Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol.41(2009) pp. 51-60

Hadamard-Type Inequalities for $s-$Convex Functions I

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Abstract. In this paper we give a refined upper bound for unit, or smaller intervals and refinement of Hermite Hadamard Inequality for $s-$convex functions in second sense. We also establish several Hadamard type Inequalities for differentiable and twice differentiable functions based on concavity and $s-$convexity with applications for some special means.

AMS (MOS) Subject Classification Codes:[2000]26D15, 26D10
Key Words: Hadamard's inequality; $s-$convex functions; concave functions; Beta function.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

$\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(t\,x + (1-t)\,y) \leq \; t\,f(x) + \; (1-t)\,f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR.
In the paper [6], H. Hudzik and L. Maligranda considered, among others, the class of

functions which are $s-$convex in the second sense. This class is defined as follows:
A function $f:[0,\infty)\to\mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(t\,x+(1-t)\,y)\leq\ t^{s}\,f(x)\ +\ (1-t)^{s}\,f(y) \tag{1.2}$$

holds for all $x,y\in[0,\infty)$, $t\in[0,1]$ and for some fixed $s\in(0,1]$. It may be noted that every 1-convex function is convex. In the same paper [6] H. Hudzik and L. Maligranda discussed a few results connecting with $s-$convex functions in second sense and some new results about Hadamard's inequality for $s-$convex functions are discussed in [1, 2, 8], while on the other hand there are many important inequalities connecting with 1-convex (convex) functions [4], but one of these is the classical Hermite-Hadamard inequality defined by [10]

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\leq\frac{f(a)+f(b)}{2}$$

for $[a,b]\subseteq\mathbb{R}$.
In [5], S. S. Dragomir et al. proved a variant of Hermite-Hadamard's inequality for $s-$convex functions in second sense.

Theorem 1. *Suppose that $f:[0,\infty)\to[0,\infty)$ is $s-$convex function in the second sense, where $s\in(0,1]$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^{1}[a,b]$, then the following inequality holds*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.3). Their result was improved in [7], where Jagers gave both the upper and lower bound for the constant $c(s)$ in the inequality

$$c(s)\,f\left(\frac{a+b}{2}\right)\leq\ \frac{1}{b-a}\int_{a}^{b}f(x)\,dx.$$

He proved that

$$\frac{2^{s+1}-1}{s+2}\leq\ c(s)\leq\ 2^{\frac{s-1}{s+1}}\left(\frac{2^{s}-1}{s}\right)^{\frac{s}{s+1}}\ \leq\ \frac{2^{s+1}-2^{s-1}-1}{s+1}.$$

In [3, 4] S. S. Dragomir et al. discussed inequalities for differentiable and twice differentiable functions connecting with the H-H Inequality on the basis of the following Lemmas.

Lemma 2. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° with $f''\in L^{1}[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx=\frac{(b-a)^{2}}{2}\int_{0}^{1}t(1-t)\,f''(ta+(1-t)b)\ dt.$$

Lemma 3. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable function on I°, $a,b\in I^{\circ}$ with $a<b$ and $f'\in L^{1}[a,b]$, then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\ \ &=\ \ \frac{(b-a)^{2}}{4}\int_{0}^{1}(1-t)\,\left[f'\left(ta+(1-t)\frac{a+b}{2}\right)+\right.\\ &\left.f'\left(tb+(1-t)\frac{a+b}{2}\right)\right]\ dt.\end{aligned}$$

We give here definition of Beta function of Euler type which will be helpful in our next discussion, which is for $x, y > 0$ defined as

$$\beta(x+1, y+1) = \int_0^1 t^x (1-t)^y \, dt.$$

This paper is organized as follows. After this Introduction, in section 2 we discuss some $s-$Hermite Hadamard type inequalities for differentiable functions, in section 3 we give applications of the results from section 2 for special means and in section 4 we will discuss refinement of $s-$Hermite Hadamard inequality and its refined upper bound for unit, or smaller, intervals.

2. Inequalities For differentiable functions

Theorem 4. *Let $f : I \to \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on I° such that $f \in L^1[a, b]$, where $a, b \in I$, $a < b$. If $|\ f'\ |^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ &\quad \leq \ 2^{-\frac{1}{p}} \frac{\{|f'(a)|^q + (s+1)|f'\left(\frac{a+b}{2}\right)|^q\}^{\frac{1}{q}}}{\{(s+1)(s+2)\}^{\frac{1}{q}}} + \\ &\qquad 2^{-\frac{1}{p}} \frac{\{|\ f'(b)\ |^q + (s+1)\ |\ f'(\frac{a+b}{2})\ |^q\}^{\frac{1}{q}}}{\{(s+1)(s+2)\}^{\frac{1}{q}}} \qquad (2.1) \\ &= \ 2^{-\frac{1}{p}} \left[\left(\beta(s+1,2)\,|f'(a)|^q \ + \ \beta(s+2,1) \left| f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} + \right. \\ &\qquad \left. \left(\beta(s+1,2)|f'(b)|^q\, \beta(s+2,1) \ + \left| f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} \right] \end{aligned}$$

Proof. By Lemma 3

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ &\quad \leq \ \frac{(b-a)^2}{4} \left[\int_0^1 (1-t) \left| f'\left(t\,a + (1-t)\frac{a+b}{2}\right)\right| dt + \right. \\ &\qquad \left. \int_0^1 (1-t) \left| f'\left(t\,b + (1-t)\frac{a+b}{2}\right)\right| dt \right] \qquad (2.2) \end{aligned}$$

$|f'|$ is $s-$convex on $[a, b]$ for $t \in [0, 1]$

$$\left| f'\left(t\,a + (1-t)\frac{a+b}{2}\right)\right| \ \leq \ t^s\,|f'(a)| + (1-t)^s \left| f'\left(\frac{a+b}{2}\right)\right|$$

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad \leq |f'(a)| \int_0^1 t^s(1-t)\,dt + \left|f'\left(\frac{a+b}{2}\right)\right| \int_0^1 (1-t)^{1+s}\,dt \\
&\qquad = \beta(s+1,2)\,|f'(a)| + \beta(1,s+2)\left|f'\left(\frac{a+b}{2}\right)\right| \\
&\qquad = \frac{|f'(a)| + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|}{(s+1)(s+2)}
\end{aligned}$$

Now

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad = \int_0^1 (1-t)^{1-\frac{1}{q}}(1-t)^{\frac{1}{q}}\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt
\end{aligned}$$

By Hölder's Inequality for $q > 1$ with $p = \frac{q}{q-1}$

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad \leq \left(\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \left(\int_0^1 (1-t)\,dt\right)^{\frac{1}{p}} \\
&\qquad = 2^{-\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad \leq 2^{-\frac{1}{p}}\left[\frac{|f'(a)|^q \,+\, (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(s+1)(s+2)}\right]^{\frac{1}{q}} \qquad (2.3) \\
&\qquad = 2^{-\frac{1}{p}}\left[|f'(a)|^q\,\beta(s+1,2) \,+\, \beta(1,s+2)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Analogously

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,b+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\qquad \leq 2^{-\frac{1}{p}}\left[\frac{|\ f'(b)\ |^q \,+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(s+1)(s+2)}\right]^{\frac{1}{q}} \qquad (2.4) \\
&\qquad = 2^{-\frac{1}{p}}\left[|f'(b)|^q\,\beta(s+1,2) \,+\, \beta(1,s+2)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}
\end{aligned}$$

By using (2.3) and (2.4) in (2.2) we get (2.1). ∎

Theorem 5. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be a differentiable function on I° such that $f' \in L^1[a,b]$, where $a, b \in I$, $a < b$. If $|\ f'\ |^q$ is concave on $[a,b]$ for $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left[f'\left(\frac{3a+b}{4}\right) + f'\left(\frac{a+3b}{4}\right)\right] \qquad (2.5)$$

Proof. Similarly as in Theorem 4 by using Hölder's Inequality for $q>1$ with $p=\frac{q}{q-1}$ we obtain

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right| dt \\
&\quad\le \left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad= (p+1)^{-\frac{1}{p}} \left(\int_0^1 \left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (2.6)
\end{aligned}$$

$|\ f'\ |^q$ is concave on $[a,b]$, by Integral Jensen's Inequality (cf. [9]) we obtain

$$\begin{aligned}
\int_0^1 \left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt &= \int_0^1 t^0\left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q dt \\
&\le \left(\int_0^1 t^0\,dt\right)\left|f'\left(\frac{\int_0^1 \left(t\,a+(1-t)\frac{a+b}{2}\right)\,dt}{\left(\int_0^1 t^0\,dt\right)}\right)\right|^q \\
&= \left|f'\left(\int_0^1 \left(t\,a+(1-t)\frac{a+b}{2}\right)\right)\right|^q \\
&= \left|f'\left(\frac{3a+b}{4}\right)\right|^q. \qquad (2.7)
\end{aligned}$$

Analogously

$$\int_0^1 \left|f'\left(t\,b+(1-t)\frac{a+b}{2}\right)\right|^q dt \ \le\ \left|f'\left(\frac{a+3b}{4}\right)\right|^q \qquad (2.8)$$

By using $(2.6)-(2.8)$ in (2.2) we get (2.5). □

Theorem 6. *Let $f:I\to\mathbb{R}$, $I\subset[0,\infty)$, be a differentiable function on I° such that $f\in L^1[a,b]$, where $a,b\in I$, $a<b$. If $|f'|^q$ is $s-$convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$ with $p=\frac{q}{q-1}$, then*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\right. \\
&\qquad\left.\left(|\ f'(b)\ |^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \qquad (2.9) \\
&\le \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\right. \\
&\qquad\left.\left(|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Proof. We proceed similar to proof of Theorem 5.
By $s-$convexity of $|f'|^q$ we obtain

$$\int_0^1 \left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q\,dt \le \frac{|f'(a)|^q\,+\,\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}. \tag{2.10}$$

Analogously

$$\int_0^1 \left|f'\left(t\,b+(1-t)\frac{a+b}{2}\right)\right|^q\,dt \le \frac{|f'(b)|^q\,+\,\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}. \tag{2.11}$$

By using (2.10), (2.11) and (2.6) in (2.2) we get (2.9).
And the second inequality follows from the facts
$s \in (0,1]$ and $q > 1$ we have $\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \le 1.$ □

Theorem 7. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be a differentiable function on I° such that $f \in L^1[a,b]$, where $a,b \in I$, $a < b$. If $\mid f' \mid^q$ is $s-$concave on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\qquad\le\ \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\,2^{\frac{s-1}{q}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right]\end{aligned} \tag{2.12}$$

Proof. we proceed similarly as in Theorem 6.
By $s-$concavity of $|f'|^q$ we obtain

$$\int_0^1 \left|f'\left(t\,a+(1-t)\frac{a+b}{2}\right)\right|^q\,dt\ \le\ 2^{s-1}\ \left|f'\left(\frac{3a+b}{4}\right)\right|^q.$$

Analogously

$$\int_0^1 \left|f'\left(t\,b+(1-t)\frac{a+b}{2}\right)\right|^q\,dt\ \le\ 2^{s-1}\ \left|f'\left(\frac{a+3b}{4}\right)\right|^q.$$

Now (2.12) is immediate from (2.2). □
Variants of these results for twice differentiable functions are given below. These can be proved in a similar way based on Lemma 2.

Theorem 8. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a,b \in I$, $a < b$. If $|f''|^q$ is $s-$convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{(b-a)^2}{2\times 6^{\frac{q-1}{q}}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}.\\ &=\frac{(b-a)^2}{2\times 6^{\frac{q-1}{q}}}\left[\beta(s+2,2)\{\mid f''(a)\mid^q+\mid f''(b)\mid^q\}\right]^{\frac{1}{q}}.\end{aligned}$$

Theorem 9. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a,b \in I$, a,b. If $\mid f'' \mid^q$ is concave on $[a,b]$ for $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{2}\left|f''\left(\frac{a+b}{2}\right)\right|[\beta(p+1,p+1)]^p.$$

Theorem 10. *Let $f : I \to \mathbb{R}$, $I \subset [0, \infty)$, be twice differentiable function on I° such that $f'' \in L^1[a, b]$, where $a, b \in I$, $a < b$. If $|f''|^q$ is $s-$convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq$$

$$\frac{(b-a)^2}{2} [|f''(a)|^q + |f''(b)|^q]^{\frac{1}{q}} (s+1)^{-\frac{1}{q}} [\beta(p+1, p+1)]^p.$$

Theorem 11. *Let $f : I \to \mathbb{R}$, $I \subset [0, \infty)$, be twice differentiable function on I° such that $f'' \in L^1[a, b]$, where $a, b \in I$, $a < b$. If $\mid f'' \mid^q$ is $s-$concave on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$ with $p = \frac{q}{q-1}$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq 2^{\frac{s-1-q}{q}} \, (b-a)^2 \left| f'' \left(\frac{a+b}{2} \right) \right| [\beta(p+1, p+1)]^p.$$

Remark 12. For $s = 1$, relations (2.1), (2.5), (2.9) and (2.12) provide the right estimate of left classical Hadmard difference, that is, the new improvements of left Hadamard inequality.

Remark 13. For $s = 1$, relations in Theorems 8-11 provide the right estimate of right classical Hadmard difference, that is, the new improvements of right Hadamard inequality.

3. Applications for special Means

Let us recall the following means for two positive numbers.

(1) *The Arithmetic mean*

$$A \equiv A(a, b) = \frac{a+b}{2}, \;\; a, b > 0$$

(2) *The Harmonic mean*

$$H \equiv H(a, b) = \frac{2ab}{a+b}, \;\; a, b > 0$$

(3) *The $p-$Logarithmic mean*

$$L_p \equiv L_p(a, b) = \begin{cases} a, & \text{if } a = b;\; a, b > 0 \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b. \end{cases}$$

(4) The $Identric\ mean$

$$I \equiv I(a, b) = \begin{cases} a, & \text{if } a = b;\; a, b > 0 \\ \frac{1}{e} \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b. \end{cases}$$

(5) *The Logarithmic mean*

$$L \equiv L(a, b) = \begin{cases} a, & \text{if } a = b;\; a, b > 0 \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b. \end{cases}$$

The following inequality is well known in the literature:

$$H \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 14. *Let* $p > 1$, $0 < a < b$ *and* $q = \frac{p}{p-1}$. *Then one has the inequality.*

$$\left|H^{-1}(a,b) \; - \; L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{6\,a^3\,b^3}\; A^{1/q}\left(a^{3q}, b^{3q}\right) \tag{3.1}$$

Proof. By Theorem 8 applied for the mapping $f(x) = \frac{1}{x}$ for $s = 1$ we have

$$\left|\frac{\frac{1}{a}+\frac{1}{b}}{2} - \frac{\ln b \; - \; \ln a}{b-a}\right| \leq \frac{(b-a)^2}{2 \times 6^{1/p}}\left[\frac{\frac{2^q}{a^{3q}} + \frac{2^q}{b^{3q}}}{12}\right]^{1/q},$$

which is equivalent to (3.1). □

Another result which is connected with $p-$Logarithmic mean $L_p(a,b)$ is the following one:

Proposition 15. *Let* $p > 1$, $0 < a < b$ *and* $q = \frac{p}{p-1}$, *then*

$$\left|A(a^p, b^p) \; - \; L_p^p(a,b)\right| \leq \frac{p\,(p-1)(b-a)^2}{12}\; A^{1/q}\left(a^{q(p-2)}, b^{q(p-2)}\right)$$

Proof. Follows by Theorem 8, setting $f(x) = x^p$ for $s = 1$. □

Another result which is connected with $p-$Logarithmic mean $L_p(a,b)$ is the following one:

Proposition 16. *Let* $p > 1$, $0 < a < b$ *and* $q = \frac{p}{p-1}$, *then*

$$\frac{A(a,b)}{I(a,b)} \leq \exp\left[\frac{3^{-1/q}}{2}\left\{\left(a^{-q} + 2A^{-q}(a,b)\right)^{1/q} + \left(b^{-q} + 2A^{-q}(a,b)\right)^{1/q}\right\}\right]$$

Proof. Follows by Theorem 4, setting $f(x) = -\ln x$ for $s = 1$. □

Remark 17. By selecting some other convex functions, in the same way as above, we can find out some new relations connecting to some special means.

4. Refinement And New Refined Upper Bound For s-Hermite Hadamard Inequality

To find new refined upper bound we integrate (1.2) w.r.t t over $[a,b] \subseteq [0,1]$

$$\frac{1}{x-y}\int_{x\,a+(1-a)\,y}^{x\,b+(1-b)\,y} f(u)\,du \leq \; (b-a)\left[L_s^s(a,b)\;f(x) \; + \; L_s^s(1-a, 1-b)\;f(y)\right],$$

where,

$$L_p^p(\alpha,\beta) = \frac{\beta^{p+1} - \alpha^{p+1}}{(\beta-\alpha)(p+1)}, \qquad \alpha \neq \beta,\; p > 0.$$

For better right bound of Hermite Hadamard Inequality for s-convex function in second, we compare the above bound with usual one, $\frac{f(a)+f(b)}{s+1}$.

Suppose the above is less than the usual upper bound, that is,

$\frac{b^{s+1}-a^{s+1}}{s+1}\,f(x) - \frac{(1-b)^{s+1}-(1-a)^{s+1}}{s+1}\,f(y) \leq \; \frac{f(x)+f(y)}{s+1}$

$or, \quad \left[b^{s+1} - a^{s+1}\right]f(x) + \left[(1-a)^{s+1} - (1-b)^{s+1}\right]f(y) \leq \; f(x) + f(y).$

Consider $b = a + \lambda$ for $\lambda > 0$ such that its cube and higher powers approaching zero.

$\left[(a+\lambda)^{s+1} - a^{s+1}\right]f(x) + \left[(1-a)^{s+1} - (1-a-\lambda)^{s+1}\right]f(y) \leq \; f(x) + f(y)$

So, all we need, for the above being true, is that

$(a+\lambda)^{s+1} - a^{s+1} \leq 1; \qquad (1-a)^{s+1} - (1-a-\lambda)^{s+1} \leq 1$

$i.e, \quad a^{s+1}\left[\left(1+\frac{\lambda}{a}\right)^{s+1} - 1\right] \leq 1; \qquad (1-a)^{s+1}\left[1 - \left(1 - \frac{\lambda}{1-a}\right)^{s+1}\right] \leq 1$

By binomial expansion,
$a^{s+1}\left[\frac{s+1}{a}\,\lambda+\,\frac{s(s+1)}{2a^2}\,\lambda^2\right]\leq 1;\qquad (1-a)^{s+1}\left[\frac{s+1}{1-a}\,\lambda-\,\frac{s(s+1)}{2(1-a)^2}\,\lambda^2\right]\leq 1$

$$\frac{s}{2}a^{s-1}\,\lambda^2+a^s\,\lambda-\frac{1}{s+1}\leq 0;\qquad ,-\frac{s}{2}(1-a)^{s-1}\,\lambda^2+(1-a)^s\,\lambda-\frac{1}{s+1}\leq 0 \quad (4.1)$$

From (4.1) we get $\lambda\leq 1$.
This means we have improved the upper bound of Hermite Hadamard inequality for $s-$convex function in second sense, when the distance between a and b is almost one. The most interesting thing is that all linking work is with the interval $[0,1]$ better than other. This discussion gives the following result.

Theorem 18. *Let $f:[a,a+\lambda]\to\mathbb{R}$ be $s-$convex function in second sense for $0<\lambda\leq 1$, $0\leq a<1$ and $s\in(0,1]$, then*

$$\begin{aligned}2^{s-1}f\left(\frac{2a+\lambda}{2}\right)\;\leq\;\frac{1}{\lambda}\int_a^{a+\lambda}f(t)\,dt&\leq\frac{1}{s+1}\left[\left\{\frac{s}{2}a^{s-1}\,\lambda^2\,+\,a^s\,\lambda\right\}f(x)+\right.\\&\left.\left\{-\frac{s}{2}(1-a)^{s-1}\,\lambda^2\,+\,(1-a)^s\,\lambda\right\}f(y)\right].\end{aligned}$$

The following result is related with the improvement of inequality (1.3).

Theorem 19. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1]$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1[a,b]$, then*

$$\begin{aligned}\frac{f(a)+f(b)}{s+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\;\geq\;&\left|\int_0^1|\,t^s\,f(a)+(1-t)^s\,f(b)|\,dt-\right.\\&\left.\int_0^1\;|\;f(t\,a+(1-t)\,b)\;|\;dt\right|\end{aligned}\quad (4.2)$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\,dx-2^{s-1}f\left(\frac{a+b}{2}\right)\\&\geq\;2^{s-1}\left|\frac{1}{2^s}\int_0^1|\,f(t\,a+(1-t)\,b)+\,f(t\,b+(1-t)\,a)|\,dt+\right.\\&\left.\left|f\left(\frac{a+b}{2}\right)\right|\right|\end{aligned}\quad (4.3)$$

Proof. From inequality (1.2)

$$\begin{aligned}&t^s\,f(a)\,+\,(1-t)^s\,f(b)-f(t\,a+(1-t)\,b)\\&=|\;t^s\,f(a)\,+\,(1-t)^s\,f(b)-f(t\,a+(1-t)\,b)\;|\\&\geq|\;|\;t^s\,f(a)\,+\,(1-t)^s\,f(b)\;|\,-\,|\;f(t\,a+(1-t)\,b)\;|\;|.\end{aligned}$$

Integrating w.r.t t over [0,1]

$$\begin{aligned}&\frac{f(a)+f(b)}{s+1}\,-\,\int_0^1 f(t\,a+(1-t)\,b)\,dt\\&\geq\int_0^1|\;|\;t^s\,f(a)\,+\,(1-t)^s\,f(b)\;|\,-\,|\;f(t\,a+(1-t)\,b)\;|\;|\;dt.\\&\geq\left|\int_0^1\{|t^s\,f(a)\,+\,(1-t)^s\,f(b)|-|f(t\,a+(1-t)\,b)|\}\,dt\right|,\end{aligned}$$

which is equivalent to (4.2).
Again by definition

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2^s}.$$

$$\begin{aligned} \frac{f(x)+f(y)}{2^s} - f\left(\frac{x+y}{2}\right) &= \left|\frac{f(x)+f(y)}{2^s} - f\left(\frac{x+y}{2}\right)\right| \\ &\geq \left|\left|\frac{f(x)+f(y)}{2^s}\right| - \left|f\left(\frac{x+y}{2}\right)\right|\right| \end{aligned}$$

By setting, $x \mapsto t\,a + (1-t)\,b \quad and \quad y \mapsto t\,b + (1-t)\,a \quad for \quad t \in [0,1]$, we have

$$\frac{f(t\,a + (1-t)\,b) + f(t\,b + (1-t)\,a)}{2^s} - f\left(\frac{a+b}{2}\right) \geq \left|\left|\frac{f(t\,a + (1-t)\,b) + f(t\,b + (1-t)\,a)}{2^s}\right| - \left|f\left(\frac{a+b}{2}\right)\right|\right|.$$

Integrating w.r.t t over [0,1]

$$\frac{1}{2^s}\left[\int_0^1 f(t\,a + (1-t)\,b)\,dt + \int_0^1 f(t\,b + (1-t)\,a)\,dt\right] - f\left(\frac{a+b}{2}\right) \geq \left|\int_0^1 \left\{\left|\frac{f(t\,a + (1-t)\,b) + f(t\,b + (1-t)\,a)}{2^s}\right| - \left|f\left(\frac{a+b}{2}\right)\right|\right\} dt\right|.$$

From here we get (4.3). □

Acknowledgement : We thank the careful referee and Editor for valuable comments and suggestions, which we have used to improve the final version of this paper

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