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On the Semilocal Convergence of Werner's Method for Solving Equations Using Recurrent Functions

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Abstract. We prove semilocal convergence of Werner's method for approximating locally unique solution of nonlinear equation in a Banach space setting. Using our new idea of recurrent functions, we provide estimates on the distances involved and information on the location of the solution.A numerical example shows that our results can apply to solve equations but the Kantorovich's sufficient convergence condition is unapplicable [7].

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.
A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear

or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.
We revisit Werner's method [9], [10]:

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1}\ F(x_n), \quad A_n = F'(\frac{x_n + y_n}{2}) \\ y_{n+1} &= x_{n+1} - A_n^{-1}\ F(x_{n+1}), \quad (n \geq 0), \quad (x_0, y_0 \in \mathcal{D}). \end{aligned} \tag{1.2}$$

The local convergence of Werner's method (1.2) was given in [9], [10] under Lipschitz conditions on the first and second Fréchet–derivatives given in non–affine invariant form (see (2.52) and (2.53)). The order of convergence of Werner's method (1.2) is $1 + \sqrt{2}$. The derivation of this method and its importance has well been explained in [9], [10] (see also [3]). The two–step method uses one inverse and two function evaluations. Note that if $x_0 = y_0$, then (1.2) becomes Newton's method [1]–[11].
We provide a semilocal convergence analysis using our new idea of recurrent functions. Our Lipschitz hypotheses are provided in affine invariant form. As far as we know the semilocal analysis of Werner's method has not been studied in this setting. We are mostly interested in finding weak sufficient convergence conditions, so as to extend the applicability of the method.
Our new approach can also be used on other one–step or two–step iterative methods [1], [3], [4]–[11].
The semilocal convergence is examined in Section 2 and a numerical example is given in Section 3.

2. Semilocal convergence analysis of Werner's method

It is convenient for us to define some auxiliary functions appearing in connection to majorizing sequences for Werner's method (1.2).
Let $\ell_0 > 0$, $\ell > 0$, $\alpha \geq 0$, $\eta > 0$, $\overline{\eta} \geq \eta$ and $\beta = 1 + \alpha$ be given constants. It is convenient for us to define function $\overline{f}_1$ on $[0, +\infty)$ by

$$\overline{f}_1(t) = \ell\ t^\beta + 4\ \ell_0\ t - 2. \tag{2.1}$$

We have:

$$\overline{f}_1(0) = -2 < 0. \tag{2.2}$$

There exists sufficiently large $u > 0$, such that:

$$\overline{f}_1(t) > 0, \quad t > u. \tag{2.3}$$

It follows from (2.2), (2.3) and the intermediate value theorem that there exists $v \in (0, u)$, such that

$$\overline{f}_1(v) = 0. \tag{2.4}$$

The number v is the unique positive zero in $(0, +\infty)$ of function $\overline{f}_1$, since

$$\overline{f}_1'(t) = \ell\ \beta\ t^\alpha + 4\ \ell_0 > 0 \quad (t \geq 0). \tag{2.5}$$

That is function $\overline{f}_1$ is increasing and as such it crosses the positive axis only once.

Moreover, define function g on $[0, +\infty)$ by

$$g(t) = 2\ \ell_0\ t^3 + 2\ \ell_0\ t^2 + \ell\ \eta^\alpha\ t - \ell\ \eta^\alpha. \tag{2.6}$$

We have as above:

$$g(0) = -\ell\ \eta^\alpha < 0 \tag{2.7}$$

and

$$g(t) > 0 \qquad (t > \zeta) \tag{2.8}$$

for sufficiently large $\zeta > 0$.
Hence, as above there exists $\delta_+ \in (0, \zeta)$, such that:

$$g(\delta_+) = 0. \tag{2.9}$$

The number δ_+ is the unique positive zero of function g on $(0, +\infty)$, since

$$g'(t) = 6\ \ell_0\ t^2 + 4\ \ell_0\ t + \ell\ \eta^\alpha > 0 \quad (t \geq 0). \tag{2.10}$$

Set

$$\delta_0 = \frac{\ell\ \eta^\beta}{1 - \ell_0\ (\eta + \overline{\eta})}, \qquad \ell_0\ (\eta + \overline{\eta}) \neq 1, \tag{2.11}$$

$$v_\infty = 1 - 2\ \ell_0\ \eta \tag{2.12}$$

and

$$\delta_1 = \max\{\frac{\delta_0}{2},\ \delta_+\}. \tag{2.13}$$

We can show the following result on majorizing sequences for Werner's method (1.2):

Lemma 1. *Let $\ell_0 > 0$, $\ell > 0$, $\alpha \geq 0$, $\eta > 0$, $\overline{\eta} \geq \eta$ and $\beta = 1 + \alpha$ be given constants. Assume:*

$$\ell_0\ (\eta + \overline{\eta}) < 1,\ \ \eta \leq v \tag{2.14}$$

and

$$\delta_1 \leq v_\infty, \tag{2.15}$$

where, v, δ_1, δ_+, v_∞ were defined by (2.4), (2.13), (2.9) and (2.12), respectively. Choose:

$$\delta \in [\delta_1,\ v_\infty]. \tag{2.16}$$

Then, sequence $\{t_n\}$ ($n \geq 0$), generated by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\ell\ (t_{n+1} - t_n)^{1+\beta}}{2\ (1 - \ell_0\ (t_{n+1} + s_{n+1}))}, \tag{2.17}$$

with,

$$s_0 = 0, \quad s_1 = \overline{\eta}, \quad s_{n+2} = t_{n+2} + \frac{\ell\ (t_{n+2} - t_{n+1})^{1+\beta}}{2\ (1 - \ell_0\ (t_{n+1} + s_{n+1}))}, \tag{2.18}$$

is non–decreasing, bounded above by:

$$t^{\star\star} = \frac{2\ \eta}{2 - \delta} \tag{2.19}$$

and converges to its unique least upper bound $t^\star$ with

$$t^\star \in [0, t^{\star\star}]. \tag{2.20}$$

Moreover the following estimates hold for all $n \geq 0$:

$$t_n \leq s_n, \tag{2.21}$$

$$0 < t_{n+2} - t_{n+1} \leq \frac{\delta}{2}\ (t_{n+1} - t_n) \leq \left(\frac{\delta}{2}\right)^{n+1}\ \eta \tag{2.22}$$

and

$$0 < s_{n+2} - t_{n+2} \leq \frac{\delta}{2}\ (s_{n+1} - t_{n+1}) \leq \left(\frac{\delta}{2}\right)^{n+2}\ \eta. \tag{2.23}$$

Proof. We shall show using induction on m:

$$\begin{aligned} 0 \leq t_{m+2} - t_{m+1} &= \frac{\ell\ (t_{m+1} - t_m)^{\beta}}{2\ (1 - \ell_0\ (t_{m+1} + s_{m+1}))}\ (t_{m+1} - t_m) \\ &\leq \frac{\delta}{2}\ (t_{m+1} - t_m), \end{aligned} \tag{2.24}$$

$$\begin{aligned} 0 \leq s_{m+2} - t_{m+2} &= \frac{\ell\ (t_{m+2} - t_{m+1})^{\beta}}{2\ (1 - \ell_0\ (t_{m+1} + s_{m+1}))}\ (t_{m+2} - t_{m+1}) \\ &\leq \frac{\delta}{2}\ (t_{m+2} - t_{m+1}) \end{aligned} \tag{2.25}$$

and

$$\ell_0\ (t_{m+1} + s_{m+1}) < 1. \tag{2.26}$$

Estimates (2.24)–(2.26) for $m = 0$ will hold if:

$$\frac{\ell\ (t_1 - t_0)^{\beta}}{1 - \ell_0\ (t_1 + s_1)} = \frac{\ell\ \eta^{\beta}}{1 - \ell_0\ (\eta + \overline{\eta})} = \delta_0 \leq \delta, \tag{2.27}$$

$$\frac{\ell\ (t_2 - t_1)^{\beta}}{1 - \ell_0\ (t_1 + s_1)} \leq \frac{\ell\ \left(\dfrac{\delta}{2}\ \eta\right)^{\beta}}{1 - \ell_0\ (\eta + \overline{\eta})} = \overline{\delta}_0 \leq \delta_0 \leq \delta \tag{2.28}$$

and

$$\ell_0\ (t_1 + s_1) = \ell_0\ (\eta + \overline{\eta}) < 1, \tag{2.29}$$

respectively, which are true by (2.16) and (2.14). Let us assume (2.21)–(2.26) hold for all $n \leq m + 1$.
Then, we get from (2.24) and (2.25):

$$t_{m+2} \leq \frac{1 - \left(\dfrac{\delta}{2}\right)^{m+2}}{1 - \dfrac{\delta}{2}}\ \eta < \frac{2\ \eta}{2 - \eta} = t^{\star\star} \tag{2.30}$$

and

$$s_{m+2} \leq t_{m+2} + \left(\frac{\delta}{2}\right)^{m+2}\ \eta \leq \left\{ \frac{1 - \left(\dfrac{\delta}{2}\right)^{m+2}}{1 - \dfrac{\delta}{2}} + \left(\frac{\delta}{2}\right)^{m+2} \right\}\ \eta. \tag{2.31}$$

We shall only show (2.24), since (2.25), will then follows (as (2.28) follows from (2.27)). Using the induction hypotheses, (2.24) certainly holds if:

$$\ell\ (t_{m+1} - t_m)^{\beta} + \ell_0\ \delta\ (t_{m+1} + s_{m+1}) - \delta \leq 0$$

or,

$$\ell\ \left\{ \left(\frac{\delta}{2}\right)^{m}\ \eta \right\}^{\beta} + \ell_0\ \delta\ \left\{ \frac{1 - \left(\dfrac{\delta}{2}\right)^{m+1}}{1 - \dfrac{\delta}{2}} + \frac{1 - \left(\dfrac{\delta}{2}\right)^{m+1}}{1 - \dfrac{\delta}{2}} + \left(\frac{\delta}{2}\right)^{m+1} \right\}\ \eta - \delta \leq 0,$$

or, since $\beta \geq 1$

$$\ell \left(\frac{\delta}{2}\right)^m \eta^\beta + \ell_0\ \delta \left\{ 2\ \frac{1 - \left(\frac{\delta}{2}\right)^{m+1}}{1 - \frac{\delta}{2}} + \left(\frac{\delta}{2}\right)^{m+1} \right\} \eta - \delta \leq 0. \tag{2.32}$$

We are motivated from (2.32) to define functions f_m $(m \geq 1)$ on $[0, +\infty)$, for $v = \frac{\delta}{2}$ and show instead of (2.32):

$$f_m(v) \ = \ \ell\ v^{m-1}\ \eta^\beta + 2\ \ell_0\ (2\ (1 + v + \cdots + v^m) + v^{m+1})\ \eta - 2 \leq 0. \tag{2.33}$$

We need a relationship between two consecutive functions f_m:

$$\begin{aligned} f_{m+1}(v) &= \ell\ v^m\ \eta^\beta + 2\ \ell_0\ (2\ (1 + v + \cdots + v^{m+1}) + v^{m+2})\ \eta - 2 \\ &= \ell\ v^m\ \eta^\beta + \ell\ v^{m-1}\ \eta^\beta - \ell\ v^{m-1}\ \eta^\beta + \\ &\quad 2\ \ell_0 (2\ (1 + v + \cdots + v^m) + v^{m+1} + v^{m+1} + v^{m+2})\ \eta - 2 \\ &= f_m(v) + \ell\ v^m\ \eta^\beta - \ell\ v^{m-1}\ \eta^\beta + 2\ \ell_0\ (v^{m+1} + v^{m+2})\ \eta \\ &= f_m(v) + g(v)\ v^{m-1}\ \eta, \end{aligned} \tag{2.34}$$

where, function g is given by (2.6).
We have by (2.33):

$$f_1(0) = \ell\ \eta^\beta + 4\ \ell_0\ \eta - 2 < 0, \tag{2.35}$$

$$f_m(0) = 4\ \ell_0\ \eta - 2 < 0 \quad (m > 1) \tag{2.36}$$

and for sufficiently large $v > 0$:

$$f_m(v) > 0. \tag{2.37}$$

It follows from (2.35)–(2.37), and the intermediate value theorem that there exists $v_m > 0$, such that $f_m(v_m) = 0$. Moreover, each v_m is the unique positive zero of f_m, since $f'_m(v) > 0$ for $v \in [0, +\infty)$.
We shall show

$$f_m(v) \leq 0 \quad \text{for all} \quad v \in [0, v_m] \quad (m \geq 1). \tag{2.38}$$

If there exists $m \geq 0$, such that $v_{m+1} \geq \frac{\delta}{2}$, then, using (2.6) and (2.34), we get:

$$f_{m+1}(v_{m+1}) = f_m(v_{m+1}) + g(v_{m+1})\ v_{m+1}^{m-1}\ \eta$$

or

$$f_m(v_{m+1}) \leq 0,$$

since $f_{m+1}(v_{m+1}) = 0$ and $g(v_{m+1})\ v_{m+1}^{m-1}\ \eta \geq 0$, which imply $v_{m+1} \leq v_m$.
We can certainly choose the last of the v_m's denoted by v_∞ (obtained from (2.32) by letting $m \longrightarrow \infty$ and given in (2.12)), to be v_{m+1}.
It then follows sequence $\{v_m\}$ is non–increasing, bounded below by zero and as such it converges to its unique maximum lowest bound $v^\star$ satisfying $v^\star \geq v_\infty$.
Then estimate (2.38) certainly holds, if

$$\frac{\delta}{2} \leq v_\infty,$$

which is true by hypothesis (2.15).
Finally, sequences $\{t_n\}$, $\{s_n\}$ are non–decreasing, bounded above by $t^{\star\star}$, given by (2.20). Hence, they converge to their common, and unique least upper bound $t^\star$ satisfying (2.20). That also completes the proof of Lemma 1. □

We can also provide a second majorizing result.
Let us define function h_m ($m \geq 1$) as f_m by:

$$h_m(s) \quad = \quad \ell\ s^{m-1}\ \eta^\beta + 4\ \ell_0\ (1 + s + \cdots + s^m)\ \eta - 2, \tag{2.39}$$

$$\overline{\delta}_+ = \frac{-\ell\ \eta^\beta + \sqrt{\ell^2\ \eta^{2\,\alpha} + 16\ \ell_0\ \ell\ \eta^\alpha}}{8\ \ell_0}, \tag{2.40}$$

$$\overline{\delta}_0 = \frac{\ell\ \eta^\beta}{1 - \ell_0\ (\eta + \overline{\overline{\eta}})}, \qquad \ell_0\ (\eta + \overline{\overline{\eta}}) \neq 1, \tag{2.41}$$

$$\overline{\delta}_1 = \max\{\frac{\overline{\delta}_0}{2}, \overline{\delta}_+\} \tag{2.42}$$

and

$$\overline{v}_\infty = v_\infty. \tag{2.43}$$

Then, with the above changes and simply following the proof of Lemma 1, we can provide another result on majorizing sequences for Werner' s method (1.2), using a different approach than in Lemma 1:

Lemma 2. *Let $\ell_0 > 0$, $\ell > 0$, $\alpha \geq 0$, $\eta > 0$, $0 < \overline{\overline{\eta}} \leq \eta$ and $\beta = 1+\alpha$ be given constants. Assume:*

$$\ell_0\ (\eta + \overline{\overline{\eta}}) < 1 \tag{2.44}$$

and

$$\overline{\delta}_1 \leq \overline{v}_\infty, \tag{2.45}$$

where $\overline{\delta}_1$, $\overline{v}_\infty$, $\overline{\delta}_+$ are given by (2.42), (2.43) and (2.40), respectively.
Choose

$$\delta \in [\overline{\delta}_1, \overline{v}_\infty]. \tag{2.46}$$

Then, scalar sequence $\{v_n\}$ ($n \geq 0$), given by

$$v_0 = 0, \quad v_1 = \eta, \quad v_{n+2} = v_{n+1} + \frac{\ell\ (v_{n+1} - v_n)^{1+\beta}}{2\ (1 - \ell_0\ (v_{n+1} + \overline{s}_{n+1}))}, \tag{2.47}$$

with,

$$\overline{s}_0 = 0, \quad \overline{s}_1 = \overline{\overline{\eta}}, \quad \overline{s}_{n+2} = v_{n+2} + \frac{\ell\ (v_{n+2} - v_{n+1})^{1+\beta}}{2\ (1 - \ell_0\ (v_{n+1} + \overline{s}_{n+1}))}, \tag{2.48}$$

is non–decreasing, bounded above by $t^{\star\star}$ and converges to its unique least upper bound $t^\star$ with $t^\star \in [0, t^{\star\star}]$, where $t^{\star\star}$ is given by (2.19).
Moreover, the following estimates hold for all $n \geq 0$:

$$\overline{s}_n \leq v_n, \tag{2.49}$$

$$0 < v_{n+2} - v_{n+1} \leq \frac{\delta}{2}\ (v_{n+1} - v_n) \leq \left(\frac{\delta}{2}\right)^{n+1} \eta \tag{2.50}$$

and

$$0 < v_{n+2} - \overline{s}_{n+2} \leq \frac{\delta}{2}\ (v_{n+1} - \overline{s}_{n+1}) \leq \left(\frac{\delta}{2}\right)^{n+2} \eta. \tag{2.51}$$

We also need a lemma due to Werner [9, Lemma 1, p. 335]:

Lemma 3. *Let $G : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a twice Fréchet differentiable operator. Assume that there exist a positive constants L_1, $L_{2,\alpha}$ and $\alpha \in [0,1]$, such that:*

$$\| G'(x) - G'(y) \| \leq L_1 \ \| x - y \| \tag{2.52}$$

and

$$\| G''(x) - G''(y) \| \leq L_{2,\alpha} \ \| x - y \|^{\alpha} \tag{2.53}$$

for all $x, y \in \mathcal{D}$.
Then, the following estimates hold:

$$\| G(x) - G(y) - G'(z)\ (x-y) \| \leq L_1 \int_0^1 \| (1-t)\ y + t\ x - z \| \ dt \ \| x - y \| \quad \text{for all } x,y,z \in \mathcal{D} \tag{2.54}$$

and
for $\theta \in [0,1]$, $x, y \in \mathcal{D}$, $z_\theta = \theta\ x + (1-\theta)\ y$:

$$\begin{aligned} \| G(x) - G(y) - G'(z_\theta)\ (x-y) \| \ \leq \ & \left(\frac{1}{4} + \left(\theta - \frac{1}{2} \right)^2 \right) \frac{L_{2,\alpha} \ \| x - y \|^{2+\alpha}}{(\alpha+1)\ (\alpha+2)} + \\ & L_1 \left| \theta - \frac{1}{2} \right| \ \| x - y \|^2 . \end{aligned} \tag{2.55}$$

We can show the following semilocal convergence result for Werner's method (1.2):

Theorem 4. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a twice Fréchet differentiable operator.*
Assume:
There exist points $x_0, y_0 \in \mathcal{D}$, $L_0 > 0$, $\alpha \in [0,1]$ and $L_{2,\alpha} > 0$, such that for all $x, y \in \mathcal{D}$:

$$A_0^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \tag{2.56}$$

$$\| A_0^{-1} \ [F'(x) - F'(\frac{x_0 + y_0}{2})] \| \leq L_0 \ \| x - \frac{x_0 + y_0}{2} \|, \tag{2.57}$$

$$\| A_0^{-1} \ [F''(x) - F''(y)] \| \leq L_{2,\alpha} \ \| x - y \|^{\alpha}, \tag{2.58}$$

$$y_0 \in \overline{U}(x_0, t^\star) = \{ x \in \mathcal{X},\ \| x - x_0 \| \leq t^\star \} \subseteq \mathcal{D}, \tag{2.59}$$

$$\| A_0^{-1} \ F(x_0) \| \leq \eta, \tag{2.60}$$

$$\| A_0^{-1} \ F(x_1) \| \leq \overline{\eta}, \tag{2.61}$$

where,

$$x_1 = x_0 - F'(\frac{x_0 + y_0}{2})^{-1} \ F(x_0) \tag{2.62}$$

and
Conditions of Lemma 1 hold, with

$$\ell_0 = \frac{L_0}{2}, \qquad \ell = \frac{L_{2,\alpha}}{2\ \beta\ (1+\beta)}. \tag{2.63}$$

Then sequence $\{x_n\}$ defined by Werner's method (1.2) is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a unique solution $x^\star$ of equation $F(x) = 0$ in $U(x_0, t^\star)$. Moreover the following estimate holds for all $n \geq 0$:

$$\| x_n - x^\star \| \leq t^\star - t_n, \tag{2.64}$$

where, sequence $\{t_n\}$ $(n \geq 0)$ is given in Lemma 1.

Proof. We shall show using induction on the integer m:

$$\| x_{m+1} - x_m \| \leq t_{m+1} - t_m \tag{2.65}$$

and

$$\| y_{m+1} - x_{m+1} \| \leq s_{m+1} - t_{m+1}. \tag{2.66}$$

Estimates (2.65) and (2.66) hold for $m = 0$ by the initial conditions.
Let us assume (2.65), (2.66) hold true and $x_m, y_m \in \overline{U}(x_0, t^\star)$ for all $n \leq m + 1$.
Using (2.58), we obtain:

$$\begin{aligned} \| A_0^{-1} \, (A_0 - A_n) \| &\leq L_0 \, \| \frac{x_n + y_n}{2} - \frac{x_0 + y_0}{2} \| \\ &\leq \frac{L_0}{2} \Big(\| x_n - x_0 \| + \| y_n - y_0 \| \Big) \\ &\leq \frac{L_0}{2} \Big((t_n - t_0) + (s_n - t_0) \Big) \\ &= \ell_0 \, (t_n + s_n) < 1 \quad (\text{by } (2.26)). \end{aligned} \tag{2.67}$$

It follows from (2.67) and the Banach lemma of invertible operators [4], [7], that A_n^{-1} exists so that

$$\| A_n^{-1} \, A_0 \| \leq \frac{1}{1 - \ell_0 \, (t_n + s_n)}. \tag{2.68}$$

In view of (1.2), we obtain the approximations:

$$F(x_{m+1}) = F(x_{m+1}) - F(x_m) - F'(\frac{x_m + y_m}{2}) \, (x_{m+1} - x_m) \tag{2.69}$$

$$F(x_{m+2}) = F(x_{m+2}) - F(x_{m+1}) - F'(\frac{x_m + y_m}{2}) \, (y_{m+1} - x_{m+1}). \tag{2.70}$$

By composing both sides of (2.69), (2.70) by A_0^{-1}, using Lemma 3 for $\theta = \dfrac{1}{2}$, $G = A_0^{-1} \, F$, we obtain:

$$\| A_0^{-1} \, F(x_{m+1}) \| \leq \frac{L_{2,\alpha}}{4 \, (\alpha + 1) \, (\alpha + 2)} \, \| x_{m+1} - x_m \|^{2+\alpha} \leq \ell \, (t_{m+1} - t_m)^{1+\beta} \tag{2.71}$$

and

$$\| A_0^{-1} \, F(x_{m+2}) \| \leq \frac{L_{2,\alpha}}{4 \, (\alpha + 1) \, (\alpha + 2)} \, \| x_{m+2} - x_{m+1} \|^{2+\alpha} \leq \ell \, (t_{m+2} - t_{m+1})^{1+\beta}, \tag{2.72}$$

respectively.
Using (1.2), (2.68), (2.17), (2.18), (2.71) and (2.72), we obtain:

$$\begin{aligned} \| x_{m+2} - x_{m+1} \| &\leq \| A_{m+1}^{-1} \, A_0 \| \, \| A_0^{-1} \, F(x_{m+1}) \| \\ &\leq \frac{\ell \, (t_{m+1} - t_m)^{1+\beta}}{2 \, (1 - \ell_0 \, (t_{m+1} + s_{m+1}))} \\ &= t_{m+2} - t_{m+1} \end{aligned}$$

and

$$\begin{aligned} \| y_{m+2} - x_{m+2} \| &\leq \| A_{m+1}^{-1} \, A_0 \| \, \| A_0^{-1} \, F(x_{m+2}) \| \\ &\leq \frac{\ell \, (t_{m+2} - t_{m+1})^{1+\beta}}{2 \, (1 - \ell_0 \, (t_{m+1} + s_{m+1}))} \\ &= s_{m+2} - t_{m+2}, \end{aligned}$$

which complete the induction for (2.65) and (2.66).

By Lemma 1, (2.65) and (2.66), sequence $\{x_n\}$ $(n \geq 0)$ is Cauchy sequence in a Banach space $\mathcal{X}$ and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $m \longrightarrow \infty$ in (2.71), we obtain $F(x^\star) = 0$.
Finally to show uniqueness, let $y^\star \in U(x_0, t^\star)$ be a solution of equation $F(x) = 0$.
Let:

$$\mathcal{M} = \int_0^1 F'(x^\star + t\ (y^\star - x^\star))\, dt. \tag{2.73}$$

Using (2.58), we obtain in turn:

$$\begin{aligned} \| A_0^{-1}\,(A_0 - \mathcal{M})\, \| &\leq L_0 \int_0^1 \| \frac{x_0 + y_0}{2} - (x^\star + t\,(y^\star - x^\star))\ \|\ dt \\ &\leq L_0 \int_0^1 \Big((1-t)\ \| \frac{(x_0 - x^\star) + (y_0 - x^\star)}{2} \| + \\ &\quad t\ \| \frac{(y_0 - y^\star) + (x_0 - y^\star)}{2} \| \Big)\, dt \\ &\leq \frac{L_0}{4} \Big(\| x_0 - x^\star \| + \| y_0 - x^\star \| + \| y^\star - x_0 \| + \\ &\quad \| y^\star - y_0 \| \Big) \\ &< \frac{L_0}{4}\ 4\ t^\star = L_0\ t^\star \leq 1 \quad (\text{by } (2.26)). \end{aligned} \tag{2.74}$$

In view of (2.74) and the Banach lemma on invertible operators, $\mathcal{M}^{-1}$ exists.
It follows from the identity:

$$0 = F(x^\star) - F(y^\star) = \mathcal{M}\,(x^\star - y^\star),$$

that

$$x^\star = y^\star.$$

That completes the proof of Theorem 4. □

Remark 5. (a) The most appropriate choices for δ in Lemmas 1 and 2 seem to be $\delta = \delta_1$ and $\delta = \overline{\delta}_1$, respectively.

(b) Note that the conclusions of Theorem 4 hold if Lemma 1 is replaced by Lemma 2 and (2.18) by (2.48).

(c) The limit point $t^\star$ (see Theorem 4) can be replaced by $t^{\star\star}$ given in closed form by (2.19).

3. Applications

Let us provided a numerical example.

Example 1. *Let* $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $\mathcal{D} = \{x\ :\ |x - x_0| \leq 1 - \gamma\}$, $\gamma \in \left[0, \frac{1}{2}\right)$, *and define function* F *on* U_0 *by*

$$F(x) = x^3 - \gamma. \tag{3.1}$$

The Kantorovich hypotheses for Newton's method are [4], [7]*:*

$$\| F'(x_0)^{-1}\ (F'(x) - F'(y))\ \| \leq K\ \ \| x - y \|, \quad \text{for all } x, y \in \mathcal{D} \tag{3.2}$$

and

$$h_K = 2\ K\ \eta \leq 1. \tag{3.3}$$

Using (3.1) and (2.60) (for $x_0 = y_0 = 1$), we obtain

$$\eta = \frac{1}{3}\,(1-\gamma) \quad and \quad K = 2\,(2-\gamma). \tag{3.4}$$

The Kantorovich condition is violated since:

$$\frac{4}{3}\,(1-\gamma)\,(2-\gamma) > 1 \quad \text{for all} \quad \gamma \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method starting at $x_0 = 1$ converges to $x^\star = \sqrt[3]{\gamma}$.

However, the condition of our Theorem 4 under the conditions of Lemma 2 are satisfied, say for $\gamma = .49$.

Indeed, using (2.1), (2.40)–(2.43), (2.60), (2.61) and (3.1), we obtain:

$$v = 2.749087577,$$

$$\overline{\delta}_+ = .0723581, \qquad \overline{\delta}_0 = .008401651,$$

$$\overline{\delta}_1 = \overline{\delta}_+, \qquad v_\infty = \overline{v}_\infty = .5733, \qquad \text{and } \delta = \delta_0.$$

Then all hypotheses of Theorem 4 hold. Hence, Werner's method (1.2) converges to $x^\star = \sqrt[3]{.49} = .788373516$.

CONCLUSION

We provided a semilocal convergence analysis for Werner's method in order to approximate a locally unique solution of an equation in a Banach space.

Using recurrent functions, a combination of Hölder condition on the second derivative and center–Lipschitz condition on the first derivative, instead of only Hölder and Lipschitz conditions [9], [10], we provided an analysis with the following advantages over the work in [9], [10]: weaker sufficient convergence conditions and larger convergence domain.

A numerical example further validating the results is also provided in this study.

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