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# Sharp Function Estimates for Multilinear Commutators of Integral Operators

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**Abstract.** In this paper, we prove the sharp function inequality for some multilinear commutators related to certain integral operators. By using the sharp inequality, we obtain the weighted $L^p$-norm inequality for the multilinear commutators. The integral operators include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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## 1. INTRODUCTION

As the development of singular integral operators, their commutators have been well studied (see [1-4]). Let $T$ be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rocherberg and Weiss (see [2]) states that commutator $[b, T](f) = T(bf) - bT(f)$( where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [4][11-13], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp function inequality for some multilinear commutators related to certain integral operators. By using the sharp inequality, we obtain the weighted $L^p$-norm inequality for the multilinear commutators. The integral operators include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

## 2. NOTATIONS AND RESULTS

First let us introduce some notations (see [3][14][15]). In this paper, $Q$ will denote a cube of $R^n$ with sides parallel to the axes, and for a cube Q let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and the sharp function of $f$ is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is well-known that (see [3])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - C| dy.$$

We say that $b$ belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $||b||_{BMO} = ||b^{\#}||_{L^\infty}$. It has been known that (see [14])

$$||b - b_{2^kQ}||_{BMO} \le Ck||b||_{BMO}.$$

Let $M$ be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)|dy.$$

We write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$. For $b_j \in BMO(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{BMO} = \prod_{j=1}^m ||b_j||_{BMO}.$$

Given some functions $b_j$ $(j = 1, \cdots, m)$ and a positive integer $m$ and $1 \le j \le m$, we denote by $C_j^m$ the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of $j$ different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

In this paper, we will study some multilinear commutators as follows.

**Definition 1.** Suppose $b_j(j = 1, \cdots, m)$ are the fixed locally integrable functions on $R^n$. Let $F_t(x, y)$ define on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y)f(y)dy$$

and

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))F_t(x, y)f(y)dy,$$

for every bounded and compactly supported function $f$. Let $H$ be the Banach space $(H, ||\cdot||)$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^{\vec{b}}(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to $H$. The multilinear commutator related to $F_t$ is defined by

$$T_{\vec{b}}(f)(x) = ||F_{(\cdot)}^{\vec{b}}(f)(x)||,$$

where $F_t$ satisfies: for fixed $\varepsilon > 0$

$$||F_t(x, y)|| \le C|x - y|^{-n}$$

and

$$||F_t(y, x) - F_t(z, x)|| + ||F_t(x, y) - F_t(x, z)|| \le C|y - z|^\varepsilon |x - z|^{-n-\varepsilon},$$

if $2|y - z| \le |x - z|$. We also define that $T(f)(x) = ||F_{(\cdot)}(f)(x)||$.

Note that when $b_1 = \cdots = b_m$, $T_{\vec{b}}$ is just the $m$ order commutator (see [1][13]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-2][4][11-13]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

Now we state our theorems as following.

**Theorem 2.** *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Suppose that $T$ is bounded on $L^q(R^n)$ for all $1 < q < \infty$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(T_{\vec{b}}(f))^{\#}(x) \le C||\vec{b}||_{BMO} \left( M_r(f)(x) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_r(T_{\vec{b}_{\sigma^c}}(f))(x) \right).$$

**Theorem 3.** *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Suppose that $T$ is bounded on $L^q(R^n)$ for all $1 < q < \infty$. Then $T_{\vec{b}}$ is bounded on $L^p(R^n)$ for $1 < p < \infty$.*

## 3. Proofs of Theorems

To prove the theorems, we need the following lemma, which is well known.

**Lemma 4.** *Let $1 < q < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$. Then*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^{k} ||b_j||_{BMO}$$

*and*

$$\left( \frac{1}{|Q|} \int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^q dy \right)^{1/q} \le C \prod_{j=1}^{k} ||b_j||_{BMO}.$$

*Proof.* **Proof of Theorem 1.** It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant $C_0$, the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T_{\vec{b}}(f)(x) - C_0| dx \le C||\vec{b}||_{BMO} \left( M_r(f)(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_r(T_{\vec{b}_{\sigma^c}}(f)(\tilde{x})) \right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$.

We first consider the **Case** $m = 1$**.** Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q}) F_t(f)(x) - F_t((b_1 - (b_1)_{2Q}) f_1)(x) - F_t((b_1 - (b_1)_{2Q}) f_2)(x).$$

Then,

$$\begin{aligned}
& |T_{b_1}(f)(x) - T(((b_1)_{2Q} - b_1) f_2)(x_0)| \\
= & \left| ||F_t^{b_1}(f)(x)|| - ||F_t(((b_1)_{2Q} - b_1) f_2)(x_0)|| \right| \\
\le & ||F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1) f_2)(x_0)|| \\
\le & ||(b_1(x) - (b_1)_{2Q}) F_t(f)(x)|| + ||F_t((b_1 - (b_1)_{2Q}) f_1)(x)|| \\
& + ||F_t((b_1 - (b_1)_{2Q}) f_2)(x) - F_t((b_1 - (b_1)_{2Q}) f_2)(x_0)|| \\
= & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by Hölder's inequality with exponent $1/r + 1/r' = 1$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q A(x)dx \\
= \ & \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}||T(f)(x)|dx \\
\le \ & C\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r} \\
\le \ & C||b_1||_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose $s,q$ such that $1<s,q<\infty$ and $r=qs$, by the boundedness of $T$ on $L^q(R^n)$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q B(x)dx \\
= \ & \frac{1}{|Q|}\int_Q [T((b_1-(b_1)_{2Q})f_1)(x)]dx \\
\le \ & \left(\frac{1}{|Q|}\int_{R^n}[T((b_1-(b_1)_{2Q})f\chi_{2Q})(x)]^q dx\right)^{1/q} \\
\le \ & C\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|b_1(x)-(b_1)_{2Q}|^q|f(x)\chi_{2Q}(x)|^q dx\right)^{1/q} \\
\le \ & C|Q|^{-1/q+1/qs'+1/qs}\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{qs'}dx\right)^{1/qs'} \times \\
& \left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^{qs}dx\right)^{1/qs} \\
\le \ & C||b_1||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, by Minkowski's inequality, we obtain, for $x \in Q$,

$$\begin{aligned}
C(x) &= \left|\left|\int_{R^n}(b_1(y)-(b_1)_{2Q})f_2(y)(F_t(x,y)-F_t(x_0,y))dy\right|\right| \\
&\le \int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}||f(y)|||F_t(x,y)-F_t(x_0,y)|||dy \\
&\le C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}||f(y)|\frac{|x_0-x|^\varepsilon}{|x_0-y|^{n+\varepsilon}}dy \\
&\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_{2Q}||f(y)|\frac{|x_0-x|^\varepsilon}{|x_0-y|^{n+\varepsilon}}dy \\
&\le C\sum_{k=1}^{\infty}2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r} \times \\
&\quad \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}|^{r'}dy\right)^{r'} \\
&\le C\sum_{k=1}^{\infty}k2^{-k\varepsilon}||b_1||_{BMO}M_r(f)(\tilde{x}) \\
&\le C||b_1||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}$$

thus

$$\frac{1}{|Q|}\int_Q C(x)dx \le C||b_1||_{BMO}M_r(f)(\tilde{x}).$$

Now, we consider the **Case** $m \ge 2$, we have known that, for $b = (b_1, \cdots, b_m)$,

$$\begin{aligned}
&F_t^{\vec{b}}(f)(x) = \int_{R^n}\left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] F_t(x,y)f(y)dy \\
= \;& \int_{R^n}[(b_1(x) - (b_1)_{2Q}) - (b_1(y) - (b_1)_{2Q})] \cdots [(b_m(x) - (b_m)_{2Q}) - (b_m(y) - \\
& (b_m)_{2Q})]F_t(x,y)f(y)dy \\
= \;& \sum_{j=0}^m \sum_{\sigma\in C_j^m} (-1)^{m-j}(b(x) - (b)_{2Q})_\sigma \int_{R^n}(b(y) - (b)_{2Q})_\sigma F_t(x,y)f(y)dy \\
= \;& (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q})F_t(f)(x) \\
& +(-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})f)(x) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma \int_{R^n}(b(y) - b(x))_{\sigma^c}F_t(x,y)f(y)dy \\
= \;& (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q})F_t(f)(x) \\
& +(-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})f)(x) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x),
\end{aligned}$$

thus

$$\begin{aligned}
& |T_{\vec{b}}(f)(x) - T(((b_1)_{2Q} - b_1)\cdots((b_m)_{2Q} - b_m))f_2)(x_0)| \\
= \;& \Big| ||F_t^{\vec{b}}(f)(x)|| - ||F_t(((b_1)_{2Q} - b_1)\cdots((b_m)2Q - b_m)f_2)(x_0)|| \Big| \\
\le \;& ||F_t^{\vec{b}}(f)(x) - F_t(((b_1)_{2Q} - b_1)\cdots((b_m)_{2Q} - b_m)f_2)(x_0)|| \\
\le \;& ||(b_1(x) - (b_1)_{2Q})\cdots \\
& (b_m(x) - (b_m)_{2Q})F_t(f)(x)|| \\
+ \;& \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(\vec{b}(x) - (b_m)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)|| \\
+ \;& ||F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_1)(x)|| \\
+ \;& ||F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})\cdots \\
& (b_m - (b_m)_{2Q})f_2)(x_0)|| \\
= \;& I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

For $I_1(x)$, by Hölder's inequality with exponent $1/p_1+\cdots+1/p_m+1/r=1$, where $1<p_j<\infty,\ j=1,\cdots,m$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_1(x)dx\\
\le\ &\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||T(f)(x)|dx\\
\le\ &\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q|b_j(x)-(b_j)_{2Q}|^{p_j}\right)^{1/p_j}\left(\frac{1}{|Q|}\int_Q|T(f)(x)|^r dx\right)^{1/r}\\
\le\ &C||\vec{b}||_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}$$

For $I_2(x)$, by the Minkowski's and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_2(x)dx\\
=\ &\frac{1}{|Q|}\int_Q\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||dx\\
\le\ &\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-(b)_{2Q})_\sigma||T_{\vec{b}_{\sigma^c}}(f)(x)|dx\\
\le\ &C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q|T_{\vec{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r}\\
\le\ &C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_r(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}).
\end{aligned}$$

For $I_3(x)$, choose $1<s,q<\infty$ with $r=qs$, by the boundedness of $T$ on $L^q(R^n)$ and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx\\
\le\ &\left(\frac{1}{|Q|}\int_{R^n}|T(\prod_{j=1}^m(b_j-(b_j)_{2Q})f\chi_{2Q})(x)|^q dx\right)^{1/q}\\
\le\ &C\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^q|f(x)\chi_{2Q}(x)|^q dx\right)^{1/q}\\
\le\ &C|Q|^{-1/q+1/qs'+1/qs}\left(\frac{1}{|2Q|}\int_{2Q}|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})|^{qs'}dx\right)^{1/qs'}\\
&\times\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^{qs}dx\right)^{1/qs}\\
\le\ &C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, choose $1 < p_j < \infty$ $j = 1, \cdots, m$ such that $1/p_1 + \cdots + 1/p_m + 1/r = 1$, we obtain, by Hölder's inequality,

$$\begin{aligned}
I_4(x) &= ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x) - F_t((b_1-(b_1)_{2Q})\cdots \\
&\quad (b_m-(b_m)_{2Q})f_2)(x_0)|| \\
&\leq \int_{R^n} |\prod_{j=1}^m (b_j(y)-b(b_j)_{2Q})||f_2(y)\chi_{(2Q)^c}(y)|||F_t(x,y)-F_t(x_0,y)||dy \\
&\leq C\int_{(2Q)^c} |\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f(y)|\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}dy \\
&\leq C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |x-x_0|^\varepsilon |x_0-y|^{-(n+\varepsilon)}|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f(y)|dy \\
&\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r} \\
&\quad \prod_{j=1}^m \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_j(y)-(b_j)_{2Q}|^{p_j}dy\right)^{1/p_j} \\
&\leq C\sum_{k=1}^\infty k^m 2^{-k\varepsilon}\prod_{j=1}^m ||b_j||_{BMO}M_r(f)(\tilde{x}) \\
&\leq C||\vec{b}||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}$$

thus

$$\frac{1}{|Q|}\int_Q I_4(x)dx \leq C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).$$

This completes the proof of the theorem.

□

*Proof.* **Proof of Theorem 2.** Choose $1 < r < p$ in Theorem 1. We first consider the case m=1, we have

$$\begin{aligned}
||T_{b_1}(f)||_{L^p} &\leq \|M(T_{b_1}(f))\|_{L^p} \leq C\|(T_{b_1}(f))^\#\|_{L^p} \\
&\leq C\|M_r(T(f))\|_{L^p} + C\|M_r(f)\|_{L^p} \\
&\leq C\|T(f)\|_{L^p} + C\|M_r(f)\|_{L^p} \\
&\leq C\|f\|_{L^p} + C\|f\|_{L^p} \\
&\leq C\|f\|_{L^p}.
\end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof. □

## 4. Applications

Now we give some applications of the theorems in this paper.

**Application 1.** Littlewood-Paley operator.

Fixed $\varepsilon > 0$. Let $\psi$ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)| \leq C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The Littlewood-Paley multilinear commutator is defined by

$$g_\psi^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))\psi_t(x-y)f(y)dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator(see [15]). Let $H$ be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to $H$, and it is clear that

$$g_\psi^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that $g_\psi$ satisfies the conditions of Theorems 1 and 2 (see [5-7]), thus Theorems 1 and 2 hold for $g_\psi^{\vec{b}}$.

**Application 2.** Marcinkiewicz operator.

Fixed $0 < \gamma \le 1$. Let $\Omega$ be homogeneous of degree zero on $R^n$ with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\Omega^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [16]). Let $H$ be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that $\mu_\Omega$ satisfies the conditions of Theorems 1 and 2 (see [8][16]), thus Theorems 1 and 2 hold for $\mu_\Omega^{\vec{b}}$.

**Application 3.** Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^{m}(b_j(x) - b_j(y))B_t^\delta(x-y)f(y)dy.$$

The maximal Bochner-Riesz multilinear commutator is defined by

$$B_{\delta,*}^{\vec{b}}(f)(x) = \sup_{t>0} |B_{\delta,t}^{\vec{b}}(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator(see [10]). Let $H$ be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^{\vec{b}}(f)(x) = ||B_{\delta,t}^{A}(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^{\vec{b}}$ satisfies the conditions of Theorems 1 and 2 (see [9]), thus Theorems 1 and 2 hold for $B_{\delta,*}^{\vec{b}}$.

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