Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 45 (2013) pp. 77-96

Some Results On Homomorphic Images of $\Delta(2, 3, 13)$

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Abstract. It is known that conjugacy classes of actions of $PGL(2, Z)$ on $PL(F_q)$ can be represented by coset diagrams $D(\theta, q)$, where $\theta \in F_q$ and q is a power of a prime p. In this paper, we have obtained conditions in terms of θ and q which ensure the emergence of coset diagrams representing homomorphic images of infinite triangle group $\Delta(2, 3, 13) =< x, y : x^2 = y^3 = (xy)^{13} = 1 >$ on $PL(F_q)$. We have also found conditions for existence of some special types of fragments in the coset diagrams representing homomorphic images of $\Delta(2, 3, 13)$. We have used technique devised in [7] to stitch together small coset diagrams representing homomorphic images of $\Delta(2, 3, 13)$ to obtain homomorphic images of the same triangle group but of larger size.

AMS (MOS) Subject Classification Codes: Primary 20F05, Secondary 20G40.
Key Words: Linear-fractional transformations, Non-degenerate homomorphisms, Conjugacy classes, Parametrization, Projective line.

1. Introduction

It is well known that modular group $PSL(2, Z)$ is generated by two linear-fractional transformations $x : z \longrightarrow \frac{-1}{z}$ and $y : z \longrightarrow \frac{z-1}{z}$, satisfying the relations

$$x^2 = y^3 = 1 \tag{1.1}$$

The linear-fractional transformation $t : z \longrightarrow \frac{1}{z}$ inverts x and y, that is, $t^2 = (xt)^2 = (yt)^2 = 1$, and so extends the group $PSL(2, Z)$ to $PGL(2, Z)$. The extended modular group $PGL(2, Z)$ is then generated by the transformations x, y and t and its defining relations are

$$x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 \tag{1.2}$$

Let q be a power of a prime p. Then the group $PGL(2,q)$ is the group of all transformations $z \longrightarrow \frac{az+b}{cz+d}$ where $a,\ b,\ c,\ d$ are in F_q and $ad-bc \neq 0$, while the group $PSL(2,q)$ is its subgroup consisting of all those linear-fractional transformations $z \longrightarrow \frac{az+b}{cz+d}$ where $ad-bc$ is a non-zero square in F_q.

Let $PL(F_q)$ denote the projective line over F_q, that is, $F_q \cup \{\infty\}$. If $PGL(2,Z)$ acts on $PL(F_q)$, then every element of $PGL(2,q)$ gives a permutation on the points of $PL(F_q)$. The group $PGL(2,q)$ is a subgroup of the symmetric group S_{q+1}. As the elements of $PSL(2,q)$ give only even permutations, it is therefore a subgroup of the alternating group $A_{q+1.}$

Triangle groups and their significance is well explained in [1], [2], [8], [9]and [10]. The groups are represented by

$$\Delta\left(l,m,n\right) = < x, y : x^l = y^m = (xy)^n = 1 > \tag{1.3}$$

where $l,m,n \in Z$ and $l,m,n > 1$. If we put

$$\delta(\Delta\left(l,m,n\right)) = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 \tag{1.4}$$

then $\Delta\left(l,m,n\right)$ contains the fundamental group of an orientable surface of positive genus as a subgroup of finite index whenever $\delta(\Delta\left(l,m,n\right)) \leq 0$; in particular $\Delta\left(l,m,n\right)$ is infinite.

The triangle groups $\Delta\left(2,3,n\right)$ are important especially as homomorphic images of $PSL(2,Z)$. The groups are infinite if and only if $n > 5$. The finite triangle groups $\Delta\left(2,3,n\right), n \leq 5$ are known and they are: trivial, S_3, A_4, S_4 and A_5, for $n = 1,2,3,4$ and 5 respectively. If $n = 6$, then the triangle group $\Delta\left(2,3,n\right)$ is an extension by the cyclic group C_6 of a free Abelian group of rank 2. For $n = 7$, the triangle group $\Delta\left(2,3,n\right)$ becomes a Hurwitz group which is studied in [5] and [11]. Triangle group $\Delta\left(2,3,9\right)$ is studied in [7] and [10]. A detailed study of $\Delta\left(2,3,11\right)$ can be found in [8]. Here we are interested in triangle group $\Delta\left(2,3,13\right)$.

The triangle group $\Delta\left(2,3,13\right)$ has a fundamental domain consisting of two copies of a hyperbolic triangle with angles $\frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{13}$. Let $a,\ b,\ c$ be the vertices of the hyperbolic triangle and R_i represent hyperbolic reflection in hyperbolic sides M_i $(i = 1,\ 2$ and $13)$. Let $x = R_{13}R_3$ and $y = R_2R_{13}$, so that $yx = R_2R_{13}R_{13}R_3 = R_2R_3$. Then $R_{13}R_3$ is an anticlockwise hyperbolic rotation of π about a, R_2R_{13} is an anticlockwise hyperbolic rotation of $\frac{2\pi}{3}$ about b, and R_2R_3 is an anticlockwise hyperbolic rotation of $\frac{2\pi}{13}$ about c. Hence $x^2 = y^3 = (xy)^{13} = 1$ which is $\Delta\left(2,3,13\right)$. The group acts on hyperbolic space.

There are three more sections in this paper. In first section, we define some relevant notions and discuss parametrization of actions of $PGL(2,Z)$ on $PL(F_q)$. We explicitly describe coset diagrams and construction of the diagrams through the actions. We are interested in the diagrams which are permutation representations of homomorphic images of the group $\Delta\left(2,3,13\right)$. Therefore, we obtain conditions in terms of θ and q which ensure the emergence of the required coset diagrams representing homomorphic images of the infinite triangle group $\Delta\left(2,3,13\right)$. In second section, we discuss some fragments observed in the diagrams. Then we explain how new diagrams representing the homomorphic images can be obtained just by stitching together two or more diagrams with the help of two fragments. In last section, we find conditions for existence of some special type of fragments in the diagrams. Finally, we give a list of linear fractionals transformations, which can be employed to construct homomorphic image representations of the group on prime fields F_q, where $q < 1300$.

2. Parametrization and Coset Diagrams

A homomorphism $\alpha : PGL(2,Z) \longrightarrow PGL(2,q)$ is called a non-degenerate homomorphism if none of the generators x, y and t of $PGL(2,Z)$ lies in the kernel of α, so that $\bar{x} = x\alpha$ and $\bar{y} = y\alpha$ and $\bar{t} = t\alpha$ are of orders 2 , 3 and 2 respectively. Any two non-degenerate homomorphisms α and β are called conjugate if there exists an inner automorphism ρ of $PGL(2,q)$ such that $\beta = \alpha\rho$.

In [6] it has been proved that the conjugacy classes of non-degenerate homomorphisms from $PGL(2,Z)$ into $PGL(2,q)$ correspond in a one-to-one fashion with the conjugacy classes of non-trivial elements of $PGL(2,q)$, under a correspondence which assigns to the non-degenerate homomorphism α the class containing the element $(xy)\alpha$. This of course, means that we can actually parametrize the conjugacy classes of non-degenerate homomorphism $\alpha : PGL(2,Z) \longrightarrow PGL(2,q)$, except for a few uninteresting ones, by the elements of F_q. That is, we can in fact parametrize the actions of $PGL(2,Z)$ on $PL(F_q)$.

Let α be any such non-degenerate homomorphism and X, Y and T denote elements of $GL(2,q)$ corresponding to the linear-fractional transformations $\bar{x}$, $\bar{y}$ and $\bar{t}$ in $PGL(2,q)$, where as described earlier, $\bar{x} = x\alpha$, $\bar{y} = y\alpha$ and $\bar{t} = t\alpha$, for some non-degenerate homomorphism α from the group $PGL(2,Z)$ into $PGL(2,q)$, where F_q is not of characteristic 2 or 3, then because of this and because of the fact that $\bar{x}$, $\bar{y}$ and $\bar{t}$ are of orders 2, 3, and 2 respectively, we can take the matrices X, Y and T to be $X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}$, $Y = \begin{bmatrix} d & kf \\ f & -d-1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$ where $a, c, d, f, k \in F_q$ with $k \neq 0$. We write

$$a^2 + kc^2 = -\Delta \tag{2.1}$$

and require that

$$d^2 + d + kf^2 + 1 = 0. \tag{2.2}$$

This certainly yields elements satisfying the relation $X^2 = \lambda_1 I$, $Y^3 = \lambda_2 I$ and $T^2 = \lambda_3 I$, where λ_1, λ_2 and λ_3 are some non-zero scalars and I is the identity matrix. The non-degenerate homomorphism α is determined by $\bar{x}\bar{y}$ because the one-to-one correspondence assigns to α the class containing $\bar{x}\bar{y}$. So we have to check only the conjugacy class of $\bar{x}\bar{y}$. The matrix XY has the trace

$$r = a\,(2d+1) + 2kcf \tag{2.3}$$

If $trace\,(XYT) = ks$, then

$$s = 2af - c\,(2d+1)\,, \tag{2.4}$$

so that

$$3\Delta = r^2 + ks^2 \tag{2.5}$$

and set

$$\theta = \frac{r^2}{\Delta}. \tag{2.6}$$

For given q and θ we can always find the matrices X, Y and T by using equations (2.1) to (2.6) . The action of $PGL(2,Z)$ on $PL(F_q)$ involves $PGL(2,q)$, and the corresponding coset diagram yields a permutation representation of $PGL(2,q)$. In [6] a mechanism has been developed to find a unique coset diagram $D(\theta, q)$ corresponding to each $\theta \in F_q$. It is unique in the sense that the actions corresponding to the same conjugacy class will produce the same coset diagram, except the labelling of the vertices.

We use coset diagrams for the actions of the group $PGL(2, Z)$ on $PL(F_q)$. The diagrams are defined as follows:

three cycles of y are represented by small triangles whose vertices are permuted counter-clockwise by y; any two vertices are interchanged by the involution x which is represented by an edge; action of t is represented by reflection about the vertical line of symmetry; fixed points of x and y (if exist) are denoted by heavy dots. Notice that $(yt)^2 = 1$ is equivalent to $tyt = y^{-1}$, which means that t reverses the orientation of the triangles representing three cycles of y (as reflection does); because of this, there is no need to make the diagram more complicated by introducing t-edges. These diagrams are called coset diagrams because here the vertices are identifiable with the right cosets in $PGL(2, Z)$, of the stabilizer N of any given point of $PL(F_q)$, so that x or y joins the coset Ngx or Ngy(for each $g \in PGL(2, Z)$); hence the description of a coset diagram.

For instance consider an action of $PGL(2, Z) =< x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 >$ on $PL(F_{23})$ by $x : z \longrightarrow \dfrac{-1}{z}$, $y : z \longrightarrow \dfrac{z-1}{z}$ and $t : z \longrightarrow \dfrac{1}{z}$ to give the following permutation representations:

$$\begin{aligned}
\overline{x} &: (0\ \infty)(1\ 22)(2\ 11)(3\ 15)(4\ 17)(5\ 9)(6\ 19)(7\ 13)(8\ 20)(10\ 16)(12\ 21)(14\ 18)\\
\overline{y} &: (0\ \infty\ 1)(2\ 12\ 22)(3\ 16\ 11)(4\ 18\ 15)(5\ 10\ 17)(6\ 20\ 9)(7\ 14\ 19)(8\ 21\ 13)\\
\overline{t} &: (0\ \infty)(1)(2\ 12)(3\ 8)(4\ 6)(5\ 14)(7\ 10)(9\ 18)(11\ 21)(13\ 16)(15\ 20)(17\ 19)(22).
\end{aligned}$$

The coset diagram of the action is shown in Figure 1:

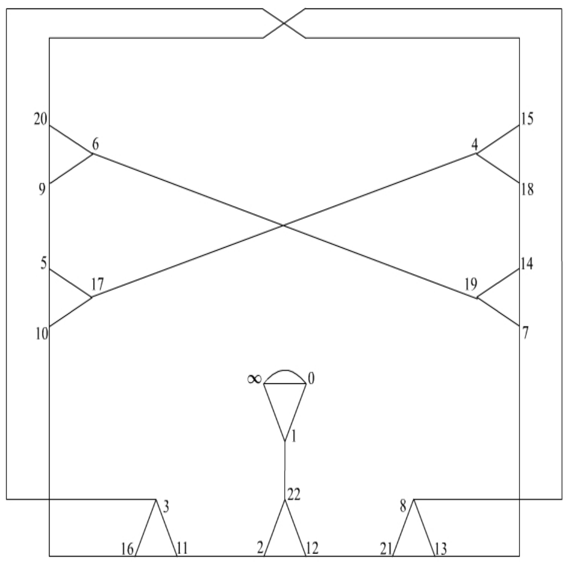

FIGURE 1

In this paper, we are interested in those conjugacy classes of non-degenerate homomorphisms α, which evolve the pairs of linear fractional transformations $\bar{x}$ and $\bar{y}$ satisfying the relation $\bar{x}^2 = \bar{y}^3 = (\bar{x}\bar{y})^{13} = 1$.

Theorem 1. *For each zero of* $f(z)=z^6-11z^5+45z^4-84z^3+70z^2-21z+1$ *in* F_p, *where* $f(z)\in Z[z]$ *and* p *is a prime number such that* $p\equiv\pm1\ (\mathrm{mod}\ 13)$, *there exists a conjugacy class of non-degenerate homomorphisms* α *such that*
$\alpha(PGL(2,\ Z))=<\bar{x},\bar{y}:\bar{x}^2=\bar{y}^3=(\bar{x}\bar{y})^{13}=1>$.

Proof. Since $p\equiv\pm1\ (\mathrm{mod}\ 13)$, therefore, due to a result of A. M. Macbeath [3], there are two distinct traces r_1, r_2 of elements of the group $SL(2,p)$, that yield elements of order 13 in $PGL(2,\ p)$. Thus there are two conjugacy classes of non-degenerate homomorphisms from $\Delta(2,\ 3,\ 13)$ into $PGL(2,\ p)$. Every element of $PSL(2,p)$ that comes from an element of $SL(2,p)$ with trace r_1 or r_2 must have order 13. Now suppose A is any element of $SL(2,p)$ which has the trace r_1 or r_2. As A is a conjugate in $GL(2,\ p^2)$ to a matrix B of the form $\begin{bmatrix}\rho & 0\\ 0 & \rho^{-1}\end{bmatrix}$, where ρ is primitive 26-th root of unity in F_{p^2}, we have $r=tr(A)=tr(B)=\rho+\rho^{-1}$. Next $r^2=\left(\rho+\rho^{-1}\right)^2=\rho^2+\rho^{-2}+2$, so $r^2-2=\rho^2+\rho^{-2}$, which is the trace of B^2 and $r^3=\left(\rho+\rho^{-1}\right)^3=\rho^3+3\rho+3\rho^{-1}+\rho^{-3}$, so $r^3-3r=\rho^3+\rho^{-3}$, which is the trace of B^3 and $\left(r^2-2\right)^2=\left(\rho^2+\rho^{-2}\right)^2$ implies that $r^4-4r^2+4=\rho^4+\rho^{-4}+2$, so $r^4-4r^2+2=\rho^4+\rho^{-4}$, which is the trace of B^4. Similarly we get $r^5-5r^3+5r=\rho^5+\rho^{-5}$ and $r^6-6r^4+9r^2-2=\rho^6+\rho^{-6}$.

Since $\rho^{26}=1$, so $\left(\rho^{13}-1\right)\left(\rho^{13}+1\right)=0$, but $\rho^{13}\neq1$, which implies that $\rho^{13}+1=0$, so $(\rho+1)\left(\rho^{12}-\rho^{11}+\rho^{10}-\rho^9+\rho^8-\rho^7+\rho^6-\rho^5+\rho^4-\rho^3+\rho^2-\rho+1\right)=0$. But $\rho\neq-1$, which implies that $\rho^{12}-\rho^{11}+\rho^{10}-\rho^9+\rho^8-\rho^7+\rho^6-\rho^5+\rho^4-\rho^3+\rho^2-\rho+1=0$. Now since $\rho^{13}=-1$, we have $\rho^{12}=-\rho^{-1}$, $\rho^{11}=-\rho^{-2}$, $\rho^{10}=-\rho^{-3}$, $\rho^9=-\rho^{-4}$, $\rho^8=-\rho^{-5}$, $\rho^7=-\rho^{-6}$. So by substituting in equation $\rho^{12}-\rho^{11}+\rho^{10}-\rho^9+\rho^8-\rho^7+\rho^6-\rho^5+\rho^4-\rho^3+\rho^2-\rho+1=0$, we get $(\rho^6+\rho^{-6})-(\rho^5+\rho^{-5})+(\rho^4+\rho^{-4})-(\rho^3+\rho^{-3})+(\rho^2+\rho^{-2})-(\rho+\rho^{-1})+1=0$, which implies that $r^6-r^5-5r^4+4r^3+6r^2-3r-1=0$. Since $tr(B)=r$ and $det(B)=\Delta=1$, we can put $\theta=r^2$ in the equation to obtain $\theta^6-11\theta^5+45\theta^4-84\theta^3+70\theta^2-21\theta+1=0$. Thus $\bar{x}\bar{y}$ has order 13 if and only if $f(\theta)=\theta^6-11\theta^5+45\theta^4-84\theta^3+70\theta^2-21\theta+1=0$. □

Above theorem can be proved alternatively as:

Let X, Y and XY be the matrices in $GL(2,\ p)$ corresponding to the elements $\bar{x}$, $\bar{y}$ and $\bar{x}\bar{y}$ respectively. Now characteristic equation of XY can be written as:

$$(XY)^2-rXY+\Delta I=0.$$

This implies that:

$$(XY)^2=rXY-\Delta I$$

$$\begin{aligned}(XY)^4 &= \left(r^2-\Delta\right)(XY)^2-r\Delta XY\\ &= \left(r^2-\Delta\right)(rXY-\Delta I)-r\Delta XY\\ &= \left(r^3-2r\Delta\right)XY+\left(-\Delta r^2+\Delta^2\right)I\end{aligned}$$

$$\begin{aligned}(XY)^5 &= \left(r^3-2r\Delta\right)(XY)^2+\left(-\Delta r^2+\Delta^2\right)XY\\ &= \left(r^3-2r\Delta\right)(rXY-\Delta I)+\left(-\Delta r^2+\Delta^2\right)XY\\ &= \left(r^4-3\Delta r^2+\Delta^2\right)XY+\left(-r^3\Delta+2r\Delta^2\right)I.\end{aligned}$$

Continuing in similar way we get:

$$(XY)^6 = \left(r^5 - 4r^3\Delta + 3r\Delta^2\right) XY + \left(-r^4\Delta + 3r^2\Delta^2 - \Delta^3\right) I$$

$$(XY)^7 = \left(r^6 - 5r^4\Delta + 6r^2\Delta^2 - \Delta^3\right) XY + \left(-r^5\Delta + 4r^3\Delta^2 - 3r\Delta^3\right) I$$

$$\begin{aligned}
(XY)^8 &= \left(r^7 - 6r^5\Delta + 10r^3\Delta^2 - 4r\Delta^3\right) XY + (-\Delta r^6 + 5r^4\Delta^2 - 6r^2\Delta^3 \\
&\quad +\Delta^4)I \\
(XY)^9 &= \left(r^8 - 7\Delta r^6 + 15r^4\Delta^2 - 10r^2\Delta^3 + \Delta^4\right) XY + (-r^7\Delta + 6r^5\Delta^2 \\
&\quad -10r^3\Delta^3 + 4r\Delta^4)I \\
(XY)^{10} &= \left(r^9 - 8r^7\Delta + 21r^5\Delta^2 - 20r^3\Delta^3 + 5r\Delta^4\right) XY + (-r^8\Delta + 7\Delta^2 r^6 \\
&\quad -15r^4\Delta^3 + 10r^2\Delta^4 - \Delta^5)I \\
(XY)^{11} &= \left(r^{10} - 9r^8\Delta + 28\Delta^2 r^6 - 35r^4\Delta^3 + 15r^2\Delta^4 - \Delta^5\right) XY + (-r^9\Delta \\
&\quad +8r^7\Delta^2 - 21r^5\Delta^3 + 20r^3\Delta^4 - 5r\Delta^5)I \\
(XY)^{12} &= \left(r^{11} - 10r^9\Delta + 36r^7\Delta^2 - 56r^5\Delta^3 + 35r^3\Delta^4 - 6r\Delta^5\right) XY + \\
&\quad \left(-r^{10}\Delta + 9r^8\Delta^2 - 28\Delta^3 r^6 + 35r^4\Delta^4 - 15r^2\Delta^5 + \Delta^6\right) I
\end{aligned}$$

$$\begin{aligned}
(XY)^{13} &= \left(r^{12} - 11r^{10}\Delta + 45r^8\Delta^2 - 84r^6\Delta^3 + 70r^4\Delta^4 - 21r^2\Delta^5 + \Delta^6\right) XY - \\
&\quad \left(r^{11}\Delta - 10r^9\Delta^2 + 36r^7\Delta^3 - 56r^5\Delta^4 + 35r^3\Delta^5 - 6r\Delta^6\right) I.
\end{aligned}$$

But $(XY)^{13} = \lambda I$, so we must have:

$$r^{12} - 11r^{10}\Delta + 45r^8\Delta^2 - 84r^6\Delta^3 + 70r^4\Delta^4 - 21r^2\Delta^5 + \Delta^6 = 0.$$

But $r^2 = \Delta\theta$, so

$$\theta^6\Delta^6 - 11\theta^5\Delta^6 + 45\theta^4\Delta^6 - 84\theta^3\Delta^6 + 70\theta^2\Delta^6 - 21\theta\Delta^6 + \Delta^6 = 0$$

or

$$\theta^6 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1 = 0,$$

which is the required condition for $(\bar{x}\bar{y})^{13} = 1$.

The six zeros of $f(\theta) = \theta^6 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1$ lie in F_p if $p = 13m \pm 1$, where $m \in Z$. Corresponding to each zero of $f(\theta) = 0$ in F_p, we can construct a coset diagram using the technique devised in [6]. For instance $f(\theta) = 0$ has six roots $10, 11, 13, 15, 28$ and 40 in F_{53}. If we take $\theta = 28$, then the linear-fractional transformations $\bar{x}$, $\bar{y}$ and $\bar{t}$ are $\dfrac{35z+29}{29z-35}$, $\dfrac{23}{23z-1}$ and $\dfrac{-1}{z}$ respectively. Linear fractional transformations $\bar{x}$, $\bar{y}$ and $\bar{t}$ have permutation representations as:

$\bar{x}$: (0 34) (1 7) (2) (3 25) (4 42) (5 51) (6 49) (8 43) (9 17) (10 31) (11 19) (12 48) (13 29) (14 ∞) (15 52) (16 33) (18 50) (20 38) (21 27) (22 32) (23 30) (24 39) (26) (28 47) (35 36) (37 41) (40 44) (45 46)

$\bar{y}$: (0 30 ∞) (1 42 31) (2 17 4) (3 51 48) (5 36 9) (6 11 39) (7 23 15) (8 12 50) (10 45 46) (13 28 26) (14 43 49) (16 34 40) (18 22 33) (19 24 44) (20 37 38) (21 47 25) (27 35 32) (29 52 41) (30 ∞ 0)

$\bar{t}$: (0 ∞) (1 52) (2 26) (3 35) (4 13) (5 21) (6 44) (7 15) (8 33) (9 47) (10 37) (11 24) (12 22) (14 34) (16 43) (17 28) (18 50) (19 39) (20 45) (23) (25 36) (27 51) (29 42)

$(30)\,(31\;41)\,(32\;48)\,(38\;46)\,(40\;49)$.
Therefore the coset diagram will be $D(28,53)$ in which each vertex is fixed by $(\bar{x}\bar{y})^{13}$. Thus it will be a diagram depicting $\alpha\,(\Delta\,(2,3,13))$.

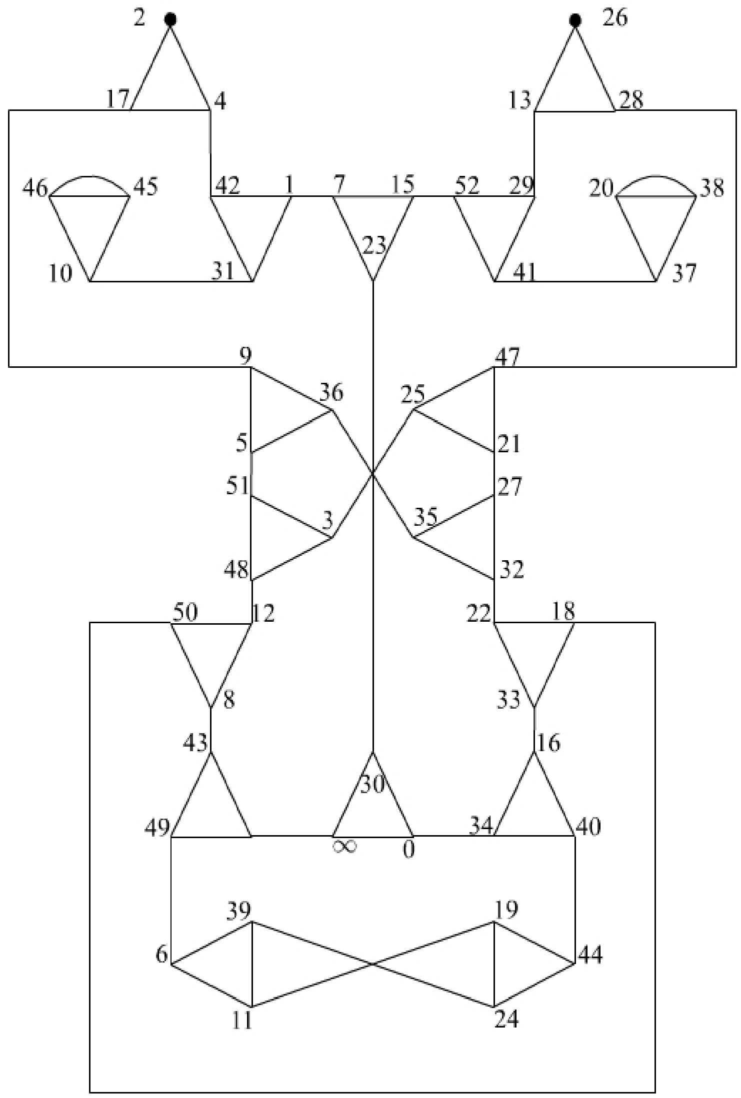

FIGURE 2

3. Fragments and Connecting Coset Diagrams

We can use Theorem1 together with the technique described in [6] to obtain coset diagrams depicting those actions of $PGL\,(2,Z)$ on $PL\,(F_q)$, which, for a suitable q, evolve homomorphic images of $\Delta\,(2,3,13)$ as subgroups of $PGL\,(2,q)$. Comparatively, it is a better procedure for obtaining homomorphic images of $\Delta(2,3,13)$ under non degenerate homomorphism $\alpha : PGL(2,Z) \longrightarrow PGL(2,q)$.

By connecting smaller graphs representing groups of smaller degree we can obtain a bigger graph representing a group of larger degree. It is then easy to study the properties of a new group just by studying its graph. We shall use some special types of fragments of coset diagrams depicting $\alpha(\Delta(2,3,13))$ to stitch together two or more coset diagrams still depicting $\alpha(\Delta(2,3,13))$. Hence, it is useful to find conditions for the existence of these special fragments in a coset diagrams representing $\alpha(\Delta(2,3,13))$. We shall find these conditions in next section.

Any two coset diagrams can be joined together to obtain a coset diagram of an arbitrary size provided they are joined together in a special way. The new coset diagram thus produced will still preserve all the inherited properties of the component coset diagrams. To join coset diagrams together one needs coset diagrams containing fragments γ_1 (Figure 3) and γ_2 (Figure 4).

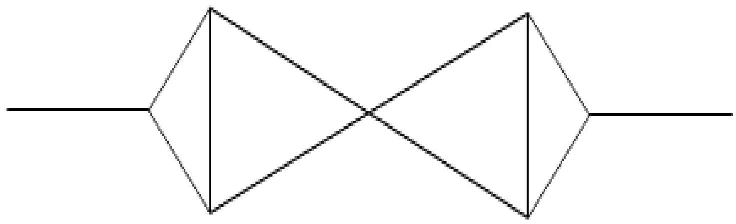

FIGURE 3. (γ_1)

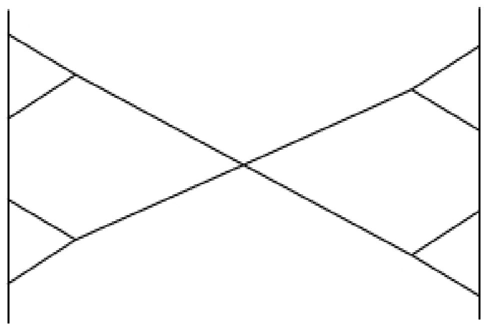

FIGURE 4. (γ_2)

We note that $D(11,53)$ and $D(36,79)$ are the coset diagrams containing both fragments γ_1 and γ_2 such that each vertex of these diagrams is fixed by $(\bar{x}\bar{y})^{13}$.

By $|D(\theta,q)|$ we shall mean the number of vertices in a coset diagram $D(\theta,q)$ and by $PL(F_{q_1}) \cup PL(F_{q_2})$ we shall mean a set having $q_1 + q_2 + 2$ elements.

Theorem 2. *Let l and m be the number of copies of the coset diagrams $D\left(11,53\right)$ and $D\left(36,79\right)$ respectively containing both γ_1 and γ_2. Then for all n which are expressible as $n = l\left|D\left(11,53\right)\right| + m\left|D\left(36,79\right)\right|$ the coset diagram $D\left(n\right)$ of n vertices depicts $\alpha\left(\Delta\left(2,3,13\right)\right)$.*

Proof. We choose either of the coset diagrams $D\left(11,53\right)$ or $D\left(36,79\right)$ and label vertices of the fragment $\gamma_1$with a, b, c and d. We place another copy of $D\left(11,53\right)$ or $D\left(36,79\right)$ with vertices $\acute{a}, \acute{b}, \acute{c}$ and $\acute{d}$ on a common vertical axis of symmetry, one above the other, and join the vertices a to $\acute{d}$, b to $\acute{c}$, c to $\acute{b}$ and d to $\acute{a}$ by x $-$edges. That is,

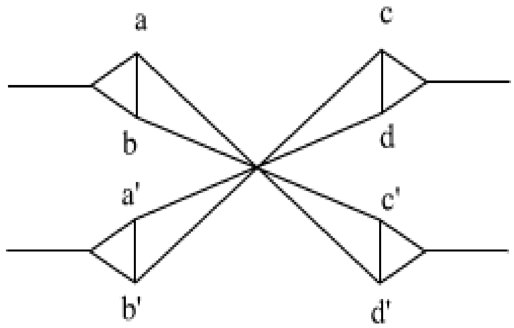

FIGURE 5

Similarly for the coset diagram $D\left(11,53\right)$ or $D\left(36,79\right)$ containing fragment γ_2 labelled with a, b, c and d, we take another copy of $D\left(11,53\right)$ or $D\left(36,79\right)$ containing fragment γ_2 labelled with $\acute{a}$, $\acute{b}$, $\acute{c}$ and $\acute{d}$ on a common vertical axis of symmetry, one above the other, and join the vertices a to $\acute{d}$, b to $\acute{c}$, c to $\acute{b}$ and d to $\acute{a}$ by x $-$edges. That is,

The resulting coset diagram will be one of the following

1. $D\left(11,53\right) + D\left(11,53\right)$
2. $D\left(11,53\right) + D\left(36,79\right)$
3. $D\left(36,79\right) + D\left(36,79\right)$

depicting an action of $PGL\left(2,Z\right)$ on $PL\left(F_{53}\right) \cup PL\left(F_{53}\right)$ of 108 elements or on $PL\left(F_{53}\right) \cup PL\left(F_{79}\right)$ of 134 elements or on $PL\left(F_{79}\right) \cup PL\left(F_{79}\right)$ of 160 elements respectively. Let $\left(\tau, c_1, c_2, ..., c_{j-1}, \sigma, c_j, ..., c_{q-2}\right)$ and $\left(\mu, d_1, d_2, ..., d_{j-1}, \lambda, d_j, ..., d_{q-2}\right)$ be cycles of $\bar{x}\bar{y}$ in the representations of $\alpha\left(\Delta\left(2,3,13\right)\right)$ depicted by $D\left(11,53\right) + D\left(11,53\right)$ or $D\left(11,53\right) + D\left(36,79\right)$ or $D\left(36,79\right) + D\left(36,79\right)$.

Here $\left(\mu, c_1, c_2, ..., c_{j-1}, \sigma, c_j, ..., c_{q-2}\right)$ and $\left(\tau, d_1, d_2, ..., d_{j-1}, \lambda, d_j, ..., d_{q-2}\right)$ are the cycles of the element $\bar{x}\bar{y}$. The rest of the cycles of $\bar{x}\bar{y}$ remain the same, that is, of the length 13. Thus resulting coset diagram $D\left(11,53\right) + D\left(11,53\right)$ or $D\left(11,53\right) + D\left(36,79\right)$ or $D\left(36,79\right) + D\left(36,79\right)$ will again be a coset diagram for $\alpha\left(\Delta\left(2,3,13\right)\right)$.

We can join l copies of $D\left(11,53\right)$ and m copies of $D\left(36,79\right)$ in a similar way. After joining these coset diagrams together, we obtain a coset diagram $D\left(n\right)$ with $l\left|D\left(11,53\right)\right| + m\left|D\left(36,79\right)\right|$ vertices representing $\alpha\left(\Delta\left(2,3,13\right)\right)$. □

Remark 1. The vertices of the coset diagram $D\left(\theta, q\right)$ can be relabelled by different symbols, because any two finite fields of the same size are isomorphic. Therefore, the vertices of coset diagram $D\left(n\right)$ can be relabeled avoiding repetition of labels.

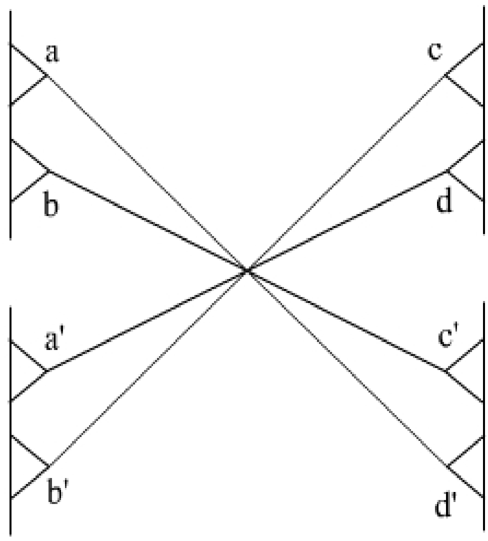

FIGURE 6

Remark 2. Note that $D(n)$ still has γ_1 and γ_2. Therefore one can continue stitching more diagrams containing γ_1 or γ_2 or both but possibly different from $D(11,53)$ and $D(36,79)$.

4. EXISTENCE OF FRAGMENTS IN COSET DIAGRAMS

In the coset diagrams for the actions of $PGL(2,Z)$ on $PL(F_q)$, where q is a power of a prime, no non trivial linear fractional transformation fixes more than two vertices. Thus, we should look for fragments of coset diagrams in which more than two vertices are fixed by a linear fractional transformation so that their existence in a coset diagram for the action of $PGL(2,Z)$ on $PL(F_q)$ ensures that the linear fractional transformation $(\bar{x}\bar{y})^{13}$ is trivial. Their existence, therefore, ensures coset diagrams for the homomorphic images of $\Delta(2,3,13)$. Since some of these fragments contain γ_1 or γ_2 or both, therefore, we are enabled to join them together to obtain a larger coset diagram of size $n = l\,|D(11,53)| + m\,|D(36,79)|$.

Existence of some types of fragments in $D(\theta,q)$ for an action of the extended modular group on the projective line over a finite field has been discussed in [4]. The coset diagrams for $\alpha(\Delta(2,3,13))$ also contain some special and useful fragments. In these fragments more than two vertices are fixed by $(\bar{x}\bar{y})^{13}$. In the following we determine conditions in terms of θ and q, for the existence of the fragments in $D(\theta,q)$ depicting homomorphic images of $\Delta(2,3,13)$.

Theorem 3. *(i) The fragment γ_3 (Figure. 7) will occur in $D(\theta,q)$ if $\theta^2-2\theta-3$ is a square in F_q. (ii) The fragment γ_4 (Figure. 8) will occur in $D(\theta,q)$ if $\theta(\theta-3)\left(\theta^2-3\theta+4\right)$ is a square in F_q. (iii) The fragment γ_5 (Figure. 9) will occur in $D(\theta,q)$ if $(\theta-1)^2(\theta-3)(\theta^3-5\theta^2+7\theta+1)$ is a square in F_q. (iv) The fragment γ_6 (Figure. 10) will occur in $D(\theta,q)$ if $\theta(\theta-4)$ is a square in F_q. (v) The fragment γ_7 (Figure. 11) will occur in $D(\theta,q)$ if $\left(\theta^4-6\theta^3+11\theta^2-6\theta-3\right)$ is a square in F_q.*

Proof. The vertices v_1, v_2, v_3, v_4 and v_5 are fixed by the elements
$\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}$ and
, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$of $\alpha\left(PGL\left(2, Z\right)\right)$. Notice that $\det\left(X\right) = \Delta$, trace$(X) = 0$, $\det\left(Y\right) = 1$, trace$(Y) = -1$, $\det\left(XY\right) = \Delta$, and trace$(XY) = r$. Following equations can be easily derived,

$$XYX = rX + \Delta I + \Delta Y \tag{4.1}$$

$$YXY = rY + X \tag{4.2}$$

$$YX = rI - X - XY \tag{4.3}$$

from the equations

$$X^2 + \Delta I = 0$$

$$\left(XY\right)^2 - r\left(XY\right) + \Delta I = 0$$

$$Y^2 + Y + I = 0$$

where X and Y are the matrices corresponding to the linear fractional transformations $\bar{x}$ and $\bar{y}$ respectively.

In fragment γ_3 vertex v_1 is fixed by $\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$ and its corresponding matrix will be $M_1 = XYXY^{-1}$ and $\det\left(M_1\right) = \det\left(XYXY^{-1}\right) = \det\left(XYXY^2\right) = \det\left(X\right)\,\det\left(Y\right)\,\det\left(X\right)\det\left(Y^2\right) = \det\left(X\right)\det\left(Y\right)\,\det\left(X\right)\left(\det\left(Y\right)\right)^2 = \Delta^2$. As $Y^{-1} = Y^2$ so M_1 can be written as $M_1 = XYXY^2$. On substituting the value of Y^2 in M_1 we get $M_1 = XYX\left(-Y - I\right) = -\left(XY\right)^2 - XYX$. Now by putting values of $\left(XY\right)^2$ and XYX from the above equations in $M_1 = -\left(XY\right)^2 - XYX$ we get $M_1 = -rXY + \Delta I - rX - \Delta I - \Delta Y = -rXY - rX - \Delta Y$. So the trace M_1 =trace$(-rXY)$ −trace(rX) −trace(ΔY) , that is, trace $\left(M_1\right) = -r^2 + \Delta$.

So the discriminant of the characteristic equation of M_1 will be $\left(-r^2 + \Delta\right)^2 - 4\Delta^2 = r^4 - 3\Delta^2 - 2r^2\Delta$. But $r^2 = \Delta\theta$. That is, the discriminant will be $\theta^2\Delta^2 - 3\Delta^2 - 2\theta\Delta^2$. Since Δ is a square if and only if θ is, we can eliminate Δ^2, as we are in the field F_q. So the discriminant of the characteristic equation of M_1 will be $\theta^2 - 2\theta - 3$.

Similarly, we can calculate the discriminants of the characteristic equations for the matrices corresponding to the elements $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}\bar{x}\bar{y}^{-1}$, $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}$ and $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}^{-1}$ as $\theta\left(\theta - 3\right)\left(\theta^2 - 3\theta + 4\right)$, $\left(\theta - 1\right)^2\left(\theta - 3\right)\left(\theta^3 - 5\theta^2 + 7\theta + 1\right)$, $\theta\left(\theta - 4\right)$ and $\left(\theta^4 - 6\theta^3 + 11\theta^2 - 6\theta - 3\right)$ respectively.

(i) The fragment γ_3 (Figure 7) will occur in $D\left(\theta, q\right)$ if $\left(\theta - 3\right)\left(\theta + 1\right)$ is a square in F_q.

(ii) The fragment γ_4 (Figure 8) will occur in $D\left(\theta, q\right)$ if $\theta\left(\theta - 3\right)\left(\theta^2 - 3\theta + 4\right)$ is a square in F_q.

(iii) The fragment γ_5 (Figure 9) will occur in $D\left(\theta, q\right)$ if $\left(\theta - 1\right)^2\left(\theta - 3\right)\left(\theta^3 - 5\theta^2 + 7\theta + 1\right)$ is a square in F_q.

(iv) The fragment γ_6 (Figure 10) will occur in $D\left(\theta, q\right)$ if $\theta\left(\theta - 4\right)$ is a square in F_q.

(v) The fragment γ_7 (Figure 11) will occur in $D\left(\theta, q\right)$ if $\left(\theta^4 - 6\theta^3 + 11\theta^2 - 6\theta - 3\right)$ is a square in F_q. □

In the following, we give a list of triplets $\bar{x}$, $\bar{y}$, $\bar{t}$ such that $\bar{x}^2 = \bar{y}^3 = \bar{t}^2 = \left(\bar{x}\bar{y}\right)^{13} = \left(\bar{x}\bar{t}\right)^2 = \left(\bar{y}\bar{t}\right)^2 = 1$ for all $q < 1300$. Here q is a prime such that $q \equiv \pm 1 \pmod{13}$ and $f\left(\theta\right)$ denotes the polynomial $\theta^6 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1$.

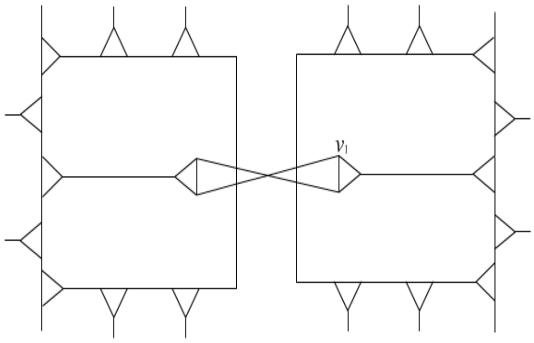

FIGURE 7. (γ_3)

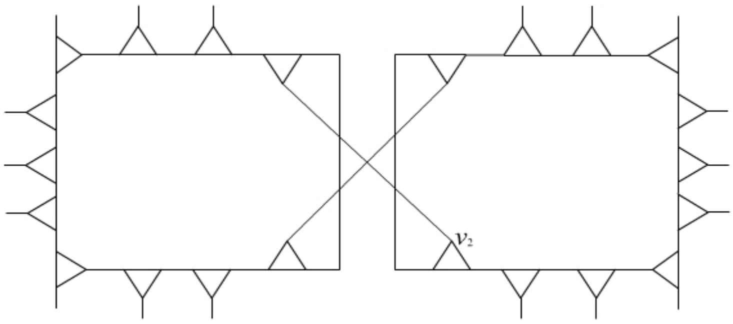

FIGURE 8. (γ_4)

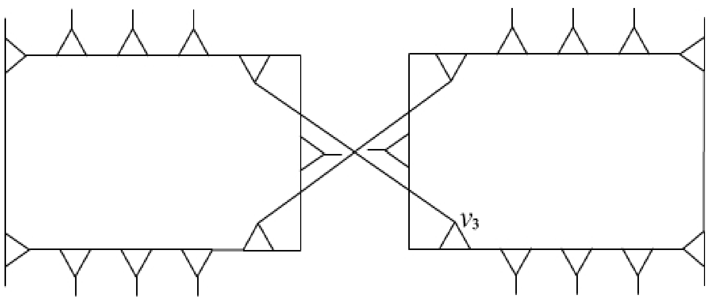

FIGURE 9. (γ_5)

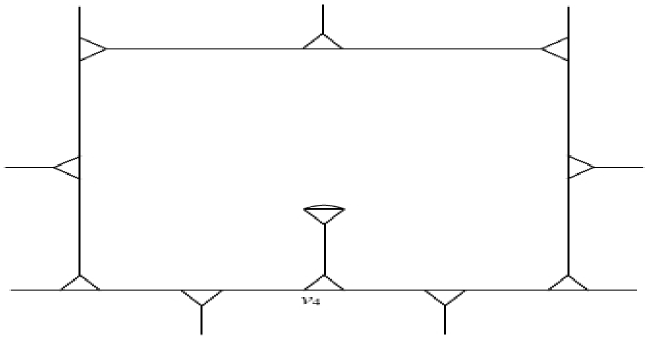

FIGURE 10. (γ_6)

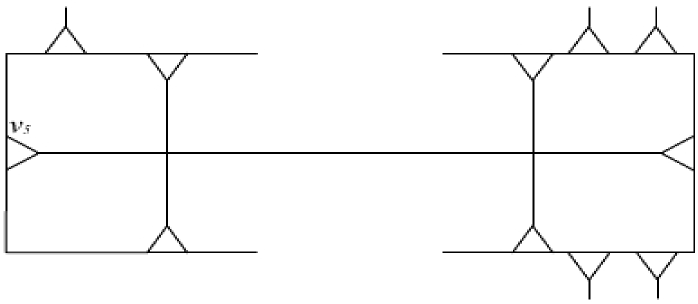

FIGURE 11. (γ_7)

q	Roots of $f(\theta)$	$\bar{x}$	$\bar{y}$	$\bar{t}$
13	4	$\frac{34z+3}{3z-34}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
53	10	$\frac{34z+3}{3z-34}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
	11	$\frac{31z-4}{2z-31}$	$\frac{4z-28}{14z-5}$	$\frac{2}{z}$
	13	$\frac{4z+6}{6z-4}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
	15	$\frac{46z+88}{44z-46}$	$\frac{z+10}{5z-2}$	$\frac{-2}{z}$
	28	$\frac{35z+29}{29z-35}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
	40	$\frac{36z+9}{9z-36}$	$\frac{23}{23z-1}$	$\frac{-1}{z}$
79	8	$\frac{60z+198}{66z-60}$	$\frac{111}{37z-1}$	$\frac{-3}{z}$
	13	$\frac{57z+21}{7z-57}$	$\frac{111}{37z-1}$	$\frac{-3}{z}$
	20	$\frac{47z+70}{70z-47}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$
	36	$\frac{20z+51}{51z-20}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$
	42	$\frac{6z+11}{11z-6}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$
	50	$\frac{67z+48}{48z-67}$	$\frac{z+32}{32z-2}$	$\frac{-1}{z}$

103	14	$\frac{87z+19}{19z-87}$	$\frac{z+10}{10z-2}$	$\frac{-1}{z}$
	41	$\frac{28z+126}{42z-28}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
	46	$\frac{22z+66}{66z-22}$	$\frac{z+10}{10z-2}$	$\frac{-1}{z}$
	58	$\frac{18z+246}{82z-18}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
	79	$\frac{7z+57}{19z-7}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
	82	$\frac{28z+183}{61z-28}$	$\frac{93}{31z-1}$	$\frac{-3}{z}$
131	15	$\frac{80z+256}{128z-80}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	38	$\frac{84z+208}{104z-84}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	45	$\frac{41z+105}{105z-41}$	$\frac{5z+10}{10z-6}$	$\frac{-1}{z}$
	64	$\frac{69z+68}{34z-69}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	117	$\frac{8z+132}{66z-8}$	$\frac{28}{14z-1}$	$\frac{-2}{z}$
	125	$\frac{96z+79}{79z-96}$	$\frac{5z+10}{10z-6}$	$\frac{-1}{z}$
157	3	$\frac{81z+8}{4z-81}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	68	$\frac{5z+290}{145z-5}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	117	$\frac{114z+172}{86z-114}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	126	$\frac{140z+288}{144z-140}$	$\frac{2z+92}{46z-3}$	$\frac{-2}{z}$
	144	$\frac{26z+39}{39z-26}$	$\frac{28}{28z-1}$	$\frac{-1}{z}$
	147	$\frac{154z+124}{124z-154}$	$\frac{28}{28z-1}$	$\frac{-1}{z}$
181	34	$\frac{110z+140}{70z-110}$	$\frac{2z+158}{79z-3}$	$\frac{-2}{z}$
	55	$\frac{125z+170}{170z-125}$	$\frac{19}{19z-1}$	$\frac{-1}{z}$
	94	$\frac{128z+238}{119z-128}$	$\frac{2z+158}{79z-3}$	$\frac{-2}{z}$
	114	$\frac{91z+166}{166z-91}$	$\frac{19}{19z-1}$	$\frac{-1}{z}$
	119	$\frac{128z+238}{119z-128}$	$\frac{2z+158}{79z-3}$	$\frac{-2}{z}$
	138	$\frac{119z+104}{104z-119}$	$\frac{19}{19z-1}$	$\frac{-1}{z}$
233	9	$\frac{167z+492}{164z-167}$	$\frac{z+267}{89z-2}$	$\frac{-3}{z}$
	49	$\frac{50z+166}{166z-50}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$
	91	$\frac{94z+564}{188z-94}$	$\frac{z+267}{89z-2}$	$\frac{-3}{z}$
	112	$\frac{164z+197}{197z-164}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$
	217	$\frac{30z+168}{168z-30}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$
	232	$\frac{7z+79}{79z-7}$	$\frac{89}{89z-1}$	$\frac{-1}{z}$

311	42	$\frac{194z+1441}{131z-194}$	$\frac{561}{51z-1}$	$\frac{-11}{z}$
	45	$\frac{249z+2178}{198z-249}$	$\frac{561}{51z-1}$	$\frac{-11}{z}$
	50	$\frac{42z+1320}{120z-42}$	$\frac{561}{51z-1}$	$\frac{-11}{z}$
	75	$\frac{109z+2354}{214z-109}$	$\frac{561}{51z-1}$	$\frac{-11}{z}$
	127	$\frac{301z+167}{167z-301}$	$\frac{5z+88}{88z-6}$	$\frac{-1}{z}$
	294	$\frac{301z+144}{144z-301}$	$\frac{5z+88}{88z-6}$	$\frac{-1}{z}$
313	26	$\frac{187z+1215}{243z-187}$	$\frac{2z+565}{113z-3}$	$\frac{-5}{z}$
	79	$\frac{91z+13}{13z-91}$	$\frac{25}{25z-1}$	$\frac{-1}{z}$
	87	$\frac{131z+475}{95z-131}$	$\frac{2z+565}{113z-3}$	$\frac{-5}{z}$
	200	$\frac{134z+86}{86z-134}$	$\frac{25}{25z-1}$	$\frac{-1}{z}$
	263	$\frac{276z+620}{124z-276}$	$\frac{2z+565}{113z-3}$	$\frac{-5}{z}$
	295	$\frac{165z+5}{z-165}$	$\frac{2z+565}{113z-3}$	$\frac{-5}{z}$
337	26	$\frac{258z+145}{29z-258}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$
	75	$\frac{308z+13}{13z-308}$	$\frac{148}{148z-1}$	$\frac{-1}{z}$
	181	$\frac{333z+40}{8z-333}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$
	227	$\frac{17z+261}{261z-17}$	$\frac{148}{148z-1}$	$\frac{-1}{z}$
	239	$\frac{233z+475}{95z-233}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$
	274	$\frac{259z+960}{192z-259}$	$\frac{5z+70}{14z-6}$	$\frac{-5}{z}$
389	178	$\frac{371z+381}{381z-371}$	$\frac{115}{115z-1}$	$\frac{-1}{z}$
	193	$\frac{54z+388}{194z-54}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	245	$\frac{159z+750}{375z-159}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	304	$\frac{41z+332}{166z-41}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	310	$\frac{103z+600}{300z-103}$	$\frac{z+108}{54z-2}$	$\frac{-2}{z}$
	337	$\frac{107z+328}{328z-107}$	$\frac{115}{115z-1}$	$\frac{-1}{z}$
443	13	$\frac{431z+228}{114z-431}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$
	75	$\frac{419z+75}{75z-419}$	$\frac{2z+128}{128z-3}$	$\frac{-1}{z}$
	121	$\frac{266z+225}{225z-266}$	$\frac{2z+128}{128z-3}$	$\frac{-1}{z}$
	289	$\frac{124z+870}{435z-124}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$
	414	$\frac{4z+710}{355z-4}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$
	428	$\frac{150z+782}{391z-150}$	$\frac{422}{211z-1}$	$\frac{-2}{z}$

467	23	$\frac{214z+419}{419z-214}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	83	$\frac{190z+77}{77z-190}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	221	$\frac{122z+153}{153z-122}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	317	$\frac{276z+280}{280z-276}$	$\frac{5z+48}{48z-6}$	$\frac{-1}{z}$
	327	$\frac{406z+106}{53z-406}$	$\frac{126}{63z-1}$	$\frac{-2}{z}$
	441	$\frac{332z+446}{223z-332}$	$\frac{126}{63z-1}$	$\frac{-2}{z}$
521	5	$\frac{447z+485}{485z-447}$	$\frac{235}{235z-1}$	$\frac{-1}{z}$
	9	$\frac{495z+642}{214z-495}$	$\frac{z+705}{235z-2}$	$\frac{-3}{z}$
	20	$\frac{33z+306}{102z-33}$	$\frac{z+705}{235z-2}$	$\frac{-3}{z}$
	49	$\frac{199z+1074}{358z-199}$	$\frac{z+705}{235z-2}$	$\frac{-3}{z}$
	125	$\frac{32z+1329}{443z-32}$	$\frac{z+705}{235z-2}$	$\frac{-3}{z}$
	324	$\frac{386z+518}{518z-386}$	$\frac{235}{235z-1}$	$\frac{-1}{z}$
547	66	$\frac{343z+99}{99z-343}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
	183	$\frac{41z+507}{507z-41}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
	209	$\frac{348z+774}{387z-348}$	$\frac{190}{95z-1}$	$\frac{-2}{z}$
	267	$\frac{490z+1086}{543z-490}$	$\frac{190}{95z-1}$	$\frac{-2}{z}$
	439	$\frac{262z+215}{215z-262}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
	488	$\frac{171z+106}{106z-171}$	$\frac{z+81}{81z-2}$	$\frac{-1}{z}$
571	66	$\frac{17z+515}{515z-17}$	$\frac{z+219}{219z-2}$	$\frac{-1}{z}$
	99	$\frac{464z+234}{117z-464}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
	273	$\frac{329z+524}{262z-329}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
	353	$\frac{23z+700}{350z-23}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
	436	$\frac{457z+43}{43z-457}$	$\frac{z+219}{219z-2}$	$\frac{-1}{z}$
	497	$\frac{458z+282}{141z-458}$	$\frac{418}{209z-1}$	$\frac{-2}{z}$
599	8	$\frac{343z+1456}{208z-343}$	$\frac{259}{37z-1}$	$\frac{-7}{z}$
	36	$\frac{525z+232}{232z-525}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$
	139	$\frac{246z+1001}{143z-246}$	$\frac{259}{37z-1}$	$\frac{-7}{z}$
	200	$\frac{53z+571}{571z-53}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$
	269	$\frac{565z+380}{380z-565}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$
	557	$\frac{518z+564}{564z-518}$	$\frac{2z+259}{259z-3}$	$\frac{-1}{z}$

677	126	$\frac{550z+546}{273z-550}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$
	134	$\frac{545z+1220}{610z-545}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$
	220	$\frac{25z+472}{472z-25}$	$\frac{26}{26z-1}$	$\frac{-1}{z}$
	482	$\frac{207z+402}{402z-207}$	$\frac{26}{26z-1}$	$\frac{-1}{z}$
	499	$\frac{174z+460}{230z-174}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$
	581	$\frac{100z+454}{227z-100}$	$\frac{z+632}{316z-2}$	$\frac{-2}{z}$
701	76	$\frac{198z+412}{206z-198}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
	132	$\frac{224z+102}{51z-224}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
	379	$\frac{473z+599}{599z-473}$	$\frac{135}{135z-1}$	$\frac{-1}{z}$
	431	$\frac{436z+982}{491z-436}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
	527	$\frac{336z+109}{109z-336}$	$\frac{135}{135z-1}$	$\frac{-1}{z}$
	569	$\frac{458z+550}{275z-458}$	$\frac{z+380}{190z-2}$	$\frac{-2}{z}$
727	15	$\frac{531z+29}{29z-531}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	169	$\frac{495z+492}{492z-495}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	263	$\frac{437z+493}{493z-437}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	510	$\frac{440z+593}{593z-440}$	$\frac{z+164}{164z-2}$	$\frac{-1}{z}$
	529	$\frac{575z+1644}{548z-575}$	$\frac{891}{297z-1}$	$\frac{-3}{z}$
	706	$\frac{566z+828}{276z-566}$	$\frac{891}{297z-1}$	$\frac{-3}{z}$
857	92	$\frac{678z+2151}{717z-678}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
	282	$\frac{19z+810}{810z-19}$	$\frac{207}{207z-1}$	$\frac{-1}{z}$
	387	$\frac{730z+1386}{462z-730}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
	413	$\frac{112z+654}{218z-112}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
	587	$\frac{147z+569}{569z-147}$	$\frac{207}{207z-1}$	$\frac{-1}{z}$
	821	$\frac{685z+588}{196z-685}$	$\frac{z+621}{207z-2}$	$\frac{-3}{z}$
859	20	$\frac{832z+680}{340z-832}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$
	249	$\frac{557z+1270}{635z-557}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$
	324	$\frac{142z+1172}{586z-142}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$
	604	$\frac{289z+64}{64z-289}$	$\frac{z+338}{338z-2}$	$\frac{-2}{z}$
	626	$\frac{83z+29}{29z-83}$	$\frac{z+338}{338z-2}$	$\frac{-1}{z}$
	765	$\frac{512z+1662}{831z-512}$	$\frac{296}{148z-1}$	$\frac{-2}{z}$

883	38	$\frac{240z+380}{190z-240}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
	116	$\frac{789z+723}{723z-789}$	$\frac{z+208}{208z-2}$	$\frac{-1}{z}$
	268	$\frac{24z+1586}{793z-24}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
	308	$\frac{29z+636}{318z-29}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
	413	$\frac{876z+673}{673z-876}$	$\frac{z+208}{208z-2}$	$\frac{-1}{z}$
	634	$\frac{205z+1028}{514z-205}$	$\frac{42}{21z-1}$	$\frac{-2}{z}$
911	53	$\frac{200z+1757}{251z-200}$	$\frac{2401}{343z-1}$	$\frac{-7}{z}$
	251	$\frac{476z+1547}{221z-476}$	$\frac{2401}{343z-1}$	$\frac{-7}{z}$
	405	$\frac{702z+415}{415z-702}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
	609	$\frac{6z+390}{390z-6}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
	647	$\frac{827z+267}{267z-827}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
	779	$\frac{496z+209}{209z-496}$	$\frac{2z+332}{332z-3}$	$\frac{-1}{z}$
937	12	$\frac{885z+818}{818z-885}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	35	$\frac{271z+139}{139z-271}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	100	$\frac{377z+360}{360z-377}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	152	$\frac{649z+3255}{651z-649}$	$\frac{2z+2240}{448z-3}$	$\frac{-5}{z}$
	234	$\frac{360z+377}{377z-360}$	$\frac{196}{196z-1}$	$\frac{-1}{z}$
	415	$\frac{180z+3810}{762z-180}$	$\frac{2z+2240}{448z-3}$	$\frac{-5}{z}$
1013	240	$\frac{360z+1692}{846z-360}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	305	$\frac{55z+822}{822z-55}$	$\frac{45}{45z-1}$	$\frac{-1}{z}$
	368	$\frac{141z+286}{143z-141}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	569	$\frac{483z+1368}{684z-483}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	639	$\frac{780z+656}{328z-780}$	$\frac{z+542}{271z-2}$	$\frac{-2}{z}$
	929	$\frac{113z+733}{733z-113}$	$\frac{45}{45z-1}$	$\frac{-1}{z}$
1039	395	$\frac{622z+2133}{711z-622}$	$\frac{1320}{440z-1}$	$\frac{-3}{z}$
	543	$\frac{33z+923}{923z-33}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$
	677	$\frac{291z+1254}{418z-291}$	$\frac{1320}{440z-1}$	$\frac{-3}{z}$
	722	$\frac{19z+646}{646z-19}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$
	852	$\frac{572z+10}{10z-572}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$
	978	$\frac{953z+857}{857z-953}$	$\frac{z+281}{281z-2}$	$\frac{-1}{z}$

1091	21	$\frac{863z+535}{535z-863}$	$\frac{6z+219}{219z-7}$	$\frac{-1}{z}$
	76	$\frac{431z+2022}{1011z-431}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
	143	$\frac{853z+1668}{834z-853}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
	243	$\frac{169z+1720}{860z-169}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
	258	$\frac{253z+277}{277z-253}$	$\frac{6z+219}{219z-7}$	$\frac{-1}{z}$
	361	$\frac{630z+1374}{687z-630}$	$\frac{1058}{529z-1}$	$\frac{-2}{z}$
1093	103	$\frac{936z+644}{644z-936}$	$\frac{530}{530z-1}$	$\frac{-1}{z}$
	364	$\frac{1072z+460}{460z-1072}$	$\frac{530}{530z-1}$	$\frac{-1}{z}$
	394	$\frac{827z+2162}{1081z-827}$	$\frac{6z+436}{1081z-827}$	$\frac{-2}{z}$
	644	$\frac{1030z+266}{133z-1030}$	$\frac{6z+436}{218z-7}$	$\frac{-2}{z}$
	808	$\frac{737z+1510}{755z-737}$	$\frac{6z+436}{218z-7}$	$\frac{-2}{z}$
	977	$\frac{360z+1048}{524z-360}$	$\frac{6z+436}{218z-7}$	$\frac{-2}{z}$
1117	12	$\frac{987z+852}{852z-987}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$
	100	$\frac{61z+989}{989z-61}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$
	107	$\frac{1023z+246}{123z-1023}$	$\frac{6z+420}{210z-7}$	$\frac{-2}{z}$
	386	$\frac{721z+1728}{864z-721}$	$\frac{6z+420}{210z-7}$	$\frac{-2}{z}$
	668	$\frac{87z+1057}{1057z-87}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$
	972	$\frac{449z+1064}{1064z-449}$	$\frac{214}{214z-1}$	$\frac{-1}{z}$
1171	84	$\frac{366z+1924}{962z-366}$	$\frac{278}{139z-1}$	$\frac{-2}{z}$
	754	$\frac{715z+636}{636z-715}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
	828	$\frac{425z+1372}{686z-425}$	$\frac{278}{139z-1}$	$\frac{-2}{z}$
	869	$\frac{688z+107}{107z-688}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
	1078	$\frac{749z+304}{304z-749}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
	1082	$\frac{421z+751}{751z-421}$	$\frac{z+330}{330z-2}$	$\frac{-1}{z}$
1223	157	$\frac{432z+2005}{401z-432}$	$\frac{2870}{574z-1}$	$\frac{-5}{z}$
	181	$\frac{57z+673}{673z-57}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$
	243	$\frac{36z+273}{273z-36}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$
	488	$\frac{1109z+5600}{1120z-1109}$	$\frac{2870}{574z-1}$	$\frac{-5}{z}$
	600	$\frac{957z+671}{671z-957}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$
	788	$\frac{845z+255}{255z-845}$	$\frac{5z+78}{78z-6}$	$\frac{-1}{z}$

1249	225	$\frac{1202z+600}{600z-1202}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$
	278	$\frac{155z+4795}{685z-155}$	$\frac{2z+4095}{585z-3}$	$\frac{-7}{z}$
	419	$\frac{1236z+783}{783z-1236}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$
	582	$\frac{302z+8190}{1170z-302}$	$\frac{2z+4095}{585z-3}$	$\frac{-7}{z}$
	1018	$\frac{220z+996}{996z-220}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$
	1236	$\frac{846z+608}{608z-846}$	$\frac{585}{585z-1}$	$\frac{-1}{z}$

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