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Maximum Principles For Parabolic Systems

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Abstract. In this paper we introduce a strong maximum principle for some nonlinear parabolic systems with convex invariant regions. We also obtain a version of the Hopf boundary lemma for the systems.

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1. Introduction

Consider the parabolic system

$$\frac{\partial u}{\partial t} - A(x,t,u)\sum_{i,j=1}^{n} a_{ij}(x,t,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x,t,u)\frac{\partial u}{\partial x_i} = f(x,t,u) \qquad (1.1)$$

on $D \times (0,T)$,where $u = \begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_m \end{bmatrix}$, D is a domain in $\mathbb{R}^n$, $A(x,t,u)$, and $B_i\ (x,t,u)$ $(i = 1,2,...,n)$ are $m \times m$ matrix- valued functions on $D \times (0 \times T) \times \mathbb{R}^m$, $a_{ij}(x,t,u)$ $(i,j = 1,...,n)$ are real valued functions.

Under the hypothesis that the differential operator on the left- hand side of (1.1) is locally uniformly parabolic on $D \times (0,T)$, that (1.1) has a C^2 convex invariant region $S \subset \mathbb{R}^m$, and under some regularity conditions, we will show that, for (1.1) Weinberger's version of strong maximum principles holds, which says that if there exist a $(x^*,t^*) \in D \times (0,T)$ such that $u(x^*,t^*) \in \partial S$, then $u(D \times (0,t^*]) \subset \partial S$. Moreover, if in addition that D satisfies the interior sphere condition, we will prove that a version of the Hopf boundary lemma holds for (1.1).

The weak and strong maximum principles for the case that in (1.1), $A(x,t,u) \equiv I$ and B_i $(i = 1,2,...,n)$ are real valued functions are studied by [6], the boundary point lemma, however, was not mentioned in [6] (see the main theorem in [3]). Our basic method is the same as Weinberger's. The local defining functions of ∂S plays an important in [6], we prefer the distance function of ∂S, making the proofs more geometric.

An extension of the boundary lemma was found by W. Troy [11] for nonnegative solution of the elliptic system

$$\sum_{i,k=1}^{n} a_{jk}^{i}(x)\frac{\partial^2 u_i}{\partial_{x_j}\partial_{x_k}} + \sum_{j=1}^{n} b_j^i(x)\frac{\partial u_i}{\partial_{x_j}} + \sum_{j=1}^{n} C_{ij}(x)u_j = 0$$

on D, where $i = 1, ..., m$. $C_{ij}(x) \geq 0$ on D for $i \neq j, 1 \leq i, j \leq m$.

The weak maximum principles for (1.1) also has been studied by K. Chueh, C. Conley and J. Smoller [1]. Their results show that for a C^1 domain $S \subset \mathbb{R}^m$ to be an invariant region we need the following condition which we assume that it holds through this paper.

Condition 1. *S is convex and for any u in ∂S, the inward unit normal $v(u)$ at u is a left eigenvector of A and $B_i(i = 1, 2, ..., n)$, and $v(x).f(x, t, u) \geq 0$ for all (x, t) in $D \times (0, T)$.*

Weakly coupled semilinear parabolic systems in unbounded domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ with polynomial nonlinearities are investigated in [2], and three conditions to insure the stability of the zero solution with respect to nonnegative H^2-perturbations are given.[4] formulate a criteria for validity of the maximum norm principle as a number of equivalent algebraic conditions describing the relation between the geometry of the unit sphere of the given norm and coefficients of the system under consideration.

In this paper we use the distance function of ∂S instead of choosing a general defining function as in [6] since it makes the proofs more geometric.

2. Preliminaries

All materials discussed in this section can be found in the Appendix of chapter 14 of [10], and they are included here for the reader's convenience.

First recall some classical definitions. Let S be a C^2 domain in $\mathbb{R}^m$ with $\partial S \neq \varnothing$. For any $u \in \partial S$, let $v(u)$ denote the unit inner normal to ∂S at u. For a fixed $u_0 \in \partial S$, construct a coordinate system $(u_1, ..., u_m)$ such that u_m- axis lies in the direction $v(u_0)$ and the origin is at u_0. Near u_0, ∂S can be expressed by $u_m = \varphi(u_1, ..., u_{m-1})$. Then the Gaussian curvature of ∂S at u_0 is $det[A^2\varphi(0)]$ and the principal curvatures of ∂S at u_0 are the eigenvalues $k_1, ..., k_{m-1}$ of the matrix $[A^2\varphi(0)]$. Now if we rotate the coordinate frame with respect to the u_m axis, we can let $u_1, .., u_m$ axes lie on eigenvector directions corresponding to $k_1, ..., k_{m-1}$, respectively. We call such a new coordinate system *a principal coordinate system at* u_0. In this system $[A^2\varphi(0)] = diag[k_1, ..., k_{m-1}]$.

For $u \in \mathbb{R}^m$, the distance function d is defined by $d(u) = dist(u, \partial S)$.

Lemma 2. *Let S be a C^k domain in $R^m.k \geq 2$ and $\partial S \neq \varnothing$. Then there exists an open (w.r.t the topology of $\overline{S}$) subset G of $\overline{S}$ such that $\partial D \subset G$, d in $C^2(G)$, and for any u in G, exists unique $y(u)$ in ∂S such that*

$$|u - y(u)| = d(u)(i.e\ u = y(u) + v(y(u))d(u)),$$

$$Ad(u) = v(y(u)), 1 - k_i(y(u))d(u) > 0,\ (i = 1, ..., m - 1)$$

where $k_i(y(u))(i = 1, ...m - 1)$ are principal curvatures of ∂S at $y(u)$. Moreover, for $u \in G$, at a principal coordinate system at $y(u)$,

$$[A^2 d(u)] = diag[\frac{-k_1}{1 - k_1 d}, ..., \frac{-k_{m-1}}{1 - k_{m-1} d}, 0]$$

3. The Main Result And Its Proof

We assume that u is a solution of (1.1) and A, a_{ij} , and B_i are functions of (x,t) only due to the compositions.

Theorem 3. *Suppose that A, a_{ij}, and $B_i(1 \le i,j \le n)$ are locally bounded on $D \times (0,T)$, $A_{m\times m}$ and $(a_{ij})_{n\times n}$ locally uniformly positive - definite on $D \times (0,T)$ and$f(x,t,u)$ is Lipschitz continuous in u locally uniformly with respect to (x,t) on $D \times (0,T)$. Assume also that there exist a C^2 domain S in R^m s.t condition (1) is satisfied. Then if $u(D \times (0,T)) \subset \overline{S}$ and there exists $(x^*,t^*) \in D \times (0,T)$ s.t $u^* = u(x^*,t^*) \in \partial S$, then $u(D \times (0,t^*]) \subset \partial S$. Furthermore , if there exist a $x_0 \in \partial D$ and $0 < t_0 < T$ s.t D satisfies the interior sphere condition at x_0 and u is continuous at (x_0,t_0) with $u(x_0,t_0) \in \partial S$, then either $u(D \times (0,t_0]) \subset \partial S$ or $v(u(x_0,t_0)).\partial u/\partial\eta < 0$.(if the directional derivative exists), where η is any outward pointing direction to $(\partial D \times (0,T)$ at (x_0,t_0),*[3].

Proof. Let us take a bounded open neighborhood $D_1 \subset D$ of x^* and $0 < t_1 < t^*$ s.t $u(D_1 \times [t_1,t^*]) \subset G$ where G is defined in Lemma (2). let $\mu(x,t,v)$ be the eigenvalue corresponding to the eigenvector v of $A(x,t)$ and $\lambda_i(x,t,v)$ be the eigenvalue of $B_i(x,t)$.Then on $D_1 \times [t_1,t^*]$

$$L = \frac{\partial}{\partial t} - \mu(x,t,v(y(u(x,t))) \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} \lambda_i(x,t,v(y(x,t))))\frac{\partial}{\partial x_i}$$

is uniformly parabolic (for definitions of v and $y(u)$), [1].
let $\overline{d}(x,t) = d(u(x,t))$. then on $D_1 \times [t_1,t^*]$ we have

$$\begin{aligned} L\overline{d} &= A_u d(u)\frac{\partial u}{\partial t} - \mu(x,t,v(y(u))) \\ &\quad \times \sum_{i,j=1}^{n} a_{ij}(x,t)\left(\sum_{\alpha,\beta=1}^{m} \frac{\partial^2 d(u)}{\partial u_\alpha \partial u_\beta}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_j} + \sum_{\alpha=1}^{m}\frac{\partial d(u)}{\partial u_\alpha}.\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\right) \\ &\quad + \sum_{i=1}^{n} \lambda_i(x,t,v(y(u))) \sum_{\alpha=1}^{m} \frac{\partial d(u)}{\partial u_\alpha}.\frac{\partial u_\alpha}{\partial x_i} \\ &= A_u d(u)\frac{\partial u}{\partial t} - I(x,t) - \mu(x,t,v(y(u)))A_u d(u) \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} \\ &\quad + \sum_{i=1}^{n} \lambda_i(x,t,v(y(u)))A_u d(u)\frac{\partial u}{\partial x_i} \\ &= A_u d(u)\frac{\partial u}{\partial t} - A_u d(u) A(x,t) \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + A_u d(u)\sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} \\ &\quad - I(x,t) \\ &= A_u d(u) f(x,t,u) - I(x,t), \end{aligned}$$

where I is defined by the second equality and in the third step we use the fact that $A_u d(u) = v(y(u))$ and condition (1).

Now by condition (1) again,

$$v(y(u))f(x,t,y(u)) \geq 0, \text{ i.e. } A_u d(y(u(x,t))).f(x,t,y(u(x,t))) \geq 0 \text{ on } A_1 \times [t_1,t^*].$$

Hence we have

$$Ld \geq A_u d(u(x,t))f(x,t,u(x,t)) - A_u d(y(u(x,t))).f(x,t,y(u(x,t))) - I(x,t)$$

$$= c(x,t).(u(x,t) - y(u(x,t))) - I(x,t),$$

where $c(x,t)$ is a vector function in $\mathbb{R}^m$ and is obtained by noticing $d \in C^2(G)$ and f is Lipschitz in $u.c(x,t)$ is bounded on $D_1 \times [t_1,t^*]$.

Since

$$u = y(u) + v(y(u))d(u),$$

we have

$$Ld \geq c(x,t)v(y(u(x,t)))d(u(x,t)) - I(x,t),$$

i.e.

$$L\overline{d} \geq c(x,t)d - I(x,t) \quad \text{On} \quad D_1 \times [t_1,t^*] \tag{3.2}$$

where c is bounded.

Next, we prove that $I \leq 0$ on $D_1 \times [t_1,t^*]$.

Fix $(x_0,t_0) \in D_1 \times [t_1,t^*]$.Since

$$\sum_{\alpha,\beta=1}^{m} \frac{\partial^2 d(u)}{\partial u_\alpha \partial u_\beta} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\beta}{\partial x_j}$$

is invariant under any parallel translation and rotation of u coordinate system, we assume that we work in a principal coordinate system at $y(u(x_0,t_0)) \in \partial S$. Then by Lemma (2) we have

$$D_u^2 d(u(x_0,t_0)) = diag\left[\frac{-k_1}{1-k_1 d(u(x_0,t_0))}, ..., \frac{-k_1}{1-k_{m-1}d(u(x_0,t_0)}, 0\right]$$

where $k_1,...,k_{m-1}$ are the principal curvatures of ∂S at $y(u(x_0,t_0))$. Thus

$$\frac{I}{\mu}(x_0,t_0) = \sum_{i,j=1}^{n} a_{ij}(x_0,t_0) \sum_{\alpha=1}^{m-1} \frac{-k_\alpha}{1-k_\alpha d(u(x_0,t_0))} \frac{\partial u_\alpha}{\partial x_i}(x_0,t_0)\frac{\partial u_\alpha}{\partial x_j}(x_0,t_0),$$

i.e.

$$\frac{I}{\mu}(x_0 t_0) = \sum_{\alpha=1}^{m-1} \frac{-k_\alpha}{1-k_\alpha d(u(x_0,t_0)} \sum_{i,j=1}^{n} a_{ij}(x_0,t_0)\frac{\partial u_\alpha}{\partial x_i}(x_0,t_0)\frac{\partial u_\alpha}{\partial x_i}(x_0,t_0). \tag{3.3}$$

Since S is convex, $k_\alpha \geq 0, 1 \leq \alpha \leq m-1$. Recall in the Lemma (2) that $1 - k_\alpha(y(u))d(u) > 0$ for $u \in G \quad (\alpha = 1.2, ...m-1)$, so

$$\frac{I}{\mu}(x_0, t_0) \leq 0 \quad on D_1 \times [t_1, t^*].$$

In view of (3.2), we have

$$L\overline{d} \geq c(x,t) \; on \; D_1 \times [t_1, t^*].$$

By the classical strong maximum principle we have, $\overline{d} = 0$ on $D_1 \times [t_1, t^*]$, that is $u(D_1 \times [t_1, t^*] \subset \partial S$. Thus we have proved that $u^{-1}(\partial S)$ is relatively open in $D \times (0, t^*]$. Obviously $u^{-1}(\partial S)$ is relatively closed in$D \times (0, t^*]$. Hence $u(D \times (0, t^*]) \subset \partial S$.

To prove the remaining part of the theorem, we choose a bounded neighborhood D_2 of x_0 which is relatively open in $\overline{D}$ as well as a small $\delta > 0$ such that $u(D_2 \times (t_0 - \delta, t_0 + \delta) \subset G$. By the same way as above we have for some bounded c_0

$$L\overline{d} \geq c_0(x,t)\overline{d} \; on \; D_2 \times (t_0 - \delta, t_0 + \delta).$$

Thus the classical boundary point lemma gives the result. □

4. Concluding Remarks

If the strict inequality in condition (1) holds for all $(x, t) \in D \times (0, t)$, then there is no $(x^*, t^*) \in D \times (0, T)$. In theorem (3), S can be the intersection of several C^2 domains S_j which satisfy condition (1). (In the case that S_j's meet at angles less than $\pi/2$, by this theorem proof, we just need S to satisfy condition (1). Combining (3.3) with $d \equiv 0$, we have $I \geq 0$. In view of (3.3) we have that $k_\alpha > 0$ for all $\alpha = 1, ..., m-1$. Thus we can add to the theorem that if ∂S has positive Gaussian curvature everywhere, then u is independent of x when $0 < t \leq t^*$. Theorem (3) holds for elliptic systems corresponding to (1.1) with some modifications.Furthermore, it's also possible to extend the boundary point lemma for domains with corners, [3], [5].

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