Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 40 (2008) pp. 49-55

On Products of Ordered Normed Spaces

Muhammad Iqbal Bhatti
Muhammad Nasir Chaudhary
Department of Mathematics
University of Engineering and Technology
Lahore, Pakistan.
E-mail: iqbalbhatti@uet.edu.pk.

Abstract. In this paper certain order properties are investigated which the products of ordered linear spaces and ordered normed spaces inherit from their component spaces.

1. Introduction

F. Riesz, H. Freudenthal, L.V. Kantorovitch, Kakutani and others initiated the study of ordered linear spaces in the late 1930's. The theory developed into a mathematical discipline around 1950's. It is now one of the most important branches of functional analysis being effectively used to solve such problems, which are posed in more general setting. Several authors have studied regarding products of ordered linear spaces. Bonsall [5], Peressini [12], Jameson [9], Dalen [7], Cristescu [6], Martin [11], Karim [10] and Hong [8] have significant contributions in this area. The authors investigated [1] the inheritance of order properties from ordered product spaces to their component spaces through projections. In [2] relatively uniform convergence, order convergence etc., in product spaces of ordered linear spaces have been discussed. Order-completeness and order-separability in product spaces of ordered linear spaces were studied in [3].

A little work is done with reference to the order properties of ordered normed spaces. The first author attempted to investigate the mutual relationship of ordered product normed spaces and their component spaces in [4]. In section 3 of this article we mainly discuss how a base for the wedge W of a product ordered linear space can be investigated. Section 4 includes some order properties, which are closed under the formation of products of ordered normed spaces.

2. Preliminaries

A wedge is a non-empty subset W of a real linear space X such that

(W_1) $\quad W + W \subseteq W,$
(W_2) $\quad \alpha W \subseteq W$ for $\alpha \geq 0,$

The wedge W in X defines an ordering or preorder relation (a reflexive and transitive relation) " $\leq$ " on X by

$$x \leq y \Leftrightarrow y - x \in W$$

which is compatible with the linear structure of X , that is, " $\leq$ " satisfies the conditions:

(O_1) $x, y \in X, x \leq y \Rightarrow x + z \leq y + z$ for all $z \in X$

(O_2) $x, y \in X, x \leq y, \alpha \geq 0 \Rightarrow \alpha x \leq \alpha y$

A wedge W in X is said to be a cone if $W \cap (-W) = 0$ i.e., $x, -x \in W \Rightarrow x = 0$. A cone C in X defines a partial order relation " $\leq$ " on X. If the partial order (resp: ordering) on the real linear space X is due to a cone C (resp: a wedge W) then we call X an ordered (resp: a pre-ordered) linear space with cone C(resp: wedge W). The element x of an ordered linear space $(X, \leq)$ is said to be a positive element if $x \geq 0$ and the set $X_+ = \{x \in X : x \geq 0\}$is referred to as a positive cone.

3. Order Properties of Product Spaces of Ordered Linear Spaces

It is well known that if W_α (resp: C_α) is a wedge (resp: cone) in the real linear space X_α where $\alpha \in I$ then $W = \prod_{\alpha \in I} W_\alpha$ (resp: $C = \prod_{\alpha \in I} C_\alpha$) is a wedge (resp: cone) in the product linear space $X = \prod_{\alpha \in I} X_\alpha$. Thus the product linear space X is a preordered (resp: ordered) linear space for the ordering (resp: order) generated by the wedge W (resp: cone C).

Using the same order notation in each space, the ordering (resp: partial order) associated with the wedge W (resp: the cone C) is given by

$$(x_1, x_2, ..., x_n, ...) \leq (y_1, y_2, ..., y_n, ...) \Leftrightarrow x_\alpha \leq y_\alpha \quad forall \quad \alpha \in I.$$

Theorem 1. *Let $X_1, X_2, ..., X_n$ be preordered linear spaces with wedges $W_1, W_2, ..., W_n$ respectively. Let $X = \prod_{i=1}^{n} X_\alpha$ and $W = \prod_{i=1}^{n} W_\alpha$, then*

(1) $W - W = X \Leftrightarrow W_\alpha - W_\alpha = X_\alpha, \alpha = 1, 2, ..., n.$

(2) $(e_1, e_2, ..., e_n)$ *is an order unit in* $X \Leftrightarrow e_\alpha$ *is an order unit in* X_α *for every* $\alpha = 1, 2, ..., n.$

(3) W *is Archimedean* $\Leftrightarrow$ *each* W_α $(\alpha = 1, 2, ..., n)$ *is.*

Definition 2. Let X be a preordered linear space with wedge W. A base for the wedge W is a convex subset B such that for each x in $W - \{0\}$, there exists $\lambda > 0$ and $b \in B$ such that the representation $x = \lambda b$ is unique.

Let X and Y be preordered linear spaces with wedges W and W' respectively. Given the bases B of W and B' of W' , what the basis is of $W \times W'$. A natural suggestion would be to try $B_o = B \times B'$. Then given $(x, y) \in X \times Y$ there are unique $\lambda, \sigma > 0$ and $b \in B, b' \in B'$ such that $x = \lambda b$ and $y = \sigma b'$ so that $(x, y) = (\lambda b, \sigma b')$. Obviously, this construction does not give us a base. For example, consider R^2 with the usual positive cone. A base B is the line segment joining the points $(1, 0)$ and $(0, 1)$: a typical element has coordinates $(x, 1 - x)$ with $0 < x < 1$. A base B' for R is the point 1, so that $B \times B' = \{(x, 1 - x, 1) : 0 < x < 1\}$ which cannot be a base for a cone in R^3. On the other hand convex cover of the three points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ is a base for the usual positive cone of R^3. This leads us to :

Theorem 3. *Let X and Y be preordered linear spaces with wedges W and W' respectively and B , B' be the bases for W and W' respectively. If $0 \leq \lambda \leq 1$, then*

$$B_o = \{(\lambda b, (1-\lambda)b') : b \in B, b' \in B'\}$$

is a base for $W \times W'$.

Proof. Let $A = (x, y) \in W \times W'$ then $x \in W$ and $y \in W'$. By the definition of B and B' there are unique $k, h > 0$ and $b \in B, b' \in B'$ such that $x = kb$ and $y = hb'$. Put $\lambda = \frac{k}{k+h}$ so that $1 - \lambda = \frac{h}{k+h}$. Thus

$$b_o = (\lambda b, (1-\lambda)b') \in B_o \text{ and } (x, y) = (kb, hb').$$

Now

$$(x, y) = (k+h)(\tfrac{k}{k+h}b, \tfrac{h}{k+h}b') = (k+h)(\lambda b, (1-\lambda)b') = (k+h)b_o$$

By construction b_o and $k + h$ are uniquely determined from (x, y). Hence B_o is a base for $W \times W'$. □

Definition 4. Bonsall [5] defines a perfect subspace as a subspace E of an ordered linear space X with an order-unit e, which satisfies the following condition:

"given x in E and $\varepsilon > 0$, there exists y in E such that $y + \varepsilon e \geq x$ and $y + \varepsilon e \geq 0$"

Theorem 5. *Let E_i $(i = 1, 2, ..., n)$ be perfect subspace of ordered linear space X_i with order-unit e_i. If $E = \prod_{i=1}^{n} E_i$ and $X = \prod_{i=1}^{n} X_i$, then E is a perfect subspace of X.*

Proof. Since $e = (e_1, e_2, ..., e_n)$ is an order-unit in X, therefore by Theorem 1(2), the result follows. □

4. Order Properties of Product Spaces of Ordered Normed Spaces

If $X_i (i = 1, 2, ..., n)$ is a normed linear space then $X = \prod_{i=1}^{n} X_i$ is also a normed linear space with norm defined by

$$\|(x_1, x_2, ..., x_n)\| = \|x_1\| + \|x_2\| + ... + \|x_n\|$$

OR

$$\|(x_1, x_2, ..., x_n)\| = \max\ \{\|x_1\|, \|x_2\|, ..., \|x_n\|\} \text{ or } \|x_1\| \vee \|x_2\| \vee\ ...\ \vee \|x_n\|$$

The two norms are equivalent and the choice as to which of the two is to be used depends on the context.

In [4] the first author has studied inheritance of certain order properties from product ordered normed spaces to their component spaces. In this section some order properties have been studied which the product normed spaces inherit from their component spaces.

Definition 6. A preordered normed linear space X is said to have the property (R_1) if given $x, y \in X$, $\|x\| \leq \|y\|$ whenever $-y \leq x \leq y$.

Theorem 7. *Let $X_1, X_2, ..., X_n$ be preordered normed linear spaces with wedges $W_1, W_2, ..., W_n$ respectively. Let $X = \prod_{\alpha=1}^{n} X_\alpha$ and $W = \prod_{\alpha=1}^{n} W_\alpha$, then*

1. *X has the property: "Given $x \in X$ with $\|x\| \leq 1$, there is $y \geq x, -x$ with $\|y\| \leq \alpha$" if each X_i has this property.*
2. *X has property (R_1) if each X_i has property (R_1);*

Proof. (1) If $x = (x_1, x_2, ..., x_n) \in X$ with $\|x\| \leq 1$, then for every x_i where $i = 1, 2, 3, ..., n$, $\|x_i\| \leq \max\{\|x_1\|, \|x_2\|, ..., \|x_n\|\} = \|x\| \leq 1$. Therefore, by hypothesis there exists y_i in X_i with $y_i \geq x_i, -x_i$ and $\|y_i\| \leq \alpha$. Taking $y = (y_1, y_2, ..., y_n)$, we get
$y \geq x, -x$ and $\|y\| = \max\{\|y_1\|, \|y_2\|, ..., \|y_n\|\} \leq \alpha$.

(2) Let $x, y \in X$ be such that $-y \leq x \leq y$. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ then $-y_i \leq x_i \leq y_i$ where $i = 1, 2, 3, ..., n$. Since each X_i has property (R_1), therefore, $\|x_i\| \leq \|y_i\|$ for every $i = 1, 2, 3, , n$. Now, $\|x\| = \|x_1\| + \|x_2\| + ... + \|x_n\| \leq \|y_1\| + \|y_2\| + ... + \|y_n\| = \|y\|$.

□

Corollary 8. (1) *Part (1) implies that X has the property: "Given $x \in X$ with $\|x\| \leq 1$, there is $y \geq x, 0$ with $\|y\| \leq \alpha$" if each X_i has this property.*
(2) *Part (2) implies that norm on X is monotonic if norm on each X_i is monotonic;*

Definition 9. A subset D of a preordered linear space $(X, \leq)$ is said to be directed if for every piar of elements x, y from D there exist elements u, v in D such that $u \geq x, y$ and $v \leq x, y$.

Theorem 10. *Let X_i be a preordered normed linar space with wedge $W_i, i = 1, 2, ..., n$. Let $X = \prod_{i=1}^{n} X_i$ and $W = \prod_{i=1}^{n} W_i$, then*
(1) *the open(closed) unit ball in X is directed if open(closed) unit ball in each X_i is directed;*
(2) *norm is additive on W if norm is additive on each W_i.*

Proof. (1) We show that open unit ball in X is directed whenever the open unit ball in each of its component spaces is directed. The case for the closed unit ball is similar. Let $x, y \in X$ with $\|x\| < 1, \|y\| < 1$. If $x = (x_1, x_2, ..., x_n)$ then for every $i = 1, 2, 3, ..., n$, $\|x_i\| \leq \max\{\|x_1\|, \|x_2\|, ..., \|x_n\|\} = \|x\| < 1$. Similarly, if $y = (y_1, y_2, ..., y_n)$ then $\|y_i\| < 1$. Since unit ball in each X_i is directed therefore for each x_i, y_i of the unit ball in X_i there exist u_i and v_i in X_i with $\|u_i\| < 1$ and $\|v_i\| < 1$ such that $u_i \geq x_i, y_i$ and $v_i \leq x_i, y_i$. Taking $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ we get $\|u\| < 1$ and $\|v\| < 1$ such that $u \geq x, y$ and $v \leq x, y$.

(2) Let $x, y \in W$. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ then $x_i, y_i \in W_i$ where $i = 1, 2, 3, ..., n$. Since norm on each W_i is additive, therefore, for every $i = 1, 2, 3, ..., n$, $\|x_i + y_i\| = \|x_i\| + \|y_i\|$.

Now

$$\begin{aligned} \|x + y\| &= \|(x_1 + y_1, x_2 + y_2, ..., x_n + y_n)\| \\ &= \|x_1 + y_1\| + \|x_2 + y_2\| + ... + \|x_n + y_n\| \\ &= \|x\| + \|y\| \end{aligned}$$

which shows that the norm is additive on W.

□

Definition 11. Let $(X, \|.\|)$ be an ordered normed space with cone C.

(1) The cone C is said to be α -normal if $\|u\|, \|v\| \leq 1$ and $u \leq x \leq v$ imply $\|x\| \leq \alpha$. In other words, the cone C is said to be α-normal if $u \leq x \leq v$ implies that $\|x\| \leq \alpha \max\{\|u\|, \|v\|\}$.
(2) The cone C is said to be α -generating if given $x \in X$ there are $u, v \in W$ such that $x = u - v$ with $\|u\| + \|v\| \leq \alpha\|x\|$.
(3) X is said to be (α, n)-generating if given $x_1, x_2, ..., x_n \in X$ with $\|x_j\| \leq 1$ $(j = 1, 2, 3, ..., n)$ we have $x \geq x_j$ such that $\|x\| \leq \alpha$.
(4) X is said to be (α, n)-additive if given $x_1, x_2, ..., x_n \in X$ we have

$$\sum_{j=1}^{n} \|x_j\| \leq \alpha \left\| \sum_{j=1}^{n} x_j \right\|$$

Theorem 12. *Let $X_1, X_2, ..., X_n$ be ordered normed linear spaces with cones $C_1, C_2, ..., C_n$ respectively. Let $X = \prod_{i=1}^{n} X_i$ and $C = \prod_{i=1}^{n} C_i$, then*
(1) *C is α -normal if each C_i is α -normal,*
(2) *C is α -generating if each C_i is α -generating*
(3) *X is (α, n)-generating if each X_i is (α, n)-generating,*
(4) *X is (α, n)-additive if each X_i is (α, n)-additive.*

Proof. (1) Let $x, y, z \in X$ be such that $y \leq x \leq z$. If $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ then $y_i \leq x_i \leq z_i$ where $i = 1, 2, 3, ..., n$. Since each C_i is α -normal, so $\|x_i\| \leq \alpha \max\{\|y_i\|, \|z_i\|\}$. Now,

$$\begin{aligned}
\|x\| &= \|x_1\| \vee \|x_2\| \vee ... \vee \|x_n\| \\
&\leq \alpha\ (\|y_1\| \vee \|z_1\|) \vee \alpha\ (\|y_2\| \vee \|z_2\|) \vee ... \vee \alpha\ (\|y_n\| \vee \|z_n\|) \\
&= \alpha\ [\ \max\ \{\|y_1\|, \|y_2\|, ..., \|y_n\|\} \vee\ \max\ \{\|z_1\|, \|z_2\|, ..., \|z_n\|\}] \\
&= \alpha\ \max\ \{\|y\|, \|z\|\}
\end{aligned}$$

which shows that the cone C is α -normal.

(2) Let $x = (x_1, x_2, ..., x_n)$ be an element of X. Since each C_i is α -generating, therefore for x_i in $X_i (i = 1, 2, 3, ..., n)$ there are u_i, v_i in C_i such that $x_i = u_i - v_i$ and $\|u_i\| + \|v_i\| \leq \alpha\ \|x_i\|$. Taking $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ then $u, v \in C$ for which $x = u - v$. Further,

$$\|u\| + \|v\| = \sum_{i=1}^{n} \|u_i\| + \sum_{i=1}^{n} \|v_i\| = \sum_{i=1}^{n} (\|u_i\| + \|v_i\|) \leq \alpha\ \sum_{i=1}^{n} \|x_i\| = \alpha\ \|x\|$$

which shows that the cone C is α -generating.

(3) Let $x_1, x_2, ..., x_n \in X$ with $\|x_j\| \leq 1$ where $j = 1, 2, 3, ..., n$. Then $x_j = (\varsigma_1^j, \varsigma_2^j, ..., \varsigma_n^j)$ where each $\varsigma_i^j (i = 1, 2, 3, ..., n)$ is an element of X_i. Since for each fixed $j = 1, 2, 3, ..., n, \|x_j\| = \max_{1 \leq i \leq n} \|\varsigma_i^j\| \leq 1$, therefore, $\|\varsigma_i^j\| \leq 1$ for every $i = 1, 2, 3, ..., n$. Also, since each X_i is (α, n)-generating therefore for each fixed $i = 1, 2, 3, ..., n$ there exists η_i in X_i such that $\eta_i \geq \varsigma_i^j$ for all $j = 1, 2, 3, ..., n$, and $\|\eta_i\| \leq \alpha$. Taking, $x = (\eta_1, \eta_2, ..., \eta_n)$ we have, $x \geq x_j$ for all $j = 1, 2, 3, ..., n$.
Furthermore,

$$\|x\| = \max_{1 \leq i \leq n} \|\eta_i\| \leq \alpha.$$

showing that X is (α, n)-generating.

(4) Let $x_1, x_2, ..., x_n \in X$. Then for each $j = 1, 2, 3, ..., n$, $x_j = (\varsigma_1^j, \varsigma_2^j, ..., \varsigma_n^j)$, where $\varsigma_i^j \in X_i (i = 1, 2, 3, ..., n)$. Since each X_i is (α, n)-additive, therefore, for each fixed $i = 1, 2, 3, ..., n$,

$$\sum_{j=1}^n \|\varsigma_i^j\| \le \alpha \; \|\sum_{j=1}^n \varsigma_i^j\|$$

Now,

$$\sum_{j=1}^n \|x_j\| = \sum_{j=1}^n (\sum_{i=1}^n \left\|\varsigma_i^j\right\|) = \sum_{i=1}^n (\sum_{j=1}^n \left\|\varsigma_i^j\right\|) \le \alpha \sum_{i=1}^n \left\|\sum_{j=1}^n \varsigma_i^j\right\| = \alpha \left\|\sum_{j=1}^n x_j\right\|$$

which shows that X is (α, n)-additive.

□

Definition 13. Let $(X, \|.\|)$ be a normed linear space;

(1) A wedge W in X is said to have the property (G) with constant α if, given x in X there exists y in X such that $-y \le x \le y$ and $\|y\| \le \alpha\|x\|$.
(2) A wedge W in X is said to have the property (N) with constant α if, $-x \le y \le x$ implies that $\|y\| \le \alpha\|x\|$.

Theorem 14. *Let X_i be a preordered normed space with wedge $W_i, i = 1, 2, ..., n$. Let $X = \prod_{i=1}^n X_i$ and $W = \prod_{i=1}^n W_i$, then*

(1) *the wedge W has the property (N) with some constant α if each W_i has the property (N) with constant α_i.*
(2) *the wedge W has the property (G) with some constant α if each W_i has the property (G) with constant α_i.*

Proof. (1) Let $x, y \in X$ be such that $-x \le y \le x$. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ where $x_i, y_i \in X_i$ then $-x_i \le y_i \le x_i$, for every $i = 1, 2, ..., n$. Since each X_i has property (N) with constant α_i, therefore $\|y_i\| \le \alpha_i \|xi\|$. Now since,
$\|y_1\| + \|y_2\| + ... + \|y_n\| \le \alpha_1\|x_1\| + \alpha_2\|x_2\| + ... + \alpha_n\|x_n\|$
therefore, $\|y\| \le \alpha\|x\|$ where $\alpha = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$.

(2) Let $x \in X$ then $x = (x_1, x_2, ..., x_n)$ where $x_i \in X_i$ for every $i = 1, 2, ..., n$. Since each X_i has property (G) with constant α_i, therefore, there exist y_i in X_i such that $-y_i \le x_i \le y_i$, and $\|y_i\| \le \alpha_i\|x_i\|$.

Taking $\alpha = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$, we obtain a point $y = (y_1, y_2, ..., y_n)$ in X such that , $-x \le y \le x$, and $\|y\| \le \alpha\|x\|$.

□

Definition 15. Let X be a preordered linear space with wedge W. A subset D of X is said to be *decomposable* if for each u in D there exist u_1, u_2 in $D \cap W$ such that $u = \alpha_1 u_1 - \alpha_2 u_2$ for $\alpha_1, \alpha_2 \ge 0$ with $\alpha_1 + \alpha_2 = 1$. A wedge W in a normed linear space $(X, \|.\|)$ gives an open *decomposition* of X if given x in X there exist, $\alpha > 0$ and u_1, u_2 in W such that $x = u_1 - u_2$ and $\|u_1\|, \|u_2\| \le \alpha\|x\|$.

A wedge W in a normed linear space $(X, \|.\|)$ gives a *bounded decomposition* of X if it gives an open decomposition of X [9].

Theorem 16. *If the wedge W_i in a preordered normed space X_i gives a bounded decomposition of X_i for every $i = 1, 2, ..., n$, then the wedge $W = \prod_{i=1}^{n} W_i$ in $X = \prod_{i=1}^{n} X_i$ gives a bounded decomposition of X.*

Proof. Let $x \in X$ then $x = (x_1, x_2, ..., x_n)$ where $x_i \in X_i$ for every $i = 1, 2, ..., n$. Since each W_i gives a bounded decomposition of X_i , therefore, there exist $\alpha > 0$ and $w_i^{(1)}, w_i^{(2)}$ in W_i such that $x_i = w_i^{(1)} - w_i^{(2)}$ and $\|w_i^{(1)}\|, \|w_i^{(2)}\| \leq \alpha_i \|x_i\|$. Let $w_1 = (w_1^{(1)}, w_2^{(1)}, ..., w_n^{(1)})$ and $w_2 = (w_1^{(2)}, w_2^{(2)}, ..., w_n^{(2)})$ then $w_1, w_2 \in W$ for which $x = w_1 - w_2$ and $\|w_1\|, \|w_2\| \leq \alpha \|x\|$ where $\alpha = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$. □

Conclusion

Order properties are of much importance when studied with reference to the context. For example, first duality theory in ordered linear spaces is due to α - normal and α-generating wedges whereas second duality theory, which is concerned with order-intervals of the form $[-x, x]$, is due to order property (G) and order property (N) of wedges. A rich theory of ordered linear spaces grows through these order properties. This article discusses various order properties of ordered normed spaces, which are closed under the formation of products. We also investigate how product of two base-normed spaces would be a base-normed space.

Acknowledgment

The authors are thankful to the referee for careful reading of this paper and providing valuable suggestions to improve the presentation of the paper.

References

[1] M. I. Bhatti and M.N. Chaudhary, *Cones in Product Spaces and their Order Properties through Projections*, Pakistan Journal of Scientific Research **54 (1-2)** (2002).

[2] M. I. Bhatti, *Order Convergence in Product Spaces of Ordered Linear Spaces*, Research journal of University of Engineering and Technology, **14 (1-2)** (2003).

[3] M. I. Bhatti, *Order Complete Spaces, Order Separable Spaces and their Products*, Pakistan Journal of Scientific Research, **56 (1-2)** (2004).

[4] M. I. Bhatti, *Inheritance of Order Properties from Product Ordered Normed Spaces to their Component Spaces*, Pakistan Journal of Science, **56 (3-4)** (2004).

[5] F.F. Bonsall, *Extreme Maximal Ideals of a Partially Ordered Vector Space*, proc. Amer. Math. Soc., **7** (1956), 831-837.

[6] R. Cristescu, *Ordered Vector Spaces and Linear Operators*, Editura Academiei Bucuresti, Romania (1976).

[7] J. V. Dalen, *Finite Products of Locally Compact Ordered Spaces*, Ph.D thesis, Vrije Universiteit, Amsterdam, (1972).

[8] S. Hong, J. Y. Hyun, H-K Kim and S-M Kim, *Linear Discrepancy of the Product of Two Chains*, Order 22, 63-72, Kluwer Academic Publishers, 2005.

[9] G. J. O. Jameson, *Ordered Linear Spaces, Lecture Notes in Mathematics*, Springer-Verlag Berlin-Heidelberg New York, 1970.

[10] B. Karim, *On Products in Lattice-ordered Algebras*, J. Aust. Math. Soc. **75** (2003), 23-40.

[11] Martin, G. and S. Shelah, *Antichains in Products of Linear Orders*, Order 19, 213 - 222, Kluwer Academic Publishers, 2002.

[12] A. L. Peressini, *Ordered Topological vector spaces*, Harper and Row, New York, 1967.