Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 39 (2007) pp. 19-27

Almost Periodic Functions Defined on $\mathbb{R}^n$ With Values in Fuzzy Setting

Muhammad Amer Latif
P.O. Box. 220871, Riyadh 11311, Saudi Arabia.
E-mail: m_amer_latif@hotmail.com.

Muhammad Iqball Bhatti
Department of Mathematics
University of Engineering and Technology
Lahore - Pakistan
E-mail: iqballmbhatti@yahoo.com

Abstract. In this paper we develop the theory of almost periodic functions defined on $\mathbb{R}^n$ with values in fuzzy setting.

AMS (MOS) Subject Classification: Primary 05C38, 15A15; Secondary 05A15, 15A18.
Keywords and Phrases: Almost Periodic Functions, Fuzzy subset, Fuzzy-number-type spaces, Relatively dense set

1. Introduction

The theory of almost periodic functions was developed in its main features by Bohr [5] in three rather long papers in the Acta Mathematica (Volumes 45, 46 and 47) under the common title "Zur theorie der Fast Periodische Funktionen" in 1923; the first of these deals with the almost periodic functions of a real variable, while the third takes up the case of a complex variable. Afterwards theory was continuously getting established by several mathematicians like Besicovitch [3], Bochner [4], Amerio and Prouse [1], Levitan [8], Levitan and Zhikov [9], Corduneaue [6], Fink [7] and Zaidman [11] etc. In 1933, Bochner defined and studied the almost periodic functions with values in Banach spaces. He showed that these functions include certain earlier generalizations of the notion of almost periodic functions. The theory of almost periodic functions was further developed by replacing Banach spaces by complete Hausdroff locally convex spaces and Fréchet spaces by N'Guérékata [10]. The theory of almost periodicity as known in Banach spaces, is studied in fuzzy setting that is based on the work of Bede and Gal [2]. The theory of almost periodic functions defined on $\mathbb{R}^n$ with values in Banach spaces is given in monograph of Zaidman [11]. However the theory of almost periodic functions defined on $\mathbb{R}^n$ with values in fuzzy-number-type spaces was not yet developed. It is the main goal of this present paper to develop this theory in section 3.

To this end we first recall the following:

2. Preliminaries

Definition 1. Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets of real axis $\mathbb{R}$ (i.e. $u : \mathbb{R} \longrightarrow [0,1]$), satisfying the following properties:

(i) $\forall u \in \mathbb{R}_F$, u is normal i.e.with $u(x) = 1$.

(ii) $\forall u \in \mathbb{R}_F$, u is convex fuzzy set i.e.
$u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1]$.

(iii) $\forall\ u \in \mathbb{R}_F$, u is upper semi-continuous on $\mathbb{R}$.

(iv) $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact.

The set $\mathbb{R}_F$ is called the space of fuzzy real numbers.

Remark 2. It is clear that $\mathbb{R} \subset \mathbb{R}_F$, because any real number $x_0 \in \mathbb{R}$, can be described as the fuzzy number whose value is 1 for $x = x_0$ and zero otherwise.

We will collect some other definitions and notations needed in the sequel. For $0 < r \leq 1$ and $u \in \mathbb{R}_F$, we define

$[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$

$[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$

Now it is well known that for each $r \in [0,1]$, $[u]^r$, is bounded closed interval. For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we have the sum $u \oplus v$ and the product $\lambda \odot u$ are defined by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \odot u]^r = \lambda [u]^r$, $\forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals as subsets of $\mathbb{R}$ and $\lambda [u]^r$means the usual product between a scalar and a subset of $\mathbb{R}$.

Now we define $D : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R} \cup \{0\}$ by

$$D(u,v) = \sup_{r \in [0,1]} \left(\max\left\{\left|u_-^r - v_-^r\right|, \left|u_+^r + u_+^r\right|\right\}\right)$$

where $[u]^r = \left[u_-^r, u_+^r\right]$, $[v]^r = \left[v_-^r, v_+^r\right]$ then $(D, \mathbb{R}_F)$ is a metric space and it possesses the following properties:

(i) $D(u \oplus w, v \oplus w) = D(u,v)$, $\forall u, v, w \in \mathbb{R}_F$.

(ii) $D(\lambda \odot\ u,\ \lambda \odot v) = \lambda D(u,v), \forall u, v \in \mathbb{R}_F, \forall \lambda \in \mathbb{R}$.

(iii) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)$, $\forall u, v, w, e \in \mathbb{R}_F$ and $(\mathbb{R}_F, D)$ is a complete metric space.

Also we have the following theorem.

Theorem 3. (i) *If we denote* $\widetilde{0} = \mathcal{X}_{\{0\}}$ *then* $\widetilde{0} \in \mathbb{R}_F$ *is neutral element with respect to* $\oplus$, *i.e.* $u \oplus \widetilde{0} = \widetilde{0} \oplus u$, *for all* $u \in \mathbb{R}_F$.

(ii) *With respect to* $\widetilde{0}$ *none of* $u \in \mathbb{R}_F \backslash \mathbb{R}$ *has opposite in* $\mathbb{R}_F$ *with respect to* $\oplus$.

(iii) *For any* $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$, *any* $u \in \mathbb{R}_F$, *we have* $(a+b) \odot u = a \odot u \oplus b \odot\ u . \forall a, b \in \mathbb{R}$ *the above property does not hold.*

(iv) *For any* $\lambda \in \mathbb{R}$*and any* $u, v \in \mathbb{R}_F$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) *For any* $\lambda, \mu \in \mathbb{R}$ *and any* $u \in \mathbb{R}$, *we have* $\lambda \odot (\lambda \odot v) = (\lambda . \mu) \odot v$.

(vi) *If we denote*$\|u\|_F = D(u, \widetilde{0})$, $\forall u \in \mathbb{R}_F$ *then* $\|.\|_F$ *has the properties of a usual norm on* $\mathbb{R}_F$, *i.e.* $\|u\|_F = 0$ *if and only if* $u = \widetilde{0}$,

$\|\lambda \odot u\|_F = |\lambda| . \|u\|_F$ *and* $\|u \oplus v\|_F \leq \|u\|_F + \|v\|_F$, $|\|u\|_F + \|v\|_F| \leq D(u,v)$.

Remark 4. The propositions (ii) and (iii) in theorem show us that $(\mathbb{R}_F, \oplus, \odot)$ is not a linear space over $\mathbb{R}$ and consequently $(\mathbb{R}_F, \|.\|_F)$ cannot be a normed space. However, the properties of D and those in theorem (iv)-(vi), have as an effect that most of the metric properties of a functions defined on $\mathbb{R}$ with values in a Banach space, can be extended to functions $f : \mathbb{R} \longrightarrow \mathbb{R}_F$, called fuzzy functions.

We now recall the following definitions and theorems

Definition 5. A function $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is said to be continuous at $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ we can find $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$, whenever $|x - x_0| < \delta$. f is said to be continuous on $\mathbb{R}$ if it is continuous at every $x \in \mathbb{R}$.

Definition 6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ be continuous on $\mathbb{R}$. We say that f is B-almost periodic if $\forall \varepsilon > 0, \exists\, l > 0$ such that any interval $[a, a + l]$ of length l contains at least one point τ with

$D(f(t + \tau), f(t)) < \varepsilon, \forall t \in \mathbb{R}$.

Definition 7. We say that f is normal if for any sequence $F_n : \mathbb{R} \longrightarrow \mathbb{R}_F$ of the form

$F_n(x) = f(x + h_n), n \in \mathbb{N}$, where $(h_n)_n$ is a sequence of real numbers, one can extract a subsequence of $(F_n)_n$, converging uniformly on $\mathbb{R}$ i.e. for every sequence $(h_n)_n$ of real numbers there exists a subsequence $(h_{n_k})_{n_k}$, and $F : \mathbb{R} \longrightarrow \mathbb{R}_F$ which may depend on $(h_n)_n$, such that

$D(F_{n_k}(x), F(x)) \longrightarrow 0$ as $k \longrightarrow \infty$, uniformly with respect to x .

Theorem 8. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost periodic then f is bounded i.e.$\exists$, $M > 0$ with*

$$D(f(x), f(y)) < M, \forall x, y \in \mathbb{R}.$$

Theorem 9. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost-periodic then f is uniformly continuous on $\mathbb{R}$.*

Theorem 10. *If $f_n : \mathbb{R} \longrightarrow \mathbb{R}_F$, $n \in \mathbb{N}$ are B-almost periodic and $f_n \longrightarrow f$ as $n \longrightarrow \infty$ uniformly on $\mathbb{R}$, then f is B-almost periodic.*

Theorem 11. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost periodic, then the set of values of f is relatively compact in the complete metric space $(\mathbb{R}_F, D)$.*

Theorem 12. *If $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is B-almost periodic, then $\lambda \odot f, \lambda \in \mathbb{R}$, $F_h(x) = f(x+h)$, and $G(x) = \|f(x)\|_F$, $x \in \mathbb{R}$ are* B-*almost periodic functions.*

Theorem 13. *The sum $\oplus$ of two B-almost periodic functions is B-almost periodic.*

Remark 14. Let us denote $AP(\mathbb{R}_F) = \{f : \mathbb{R} \longrightarrow \mathbb{R}_F : f \text{ is B-almost periodic}\}$, and for $f \in AP(\mathbb{R}_F)$, let us define$\|f\| = \sup \{\|f(x)\|_F : x \in \mathbb{R}\}$. By theorem 8 we get $\|f\| < +\infty$. Also by theorems 3, 12 and 13 $AP(\mathbb{R}_F, \oplus, \odot)$, is not a linear space, and consequently $AP(\mathbb{R}_F, \|.\|_F)$ is not a normed space. However, endowed with the metric

$D^* : AP(\mathbb{R}_F) \times AP(\mathbb{R}_F) \longrightarrow \mathbb{R}_+ \cup \{0\}$ defined by

$D^*(f, g) = \sup_{x\in\mathbb{R}} D(f(x), g(x)), f, g \in AP(\mathbb{R}_F)$

becomes a complete metric space. Indeed if we denote

$C_b(\mathbb{R}_F) = \{f : \mathbb{R} \longrightarrow \mathbb{R}_F : f$ is continuous and bounded on $\mathbb{R}\}$, then because

$(\mathbb{R}_F, D)$ is a complete metric space, it follows that $(C_b(\mathbb{R}_F), D^*)$ is a complete metric space. Then theorems 8 and 11 show that $AP(\mathbb{R}_F)$ is a closed subset of $C_b(\mathbb{R}_F)$, i.e. $(AP(\mathbb{R}_F), D^*)$ is a complete metric space. For all of the above C.f [5].

3. Almost Periodic Functions Defined On $\mathbb{R}^n$ With Values In Fuzzy-Number-Type Space

Now we recall the following facts about the Euclidean n-dimensional space $\mathbb{R}^n$

Let $\mathbb{R}^n$ the usual Euclidean n-dimensional space. The elements x of $\mathbb{R}^n$ are the n-tuples $x = (x_1, x_2, ..., x_n)$ and a norm of $x \in \mathbb{R}^n$ is given by $\|x\| = (x_1 + x_2 + ... + x_n)^{\frac{1}{2}}$. A closed ball $\overline{B}(x_0; r)$ in $\mathbb{R}^n$ with center x and radius $r > 0$ is defined by the set $\overline{B}(x_0; r) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$. A set P is said to be relatively dense in $\mathbb{R}^n$ if there exists a number $r > 0$ such that $P \cap \overline{B}(x_0; r) \neq \varnothing$, for all $x \in \mathbb{R}^n$. We also have the following two important theorems for our further discussion.

Theorem 15. *A subset P of $\mathbb{R}^n$ is relatively dense in $\mathbb{R}^n$ if and only if, for some $r > 0$, we have the relation $\mathbb{R}^n = \underset{p \in P}{\cup} \overline{B}(p; r)$.*

Theorem 16. *A subset P of $\mathbb{R}^n$ is relatively dense if and only if there exists a compact set K in such that $K + P = \mathbb{R}^n$ (vector sum of K and P). Now we define almost periodic functions defined on $\mathbb{R}^n$ and taking values in $\mathbb{R}_F$ but before that we define continuity and uniform continuity of functions defined on $\mathbb{R}^n$ and taking values in $\mathbb{R}_F$ then we define the almost periodicity of functions defined on $\mathbb{R}^n$ with values in the fuzzy-number-type spaces.*

Definition 17. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is said to be continuous at $x_0 \in \mathbb{R}^n$ if for every $\varepsilon > 0$ we can find $\delta > 0$ such that $D(f(x), f\ (x_0)) < \varepsilon$, whenever $\|x - x_0\| < \delta$. f is said to be continuous on $\mathbb{R}^n$ if it is continuous at every $x \in \mathbb{R}^n$.

Definition 18. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is said to be uniformly continuous on $\mathbb{R}^n$ if such that

$D(f(x_1), f\ (x_2)) < \varepsilon$, whenever $\|x_1 - x_2\| < \delta, \forall x_1, x_2 \in \mathbb{R}^n$.

Definition 19. A continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, is said to be B-almost periodic, if for every $\varepsilon > 0$, we can find a relatively dense set which we denote by $T(f; \varepsilon)$ in $\mathbb{R}^n$ such that

$D(f(t+\tau), f\ (t)) < \varepsilon, \forall t \in \mathbb{R}^n, \tau \in T(f; \varepsilon)$. Hence to any $\varepsilon > 0$, we may associate a number, $r = r(\varepsilon) > 0$, in such manner that in any closed ball $\overline{B}(x; r)$ there exists at least one element of the set $T(f; \varepsilon)$.

Theorem 20. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic function then given any any $\varepsilon > 0$, there are two positive numbers $r_1 = r_1(\varepsilon)$ and $r_2 = r_2(\varepsilon)$ such that any ball $\overline{B}(x; r_1)$ in $\mathbb{R}^n$ contains a ball of radius r_2 which is contained in $T(f; \varepsilon)$.*

Proof. Consider the set $T(f; \frac{\varepsilon}{2})$ which is relatively dense in $\mathbb{R}^n$ ($f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is assumed to be almost periodic) and the associated number $R = R(\frac{\varepsilon}{2})$ such that $\overline{B}(a; R) \cap T(f; \frac{\varepsilon}{2}) \neq \varnothing$, $\forall a \in \mathbb{R}^n$. Using now the uniform continuity of f over $\mathbb{R}^n$ we find a number $\delta_1 = \delta_1(\frac{\varepsilon}{2})$ such that if $y \in \mathbb{R}^n$ and $\|y\| \leq \delta_1$ it follows that $D(f(x + y), f\ (x)) < \frac{\varepsilon}{2}, \forall\ x \in \mathbb{R}^n$ we say that $r_1 = R + 2\delta$ and $r_1 = \delta_1$ form required numbers. In fact, given $x \in \mathbb{R}^n$, take $z \in \mathbb{R}^n$ such that $\|z\| = r_1$. Then $\exists$ $y \in \overline{B}(x_0 + z; R) \cap T(f; \frac{\varepsilon}{2})$. Hence $\|y - x_0\| \leq R + \delta_1 < r_1$ so that $y \in \overline{B}(x_0; r_1)$. Furthermore $\forall y \in \mathbb{R}^n$, $\|y\| \leq \delta_1$,

$\left\|x' + \ y - x_0\right\| \leq R + \delta_1 + \delta_1 = R + 2\delta_1 = r_1$. Hence $x' + y \in \overline{B}(x_0; r_1)$. Therefore the whole ball $\in \overline{B}(x'; \delta_1)$ is contained in $\in \overline{B}(x_0; r_1)$. Finally any vector in this ball belong to$T(f; \varepsilon)$, this is because $x' + \ y$ with $\|y\| \leq \delta_1$ is such a vector, we have $\forall\ y \in \mathbb{R}^n$

$$D(f(x + y + x'), f\ (x))$$
$$= D(f(x + y) \oplus f(x + y + x'), f(x) \oplus f(x + y))$$
$$= D(f(x + y) \oplus f(x + y + x'), f(x + y) \oplus f(x))$$

$\leq D(f(x+y), f(x+y+x^{'})) + D(f(x+y), f(x)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$.

Here we have used the fact that $x^{'} \in T(f; \frac{\varepsilon}{2})$, $\|y\| \leq \delta_1$ and uniform continuity of f over $\mathbb{R}^n$ is proved. The following result shows the boundedness of B-almost periodic. □

Theorem 21. *If a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, is B-almost periodic, then f is bounded i.e. $\exists M > 0$ with $D(f(x), f\ (y)) < M, \forall x, y \in \mathbb{R}$.*

Proof. Because $D(f(x), f\ (y)) \leq D(f(x), \widetilde{0}) + D(\widetilde{0}, f\ (y)) = \|f(x)\|_F + \|f(y)\|_F$ it is sufficient to prove that M_1 with $\|f(x)\|_F \leq M_1$. Take any $\varepsilon > 0$ and associative relatively set $T(f; \varepsilon)$. Therefore, for some $r = r(\varepsilon) > 0$, $\mathbb{R}^n = \underset{\tau \in T(f;\varepsilon)}{\cup} \overline{B}(\tau; r)$ and consequently $\forall t^{'} \in \mathbb{R}^n, \exists\ \tau \in T(f; \varepsilon)$ such that $\left\|t^{'} - y\right\| \leq r$. Then, if t is $t^{'} - y$, we have $t^{'} = t - y$ where $\tau \in T(f; \varepsilon)$. Therefore

$$\begin{aligned}
&\left\|f(t^{'})\right\|_F \\
&= D(f(t^{'}), \widetilde{0}) \\
&= D(f(t) \oplus f(t+\tau), f(t) \oplus \widetilde{0}) \\
&\leq D(f(t), f(t+\tau)) + D(f(t) \oplus \widetilde{0}) \\
&< \varepsilon + \|f(t)\|_F \\
&< \varepsilon + \sup_{t \in \overline{B}(0;r)} (\|f(t)\|_F)
\end{aligned}$$

For instance, if we take $\varepsilon = 1$, $\left\|f(t^{'})\right\|_F < \varepsilon + \sup\limits_{\|t\| \leq r(1)} (\|f(t)\|_F)$ which gives us an upper bound for f over $\mathbb{R}^n$. □

Next theorem shows that the range of B-almost periodic functions is relatively compact.

Theorem 22. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, then the set of values of f is relatively compact in the complete metric space $(\mathbb{R}_F, D)$.*

Proof. In complete metric spaces, the relatively compact sets coincides with totally bounded sets, it is sufficient to show that the values of the functions can be embedded in a finite number of spheres of radius 2ε. Take any $\varepsilon > 0$ and the and the number $r = r(\varepsilon) > 0$. The range of f when t runs over the compact ball $\overline{B}(0; r) = \{t \in \mathbb{R}^n : \|t\| \leq r\}$ is compact in $\mathbb{R}_F$. Therefore, there are ν points $f(t_1), f(t_2), ..., f(t_\nu)$ where $\|t_i\| \leq r, 1 \leq i \leq \nu$ and $f(t) \in \overset{\nu}{\underset{i=1}{\cup}} \overline{B}(f(t_i); \varepsilon), \forall t, \|t\| \leq r$. Take now any arbitrary $t^{'} \in \mathbb{R}^n$, it can be written as $t^{'} = t + \tau$ where $\|t\| \leq r$ and $\tau \in T(f; \varepsilon)$, hence, there is an $i \in \{1, 2, 3, ..., \nu\}$ such that $D(f(t^{'}), f\ (t_i)) \leq \varepsilon$. It follows that

$$\begin{aligned}
&D(f(t^{'}), f\ (t_i)) \\
&= D(f(t+\tau), f\ (t_i)) \\
&= D(f(t+\tau) \oplus f\ (t_i), f(t_i) \oplus f(t)) \\
&\leq D(f(t+\tau), f\ (t_i)) + D(f(t_i) \oplus f(t)) \leq \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Thus $R_f \subset \overset{\nu}{\underset{i=1}{\cup}} \overline{B}(f(t_i); 2\varepsilon)$. □

Remark 23. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, and let us consider the sequence $(f(t_n))_n$of values. Denote $A = \{f(t_n) : n \in \mathbb{N}\}$,and take the closure $\overline{A} \subset$

$\overline{f(\mathbb{R}^n)} \subset \mathbb{R}_F$. It follows that $\overline{A}$ is compact, so is $\overline{A}$ sequentially compact too, which by $A \subset \overline{A}$ implies that the sequence has a convergent subsequence in $\mathbb{R}_F$.

Theorem 24. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, then it is uniformly continuous over $\mathbb{R}^n$.*

Proof. Let $\varepsilon > 0$ be given, we can find $r = r(\frac{\varepsilon}{3}) > 0$ such that for any $t^{'} \in \mathbb{R}^n$ can be represented as $t^{'} = t + \tau$ where $\tau \in T(f; \frac{\varepsilon}{3})$. Next we note the uniform continuity of the function f on the closed ball $\overline{B}(0; 2r) = \{t \in \mathbb{R}^n : \|t\| \leq 2r\}$. Therefore, there is a $\delta > 0$, $\left(\delta < r(\frac{\varepsilon}{3})\right)$ such that, if $\|t_1\| \leq 2r, \|t_2\| \leq 2r$ and $\|t_1 - t_2\| < \delta$, then we have $D(f(t_1), f\ (t_2)) \leq \frac{\varepsilon}{3}$. Take now any pair $t_1^{'}, t_2^{'} \in \mathbb{R}^n$, such that $\left\|t_1^{'} - t_2^{'}\right\| < \delta$. We can write, for some $\tau \in T(f; \frac{\varepsilon}{3})$, the decomposition $t_1^{'} = t_1 + \tau$, where $\|t_1\| \leq r$. Then define t_2 as $t_2^{'} - \tau$. It follows that $\|t_1 - t_2\| = \left\|t_1^{'} - t_2^{'}\right\| < \delta$. Also $\|t_2\| = \|t_1 - t_2\| + \|t_1\| < \delta + r \leq 2r$. From the above we derive that $D(f(t_1), f\ (t_2)) \leq \frac{\varepsilon}{3}$, and accordingly, as $\tau \in T(f; \frac{\varepsilon}{3})$, we find that

$$
\begin{aligned}
& D(f(t_2^{'}), f\ (t_1^{'}) \\
&= D(f(t_2 + \tau), f\ (t_1 + \tau)) \\
&= D(f(t_1) \oplus f(t_2 + \tau), f\ (t_1 + \tau) \oplus f(t_1)) \\
&= D(f(t_1) \oplus f(t_1 + \tau), f\ (t_1) \oplus f(t_2 + \tau)) \\
&\leq D(f(t_1), f(t_1 + \tau)) + D(f\ (t_1), f(t_2 + \tau)) \\
&= D(f(t_1), f(t_1 + \tau)) + D(f(t_1) \oplus f(t_2 + \tau), f\ (t_1) \oplus f(t_2 + \tau)) \\
&\leq D(f(t_1), f(t_1 + \tau)) + D(f(t_1), f(t_1 + \tau)) + D(f\ (t_1 + \tau), f(t_2 + \tau)) \\
&= \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$
□

The next result shows that the set $AP(\mathbb{R}_F)$ is closed with respect to uniform convergence.

Theorem 25. *If $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, $n \in \mathbb{N}$ are B-almost periodic and $f\ _k \longrightarrow f$ as $k \longrightarrow \infty$ uniformly on $\mathbb{R}^n$, then is B-almost periodic.*

Proof. Let $\varepsilon > 0$. Since $f_k(t) \longrightarrow f(t)$ uniformly over $\mathbb{R}^n$ as $k \longrightarrow \infty$, so we can find a natural number k_0 such that$\forall k \geq k_{0,}$ we have $D(f_k(t), f\ (t)) < \frac{\varepsilon}{3}$. Since $f\ _k : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is almost periodic for $k = 1, 2, 3, ...$ so for already chosen $\varepsilon > 0$, we can find a relatively dense set $T(f_k; \frac{\varepsilon}{3})$ such that

$D(f_k(t + \tau), f\ _k(t)) < \frac{\varepsilon}{3}$, $\forall \tau \in T(f_k; \frac{\varepsilon}{3})$, $t \in \mathbb{R}^n$, $k = 1, 2, 3, ...$ Now

$$
\begin{aligned}
& D(f_k(t + \tau), f\ _k(t)) \\
&= D(f(t + \tau) \oplus f_k(t + \tau), f_k(t + \tau) \oplus f\ (t)) \\
&\leq D(f(t + \tau), f_k(t + \tau)) + D(f_k(t + \tau), f\ (t)) \\
&= D(f(t + \tau), f_k(t + \tau)) + D(f_k(t + \tau) \oplus f_k(t), f_k\ (t) \oplus f(t)) \\
&\leq D(f(t + \tau), f_k(t + \tau)) + D(f_k(t + \tau), f_k(t)) + D(f_k\ (t), f(t)) \\
&= \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon, \forall \tau \in T(f_k; \tfrac{\varepsilon}{3}),\ t \in \mathbb{R}^n,\ k = 1, 2, 3, ...
\end{aligned}
$$

which implies that $\tau \in T(f; \varepsilon)$, and hence $T(f; \varepsilon)$ is a relatively dense set in $\mathbb{R}^n$ so f is proved to be almost periodic function. □

We now give the following simple theorem.

Theorem 26. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic, then $\lambda \odot f, \lambda \in \mathbb{R}$, $F_h(x) = f\ (x + h)$, and $G(x) = \|f(x)\|_F, x \in \mathbb{R}^n$ are B-almost periodic functions.*

Proof. Because $D(\lambda \odot f(t + \tau), \lambda \odot f(t)) = |\lambda| D(f(t + \tau), f(t))$, for all, it follows that λf is almost periodic whenever f is B-almost periodic. And since

$|\|f(t+\tau)\|_F - \|f(t)\|_F| \leq D(f(t+\tau), f(t))$, then it is immediate that $G(x) = \|f(x)\|_F, x \in \mathbb{R}^n$, is B-almost periodic. Now at the last let h be fixed and for every $\varepsilon > 0$, let $r = r(\varepsilon) > 0$ be attached to f in the definition of B-almost periodic.

By $D(f(t+\tau), f\ (t)) < \varepsilon$, $\forall \tau \in T(f; \varepsilon)$ and $\forall t \in \mathbb{R}^n$, we get (by taking $t = u+h$),

$D(f(u+h+\tau), f\ (u+h)) < \varepsilon$, $\forall \tau \in T(f; \varepsilon)$ and $\forall t \in \mathbb{R}^n$.

$D(F_h(u+\tau), F(u)) < \varepsilon$, $\forall \tau \in T(f; \varepsilon)$ and $\forall t \in \mathbb{R}^n$. This implies that $T(f; \varepsilon) \subset T(F_h; \varepsilon)$, therefore F is B-almost periodic. □

We now define normal functions and prove some results.

Definition 27. We say that f is normal if for any sequence $F_k : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ of the form

$F_k(x) = f\ (x+h_k), k \in \mathbb{N}$, where $(h_k)_k$ is a sequence of real numbers, one can extract a subsequence of $(F_k)_k$, converging uniformly on $\mathbb{R}^n$ i.e. for every sequence $(h_k)_k$ of real numbers there exists a subsequence $(h_{k_l})_{k_l}$, and $F : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ which may depend on $(h_k)_k$, such that

$D(F_{k_l}(x), F(x)) \longrightarrow 0$ as $l \longrightarrow \infty$, uniformly with respect to x.

We now apply this to prove the following theorem.

Theorem 28. *The sum of two B-almost periodic functions are B-almost periodic.*

Proof. Let f and g be two B-almost periodic functions, and let $(h_k)_k$ an arbitrary sequence in $\mathbb{R}^n$. From the sequence $(f_{h_k})_k$of translates, we can choose a uniformly convergent subsequence on $\mathbb{R}^n$ say $(f_{l_k})_k$. From the sequence $(g_{k_k})_k$,we choose a subsequence uniformly convergent on $\mathbb{R}^n$, say $(g_{l_k})_k$. Then the sequence $(f_{l_k} + g_{k_k})_k$, which is a subsequence of the sequence $(f_{h_k} + g_{h_k})_k$, is uniformly convergent on $\mathbb{R}^n$. □

To prove the equivalence between the normal functions and B-almost periodic functions we need the following lemma.

Lemma 29. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ *be a B-almost periodic function and a sequence* $(x_k)_k \subset \mathbb{R}^n$ *be given then to any* $\varepsilon > 0$ *we may associate a subsequence* $(x_{k_i})_{k_i}$ *such that the inequality*

$\underset{x \in \mathbb{R}^n}{sup}\, D(f(x+x_{k_p}), f(x+x_{k_q})) < \varepsilon, \forall p, q \in \mathbb{N}$, *is satisfied.*

Proof. We know that any vector $x_k \in \mathbb{R}^n$, can be written as $x_k = y_k + z_k$, where $z_k \in T(f; \frac{\varepsilon}{2})$ and $\|y_k\| \leq r(\frac{\varepsilon}{2}) = r > 0$ (r is independent of k). Let y be the limit point of the sequence $(y_k)_k$ then $\|y\| \leq r$. Since $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ be a B-almost periodic function so it is uniformly continuous over $\mathbb{R}^n$ so we can find $\delta > 0$ such that $\|x_1 - x_2\| < \delta \Longrightarrow \|f(x_1) - f(x_2)\| < \frac{\varepsilon}{2}$. Then in the ball $\{x \in \mathbb{R}^n : \|x\| \leq \delta\}$ we find an infinite sequence of y_k, s which we denote by $(x_{k_i})_{k_i}$. Take now two vectors x_{k_p} and x_{k_q} then $x_{k_p} = y_{k_p} + z_{k_p}$, $x_{k_q} = y_{k_q} + z_{k_q}$. We deduce that

$$\sup_{x \in \mathbb{R}^n} D(f(x+x_{k_p}), f(x+x_{k_q})) =$$
$$\sup_{x \in \mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}), f(x+y_{k_q}+z_{k_q}))$$
$$= \sup_{x \in \mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}-y_{k_q}-z_{k_q}), f(x))$$
$$= \sup_{x \in \mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}-y_{k_q}-z_{k_q}) \oplus f(x+y_{k_p}-y_{k_q}), f(x) \oplus f(x+y_{k_p}-y_{k_q}))$$
$$\leq \sup_{x \in \mathbb{R}^n} D(f(x+y_{k_p}+z_{k_p}-y_{k_q}-z_{k_q}), f(x+y_{k_p}-y_{k_q})) + D(f(x), f(x+y_{k_p}-y_{k_q}))$$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$
The last inequality is a consequence of fact that
$y_{k_p} - y_{k_q} \in T(f; \frac{\varepsilon}{2})$ and $\left\|y_{k_p} - y_{k_q}\right\| \leq \left\|y_{k_p} - y\right\| + \left\|y - y_{k_q}\right\| \leq \delta + \delta = 2\delta.$ □

Theorem 30. *Any B-almost periodic $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$, is normal.*

Proof. Let $(x_k)_k \subset \mathbb{R}^n$ be a given sequence. Then by Lemma 29 we can find a subsequence $(x_{k_i,1})_i$such that
$\sup\limits_{x\in\mathbb{R}^n} D(f(x + x_{k_p,1}), f(x + x_{k_q,1})) < 1, \forall p, q \in \mathbb{N}.$
$D(f(x + x_{k_p,1}), f(x + x_{k_q,1})) < 1, \forall p, q \in \mathbb{N}, \forall x \in \mathbb{R}^n.$
Next we choose a subsequence $(x_{k_i,2})_i \subset (x_{k_i,1})_i$ with the property
$D(f(x + x_{k_p,2}), f(x + x_{k_q,2})) < \frac{1}{2}, \forall p, q \in \mathbb{N}, \forall x \in \mathbb{R}^n.$
We can choose a further subsequence $(x_{k_i,3})_i \subset (x_{k_i,2})_i$ with the property. $D(f(x + x_{k_p,3}), f(x + x_{k_q,3})) < \frac{1}{3}, \forall p, q \in \mathbb{N}, \forall x \in \mathbb{R}^n.$
And so on. Consider now a diagonal sequence, $(f(x + x_{k_i,i})_i$ of translated functions. Now if $p, q \in \mathbb{N}$, with $p \leq q$, then we have
$\sup\limits_{x\in\mathbb{R}^n} D(f(x + x_{k_p,p}), f(x + x_{k_q,q})) < \frac{1}{3}, \forall p, q \in \mathbb{N}.$ This proves that the sequence $(f(x + x_{k_i,i}))_i$ is uniformly convergent over $\mathbb{R}^n$ this also proves the normality of f. □

The converse of this theorem also holds true as proved in the following theorem.

Theorem 31. *Any normal function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_F$ is B-almost periodic.*

Proof. On contrary suppose that f is not B-almost periodic function then there exists an $\varepsilon > 0$ such that the set $T(f; \varepsilon)$ is not relatively dense in $\mathbb{R}^n$. This implies that for all $r(\varepsilon) = r > 0$ there is a ball $\overline{B}(a; r)$ which contains no element of the set $T(f; \varepsilon)$. Consider now an arbitrary element $x_1 \in \mathbb{R}^n$ and take $r_1 > \|x_1\|$, hence there exists a ball $\overline{B}(x_2; r_2)$ which is disjoint with $T(f; \varepsilon)$. Since $x_2 - x_1 \in \overline{B}(x_2; r_2) \Longrightarrow x_2 - x_1 \notin T(f; \varepsilon)$. Next take $r_1 > \|x_1\| + \|x_2\|$ and find a ball $\overline{B}(x_3; r_3)$ which is disjoint of $T(f; \varepsilon)$. Now both the vectors $x_2 - x_1$ and $x_2 - x_3$ belong to $\overline{B}(x_3; r_3)$ but $x_2 - x_1 \notin T(f; \varepsilon)$ and $x_2 - x_3 \notin T(f; \varepsilon)$. Continuing this procedure, we can find an infinite sequence $(x_k)_k \subset \mathbb{R}^n$ such that $\forall k, l \in \mathbb{N}, k \neq l \Longrightarrow x_k - x_l \notin T(f; \varepsilon)$. It follows that, by replacing x by $x - x_k$

$$D(f(x + x_k), f(x + x_l)) = D(f(x + x_k - x_l), f(x)) > \varepsilon, \forall x \in \mathbb{R}^n.$$

This shows that the sequence, $(f(x + x_k))_k$, $x \in \mathbb{R}^n$ contains no subsequence which converges uniformly over $\mathbb{R}^n$. A contradiction to the fact that f is normal function. So our assumption that f is not B-almost periodic function is wrong. Therefore f is proved to be almost periodic. □

Remark 32. Let us denote
$AP(\mathbb{R}_F) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}_F : f \text{ is B-almost periodic}\}$, and for $f \in AP(\mathbb{R}_F)$, let us define $\|f\| = \sup \{\|f(x)\|_F : x \in \mathbb{R}^n\}$. By theorem 20 we get $\|f\| < +\infty$. $AP(\mathbb{R}_F)$, is a complete metric space with respect to the metric
$D^* : AP(\mathbb{R}_F) \times AP(\mathbb{R}_F) \longrightarrow \mathbb{R}_+ \cup \{0\}$ defined by
$D^*(f, g) = \sup\limits_{x\in\mathbb{R}^n} D(f(x), g(x)),\ f, g \in AP(\mathbb{R}_F)$
Let us denote
$C_b(\mathbb{R}_F) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}_F : f$ is continuous and bounded on $\mathbb{R}^n\}$ Then because

$(\mathbb{R}_F, D)$ is a complete metric space, it follows that $(C_b(\mathbb{R}_F), D^*)$ is a complete metric space.

Then theorems 20 and 23 show that $AP(\mathbb{R}_F)$ is a closed subset of $C_b(\mathbb{R}_F)$, i.e. $(AP(\mathbb{R}_F), D^*)$ is a complete metric space.

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