Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 45 (2013) pp. 97- 111

Improvement Of Jensen'S Inequality For The Quasi Arithmetic Mean With Some Applications

M. Adil Khan
Department of Mathematics
University of Peshawar
Pakistan
Email: adilbandai@yahoo.com

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a, 10000
Zagreb, Croatia

Abdus Salam School of Mathematical Sciences
GC University, Lahore
Pakistan
Email: pecaric@mahazu.hazu.hr

Abstract. Improvement of Jensen's inequality for the Quasi arithmetic mean for convex and monotone convex functions is given as well as various applications for some other means are also given.

AMS (MOS) Subject Classification Codes: 26A51, 26A24, 26A48
Key Words: Jensen's inequality, Convex function, Monotone Convex function, Mean.

1. INTRODUCTION

The well known Jensen inequality for convex function is given by

Theorem 1. *If* (Ω, A, μ) *is a measure space with* $0 < \mu(\Omega) < \infty$ *and if* $f \in L^1(\mu)$ *is such that* $a \le f(t) \le b$ *for all* $t \in \Omega, -\infty \le a < b \le \infty$, *then*

$$\phi\left(\frac{1}{\mu(\Omega)}\int_{\Omega} f(t)d\mu(t)\right) \le \frac{1}{\mu(\Omega)}\int_{\Omega} \phi(f(t))d\mu(t) \tag{1.1}$$

is valid for any convex function $\phi : [a, b] \to \mathbb{R}$. *In the case when* ϕ *is strictly convex on* $[a, b]$ *we have equality in* (1) *iff* f *is constant* μ*-almost every where on* Ω.

The following improvement of (1) is valid ([1, 10]).

Theorem 2. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$. *Let* $f \in L^1(\mu)$ *be such that* $a \le f(t) \le b$ *for all* $t \in \Omega, -\infty \le a < b \le \infty$.

(i) If $\phi : [a,b] \to \mathbb{R}$ is convex, then

$$\frac{1}{\mu(\Omega)} \int_\Omega \phi(f(t))d\mu(t) - \phi\left(\overline{f}\right) \geq \left| \frac{1}{\mu(\Omega)} \int_\Omega |\phi(f(t)) - \phi(\overline{f})| d\mu(t) - \frac{|\phi'_+(\overline{f})|}{\mu(\Omega)} \int_\Omega |f(t) - \overline{f}| d\mu(t) \right| \quad (1.2)$$

(ii) If $\phi : [a,b] \to \mathbb{R}$ is monotone convex and $\Omega' = \{t \in \Omega : f(t) \geq \overline{f}\}$, then

$$\begin{aligned} \frac{1}{\mu(\Omega)} \int_\Omega \phi(f(t))\, d\mu(t) \, - \, \phi\left(\overline{f}\right) &\geq \left| \frac{1}{\mu(\Omega)} \int_\Omega sgn\left(f(t) - \overline{f}\right) \left[\phi(f(t)) - \phi'_+(\overline{f}) f(t)\right] d\mu(t) \right. \\ &\quad \left. + \left[\phi(\overline{f}) - \overline{f}\phi'_+(\overline{f})\right] \left[1 - \frac{2\mu(\Omega')}{\mu(\Omega)}\right] \right|, \quad (1.3) \end{aligned}$$

where $\phi'_+(x)$ represents the right hand derivative of ϕ and

$$\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f(t) d\mu(t).$$

If the function $\phi(t)$ is concave (monotone concave), then the left-hand side of (1.2) and (1.3) should be $\phi\left(\overline{f}\right) - \frac{1}{\mu(\Omega)} \int_\Omega \phi(f(t)) d\mu(t)$.

Remark 1. Theorem $2(i)$ has been proved in [10] and Theorem $2(ii)$ has been proved in [1]

Discrete inequalities are simple consequences of Theorem 2.

Theorem 3. *Let $\phi : [a,b] \to \mathbb{R}$ be a convex function, $x_1, x_2, .., x_n \in [a,b]$ and $p_1, p_2.., p_n$ positive real numbers with $P_n = \sum_{i=1}^n p_i$, then*
(i)

$$\frac{1}{P_n} \sum_{i=1}^n p_i \phi(x_i) - \phi\left(\overline{x}\right) \; \geq \; \left| \frac{1}{P_n} \sum_{i=1}^n p_i \left|\phi(x_i) - \phi(\overline{x})\right| - \left|\phi'_+(\overline{x})\right| \frac{1}{P_n} \sum_{i=1}^n p_i \left|x_i - \overline{x}\right| \right| \quad (1.4)$$

(ii) If ϕ is monotone convex and $I = \{i \in I_n = \{1, 2, ..., n\} : x_i \geq \overline{x}\}$, then

$$\begin{aligned} \frac{1}{P_n} \sum_{i=1}^n p_i \phi(x_i) \, - \, \phi\left(\overline{x}\right) &\geq \left| \frac{1}{P_n} \sum_{i=1}^n p_i sgn\left(x_i - \overline{x}\right) \left[\phi(x_i) - x_i \, \phi'_+(\overline{x})\right] \right. \\ &\quad \left. + \left[\phi(\overline{x}) - \overline{x}\phi'_+(\overline{x})\right] \left[1 - \frac{2P_I}{P_n}\right] \right|, \quad (1.5) \end{aligned}$$

where, $\overline{x} = \frac{1}{P_n} \sum_{i=1}^n p_i \, x_i$ and $P_I = \sum_{i \in I} p_i$

If the function $\phi(t)$ is concave (monotone concave), then the left-hand side of (1.4) and (1.5) should be $\phi\left(\overline{x}\right) - \frac{1}{P_n} \sum_{i=1}^n p_i \phi(x_i)$.

In this paper we will give further extension and application of Theorem 2.

2. Main Results

We will give Jensen's inequalities for the quasi arithmetic mean.

Let (Ω, A, μ) be a probability space and $f : \Omega \to \mathbb{R}$ be a continuous, g strictly monotone function defined on the image of f, then the quasi-arithmetic mean $M_g(f;\mu)$ is defined as follows:

$$M_g(f;\mu) = g^{-1}\left(\int_\Omega (g \circ f)(t)\, d\mu(t)\right).$$

Theorem 4. *Let $g : \Omega \to \mathbb{R}$ be a continuous, h be real valued strictly monotone differentiable function defined on the image of g and f a real valued differentiable function defined on the image of g.*
(i) If $k(t) = \left(f \circ h^{-1}\right)(t)$ is convex function, then

$$\int_\Omega f(g(t))\, d\mu(t) - f\left(M_h(g;\mu)\right) \geq \left|\int_\Omega \left|f(g(t)) - f\left(M_h(g;\mu)\right)\right| d\mu(t) - \left|\left(\frac{f'}{h'}\right) \circ \left(M_h(g;\mu)\right)\right| \times \int_\Omega \left|h(g(t)) - h\left(M_h(g;\mu)\right)\right| d\mu(t)\right|. \quad (2.1)$$

(ii) If $k(t) = (f \circ h^{-1})(t)$ is monotone convex and
$\Omega' = \{t \in \Omega : h(g(t)) \geq h(M_h(g;\mu))\}$, then

$$\int_\Omega f(g(t))\, d\mu(t) - f\left(M_h(g;\mu)\right) \geq \left|\int_\Omega sgn\left(h(g(t)) - h(M_h(g;\mu))\right)\left[f(g(t)) - h \circ g(t)\left(\frac{f'}{h'}\right) \circ \left(M_h(g;\mu)\right)\right] d\mu(t) + \left[f\left(M_h(g;\mu)\right) - h\left(M_h(g;\mu)\right)\left(\frac{f'}{h'}\right) \circ \left(M_h(g;\mu)\right)\right]\left(1 - 2\mu(\Omega')\right)\right|. \quad (2.2)$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (2.1) and (2.2) should be $f\left(M_h(g;\mu)\right) - \int_\Omega f(g(t))\, d\mu(t)$.

Proof. The proofs of (2.1) and (2.2) follow by setting $\phi = f \circ h^{-1}$ and $f = h \circ g$ in (2) and in (3) respectively. ☐

Remark 2. For the functions f, g, h defined as in Theorem 4, the function $k(t)$ is convex(concave) if any of the following cases occur:
(i) f is strictly increasing, h strictly increasing and $h \circ f^{-1}$ concave(convex).
(ii) f is strictly increasing, h strictly decreasing and $h \circ f^{-1}$ convex(concave).
(iii) f is strictly decreasing, h strictly increasing and $h \circ f^{-1}$ convex(concave).
(iv) f is strictly decreasing, h strictly decreasing and $h \circ f^{-1}$ concave(convex).

Remark 3. If f^{-1} exists then the left hand side of (2.1) and (2.2) becomes $f\left(M_f(g;\mu)\right) - f\left(M_h(g;\mu)\right)$

3. Applications for Means

3.1. **Jensen's inequalities for Power mean.** Let Ω be a set equipped with probability measure μ. For $r \in \mathbb{R}$, the integral power mean of positive continuous function g is defined as follows:

$$M_r(g;\mu) = \begin{cases} \left[\int_\Omega (g(t))^r \, d\mu(t)\right]^{\frac{1}{r}}, & \text{for } r \neq 0; \\ \exp\left(\int_\Omega \ln(g(t))\right) d\mu(t), & \text{for } r = 0. \end{cases}$$

It is well-known that for $r > s$ we have $M_s(g;\mu) \leq M_r(g;\mu)$.

Theorem 5. *Let $g : \Omega \to \mathbb{R}^+$ be a continuous function and f a real valued differentiable function defined on the image of g. Let*

$$k(t) = \begin{cases} f(t^{\frac{1}{r}}), & r \neq 0, \\ f(e^t), & r = 0. \end{cases}$$

(i) If $k(t)$ is convex function, then

$$\int_\Omega f(g(t))\, d\mu(t) - f(M_r(g;\mu)) \geq \begin{cases} \Big| \int_\Omega |f(g(t)) - f(M_r(g;\mu))|\, d\mu(t) \\ \quad - \left|\frac{f'(M_r(g;\mu))}{r\, M_r^{r-1}(g;\mu)}\right| \int_\Omega |(g(t))^r - M_r^r(g;\mu)|\, d\mu(t) \Big|, \\ \qquad \text{for } r \neq 0, \\ \Big| \int_\Omega |f(g(t)) - f(M_r(g;\mu))|\, d\mu(t) \\ \quad - |M_r(g;\mu)\, f'(M_r(g;\mu))| \int_\Omega \left|\ln \frac{g(t)}{M_r(g;\mu)}\right| d\mu(t) \Big| \\ \qquad \text{for } r = 0. \end{cases} \tag{3.1}$$

(ii) If $k(t)$ is monotone convex function and $\Omega' = \{t \in \Omega : (g(t))^r \geq M_r^r(g;\mu)$ for $r \neq 0$ and $\ln\left(\frac{g(t)}{M_r(g;\mu)}\right) \geq 0$ for $r = 0\}$, then

$$\int_\Omega f(g(t))\, d\mu(t) - f(M_r(g;\mu)) \geq \begin{cases} \Big| \int_\Omega sgn\left((g(t))^r - M_r^r(g;\mu)\right) \Big[f(g(t)) \\ \quad - \frac{(g(t))^r f'(M_r(g;\mu))}{r\, M_r^{r-1}(g;\mu)}\Big]\, d\mu(t) \\ \quad + \left[f(M_r(g;\mu)) - \frac{M_r(g;\mu) f'(M_r(g;\mu))}{r}\right][1 - 2\mu(\Omega')] \Big|, \\ \qquad for\ r \neq 0 \end{cases}$$

$$\int_\Omega f(g(t))\, d\mu(t) - f(M_r(g;\mu)) \geq \begin{cases} \Big| \int_\Omega sgn\left(\ln\left(\frac{g(t)}{M_r(g;\mu)}\right)\right) [f(g(t)) \\ \quad - M_r(g;\mu) \ln g(t) f'(M_r(g;\mu))]\, d\mu(t) + [f(M_r(g;\mu)) \\ \quad - \ln(M_r(g;\mu)) M_r(g;\mu) f'(M_r(g;\mu))]\,[1 - 2\mu(\Omega')] \Big| \\ \qquad \text{for } r = 0. \end{cases} \tag{3.2}$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.1) and (3.2) should be $f(M_r(g;\mu)) - \int_\Omega f(g(t))\, d\mu(t)$.

Proof. The proofs of (3.1) and (3.2) follow by setting

$$h(t) = \begin{cases} t^r, r \neq 0, \\ \ln t, r = 0. \end{cases}$$

in (2.1) and in (2.2) respectively. □

Definition 1. A function $\phi : [a, b] \to \mathbb{R}^+$ is said to be log-convex if for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$, we have

$$\phi(\lambda x + (1 - \lambda)y) \leq \phi^{\lambda}(x)\phi^{1-\lambda}(y). \tag{3.3}$$

If reverse inequality holds in (3.3), then ϕ is said to be log-concave.

Remark 4. For the functions f, g defined as in the Theorem 5, the function $k(t)$ is convex (concave) if any of the following cases occur:
(i) f is strictly increasing, $r > 0$ and $\left(f^{-1}\right)^r$ concave(convex).
(ii) f is strictly increasing, $r < 0$ and $\left(f^{-1}\right)^r$ convex(concave).
(iii) f is strictly decreasing, $r > 0$ and $\left(f^{-1}\right)^r$ convex(concave).
(iv) f is strictly decreasing, $r < 0$ and $\left(f^{-1}\right)^r$ concave(convex).
(v) f is strictly increasing, $r = 0$ and f^{-1} $\log$-concave($\log$-convex).
(vi) f is strictly decreasing, $r = 0$ and f^{-1} $\log$-convex($\log$-concave).

3.2. **Jensen's inequalities for Tobey mean.** In [6] H. J. Seiffert has consider the identric mean $I(a, b)$ of two points $a\ and\ b$ $(a, b > 0)$ that is

$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & , \quad a \neq b, \\ a & , \quad a = b. \end{cases} \tag{3.4}$$

and he proved:

Theorem 6. *If f is a strictly increasing continuous function on $[a, b]$, $0 < a < b$, having a logarithmically convex inverse function, then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq f(I(a,b)), \tag{3.5}$$

while the inequality in (3.5) is reversed if f is strictly decreasing.

A related result is given by H. Alzer ([2]), that is

$$f(L(a,b)) \leq \frac{1}{b-a}\int_a^b f(t)dt, \qquad a, b > 0 \tag{3.6}$$

if $f \in C([a, b])$ with $a, b > 0$, is strictly increasing, $1/f^{-1}$ is convex and $L(a, b)$ is the logarithmic mean defined by

$$L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a} & , \quad a \neq b, \\ a & , \quad a = b, \end{cases} \tag{3.7}$$

while the inequality in (3.6) is reversed if f is strictly decreasing.

The identric and the logarithmic means of two positive real numbers a, b are rather special cases of the generalized logarithmic mean defined by

$$L_r(a,b) = \begin{cases} \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{\frac{1}{r}} & , \quad r \neq -1, 0\ ,\ a \neq b; \\ \frac{b-a}{\ln b - \ln a}\ , & \quad r = -1,\ a \neq b; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & , \quad r = 0,\ a \neq b; \\ a\ , & \quad a = b. \end{cases} \tag{3.8}$$

In [4] authors gave the analogous result for this generalized logarithmic mean ([4],Theorem 2.1.):

Theorem 7. *Let a, b be positive numbers and $f : [a,b] \to \mathbb{R}$ be a function. If $r \neq 0$ and $k(t) = f\left(t^{\frac{1}{r}}\right)$ is convex function, or $r = 0$ and $k(t) = f\left(e^t\right)$ is convex, then*

$$f\left(L_r(a,b)\right) \leq \frac{1}{b-a}\int_a^b f(t)dt. \tag{3.9}$$

If $r \neq 0$ and $k(t) = f\left(t^{\frac{1}{r}}\right)$ is concave, or $r = 0$ and $k(t) = f\left(e^t\right)$ is concave, then (3.9) is reversed.

This result is the generalization of Seiffert's and Alzer's result, what can be easily seen by a short calculation.
Now we give the improvements of (3.9).

Theorem 8. *Let a, b be positive real numbers and $f : [a,b] \to \mathbb{R}$ be differentiable function. Let*

$$k(t) = \begin{cases} f(t^{\frac{1}{r}}), & r \neq 0, \\ f(e^t), & r = 0. \end{cases}$$

(i) If $k(t)$ is convex function, then

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f(L_r(a,b)) \geq$$

$$\begin{cases} \Bigg| \frac{1}{b-a}\int_a^b \left|f(t) - f\left(L_r(a,b)\right)\right| dt \\ - \left|\frac{f'(L_r(a,b))}{(b-a)r\, L_r^{r-1}(a,b)}\right| \int_a^b |t^r - L_r^r(a,b)| dt \Bigg|, \\ \qquad\qquad\qquad \text{for } r \neq 0 \\ \Bigg| \frac{1}{b-a}\int_a^b \left|f(t) - f\left(L_r(a,b)\right)\right| dt \\ - \left|\frac{L_r(a,b)\, f'(L_r(a,b))}{b-a}\right| \int_a^b \left|\ln \frac{t}{L_r(a,b)}\right| dt \Bigg|, \\ \qquad\qquad\qquad \text{for } r = 0. \end{cases} \tag{3.10}$$

(ii) If $k(t)$ is monotone convex function, then

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f(L_r(a,b)) \geq$$

$$\begin{cases} \Bigg| \frac{1}{b-a}\int_a^b sgn\left(t^r - L_r^r(a,b)\right)[f(t) \\ - \frac{t^r f'(L_r(a,b))}{r\, L_r^{r-1}(a,b)}\Big] dt \\ + \left[f\left(L_r(a,b)\right) - \frac{L_r(a,b) f'(L_r(a,b)}{r}\right][1 - \frac{2(b - L_r(a,b)))}{b-a}]\Bigg|, \\ \qquad\qquad\qquad for\ r \neq 0 \\ \Bigg| \frac{1}{b-a}\int_a^b sgn\left(\ln\left(\frac{t}{L_r(a,b)}\right)\right)[f(t) \\ - L_r(a,b)\ln(t) f'(L_r(a,b))]\, dt + [f(L_r(a,b)) \\ - \ln(L_r(a,b)) L_r(a,b) f'(L_r(a,b))]\left[1 - \frac{2(b - L_r(a,b))}{b-a}\right]\Bigg|, \\ \qquad\qquad\qquad for\ r = 0 \end{cases} \tag{3.11}$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.10) and (3.11) should be $f(L_r(a,b)) - \frac{1}{b-a}\int_a^b f(t)\,dt$.

Proof. The proofs of (3.10) and (3.11) follow by setting , $\Omega = [a,b]$, $d\mu(t) = dt$, $\phi(t) = f\left(t^{\frac{1}{r}}\right)$, $f(t) = t^r$ for $r \neq 0$ and $\phi(t) = f(e^t)$, $f(t) = \ln t$ for $r = 0$ in (1.2) and in (1.3) respectively. □

Let us note that multidimentional generalization of (3.5), (3.6) and (3.9) were considered in [4] and [8]. In this paper we shall give related improvements of these results.
Let E_{n-1} represents $(n-1)$-dimensional Euclidean simplex given by

$$E_{n-1} := \left\{(u_1, u_2, ...u_{n-1}) :\ u_i \geq 0,\ 1 \leq i \leq n-1\ and\ \sum_{i=1}^{n-1} u_i \leq 1\right\}$$

with

$$u_n = 1 - \sum_{i=1}^{n-1} u_i.$$

Let $\mathbf{u} = (u_1, ..., u_n)$ and μ be a probability measure on E_{n-1}, then the power mean of order p ($p \in \mathbb{R}$) of the positive n-tuple

$$\mathbf{x} = (x_1, ...x_n) \in \mathbb{R}^n_+,$$

with the weights $\mathbf{u} = (u_1, ...u_n)$, is defined as

$$\overline{M}_p(\mathbf{x}, \mathbf{u}) = \begin{cases} \left(\sum_{i=1}^n u_i\, x_i^p\right)^{\frac{1}{p}}, & \text{for } p \neq 0; \\ \prod_{i=1}^n x_i^{u_i}, & \text{for } p = 0. \end{cases}$$

For $p = 1$ we denote $\overline{M}_1(\mathbf{x}, \mathbf{u}) = \mathbf{x} \cdot \mathbf{u}$.

The Tobey mean, $L_{p,r}(\mathbf{x};\, \mu)$, is defined as

$$L_{p,r}(\mathbf{x};\, \mu) = M_r(\overline{M}_p(\mathbf{x}, \mathbf{u}); \mu),$$

where $M_r(.;\, \mu)$ is the integral power mean in which Ω is $(n-1)$-dimensional Euclidean simplex E_{n-1}. The following results are valid.

Theorem 9. *Let $[a,b]$ be positive interval containing all x_i $(i = 1, 2, .., n)$ and $f : [a,b] \to \mathbb{R}$ be differentiable function. Let*

$$k(t) = \begin{cases} f(t^{\frac{1}{r}}), & r \neq 0, \\ f(e^t), & r = 0. \end{cases}$$

(i) If $k(t)$ is convex function, then

$$\int_{E_{n-1}} f(\overline{M}_p(\mathbf{x}, \mathbf{u}))\, d\mu(\mathbf{u}) - f(L_{p,r}(\mathbf{x};\, \mu)) \geq$$
$$\begin{cases} \Big|\int_{E_{n-1}} |f(\overline{M}_p(\mathbf{x}, \mathbf{u})) - f\left(L_{p,r}(\mathbf{x};\, \mu)\right)| d\mu(\mathbf{u}) \\ - \left|\frac{f'(L_{p,r}(\mathbf{x};\, \mu))}{r\, L_{p,r}^{r-1}(\mathbf{x};\, \mu)}\right| \int_{E_{n-1}} |\overline{M}_p^r(\mathbf{x}, \mathbf{u}) - L_{p,r}^r(\mathbf{x};\, \mu)| d\mu(\mathbf{u})\Big|, \\ \qquad\qquad for\ r \neq 0, \\ \Big|\int_{E_{n-1}} |f(\overline{M}_p(\mathbf{x}, \mathbf{u})) - f\left(L_{p,r}(\mathbf{x};\, \mu)\right)| d\mu(\mathbf{u}) \\ - |L_{p,r}(\mathbf{x};\, \mu)\, f'\left(L_{p,r}(\mathbf{x};\, \mu)\right)| \\ \times \int_{E_{n-1}} |\ln \frac{\overline{M}_p(\mathbf{x},\mathbf{u})}{L_{p,r}(\mathbf{x};\, \mu)}| d\mu(\mathbf{u})\Big|, \\ \qquad\qquad for\ r = 0, \end{cases} \tag{3.12}$$

(ii) If $k(t)$ is monotone convex function and $E'_{n-1} = \{(u_1, u_2, ...u_{n-1}) \in E_{n-1} : \overline{M}_p^r(\mathbf{x}, \mathbf{u}) \geq L_{p,r}^r(\mathbf{x};\, \mu)$ *for* $r \neq 0$ *and* $\ln\left(\frac{\overline{M}_p(\mathbf{x},\mathbf{u})}{L_{p,r}(\mathbf{x};\,\mu)}\right) \geq 0$ *for* $r = 0\}$*, then*

$$\int_{E_{n-1}} f(\overline{M}_p(\mathbf{x},\mathbf{u}))\, d\mu(\mathbf{u}) - f(L_{p,r}(\mathbf{x};\, \mu)) \geq$$

$$\begin{cases} \Bigg| \int_{E_{n-1}} sgn\left(\overline{M}_p^r(\mathbf{x},\mathbf{u}) - L_{p,r}^r(\mathbf{x};\,\mu)\right)\left[f\left(\overline{M}_p(\mathbf{x},\mathbf{u})\right) - \frac{\overline{M}_p^r(\mathbf{x},\mathbf{u}) f'(L_{p,r}(\mathbf{x};\,\mu))}{rL_{p,r}^{r-1}(\mathbf{x};\,\mu)}\right] d\mu(\mathbf{u}) \\ \quad + \left[f\left(L_{p,r}(\mathbf{x};\,\mu)\right) - \frac{L_{p,r}(\mathbf{x};\,\mu) f'(L_{p,r}(\mathbf{x};\,\mu))}{r}\right]\left[1 - 2\mu(E'_{n-1})\right]\Bigg|, \\ \qquad\qquad for\ r \neq 0, \\ \Bigg| \int_{E_{n-1}} sgn\left(\ln\left(\frac{\overline{M}_p(\mathbf{x},\mathbf{u})}{L_{p,r}(\mathbf{x};\,\mu)}\right)\right)\left[f\left(\overline{M}_p(\mathbf{x},\mathbf{u})\right) - L_{p,r}(\mathbf{x};\,\mu)\ln \overline{M}_p(\mathbf{x},\mathbf{u}) f'(L_{p,r}(\mathbf{x};\,\mu))\right] d\mu(\mathbf{u}) \\ \quad + \left[f(L_{p,r}(\mathbf{x};\,\mu)) - \ln(L_{p,r}(\mathbf{x};\,\mu))L_{p,r}(\mathbf{x};\,\mu) f'(L_{p,r}(\mathbf{x};\,\mu))\right]\left[1 - 2\mu(E'_{n-1})\right]\Bigg|, \\ \qquad\qquad for\ r = 0. \end{cases} \tag{3.13}$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.12) *and* (3.13) *should be* $f(L_{p,r}(\mathbf{x};\,\mu)) - \int_{E_{n-1}} f(\overline{M}_p(\mathbf{x},\mathbf{u}))\, d\mu(\mathbf{u})$.

Proof. The proofs of (3.12) and (3.13) follow by setting $\Omega = E_{n-1}, \Omega' = E'_{n-1}$ and $g(\mathbf{u}) = \overline{M}_p(\mathbf{x},\mathbf{u})$ in (3.1) and in (3.2) respectively. □

Remark 5. For strictly monotone function $f : [a,b] \to \mathbb{R}$, the function $k(t)$ is convex(concave) if any of the cases $(i) - (vi)$ from the Remark 4 occurs.

3.3. **Jensen's inequalities for Stolarsky-Tobey mean:** For $\mathbf{x} = (x_1, ...x_n) \in \mathbb{R}_+^n$, and $p, q \in \mathbb{R}$ the Stolarsky-Tobey mean $\varepsilon_{p,q}(\mathbf{x};\mu)$ [7] is defined by

$$\varepsilon_{p,q}(\mathbf{x};\mu) = \begin{cases} \left(\int_{E_{n-1}} \left(\sum_{i=1}^n u_i x_i^p\right)^{\frac{q-p}{p}} d\mu(\mathbf{u})\right)^{\frac{1}{q-p}}, & \text{for } p(q-p) \neq 0; \\ \exp\left(\int_{E_{n-1}} \ln\left(\sum_{i=1}^n u_i\, x_i^p\right)^{\frac{1}{p}} d\mu(\mathbf{u})\right), & \text{for } p = q \neq 0; \\ \left(\int_{E_{n-1}} \left(\prod_{i=1}^n x_i^{u_i}\right)^q d\mu(\mathbf{u})\right)^{\frac{1}{q}}, & \text{for } p = 0;\ q \neq 0; \\ \exp\left(\int_{E_{n-1}} \ln\left(\prod_{i=1}^n x_i^{u_i}\right) d\mu(\mathbf{u})\right), & \text{for } p = q = 0. \end{cases}$$

or, alternatively, by

$$\varepsilon_{p,q}(\mathbf{x};\mu) = L_{p,q-p}(\mathbf{x};\mu) = M_{q-p}(\overline{M}_p(\mathbf{x},\mathbf{u});\mu),$$

where $L_{p,r}(\mathbf{x};\mu)$ is the Tobey mean.

Theorem 10. *Let $[a, b]$ be positive interval containing all x_i $(i = 1, 2, .., n)$ and $f : [a, b] \to \mathbb{R}$ be differentiable function and $p, q \in \mathbb{R}$. Let*

$$k(t) = \begin{cases} f(t^{\frac{1}{q-p}}), & q - p \neq 0, \\ f(e^t), & q - p = 0. \end{cases}$$

(i) If $k(t)$ is convex function, then

$$\int_{E_{n-1}} f(\overline{M}_p(\mathbf{x}, \mathbf{u}))\, d\mu(\mathbf{u}) - f(\varepsilon_{p,q}(\mathbf{x}; \mu)) \geq$$
$$\begin{cases} \Big| \int_{E_{n-1}} |f(\overline{M}_p(\mathbf{x}, \mathbf{u})) - f\left(\varepsilon_{p,q}(\mathbf{x}; \mu)\right)| d\mu(\mathbf{u}) \\ - \left| \frac{f'(\varepsilon_{p,q}(\mathbf{x};\mu))}{(q-p)\, \varepsilon_{p,q}^{q-p-1}(\mathbf{x};\mu)} \right| \int_{E_{n-1}} |\overline{M}_p^{q-p}(\mathbf{x}, \mathbf{u}) - \varepsilon_{p,q}^{q-p}(\mathbf{x}; \mu)| d\mu(\mathbf{u}), \Big| \\ \qquad for\ q - p \neq 0, \\ \Big| \int_{E_{n-1}} |f(\overline{M}_p(\mathbf{x}, \mathbf{u})) - f\left(\varepsilon_{p,q}(\mathbf{x}; \mu)\right)| d\mu(\mathbf{u}) \\ - |\varepsilon_{p,q}(\mathbf{x}; \mu)\, f'\left(\varepsilon_{p,q}(\mathbf{x}; \mu)\right)| \int_{E_{n-1}} |\ln \frac{\overline{M}_p(\mathbf{x},\mathbf{u})}{\varepsilon_{p,q}(\mathbf{x};\mu)}| d\mu(\mathbf{u}) \Big|, \\ \qquad for\ q - p = 0, \end{cases} \tag{3.14}$$

(ii) If $k(t)$ is monotone convex function and $E'_{n-1} = \{(u_1, u_2, ...u_{n-1}) \in E_{n-1} : \overline{M}_p^{q-p}(\mathbf{x}, \mathbf{u}) \geq \varepsilon_{p,q}^{q-p}(\mathbf{x}; \mu)$ for $q - p \neq 0$ and $\ln\left(\frac{\overline{M}_p(\mathbf{x},\mathbf{u})}{\varepsilon_{p,q}(\mathbf{x};\mu)}\right) \geq 0$ for $q - p = 0\}$, then

$$\int_{E_{n-1}} f(\overline{M}_p(\mathbf{x}, \mathbf{u}))\, d\mu(\mathbf{u}) - f(\varepsilon_{p,q}(\mathbf{x}; \mu)) \geq$$
$$\begin{cases} \Big| \int_{E_{n-1}} sgn\left(\overline{M}_p^{q-p}(\mathbf{x}, \mathbf{u}) - \varepsilon_{p,q}^{q-p}(\mathbf{x}; \mu)\right) \Big[f\left(\overline{M}_p(\mathbf{x}, \mathbf{u})\right) \\ - \frac{\overline{M}_p^{q-p}(\mathbf{x},\mathbf{u}) f'(\varepsilon_{p,q}(\mathbf{x};\mu))}{(q-p)\varepsilon_{p,q}^{q-p}(\mathbf{x};\mu)} \Big] d\mu(\mathbf{u}) \\ + \left[f\left(\varepsilon_{p,q}(\mathbf{x}; \mu)\right) - \frac{\varepsilon_{p,q}(\mathbf{x};\mu) f'(\varepsilon_{p,q}(\mathbf{x};\mu))}{r} \right] [1 - 2\mu(E'_{n-1})] \Big|, \\ \qquad for\ q - p \neq 0 \\ \Big| \int_{E_{n-1}} sgn\left(\ln\left(\frac{\overline{M}_p(\mathbf{x},\mathbf{u})}{\varepsilon_{p,q}(\mathbf{x};\mu)}\right)\right) \Big[f\left(\overline{M}_p(\mathbf{x}, \mathbf{u})\right) \\ - \varepsilon_{p,q}(\mathbf{x}; \mu) \ln \overline{M}_p(\mathbf{x}, \mathbf{u}) f'(\varepsilon_{p,q}(\mathbf{x}; \mu)) \Big] d\mu(\mathbf{u}) \\ + [f(\varepsilon_{p,q}(\mathbf{x}; \mu)) \\ - \ln(\varepsilon_{p,q}(\mathbf{x}; \mu))\varepsilon_{p,q}(\mathbf{x}; \mu) f'(\varepsilon_{p,q}(\mathbf{x}; \mu))] [1 - 2\mu(E'_{n-1})] \Big|. \\ \qquad for\ q - p = 0. \end{cases} \tag{3.15}$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.14) and (3.15) should be $f(\varepsilon_{p,q}(\mathbf{x}; \mu)) - \int_{E_{n-1}} f(\overline{M}_p(\mathbf{x}, \mathbf{u}))\, d\mu(\mathbf{u})$.

Proof. The proofs of (3.14) and (3.15) follow by setting $r = q - p$ in (3.12) and in (3.13) respectively. □

Remark 6. For strictly monotone function $f : [a, b] \to \mathbb{R}$ the function $k(t)$ is convex(concave) if any of the following cases occur:
(i) f is strictly increasing, $q - p > 0$ and $\left(f^{-1}\right)^{q-p}$ concave(convex).
(ii) f is strictly increasing, $q - p < 0$ and $\left(f^{-1}\right)^{q-p}$ convex(concave).
(iii) f is strictly decreasing, $q - p > 0$ and $\left(f^{-1}\right)^{q-p}$ convex(concave).
(iv) f is strictly decreasing, $q - p < 0$ and $\left(f^{-1}\right)^{q-p}$ concave(convex).

(v) f is strictly increasing, $q-p=0$ and f^{-1} log-concave(log-convex).
(vi) f is strictly decreasing, $q-p=0$ and f^{-1} log-convex(log-concave).

Note that in the following Theorems $\mathbf{x}\cdot\mathbf{u}=\sum_{i=1}^{n} u_i\, x_i$
For $L_r(\mathbf{x};\,\mu)=\varepsilon_{1,r+1}(\mathbf{x};\mu)$ it follows:

Theorem 11. *Let $[a,b]$ be positive interval containing all x_i $(i=1,2,..,n)$ and $f:[a,b]\to\mathbb{R}$ be differentiable function. Let*

$$k(t)=\begin{cases} f(t^{\frac{1}{r}}), & r\neq 0,\\ f(e^t), & r=0.\end{cases}$$

(i) If $k(t)$ is convex function, then

$$\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u})\,d\mu(\mathbf{u})-f(L_r(\mathbf{x};\,\mu))\geq$$
$$\begin{cases} \Big|\int_{E_{n-1}}|f(\mathbf{x}\cdot\mathbf{u})-f\left(L_r(\mathbf{x};\,\mu)\right)|d\mu(\mathbf{u}) \\ -\left|\frac{f'(L_r(\mathbf{x};\,\mu))}{r\,L_r^{r-1}(\mathbf{x};\,\mu)}\right|\int_{E_{n-1}}|(\mathbf{x}\cdot\mathbf{u})^r-L_r^r(\mathbf{x};\,\mu)|d\mu(\mathbf{u})\Big|, \\ \qquad\qquad for\ r\neq 0, \\ \Big|\int_{E_{n-1}}|f((\mathbf{x}\cdot\mathbf{u}))-f\left(L_r(\mathbf{x};\,\mu)\right)|d\mu(\mathbf{u}) \\ -|L_r(\mathbf{x};\,\mu)\,f'\left(L_r(\mathbf{x};\,\mu)\right)|\int_{E_{n-1}}|\ln\frac{(\mathbf{x}\cdot\mathbf{u})}{L_r(\mathbf{x};\,\mu)}|d\mu(\mathbf{u})\Big|, \\ \qquad\qquad for\ r=0. \end{cases} \tag{3.16}$$

(ii) If $k(t)$ is monotone convex function and $E'_{n-1}=\{(u_1,u_2,...u_{n-1})\in E_{n-1}:(\mathbf{x}\cdot\mathbf{u})^{\mathbf{r}}\geq L_r^r(\mathbf{x};\,\mu)$ for $r\neq 0$ and $\ln\left(\frac{(\mathbf{x}\cdot\mathbf{u})}{L_r(\mathbf{x};\,\mu)}\right)\geq 0$ for r=0$\}$, then

$$\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u})\,d\mu(\mathbf{u})-f(L_r(\mathbf{x};\,\mu))\geq$$
$$\begin{cases} \Big|\int_{E_{n-1}} sgn\left((\mathbf{x}\cdot\mathbf{u})^{\mathbf{r}}-L_r^r(\mathbf{x};\,\mu)\right)[f\left(\mathbf{x}\cdot\mathbf{u}\right) \\ -\frac{(\mathbf{x}\cdot\mathbf{u})^{\mathbf{r}}f'(L_r(\mathbf{x};\,\mu))}{rL_r^{r-1}(\mathbf{x};\,\mu)}\Big]\,d\mu(\mathbf{u}) \\ +\left[f\left(L_r(\mathbf{x};\,\mu)\right)-\frac{L_r(\mathbf{x};\,\mu)f'(L_r(\mathbf{x};\,\mu))}{r}\right][1-2\mu(E'_{n-1})]\Big|, \\ \qquad\qquad for\ r\neq 0 \\ \Big|\int_{E_{n-1}} sgn\left(\ln\left(\frac{(\mathbf{x}\cdot\mathbf{u})}{L_r(\mathbf{x};\,\mu)}\right)\right)[f\left(\mathbf{x}\cdot\mathbf{u}\right) \\ -L_r(\mathbf{x};\,\mu)\ln\left(\mathbf{x}\cdot\mathbf{u}\right)f'(L_r(\mathbf{x};\,\mu))]\,d\mu(\mathbf{u}) \\ +[f(L_r(\mathbf{x};\,\mu)) \\ -\ln(L_r(\mathbf{x};\,\mu))L_r(\mathbf{x};\,\mu)f'(L_r(\mathbf{x};\,\mu))]\,[1-2\mu(E'_{n-1})]\Big|. \\ \qquad\qquad for\ r=0. \end{cases} \tag{3.17}$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.16) and (3.17) should be $f(L_r(\mathbf{x};\,\mu))-\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u})\,d\mu(\mathbf{u})$.

Proof. The proofs of (3.16) and (3.17) follow by setting $p=1, q=r+1$ in (3.14) and in (3.15) respectively. □

Remark 7. For strictly monotone function $f:[a,b]\to\mathbb{R}$, the function $k(t)$ is convex(concave) if any of the cases $(i)-(vi)$ from the Remark 4 occurs.

3.4. **Jensen's inequalities for functional Stolarsky means.** For strictly monotone continuous functions f and g, the functional Stolarsky means are defined by [5]

$$m_{f,g}(\mathbf{x};\mu) = f^{-1}\left(\int_{E_{n-1}} (f\circ g^{-1})(\mathbf{u}\cdot\mathbf{g})\,d\mu(\mathbf{u})\right),$$

where

$$\mathbf{g} = (g(x_1), ..., g(x_n))$$

and μ is a probability measure on E_{n-1}.

Theorem 12. *Let $[a,b]$ be positive interval containing all x_i $(i = 1,2,..,n)$ and $f : [a,b] \to \mathbb{R}$ be differentiable function and $h : [a,b] \to \mathbb{R}$ be strictly monotone differentiable function.*
(i) If $k(t) = \left(f\circ h^{-1}\right)(t)$ is convex function, then

$$\int_{E_{n-1}} f(g^{-1}(\mathbf{u}\cdot\mathbf{g})\,d\mu(\mathbf{u}) - f\left(m_{h,g}(\mathbf{x};\mu)\right) \geq \left|\int_{E_{n-1}} \left|f(g^{-1}(\mathbf{u}\cdot\mathbf{g})) - f\left(m_{h,g}(\mathbf{x};\mu)\right)\right| d\mu(\mathbf{u}) - \left|\left(\frac{f'}{h'}\right)\circ\left(m_{h,g}(\mathbf{x};\mu)\right)\right| \times \int_{E_{n-1}} \left|h(g^{-1}(\mathbf{u}\cdot\mathbf{g})) - h\left(m_{h,g}(\mathbf{x};\mu)\right)\right| d\mu(\mathbf{u})\right| \quad (3.18)$$

(ii) If $k(t) = (f\circ h^{-1})(t)$ is monotone convex and $E'_{n-1} = \{(u_1,u_2,...u_{n-1}) \in E_{n-1} : h(g^{-1}(\mathbf{u}\cdot\mathbf{g})) \geq h(m_{h,g}(\mathbf{x};\mu))\}$, then

$$\int_{E_{n-1}} f(g^{-1}(\mathbf{u}\cdot\mathbf{g})\,d\mu(\mathbf{u}) - f\left(m_{h,g}(\mathbf{x};\mu)\right) \geq \left|\int_{E_{n-1}} sgn\left(h(g^{-1}(\mathbf{u}\cdot\mathbf{g}) - h(m_{h,g}(\mathbf{x};\mu))\right)\left[f(g^{-1}(\mathbf{u}\cdot\mathbf{g})) - h\circ g^{-1}(\mathbf{u}\cdot\mathbf{g})\left(\frac{f'}{h'}\right)\circ\left(m_{h,g}(\mathbf{x};\mu)\right)\right] d\mu(\mathbf{u}) + \left[f\left(m_{h,g}(\mathbf{x};\mu)\right) - h\left(m_{h,g}(\mathbf{x};\mu)\right)\left(\frac{f'}{h'}\right)\circ\left(m_{h,g}(\mathbf{x};\mu)\right)\right]\left(1 - 2\mu(E'_{n-1})\right)\right|. \quad (3.19)$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.18) and (3.19) should be $f\left(m_{h,g}(\mathbf{x};\mu)\right) - \int_{E_{n-1}} f(g^{-1}(\mathbf{u}\cdot\mathbf{g}))\,d\mu(\mathbf{u})$.

Proof. The proof is analogous to that of Theorem 4; we just consider the function $g^{-1}(\mathbf{u}\cdot\mathbf{g})$ instead of $g(\mathbf{u})$. □

Remark 8. For the functions f, g, h defined as in the Theorem 12, the function $k(t)$ is convex(concave) if any of the cases $(i) - (vi)$ from the Remark 1.5 occurs.

3.5. **Jensen's inequalities for Complete symmetric polynomial means.** The rth complete symmetric polynomial mean (or, simply, the complete symmetric mean) of the positive real n-tuple $\mathbf{x}$ is defined by [7]

$$Q_n^{[r]}(\mathbf{x}) = \left(q_n^{[r]}(\mathbf{x})\right)^{\frac{1}{r}} = \left(\frac{c_n^{[r]}(\mathbf{x})}{\binom{n+r-1}{r}}\right)^{\frac{1}{r}},$$

where

$$c_n^{[0]}(\mathbf{x}) = 1 \quad and \quad c_n^{[r]}(\mathbf{x}) = \sum\left(\prod_{i=1}^{n} x_i^{i_j}\right)$$

and the sum is taken over all

$$\binom{n+r-1}{r}$$

non-negative integral n-tuples $(i_1, i_2, ..., i_n)$ with

$$\sum_{j=1}^{n} i_j = r \quad (r \neq 0).$$

The complete symmetric polynomial mean can also be written in an integral form as follows

$$Q_n^{[r]}(\mathbf{x}) = \left(\int_{E_{n-1}} \left(\sum_{i=1}^{n} x_i\, u_i \right)^r d\mu(\mathbf{u}) \right)^{\frac{1}{r}},$$

where μ represents a probability measure such that

$$d\mu(\mathbf{u}) = (n-1)!\, du_1....du_{n-1}.$$

It may be noted that, this is a special case of the integral power mean $M_r(\nu;\mu)$, where

$$\nu(\mathbf{u}) = \sum_{i=1}^{n} x_i\, u_i,$$

μ is a probability measure such that

$$d\mu(\mathbf{u}) = (n-1)!\, du_1....du_{n-1},$$

and Ω is the before defined $(n-1)$-dimensional simplex E_{n-1}.

Theorem 13. *Let $[a,b]$ be positive interval containing all $x_i (i = 1, 2, ..., n)$ and let $f : [a,b] \to \mathbb{R}$ be differentiable function. For $r \neq 0$ define the function $k(t) = f\left(t^{\frac{1}{r}}\right)$.*
(i) If $k(t)$ is convex, then

$$\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u}) - f\left(Q_n^{[r]}(\mathbf{x})\right) \geq$$

$$\left\{ \begin{array}{l} \left| \int_{E_{n-1}} \left| f(\mathbf{x}\cdot\mathbf{u}) - f\left(Q_n^{[r]}(\mathbf{x})\right) \right| d\mu(\mathbf{u}) \right. \\ \left. - \left| \frac{f'\left(Q_n^{[r]}(\mathbf{x})\right)}{r\ Q_n^{[r]}(\mathbf{x})^{\ r-1}} \right| \int_{E_{n-1}} \left| f(\mathbf{x}\cdot\mathbf{u})^r - \left(Q_n^{[r]}(\mathbf{x})\right)^r \right| d\mu(\mathbf{u}) \right|. \end{array} \right. \tag{3.20}$$

(ii) If $k(t)$ is monotone convex and $E'_{n-1} = \{(u_1, u_2, ...u_{n-1}) \in E_{n-1} : (\mathbf{x}\cdot\mathbf{u})^r \geq \left(Q_n^{[r]}(\mathbf{x})\right)^r\}$, then

$$\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u}) d\mu(\mathbf{u}) - f\left(Q_n^{[r]}(\mathbf{x})\right) \geq$$

$$\left\{ \begin{array}{l} \left| \int_{E_{n-1}} sgn\left((\mathbf{x}\cdot\mathbf{u})^r - \left(Q_n^{[r]}(\mathbf{x})\right)^r \right) \right. \\ \times \left[f(\mathbf{x}\cdot\mathbf{u}) - \frac{(\mathbf{x}\cdot\mathbf{u})^r f'\left(Q_n^{[r]}(\mathbf{x})\right)}{r\ Q_n^{[r]}(\mathbf{x})^{\ r-1}} \right] d\mu(\mathbf{u}) \\ \left. + \left[f\left(Q_n^{[r]}(\mathbf{x})\right) - \frac{\left(Q_n^{[r]}(\mathbf{x})\right) f'\left(Q_n^{[r]}(\mathbf{x})\right)}{r} \right] \left[1 - 2\mu(E'_{n-1})\right] \right|. \end{array} \right. \tag{3.21}$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.20) *and* (3.21) *should be* $f\left(Q_n^{[r]}(\mathbf{x})\right) - \int_{E_{n-1}} f\left(\sum_{i=1}^{n} x_i u_i\right) d\mu(\mathbf{u})$.

Proof. The proofs of (3.20) and (3.21) follow by setting $p = 1$ in (3.12) and in (3.13) respectively. □

Remark 9. For strictly monotone function $f : [a, b] \to \mathbb{R}$, the function $k(t)$ is convex(concave) if any of the cases $(i)-$(vi) from the Remark 1.5 occurs.

3.6. **Jensen's inequalities for Whiteley means.** Let **x** be a positive real n-tuple, $s \in \mathbb{R}(s \neq 0)$ and $r \in \mathbb{N}$. Then the sth function of degree r is defined by the following generating function [3].

$$\sum_{r=0}^{\infty} t_n^{[r,s]}(\mathbf{x})\, t^r = \begin{cases} \prod_{i=1}^{n}(1 + x_i t)^s, & s > 0; \\ \prod_{i=1}^{n}(1 - x_i t)^s, & s < 0. \end{cases}$$

The Whiteley mean is now defined by

$$\mathcal{W}_n^{[r,s]}(\mathbf{x}) = \left(w_n^{[r,s]}(\mathbf{x})\right)^{\frac{1}{r}} = \begin{cases} \left(\frac{t_n^{[r,s]}(\mathbf{x})}{\binom{ns}{r}}\right)^{\frac{1}{r}}, & s > 0; \\ \left(\frac{t_n^{[r,s]}(\mathbf{x})}{(-1)^r\binom{ns}{r}}\right)^{\frac{1}{r}}, & s < 0. \end{cases}$$

For $s < 0$, the Whiteley mean can be further generalized if we slightly change the definition of $t_n^{[r,s]}(\mathbf{x})$ and define $h_n^{[r,\sigma]}(\mathbf{x})$ as follows

$$\sum_{r=0}^{\infty} h_n^{[r,\sigma]}(\mathbf{x})\, t^r = \prod_{i=1}^{n} \frac{1}{(1 - x_i t)^{\sigma_i}},$$

where $\sigma = (\sigma_1, ..., \sigma_n);\ \sigma \in \mathbb{R}_+$, $i = 1, ..n$.
The following generalization of the Whiteley mean for $s < 0$ is defined by [9]

$$\mathcal{H}_n^{[r,\sigma]}(\mathbf{x}) = \left(\frac{h_n^{[r,\sigma]}(\mathbf{x})}{\binom{\sum_{i=1}^{n}\sigma_i + r - 1}{r}}\right)^{\frac{1}{r}}$$

If we denote by μ a measure on the simplex

$$E_{n-1} = \{(u_1, ..., u_{n-1}) : u_i \geq 0, i = 1, ..., n-1, \sum_{i=1}^{n-1} u_i \leq 1\}$$

such that

$$d\mu(\mathbf{u}) = \frac{\Gamma\left(\sum_{i=1}^{n}\sigma_i\right)}{\prod_{i=1}^{n}\Gamma(\sigma_i)} \prod_{i=1}^{n} u_i^{\sigma_i - 1} du_1 ... du_{n-1},$$

where $u_n = 1 - \sum_{i=1}^{n-1}$, then we have that μ is a probability measure and we can also write the mean $\mathcal{H}_n^{[r,\sigma]}(x)$ in integral form as follows

$$\mathcal{H}_n^{[r,\sigma]}(\mathbf{x}) = \left(\int_{E_{n-1}} \left(\sum_{i=1}^{n} x_i\, u_i\right)^r d\mu(\mathbf{u})\right)^{\frac{1}{r}}.$$

Theorem 14. *Let $[a, b]$ be positive interval containing all $x_i (i = 1, 2, ..., n)$ and let $f : [a, b] \to \mathbb{R}$ be differentiable function. For $r \neq 0$ define the function $k(t) = f\left(t^{\frac{1}{r}}\right)$.*

(i) If $k(t)$ is convex, then

$$\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u})d\mu(\mathbf{u}) - f\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)$$
$$\geq \left| \int_{E_{n-1}} \left| f(\mathbf{x}\cdot\mathbf{u}) - f\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)\right| d\mu(\mathbf{u}) - \left|\frac{f'\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)}{r\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})}\right| \int_{E_{n-1}} \left|(\mathbf{x}\cdot\mathbf{u})^r - \left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r\right| d\mu(\mathbf{u})\right|. \quad (3.22)$$

(ii) If $k(t)$ is monotone convex and $E'_{n-1} = \{(u_1, u_2, ...u_{n-1}) \in E_{n-1} : (\mathbf{x}\cdot\mathbf{u})^r \geq \left(\mathcal{H}_n^{[r,\sigma]}\right)^r\}$, then

$$\int_{E_{n-1}} f(\mathbf{x}\cdot\mathbf{u})d\mu(\mathbf{u}) - f\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)$$
$$\geq \left| \int_{E_{n-1}} sgn\left((\mathbf{x}\cdot\mathbf{u})^r - \left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r\right)\left[f(\mathbf{x}\cdot\mathbf{u}) - \frac{(\mathbf{x}\cdot\mathbf{u})^r f'\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)}{r\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)^{r-1}}\right] d\mu(\mathbf{u}) + \left[f\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right) - \frac{\mathcal{H}_n^{[r,\sigma]}(\mathbf{x}) f'\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)}{r}\right][1 - 2\mu(E'_{n-1})]\right|. \quad (3.23)$$

If the function $k(t)$ is concave (monotone concave), then the left-hand side of (3.22) and (3.23) should be $f\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right) - \int_{E_{n-1}} f\left(\sum_{i=1}^{n} x_i\, u_i\right) d\mu(\mathbf{u})$.

Proof. The proofs of (3.22) and (3.23) follow by setting $p = 1$ in (3.12) and in (3.13) respectively. □

Remark 10. For strictly monotone function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, the function $k(t)$ is convex(concave) if any of the cases $(i)-$(vi) from the Remark 3.5 occurs.

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