

On θ -Euclidean Fuzzy k -ideals of Semirings

Prince Williams
Department of Information Technology
Salalah College of Technology
Post Box:608,Salalah-211
Sultanate of Oman
E-mail: *princeshree1@gmail.com*

Abstract. The concept of fuzzy ideals is extended by introducing θ -Euclidean fuzzy k -ideals in semirings. In this paper, factorization theorem of homomorphism on θ -Euclidean fuzzy k -ideal of semiring R is proved.

AMS (MOS) Subject Classification Codes: 08A72, 03E72.

Key Words: Fuzzy quotient semiring, semiring, θ -Euclidean L -fuzzy ideal, θ -Euclidean fuzzy k -ideal, Fuzzy k -ideal.

1. INTRODUCTION

The concept of fuzzy subset was introduced by Zadeh [15]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The concept of fuzzy subgroup was introduced by Rosenfeld [11] and was further studied by several authors [9, 6]. Liu [8] studied fuzzy ideals in rings. Subsequently, many authors [10, 12, 14] fuzzified certain concepts on rings and ideals. The theory of semirings has been studied by many authors [1, 3]. The fuzzy ideal of semiring is a good tool for us to study the fuzzy algebraic structure. Dutta and Biswas [4] studied different kinds of fuzzy ideals such as fuzzy k -ideals and fuzzy prime k -ideals of semirings and characterized fuzzy prime k -ideals of semirings of non-negative integers and determined all its prime k -ideals. Baik *et al.* [2] studied further results on fuzzy k -ideals of semirings. Further, Zhan *et al.* [19] fuzzified the concept of left k -ideals of semirings over t -norm and studied their related results. Then, Jun *et al.* [13] extended the concept of L -fuzzy ideal of a ring to a semiring. Moreover, Zhan *et al.* [17] fuzzified the concept h -ideals of hemirings and studied several results. Zhan [16] extended the concept of fuzzy left h -ideals of hemirings to fuzzy left h -ideals of hemirings over t -norm and studied many properties. Also, Zhan [18] introduced intuitionistic M -fuzzy h -ideals in M -hemirings. Koç and Balkanay [5] introduced the concept of θ -Euclidean L -fuzzy ideals of rings. Latha and Williams [7] introduced a notion of θ -Euclidean fuzzy k -ideals of semirings. In

this paper, we study the factorization theorem on homomorphism for a θ -Euclidean fuzzy k -ideals in semirings.

2. PRELIMINARIES

In this section, we cite the fundamental definitions that are used in the sequel:

Definition 1. [1] An algebraic system $(R; +, \cdot)$ is said to be a *semiring* if it satisfies the following conditions:

- (1): $(R; +)$ and $(R; \cdot)$ are semigroups,
- (2): $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$.

A semiring R may have an identity 1, defined by $1 \cdot a = a = a \cdot 1$ and a zero 0, defined by $0 + a = a = a + 0$ and $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in R$.

In what follows, we denote $x \cdot y = xy$, for all $x, y \in R$.

Definition 2. [9] A non-empty subset I of R is said to be a *left ideal* (resp., *right ideal*) if $x, y \in I$ and $r \in R$ imply that $x + y \in I$ and $rx \in I$ (resp., $xy \in I$).

If I is both left and right ideal of R , we say I is a *two-sided ideal*, or *simply ideal*, of R .

Definition 3. [4] An ideal I of a semiring R is said to be a *k-ideal* if $a \in I$ and $x \in R$, and if $x + a \in I$ or $a + x \in I$ then $x \in I$.

Definition 4. [10] A fuzzy subset $\mu : R \rightarrow [0, 1]$ of a semiring R is said to be a fuzzy left (resp., *right*) ideal of R if

- (F1): $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
- (F2): $\mu(xy) \geq \mu(y)$ (resp. *right*, $\mu(xy) \geq \mu(x)$) for all $x, y \in R$.

Definition 5. [4] A fuzzy ideal of a semiring R is said to be a fuzzy k -ideal of R if

$$\mu(x) \geq \min\{\max\{\mu(x + y), \mu(y + x)\}, \mu(y)\},$$

for all $x, y \in R$.

If R is an additively commutative semiring then the condition reduces to

$$\mu(x) \geq \min\{\mu(x + y), \mu(y)\}$$

for all $x, y \in R$.

Definition 6. [7] Let R be a semiring and let $\theta : R \rightarrow [0, 1]$ be a non-constant fuzzy subset of R . A fuzzy ideal $\mu : R \rightarrow [0, 1]$ is called a θ -Euclidean k -fuzzy ideal if μ satisfies the following axioms:

- (F3): $\mu(x) \geq \min\{\max\{\mu(x + y), \mu(y + x)\}, \mu(y)\}$, for all $x, y \in R$;
- (F4): For any $x, y \in R$ with $y \neq 0$, there exist elements $q, r \in R$ such that $x = yq + r$, where either $r = 0$ or else $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(y), \theta(y)\}$ for all $x, y \in R$.

Example

Let R be the set of Natural Numbers including zero and $\phi : R \rightarrow [0, 1]$ be a fuzzy subset defined by

$$\phi(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even,} \\ 0 & \text{if } a \text{ is odd.} \end{cases}$$

Let $\theta : R \rightarrow [0, 1]$ be a fuzzy subset defined by

$$\theta(a) = \begin{cases} 0 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a = 3, 5, 7, \dots, \\ \frac{1}{|a|} & \text{otherwise.} \end{cases}$$

Clearly ϕ is a fuzzy k -ideal of R , also ϕ is a θ -Euclidean fuzzy k -ideal of R .

Example

Let R be the set of Natural Numbers including zero and $\phi : R \rightarrow [0, 1]$ be a fuzzy subset defined by

$$\phi(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even,} \\ 0 & \text{if } a \text{ is odd.} \end{cases}$$

Let $\theta_1 : R \rightarrow [0, 1]$ be a fuzzy subset defined by

$$\theta_1(a) = \begin{cases} 0 & \text{if } a = 0 \\ \frac{1}{|a|} & \text{otherwise.} \end{cases}$$

So ϕ is a fuzzy k -ideal but θ_1 is not a θ -Euclidean fuzzy k -ideal of R .

3. FUZZY QUOTIENT SEMIRING

In this section, we introduce the notion of *fuzzy quotient semiring* and study the factorization theorem of homomorphism on θ -Euclidean fuzzy k -ideal of semiring R .

Definition 7. Let μ be a fuzzy ideal of R . For all $x \in R$, let $x + \mu$ be the fuzzy subset of R defined by

$$(x + \mu)(y) = \mu(y - x)$$

for all $y \in R$. The fuzzy subset $x + \mu$ is called a fuzzy coset of the fuzzy ideal R .

The set of all such fuzzy cosets will be denoted by $\frac{R}{\mu}$. Two binary operations (denoted by $+$ and \cdot) on $\frac{R}{\mu}$ are defined as follows:

$$(x + \mu) + (y + \mu) = (x + y) + \mu$$

and

$$(x + \mu) \cdot (y + \mu) = (x \cdot y) + \mu$$

for all $x, y \in R$. The above two operations are well defined and makes $\frac{R}{\mu}$ into a semiring, called the *fuzzy quotient semiring* of R by μ [5].

Theorem 8. Let R be a semiring and $\mu : R \rightarrow [0, 1]$ be a θ -Euclidean k -fuzzy ideal, $n : R \rightarrow \frac{R}{R_\mu}$ be the natural homomorphism. Also suppose that $\theta(a) = \theta(b)$ when $a - b \in \ker n$, for all $a, b \in R$. Let $\phi : \frac{R}{R_\mu} \rightarrow [0, 1]$ be defined by

$$\phi(a + R_\mu) = \mu(a).$$

Then there exists a unique θ^* ($= n(\theta)$)-Euclidean k -fuzzy ideal $\phi : \frac{R}{R_\mu} \rightarrow [0, 1]$ with the property that $\mu = \phi \circ n$.

Proof: First we shall show that the function ϕ is well defined. For all $a + R_\mu, b + R_\mu \in \frac{R}{R_\mu}$ and $a + R_\mu = b + R_\mu$, then there exists $x \in R_\mu$ such that $a - b = x$. Using the definition of R_μ we obtain

$$\begin{aligned}
\mu(x) &= \mu(0) \\
\mu(0) &= \mu(x) \\
\mu(0) &= \mu(a-b) \\
0 &= \mu(a) - \mu(b) \\
\mu(a) &= \mu(b) \\
\phi(a + R_\mu) &= \phi(b + R_\mu)
\end{aligned}$$

Hence ϕ is well defined.

(i) For any $a + R_\mu, b + R_\mu \in \frac{R}{R_\mu}$, for all $a, b \in R$, then

$$\begin{aligned}
\phi[(a + R_\mu) + (b + R_\mu)] &= \phi[(a + b) + R_\mu] \\
&= \mu(a + b) \\
&\geq \min[\mu(a), \mu(b)], \text{ by (F1)}
\end{aligned}$$

Thus

$$\phi[(a + R_\mu) + (b + R_\mu)] \geq \min[\phi(a + R_\mu) + \phi(b + R_\mu)].$$

Also, we have

$$\phi[(a + R_\mu) \cdot (b + R_\mu)] = \phi[(ab) + R_\mu] = \mu(ab) \geq \mu(b), \text{ by (F2)}$$

Thus

$$\phi[(a + R_\mu) \cdot (b + R_\mu)] \geq \phi(b + R_\mu).$$

(ii) For any $a + R_\mu, b + R_\mu \in \frac{R}{R_\mu}$, for all $a, b \in R$, then

$$\begin{aligned}
\phi[(a + R_\mu)] &= \mu(a) \\
&\geq \min\{\max\{\mu(a + b), \mu(b + a)\}, \mu(b)\},
\end{aligned}$$

since μ is a θ -Euclidean k -fuzzy ideal

$$\begin{aligned}
&= \min\{\max\{\phi(a + b + R_\mu), \phi(b + a + R_\mu)\}, \phi(b + R_\mu)\} \\
&= \min\{\max\{\phi[(a + R_\mu) + (b + R_\mu)], \phi[(b + R_\mu) + (a + R_\mu)]\}, \phi(b + R_\mu)\}
\end{aligned}$$

Thus, ϕ is a k -fuzzy ideal.

(iii) For any $a + R_\mu, b + R_\mu \in \frac{R}{R_\mu}$ such that $R_\mu \neq b + R_\mu$, for all $a, b \in R$, then

$$\begin{aligned}
&\Rightarrow b \notin R_\mu \Rightarrow \mu(b) \neq \mu(0) \\
&\Rightarrow b \neq 0.
\end{aligned}$$

So $0 \neq b \in R$. Since μ is a θ -Euclidean fuzzy k -ideal of R , then there exist elements $q, r \in R$ such that $a = bq + r$ where $r = 0$ or else $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(b), \theta(b)\}$.

$$\begin{aligned}
a &= bq + r \\
a + R_\mu &= bq + r + R_\mu \\
a + R_\mu &= (bq + R_\mu) + (r + R_\mu) \\
a + R_\mu &= (b + R_\mu)(q + R_\mu) + (r + R_\mu)
\end{aligned}$$

If $r = 0$ then $r + R_\mu = 0 + R_\mu$, thus $r + R_\mu = R_\mu$. For all $r, q \in R$, then $r + R_\mu, q + R_\mu \in$

$\frac{R}{R_\mu}$.

Let $r + R_\mu = r'$. If $n(z) = r'$ and $n(r) = r'$ then $n(z - r) = 0$. This means $z - r \in \ker n$. we get $\theta(z) = \theta(r)$. Thus

$$n(\theta)(r') = \bigcup \left\{ \theta(z) \mid z \in n^{-1}(r') \right\} = \theta(r).$$

If $\max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(b), \theta(b) \}$, then

$$\max \{ \phi(r + R_\mu), n(\theta)(r') \} \geq \max \{ \phi(b + R_\mu), n(\theta)(b') \}.$$

Thus

$$\max \{ \phi(r + R_\mu), \theta^*(r + R_\mu) \} \geq \max \{ \phi(b + R_\mu), \theta^*(b + R_\mu) \}.$$

Hence, μ is θ^* -Euclidean fuzzy k -ideal from $\frac{R}{R_\mu}$ to $[0, 1]$.

Also, for all $a \in \mu, \mu(a) = \phi(a + R_\mu) = \phi(n(a)) = (\phi \circ n)(a)$. It means that $\mu = \phi \circ n$. Now, we need to show that this factorization is unique. Suppose $\phi' \circ n = \mu$, for some other $\theta^*(=n(\theta))$ -Euclidean fuzzy k -ideal $\phi' : \frac{R}{R_\mu} \rightarrow [0, 1]$.

If $a + R_\mu \in \frac{R}{R_\mu}$, then

$$\phi(a + R_\mu) = \mu(a) = (\phi' \circ n)(a) = \phi'(a + R_\mu),$$

for all $a \in R$. Thus $\phi = \phi'$. Hence, ϕ is unique $\theta^*(=n(\theta))$ -Euclidean fuzzy k -ideal from $\frac{R}{R_\mu}$ into $[0, 1]$ with the property that $\mu = \phi \circ n$.

Corollary 9. Let μ be a θ -Euclidean fuzzy k -ideal, $n : R \rightarrow \frac{R}{R_\mu}$ be the natural homomorphism.. Suppose that $\theta(a) = \theta(b)$ when $a - b \in \ker n$. Then there exists a θ^* -Euclidean fuzzy k -ideal from $\frac{R}{R_\mu} \rightarrow [0, 1]$.

Proof: Since $\mu : R \rightarrow [0, 1]$ is a θ -Euclidean fuzzy k -ideal from Theorem 8, $\phi : \frac{R}{R_\mu} \rightarrow [0, 1]$ is a θ^* -Euclidean fuzzy k -ideal. Also, the semirings $\frac{R}{R_\mu}$ and $\frac{R}{R_\mu}$ are isomorphic. So there exists a θ^* -Euclidean fuzzy k -ideal from $\frac{R}{R_\mu}$ to $[0, 1]$.

Acknowledgement: The author is very grateful to the referees for their valuable comments and suggestions on this work.

REFERENCES

1. P. J. Allen, *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc **21** (1969), 412–416.
2. S. I. Baik and H. S. Kim, *On fuzzy k -ideals in semiring*, Kongweon-Kyungki. Math. Jour. **8** (2000), 147–154.
3. L. Dale, *Direct sums of semirings and the krull-schmidt theorem*, Kyungpook Math. **17** (1977), 135–141.
4. T. K. Dutta and B. K. Biswas, *Fuzzy k -ideals of semirings*, Bull. Calcutta Math. Soc. **87** (1995), 91–96.
5. Ayten Koc and Erol Balkanay, *θ -euclidean l -fuzzy ideals of rings*, Turk. J. Math. **26** (2002), 149–158.
6. N. Kuroki, *Fuzzy bi-ideals in semigroups*, Comment. Math. Univ. St. Pauli **28** (1979), 17–21.
7. K. B. Latha and D. R. Prince Williams, *θ -euclidean fuzzy k -ideals of semirings*, Global Journal of Pure and Applied Mathematics **2** (2006), 55–60.
8. W. J. Liu, *Fuzzy invariants subgroups and fuzzy ideals*, Fuzzy sets and Sys. **8** (1987), 219–226.
9. D. S. Malik and H. Kummer, *Fuzzy ideals in semigroups*, Fuzzy Sets and Systems **48** (1992), 137–140.
10. T. K. Mukherjee and M. K. Sen, *On fuzzy ideals of a semirings i* , Fuzzy Sets and Systems **21** (1987), 99–104.
11. A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
12. K. L. N. Swamy and U. M. Swamy, *Fuzzy prime ideals of rings*, J. Math. Anal. Appl. **134** (1988), 94–103.
13. J. Neggers Y. B. Jun and H. S. Kim, *On l -fuzzy ideals in semirings i* , Czech. Math. J. **48** (1998), 669–675.
14. Zhang Yue, *Prime l -fuzzy ideals of rings*, Fuzzy Sets and Systems **27** (1988), 345–350.
15. L. A. Zadeh, *Fuzzy sets*, Inform and Control **8** (1965), 338–353.

16. Jianming Zhan, *On properties of fuzzy left h-ideals in hemirings with t-norms*, Int. J. Math. Math. Sci. (2005), 3127–3144.
17. Jianming Zhan and W. A. Dudek, *Fuzzy h-ideals of hemirings*, Inform. Sci. **117** (2007), 876–886.
18. Jianming Zhan and K. P. Shum, *Intuitionistic m-fuzzy h-ideal in m-hemiring*, J.Fuzzy Math. **14** (2006), 929–945.
19. Jianming Zhan and Zhisong Tan, *T-fuzzy left k-ideals of semiring*, Sci. Math.Jpn **58** (2003), 597–601.