotone close to a for $h(t) = t^\alpha$.

It is clear, that every monotone in the Lebesgue sense function is weakly (α,ν)-monotone, $\alpha = 1$, close to every point.

3.2 For an arbitrary continuous mapping $y = \varphi(x) : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and for a set $A \subset D$ by $N(y;\varphi,A)$, we shall denote the number of preimages of a point $y \in \mathbb{R}^n$ in A. Next we put

$$n(x;\varphi,A) = N(y;\varphi,A)\,, \quad \text{where} \quad y = \varphi(x)\,.$$

3.3. The following statement is the main result of this paper

Theorem 3. *Suppose that a vector-function $f : D \to \mathbb{R}^n$ is an almost quasiconformal mapping of a domain $D \subset \mathbb{R}^n$ in the sense* (1. 2)*, for which*

$$\int\limits_D \frac{\delta(x)\,dx}{K(x)} < \infty\,. \tag{3. 11}$$

Then for every subdomain $A \subset D$ the following inequality holds

$$\int\limits_A \frac{\|f'(x)\|^n\,dx}{K(x)\,n(x;f,A)} \le \operatorname{mes}_n(f(A)) + \int\limits_A \frac{\delta(x)\,dx}{K(x)\,n(x;f,A)}\,. \tag{3. 12}$$

On the other hand, let $a = a_\xi \in \overline{D}$ be an interior or boundary point of the domain and let $\nu : \overline{D} \to \mathbb{R}^k$ be a continuous vector-function satisfying (2. 6)*. If*

$i)$ *for some $p > n$ and some constant matrix $C = (C_{ij})^n_{i,j=1}$ the following assumption holds*

$$\frac{\limsup_{\substack{y\to a\\ y\in D}} \displaystyle\int\limits_{B^\nu_D(a,r(a,y))} \frac{\|f'(x) - C\|^p\,dx}{K(x)\,n(x;f,B^\nu_D(a,r(a,y)))}}{r^p \operatorname{mod}_{p,\sigma_r}\Gamma(y,a;B^\nu_D(a,r(a,y)))} = 0\,, \tag{3. 13}$$

where

$$r(a,y) = \inf\{t > 0 : y \in B^\nu_D(a,t)\}, \quad \sigma_r(x) = \frac{1}{K(x)\,n(x;f,B^\nu_D(a,r))}\,, \tag{3. 14}$$

or

$ii)$ *the vector-function* $f(x) - C \cdot x$ *is weakly* (α, ν)*-monotone close to* a *and there is a constant* $\lambda > 1$*, for which*

$$\limsup_{\substack{y \to a \\ y \in D}} \int\limits_{B_D^\nu(a,\lambda r(a,y))} \frac{\|f'(x) - C\|^n \, dx}{K(x)\, n(x; f, B_D^\nu(a, r(a,y)))} \Bigg/ r^{n\alpha}(a,y) \int\limits_{r(a,y)}^{\lambda r(a,y)} \kappa^\nu(a,t) \frac{dt}{t} = 0, \tag{3. 15}$$

then f *has at the point* $a = a_\xi$ *the total differential* $C \cdot dx$.

3.4. Consider some particular cases of Theorem. Let $w = f(z) : D \subset \mathbb{C}^1 \to \mathbb{C}^1$ be a generalized solution of a Beltrami equation

$$f_{\overline{z}} = \mu(z)\, f_z\,, \tag{3. 16}$$

where $\mu(z)$ is a measurable complex-valued function, and by symbols

$$f_z = \frac{1}{2}(f_x - i\, f_y)\,, \quad f_{\overline{z}} = \frac{1}{2}(f_x + i\, f_y)$$

we denote formal derivatives.

Note that in contrast to the traditional case (see, for example, [10, Ch. V], [11, Ch. 1]) we do not assume here, that $|\mu(z)| < 1$.

We have

$$J(z,f) = |f_z|^2 - |f_{\overline{z}}|^2 = (1 - |\mu(z)|^2)|f_z|^2$$

and

$$\|f'\|^2 = |f_z|^2 + |f_{\overline{z}}|^2 = (1 + |\mu(z)|^2)|f_z|^2\,.$$

Thus,

$$\|f'\|^2 \le \frac{1 + |\mu|^2}{|1 - |\mu|^2|}\, |J(z,f)|$$

and (1. 2) holds with

$$K(z) = \frac{1 + |\mu(z)|^2}{|1 - |\mu(z)|^2|}\,, \quad \delta(z) \equiv 0\,.$$

The assumption (3. 11) holds always.

Here we have also

$$\sigma_r(z) = \frac{1}{K(x)\, n(x; f, B_D^\nu(a,r))} = \frac{|1 - |\mu(z)|^2|}{(1 + |\mu(z)|^2)\, n(x; f, B_D^\nu(a,r))}.$$

For schlicht maps $n(x; f, B_D^\nu(a,r)) \equiv 1$ and

$$\sigma_r(z) = \frac{|1 - |\mu(z)|^2|}{1 + |\mu(z)|^2}\,.$$

Theorem connects the differentiability of f at a singular point $a = a_\xi$ with the behavior of the characteristic $\mu(z)$ close to its neighborhood. In the case, if a is an inner point, see [1], [2], [3, Ch. VI], [4, Ch. 11]. In the case, if $a = a_\xi$ is a boundary point and $\mu(z) \equiv 0$, see related results in [5, Ch. 11].

In the case, if the matrix C is orthogonal, Theorem gives conditions, under which maps $f : D \to \mathbb{R}^n$ are conformal at $a = a_\xi$.

For space quasiregular maps, near questions were being considered in [24, Ch. VI].

4. Proof of Theorem

Let $\varphi : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping. This mapping φ is called *absolutely continuous*, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for an arbitrary measurable set $E \subset D$, $\mathrm{mes}_n E < \delta$, we have $\mathrm{mes}_n \varphi(E) < \varepsilon$. In particular, every absolutely continuous mapping possesses the Lusin N-property.

Lemma 4. [20] *If a mapping φ is continuous and belongs to the class $\mathcal{W}^{1,n}_{\mathrm{loc}}(D)$, then φ is absolutely continuous on every subdomain $D' \Subset D$.*

Applying Lemma 2, we conclude that the following statement holds (see, for example, [22]).

Lemma 5. *If a mapping φ is continuous and belongs to $\mathcal{W}^{1,n}_{\mathrm{loc}}(D)$, then for an arbitrary integrable in $\varphi(D)$ function $u(y)$, the function $(u \circ \varphi)(x)|J(x,\varphi)|$ is integrable in D, and moreover*

$$\int\limits_{\varphi(D)} u(y)\, N(y;\varphi,A)\, dy = \int\limits_{A} (u \circ \varphi)(x)\, |J(x,\varphi)|\, dx\,. \tag{4. 17}$$

In particular, if we observe that

$$J(x,\varphi)\, J(y,\varphi^{-1}) = 1\,, \quad y = \varphi(x)\,,$$

and set

$$u(y) = \frac{1}{N(y;\varphi,A)}\,,$$

then using (4. 17) we have

$$\mathrm{mes}_n(\varphi(A)) = \int\limits_{\varphi(A)} dy = \int\limits_{A} \frac{|J(x,\varphi)|}{n(x;\varphi,A)}\, dx\,.$$

From this, using (1. 2) we conclude that

$$\int\limits_{A} \frac{\|f'(x)\|^n - \delta(x)}{K(x)\, n(x;f,A)}\, dx \le \mathrm{mes}_n(f(A))\,,$$

and, thus, we obtain (3. 12).

Consider the family of locally rectifiable arcs $\Gamma(y,a;B^{\nu}_{D}(a,|y-a|))$, lying in $B^{\nu}_{D}(a,|y-a|)$ and joining the point $y \in B^{\nu}_{D}(a,|y-a|)$ with the point a. Choose in (2. 4) the function $\rho(x) = \|f'(x) - C\|$. We find

$$\mathrm{mod}_{p,\sigma}\Gamma(y,a;B^{\nu}_{D}(a,|y-a|)) \le \frac{\displaystyle\int\limits_{B^{\nu}_{D}(a,|y-a|)} \|f'(x) - C\|^p \sigma(x)\, d\mathcal{H}^n}{\displaystyle\inf_{\gamma \in \Gamma(y,a;B^{\nu}_{D}(a,|y-a|))} \left(\int\limits_{\gamma} \|f'(x) - C\|\, |dx| \right)^p}\,. \tag{4. 18}$$

If γ is an arc of the family $\Gamma(y,a;B^{\nu}_{D}(a,|y-a|))$, then

$$|f(y) - f(a) - C \cdot (y-a)| \le \int\limits_{\gamma} \|f'(x) - C\|\, d\mathcal{H}^1\,.$$

Thus using (4. 18), for every point $y \in D$ we have

$$|f(y) - f(a) - C \cdot (y-a)|^p \leq \frac{\displaystyle\int\limits_{B_D^{\nu}(a,|y-a|)} \|f'(x) - C\|^p \sigma(x)\, d\mathcal{H}^n}{\operatorname{mod}_{p,\sigma}\Gamma(y,a;B_D^{\nu}(a,|y-a|))}. \qquad (4.\ 19)$$

By virtue of (4. 19) the assumption (3. 13) implies realization (1. 1) (respectively, (1. 3)) and, thus, the existence in the case $i)$ of the total differential at $a = a_{\xi}$.

We first prove the statement in the case $ii)$. Fix a point $y \in D$ and consider the subdomain

$$B_D^{\nu}(a,r)\,, \quad a = a_{\xi}\,, \quad r = r(a,y)\,,$$

adjoining at the and ξ and containing y in its closure. We put $f^*(x) = f(x) - f(a) - C \cdot (x-a)$. Applying Lemma 1 by virtue of (2. 9), we have

$$\int\limits_{r(a,y)}^{\lambda r(a,y)} \Omega^n\left(f^*, S_D^{\nu}(a,t)\right) \kappa^{\nu}(a,t)\, dt \leq \int\limits_{D(r(a,y),\lambda r(a,y))} \|(f^*)'(x)\|^n\, \sigma_{\lambda r(a,y)}(x)\, d\mathcal{H}^n(x)\,,$$

where $\sigma_{\lambda r(a,y)}(x)$ is defined in (3. 14).

The mapping $f^*(x)$ is weakly (α,ν)-monotone close to a and by virtue of (3. 10) for every t, $r(a,y) < t < \lambda\, r(a,y)$, and some constant $A < \infty$ the following estimates hold

$$|f^*(y) - f^*(a)|^{\alpha} \leq \operatorname{osc}^{\alpha}(f^*, B_D^{\nu}(a,t)) \leq A\, \Omega\left(f^*, S_D^{\nu}(a,t)\right).$$

From this we obtain

$$|f^*(y) - f^*(a)|^{\alpha n} \int\limits_{r(a,y)}^{\lambda r(a,y)} \kappa^{\nu}(a,t)\, dt \leq$$

$$\leq A^n \int\limits_{D(r(a,y),\lambda r(a,y))} \|f'(x) - C\|^n\, \sigma_{\lambda r(a,y)}(x)\, d\mathcal{H}^n(x)\,.$$

The assumption (3. 15) implies (1. 1) (and, respectively, (1. 3)). Theorem is proved. □

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