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Common Fixed Point Theorem for Weakly Compatible Maps Satisfying E.A. Property in Intuitionistic Fuzzy Metric Spaces

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Abstract. In this paper, we use the notion of E.A. property in intuitionistic fuzzy metric space and prove a common fixed point theorem for weakly compatible mappings using this property.

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1. Introduction

Atanassove[3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park[9] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et al.[2] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t- conorms as a generalization of fuzzy metric space due to Kramosil and Michalek[7]. In 2006, Turkoglu[12] proved Jungck's[6] common fixed point theorem in the setting of intuitionistic fuzzy metric spaces for commuting mappings. In this paper, we use the notion of E.A. property in intuitionistic fuzzy metric space and prove a common fixed point theorem for weakly compatible mappings using this property.

2. Preliminaries

The concepts of triangular norms (t -norm) and triangular conorms (t -conorm) are known as the axiomatic skelton that we use, are characterization of fuzzy intersections and union respectively. These concepts were originally introduced by Menger [8] in study of statistical metric spaces.

Definition 1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if * is satisfies the following conditions:

(i) * is commutative and associative;
(ii)* is continuous;
(iii) $a * 1 = a$ for all a in $[0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2. A binary operation $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$is continuous t-conorm if $\diamond$ is satisfies the following conditions:

(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a \diamond 0 = a$ for all a $\in [0, 1]$;
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Alaca et al.[2] defined the notion of intuitionistic fuzzy metric space as follows :

Definition 3. A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all x, y in X and $t > 0$;
(ii) $M(x, y, 0) = 0$ for all x, y in X;
(iii) $M(x, y, t) = 1$ for all x, y in X and $t > 0$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$ for all x, y in X and $t > 0$;
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all x, y, z in X and $s, t > 0$;
(vi) for all x, y in X, $M(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is left continuous;
(vii) $\lim_{t \to \infty} M(x, y, t) = 1$ for all x, y in X and $t > 0$;
(viii) $N(x, y, 0) = 1$ for all x, y in X;
(ix) $N(x, y, t) = 0$ for all x, y in X and $t > 0$ if and only if $x = y$;
(x) $N(x, y, t) = N(y, x, t)$ for all x, y in X and $t > 0$;
(xi) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all x, y, z in X and $s, t > 0$;
(xii) for all x, y in X, $N(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is right continuous;
(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$ for all x, y in X.

Then (M, N) is called an intuitionistic fuzzy metric. The functions M(x, y, t) and N(x, y,t) denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t respectively.

Remark 4. Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm * and t-conorm $\diamond$ are associated as $x \diamond y$ = $1 - ((1 - x) * (1 - y))$ for all x, y in X.

Remark 5. [2] In intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, M(x, y, *) is non-decreasing and $N(x, y, \diamond)$ is non-increasing for all x, y in X.

Definition 6. [2] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then
(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0$.
(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$, $\lim_{n \to \infty} M(x_n, x, t) = 1$ and $\lim_{n \to \infty} N(x_n, x, t) = 0$.

(c) $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 1. Let X = $\{1/n : n = 1, 2, 3, ...\} \cup \{0\}$ and let * be the continuous t-norm and $\diamond$ be the continuous t-conorm defined by a * b = ab and $a \diamond b = min\{1, a + b\}$ respectively, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and x, y in X, define (M, N) by M(x, y, t) = $\frac{t}{t+|x-y|}$ if $t > 0$, M(x, y, t)= 0 if $t = 0$, and N(x, y, t) = $\frac{|x-y|}{t+|x-y|}$ if $t > 0$, N(x, y, t)= 1 if $t = 0$. Clearly, $(X, M, N, *, \diamond)$ is complete intuitionistic fuzzy metric space.

Definition 7. [12] A pair of self mappings (f , g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be commuting if
M(fgx , gfx , t) = 1 and N(fgx , gfx , t) = 0 for all x in X .

Recently, Amari and Moutawakil[1] introduced a generalization of non compatible maps as E.A. property.

Definition 8. [1] Let A and S be two self-maps of a metric space (X,d). The pair (A,S) is said to satisfy E.A. property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some t in X.

Example 2. Let X = $[0, +\infty)$. Define S, T : X $\to$ X by Tx = $\frac{x}{4}$ and Sx = $\frac{3x}{4}$, for all x in X. Consider the sequence $\{x_n\}$ = $\{1/n\}$. Clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then S and T satisfy E.A. property.

Example 3. Let X = [2, $+\infty$). Define S, T : X$\to$X by Tx = x+1 and Sx = 2x+1 for all x in X. Suppose that the E.A. property holds. Then, there exists in X, a sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X. Therefore, $\lim_{n\to\infty} x_n = z - 1$ and $\lim_{n\to\infty} x_n = \dfrac{z-1}{2}$. Thus, z = 1, which is a contradiction, is said to satisfy the E.A. property if there exist a sequence $\{x_n\}$ in X such that since 1 is not contained in X. Hence S and T do not satisfy E.A. property.

Definition 9. A pair of self mappings (f , g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to satisfy the E.A. property if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} M(fx_n, gx_n, t) = 1$ and $\lim_{n\to\infty} N(fx_n, gx_n, t) = 0$.

Example 4. Let X = $[0, \infty)$. Let us consider $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space as in example 1. Define T, S : X $\to$ $[0, \infty)$ by Tx = $\frac{x}{5}$ and Sx = $\frac{2x}{5}$. Now, $\lim_{n\to\infty} M(Sx_n, Tx_n, t) = 1$ and $\lim_{n\to\infty} N(Sx_n, Tx_n, t) = 0$. Clearly S and T satisfies E.A. property.
Jungck [6] introduced the notion of weakly compatible maps as follows:

Definition 10. A pair of self mappings (f , g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be weakly compatible if they commute at the coincidence points i.e. Tu = Su for some u in X , then TSu = STu.

Alaca [2] proved the following result:

Lemma 11. *Let $(X, M, N, *, \diamond)$ be intuitionistic fuzzy metric space and for all x , y in X , $t > 0$ and if for a number $k \in (0, 1)$, $M(x, y, kt) \geq M(x, y, t)$ and $N(x, y, kt) \leq N(x, y, t)$ then x = y.*

3. Weakly compatible maps and E.A. property

Turkoglu et al.[12] proved the following Theorem:

Theorem 12. *Let A, B, S and T be self maps of complete intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ with continuous t-norm and continuous t- conorm defined by $a * a \geq a$ and $(1 - a) \diamond (1 - a) \leq (1 - a)$ for all a in $[0, 1]$, satisfying the following conditions:*

(1) $A(X) \subset T(X), B(X) \subset S(X)$,
(2) S and T are continuous,
(3) The pairs (A , S) and (B , T) are compatible maps,
(4) for all x, y in X, k in $(0, 1), t > 0$
$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, 2t) * M(Ax, Ty, t)$ *and*
$N(Ax, By, kt) \leq N(Sx, Ty, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t) \diamond N(By, Sx, 2t) \diamond N(Ax, Ty, t)$,
Then A, B, S and T have a unique common fixed point in X.

Now,we generalize theorem 12 for weakly compatible maps using E.A property. Our theorem generalise theorem 12 in the following way:

(a) relaxing the continuity requirement of maps and,
(b) relaxing the completeness of the space X.

Theorem 13. *Let A, B, S and T be self maps of intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ with continuous t-norm and continuous t- conorm defined by $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$ for all a, b in $[0, 1]$ satisfying the following conditions:*

(i) for all x, y in X, k in $(0, 1), t > 0$
(ii) (A, S) and (B, T) are weakly compatible,
(iii) (A, S) or (B, T) satisfies E.A. property,
(iv) $A(X) \subset T(X), B(X) \subset S(X)$,
If one of A, B, S and T is a complete subspace of X then A, B, S and T have unique common fixed point in X.

Proof: Suppose the pair (B, T) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$ for some p in X. Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n = p$. Hence $\lim_{n\to\infty} Sy_n = p$. We shall show that $\lim_{n\to\infty} Ay_n = p$. From (i), we have
$M(Ay_n, Bx_n, kt) \geq M(Sy_n, Tx_n, t) * M(Ay_n, Sy_n, t) * M(By_n, Ty_n, t) *$
$M(Bx_n, Sy_n, 2t) * M(Ay_n, Tx_n, t)$ and

$N(Ay_n, Bx_n, kt) \leq N(Sy_n, Tx_n, t) \diamond N(Ay_n, Sy_n, t) \diamond N(Bx_n, Tx_n, t) \diamond$
$N(Bx_n, Sy_n, 2t) \diamond N(Ay_n, Tx_n, t)$,
Taking limit as $n \to \infty$, we get $M(Ay_n, p, kt) \geq M(Ay_n, p, t)$ and $N(Ay_n, p, kt) \leq N(Ay_n, p, t)$.
Using Lemma 11, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sx_n = p$. Suppose that S(X) is a complete subspace of X. Then p = Su for some u in X. Subsequently, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p = Su$. Now, we shall show that Au = Su. From (4), we have
$M(Au, Bx_n, kt) \geq M(Su, Tx_n, t) * M(Au, Su, t) * M(Bx_n, Tx_n, t) * M(Bx_n, Su, 2t) *$
$M(Au, Tx_n, t)$ and
$N(Au, Bx_n, kt) \leq N(Su, Tx_n, t) \diamond N(Au, Su, t) \diamond N(By_n, Tx_n, t) \diamond N(Bx_n, Su, 2t) \diamond$
$N(Au, Tx_n, t)$,
Taking limit as $n \to \infty$ we get $M(Au, Su, kt) \geq M(Au, Su, t)$ and $N(Au, Su, kt) \leq N(Au, Su, t)$.
Now by using Lemma 11, we have Au = Su. Therefore (A, S) have coincidence point. The weak compatibility of A and S implies that ASu = SAu and thus AAu = ASu = SAu = SSu. As $A(X) \subset T(X)$, there exists v in X such that Au = Tv. We claim that Tv = Bv. From (i), we have M(Au, Bv, kt) $\geq M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) * M(Bv, Su, 2t) * M(Au, Tv, t)$ and $N(Au, Bv, kt) \leq N(Su, Tv, t) \diamond N(Au, Su, t) \diamond N(Bv, Tv, t) \diamond N(Bv, Su, 2t) \diamond N(Au, Tv, t)$, Now by using Lemma 11, we have Au = Bv. Hence, Tv = Bv. Thus we have Au = Su = Tv = Bv. The weak compatibility of B and T implies that BTv = TBv = TTv = BBv. Finally, we show that Au is the common fixed point of A, B, S and T. From (i), we have
$M(Au, AAu, kt) = M(AAu, Bv, kt) \geq M(SAu, Tv, t) * M(AAu, SAu, t) * M(Bv, Tv, t) *$
$M(Bv, SAu, 2t) * M(AAu, Tv, t)$ and $N(Au, AAu, kt) = N(AAu, Bv, kt)$
$\leq N(SAu, Tv, t) \diamond N(AAu, SAu, t) \diamond N(Bv, Tv, t) \diamond N(Bv, SAu, 2t) \diamond N(AAu, Tv, t)$,
we have $M(Au, AAu, kt) = M(AAu, Bv, kt) \geq M(AAu, Bv, t)$
and $N(Au, AAu, kt) = N(AAu, Bv, kt) \leq N(AAu, Bv, t)$.
Now, the use of Lemma 11 gives AAu = Bv =Au and thus, AAu = Au. Therefore, Au = AAu = SAu is the common fixed point of A and S. Similarly, we prove that Bv is the common fixed point of B and T. Since Au = Bv, Au is common fixed point of A, B, S, and T. The proof is similar when T(X) is assumed to be a complete subspace of X. The cases in which A(X) or B(X) is a complete subspace of X are similar to the cases in which T(X) or S(X), respectively is complete subspace of X as $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Finally now we show that the common fixed point is unique. If possible, let x_0 and y_0 be two common fixed points of A, B, S, and T. Then by condition (i),
$M(x_0, y_0, kt) = M(Tx_0, By_0, kt)$
$\geq M(Sx_0, Ty_0, t) * M(Ax_0, Sx_0, t) * M(By_0, Ty_0, t) * M(By_0, Sx_0, 2t) * M(Ax_0, Ty_0, t)$
and $N(x_0, y_0, kt) = N(Tx_0, By_0, kt)$
$\leq N(Sx_0, Ty_0, t) \diamond N(Ax_0, Sx_0, t) \diamond N(By_0, Ty_0, t) \diamond N(By_0, Sx_0, 2t) \diamond N(Ax_0, Ty_0, t)$,
By fixed point property and using intuitionistic fuzzy metric space, we get
$M(x_0, y_0, kt) \geq M(x_0, y_0, t)$ and $N(x_0, y_0, kt) \leq N(x_0, y_0, t)$.
This implies, by using Lemma 11 that $x_0 = y_0$. Therefore, the mappings A, B, S, and T have a unique common fixed point.

Example 5. Let $(X, M, N, *, \diamond)$ be a intuitionistic fuzzy metric space with X = $[0, 1]$, t-norm * and t-conorm $\diamond$ defined by $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$ where a, b in $[0, 1]$, respectively. Let (M, N) is the intuitionistic fuzzy set on $X^2 \times (0, \infty)$, defined

by M(x, y, t)= $(\exp(\frac{|x-y|}{t}))^{-1}$ for all $t > 0$, M(x, y, t)= 0 when $t = 0$ and N(x, y, t) =$[(\exp(\frac{|x-y|}{t}))-1][(\exp(\frac{|x-y|}{t}))^{-1}]$, N(x, y, t) = 1 when $t = 0$. Then it is well known that $(X, M, N, *, \diamond)$ is a intuitionistic fuzzy metric space. Let us define self maps A, B, S, and T on X such that Ax = $\frac{x}{64}$, Tx = $\frac{x}{2}$, Bx = $\frac{x}{32}$, Sx = $\frac{x}{4}$ then for $k \in [1/16, 1)$
$M(Ax, By, kt)$ = $(\exp(\frac{|x/64-y/32|}{kt}))^{-1}$
$\geq(\exp(\frac{|x/4-y/2|}{t}))^{-1}$ = M(Sx, Tx, t)
$\geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Bx, Ty, t) * M(By, Sx, 2t) * M(Ax, Ty, t)$ and
$N(Ax, By, kt)$=$[(\exp(\frac{|x/64-y/32|}{t}))-1][(\exp(\frac{|x/64-y/32|}{t}))^{-1}]$
$\leq[(\exp(\frac{|x/4-y/2|}{t}))-1][(\exp(\frac{|x/4-y/2|}{t}))^{-1}]$ = $N(Sx, Ty, t)$
$\leq$ N(Sx, Ty, t) $\diamond$ N(Ax, Sx, t) $\diamond$ N(Bx, Ty, t) $\diamond$ N(By, Sx, 2t) $\diamond$ N(Ax, Ty, t).

Clearly,
(a) condition (i) of above theorem holds,
(b) for sequence $\{x_n\}$ = $\{1/n\}$, pairs (A, S) and (B,T) satisfies E.A. property,
(c) $A(X) \subset T(X), B(X) \subset S(X)$,
(d) one of A(X), B(X),S(X) or T(X) is complete subsets of X,
(e) the pairs (A , S) and (B , T) are weakly compatible at x = 0 which is the coincident point of the maps A , B, S and T.
Thus all the conditions of Theorem 13 are satisfied and also x = 0 is the unique common fixed point of A, B, S and T.

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