

On γ -Semi-Continuous Functions

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Abstract. In this paper, we continue studying the properties of γ -semi-continuous and γ -semi-open functions introduced in [5].

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1. INTRODUCTION

N. Levine [11] introduced the notion of semi-open sets in topological spaces. A. Csaszar [7,8] defined generalized open sets in generalized topological spaces. In 1975, Maheshwari and Prasad [12] introduced concepts of semi- T_1 -spaces and semi- R_0 -spaces. In 1979, S. Kasahara [10] defined an operation α on topological spaces. Carpintero, et. al [6] introduced the notion of α -semi-open sets as a generalization of semi-open sets. B. Ahmad and F.U. Rehman [1, 14] introduced the notions of γ -interior, γ -boundary and γ -exterior points in topological spaces. They also studied properties and characterizations of (γ, β) -continuous mappings introduced by H. Ogata [13]. In [2-4], B. Ahmad and S. Hussain introduced the concept of γ_0 -compact, γ^* -regular, γ -normal spaces and explored their many interesting properties. They initiated and discussed the concepts of γ^* -semi-open sets , γ^* -semi-closed sets, γ^* -semi-closure, γ^* -semi-interior points in topological spaces [5,9]. In [9], they introduced Λ_s^γ -set and $\Lambda^{s\gamma}$ -set by using γ^* -semi-open sets. Moreover, they also

introduced γ -semi-continuous function and γ -semi-open (closed) functions in topological spaces and established several interesting properties.

In this paper, we continue studying the properties of γ -semi-continuous functions and γ -semi-open function introduced by B. Ahmad and S. Hussain [5].

Hereafter, we shall write space in place of topological space in the sequel.

2. PRELIMINARIES

We recall some definitions and results used in this paper to make it self-contained.

Definition 1. [13] Let (X, τ) be a space. An operation $\gamma : \tau \rightarrow P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V . The operations defined by $\gamma(G) = G$, $\gamma(G) = \text{cl}(G)$ and $\gamma(G) = \text{intcl}(G)$ are examples of operation γ .

Definition 2. [13] Let A be a subset of a space X . A point $x \in A$ is said to be γ -interior point of A , if there exists an open nbd N of x such that $N^\gamma \subseteq A$. The set of all such points is denoted by $\text{int}_\gamma(A)$. Thus

$$\text{int}_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ -open [13] iff $A = \text{int}_\gamma(A)$. A set A is called γ -closed [13] iff $X - A$ is γ -open.

Definition 3. [10] A point $x \in X$ is called a γ -closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x . The set of all γ -closure points of A is called γ -closure of A and is denoted by $\text{cl}_\gamma(A)$. A subset A of X is called γ -closed, if $\text{cl}_\gamma(A) \subseteq A$. Note that $\text{cl}_\gamma(A)$ is contained in every γ -closed superset of A .

Definition 4. [14] The γ -exterior of A , written $\text{ext}_\gamma(A)$ is defined as the γ -interior of $(X - A)$. That is, $\text{int}_\gamma(A) = \text{ext}_\gamma(X - A)$.

Definition 5. [14] The γ -boundary of A , written $\text{bd}_\gamma(A)$ is defined as the set of points which do not belong to γ -interior or the γ -exterior of A .

Definition 6. [13] An operation γ on τ is said be regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition 7. [13] An operation γ on τ is said to be open, if for every open nbd U of each $x \in X$, there exists γ -open set B such that $x \in B$ and $U^\gamma \subseteq B$.

Definition 8. [2] Let $A \subseteq X$. A point $x \in X$ is said to be γ -limit point of A , if $U \cap \{A - \{x\}\} \neq \phi$, where U is a γ -open set in X . The set of all γ -limit points of A denoted A_γ^d is called γ -derived set.

Definition 9. [9] A subset A of a space (X, τ) is said to be a γ^* -semi-open set, if there exists a γ -open set O such that $O \subseteq A \subseteq \text{cl}_\gamma(O)$. The set of all γ^* -semi-open sets is denoted by $SO_{\gamma^*}(X)$.

Definition 10. [5] A function $f : (X, \tau) \rightarrow (Y, \tau)$ is said to be γ -semi-continuous, if for any γ -open B of Y , $f^{-1}(B)$ is γ^* -semi-open in X .

Definition 11. [5] A function $f : (X, \tau) \rightarrow (Y, \tau)$ is said to be γ -semi-open (closed), if for each γ -open (closed) set U in X , $f(U)$ is γ^* -semi-open (closed) in Y .

Definition 12. [5] A set A in a space X is said to be γ^* -semi-closed, if there exists a γ -closed set F such that $int_\gamma(F) \subseteq A \subseteq F$.

Proposition 13. [5] A subset A of X is γ^* -semi-closed if and only if $X - A$ is γ^* -semi-open.

Definition 14. A subset A of X is said to be γ -semi-nbd of a point $x \in X$, if there exists a γ^* -semi-open set U such that $x \in U \subseteq A$.

Definition 15. [9] Let A be a subset of space X . The intersection of all γ^* -semi-closed sets containing A is called γ^* -semi-closure of A and is denoted by $scl_{\gamma^*}(A)$. Note that A is γ^* -semi-closed if and only if $scl_{\gamma^*}(A) = A$.

Definition 16. [5] Let A be a subset of a space X . The union of all γ^* -semi-open sets of X contained in A is called γ^* -semi-interior of A and is denoted by $sint_{\gamma^*}(A)$.

Lemma 17. Let A be a subset of a space X . Then $x \in scl_{\gamma^*}(A)$ if and only if for any γ -semi-nbd N_x of x in X , $A \cap N_x \neq \phi$.

Proof. Let $x \in scl_{\gamma^*}(A)$. Suppose on the contrary, there exists a γ -semi-nbd N_x of x in X such that $A \cap N_x = \phi$. Then there exists $U \in SO_{\gamma^*}(A)$ such that $x \in U \subseteq N_x$. Therefore, $U \cap A = \phi$, so that $A \subseteq X - U$. Clearly $X - U$ is γ^* -semi-closed in X and hence $scl_{\gamma^*}(A) \subseteq X - U$. Since $x \notin X - U$, we obtain $x \notin scl_{\gamma^*}(A)$. This is contradiction to the hypothesis. This proves the necessity.

Conversely, suppose that every γ -semi-nbd of x in X meets A . If $x \notin scl_{\gamma^*}(A)$, then by definition there exists a γ^* -semi-closed F of X such that $A \subseteq F$ and $x \notin F$. Therefore we have $x \in X - F \in SO_{\gamma^*}(X)$. Hence $X - F$ is γ -semi-nbd of x in X . But $(X - F) \cap A = \phi$. This is contradiction to the hypothesis. Thus $x \in scl_{\gamma^*}(A)$. \square

3. γ -SEMI-OPEN FUNCTIONS

Theorem 18. Let $f : X \rightarrow Y$ be a function from a space X into a space Y and γ is an open, monotone and regular operation. Then the following statements are equivalent:

- (1) f is γ -semi-open.
- (2) $f(int_\gamma(A)) \subseteq sin_{\gamma^*}(f(A))$ for each subset A of X .
- (3) For each $x \in X$ and each γ -open-nbd U of x , there exists a γ -semi-nbd V of $f(x)$ such that $V \subseteq f(U)$.

Proof. (1) \Rightarrow (2). Suppose that f is γ -semi-open, and let A be an arbitrary subset of X . Since $f(int_\gamma(A))$ is γ^* -semi-open and $f(int_\gamma(A)) \subseteq f(A)$, then $f(int_\gamma(A)) \subseteq sin_{\gamma^*}(f(A))$.

(2) \Rightarrow (3). Let U be an arbitrary γ -open-nbd of $x \in X$. Then there exists γ -open set O such that $x \in O \subseteq U$. By hypothesis, we have $f(O) = f(int_\gamma(O)) \subseteq sin_{\gamma^*}(f(O))$ and hence $f(O) \subseteq sin_{\gamma^*}(f(O))$. Therefore it follows that $f(O)$ is γ -semi-open-nbd in Y such that $f(x) \in f(O) \subseteq f(U)$. This proves (3).

(3) \Rightarrow (1). Let U be an arbitrary γ -open set in X . For each $y \in f(U)$, by hypothesis there exists a γ -semi-nbd V_y of y in Y such that $V_y \subseteq f(U)$. Since V_y is a γ -semi-nbd of y , there exists a γ^* -semi-open set A_y in Y such that $y \in A_y \subseteq V_y$. Therefore $f(U) = \bigcup \{A_y : y \in f(U)\}$ is a γ^* -semi-open in Y , since is γ regular [9]. This shows that f is a γ -semi-open function. \square

Theorem 19. A bijective function $f : X \rightarrow Y$ is γ -semi-open if and only if $f^{-1}(scl_{\gamma^*}(B)) \subseteq cl_\gamma(f^{-1}(B))$ for every subset B of Y , where γ is an open operation.

Proof. Let B be an arbitrary subset of Y . Put

$$U = X - cl_\gamma(f^{-1}(B)) \quad (3. 1)$$

Clearly U is a γ -open set in X . Then by hypothesis, $f(U)$ is a γ^* -semi-open set in Y , or $Y - f(U)$ is γ^* -semi-closed set in Y . Since f is onto, from (3. 1), it follows $B \subseteq Y - f(U)$. Thus we have $scl_{\gamma^*}(B) \subseteq Y - f(U)$. Since f is one-one, we have $f^{-1}(scl_{\gamma^*}(B)) \subseteq f^{-1}(Y) - f^{-1}f(U) = X - f^{-1}f(U) \subseteq X - U = cl_\gamma(f^{-1}(B))$. This proves the necessity.

Conversely, let U be an arbitrary γ -open set in X . Put $B = Y - f(U)$. Since f is bijective, therefore by hypothesis, $f(U) \cap scl_{\gamma^*}(B) = f(U \cap f^{-1}(scl_{\gamma^*}(B))) \subseteq f(U \cap cl_\gamma(f^{-1}(B)))$. Since U is γ -open, therefore by Lemma 2(3) [14], we have $U \cap cl_\gamma(f^{-1}(B)) \subseteq cl_\gamma(U \cap f^{-1}(B))$. Moreover, it is obvious that $U \cap f^{-1}(B) = \phi$. Thus we have $f(U) \cap scl_{\gamma^*}(B) = \phi$ and hence $scl_{\gamma^*}(B) \subseteq Y - f(U) = B$. Therefore B is a γ^* -semi-closed in Y and hence $f(U)$ is a γ^* -semi-open set in Y . This proves that f is a γ -semi-open mapping. \square

Definition 20. [13] A function $f : (X, \tau, \gamma) \rightarrow (Y, \delta, \beta)$ is said to be (γ, β) -continuous, if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U such that $x \in U$ and $f(U^\gamma) \subseteq V^\beta$, where γ and β are operations on τ and δ respectively.

Definition 21. [13] A function $f : (X, \tau, \gamma) \rightarrow (Y, \delta, \beta)$ is said to be (γ, β) -open (closed), if for any γ -open (closed) set A of X , $f(A)$ is γ -open (closed) in Y .

Theorem 22. [1] Let $f : (X, \tau, \gamma) \rightarrow (Y, \delta, \beta)$ be a function and β be an open operation on Y . Then f is (γ, β) -continuous if and only if for each β -open set V in Y , $f^{-1}(V)$ is γ -open in X .

Theorem 23. [1] Let $f : (X, \tau, \gamma) \rightarrow (Y, \delta, \beta)$ be a function and β be an open operation on Y . Then the following are equivalent:

- (1) f is (γ, β) -open.
- (2) $f^{-1}(cl_\beta(B)) \subseteq cl_\gamma(f^{-1}(B))$.
- (3) $f^{-1}(bd_\beta(B)) \subseteq bd_\gamma(f^{-1}(B))$ for any subset B of Y .

Theorem 24. If a function $f : (X, \tau, \gamma) \rightarrow (Y, \delta, \beta)$ is a (γ, β) -open and a (γ, β) -continuous, then the inverse image $f^{-1}(B)$ of each β^* -semi-open set B in Y is γ^* -semi-open in X , where β is an open operation on Y .

Proof. Let B be an arbitrary β^* -semi-open set in Y . Then there exists β -open set V in Y such that $V \subseteq B \subseteq cl_\beta(V)$. Since f is (γ, β) -open, using Theorem 23, we have $f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}(cl_\beta(V)) \subseteq cl_\gamma(f^{-1}(V))$. Since f is (γ, β) -continuous and V is β -open in Y , by Theorem 22, $f^{-1}(V)$ is γ -open in X . This shows that $f^{-1}(B)$ is γ^* -semi-open set in X . \square

Theorem 25. Let X, Y and Z be three spaces and let $f : X \rightarrow Y$ be a function, $g : Y \rightarrow Z$ be an injective function and $gof : X \rightarrow Z$ is a γ -semi-open function. Then we have:

- (1) If f is (γ, β) -continuous and surjective, then g is γ -semi-open.
- (2) If g is (β, α) -open, (β, α) -continuous and injective, then f is γ -semi-open, where β is open operation on Y .

Proof. (1) Let V be a β -open set in Y . Then $f^{-1}(V)$ is γ -open in X , because f is (γ, β) -continuous. Since gof is γ -semi-open and f is surjective, therefore

$g(V) = (g \circ f)(f^{-1}(V))$ is α^* -semi-open in Z . This shows that g is a γ -semi-open function.

(2) Since g is injective, therefore for $A \subseteq X$, $f(A) = g^{-1}(g(f(A)))$. Let U be a γ -open set in X , then $g \circ f(U)$ is α^* -semi-open. Thus by Theorem 24, $g^{-1}(g(f(U))) = f(U)$ is β^* -semi-open in Y . This shows that f is a γ -semi-open function. \square

Let $B \subseteq X$, $\gamma : \tau \rightarrow P(X)$ be an operation. We define $\gamma_B : \tau_B \rightarrow P(X)$ as $\gamma_B(U \cap B) = \gamma(U) \cap B$. From here γ_B is an operation and satisfies that $cl_{\gamma_B}(U \cap B) \subseteq cl_\gamma(U \cap B) \subseteq cl_\gamma(U) \cap cl_\gamma(B)$. Using this fact we prove the following:

Theorem 26. *Let X be a space and B a γ^* -semi-open set in X containing a subset A of X . If A is γ^* -semi-open in the subspace B , then A is γ^* -semi-open in X , where γ is a regular operation.*

Proof. Let A be γ_B^* -semi-open in the subspace B . Then there exists a γ_B -open set U_B in B such that $U_B \subseteq A \subseteq cl_{\gamma_B}(U_B)$. Since U_B is γ_B -open in B , there exists a γ -open set U in X such that $U_B = U \cap B$ [4]. Thus we have $U \cap B \subseteq A \subseteq cl_{\gamma_B}(U \cap B) \subseteq cl_\gamma(U \cap B) = cl_\gamma(A) \cap cl_\gamma(B)$. Since B is γ^* -semi-open set in X and U is γ -open in X , therefore $U \cap B$ is γ -open in X . Consequently, A is a γ^* -semi-open set in X . \square

Theorem 27. *Let X and Y be spaces. If a bijective function $f : X \rightarrow Y$ is a γ -semi-open, then for each γ -open set $V (\neq \phi)$ in Y $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ is γ -semi-open, where γ is a regular operation.*

Proof. Let U_V be an arbitrary $\gamma_{f^{-1}(V)}$ -open set in $f^{-1}(V)$. Then there exists a γ -open set U in X such that $U_V = U \cap f^{-1}(V)$. Now we have $[f|_{f^{-1}(V)}](U_V) = f(U \cap f^{-1}(V)) = f(U) \cap V$. Since $f(U)$ is γ^* -semi-open and V is γ -open, $f(U) \cap V$ is γ^* -semi-open. Hence $[f|_{f^{-1}(V)}](U_V)$ is also γ_V^* -semi-open in V . This shows that $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ is a γ -semi-open mapping. \square

Theorem 28. *A bijective function $f : X \rightarrow Y$ is γ -semi-open if and only if for any subset V of Y and for any γ -closed set F of X containing $f^{-1}(V)$, there exists a γ^* -semi-closed set G of Y containing V such that $f^{-1}(G) \subseteq F$.*

Proof. Let $V \subseteq Y$ and F be a γ -closed set of X containing $f^{-1}(V)$. Put $G = Y - f(X - F)$. Since f is γ -semi-open, so G is γ^* -semi-closed sets in Y . As f is bijective, it follows from $f^{-1}(V) \subseteq F$ that $V \subseteq G$. Calculations give $f^{-1}(G) \subseteq F$.

Conversely, suppose U is γ -open set. Put $V = Y - f(U)$. Then $X - U$ is γ -closed set in X containing $f^{-1}(V)$. By hypothesis, there exists a γ^* -semi-closed set G of Y such that $V \subseteq G$ and $f^{-1}(G) \subseteq (X - U)$. On the other hand, it follows from $V \subseteq G$ that $f(U) = (Y - V) \subseteq (Y - G)$. Therefore, we obtain $f(U) = (Y - G) \in SO_{\gamma^*}(Y)$. This shows that f is γ -semi-open. \square

Lemma 29. [5] *The following properties of a subset A of X are equivalent:*

- (1) A is γ^* -semi-closed.
- (2) $int_\gamma(cl_\gamma(A)) \subseteq A$.
- (3) $X - A$ is γ^* -semi-open.

Theorem 30. *If $f : X \rightarrow Y$ is (γ, β) -open and (γ, β) -continuous mapping, then the inverse image $f^{-1}(B)$ of each β^* -semi-closed B in Y is γ^* -semi-closed in X , where β is an open operation on Y .*

Proof. This follows from Theorem 24 and Lemma 29. \square

Theorem 31. Let $f : X \rightarrow Y$ be surjective and $g : Y \rightarrow Z$ be an injective function and let $gof : X \rightarrow Z$ be a γ -semi-closed function. Then

(1) If f is (γ, β) -continuous and surjective, then g is β -semi-closed.

(2) If g is (β, α) -open, (β, α) -continuous and injective, then f is γ -semi-closed, where β is an open operation on Y .

Proof. (1) Suppose H is an arbitrary β -closed set in Y . Then $f^{-1}(H)$ is γ -closed in X because f is (γ, β) -continuous. Since gof is γ -semi-closed and f is surjective, $gof(f^{-1}(H)) \subseteq g(f(f^{-1}(H))) = g(H)$, is α^* -semi-closed in Z . This implies that g is β -semi-open function. This proves (1).

(2) Since g is injective so for every subset A of X , $f(A) = g^{-1}(g(f(A)))$. Let F be an arbitrary γ -closed set in X . Then $gof(F)$ is γ^* -semi-closed. It follows immediately from Theorem 30 that $f(F)$ is γ^* -semi-closed set in Y . This implies that f is γ -semi-closed. \square

4. γ -SEMI-CLOSED FUNCTIONS

Theorem 32. Let γ be an open and monotone operation. A function $f : X \rightarrow Y$ is γ -semi-closed if and only if $f(cl_\gamma(A)) \supseteq int_\gamma(cl_\gamma(f(A)))$ for every subset A of X .

Proof. Suppose f is a γ -semi-closed mapping and A is an arbitrary subset of X . Then $f(cl_\gamma(A))$ is γ^* -semi-closed in Y . Then by Lemma 29, we obtain $f(cl_\gamma(A)) \supseteq int_\gamma(cl_\gamma(f(cl_\gamma(A)))) \supseteq int_\gamma(cl_\gamma(f(A)))$. This implies that $f(cl_\gamma(A)) \supseteq int_\gamma(cl_\gamma(f(A)))$.

Conversely, suppose that F is an arbitrary γ -closed set in X . Then by hypothesis, we have $int_\gamma(cl_\gamma(f(F))) \subseteq f(cl_\gamma(F)) = f(F)$. By Lemma 29, $f(F)$ is γ^* -semi-closed in Y . This implies that f is γ -semi-closed. \square

Recall [9] that the intersection of all γ^* -semi-closed sets containing A is called γ -semi-closure of A and is denoted by $scl_{\gamma^*}(A)$. Clearly A is γ^* -semi-closed if and only if $scl_{\gamma^*}(A) = A$.

Theorem 33. Let γ be an open and monotone operation. A function $f : X \rightarrow Y$ is γ -semi-closed if and only if $scl_{\gamma^*}(A) \subseteq f(cl_\gamma(A))$ for every subset A of X .

Proof. Suppose f is a γ -semi-closed mapping and A is an arbitrary subset of X . Then $f(cl_\gamma(A))$ is γ^* -semi-closed. Since $f(A) \subseteq f(cl_\gamma(A))$, we obtain $scl_{\gamma^*}(f(A)) \subseteq f(cl_\gamma(A))$. This implies $scl_{\gamma^*}(f(A)) \subseteq f(cl_\gamma(A))$. \square

Sufficiency follows from Theorem 32.

Theorem 34. A surjective function $f : X \rightarrow Y$ is γ -semi-closed if and only if for each subset B in Y and each γ -open set U in X containing $f^{-1}(B)$, there exists a γ^* -semi-open set V in Y containing B such that $f^{-1}(V) \subseteq U$, where γ is a monotone and regular operation.

Proof. Suppose B is an arbitrary subset in Y and U is an arbitrary γ -open set in X containing $f^{-1}(B)$. We put

$$V = Y - f(X - U) \quad (4.2)$$

Then V is γ^* -semi-open set in Y . Since $f^{-1}(B) \subseteq U$, calculations give $B \subseteq V$. Moreover, by 4.2, we have $f^{-1}(V) = f^{-1}(Y) - f^{-1}(f(X - U)) = X - f^{-1}(f(X - U)) \subseteq X - (X - U) = U$.

Conversely, suppose that F is an arbitrary γ -closed set in X . Let y be an arbitrary point in $Y - f(F)$, then $f^{-1}(y) \subseteq X - f^{-1}(f(F)) \subseteq X - F$, and $X - F$ is γ -open in

X. Hence by the hypothesis, there exists a γ^* -semi-open set V_y containing y such that $f^{-1}(V_y) \subseteq X - F$. This implies that $v \in V_y \subseteq Y - f(F)$. We obtain that $Y - f(F) = \bigcup \{V_y : y \in Y - f(F)\}$ is γ^* -semi-open in Y, since union of any collection of γ^* -semi-open sets is γ^* -semi-open. Therefore $f(F)$ is γ^* -semi-closed. \square

5. γ -SEMI-CONTINUOUS FUNCTIONS

Theorem 35. *Let $f : X \rightarrow Y$ be a function and γ is an open operation. Then the following are equivalent:*

- (1) f is γ -semi-continuous.
- (2) $int_\gamma(cl_\gamma(f^{-1}(B))) \subseteq f^{-1}(cl_\gamma(B))$ for each subset B of Y .
- (3) $f(int_\gamma(cl_\gamma(A))) \subseteq cl_\gamma(f(A))$ for each subset A of X .

Proof. (1) \Rightarrow (2). Let B be an arbitrary subset of Y . Then by (1), $f^{-1}(cl_\gamma(B))$ is a γ^* -semi-closed set of X . Since $B \subseteq cl_\gamma(B)$, by Lemma 29, we get $f^{-1}(cl_\gamma(B)) \supseteq int_\gamma(cl_\gamma(f^{-1}(cl_\gamma(B)))) \supseteq int_\gamma(cl_\gamma(f^{-1}(B)))$ implies that $int_\gamma(cl_\gamma(f^{-1}(B))) \subseteq f^{-1}(cl_\gamma(B))$.

(2) \Rightarrow (3). Let A be an arbitrary subset of X . Put $B = f(A)$. Then $A \subseteq f^{-1}(B)$. Therefore by hypothesis, we have $int_\gamma(cl_\gamma(A)) \subseteq int_\gamma(cl_\gamma(f^{-1}(B))) \subseteq f^{-1}(cl_\gamma(B))$. Consequently, we have $f(int_\gamma(cl_\gamma(A))) \subseteq f f^{-1}(cl_\gamma(B)) \subseteq cl_\gamma(B) = cl_\gamma(f(A))$. This gives (3).

(3) \Rightarrow (1). Let F be an arbitrary γ -closed set of Y . Put $A = f^{-1}(F)$, then $f(A) \subseteq F$. Therefore by hypothesis, we have

$$f(int_\gamma(cl_\gamma(A))) \subseteq cl_\gamma(f(A)) \subseteq cl_\gamma(F) = F \quad (5.3)$$

By 5.3, we have $int_\gamma(cl_\gamma(A)) \subseteq f^{-1}f(int_\gamma(cl_\gamma(A))) \subseteq f^{-1}(cl_\gamma(f(A))) \subseteq f^{-1}(cl_\gamma(F)) = f^{-1}(F)$, or $int_\gamma(cl_\gamma(A)) \subseteq f^{-1}(F)$. By Lemma 29, $f^{-1}(F)$ is a γ^* -semi-closed set in X . This implies that f is γ -semi-continuous. \square

Definition 36. Let X be a space $A \subseteq X$ and $p \in X$. Then p is a γ^* -semi-limit point of A , for all γ^* -semi-open set U containing p such that $U \cap (A - \{p\}) \neq \emptyset$. The set of all γ^* -semi-limit point of A is said to be γ^* -semi-derived set of A and is denoted by $sd_{\gamma^*}(A)$.

Clearly if $A \subseteq B$ then

$$sd_{\gamma^*}(A) \subseteq sd_{\gamma^*}(B) \quad (5.4)$$

Remark 37. From the definition, it follows that p is a γ^* -semi-limit point of A if and only if $p \in scl_{\gamma^*}(A - \{p\})$.

Theorem 38. *The γ^* -semi-derived set, sd_{γ^*} , has the following properties:*

- (1) $scl_{\gamma^*}(A) = A \cup sd_{\gamma^*}(A)$.
- (2) $sd_{\gamma^*}(A \cup B) = sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$. In general
- (3) $\bigcup_i sd_{\gamma^*}(A_i) = sd_{\gamma^*}(\bigcup_i (A_i))$.
- (4) $sd_{\gamma^*}(sd_{\gamma^*}(A)) \subseteq sd_{\gamma^*}(A)$.
- (5) $scl_{\gamma^*}(sd_{\gamma^*}(A)) = sd_{\gamma^*}(A)$.

Proof. (1) Let $x \in scl_{\gamma^*}(A)$. Then $x \in C$, for every γ^* -semi-closed superset C of A . Now

(i) If $x \in A$, then $x \in A \cup sd_{\gamma^*}(A)$.

(ii) If $x \notin A$, then we prove that $x \in scl_{\gamma^*}(A)$.

To prove (ii), suppose U is γ^* -semi-open set containing x . Then $U \cap A \neq \emptyset$, for otherwise, $A \subseteq X - U = C$, where C is a γ^* -semi-closed superset of A not containing x . This contradicts the fact that x belongs to every γ^* -semi-closed superset C of A . Therefore $x \in sd_{\gamma^*}(A)$ gives $x \in A \cup sd_{\gamma^*}(A)$.

Conversely, suppose that $x \in A \cup sd_{\gamma^*}(A)$, we show that $x \in scl_{\gamma^*}(A)$. If $x \in A$ then $x \in scl_{\gamma^*}(A)$. If $x \in sd_{\gamma^*}(A)$, then we show that x is in every γ^* -semi-closed superset of A . We suppose otherwise that there is γ^* -semi-closed superset C of A not containing x . Then $x \in X - C = U$ (say), which is γ^* -semi-open and $U \cap A = \phi$. This implies that $x \notin sd_{\gamma^*}(A)$. This contradiction proves that $x \in scl_{\gamma^*}(A)$. Consequently $scl_{\gamma^*}(A) = A \cup sd_{\gamma^*}(A)$. This proves (1).

(2) $sd_{\gamma^*}(A \cup B) \subseteq sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$.

Let $x \in sd_{\gamma^*}(A \cup B)$. Then $x \in scl_{\gamma^*}((A \cup B) - \{x\})$ or $x \in scl_{\gamma^*}((A - \{x\}) \cup (B - \{x\}))$ implies $x \in scl_{\gamma^*}(A - \{x\})$ or $x \in scl_{\gamma^*}(B - \{x\})$. This gives $x \in sd_{\gamma^*}(A)$ or $x \in sd_{\gamma^*}(B)$. Therefore $x \in sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$. This proves $sd_{\gamma^*}(A \cup B) \subseteq sd_{\gamma^*}(A) \cup sd_{\gamma^*}(B)$.

Converse follows directly by using the property 5.4.

(3) The proof is immediate by property 5.4.

(4) Suppose that $x \notin sd_{\gamma^*}(A)$. Then $x \notin scl_{\gamma^*}(A - \{x\})$. This implies that there is γ^* -semi-open set U such that $x \in U$ and $U \cap (A - \{x\}) = \phi$. We prove that $x \notin sd_{\gamma^*}(sd_{\gamma^*}(A))$. Suppose on the contrary that $x \in sd_{\gamma^*}(sd_{\gamma^*}(A))$. Then $x \in scl_{\gamma^*}(sd_{\gamma^*}(A) - \{x\})$. Since $x \in U$, we have $U \cap (sd_{\gamma^*}(A) - \{x\}) \neq \phi$. Therefore there is a $q \neq x$ such that $q \in U \cap (sd_{\gamma^*}(A))$. It follows that $q \in (U - \{x\}) \cap (sd_{\gamma^*}(A) - \{x\})$. Hence $(U - \{x\}) \cap (sd_{\gamma^*}(A) - \{x\}) \neq \phi$, a contradiction to the fact that $(U \cap (sd_{\gamma^*}(A) - \{x\})) = \phi$. This implies that $x \notin sd_{\gamma^*}(sd_{\gamma^*}(A))$ and so $sd_{\gamma^*}(sd_{\gamma^*}(A)) \subseteq sd_{\gamma^*}(A)$. This proves (4).

(5) This is a consequence of (1), (2) and (4). \square

Theorem 39. [5] *Let $f : X \rightarrow Y$ be a function. Then the following are equivalent:*

- (1) $f : X \rightarrow Y$ is γ -semi-continuous.
- (2) $scl_{\gamma^*}(f^{-1}(A)) \subseteq f^{-1}(cl_{\gamma}(A))$ for each subset A of Y .

Theorem 40. *Let $f : X \rightarrow Y$ be a function and γ is an open operation. Then the following are equivalent:*

- (1) $f : X \rightarrow Y$ is γ -semi-continuous.
- (2) $f(sd_{\gamma^*}(A)) \subseteq cl_{\gamma}(f(A))$ for any subset A of X .

Proof. (1) \Rightarrow (2). Suppose that f is γ -semi-continuous. Let A be any set in X . Since $cl_{\gamma}(f(A))$ is γ -closed in Y , $f^{-1}(cl_{\gamma}(f(A)))$ is γ^* -semi-closed in X . $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl_{\gamma}(f(A)))$ gives $scl_{\gamma^*}(A) \subseteq scl_{\gamma^*}(f^{-1}(cl_{\gamma}(f(A)))) = f^{-1}(cl_{\gamma}(f(A)))$. Therefore $f(sd_{\gamma^*}(A)) \subseteq f(scl_{\gamma^*}(A)) \subseteq f f^{-1}(cl_{\gamma}(f(A))) \subseteq cl_{\gamma}(f(A))$. Consequently, $f(sd_{\gamma^*}(A)) \subseteq cl_{\gamma}(f(A))$.

(2) \Rightarrow (1). Suppose that $f(sd_{\gamma^*}(A)) \subseteq cl_{\gamma}(f(A))$, for $A \subseteq X$. Let B be any γ -closed subset of Y . We show that $f^{-1}(B)$ is γ^* -semi-closed in X . By hypothesis, $f(sd_{\gamma^*}(f^{-1}(B))) \subseteq cl_{\gamma}(f(f^{-1}(B))) \subseteq cl_{\gamma}(B) = B$ or $f(sd_{\gamma^*}(f^{-1}(B))) \subseteq B$ gives $sd_{\gamma^*}(f^{-1}(B)) \subseteq f^{-1}f(sd_{\gamma^*}(f^{-1}(B))) \subseteq f^{-1}(B)$ or $sd_{\gamma^*}(f^{-1}(B)) \subseteq f^{-1}(B)$ implies $f^{-1}(B)$ is γ^* -semi-closed in X . Thus f is γ -semi-continuous. \square

Theorem 41. [5] *Let $f : X \rightarrow Y$ be a function and $x \in X$. Then f is γ -semi-continuous if and only if for each γ -open set B containing $f(x)$, there exists $A \in SO_{\gamma^*}(X)$ such that $x \in A$ and $f(A) \subseteq B$, where γ is a regular operation.*

We use Theorem 41 and prove the following:

Theorem 42. *Let $f : X \rightarrow Y$ be an injective function. If f is γ -semi-continuous then $f(sd_{\gamma^*}(A)) \subseteq (f(A))_{\gamma}^d$ for every $A \subseteq X$, where γ is a regular operation.*

Proof. Suppose that f is γ -semi-continuous. Let $A \subseteq X$, $a \in sd_{\gamma^*}(X)$ and V be a γ -open-ncbd of $f(a)$. Since f is γ -semi-continuous then by Theorem 41, there exists a γ -semi-open-ncbd U of a such that $f(U) \subseteq V$. But $a \in sd_{\gamma^*}(A)$, therefore there exists an element $a_1 \in U \cap A$ such that $a \neq a_1$; then $f(a_1) \in f(A)$ and since f is an injection $f(a) \neq f(a_1)$. Thus every γ -open-ncbd V of $f(a)$ contains an element $f(a_1)$ of $f(A)$ different from $f(a)$. Consequently $f(a) \in (f(A))_{\gamma}^d$. We have therefore, $f(sd_{\gamma^*}(A)) \subseteq (f(A))_{\gamma}^d$. \square

The following theorem follows from Theorem 40:

Theorem 43. *Let $f : X \rightarrow Y$ be a function. If for every $A \subseteq X$, $f(sd_{\gamma^*}(A)) \subseteq (f(A))_{\gamma}^d$, then f is γ -semi-continuous, where γ is an open operation.*

Theorem 44. *A function $f : X \rightarrow Y$ is γ -semi-continuous if and only if $f^{-1}(int_{\gamma}(B)) \subseteq sint_{\gamma^*}(f^{-1}(B))$, for each $B \subseteq Y$, where γ is a regular operation.*

Proof. For any $B \subseteq Y$, $int_{\gamma}(B) = Y - cl_{\gamma}(Y - B)$ [14]. This implies $f^{-1}(int_{\gamma}(B)) = f^{-1}(Y - cl_{\gamma}(Y - B)) = X - f^{-1}(cl_{\gamma}(Y - B))$. Since f is γ -semi-continuous, by Theorem 39 we have $scl_{\gamma^*}(f^{-1}(Y - B)) \subseteq f^{-1}(cl_{\gamma}(Y - B))$. Hence $f^{-1}(int_{\gamma}(B)) \subseteq X - scl_{\gamma^*}(f^{-1}(Y - B))$. Thus $f^{-1}(int_{\gamma}(B)) \subseteq X - scl_{\gamma^*}(X - f^{-1}(B))$. Hence $f^{-1}(int_{\gamma}(B)) \subseteq X - scl_{\gamma^*}(X - f^{-1}(B)) = sint_{\gamma^*}(f^{-1}(B))$.

Conversely, let B be an arbitrary γ -open set in Y . Then $int_{\gamma}(B) = B$. By hypothesis $f^{-1}(B) = f^{-1}(int_{\gamma}(B)) \subseteq sint_{\gamma^*}(f^{-1}(B))$ implies $f^{-1}(B) \subseteq sint_{\gamma^*}(f^{-1}(B))$. But $sint_{\gamma^*}(f^{-1}(B)) \subseteq f^{-1}(B)$. Therefore, $f^{-1}(B) = sint_{\gamma^*}(f^{-1}(B))$. Thus $f^{-1}(B)$ is γ^* -semi-open. Consequently, f is γ -semi-continuous. \square

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