

On Stratification Of L -Fuzzy Topological Structures

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Abstract. We introduce the stratifications of an L -fuzzy topogenous, proximity and uniformity. We investigate the relationships between these structures and the stratifications of them.

Keywords and Phrases: Stratified L -fuzzy topogeneous, proximity, uniformity, stratifications

1. INTRODUCTION AND PRELIMINARIES

In [25] Čsaszar introduce the concept of a syntopogenous structure to develop a unified approach to the three main structures of set-theoretic topology: topologies, uniformities and proximities. This enabled him to evolve a theory including the foundations of three classic theories of topological spaces, uniform spaces and proximity spaces. In the case of the fuzzy structures there are at least two notions of fuzzy syntopogenous structures, the first notion worked out in [13, 10] presents a unified approach to the theories of Chang fuzzy topological spaces [4], Hutteon fuzzy uniform spaces [7], Katsaras fuzzy proximity spaces [9, 8] and Artico fuzzy proximity spaces [1]. The second notion worked out in [11, 12] agree very well with Lowen fuzzy topological spaces [18], Lowen-Höhle fuzzy uniform spaces [19] and Artico-Moresco fuzzy proximity spaces [1]. Šostak [26] introduced the notion of (L -)fuzzy topological spaces as a generalization of L -topological spaces (originally called (L -)fuzzy topological spaces by Chang [4]). It is the grade of openness of L -fuzzy set. Badard introduced the concept of smooth structure and gives some rules and shows how such an extension can be realized [2]. In [22], Ramadan introduced the similar definition of Šostak (L -)fuzzy topology [26] under the name "smooth topology". Also, in [21] Badard et al. introduced the concept of smooth preuniform and preproximity spaces.

In this paper, we can obtain the stratified L -fuzzy topogenous (resp. proximity, uniformity) from an L -fuzzy topogenous (resp. proximity, uniformity). It is called stratification. We prove that the L -fuzzy topogenous (resp. proximity) associated to a stratified L -fuzzy uniformity U (resp. topogenous N) also are stratified. Moreover the stratification of the L -fuzzy topology (resp. proximity) associated to U (resp. N) coincide with the L -fuzzy topology (resp. proximity) associated to the stratification of U (resp. N).

Throughout this paper, let X be a nonempty set. Let a complete lattice $L = (L, \leq, \vee, \wedge, ')$ be a completely distributive complete lattice with an order-reversing involution on it, and with a smallest element 0 and largest element 1

($0 \neq 1$) [27]. Let X be a non-empty set L^X denotes the collection of all mappings from X into L . The elements of L^X are called L-fuzzy sets on X . L^X can be made into a fuzzy lattice by including the order and involution from $L = (L, \leq, \vee, \wedge, ')$. We say that the fuzzy points x_α belongs to a fuzzy set λ i.e., $x_\alpha \in \lambda$ iff $\alpha \leq \lambda(x)$, for $\alpha \in L$, $\underline{\alpha}(x) = \alpha$, for all $x \in X$, and the set of all fuzzy points in L^X is denoted by $\text{Pt}(L^X)$. Let Ω_X denote the family of all mapping $a : L^X \rightarrow L^X$ with the following properties:

- (i) $a(\underline{0}) = \underline{0}$, $\lambda \leq a(\lambda)$ for each $\lambda \in L^X$,
- (ii) $a(\bigvee_{i \in \Gamma} \lambda_i) = \bigvee_{i \in \Gamma} a(\lambda_i)$, for $\lambda_i \in L^X$.

For $a \in \Omega_X$, the mapping $a^{-1} \in \Omega_X$ and $a \wedge b : L^X \rightarrow L^X$ are defined by $a^{-1}(\lambda) = \bigwedge \{\mu : a(\mu) \leq \lambda\}$ and $(a \wedge b)(\mu) = \bigwedge \{a(\mu_1) \vee b(\mu_2) : \mu_1 \vee \mu_2 = \mu\}$. Then $(a^{-1})^{-1} = a$ and $a_1 \leq a_2$ iff $(a_1)^{-1} \leq (a_2)^{-1}$. For any $a, b \in \Omega_X$ and $\mu \in L^X$, $(a \circ b)(\mu) = a(b(\mu))$ [16]. Notation and notion not described in this paper are standard and usual.

Proposition 1. [3] For each $\alpha \in L$, define the mapping $\hat{\alpha} : L^X \rightarrow L^X$ by

$$\hat{\alpha}(\lambda) = \begin{cases} \sup(\lambda) & \text{if } \sup(\lambda) \leq \alpha, \\ \underline{1} & \text{otherwise.} \end{cases}$$

for each $\lambda \in L^X$, where $\sup(\lambda) = \bigvee_{x \in X} \lambda(x)$. Then

- (i) $\hat{\alpha} \in \Omega_X$ (ii) $\hat{\alpha} \circ \hat{\alpha} = \hat{\alpha}$ for each $\alpha \in L$.
- (iii) If $\alpha \leq \beta$, then $\hat{\alpha} \geq \hat{\beta}$ for each $\alpha, \beta \in L$. (iv) $a \leq \hat{0}$ for each $a \in \Omega_X$.

Lemma 2. [3] Let $a_\lambda : L^X \rightarrow L^X$ be a mapping define as follows:

$$a_\lambda(\gamma) = \begin{cases} \lambda & \text{if } \gamma \leq \lambda, \\ \underline{1} & \text{otherwise.} \end{cases}$$

Then $a_\lambda \in \Omega_X$ and $a_\lambda \circ a_\lambda = a_\lambda$ for any $\lambda \in L^X$

Definition 3. [6, 20] A mapping $T : L^X \rightarrow L$ is called an L-fuzzy topology on X if it satisfies the following conditions:

- (O₁) $T(\underline{0}) = T(\underline{1}) = 1$,
- (O₂) $T(\lambda \wedge \mu) \geq T(\lambda) \wedge T(\mu)$ for any $\lambda, \mu \in L^X$,
- (O₃) $T(\bigvee_{k \in \Gamma} \lambda_k) \geq \bigwedge_{k \in \Gamma} T(\lambda_k)$ for any $\{\lambda_k\}_{k \in \Gamma} \subset L^X$.

The pair (X, T) is called an L-fuzzy topological space (L-fts, for short). An L-fts (X, T) is called stratified if $T(\underline{\alpha}) = 1$ for each $\alpha \in L$ [?].

Let (X, T) and (Y, η) be L-fts's. A mapping $f : (X, T) \rightarrow (Y, \eta)$ is called L-fuzzy continuous if $T(f^{-1}(\mu)) \geq \eta(\mu)$ for all $\mu \in L^Y$. Let T_1 and T_2 be L-fuzzy topologies on X . We say that T_1 is finer than T_2 (T_2 is coarser than T_1) if $T_2(\lambda) \leq T_1(\lambda)$, for all $\lambda \in L^X$.

Theorem 4. [15] Let (X, T) be L-fts. Define the mapping $T_s : L^X \rightarrow L$ as follows: for each $\lambda \in L^X$,

$$T^{st}(\lambda) = \bigvee_{\{(\lambda_k, \underline{\alpha}_k) \mid k \in \Gamma\} \in N(\lambda)} \bigwedge_{(\lambda_i, \underline{\alpha}_i) \in \{(\lambda_k, \underline{\alpha}_k) \mid k \in \Gamma\}} T(\lambda_i),$$

where $N(\lambda) = \{ \{(\lambda_k, \underline{\alpha}_k) \mid k \in \Gamma\} \mid \lambda = \bigvee_{k \in \Gamma} (\lambda_k \wedge \underline{\alpha}_k) \}$. Then T_s the coarsest stratified L-fuzzy topology on X which is finer than T . And T_s is called the stratification of an L-fuzzy topology T on X .

Definition 5. [5] A mapping $\delta : L^X \times L^X \longrightarrow L$ is said to be L-fuzzy quasi-proximity on X , which satisfied the following conditions:

(FP1) $\delta(\underline{0}, \underline{1}) = 0$,

(FP2) $\delta(\lambda \vee \rho, \mu) = \delta(\lambda, \mu) \vee \delta(\rho, \mu)$ and $\delta(\lambda, \mu \vee \nu) = \delta(\lambda, \mu) \vee \delta(\lambda, \nu)$.

(FP3) For any $\lambda, \mu \in L^X$, there exist $\rho \in L^X$ such that

$$\delta(\lambda, \mu) \geq \bigwedge_{\rho \in L^X} (\delta(\lambda, \rho) \vee \delta(\rho, \mu)).$$

(FP4) $\delta(\lambda, \mu) \neq 1$, then $\lambda \leq \mu$.

The pair (X, δ) is called L-fuzzy quasi-proximity space. An L-fuzzy quasi-proximity space (X, δ) is called L-fuzzy proximity space if (FP) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$ for any $\mu, \lambda \in L^X$. Let (X, δ_1) and (X, δ_2) be L-fuzzy proximity space. A mapping $f : (X, \delta_1) \longrightarrow (Y, \delta_2)$ is L-fuzzy proximity continuous if satisfies $\delta_1(\lambda, \mu) \leq \delta_2(f(\lambda), f(\mu))$, for any $\lambda, \mu \in L^X$.

Theorem 6. [17] Let $\{(X_i, \delta_i)\}_{i \in \Gamma}$ be a family of L-fuzzy quasi-proximity spaces, $X = \prod_{i \in \Gamma} X_i$ a product set and, for each $i \in \Gamma$, $\pi_i : X \longrightarrow X_i$ a projection mapping. Define the mapping $\prod_{i \in \Gamma} \delta_i : L^X \times L^X \longrightarrow L$ on X by for any $\lambda, \mu \in L^X$,

$$\prod_{i \in \Gamma} \delta_i(\lambda, \mu) = \bigwedge_{j, k \in \Gamma} \left\{ \bigvee_{i \in \Gamma} \delta_i(\pi_i(\lambda_j), \pi_i(\mu_k)) \right\}$$

Where \bigwedge is taken over all finite families $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ and $\{\mu_k : \mu = \bigvee_k \mu_k\}$. Then $\prod_{i \in \Gamma} \delta_i$ is the coarsest L-fuzzy quasi-proximity on X which all π_i is L-fuzzy quasi-proximity continuous. $\prod_{i \in \Gamma} \delta_i$ is called the product L-fuzzy quasi-proximity structure with respect to a family $\{\pi_i : X \longrightarrow X_i : i \in \Gamma\}$.

Definition 7. [23] A mapping $N : L^X \times L^X \longrightarrow L$ is said to be L-fuzzy topogenous order on X . Which satisfied the following conditions:

(FN1) $N(\underline{1}, \underline{1}) = N(\underline{0}, \underline{0}) = 1$,

(FN2) If $N(\lambda, \mu) \neq 0$, then $\lambda \leq \mu$,

(FN3) If $\lambda \leq \lambda_1, \mu_1 \leq \mu$, then $N(\lambda_1, \mu_1) \leq N(\lambda, \mu)$,

(FN4) (i) $N(\lambda_1, \mu_1) \wedge N(\lambda_2, \mu_2) \leq N(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2)$,

(ii) $N(\lambda_1, \mu_1) \wedge N(\lambda_2, \mu_2) \leq N(\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2)$.

The pair (X, N) is called L-fuzzy topogenous order space. The L-fuzzy topogenous order N is called

(i) Symmetrical iff $N(\lambda, \mu) = N(\mu, \lambda)$.

(ii) Perfect iff $N(\bigvee_{i \in \Gamma} \lambda_i, \bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} N(\lambda_i, \mu_i)$.

Let (X, N_1) and (Y, N_2) be L-fuzzy topogenous order spaces. A mapping $f : (X, N_1) \longrightarrow (Y, N_2)$ is L-fuzzy topogenous continuous if satisfies $N_1(f^{-1}(\lambda), f^{-1}(\mu)) \geq N_2(\lambda, \mu)$, for any $\lambda, \mu \in L^Y$.

Theorem 8. [24] (i) Let N be a perfect L-fuzzy topogenous structure on X . Define a mapping $T_N : L^X \longrightarrow L$ by $T_N(\lambda) = N(\lambda, \lambda)$. Then T_N is L-fuzzy topology on X .

(ii) Let T be an L-fuzzy topology on X . Define a mapping $N_T : L^X \times L^X \longrightarrow L$ by

$$N_T(\lambda, \mu) = \bigvee \{T(\nu) : \lambda \leq \nu \leq \mu\}.$$

Then N_T is perfect L -fuzzy topogenous order on X .

(iii) Let N be a symmetrical L -fuzzy topogenous on X . Define a mapping $\delta_N : L^X \times L^X \rightarrow L$ by $\delta_N(\lambda, \mu) = (N(\lambda, \mu))$. Then δ_N is L -fuzzy proximity on X .

(iv) Let δ be an L -fuzzy proximity on X . Define a mapping $N_\delta : L^X \times L^X \rightarrow L$ by $N_\delta(\lambda, \mu) = (\delta(\lambda, \mu))$. Then N_δ is a symmetrical L -fuzzy topogenous on X .

Theorem 9. [24] Let $\{(X_i, N_i)\}_{i \in \Gamma}$ be a family of L -fuzzy topogenous order spaces, $X = \prod_{i \in \Gamma} X_i$ a product set and, for each $i \in \Gamma$, $\pi_i : X \rightarrow X_i$ a projection mapping. Define the mapping $\prod_{i \in \Gamma} N_i : L^X \times L^X \rightarrow L$ on X by for any $\mu, \lambda \in L^X$,

$$\prod_{i \in \Gamma} N_i(\lambda, \mu) = \bigvee \left\{ \bigwedge_{j, k \in \Gamma} N_i(\pi_i(\lambda_j), (\pi_i(\mu_k))) \right\},$$

Where \bigvee is taken over all finite families $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ and $\{\mu_k : \mu = \bigwedge_k \mu_k\}$. Then $\prod_{i \in \Gamma} N_i$ is the coarsest L -fuzzy topogenous on X which all π_i is L -fuzzy topogenous continuous. $\prod_{i \in \Gamma} N_i$ is called the product L -fuzzy topogenous structure with respect to a family $\{\pi_i : X \rightarrow X_i : i \in \Gamma\}$.

Definition 10. [16] A mapping $U : \Omega_X \rightarrow L$ is said to be an L -fuzzy quasi-uniformity on X if it satisfied the following conditions:

- (FU1) for $a, b \in \Omega_X$, $U(a \wedge b) \geq U(a \wedge b)$,
- (FU2) for $a \in \Omega_X$, there exists $a_1 \in \Omega_X$ with $a_1 \circ a_1 \leq a$ such that $U(a_1) \geq U(a)$,
- (FU3) If $a \geq a_1$ such that $U(a) \geq U(a_1)$,
- (FU4) there exists $a \in \Omega_X$ such that $U(a) = 1$.

The pair (X, U) is said to be an L -fuzzy quasi-uniform space. An L -fuzzy quasi-uniform space (X, U) is called L -fuzzy uniform space if the following is satisfies:

(FU) for $a \in \Omega_X$, there exists $a_1 \in \Omega_X$ with $a_1 \leq a^{-1}$ such that $U(a_1) \geq U(a)$.

Theorem 11. [14] Let (X, U) be an L -fuzzy uniform space. Define $\delta_U(\lambda, \mu) = \bigwedge \{U(a) : a(\mu) \leq \lambda\}$. Then (X, δ_U) is L -fuzzy proximity space.

Theorem 12. [16] Let (X, U) be an L -fuzzy quasi-uniform space. For each $\alpha \in L$, $\lambda \in L^X$, $C_U(\lambda, \alpha) = \bigwedge \{a^{-1}(\lambda) : U(a) > \alpha\}$ and $I_U(\lambda, \alpha) = \bigvee \{\mu : a(\mu) \leq \lambda \text{ for some } U(a) > \alpha\}$. Then $I_U(\lambda, \alpha) = (C_U(\lambda, \alpha))$. for any $\lambda \in L^X$.

Consider the mapping $T_U : L^X \rightarrow L$ define by

$$T_U(\lambda) = \bigvee \{\alpha : I_U(\lambda, \alpha) = \lambda\}.$$

Then T_U is an L -fuzzy topology on X .

2. Stratifications of L -fuzzy topogenous order spaces

An L -fuzzy topogenous order $N : L^X \times L^X \rightarrow L$ is said to be stratified L -fuzzy topogenous order on X , iff N satisfied the following conditions: $(FNS) N(\underline{\alpha}, \underline{\alpha}) = 1$, for each $\alpha \in L$. And the pair (X, N) is called stratified L -fuzzy topogenous order space.

Theorem 13. Let (X, N) be an L-fuzzy topogenous order space. We define for all $\mu, \lambda \in L^X$,

$$N^s(\lambda, \mu) = \bigvee_{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N \in M(\lambda, \mu)} \left\{ \bigwedge_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\}} N(\lambda_i, \mu_i) \right\},$$

where

$$M(\lambda, \mu) = \left\{ \left\{ (\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N, N \text{ finite index set} \right\} \mid \lambda \leq \bigvee_{i \in N} (\lambda_i \wedge \underline{\alpha}_i), \right. \\ \left. \mu \geq \bigvee_{i \in N} (\mu_i \wedge \underline{\alpha}_i) \right\}.$$

Then N^s is the coarsest stratified L-fuzzy topogenous order on X which finer than N .

Proof. (FN1), (FN2) and (FN3) Obvious.

(FN4) (i) Suppose that there exist $\mu_1, \mu_2, \lambda_1, \lambda_2 \in L^X$, such that

$$N^s(\lambda_1, \mu_1) \wedge N^s(\lambda_2, \mu_2) \not\leq N^s(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2).$$

By the definition of N^s , there are $\{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\} \in M(\lambda_1, \mu_1)$ and $\{(\lambda_j, \mu_j, \underline{\beta}_j) | j \in K\} \in M(\lambda_2, \mu_2)$ such that

$$N^s(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2) \not\leq \left(\bigwedge_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\}} N(\lambda_i, \mu_i) \right) \\ \wedge \left(\bigwedge_{(\lambda_n, \mu_n, \underline{\alpha}_n) \in \{(\lambda_j, \mu_j, \underline{\beta}_j) | j \in K\}} N(\lambda_n, \mu_n) \right) \\ = \bigwedge_{(\lambda_m, \mu_m, \underline{\alpha}_m) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i), (\lambda_j, \mu_j, \underline{\beta}_j) | i \in N, j \in K\}} N(\lambda_m, \mu_m).$$

Since $\{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\} \in M(\lambda_1, \mu_1)$ and $\{(\lambda_j, \mu_j, \underline{\beta}_j) | j \in K\} \in M(\lambda_2, \mu_2)$, then we have

$$\{(\lambda_i, \mu_i, \underline{\alpha}_i), (\lambda_j, \mu_j, \underline{\beta}_j) | i \in N, j \in K\} \in M(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2).$$

By the definition of N^s we have

$$N^s(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2) \geq \bigwedge_{(\lambda_m, \mu_m, \underline{\alpha}_m) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i), (\lambda_j, \mu_j, \underline{\beta}_j) | i \in N, j \in K\}} N(\lambda_m, \mu_m).$$

It is a contradiction. Thus $N^s(\lambda_1, \mu_1) \wedge N^s(\lambda_2, \mu_2) \leq N^s(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2)$.

(ii) Similar the proof of (i)

(FNS) since $\underline{\alpha} = \underline{1} \wedge \underline{\alpha}$, then $N^s(\underline{\alpha}, \underline{\alpha}) \geq N(\underline{1}, \underline{1}) = 1$. Thus $N^s(\underline{\alpha}, \underline{\alpha}) = 1$, for each $\underline{\alpha} \in L$.

Second since $\lambda \leq \lambda \wedge \underline{1}, \mu \geq \mu \wedge \underline{1}$, then $N^s(\lambda, \mu) \geq N(\lambda, \mu)$ for any $\mu, \lambda \in L^X$. Thus N^s is the stratified L-fuzzy topogenous order on X which finer than N .

Finally, consider N^s is stratified finer than N , then $N^*(\lambda, \mu) \geq N(\lambda, \mu)$ for any $\mu, \lambda \in L^X$.

We will show $N^*(\lambda, \mu) \geq N^s(\lambda, \mu)$ for any $\mu, \lambda \in L^X$. Suppose there exist $\mu, \lambda \in L^X$ such that

$$N^*(\lambda, \mu) \not\geq N^s(\lambda, \mu).$$

By the definition of N^s , then there exists $\{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\} \in M(\lambda, \mu)$ such that

$$N^*(\lambda, \mu) \not\leq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\}} N(\lambda_l, \mu_l).$$

On the other hand, we have

$$\begin{aligned} N^*(\lambda, \mu) &\geq N^*\left(\bigvee_{i \in N} (\lambda_i \wedge \underline{\alpha}_i), \bigvee_{i \in N} (\mu_i \wedge \underline{\alpha}_i)\right) \\ &\geq \bigwedge_{i \in N} N^*((\lambda_i \wedge \underline{\alpha}_i), (\mu_i \wedge \underline{\alpha}_i)) \\ &\geq \bigwedge_{i \in N} (N^*(\lambda_i, \mu_i) \wedge N^*(\underline{\alpha}_i, \underline{\alpha}_i)) \\ &\geq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\}} N(\lambda_l, \mu_l). \end{aligned}$$

It is a contradiction. Thus N^s is the coarsest stratified L-fuzzy topogenous order on X which finer than N . \square

Example. Let $X = \{x, y\}$, be set, $L = [0, 1]$. Let $\nu, \gamma \in L^X, \nu(x) = 0.5, \nu(y) = 0.5$ and $\gamma(x) = 0.3, \gamma(y) = 0.7$. Then we define the L-fuzzy topogenous order N on X as follows:

for each $\lambda, \mu \in L^X$,

$$N(\lambda, \mu) = \begin{cases} 1 & : \lambda = \underline{0} \text{ or } \mu = \underline{1}, \\ 0.3 & : \lambda \leq \nu \leq \mu, \mu \not\leq \gamma, \\ 0.5 & : \lambda \leq \gamma \leq \mu, \lambda \not\leq \nu, \\ & : \lambda \leq \nu \vee \gamma \leq \mu, \\ 0.7 & : \lambda \leq \nu \vee \gamma \leq \mu, \\ & : \mu \not\leq \gamma, \lambda \not\leq \nu, \\ 0.6 & : \lambda \leq \nu \wedge \gamma \leq \mu, , \\ 0 & : O.w. \end{cases}$$

If $\lambda(x) \leq \alpha \leq \mu(x)$ for $0.5 < \alpha < 0.7$ and $\lambda(y) \leq \beta \leq \mu(y)$ for each $\beta \geq 0.7$, since $\lambda \leq (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge (\nu \vee \gamma)) = (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge \gamma) \leq \mu$, we have $N^s(\lambda, \mu) = (N(\underline{1}, \underline{1}) \wedge N(\nu \vee \gamma, \nu \vee \gamma)) \vee N(\underline{1}, \underline{1}) \wedge N(\gamma, \gamma) = 0.7$.

If $\lambda(x) \leq \alpha \leq \mu(x)$ for $0.5 < \alpha < 0.7$ and $\lambda(y) \leq \beta \leq \mu(y)$ for $0.5 < \alpha, \beta < 0.7$ and $\alpha < \beta$, since $\lambda \leq (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge (\nu \vee \gamma)) = (\underline{\alpha} \wedge \underline{1}) \vee (\underline{\beta} \wedge \gamma) \leq \mu$, we have $N^s(\lambda, \mu) = 0.7$. By a similar method as the above cases, we can obtain the stratification N^s as following:

$$N^s(\lambda, \mu) = \begin{cases} 1 & : \lambda = \underline{\alpha} \text{ or } \mu = \underline{\alpha}, \\ 0.7 & : \lambda(x) \leq \alpha \leq \mu(x), \lambda(y) \leq \beta \leq \mu(y), \\ & : \text{for } 0.5 < \alpha, \beta < 0.7 \text{ and } \alpha < \beta, \\ 0.5 & : \lambda(x) \leq \alpha \leq \mu(x), \lambda(y) \leq \beta \leq \mu(y), \\ & : \text{for } 0.3 \leq \alpha < 0.5, 0.5 < \beta \leq 0.7, \\ 0.6 & : \lambda(x) \leq \alpha \leq \mu(x), \lambda(y) \leq \beta \leq \mu(y), \\ & : \text{for } 0.3 \leq \alpha, \beta \leq 0.5 \text{ and } \alpha < \beta, \\ 0 & : O.w. \end{cases}$$

Theorem 14. (i) If N be an L-fuzzy topogenous order structure which is symmetrical, then the stratification N^s of N also is symmetrical.

(ii) If N be a perfect L-fuzzy topogenous order structure. T_N the L-fuzzy topology associated to N , then $T_{N^s} = (T_N)^s$.

(iii) Let T be an L-fuzzy topology, N_T the L-fuzzy topogenous associated to T . Then $N_{T^s} = (N_T)^s$ iff $(N_T)^s$ is perfect L-fuzzy topogenous order.

Proof. (i) Left to leader.

(ii) Since $T_N^s(\underline{\alpha}) = N^s(\underline{\alpha}, \underline{\alpha}) = 1$, for each $\alpha \in L$, then T_N^s is stratified finer than T_N , so $T_N^s \geq (T_N)^s$.

Conversely. suppose there exists $\lambda \in L^X$ such that

$$T_{N^s}(\lambda) = N^s(\lambda, \lambda) \not\leq (T_N)^s(\lambda).$$

By the definition of N^s , there exists a family $\{(\lambda_i, \lambda_i, \underline{\alpha}_i) | i \in N\} \in M(\lambda, \lambda)$ such that

$$\begin{aligned} (T_N)^s(\lambda) &\not\leq \bigwedge_{(\lambda_i, \lambda_i, \underline{\alpha}_i) \in \{(\lambda_i, \lambda_i, \underline{\alpha}_i) | i \in N\}} N(\lambda_i, \lambda_i) \\ &= \bigwedge_{(\lambda_i, \lambda_i, \underline{\alpha}_i) \in \{(\lambda_i, \lambda_i, \underline{\alpha}_i) | i \in N\}} T_N(\lambda_i) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (T_N)^s(\lambda) &= (T_N)^s \left(\bigvee_{i \in N} (\lambda_i \wedge \underline{\alpha}_i) \right) \\ &\geq \bigwedge_{i \in N} (T_N)^s(\lambda_i \wedge \underline{\alpha}_i) \\ &\geq \bigwedge_{i \in N} ((T_N)^s(\lambda_i) \wedge (T_N)^s(\underline{\alpha}_i)) \\ &\geq \bigwedge_{(\lambda_i, \lambda_i, \underline{\alpha}_i) \in \{(\lambda_i, \lambda_i, \underline{\alpha}_i) | i \in N\}} T_N(\lambda_i). \end{aligned}$$

It is a contradiction. Thus $T_{N^s} = (T_N)^s$.

(iii) (\Leftarrow) Suppose $(N_T)^s$ is perfect L-fuzzy topogenous order. We will show $N_{T^s} = (N_T)^s$. Since $N_{T^s}(\underline{\alpha}, \underline{\alpha}) \geq T^s(\underline{\alpha}) = 1$, for each $\alpha \in L$, then N_{T^s} is stratified finer than N_T , so $N_{T^s} \geq (N_T)^s$.

Conversely, Suppose there exist $\mu, \lambda \in L^X$ such that

$$N_{T^s}(\lambda, \mu) \leq (N_T)^s(\lambda, \mu).$$

By the definition of T^s , there exists a family $\{(\nu_i, \underline{\alpha}_i) | i \in \Gamma\} \in N(\nu)$, where $\lambda \leq \nu \leq \mu$ such that

$$\bigwedge_{(\nu_i, \underline{\alpha}_i) \in \{(\nu_i, \underline{\alpha}_i) | i \in \Gamma\}} T(\nu_i) \not\leq (N_T)^s(\lambda, \mu).$$

Since $N_T(\nu_i, \nu_i) \geq T(\nu_i)$, for each $i \in \Gamma$, then we have

$$\bigwedge_{(\nu_i, \underline{\alpha}_i) \in \{(\nu_i, \underline{\alpha}_i) | i \in \Gamma\}} N_T(\nu_i, \nu_i) \not\leq (N_T)^s(\lambda, \mu).$$

On the other hand, since $(N_T)^s$ is perfect L-fuzzy topogenous, then we have

$$\begin{aligned}
(N_T)^s(\lambda, \mu) &\geq (N_T)^s(\nu, \nu) \\
&= (N_T)^s\left(\bigvee_{i \in \Gamma} (\nu_i \wedge \underline{\alpha}_i), \bigvee_{i \in \Gamma} (\nu_i \wedge \underline{\alpha}_i)\right) \\
&\geq \bigwedge_{i \in \Gamma} (N_T)^s(\nu_i \wedge \underline{\alpha}_i, \nu_i \wedge \underline{\alpha}_i) \\
&\geq \bigwedge_{i \in \Gamma} ((N_T)^s(\nu_i, \nu_i) \wedge (N_T)^s(\underline{\alpha}_i, \underline{\alpha}_i)) \\
&\geq \bigwedge_{(\nu_i, \underline{\alpha}_i) \in \{(\nu_i, \underline{\alpha}_i) | i \in \Gamma\}} N_T(\nu_i, \nu_i).
\end{aligned}$$

It is a contradiction. Thus $N_{T^s} = (N_T)^s$.

(\Rightarrow) Obvious from Theorem 8(ii). \square

Theorem 15. Let (X, N_1) and (Y, N_2) be L-fuzzy topogenous order spaces and N_1^s and N_2^s be stratification for N_1 and N_2 respectively. If $f : (X, N_1) \rightarrow (Y, N_2)$ is L-fuzzy topogenous continuous, then $f : (X, N_1^s) \rightarrow (Y, N_2^s)$ is L-fuzzy topogenous continuous.

Proof. Suppose there exist $\lambda, \mu \in L^Y$ such that

$$N_1^s(f^{-1}(\lambda), f^{-1}(\mu)) \not\geq N_2^s(\lambda, \mu).$$

From the definition of N_2^s , there exists a family $\{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\} \in M(\lambda, \mu)$ such that

$$N_1^s(f^{-1}(\lambda), f^{-1}(\mu)) \not\geq \bigwedge_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\}} N_2(\lambda_i, \mu_i).$$

Since $f : (X, N_1) \rightarrow (Y, N_2)$ is an L-fuzzy topogenous continuous, then $N_1(f^{-1}(\lambda_i), f^{-1}(\mu_i)) \geq N_2(\lambda_i, \mu_i)$, for each $i \in N$. On the other hand, we have a family $\{(f^{-1}(\lambda_i), f^{-1}(\mu_i), \underline{\alpha}_i) | i \in N\} \in M(f^{-1}(\lambda), f^{-1}(\mu))$, by the definition of N_1^s , we have

$$\begin{aligned}
N_1^s(f^{-1}(\lambda), f^{-1}(\mu)) &\geq \bigwedge_{(\lambda_i^*, \mu_i^*, \underline{\alpha}_i) \in \{(f^{-1}(\lambda_i), f^{-1}(\mu_i), \underline{\alpha}_i) | i \in N\}} N_1(\lambda_i^*, \mu_i^*) \\
&\geq \bigwedge_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) | i \in N\}} N_2(\lambda_i, \mu_i).
\end{aligned}$$

It is a contradiction. Thus $f : (X, N_1^s) \rightarrow (Y, N_2^s)$ is L-fuzzy topogenous continuous. \square

Counterexample. Let X be any set, $L = [0, 1]$ and identity mapping $Id_X : (X, N_1) \rightarrow (X, N_2)$ define the L-fuzzy topogenous N_1 and N_2 on X as follows:

$$\begin{aligned}
N_1(\lambda, \mu) &= \begin{cases} 1 : \lambda = \underline{0} \text{ or } \mu = \underline{1}, \\ 0 : O.w \end{cases} \\
N_2(\lambda, \mu) &= \begin{cases} 1 : \lambda = \underline{0} \text{ or } \mu = \underline{1}, \\ \frac{1}{3} : \underline{0} \neq \lambda \leq \underline{0.5} \leq \mu \neq \underline{1}, \\ 0 : O.w \end{cases}
\end{aligned}$$

From Theorem we obtain

$$N_1^s(\lambda, \mu) = N_2^s(\lambda, \mu) = \begin{cases} 1 : \lambda \leq \underline{\alpha} \leq \mu \text{ for each } \alpha \in L, \\ 0 : O.w \end{cases}$$

Clearly $Id_X : (X, N_1^s) \rightarrow (X, N_2^s)$ is L-fuzzy topogenous continuous. But $Id_X : (X, N_1) \rightarrow (X, N_2)$ is not L-fuzzy topogenous continuous, where $0 = N_1(\underline{0.5}, \underline{0.5}) < N_2(\underline{0.5}, \underline{0.5}) = \frac{1}{3}$.

Theorem 16. Let $\{(X_i, N_i)\}_{i \in \Gamma}$ be a family of L-fuzzy topogenous order spaces, $X = \Pi_{i \in \Gamma} X_i$ a set and, for each $i \in \Gamma$, $\pi_i : X \rightarrow X_i$ a projection mapping and N is the product L-fuzzy topogenous order structure with respect to a family $\{\pi_i : X \rightarrow X_i : i \in \Gamma\}$. If there exists a $j_0 \in \Gamma$ such that $\overline{N}_{j_0} = N_{j_0}^s$ and $\overline{N}_i = N_i$ for each $i \in \Gamma - \{j_0\}$, then:

(i) $\Pi_{i \in \Gamma} \overline{N}_i$ is the stratified L-fuzzy topogenous structure on X which is finer than N .

(ii) $\Pi_{i \in \Gamma} \overline{N}_i = N^s$.

Proof. (i) By Theorem 9, $\Pi_{i \in \Gamma} \overline{N}_i$ is the L-fuzzy topogenous on X which all π_i is L-fuzzy topogenous continuous. Since $\Pi_{i \in \Gamma} \overline{N}_i(\underline{\alpha}, \underline{\alpha}) = 1$ for each $\alpha \in L$ (Indeed, by the definition of $\Pi_{i \in \Gamma} \overline{N}_i$, there exists a family $\{\underline{\alpha}\}$,

$$\begin{aligned} \Pi_{i \in \Gamma} \overline{N}_i(\underline{\alpha}, \underline{\alpha}) &\geq \bigvee_{i \in \Gamma} \overline{N}_i(\pi_i(\underline{\alpha}), 1 - \pi_i(\underline{1} - \underline{\alpha})) \\ &= \bigvee_{i \in \Gamma} \overline{N}_i(\underline{\alpha}, \underline{\alpha}) \\ &\geq \overline{N}_{j_0}(\underline{\alpha}, \underline{\alpha}) = N_{j_0}^s(\underline{\alpha}, \underline{\alpha}) = 1, \end{aligned}$$

then $\Pi_{i \in \Gamma} \overline{N}_i$ is the stratified. But $N_{j_0}^s$ finer than N_{j_0} . Thus $\Pi_{i \in \Gamma} \overline{N}_i$ is the stratified L-fuzzy topogenous order on X which is finer than N .

(ii) Since $\Pi_{i \in \Gamma} \overline{N}_i$ is a stratified, then for any $\mu, \lambda \in L^X$, $N^s(\lambda, \mu) \leq \Pi_{i \in \Gamma} \overline{N}_i(\lambda, \mu)$. Conversely, suppose there exist $\mu, \lambda \in L^X$ such that

$$N^s(\lambda, \mu) \not\leq \Pi_{i \in \Gamma} \overline{N}_i(\lambda, \mu) \quad (2. 1)$$

From definition of $\Pi_{i \in \Gamma} \overline{N}_i$, there exist finite families $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ and $\{\mu_k : \mu = \bigwedge_k \mu_k\}$.

$$N^s(\lambda, \mu) \not\leq \bigwedge_{j,k} \bigvee_{i \in \Gamma} \overline{N}_i(\pi_i(\lambda_j), (\pi_i(\mu_k))).$$

Then there exists $n \in \Gamma$ such that

$$N^s(\lambda, \mu) \not\leq \overline{N}_n(\pi_n(\lambda), (\pi_n(\mu))).$$

On the other hand, if $n \neq j_0$, then by definition of N^s we have

$$\begin{aligned} N^s(\lambda, \mu) \geq N(\lambda, \mu) &\geq \bigwedge_{j,k} \bigvee_{i \in \Gamma} N_i(\pi_i(\lambda_j), (\pi_i(\mu_k))) \\ &= \bigwedge_{j,k} \bigvee_{i \in \Gamma} \overline{N}_i(\pi_i(\lambda_j), (\pi_i(\mu_k))) \\ &\geq \overline{N}_n(\pi_n(\lambda), (\pi_n(\mu))). \end{aligned}$$

It is a contradiction for eq. 2. 1 . If $n = j_0$, $\overline{N}_n = N_n^s$. By the definition of N_n^s , then there exists a family $\{(\lambda_k^*, \mu_k^*, \underline{\alpha}_k) \mid k \in N\} \in M(\pi_n(\lambda), (\pi_n(\mu)))$ such that

$$N^s(\lambda, \mu) \not\geq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\lambda_k^*, \mu_k^*, \underline{\alpha}_k) \mid k \in N\}} N_n(\lambda_l, \mu_l)$$

Since π_n is L-fuzzy topogenous continuous, $N(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*)) \geq N_n(\lambda_k^*, \mu_k^*)$, for each $k \in N$,

$$N^s(\lambda, \mu) \not\geq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*), \underline{\alpha}_k) \mid k \in N\}} N(\lambda_l, \mu_l)$$

On the other hand, since $\{(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*), \underline{\alpha}_k) \mid k \in N\} \in M(\lambda, \mu)$. By the definition of N^s we have

$$N^s(\lambda, \mu) \geq \bigwedge_{(\lambda_l, \mu_l, \underline{\alpha}_l) \in \{(\pi_n^{-1}(\lambda_k^*), \pi_n^{-1}(\mu_k^*), \underline{\alpha}_k) \mid k \in N\}} N(\lambda_l, \mu_l)$$

Also, it is a contradiction for eq. 2. 1 . Thus $N^s(\lambda, \mu) \geq \prod_{i \in \Gamma} \overline{N}_i(\lambda, \mu)$. \square

Corollary 17. Let (X, N_1) and (X, N_2) be L-fuzzy topogenous order spaces. We define, for any $\mu, \lambda \in L^X$

$$(N_1 \odot N_2)(\lambda, \mu) = \bigvee \bigwedge_{j,k} \{N_1(\lambda_j, \mu_k) \wedge N_2(\lambda_j, \mu_k)\}$$

where \bigvee is taken over all finite families $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ and $\{\mu_k : \mu = \bigwedge_k \mu_k\}$. Then

- (i) $N_1^s \odot N_2$ and $N_1^s \odot N_2^s$ are stratified L-fuzzy topogenous on X , which finer than $N_1 \odot N_2$.
- (ii) $N_1^s \odot N_2 = N_1^s \odot N_2^s = (N_1^s \odot N_2)^s$.

Example.

Let $X = \{x, y, z\}$, $Y = \{p, q\}$ be sets, $L = [0, 1]$ and a mapping $f : X \rightarrow Y$ defined by $f(x) = p, f(y) = f(z) = q$. Let $\nu \in L^X$, $\nu(x) = 0.2, \nu(y) = 0.4, \nu(z) = 0.5$. Then we define the L-fuzzy topogenous order N on X as follows: for each $\lambda, \mu \in L^X$.

$$N(\lambda, \mu) = \begin{cases} 1 : \lambda = \underline{0} \text{ or } \mu = \underline{1}, \\ \frac{1}{2} : \underline{0} \neq \lambda \leq \nu \leq \mu \neq \underline{1}, \\ 0 : O.w \end{cases}$$

Let $f(N)$ is the L-fuzzy topogenous order on Y , define by $f(N)(\gamma, \nu) = N(f^{-1}(\gamma), f^{-1}(\nu))$ for any $\gamma, \nu \in L^Y$. Then we obtain

$$f(N)(\gamma, \theta) = \begin{cases} 1 : \gamma = \underline{0} \text{ or } \theta = \underline{1}, \\ 0 : O.w \end{cases}$$

and from Theorem 2. we obtain

$$(f(N))^s(\gamma, \theta) = \begin{cases} 1 : \gamma \leq \underline{\alpha} \leq \theta = \underline{1} \text{ for each } \alpha \in L, \\ 0 : O.w \end{cases}$$

On the other hand, we have

$$(N)^s(\lambda, \mu) = \begin{cases} 1 : \lambda \leq \underline{\alpha} \leq \mu \text{ for each } \alpha \in L, \\ \quad : \lambda \neq \nu \wedge \underline{\beta} \text{ for } 0.2 < \beta \leq 1, \\ \frac{1}{2} : \lambda = \nu \wedge \underline{\beta}, \mu \geq \nu \wedge \underline{\beta} \text{ for } 0.2 < \beta \leq 1, \\ \quad : \nu \neq \underline{\alpha}, \text{ for each } \alpha \in L, \\ 0 : O.w. \end{cases}$$

For $\lambda_1 = \nu \wedge \underline{0.3}$, there exist $\gamma_1 \in L^Y$ such that $f^{-1}(\gamma_1) = \lambda_1$ with $\gamma_1(p) = 0.2$, $\gamma_1(q) = 0.3$. Thus we have $f(N^s)(\gamma_1, \gamma_1) = N^s(f^{-1}(\gamma_1), f^{-1}(\gamma_1)) = \frac{1}{2}$, but $(f(N))^s(\gamma_1, \gamma_1) = 0$. Hence $f(N^s) \neq (f(N))^s$.

3. STRATIFICATIONS OF L-FUZZY PROXIMITY SPACES

An L-fuzzy quasi-proximity (resp. proximity) $\delta : L^X \times L^X \rightarrow L$ is said to be stratified L-fuzzy quasi-proximity (resp. proximity) on X , iff δ satisfied the following conditions: (FPS) $\delta(\underline{\alpha}, 1 - \underline{\alpha}) = 0$, for each $\alpha \in L$. The pair (X, δ) is called stratified L-fuzzy quasi-proximity (resp. proximity) space.

Theorem 18. *Let (X, δ) be L-fuzzy proximity space. We define for all $\lambda, \mu \in L^X$,*

$$\delta^s(\lambda, \mu) = \bigwedge_{\{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\} \in N(\lambda, \mu)} \left\{ \bigwedge_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\}} \delta(\lambda_i, \mu_i) \right\}.$$

where $W(\lambda, \mu) = \{ \{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N, N \text{ finite index set}\} \mid \lambda \leq \bigvee_{i \in N} (\lambda_i \wedge \underline{\alpha}_i), \mu \leq \bigwedge_{i \in N} (\mu_i \vee \underline{\alpha}_i) \}$. Then δ^s is the coarsest stratified L-fuzzy proximity on X which finer than δ .

Proof. (FP1), (FP2), (FP4) and (FPS) are similar to the proof of Theorem 2. And (FP) Obvious.

(FP3) Suppose there exists $\lambda, \mu \in L^X$ such that

$$\delta^s(\lambda, \mu) \not\leq \bigwedge_{\rho \in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)).$$

By the definition of δ^s there exists a family $\{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\} \in W(\lambda, \mu)$, such that

$$\bigvee_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\}} \delta(\lambda_i, \mu_i) \not\leq \bigwedge_{\rho \in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)).$$

Since δ is L-fuzzy proximity. $\delta(\lambda_i, \mu_i) \geq \bigwedge_{\rho_i \in L^X} (\delta(\lambda_i, \rho_i) \vee \delta(\rho_i, \mu_i))$. Then we have

$$\bigvee_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\}} (\delta(\lambda_i, \rho_i) \vee \delta(\rho_i, \mu_i)) \not\leq \bigwedge_{\rho \in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)).$$

On the other hand, put $\rho = \bigwedge_{i \in N} (\rho_i \vee \underline{\alpha}_i)$. Then by the definition of δ^s

$$\begin{aligned} \delta^s(\lambda, \rho) &\leq \bigvee_{(\lambda_i, \rho_i, \underline{\alpha}_i) \in \{(\lambda_i, \rho_i, \underline{\alpha}_i) \mid i \in N\}} (\delta(\lambda_i, \rho_i), \delta^s(\rho, \mu)) \\ &\leq \bigvee_{(\rho_i, \mu_i, \underline{\alpha}_i) \in \{(\rho_i, \mu_i, \underline{\alpha}_i) \mid i \in N\}} \delta(\rho_i, \mu_i). \end{aligned}$$

Thus we have

$$\begin{aligned} \bigwedge_{\rho \in L^X} (\delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu)) &\leq \delta^s(\lambda, \rho) \vee \delta^s(\rho, \mu) \\ &\leq \bigvee_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\}} (\delta(\lambda_i, \rho_i) \vee \delta(\rho_i, \mu_i)). \end{aligned}$$

It is a contradiction. \square

Example Let $X = \{x, y\}$, be set, $L = [0, 1]$. Let $\nu, \gamma \in L^X$, $\nu(x) = 0.5$, $\nu(y) = 0.5$ and $\gamma(x) = 0.4$, $\gamma(y) = 0.6$. Then we define the L-fuzzy proximity δ on X as follows: for each $\lambda, \mu \in L^X$,

$$\delta(\lambda, \mu) = \begin{cases} 1 & : \lambda = \underline{0} \text{ or } \mu = \underline{1}, \\ 0.7 & : \lambda \leq \nu, \mu \leq \nu, \mu \not\leq \gamma, \\ 0.5 & : \lambda \leq \gamma, \mu \leq \gamma, \lambda \not\leq \nu, \\ 0.3 & : \lambda \leq \nu \vee \gamma, \mu \leq (\nu \vee \gamma), \\ & : \lambda \not\leq \nu, \mu \not\leq \gamma, \\ 0.4 & : \lambda \leq \nu \wedge \gamma, \mu \leq (\nu \wedge \gamma), \\ 0 & : O.w \end{cases}$$

We can obtain the stratification δ^s as following:

$$N^s(\lambda, \mu) = \begin{cases} 1 & : \lambda = \underline{\alpha} \text{ or } \mu = \underline{\alpha}, \\ 0.3 & : \lambda(x) \leq \alpha, \mu(x) \leq \alpha, \lambda(y) \leq \beta, \mu(y) \leq \beta, \\ & : \text{for } 0.5 < \alpha, \beta < 0.6 \text{ and } \alpha < \beta, \\ 0.5 & : \lambda(x) \leq \alpha, \mu(x) \leq \alpha, \lambda(y) \leq \beta, \mu(y) \leq \beta, \\ & : \text{for } 0.4 \leq \alpha < 0.5, 0.5 < \beta \leq 0.6, \\ 0.6 & : \lambda(x) \leq \alpha, \mu(x) \leq \alpha, \lambda(y) \leq \beta, \mu(y) \leq \beta, \\ & : \text{for } 0.4 \leq \alpha, \beta \leq 0.5 \text{ and } \alpha < \beta, \\ 0 & : O.w \end{cases}$$

Theorem 19. (i) Let δ be an L-fuzzy proximity structure, N_δ symmetrical L-fuzzy topogenous order associated to δ . Then $N_{\delta^s} = (N_\delta)^s$.

(ii) Let N be a symmetrical L-fuzzy topogenous order structure, δ_N the L-fuzzy proximity associated to N . Then $\delta_{N^s} = (\delta_N)^s$.

Proof. (i) Since $N_{\delta^s}(\underline{\alpha}, \underline{\alpha}) = (\delta^s(\underline{\alpha}, \underline{\alpha})) = 1$, for each $\alpha \in L$, then N_{δ^s} is stratified finer than N_δ , so $N_{\delta^s} \geq (N_\delta)^s$.

Conversely, suppose there exist $\mu, \lambda \in L^X$ and $t \in L - \{0, 1\}$ such that

$$N_{\delta^s}(\lambda, \mu) = (\delta^s(\lambda, \mu)) > t > (N_\delta)^s(\lambda, \mu).$$

Since $\delta^s(\lambda, \mu) < t$, then there exists a family $\{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\} \in W(\lambda, \mu)$, such that $\delta(\lambda_i, \mu_i) < t$ for each $i \in N$. Then $N_\delta(\lambda_i, \mu_i) = (\delta(\lambda_i, \mu_i)) > t$ for each $i \in N$. On the other hand, since $\{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\} \in M(\lambda, \mu)$, then

$$(N_\delta)^s(\lambda, \mu) \geq \bigwedge_{(\lambda_i, \mu_i, \underline{\alpha}_i) \in \{(\lambda_i, \mu_i, \underline{\alpha}_i) \mid i \in N\}} N_\delta(\lambda_i, \mu_i) > t.$$

It is a contradiction. Thus $N_{\delta^s} = (N_\delta)^s$.

(ii) Similar (i). □

Theorem 20. Let (X, δ_1) and (Y, δ_2) be L-fuzzy quasi-proximity spaces, δ_1^s and δ_2^s be stratification for δ_1 and δ_2 respectively. If $f : (X, \delta_1) \rightarrow (Y, \delta_2)$ is L-fuzzy proximity continuous, then $f : (X, \delta_1^s) \rightarrow (Y, \delta_2^s)$ is L-fuzzy proximity continuous.

Proof. Similar of the proof of Theorem 15. □

Theorem 21. Let $\{(X_i, \delta_i)\}_{i \in \Gamma}$ be a family of L-fuzzy quasi-proximity spaces, $X = \prod_{i \in \Gamma} X_i$ a set and, for each $i \in \Gamma$, $\pi_i : X \rightarrow X_i$ a projection mapping and δ is the product L-fuzzy quasi-proximity structure with respect to a family $\{\pi_i : X \rightarrow X_i : i \in \Gamma\}$. If there exists a $j_0 \in \Gamma$ such that $\bar{\delta}_{j_0} = \delta_{j_0}^s$ and $\bar{\delta}_i = \delta_i$ for each $i \in \Gamma - \{j_0\}$, then:

- (i) $\Pi_{i \in \Gamma} \bar{\delta}_i$ is the stratified L-fuzzy quasi-proximity structure on X which is finer than δ .
(ii) $\Pi_{i \in \Gamma} \bar{\delta}_i = \delta^s$.

Proof. Similar of the proof of Theorem 16. \square

Corollary 22. Let (X, δ_1) and (X, δ_2) be L-fuzzy quasi-proximity spaces. We define, for any $\mu, \lambda \in L^X$

$$(\delta_1 \sqcup \delta_2)(\lambda, \mu) = \bigwedge \bigvee_{j,k} \{\delta_1(\lambda_j, \mu_k) \wedge \delta_2(\lambda_j, \mu_k)\},$$

where \bigwedge is taken over all finite families $\{\lambda_j : \lambda = \bigvee_j \lambda_j\}$ and $\{\mu_k : \mu = \bigvee_k \mu_k\}$.
Then:

- (i) $\delta_1^s \sqcup \delta_2$ and $\delta_1 \sqcup \delta_2^s$ are stratified L-fuzzy quasi-proximity on X , which finer than $\delta_1 \sqcup \delta_2$.
(ii) $\delta_1^s \sqcup \delta_2 = \delta_1 \sqcup \delta_2^s = (\delta_1 \sqcup \delta_2)^s$.

4. STRATIFICATIONS OF L-FUZZY UNIFORM SPACES

an L-fuzzy quasi-uniformity (resp. uniformity) $U : \Omega_X \rightarrow L$ is said to be stratified L-fuzzy quasi-uniformity (resp. uniformity) on X , iff U satisfied the following conditions: (FUS) $U(\hat{\alpha}) = 1$, for each $\alpha \in L$. The pair (X, U) is called stratified L-fuzzy quasi-uniform (resp. uniform) space.

Proposition 23. If (X, U) is stratified L-fuzzy uniform space and δ_U L-fuzzy proximity associated to U , then (X, δ_U) is stratified L-fuzzy proximity space.

Proof. Since U is stratified L-fuzzy uniformity, then $U(\hat{\alpha}) = 1$, $\hat{\alpha}(\underline{\alpha}) = \underline{\alpha}$ for each $\alpha \in L$. By Theorem 11. $\delta_U(\underline{\alpha}, \underline{\alpha}) = \bigwedge \{U(a) : a(\underline{\alpha}) \leq \underline{\alpha}\} = 0$ for each $\alpha \in L$. That is δ_U is stratified. \square

Theorem 24. Let (X, U) be an L-fuzzy quasi-uniform space. Define for every $a \in \Omega_X$,

$$U^s(a) = \begin{cases} \bigvee \{U(b) : a \geq b \wedge \hat{\alpha}, \alpha \in L\}, \\ 0 : O.w. \end{cases}$$

Then U^s is the coarsest stratified L-fuzzy quasi-uniformity finer than U .

Proof. (FU1) Suppose that there exist $a_1, a_2 \in \Omega_X$ such that

$$U^s(a_1 \wedge a_2) \not\geq U^s(a_1) \wedge U^s(a_2).$$

Form definition of U^s there exist $b_i \in \Omega_X$, $\alpha_i \in L$ with $a_i \geq b_i \wedge \hat{\alpha}_i, i = 1, 2$ such that

$$U^s(a_1 \wedge a_2) \not\geq U(b_1) \wedge U(b_2).$$

On the other hand, since $a_1 \wedge a_2 \geq (b_1 \wedge b_2) \wedge \hat{\alpha}_1$ or $a_1 \wedge a_2 \geq (b_1 \wedge b_2) \wedge \hat{\alpha}_2$, then we have

$$\begin{aligned} U^s(a_1 \wedge a_2) &\geq U(b_1 \wedge b_2) \\ &\geq U(b_1) \wedge U(b_2). \end{aligned}$$

It is a contradiction. Thus $U^s(a_1 \wedge a_2) \geq U^s(a_1) \wedge U^s(a_2)$

(FU2) Let $a_1 \in \Omega_X$. Suppose that there exist $a_1 \in \Omega_X$ such that

$$U^s(a) \not\geq U^s(a_1).$$

By the definition of U^s , there exist $b \in \Omega_X$, $\alpha \in L$ with $a \geq b \wedge \hat{\alpha}$ such that $U(b) \not\leq U^s(a_1)$. Then there exists $c \in \Omega_X$ such that $c \circ c \leq b$, $U(c) \geq U(b)$. By Proposition 1(ii),

$$(c \wedge \hat{\alpha}) \circ (c \wedge \hat{\alpha}) \leq (c \circ c) \wedge \hat{\alpha} \leq b \wedge \hat{\alpha} \leq a.$$

This means that there is $c \wedge \hat{\alpha} = a_1$, with $a_1 \circ a_1 \leq a$, $U^s(a_1) \geq U(c) \geq U(b)$. It is a contradiction. Thus $U^s(a) \leq U^s(a_1)$.

(FU3) Obvious.

(FU4) There exists $b \in \Omega_X$ such that $U(b) = 1$. Since $b = b \wedge \hat{0}$, by Proposition 1(iv), then $U^s(b) = 1$.

(FUS) Since $\hat{\alpha} \wedge \hat{0} = \hat{\alpha}$ for each $\alpha \in L$, then $U^s(\hat{\alpha}) = 1$ for each $\alpha \in L$. Thus U^s is stratified.

For each $b \in \Omega_X$, $b \wedge \hat{0} = b$. Then $U^s(b) \geq U(b)$ for any $b \in \Omega_X$. Thus U^s is finer than U .

Finally, consider U^* be stratified L-fuzzy quasi-uniformity finer than U . Suppose there exists $a \in \Omega_X$,

$$U^*(a) \not\leq U^s(a).$$

from definition of U^s , there exist $b \in \Omega_X$, $\alpha \in L$ with $a \geq b \wedge \hat{\alpha}$ such that $U^*(a) \not\leq U(b)$. Since U^* is stratified, then

$$\begin{aligned} U^*(a) &\geq U^*(b \wedge \hat{\alpha}) \geq U^*(b) \wedge U^*(\hat{\alpha}) \\ &= U^*(b) \geq U(b) \end{aligned}$$

It is a contradiction. Hence $U^*(a) \geq U^s(a)$ for any $a \in \Omega_X$. Thus U^s is the coarsest stratified L-fuzzy quasi-uniformity finer than U . U^s is called the stratification of an L-fuzzy quasi-uniformity of U on X . \square

Example. Let X be any set, $L = [0, 1]$, define the L-fuzzy quasi-uniformity U on X as follows:

$$U(a) = \begin{cases} 1 : a = a_0, \\ \frac{1}{4} : a_{\alpha} \leq a < a_0, \\ 0 : O.w \end{cases}$$

where $a_{\lambda} : L^X \rightarrow L^X$ defined by Theorem 6. Since $a_{\alpha} = \hat{\alpha}$ for any $\alpha \in L$, we obtain

$$U_2(a) = \begin{cases} 1 : a \geq \hat{\alpha}, \text{ for each } \alpha \in L \\ 0 : O.w \end{cases}$$

Theorem 25. Let (X, U) be an L-fuzzy uniform space, δ_U the L-fuzzy proximity associated to U . Then $(\delta_U)^s = \delta_{U^s}$.

Proof. Since U^s is stratified finer than U , then δ_{U^s} is stratified finer than δ_U . Thus $\delta_{U^s}(\lambda, \mu) \leq (\delta_U)^s(\lambda, \mu)$. for any $\mu, \lambda \in L^X$.

Conversely, suppose there exist $\mu, \lambda \in L^X$ and $t \in L - \{0, 1\}$ such that

$$\delta_{U^s}(\lambda, \mu) < t < (\delta_U)^s(\lambda, \mu).$$

By Theorem 11. there exists $a \in \Omega_X$ such that $\delta_{U^s}(\lambda, \mu) \leq (U^s(a))' < t$ and $a(\mu) \leq \lambda'$. Then there exists $b \in \Omega_X$ with $a \geq b \wedge \hat{\alpha}$ such that $U^s(a) \geq U(b) > t'$

and $(b \wedge \widehat{\alpha})(\mu) \leq \lambda'$. Hence

$$\lambda \leq \left(\bigwedge_{\mu=\mu_1 \vee \mu_2} (b(\mu_1) \vee \widehat{\alpha}(\mu_2)) \right) = \bigvee_{i \in N} (\lambda_i \wedge \underline{\alpha}_i)$$

where $(\widehat{\alpha}(\mu_2)) = \underline{\alpha}_i$, $(b(\mu_1)) = \lambda_i$ for $i \in N = \{1, 2, 3, \dots, n\}$. Put $\mu_1 = \nu_i$ for each $i \in N$, then $\mu \leq \bigwedge_{i \in N} (\nu_i \vee \underline{\alpha}_i)$ and hence $b(\nu_i) \leq \lambda_i$, for each $i \in N$. Then we have

$$\delta_U(\lambda_i, \nu_i) \leq (U(b)) < t, \text{ for each } i \in N.$$

On the other hand, from definition of $(\delta_U)^s$ we have

$$(\delta_U)^s(\lambda, \mu) \leq \bigwedge_{(\lambda_i, \nu_i, \underline{\alpha}_i) \in \{(\lambda_i, \nu_i, \underline{\alpha}_i) \mid i \in N\}} \delta_U(\lambda_i, \nu_i) < t.$$

It is a contradiction. Thus $\delta_{U^s}(\lambda, \mu) \geq (\delta_U)^s(\lambda, \mu)$. \square

Theorem 26. *Let (X, U) be an L-fuzzy quasi-uniform space and T_U be an L-fuzzy topology associated to U . Then the L-fuzzy topology T_{U^s} associated to U^s coincides with the stratification of T_U .*

Proof. We will show $T_{U^s} = (T_U)^s$. Since $U^s(\widehat{\alpha}) = 1$, $\widehat{\alpha}(\underline{\alpha}) = \underline{\alpha}$, for each $\alpha \in L$, then

$$I_{U^s}(\underline{\alpha}, \beta) = \bigvee \{ \mu : a(\mu) \leq \lambda \text{ for some } U^s(a) > \beta \} = \underline{\alpha},$$

for each $\beta, \alpha \in L$. Hence $T_{U^s}(\underline{\alpha}) = 1$, for each $\alpha \in L$ i.e T_{U^s} is stratified. By Theorem 24, U^s is finer than U . Then T_{U^s} is stratified finer than T_U . Thus $T_{U^s} \geq (T_U)^s$.

Conversely, Suppose there exists $\lambda \in L^X$ such that

$$T_{U^s}(\lambda) \not\leq (T_U)^s(\lambda).$$

Then from definition of T_{U^s} , there exists $\alpha \in L$, $T_{U^s}(\lambda) \geq \alpha$. That is

$$I_{U^s}(\lambda, \alpha) = \lambda = \bigvee \{ \mu : a(\mu) \leq \lambda, U^s(a) > \alpha \}.$$

Since $U^s(a) > \alpha$, then there exist $b \in \Omega_X$, $\beta \in L$ with $a \geq b \wedge \widehat{\beta}$ such that

$$U^s(a) \geq U(b) > \alpha.$$

On the other hand, by Proposition 1. $\widehat{\beta}(\mu) = \underline{1}$ for each $\mu \in L^X$ with $\text{sup}(\mu) \not\leq \beta$. Then $b(\mu) \leq \lambda$ for each μ with $\text{sup}(\mu) \not\leq \beta$ and therefore

$$\begin{aligned} \lambda &= \bigvee \{ \mu : b(\mu) \wedge \widehat{\beta}(\mu) \leq \lambda, U(b) > \alpha \} \\ &\leq \bigvee \{ \mu : b(\mu) \leq \lambda, \text{sup}(\mu) \not\leq \beta, U(b) > \alpha \} \\ &\leq I_U(\lambda, \alpha). \end{aligned}$$

Hence $I_U(\lambda, \alpha) = \lambda$ i.e $T_U(\lambda) \geq \alpha$. Hence $(T_U)^s(\lambda) \geq \alpha$. It is a contradiction. Thus $T_{U^s} \leq (T_U)^s$. \square

Theorem 27. *Let (X, T) be a stratified L-fts. Define for each $a \in \Omega_X$,*

$$U(a) = \vee \bigwedge_{i \in N} \{ T(\lambda_i) : a \geq \bigwedge_{i \in N} a_{\lambda_i} \},$$

where \vee is taken over all finite families $\{ \lambda_i : a \geq \bigwedge_{i \in N} a_{\lambda_i} \}$. Then
(i) U is stratified L-fuzzy quasi-uniformity on X . (ii) $T_U = T$.

Proof. (i) (FU1) Suppose there exist $a, b \in \Omega_X$ such that

$$U(a \wedge b) \not\geq U(a) \wedge U(b).$$

Then by the definition of U , there exist two finite families $\{\lambda_i : a \geq \bigwedge_{i \in N} a_{\lambda_i}\}$, $\{\gamma_j : b \geq \bigwedge_{j \in K} a_{\gamma_j}\}$, such that

$$U(a \wedge b) \not\geq \bigwedge_{i \in N} T(\lambda_i) \wedge \bigwedge_{j \in K} T(\gamma_j).$$

On the other hand, we have

$$a \wedge b \geq \left(\bigwedge_{i \in N} a_{\lambda_i} \right) \wedge \left(\bigwedge_{j \in K} a_{\gamma_j} \right).$$

Then by definition of U ,

$$U(a \wedge b) \geq \left(\bigwedge_{i \in N} T(\lambda_i) \right) \wedge \left(\bigwedge_{j \in K} T(\gamma_j) \right).$$

It is a contradiction.

(FU2) Since $a_\lambda \circ a_\lambda = a_\lambda$ by Lemma 2, then (FU2) holed.

(FU3) Obivous.

(FU4) There exists a (say) $= a_{\underline{1}} \in \Omega_X$ such that $U(a) \geq T(\underline{1}) = 1$.

(FUS) Since $a_\alpha = \hat{\alpha}$, then $U(\hat{\alpha}) \geq T(\underline{\alpha}) = 1$. Hence $U(\hat{\alpha}) = \underline{1}$ for each $\alpha \in L$.

Thus U is stratified.

(ii) Obvious $T_U \leq T$, by definition of U .

Conversely, suppose there exist $\lambda \in L^X$ and $\alpha \in L - \{0, 1\}$ such that

$$T_U(\lambda) < \alpha < T(\lambda).$$

Since $a_\lambda(\lambda) = \lambda$, then $U(a_\lambda) \geq T(\lambda) > \alpha$ and then $I_U(\lambda, \alpha) = \bigvee \{\mu : a(\mu) \leq \lambda, \text{ for some } U(a) > \alpha\} = \lambda$. Thus $T_U(\lambda) \geq \alpha$. It is a contradiction. Thus $T_U \geq T$. \square

Theorem 28. Let (X, U_1) and (Y, U_2) be L-fuzzy quasi-uniform spaces and U_1^s and U_2^s be stratification for U_1 and U_2 respectively. If $f : (X, U_1) \rightarrow (Y, U_2)$ is L-fuzzy uniformly continuous, then $f : (X, U_1^s) \rightarrow (Y, U_2^s)$ is L-fuzzy uniformly continuous.

Proof. We will show $U_1^s(f^l(b)) \geq U_2^s(b)$ for each $b \in \Omega_Y$, If $U_2^s(b) = 0$, trivial. If $U_2^s \neq 0$, suppose that

$$U_1^s(f^l(b)) \not\geq U_2^s(b)$$

From definition of U_2^s , there exist $c \in \Omega_Y$, $\alpha \in L$ with $b \geq c \wedge \hat{\alpha}$,

$$U_1^s(f^l(b)) \not\geq U_2(c).$$

Since $f : (X, U_1) \rightarrow (Y, U_2)$ is L-fuzzy uniformly continuous, $U_1(f^l(c)) \geq U_2(c)$. From the definition of U_1^s , we have

$$U_1^s(f^l(b)) \geq U_1(f^l(c)) \geq U_2(c)$$

It is a contradiction. Hence $U_1^s(f^l(b)) \geq U_2^s(b)$ for any $b \in \Omega_Y$. \square

Counterexample.

Let X be any set, $L = [0, 1]$ and identity mapping $Id_X : (X, U_1) \rightarrow (X, U_2)$, define the L-fuzzy quasi-uniformity U_1 and U_2 on X as follows:

$$U_1(a) = \begin{cases} 1 & : a = a_0, \\ 0 & : O.w \end{cases}$$

$$U_2(a) = \begin{cases} 1 & : a = a_0, \\ \frac{1}{3} & : a_{0.5} \leq a < a_0, \\ 0 & : O.w \end{cases}$$

where $a_\alpha : L^X \rightarrow L^X$ defined by Theorem 6. Since $a_{\underline{\alpha}} = \widehat{\alpha}$ for any $\alpha \in L$, then from Theorem 2, we obtain

$$U_1^s(a) = U_2^s(a) = \begin{cases} 1 & : a \geq \widehat{\alpha}, \text{ for each } \alpha \in L \\ 0 & : O.w \end{cases}$$

Clearly $Id_X : (X, U_1^s) \rightarrow (X, U_2^s)$, is L-fuzzy uniformly continuous. But $Id_X : (X, U_1^s) \rightarrow (X, U_2^s)$, is not L-fuzzy uniformly continuous, where $0 = U_1(f^l(a_{0.5})) < U_1(a_{0.5}) = \frac{1}{3}$.

Theorem 29. Let $\{(X_i, U_i)\}_{i \in \Gamma}$ be a family of L-fuzzy uniform spaces, X a set and, for each $i \in \Gamma$, $f_i : X \rightarrow X_i$ a mapping. We define, for each $a \in \Omega_X$

$$U(a) = \bigvee_{j \in N} \bigwedge \{U_j(b_j) : a \geq \bigwedge_{j \in N} f_j^i(b_j)\},$$

where \bigvee is taken over all finite index $N \subset \Gamma$. Then

(i) U is the coarsest L-fuzzy uniformity on X for which all f_i , are L-fuzzy uniformity continuous.

(ii) A mapping $f : (X^*, U^*) \rightarrow (X, U)$ is L-fuzzy uniformity continuous iff for each $i \in \Gamma$, $f_i \circ f$ is L-fuzzy uniformity continuous.

Proof. (i) (FU1) Suppose that there exist $a, b \in \Omega_X$ such that

$$U(a \wedge b) \not\geq U(a) \wedge U(b).$$

By the definition of U there exist two finite families $\{a_n : a \geq \bigwedge_{n \in N} f_n^l(a_n)\}$, $\{b_j : b \geq \bigwedge_{j \in K} f_j^l(b_j)\}$ such that

$$U(a \wedge b) \not\geq \left(\bigwedge_{n \in N} U_n(a_n) \right) \wedge \left(\bigwedge_{j \in K} U_j(b_j) \right).$$

On the other hand, since $a \wedge b \geq (\bigwedge_{n \in N} f_n^l(a_n)) \wedge (\bigwedge_{j \in K} f_j^l(b_j))$, then

$$U(a \wedge b) \geq \left(\bigwedge_{n \in N} U_n(a_n) \right) \wedge \left(\bigwedge_{j \in J} U_j(b_j) \right).$$

It is a contradiction. Thus $U(a \wedge b) \geq U(a) \wedge U(b)$.

(FU2) Let $a \in \Omega_X$. Suppose that there exist $b \in \Omega_X$ such that

$$U(a) \not\geq U(b).$$

By the definition of U , there exists finite family $\{a_n : a \geq \bigwedge_{n \in N} f_n^l(a_n)\}$ such that

$$\bigwedge_{n \in N} U_n(a_n) \not\geq U(b).$$

Since $U_n(a_n) > 0$, for each $n \in N$, then there exists $b_n \in \Omega_{X_n}$, $b_n \circ b_n \leq a_n$ such that $U_n(b_n) \geq U_n(a_n)$. Put $b = \bigwedge_{n \in N} f_n^l(b_n)$. Hence

$$\begin{aligned} b \circ b &= \bigwedge_{n \in N} (f_n^l(b_n) \circ f_n^l(b_n)) \\ &\leq \bigwedge_{n \in N} f_n^l(b_n \circ b_n) \\ &\leq \bigwedge_{n \in N} f_n^l(a_n) \leq a. \end{aligned}$$

Then there exists $b \in \Omega_X$ such that $b \circ b \leq a$ and

$$U(b) \geq \bigwedge_{n \in N} U_n(b_n) \geq \bigwedge_{n \in N} U_n(a_n).$$

It is a contradiction. Thus $U(b) \geq U(a)$.

(FU3) Obvious.

(FU4) There exists $b \in \Omega_{X_j}$, $j \in \Gamma$ such that $U_j(b) = 1$, put $f_j^l(b) = a$. Then $U(a) = 1$.

(FU) Let $a \in \Omega_X$. Suppose that there exist $b \in \Omega_X$ such that

$$U(a) \not\leq U(b).$$

By the definition of U , there exists finite family $\{a_n : a \geq \bigwedge_{n \in N} f_n^l(a_n)\}$ such that

$$\bigwedge_{n \in N} U_n(a_n) \not\leq U(b).$$

Since $U_n(a_n) > 0$, for each $n \in N$, then there exists $b_n \in \Omega_{X_n}$, $b_n \leq a_n^{-1}$ such that $U_n(b_n) \geq U_n(a_n)$ for each $n \in N$. Put $b = \bigwedge_{n \in N} f_n^l(b_n)$. Hence

$$\begin{aligned} b^{-1} &= \left(\bigwedge_{n \in N} f_n^l(b_n) \right)^{-1} \leq \bigwedge_{n \in N} (f_n^l(b_n))^{-1} \leq \bigwedge_{n \in N} f_n^l(b_n^{-1}) \\ &\leq \bigwedge_{n \in N} f_n^l((a_n^{-1})^{-1}) = \bigwedge_{n \in N} f_n^l(a_n) \\ &\leq a. \end{aligned}$$

Then there exists $b \in \Omega_X$ such that $b \leq a^{-1}$ and

$$U(b) \geq \bigwedge_{n \in N} U_n(b_n) \geq \bigwedge_{n \in N} U_n(a_n)$$

It is a contradiction. Thus $U(b) \geq U(a)$.

Second, it is easily proved that, by the definition of U , for all $i \in \Gamma$,

$$U(f_i^l(b)) \geq U_i(b) \text{ for each } b \in \Omega_{X_i}.$$

Hence, for each $f_i : X \rightarrow X_i$ is L-fuzzy uniformity continuous.

Finally, if $f_i : (X, U^*) \rightarrow (X_i, U_i)$ is L-fuzzy uniformity continuous, then $U^*(f_i^l(b)) \geq$

$U_i(b)$ for each $j \in \Gamma$. We will show $U^* \geq U$, from the following:

$$\begin{aligned} U(a) &= \vee \bigwedge_{j \in N} \{U_j(b_j) : a \geq \bigwedge_{j \in N} f_j^l(b_j)\} \\ &\leq \vee \bigwedge_{j \in N} \{U^*(f_j^l(b)) : a \geq \bigwedge_{j \in N} f_j^l(b_j)\} \\ &\leq \vee \{U^*(\bigwedge_{j \in N} f_j^l(b)) : a \geq \bigwedge_{j \in N} f_j^l(b_j)\} \\ &\leq U^*(a). \end{aligned}$$

(ii) Obvious. □

Definition 30. In a above theorem. U is called the initial L-fuzzy uniformity structure on X with respect to a family $\{(X_i, U_i)\}_{i \in \Gamma}$.

Corollary 31. Let $\{(X_i, U_i)\}_{i \in \Gamma}$ be a family of L-fuzzy uniform spaces, $X = \prod_{i \in \Gamma} X_i$ a set and, for each $i \in \Gamma, \pi_i : X \rightarrow X_i$ a projection mapping . We define, for each $a \in \Omega_X$

$$\prod_{i \in \Gamma} U_i(a) = \vee \bigwedge_{j \in N} \{U_j(b_j) : a \geq \bigwedge_{j \in N} \pi_j^l(b_j)\},$$

Where \bigwedge is taken over all finite index $N \subset \Gamma$. Then

(i) $\prod_{i \in \Gamma} U_i$ is the coarsest L-fuzzy uniformity on X for which all π_i , are L-fuzzy uniformity continuous.

(ii) A mapping $f : (X^*, U^*) \rightarrow (X, \prod_{i \in \Gamma} U_i)$ is L-fuzzy uniformity continuous iff for each $i \in \Gamma, \pi_i \circ f$ is L-fuzzy uniformity continuous.

The initial L-fuzzy uniformity structure $\prod_{i \in \Gamma} U_i$ is called the product L-fuzzy uniformity structure of a family $\{(X_i, U_i)\}_{i \in \Gamma}$, and $(X, \prod_{i \in \Gamma} U_i)$ is called product uniform space.

Theorem 32. Let U be a product L-fuzzy uniformity structure of a family $\{(X_i, U_i)\}_{i \in \Gamma}$ and $X = \prod_{i \in \Gamma} X_i$. If there exists a $j_0 \in \Gamma$ such that $\bar{U}_{j_0} = U_{j_0}^s$ and $\bar{U}_i = U_i$ for each $i \in \Gamma - \{j_0\}$, then:

(i) $\prod_{i \in \Gamma} \bar{U}_i$ is the stratified L-fuzzy quasi -uniformity on X which is finer than U .

(ii) $\prod_{i \in \Gamma} \bar{U}_i = U^s$.

Proof. (i) From corollary 31, $\prod_{i \in \Gamma} \bar{U}_i$ is L-fuzzy quasi -uniformity on X for which all $\pi_i, i \in \Gamma$ are L-fuzzy uniformity continuous. For each $\alpha \in L$, we have

$$\prod_{i \in \Gamma} \bar{U}_i(\hat{\alpha}) \geq \bar{U}_{j_0}(\hat{\alpha}) = U_{j_0}^s(\hat{\alpha}) = 1.$$

Thus $\prod_{i \in \Gamma} \bar{U}_i(\hat{\alpha}) = 1$. Moreover, since $U_{j_0}^s$ is finer than $\prod_{i \in \Gamma} \bar{U}_i$ is the stratified L-fuzzy quasi-uniformity on X which is finer than U .

(ii) Since $\prod_{i \in \Gamma} \bar{U}_i$ is stratified, then by Theorem 24, $U^s \preceq \prod_{i \in \Gamma} \bar{U}_i$.

Conversely, suppose there exist $a \in \Omega_X$ such that

$$U^s(a) \not\preceq \prod_{i \in \Gamma} \bar{U}_i(a). \quad (4. 2)$$

By the definition of $\Pi_{i \in \Gamma} \bar{U}_i$, there exists a finite family $\{a_n : a \geq \bigwedge_{n \in N} \pi_n^l(a_n)\}$ such that

$$U^s(a) \not\geq \bigwedge_{n \in N} \bar{U}_n(a_n).$$

On the other hand, if $j_0 \notin N$, then

$$U^s(a) \geq U(a) \geq \bigwedge_{n \in N} U_n(a_n) = \bigwedge_{n \in N} \bar{U}_n(a_n).$$

It is a contradiction for Eq. 4. 2. If $j_0 \in N$, then $\bar{U}_{j_0}(a_{j_0}) = U_{j_0}^s(a_{j_0})$. From definition of $U_{j_0}^s$, there exists $b_{j_0} \in \Omega_{X_{j_0}}$ with $a_{j_0} \geq b_{j_0} \wedge \hat{\alpha}_{j_0}$ such that $U_{j_0}^s(a_{j_0}) \geq U_{j_0}(b_{j_0})$. Since $\pi_{j_0} : X \rightarrow X_{j_0}$ is L-fuzzy uniformity continuous, then $U(\pi_{j_0}^l(b_{j_0})) \geq U_{j_0}(b_{j_0})$. Now,

$$\begin{aligned} a &\geq \pi_{j_0}^l(a_{j_0}) \wedge \left(\bigwedge_{n \in N - \{j_0\}} \pi_n^l(a_n) \right) \\ &\geq \pi_{j_0}^l(b_{j_0} \wedge \hat{\alpha}_{j_0}) \wedge \left(\bigwedge_{n \in N - \{j_0\}} \pi_n^l(a_n) \right) \\ &\geq (\pi_{j_0}^l(b_{j_0}) \wedge \left(\bigwedge_{n \in N - \{j_0\}} \pi_n^l(a_n) \right)) \wedge \hat{\alpha}_{j_0} \end{aligned}$$

Then we have

$$\begin{aligned} U^s(a) &\geq U(\pi_{j_0}^l(b_{j_0}) \wedge \left(\bigwedge_{n \in N - \{j_0\}} \pi_n^l(a_n) \right)) \\ &\geq U(\pi_{j_0}^l(b_{j_0})) \wedge \left(\bigwedge_{n \in N - \{j_0\}} U(\pi_n^l(a_n)) \right) \\ &\geq U_{j_0}(b_{j_0}) \wedge \left(\bigwedge_{n \in N - \{j_0\}} \bar{U}_n(a_n) \right) \\ &= \bigwedge_{n \in N} \bar{U}_n(a_n). \end{aligned}$$

Also, it is a contradiction for Eq. 4. 2. Thus $U^s \succeq \Pi_{i \in \Gamma} \bar{U}_i$. \square

Corollary 33. Let (X, U_1) and (X, U_2) be L-fuzzy quasi-uniform spaces. We define, for each $a \in \Omega_X$

$$(U_1 \pitchfork U_2)(a) = \vee \{U_1(b_1) \wedge U_2(b_2) : a \geq b_1 \wedge b_2\}.$$

Then: (i) $U_1 \pitchfork U_2^s$ and $U_1^s \pitchfork U_2$ are stratified L-fuzzy quasi-uniformity on X , which finer than $U_1 \pitchfork U_2$.

(ii) $U_1 \pitchfork U_2^s = U_1^s \pitchfork U_2 = (U_1 \pitchfork U_2)^s$

(iii) $T_{U_1} U_2^s = T_{U_1^s} U_2 = T_{(U_1 \pitchfork U_2)^s} = (T_{U_1 \pitchfork U_2})^s$.

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