Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 39 (2007) pp. 47-53

Alternative Approach To The Persistence In A 3-Species Predator-Pray Modal

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Abstract. The ecological notion of system persistence in modeling the interaction of two competing predatory populations living exclusively on a common prey, is investigated.
Freedom and Waltman [3], and El-Owaidy and Ammar [4] have discussed the persistence of such models based upon the assumption of nonexistence of limit cycles. In this paper, however ; the nonexistence of limit cycles is proven, first, by global asymptotic stability of equilibria through the constriction of a suitable Lyapunov function, and second, persistence criteria of the system are obtained.

AMS (MOS) Subject Classification Codes: 92B05.
Key Words: Predator - prey system, Lyapunov function, limit cycle, asymptotic global stability.

1. Introduction

One of the most interesting questions in ecological models concerns the survival of all species of the model. This question of persistence becomes more important when two or more species compete for a single prey species. The significant concepts of permanence and persistence, both exclude extinction of species for all positive initial conditions. In biological terms, persistence means that the density of each population remains, asymptotically, above a positive bound independent of the initial conditions, i.e. all species stay away from extinction.
Mathematically, this may be stated in terms of behaviour of solutions of the modal which represents the biological phenomenon. An ecological differential system.

$$\dot{x}_i = X_i f(X_1, X_2, ..., X_n); \text{ for } i = 1, 2, ..., n$$

is said to be permanent or uniformly persistent if there exists a compact set k in the interior of $R^n = x \in R^n : x_i \geq 0$ for $1 \leq i \geq n$ such that all orbits end up in k. This guarantees that the number of each population $x_i(t)$ is bounded away from zero if $x_i(0) > 0$ for all i.
By the term weak persistence, we mean that for all i, $limsup\ x_i(t) > 0$ whenever $x(0) > 0$. Under this "lim sup" definition of persistence, a population can frequently approach extinction. By the term strong persistence, we mean that for all $i,\ t \to \infty\ liminf\ x_i(t) > 0$ whenever $x(0) > 0$. We have applied the latter definition of persistence as used in Freedman and Waltman [3]. This definition of

persistence was reformulated in [3] as follows: "A persistence with initial conditions in the positive cone will persist if there are no ω- limit point s of the solution on the boundary of the positive cone". This means if $\Box\ (X)$ be the orbit through the point $X = (x, y, z)$ with $x > 0,\ y > 0,\ z > 0$, and if $\Omega(X)$ be the $\omega - limit$ set of $\Box\ (X)$, then $\Omega(X)$ be interior to the positive cone. Since, the question of two predator populations competing for a single prey population has occupied an important place in ecological literature, so in this paper, we have considered a system modelling the interactions of two competing predator populations living exclusively on a common prey. For both the predator populations we have taken density - dependent death rates and in this sense, we have extended the result of EL-Owaidy and Ammar [4] with an alternative approach.
Freedman and Waltman [3] in theorem 4.1, obtained persistence criteria by an assumption that for such a system, there are no limit cycles surrounding an interior equilibrium in a co-ordinate plane. We have actually proved the non-existence of limit cycles by constructing Lyapunov functions for the equilibria and have obtained conditions under which the equilibria are globally asymptotically stable. This implies that all the trajectories of the system in the positive cone will spirsl toward this equilibrium point, that is non-existence of limit cycles.
Then, we have applied the persistence criteria as given in theorem 4.1 under reference, to obtain conditions for persistence of our system. thus our result of persistence seems to be more comprehensive than that in theorem 4.1 of Freedman and Waltman [3].
The organization of this paper is as follows:
In section 2, we give the model. Section 3 deals with the global asymptotic stability of the problem. Section 4 deals with persistence results and we illustrate these by an example.

2. The Model And Equilibria

A system modelling the interaction of two competing predator populations $y(t)$ and $z(t)$ living exclusively on a common prey $x(t)$ is given by

$$\begin{aligned} \dot{x} &= xf(x) - yp(x) - zq(x) \\ \dot{y} &= y[-g(y) + c_1 p(x)] \\ \dot{z} &= z[-h(z) + d_1 q(x)] \qquad (2.\ 1) \\ X(0) = x_0 > 0, &\quad Y(0) = y_0 \geq 0, \quad Z(0) = z_0 \geq 0. \end{aligned}$$

where $(. = \frac{d}{dt}),\ c_1,\ d_1$ are positive constants, known as food conversion rates. We make the following assumptions, which are consistent with models of predator - prey systems. :
(H1) : $f(x)$ [specific growth rate of prey x]:

$$f(0) > 0, f'(x) \leq 0 \text{ for all } x < 0.$$

There exists $k < 0$ (the carrying capacity) such that $f(x) > 0$ on

$$0 \leq x < k, f(k) = 0 \text{ and } f(x) < 0 \text{ on } x > k.$$

(H2) $p(x)$: the functional response of the predator y with respect to the prey x and

$$p(0) = 0, p'(x) > 0 \text{ for all } x \geq 0.$$

(H3) $q(x)$: the functional response of the predator z with respect to the prey x and

$$q(0) = 0, q'(x) > 0 \text{ for all } x \geq 0.$$

(H4) : we assume $q(x) = p(x)$, where is a positive constant
(H5) : $g(y)$; density - dependent death rate of the predatory and

$$g(y) := g'(y) \geq 0 \text{ for all } y > 0.$$

(H6) : $h(z)$;density -dependent death rate of the predator z and

$$h(0) > 0, h'(z) > 0 \text{ for all } z > 0.$$

Clearly the system (2.1) has equilibrium point $E_0(0, 0, 0)$. By assumption (H1), $E_1(k, 0, 0)$ is also an equilibrium point. We assume that each of the predators y and z can survive on the prey x, that is there exist equilibrium points

$$E^*(x^*, y^*, 0) \text{ and } \hat{E}(\hat{x}, 0, \hat{z}) \text{ such that}$$

$$\begin{aligned} \dot{x}f(\dot{x}) - \dot{y}p(\dot{x}) &= 0 \\ -g(\dot{y}) + c_1 p(\dot{x}) &= 0 \end{aligned} \tag{2. 2}$$

and

$$\begin{aligned} \hat{x}f(\hat{x}g - \hat{z}q(\hat{x}) &= 0 \\ -h(\hat{z}) + d_1 q(\hat{x}) &= 0 \end{aligned} \tag{2. 3}$$

where $x^*, y^*, \hat{x}, \hat{z}, > 0$ and $x^* < k, \hat{x} < k$.

3. Global Asymptotic Stability Of Equilibria

$$E^*(x^*, y^*, 0) \text{ and } E^*(x^*, y^*, 0) \text{ and } \hat{E}(\hat{x}, 0, \hat{z})$$

Lemma 1. *Assume that (H1) -(H6) hold for the system (2.1) and in a neighborhood of* $(x^*, y^*, 0)$ *in the positive cone, the function* $\frac{xf(x)}{p(x)}$ *is strictly decreasing. Then the equilibrium point* $E^*(x^*, y^*, 0)$ *is globally asymptotically stable.*

Proof. Define a Lyapunov function $V(x, y, z)$ as [see [1]]:

$$V(x, y, z) = \int_{x^*}^{x} [c_1(1 - \frac{p(x^*)}{p(w)}) + d_1(1 - \frac{q(x^*)}{q(w)})]dw + \int_{y^*}^{y} \frac{w - y^*}{w} dw + z$$

Now $E^*(x^*, y^*, 0) = 0$ and due to (H2) and (H3), (x, y, z) is positive in the region: $0 < x < x^* < k,\ 0 < y^* < y < \tilde{\beta_1}, 0 < z < \beta_2,$ where $\beta_1,\ \beta_2$ are positive constants.

$$+d_1[q(x) - q(\dot{x})][\frac{xf(x)}{q(x)} - \frac{yp(x)}{q(x)} - z]$$

$$+(y - y^*)[-g(y) + c_1 p(x)] + z[-h(z) + d_1 q(x)]$$

using (2.2), (H4) and with some algebraic manipulations we get

$$\begin{aligned}\dot{V}(x,y,z) &= [c_1(p(x)-p(\dot{x}))+\frac{1}{\alpha}d_1(q(x)-q(\dot{x}))][\frac{xf(x)}{p(x)}-\frac{\dot{x}f(x^*)}{p(x^*)}]\\ &+(y-y^*)[(g(y^*)-g(y))+\frac{d_1}{\alpha}(q(x^*)-q(x))]\\ &+z[\alpha c_1(p(x^*)-p(x))+(h(0)-h(z))]<0\end{aligned}$$

Thus $E^*\left(\ x^*, y^*, 0\ \right)$ is globally asymptotically stable. □

Lemma 2. *Assume that (H1)-(H6) holds for the system(2.1) and in a neighbourhood of* $x(\hat{x},0,\hat{z})$ *in the positive cone, the function* $\frac{xf(x)}{p(x)}$ *is strictly decrictly decreasing. Then the equilibrium point* $\hat{E}(\hat{x},0,\hat{z})$*is globally asymptotically stable.*

Proof. Define Lypunov function (x, y, z) as:

$$V(x,y,z)=\int_x^x [c_1(1-\frac{p(\hat{x})}{p(w)})+d_1(1-\frac{q(\hat{x})}{q(w)})]dw+y+\int_{\hat{z}}^x \frac{w-\hat{z}}{w}dw$$

Rest of the proof follows as in Lemma 1. □

Remark 3. We consider equilibrium points $E_0(0,0,0)$ and $E_1(k,0,0)$. The eigenvalues of the variational matrix (E_0) of the system (2.1) about $E_0(0,0,0)$ are:

$$\lambda_1=f(0)>0, \lambda_2=-g(0)<0, \text{ and } \lambda_3=-h(0)<0.$$

Clearly $E_0(0,0,0)$ is a hyperbolic point and is unstable along the x-axis.This implies that the prey population x grows near E_0. The eigenvalues of the variational matrix (E_1) of the system (2.1) about $E_1(k,0,0)$ are:

$$\lambda_1=kf'(k)<0, \lambda_2=-g(0)+c_1p(k),\ \lambda_3=-h(0)+d_1q(k).$$

Thus $E_1(k,0,0)$ is asymptotically stable along the x-axis.This implies that the prey population x remains in neighbourhood of k.

Remark 4. For existence of $E^*(x^*,y^*,0)$ and $\hat{E}(\hat{x},0,\hat{z})$ it is necessary that

$$g(0)+c_1p(k)>0 \text{ and } -h(0)+d_1q(k)>0$$

As it implies increase of predator population and predator population Z.

4. Persistence Criteria

In section 3, we have given necessary conditions for existence of equilibria $E^*(x^*,y^*,0)$ and $\hat{E}(\hat{x},0,\hat{z})$ and criteria for their global asymptotic stability.
In this section we shall assume global stability of these equilibria and obtain persistence criteria for the system (2.1). First,we prove the following two lemmas.

Lemma 5. *The equilibrium* $E^*(x^*,y^*,0)$ *in the interior of the* $x-y$ *plane is unstable in the positive direction orthogonal to* $x-y$ *plane if*

$$-h(0)+d_1q(x^*)>0 \text{ or } -h(0)+d_1\alpha p(x^*)>0$$

Proof. The proof is immediate upon computing the variational matrix (E) of system (2.1) about $E^*(x^*,y^*,0)$. We have:

$$V(E^*)\begin{pmatrix} f(x^*)+x^*f^1(x^*)-y^*p'(x^*) & -p(x^*) & -q(x^*)\\ -gc_1y^*p'(x^*) & (y^*)+c_1p(x^*)-y^*g(y^*) & 0\\ 0 & 0 & -h(0)+d_1q(x^*)\end{pmatrix}$$

Thus if $-h(0) + d_1 q(x^*) >$ or $-h(0) + \delta_{1\alpha} p(x^*) > 0$ we have the result. □

Lemma 6. *The equilibrium $\hat{E}(\hat{x}, 0, \hat{z})$ in the interior of the $x - z$ plane is unstable in the positive direction orthogonal to $x - z$ plane if $-g(0) + c_1 p(\hat{x}) > 0$.*

Proof. The proof is immediate upon computing the variational matrix $V(E^*)$ of system (2.1) about $\hat{E}(\hat{x}, 0, \hat{z})$. Now, to apply persistence criteria to our system (2.1), we have to check hypotheses $(B1) - (B4)$ of Theorem 4.1, in Freedman and Waltman [3] and boundedness of solutions. We have:

$$\begin{aligned} F(x, y, z) &= f(x) - y\frac{p(x)}{x} - z\frac{q(x)}{x} \\ G_1(x, y, z) &= -g(y) + c_1 p(x) \\ G_2(x, y, z) &= -h(z) + d_1 q(x) \end{aligned}$$

Therefore, condition (B1) is trivially satisfied due to (H1)-(H6). Also notice that $p(0) = 0$ and $p'(x) > 0$ implies $p(x)$ is strictly increasing positive function, similarly $q(x)$. Condition(B2) is true due to (H1).

$$F(0, 0, 0) = f(0) > 0, F(k, 0, 0) = f(k) = 0$$

$$\frac{\partial f}{\partial x}(x, 0, 0) = f'(x) \leq 0,$$

satisfying (B3) there are no equilibria on y-axis or z-axis in $y - z$ plane, for if we suppose that there exists an equilibrium $E(0, y_1, z_1)$ in $y - z$ plane which is given by:

$$g(y_1) = 0 \textit{ and } h(z_1) = 0,$$

then this contradicts (H5) and(H6), and satisfying (B4) each predator can survive on prey, that is, there exist points $E^*(x^*, y^*, 0)$ and $\hat{E}(\hat{x}, 0, \hat{z})$ such that (2.2) and (2.3) hold. Also see Remark 4. Moreover, we require;
5) : Boundedness of solutions of system (2.1).We suppose that the functions $f(x)$, $p(x)$, $q(x)$, $g(y)$ and $h(z)$ are sufficiently smooth so that the solution to initial value problem (2.1) exists, is unique and continuable for positive values of t. Regarding boundedness of solution, see Freedman and Waltman [2] and [3]. □

We state and prove the main theorem.

Theorem 7. *Let (B1)-(B5) hold and $\frac{xf(x)}{p(x)}$ be strictly decreasing function and*

$$-g(0) + c_1 p(x) > 0 \text{ and } -h(0) + d_1 \alpha p(x) > 0.$$

Then the system (2.1) persists.

Proof. By (B5) solutions are bounded. By Remark 3 the equilibrium $E_0(0, 0, 0)$ is unstable along the x-axis and unstable manifold of $E_1(k, 0, 0)$ is two dimensional. Conditions (4.1) follow from Lemmas 5 and 6. Non-existence of limit cycles follows from Lemmas 1 and 2. This completes the proof. □

Remark 8. From (H4), we have

$$\alpha = \frac{q(x)}{p(x)} = \frac{\text{rate of prey consuon per predator } z \text{ at prey density } x}{\text{rate of prey consuon per predator } y \text{ at prey density } x}$$

Thus, if we consider α as parameter, the system (2.1) will persist provided

$$\alpha > \frac{h(0)}{\alpha_1 d_1 p(\dot{x})}$$

Remark 9. We have discussed persistence criteria for a system modeling the interaction of two competing predator populations living exclusively on a common prey. But in the same way, the persistence criteria can be obtained for the system modeling interactions between two prey populations and one predator population, that is, for construction of Lyapunov functions for the systems (4.2) and (4.3) see [1]. System modeling interactions between two prey populations and one predator population, that is,

$$\left.\begin{array}{l} \dot{x} = xf(x) = zq(x) \\ \dot{y} = yg(y) - zr(y) \\ \dot{z} = z[-h(z) + d_1 q(x) + d_2 r(y)] \end{array}\right\} \tag{4. 4}$$

Example 4.1

To illustrate the Theorem 7, consider the system with the Holling type II functional response.

$$\begin{aligned} \dot{x} &= x(1-x) - y\frac{2x}{1+x}z\alpha\frac{2x}{1+x} \\ \dot{y} &= y[-1(1+y) + \frac{33}{16}\frac{2x}{1+x}] \\ \dot{z} &= z[-1(1-z) + \frac{11}{q}\alpha\frac{2x}{1+x}] \end{aligned}$$

Here, $k = 1$, $\frac{xf(x)}{p(x)} = \frac{1-x^2}{2}$. Thus $\frac{xf(x)}{p(x)}$ is a strictly decreasing function for $x > 0$.

$$E^*(x^*, y^*, 0) = E^*(\frac{1}{2}, \frac{3}{8}, 0)$$

$$p(x^*) = p(\frac{1}{2}) = \frac{2}{3}$$

$$\alpha > \frac{h(0)}{d_1 p(\dot{x})} = \frac{27}{22}.$$

Thus take $\alpha = 2$

$$\hat{E}(\hat{x}, 0, \hat{z}) = \hat{E}(\frac{1}{3}, 0, \frac{2}{9})$$

To check condition (4.4)

$$-g(0) + c_1 p(\hat{x}) = -1 + \frac{33}{16} \times \frac{1}{2} = \frac{1}{32} > 0$$

$$-h(0) + d_1 \alpha p(x^*) = -1 + \frac{11}{9} \times 2 \times \frac{2}{3} = -1 + \frac{44}{27} = \frac{17}{27} > 0$$

Theorem 4.1 applies and hence the system (4.4) is persistent.

References

1. H. L. Arora, G. Hamad, and I. H. Elmabruk, *Global stability of three-species predator-prey system*, To appear.
2. H. I. Freedman and P. Waltman, *Mathematical analysis of some three-species food-chain models*, Math. Biosci **33** (1977), 257–276.
3. ______, *Persistence in models of three integrating predator-prey populations*, Math. Biosci **68** (1984), 213–231.
4. H. EL. Owaidy and A. A. Ammar, *Mathematical analysis of a food-web models*, Math. Biosci **80** (1980), 1–15.