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Numerical Solution of Gas Dynamics Equation using Second Order Dynamic Mesh Technique

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Abstract. Second order dynamic mesh technique is employed to solve gas dynamic equations which represent the different aspects of hyperbolic (nonlinear) equations. The value of the density, velocity, pressure and internal energy of the gas at t= 0.15s are computed using dynamic mesh technique and uniform mesh method. Their graphical comparison is given in figures 1-4. We observe that the dynamic mesh graphs are more smooth than uniform mesh graphs. Therefore it is clear that the dynamic mesh technique gives better results than standard uniform mesh method.

1. INTRODUCTION

In this paper we have considered one dimensional gas dynamics equations representing laws of conservation of mass, momentum and energy along with the equation of state of the gas. This problem is considered as a case study for solving hyperbolic (non linear) conservation laws because it depicts the next level of complexity after the Berger's equations. In dynamic mesh technique a fixed number of mesh points move automatically to minimize the error in the solution. Second order dynamic mesh technique based on equidistribution principle is used here because it is more efficient than uniform mesh methods for solving time-dependent partial differential equations.

2. GAS DYNAMICS EQUATION

The one dimensional gas dynamics equations in conservation form are
Continuity equation

$$\rho_t + m_x = 0 \tag{2.1}$$

Momentum equation

$$m_t + \left[\frac{m^2}{\rho} + P\right]_x = 0 \tag{2.2}$$

Energy Equation

$$E_t + \left[\frac{m}{\rho}(E+P)\right]_x = 0 \tag{2.3}$$

In vector form, they can be written as

$$u_t + [f(u)]_x = 0 \tag{2.4}$$

Where

$$u = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, F(u) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{m}{\rho}(E+P) \end{bmatrix}$$

And ρ stands for density, m for momentum, P for pressure and E is the total energy per unit volume.
The equation of state for an ideal gas to express the internal energy per unit mass e as a function of P and ρ is given by

$$e = \frac{P}{\rho(\gamma - 1)}, \gamma > 1 \tag{2.5}$$

Where γ is the ratio of specific heats.
We take γ = 1.4, a value corresponding to a diatomic gas.
Bernoullie's equation is given by

$$E = e\rho + \frac{m^2}{2\rho}$$

Using equation 2.5 it becomes

$$E = \frac{P}{\gamma - 1} + \frac{m^2}{2\rho}$$
$$or P = (\gamma - 1)\left[E - \frac{m^2}{2\rho}\right] \tag{2.6}$$

We solve equations(2.1), (2.2), (2.3) and ((2.6)) which involves three independent variables ρ, m and E, t is time and x are the position coordinates, using the following initial and boundary conditions.

$$\rho(x,0) = \begin{bmatrix} 1.0000 & \text{if x} < 0 \\ 0.5635 & \text{if x} = 0 \\ 0.1250 & \text{if x} > 0 \end{bmatrix}$$

m(x,0) = 0 for all

$$E(x,0) = \begin{bmatrix} 2.5000 & if\ \text{x} < 0 \\ 1.375 & if\ \text{x} = 0 \\ 0.2500 & if\ \text{x} > 0 \end{bmatrix}$$

ρ (-0.5,t) = 1.0000, ρ (0.5,t) = 0.1250
m (-0.5,t) = 0.0000 = m (0.5,t)
E (-0.5,t) = 2.5000, E(0.5,t) = 0.2500

3. ADDITION OF ARTIFICIAL VISCOSITY

If we solve equation(2.4) by using finite differences for ux it reduces to a system of first order differential equations. The RKF45 program can be used to solve this system under the given boundary conditions but it has discontinuous solutions. To overcome this difficulty artificial viscosity, a term proportional to uxx is added to the system which makes the problem simpler. Then the system of equations (2.1), (2.2) and (2.3) becomes

$$\rho_t + mx = \lambda \rho_{xx} \tag{3.1}$$

$$m_t + \left[\frac{m^2}{\rho} + P\right]_x = \lambda m_{xx} \tag{3.2}$$

and

$$E_t + \left[\frac{m}{\rho}(E+P)\right]_x = \lambda E_{xx} \tag{3.3}$$

and in vector form, it can be written as

$$u_t + [F(u)]_x = \lambda u_{xx} \tag{3.4}$$

With boundary conditions as
ρ (-0.5,t) = 1.0000, ρ(0.5,t) = 0.1250
m (-0.5,t) = 0.000 = m (0.5,t) for all *t*
E (-0.5,t) = 2.5000, E(0.5,t) = 0.2500

$$u = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, F(u) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{m}{\rho}(E+P) \end{bmatrix}$$

It is well known that for hyperbolic conservation laws, even smooth initial conditions can produce solutions which eventually become discontinuous. Hence when we speak here of a solution of (2.4) we will mean a weak solution. In [6] Lax proves that if the solution u (x,t;λ) of (3.4) converges to a limit $\bar{u}$(x,t) as λ approaches 0^+, then $\bar{u}$(x,t) is a weak solution of (2.4). Further Foy [5] proves that the solution of (3.4) do indeed converge if the original shock waves are weak enough. Therefore the addition of artificial viscosity will not destroy the essential character of hyperbolic equations ((2.1)-(2.3)). We take λ= 5 x 10^{-4}.

4. DYNAMIC MESH:

Case Study.
To apply dynamic mesh technique we write the gas dynamics equation 3.4 as

$$u_t = G \tag{4.1}$$

where $G = \lambda u_{xx} - [F(u)]_x$
and we take λ = 5 x 10^{-4}
Finite difference approximation for the first and second order derivatives on a moving grid are given by

$$u_x = \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}} \tag{4.2}$$

$$u_{xx} = \left[\frac{u_{i+1}(x_i - x_{i-1}) - u_i(x_{i+1} - x_{i-1}) + u_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_i)(x_i - x_{i-1})(x_{i+1} - x_{i-1})}\right] \tag{4.3}$$

By using transformation

x = x (ξ,t), xϵ[0,1]
x(0,t) = 0, x (1,t) = 1
equation (4.1) reduces to the quasi Lagrangian form as

$$\dot{u} - \frac{\partial u}{\partial x}\dot{x} = G \tag{4.4}$$

Using eq (4.2), it becomes

$$\dot{u} - \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}\dot{x} = G_i \tag{4.5}$$

i = 0,1,2,. . . ,n
and Gi is the discrete approximation of G. Monitor functions involving higher derivatives of u(x) are extremely complicated for use, so White [7] recommends to use the arc length function

$$M(x,t) = \sqrt{1 + u_x^2}$$

5. Discretization of moving mesh partial differential equation (MMPDE)

MMPDE [1]

$$\dot{x} = \frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right) \tag{5.1}$$

gives the node speed and it will be used to solve the system because it gives better results. For this purpose equation (5.1) is discretized in space with centered finite differences on the uniform mesh.

$$\xi_i = \frac{i}{n}, i = 0, 1..., n$$

where n is positive integers and by using method of lines.
The discrete approximation of (5.1) is then

$$\dot{x} = \frac{E_i}{\tau} \tag{5.2}$$

Where τ = 1/λ and Ei is the discrete approximation of
$E = \frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right)$at ξ = ξi given by

$$E_i = \frac{\bar{M}_{i+1} + \bar{M}_i}{2\left(\frac{1}{n}\right)^2}(x_{i+1} - x_i) - \frac{\bar{M}_i + \bar{M}_{i-1}}{2\left(\frac{1}{n}\right)^2}(x_i - x_i - 1) \tag{5.3}$$

Thus equations (5.2) becomes

$$\dot{x}_i = \frac{n^2}{2\tau}\left[(\bar{M}_i + \bar{M}_{i+1})x_{i+1} - (\bar{M}_{i-1} + 2\bar{M}_{i+1})x_i + (\bar{M}_{i-1} + \bar{M}_i)x_{i-1}\right] \tag{5.4}$$
$$i = 1, 2, \ldots, n-1$$

$$\dot{x}_1 = \frac{n^2}{2\tau}\left[(\bar{M}_i + \bar{M}_2)x_2 - (\bar{M}_0 + 2\bar{M}_1 + \bar{M}_2)x_1 - 0.5(\bar{M}_0 + \bar{M}_1)\right] \tag{5.5}$$

$$\dot{x}_i = \frac{n^2}{2\tau}\left[(\bar{M}_i + \bar{M}_{i+1})_{i+1} - (\bar{M}_{i-1} + 2\bar{M}_i + \bar{M}_{i+1})x_i + (\bar{M}_{i-1} + \bar{M}_i)x_{i-1}\right] \tag{5.6}$$

i = 2,. . . ,n-2

$$\dot{x}_{n-1} = \frac{n^2}{2\tau}[0.5(\bar{M}_{n-1} + \bar{M}_n) - (\bar{M}_{n-2} + 2\bar{M}_{n-1} + \bar{M}_n)x_{n-1} + (\bar{M}_{n-2} + \bar{M}_{n-1})x_{n-2}] \quad (5.7)$$

where $\bar{M}_i$is the smoothed form of

$$M_i = \left[1 + \left(\frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}\right)^2\right]^{1/2} \quad (5.8)$$

Mi must be smoothed in order to obtain reasonable accuracy
Let $\bar{M}_i = \left(\frac{M_i^*}{S_i^*}\right)^{1/2}$
where $M_i^* = \sum_{k=i-j}^{i+j} (M_k)^2 \left(\frac{\eta}{1+\eta}\right)^{|k-i|}$
and $S_i^* = \sum_{k=i-j}^{i+j} \left(\frac{\eta}{1+\eta}\right)^{|k-i|}$
Where $\eta > 0$ is the smoothing parameter, and j a non-negative integer, is the smoothing index. The summation is understood to contain only elements with indices between zero and n. Thus the problem is reduced to solving two sets of equations (4.5) and (5.2). The initial conditions for xi is a uniform mesh, i.e.

$$x_i(0) = \frac{i}{n},\ i = 0, 1, 2, 3..., n$$

The boundary conditions being used are $\dot{x}$(0) = 0 and $\dot{x}$(n) = 0
The systems of ordinary differential equations are solved using ordinary differential equation solver RKF45. For calculation a relative and absolute tolerance of 10^{-8} is assumed. After testing various values and combinations for the parameters; η, J and τ the following values have been chosen since they given the most accurate result:
η = 2, J = 2 and τ = 10^{-3}

6. Results

The one dimensional gas dynamics equation is solved using moving mesh method with n = 100 at t = 0.15s. Other dynamic mesh formulations have also been considered. However, better results were obtained using (5.1).
The values of density, velocity, pressure and internal energy of the gas at t = 0.15s using dynamic mesh technique and uniform mesh method are given in tables 1-4 and are plotted together in figures 1-4. It is observed that the dynamic mesh technique yields equally accurate results as the uniform mesh method for significantly smaller number of points for the gas dynamics equations considered in this paper.

7. Conclusion

In this paper we have used the dynamic (moving) mesh technique for solution of one dimensional gas dynamics equation. This technique employs the equidistribution principle. On investigation it is found that dynamic (moving) mesh technique gives the better results for gas dynamics equations.

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Table 1- Density of the gas

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.90522825	0.91599216	0.01076391
-0.100000	0.59331406	0.60146611	0.00815205
0.000000	0.43400938	0.43642349	0.00241411
0.100000	0.42314568	0.42465446	0.00150878
0.200000	0.24555786	0.26980234	0.02424448
0.300000	0.24500000	0.26450688	0.01950688
0.400000	0.12500000	0.12500000	0.00000000
0.500000	0.12500000	0.12500000	0.00000000

Table 2- Velocity of the gas

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	0.00000000	0.00000000	0.00000000
-0.400000	0.00000000	0.00000000	0.00000000
-0.300000	0.00000000	0.00000000	0.00000000
-0.200000	0.00565540	0.00000983	0.00564557
-0.100000	0.42905106	0.32891547	0.10013559
0.000000	0.90723119	0.74714813	0.16008306
0.100000	0.92746010	0.92765451	0.00019441
0.200000	0.92742284	0.92746329	0.00004045
0.300000	0.00000000	0.93505693	0.93505930
0.400000	0.00000000	0.00000003	0.00000003
0.500000	0.00000000	0.00000000	0.00000000

Table 3- Pressure of the gas

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.99332728	0.99694026	0.00361298
-0.100000	0.52936241	0.55750428	0.02814187
0.000000	0.31182478	0.3220423	0.01021752
0.100000	0.30312357	0.30311585	0.00000772
0.200000	0.30311914	0.30312362	0.00000448
0.300000	0.10000000	0.30200000	0.20200000
0.400000	0.10000000	0.10000000	0.00000000
0.500000	0.10000000	0.10000000	0.00000000

Table 4-Energy of the gas

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	2.50000000	2.50000000	0.00000000
-0.400000	2.50000000	2.50000000	0.00000000
-0.300000	2.50000000	2.50000000	0.00000000
-0.200000	2.49522477	2.49854773	0.00332296
-0.100000	2.16122261	2.21779172	0.05656911
0.000000	1.79618690	1.88824636	0.09205946
0.100000	1.79089372	1.78172459	0.00916913
0.200000	1.85360731	1.82745774	0.02614957
0.300000	2.00000000	2.85256890	0.85256890
0.400000	1.99999999	2.00000000	0.00000001
0.500000	2.00000000	2.00000000	0.00000000