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Cubic Coefficients Estimates and Asymptotic Properties of the Estimates from the Two Parameters Rayleigh Distribution

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Abstract. A particular linear estimate with four terms approximation called Cubic Coefficient Estimate (CCE) using ordered observations is derived by using general theory of linear coefficients with polynomial coefficients Downton [3]. This estimate is applicable for complete samples and is shown to yield highly efficient estimators even in the case of small samples. The asymptotic properties of the Cubic Coefficients estimates are also described. The Cubic Coefficient Estimates also applicable to, and can even be simplified for, one parameter distributions of the type $F(x/\lambda)$.

1. Introduction

A common statistical problem is to estimate the unknown location and scale parameters from the distribution of the form $F\left(\frac{x-\mu}{\lambda}\right)$. In general, μ and λ may be mean and standard deviation and this is not always necessary and in certain circumstances μ may be the percentage point of the distribution and λ may be defined as the range of variation of the variate X.
Lloyd [9] using least squares method obtained the estimates of μ and λ using ordered observations Lloyd's method gives the exact solution of the minimum variance unbiased estimation of the location and scale parameters. This method becomes impracticable for certain distribution as the ordered moments of certain distributions are not available for a sample of size more than 10.
Moreover it involves the calculation of $\frac{1}{2}n(n-1)$ double integrals and 2n single integrals. Lloyds method can be used for singly or doubly censored data.
Various approximate methods have been devised to overcome this difficulty. Blom [1] and Weiss [11] devised approximate method using ordered observations. Blom's nearly unbiased method is quite efficient for small samples but its standard error may be poor.

The approximate methods suggest that the efficiency of the determination of linear estimates does not seem particularly sensitive to the changes in the coefficients and may be chosen for convenience. Hirai [4] considered the estimation of the parameters from the Rayleigh distribution by linear coefficient method and in [6]

by the Quadratic Coefficients method as a particular case of Downtion's General Theory to estimate μ and λ from the Rayleigh distribution.
In this paper, therefore we discuss the properties of the moments of ordered random variables and then derive the Cubic Coefficient Estimate (CCE) and apply Cubic Coefficient Estimate to estimate the two parameters of the Rayleigh distribution and also discuss its asympotic properties.
To this end we first recall the following:

2. Properties of the Moments of Ordered Random Variables

If $x_1^{(n)} \leq x_2^{(n)} \leq x_3^{(n)} \leq \cdots x_n^{(n)}$ are the available ordered observations from the distribution $F\left(\frac{x-\mu}{\lambda}\right)$, where μ and λ are the unknown parameters to be estimated. We make the transformation $y_i^{(n)} = \frac{x_i^{(n)}-\mu}{\lambda}$ $(i = 1, 2, \cdots, n)$ such that $y_1^{(n)} \leq y_2^{(n)} \leq \cdots y_n^{(n)}$ are the realizations of the set of random variables $Y_1^{(n)} \leq Y_2^{(n)}, \cdots, Y_n^{(n)}$.
We denote

$$E(Y_i^{(n)}) = \alpha_i^{(n)} = \int_{-\infty}^{\infty} x \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}dF(x) \quad (2.\ 1)$$

$$E(Y_i^{(n)})^2 = W_{ii}^{(n)} = \int_{-\infty}^{\infty} x^2 \frac{n!}{(i-1)!(n-i)!}[F(x_i)]^{i-1}[1-F(x_i)]^{n-i}dF(x) \quad (2.\ 2)$$

and

$$\begin{aligned} E(Y_i^{(n)}, Y_j^{(n)}) i < j = W_{ij}^{(n)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} xy \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times \\ & [F(x)]^{i-1}[F(Y)-F(x)]^{j-i-1} \times \\ & [1-F(y)^{n-j}]dF(x)dF(y) \end{aligned} \quad (2.\ 3)$$

Also

$$Cov\ (Y_i^{(n)}, Y_i^{(n)}) = W_{ij}^{(n)} - \alpha_i^{(n)}\alpha_j^{(n)} \quad (2.\ 4)$$

We denote $m^{(r)}$ with m and r integers, as the rth factorial power of m i.e.
$m^{(r)} = \frac{m!}{(m-r)!} = m(m-1)\cdots(m-r+1)$ and introduce also the two identities

$$(a+b)^{(m)} = \sum_{r=0}^{m} \binom{m}{r} a^{(r)}b^{(m-r)} \quad (2.\ 5)$$

$$(a-b)^{(m)} = \sum_{r=0}^{m} (-1)^r (a-r)^{(m-r)} b^{(r)} \quad (2.\ 6)$$

We note that

$$\sum_{i=1}^{n} (i-1)^{(k)} = \frac{1}{k+1} n^{(k+1)} \quad (2.\ 7)$$

Property 1

$$\begin{aligned}\sum_{i=1}^{n}(i-1)^{(k)}\alpha_i^{(n)} &= \sum_{i=k+1}^{n}(i-1)^{(k)}\alpha_i^{(n)} \\ &= \int_{-\infty}^{\infty} xf(x)\sum_{i=k+1}^{n}\frac{n!(i-1)^{(k)}}{(i-1)!(n-i)!}\times \\ &\quad [F(x)]^{i-1}[1-F(x)]^{n-1}dF(x) \\ &= n^{(k+1)}\int_{-\infty}^{\infty}[F(x)]^k xf(x)\sum_{i=k+1}^{n}\begin{pmatrix} n-k-1 \\ i-k-1\end{pmatrix} \\ &\quad [F(x)]^{i-k-1}[1-F(x)]^{n-1}dx\end{aligned}$$

Setting $s = i - k - 1$ we get after simplification

$$\sum(i-1)^{(k)}\alpha_i^{(n)} = \frac{n^{(k+1)}}{(k+1)}\alpha_{k+1}^{(k+1)} \quad (2.\ 8)$$

Property 2

$$\sum_{i=1}^{n}(i-1)^{(k)}w_{ii}^{(n)} = \frac{n^{(k+1)}}{(k+1)}W_{k+1,k+1}^{(k+1)} \quad (2.\ 9)$$

The result follows immediately from property 1

Property 3

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(i-1)^{(p)}(j-1)^{(q)}W_{ij}^{(n)} &= \sum_{i=1}^{n}(i-1)^{(p)}(i-1)^{(q)}w_{ii}^{(n)} \\ &\quad +\sum_{i<j}\{(i-1)^{(p)}(j-1)^{(q)} \\ &\quad +(j-1)^{(p)}(i-1)^{(q)}\}w_{ij}^{(n)} \qquad (2.\ 10)\end{aligned}$$

Property 4

$$\sum_{i=1}^{n-1}(i-1)^{(k)}(n-i)^{(l)}W_{ij}^{(n)} = \frac{n!k!l!}{(k+l+2)!(n-k-l-1)!}W_{l+1,k+1}^{(k+l+2)} \quad (2.\ 11)$$

For the proofs of the last 3 and 4 properties we refer to Hirai [5], David [2] and Hirai [8].

3. Coraviance between Linear Function

$$\psi_k = \frac{k+1}{n^{(k+1)}}\sum_{i=1}^{n}(i-1)^{(k)}Y_i^{(n)} \quad (3.\ 12)$$

$$\psi_s = \frac{s+1}{n^{(s+1)}}\sum_{i=1}^{n}(i-1)^{(s)}Y_i^{(n)} \quad (3.\ 13)$$

By definition we have

$$Cov(\phi_i,\phi_j) = \Omega_{ij} = \Omega_{ji} = \frac{(i+1)(j+1)}{n^{(i+1)}n^{(j+1)}}\times$$

$$E\left\{\sum_{r=1}^{n}\sum_{s=1}^{n}(r-1)^{(i)}(s-1)^{(j)}Y_r^{(n)}Y_s^{(n)}\right\}-\alpha_{i+1}^{(i+1)}\alpha_{j+1}^{(j+1)} \tag{3. 14}$$

Now

$$\begin{aligned} E\sum_{r=1}^{n}\sum_{s=1}^{n}(r-1)^{(i)}(s-1)^{(j)}Y_r^{(n)}Y_s^{(n)} &= \sum_{s=r=1}^{n}(r-1)^{(i)}(r-1)^{(j)}w_{rr}^{(n)} \\ &+\sum_{r}\sum_{<s}\{(r-1)^{(i)}(s-1)^{(j)} \\ &+(s-1)^{(i)}(r-1)^{(j)}\}W_{rs}^{(n)} \end{aligned} \tag{3. 15}$$

Using the properties of ordered random variables we have

$$\begin{aligned} &= \sum_{t=0}^{i}\binom{i}{t} j^{(i-t)}(j+1)!\binom{n}{j+t+1}W_{j+t+1,j+t+1}^{(j+t+1)} \\ &+\sum_{t=0}^{i}(-1)^t\binom{i}{t}(n-t-1)^{(i-t)}j!t!\binom{n}{j+t+2}W_{j+1,j+2}^{(j+t+2)} \\ &+\sum_{t=0}^{i}(-1)^t\binom{j}{t}(n-t-1)^{(j-t)}j!t!\binom{n}{j+t+2}W_{i+1,i+2}^{(i+t+2)} \\ &= S_1+S_2+S_3 \end{aligned} \tag{3. 16}$$

In S_1 we substitute $v=j+t+1$ and obtain

$$S_1=\sum_{v=j+1}^{i+j+1}\frac{i!j!n^{(v)}}{(i+j+1-v)!(v-j-1)!}W_{u,v}^{(v)} \tag{3. 17}$$

Using the identity (2.5) we see that

$$\begin{aligned} (n-t-1)^{(i-t)} &= (n-j-t-2+j+1)^{(i-t)} \\ &= \sum_{r=0}^{i-t}\binom{i-t}{r}(n-j-t-2)^{(r)}(j+1)^{(i-t-r)} \end{aligned} \tag{3. 18}$$

and writing $t=s-r$ we obtain

$$S_2 = \sum_{t=0}^{i}\binom{i}{t}(-1)^t\frac{j!t!}{(j+t+2)}W_{j+1,j+2}^{(j+t+2)}\sum_{s=t}^{i}\binom{i-t}{s-t}n^{(j+s+2)}(j+1)^{(i-s)} \tag{3. 19}$$

Changing the order of summation and putting $v=j+s^1+2$ we have

$$S_2=\sum_{u=j+2}^{i+j+2}\sum_{t=0}^{u-j-2}\frac{(-1)^t!j!(j+1)^{(i+j+2-v)}n^{(v)}}{(v-j-2-t)!(i+j+2-v)!(j+t+2)!}W_{j+1,j+2}^{(j+t+2)} \tag{3. 20}$$

By symmetry we put $i=j$ in (3.20) then

$$S_3=\sum_{s=0}^{i}\sum_{t=0}^{s}(-1)^t\frac{i!j!(n)^{(i+s+2)}}{(i+t+2)!(s-t)!(j-2)!}(i+1)^{(j-s)}W_{i+1,i+2}^{(i+t+2)}$$

Writing the value of $W^{(i+t+2)}_{i+1,i+2}$ and after little simplification we get

$$S_3 = \sum_{s=0}^{j} \frac{j!n^{(i+s+2)}(i+1)!}{(j-s)!s!(i+1-j+s)!} \int_{-\infty}^{\infty}\int_{-\infty}^{y} xy[F(x)]^i[F(y)]^s dF(x)\ dF(y) \quad (3.\ 21)$$

Now we have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{y} xy[F(x)]^i[F(y)]^s dF(x)dF(y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy[F(x)]^i[F(y)]^s dF(x)dF(y) \\ &\quad -\int_{-\infty}^{\infty}\int_{y}^{\infty} xy[F(y)]^s[1-\overline{1-F(x)}]^i \\ &\quad dF(x)dF(y) \\ &= \frac{i!\alpha^{(i+1)}_{i+1}}{(i+1)!} - \frac{s!\alpha^{(s+1)}_{s+1}}{(s+1)!} \\ &\quad -\int_{-\infty}^{\infty}\int_{-\infty}^{y} yx[F(x)]^t \sum_{r=0}^{i}(-1)^r \binom{i}{r} \\ &\quad [1-F(y)]^r dF(x)dF(y) \qquad (3.\ 22)\end{aligned}$$

Putting (3.22) in (3.21) and writing $v = i+s+2$.

$$S_3 = \sum_{v=i+2}^{i+j+2} \frac{j!n^{(v)}(i+1)^{(i+j+2-v)}}{(i+j+2-v)!}\left[\frac{\alpha^{(i+1)}_{i+1}\alpha^{(v-i-1)}_{(v-i-1)}}{(i+1)!(v-i-1)!} - \sum_{r=u}^{i}(-1)^r i! W^{(v-1+r)}_{v-1-i,v-1}\right]. \quad (3.\ 23)$$

This vanishes for $i=j,\ v=j+1$
Adding (3.17), (3.20) and (3.23) we have factorial series of the form i.e.

$$\begin{aligned} S_1+S_2+S_3 &= \sum_{r=1}^{n}\sum_{s=1}^{n}(r-1)^i(s-1)^{(j)}W^{(n)}_{rs} \\ &= \sum_{v=j+1}^{i+j+2} a^{(v)}_{ij} n^{(v)} \\ &= \sum_{t=j+1}^{i+j+2} a^{(t)}_{ij} n^{(t)} \qquad (3.\ 24)\end{aligned}$$

Hence we get

$$\Omega_{ij} = \frac{(i+1)(j+1)\sum_{t=j+1}^{i+j+2} a^{(t)}_{ij} n^{(t)}}{n^{(i+1)}n^{(j+1)}} - \alpha^{(i+1)}_{i+1}\alpha^{(j+1)}_{j+1} \qquad (3.\ 25)$$

We can write

$$\begin{aligned} n^{(t)} &= n(n-1)\cdots(n-j)(n-j-1)\cdots(n-t+1) \\ &= n^{(j+1)}(n-j-1)^{(t-j-1)} \qquad (3.\ 26)\end{aligned}$$

Using again the identity (2.5)

$$n^{(t+1)} = (n-j-1+j+1)^{(i+1)} = \sum_{s=0}^{i+1}\binom{i+1}{s}(n-j-1)^{(s)}(j+1)^{(i+1-s)} \quad (3.\ 27)$$

Putting (3.26) and (3.27) in (3.25) we get after simplification

$$\begin{aligned}\Omega_{ij} = \Omega_{ji} &= \frac{(i+1)(j+1)}{n^{(t+1)}}\left[\sum_{s=0}^{i+1} a_{ij}^{(s+j+1)}(n-j-1)^{(s)} \times \right.\\ &\left. \frac{\alpha_{i+1}^{(i+1)}\alpha_{j+1}^{(j+1)}}{(i+1)(j+1)}\sum_{s=0}^{i+1}\begin{pmatrix} i+1 \\ s \end{pmatrix}(n-j-1)^{(s)}(j+1)^{(i+1-s)}\right],\ i \leq j \\ &= \frac{(i+1)(j+1)}{n^{(i+1)}}\sum_{S=0}^{i} b_{ij}^{(S)}(n-j-1)^{(S)} \qquad (3.\ 28)\end{aligned}$$

Using Downton's notation we have

$$\begin{aligned}b_{ij}^{(s)} &= i!j!W_{s+j+1,s+j+1}^{(s+j+1)}/(i-s)!(s+j+i)!(s+j-i)!(s+j+1)!s! \\ &+ i!j!(j+1)^{(i+1-s)}\sum_{r=0}^{s-1}(-1)^r W_{j+1,j+2}^{(j+2+r)}/(i+1-s)!(j+2+r)!(s-1-r)! \\ &+ i!j!(i+1)^{(i+1-s)}\sum_{r=0}^{i}(-1)^r W_{s+j-i,s+j-i+1}^{(s+j-i+1+r)}/(i+1-s)!(i-r)! \\ &(s+j-i+1+r)! + i!j!\alpha_{i+1}^{(i+1)}[\alpha_{s+j-i}^{(s+j-i)} - \alpha_{j+1}^{(j+1)}]/(i+1-s)!(s+j-i)!S! \\ & \qquad (3.\ 29)\end{aligned}$$

When $s = 0$, some of these terms vanish in $b_{ij}^{(s)}$. These coefficients depend only upon diagonal and next diagonal terms of relatively small variance matrices of ordered observations and expected value of the largest observations. In the evaluation of $b_{ij}^{(s)}$ we note that

$$\begin{aligned}(i) \quad & W_{12}^{(2)} = (\alpha_1^{(1)})^2 \\ (ii) \quad & W_{23}^{(4)} = 4W_{23}^{(3)} - 3(\alpha_2^{(2)})^2 \qquad (3.\ 30) \\ (iii) \quad & W_{23}^{(3)} = 3\alpha_1^{(1)}\alpha_2^{(2)} - 3(\alpha_1^{(1)})^2 + W_{12}^{(3)}\end{aligned}$$

(i) By definition we have

$$W_{12}^{(2)} = 2\int_{-\infty}^{\infty}\int_{-\infty}^{y} xy dF(x) dF(y)$$

put

$$\int_{-\infty}^{y} x f(x) dx = g(y)$$

then

$$g'(y) = y f(y)$$

thus

$$W_{12}^{(2)} = 2\int_{-\infty}^{\infty} g(y)g'(y)dy = [g(y)^2]_{-\infty}^{\infty} = [\int_{-\infty}^{\infty} x f(x) dx]^2 = (\alpha_1^{(1)})^2 \qquad (3.\ 31)$$

Similarly (i) and (ii) can be proved.
and hence $b_{ij}^{(S)}$ can further be simplified.

4. Cubic Coefficients Estimates (CCE)

In Downton's general theory if we take 4 terms approximation and call it Cubic Coefficient Estimate (CCE). Suppose we have an ordered sample $x_1^{(n)} \leq x_2^{(n)} \cdots \leq x_n^{(n)}$ from the distribution $F\left(\frac{x-\mu}{\lambda}\right)$ where μ and λ are unknown parameters. We want to estimate a parametric function $p = k_1\mu + k_2\lambda$ using the available ordered observation where k_1 and k_2 are known constants.
Let

$$y_1^{(n)} = \frac{x_i^{(n)} - \mu}{\lambda} \quad i = 1, 2, \cdots, n \tag{4. 32}$$

then $y_1^{(n)} \leq y_2^{(n)} \cdots \leq y_n^{(n)}$ are the realizations of the random variables $Y_1^{(n)} \leq Y_2^{(n)} \cdots \leq Y_n^{(n)}$.
We define an estimate function Cubic Coefficient Estimate (CCE) for p as

$$U = at + bg + cs + de \tag{4. 33}$$

where

$$t = \sum_{i=1}^{n} (i-1)^{(0)} x_i^{(n)}, \; g = \sum_{i=1}^{n} (i-1)^{(1)} x_i^{(n)}, \; s = \sum_{i=1}^{n} (i-1)^{(2)} x_i^{(n)}$$

$$e = \sum_{i=1}^{n} (i-1)^{(3)} x_i^{(n)} \tag{4. 34}$$

Now using (2.9) we see that

$$\begin{aligned} E(U) &= \left(\frac{an^{(1)}}{1} + \frac{bn^{(2)}}{2} + \frac{cn^{(3)}}{3} + \frac{dn^{(4)}}{4}\right)\mu \\ &+ \left(\frac{an^{(1)}}{1}\alpha_1^{(1)} + \frac{bn^{(2)}}{2}\alpha_2^{(2)} + \frac{cn^{(3)}}{3}\alpha_3^{(3)} + \frac{dn^{(4)}}{4}\alpha_4^{(4)}\right)\lambda \end{aligned} \tag{4. 35}$$

U will be unbiased estimator of p if the coefficients are so chosen such that

$$\vec{a}\vec{\psi} = \vec{k}$$

where

$$\vec{k} = [k_1 k_2] \tag{4. 36}$$

and

$$a = [a \; b \; c \; d] : \; \psi' = \begin{bmatrix} \frac{n^{(1)}}{1} & \frac{n^{(2)}}{2} & \frac{n^{(3)}}{3} & \frac{n^{(4)}}{4} \\ \frac{n^{(1)}}{1}\alpha_1^{(1)} & \frac{n^{(2)}}{2}\alpha_2^{(2)} & \frac{n^{(3)}}{3}\alpha_3^{(3)} & \frac{n^{(4)}}{4}\alpha_4^{(4)} \end{bmatrix} \tag{4. 37}$$

We define

$$\begin{aligned} T^* &= \sum_{i=1}^{n} (i-1)^{(0)} Y_i^{(n)}; \; G^* = \sum_{i=1}^{n} (i-1)^{(1)} Y_i^{(n)} \\ S^* &= \sum_{i=1}^{n} (i-1)^{(2)} Y_i^{(n)}; \; E = \sum_{i=1}^{n} (i-1)^{(3)} Y_i^{(n)} \end{aligned} \tag{4. 38}$$

as random variables.
Now we denote

$$Var\begin{pmatrix} * \\ T \end{pmatrix} = \Omega_{00}; \; Var\begin{pmatrix} * \\ G \end{pmatrix} = \Omega_{11}$$

$$\Omega_{01} = Cov\begin{pmatrix} * & * \\ T & G \end{pmatrix}; \cdots Cov\begin{pmatrix} * & * \\ G, & E \end{pmatrix} = \Omega_{23}$$

and thus covariance matrix is

$$\Omega = \begin{bmatrix} \Omega_{00} & \Omega_{01} & \Omega_{02} & \Omega_{03} \\ & \Omega_{11} & \Omega_{12} & \Omega_{13} \\ & & \Omega_{22} & \Omega_{23} \\ & & & \Omega_{33} \end{bmatrix}$$

Hence we have

$$Var(U) = \lambda^2 \vec{a}\vec{\Omega}\vec{a} \tag{4. 39}$$

Thus we have to minimize the expression subject to restraint given in (4.36)

$$\chi = \lambda^2 \vec{a}\vec{\Omega}\vec{a'} + \vec{\phi}\vec{\psi'}\vec{a'} \tag{4. 40}$$

Where $\vec{\phi} = [\phi_1, \phi_2]$ and ϕ_1 and ϕ_2 are the undetermined Lagrangian multipliers. The minimization of χ is obtained by solutions of the equations with $\vec{a}\vec{\psi} = \vec{k}$ yields.

$$\vec{a} = \vec{k}[\vec{\psi'}\vec{\Omega^{-1}}\vec{\psi}]^{-1}\vec{\psi'}\vec{\Omega^{-1}} \tag{4. 41}$$

and

$$Var(U) = \lambda^2 \vec{k}[\vec{\psi'}\vec{\Omega^{-1}}\vec{\psi}]^{-1}k' \tag{4. 42}$$

If $k = [1, 0]$ we get μ^* as an estimate of μ and similarly if $k = [0, 1]$ we get λ^* as an estimate of λ. We may some times be interested in other values of k_1 and k_2.

5. Estimates of μ and λ from the Rayleigh Distribution

The probability density function of the two parameters Rayleigh distribution is

$$f(x) = \frac{2(x-\mu)}{\lambda^2} e^{-(x-\mu)^2/\lambda^2};\ \mu \le x < \infty\ \lambda > 0 \tag{5. 43}$$

The M.L.E. of λ say $\hat{\lambda}$ is given by

$$\hat{\lambda} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \hat{\mu})^2}{n}} \tag{5. 44}$$

where $x_1^{(n)} = \hat{\mu}$ the smallest observation.

As the lower range depends upon μ we can not calculate the large sample variances of these estimates.

For the evaluation of the coefficients of μ^* and λ^* by Cubic Coefficient Estimate (CCE) we have

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} \frac{\theta_0}{n} & \frac{2\theta_1}{n^{(2)}} & \frac{3\theta_3}{n^{(3)}} & \frac{4\theta_4}{n^{(4)}} \\ \\ \frac{\{_0}{n} & \frac{2\{_1}{n^{(2)}} & \frac{3\{_3}{n^{(3)}} & \frac{4\{_4}{n^{(4)}} \end{bmatrix} \tag{5. 45}$$

where

$$\begin{bmatrix} \vec{\theta'_0} \\ \cdots \\ \vec{\{'_0} \end{bmatrix} = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_0 \\ \cdots & \cdots & \cdots & \cdots \\ \{_0 & \{_1 & \{_2 & \{_3 \end{bmatrix} = \Omega^{-1}[\vec{1} \vdots \vec{\alpha}] \left[\begin{bmatrix} \vec{1'} \\ \cdots \\ \vec{\alpha'} \end{bmatrix} \Omega^{-1}[\vec{1} \vdots \vec{\alpha}] \right]^{-1} \tag{5. 46}$$

and

$$\alpha' = [\alpha_1^{(1)}, \alpha_2^{(2)}, \alpha_3^{(3)}, \alpha_4^{(4)}] \text{ and } 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (5.\ 47)$$

Using the moments of order statistics Hirai [7] from the Rayleigh distribution we have the elements of $\vec{\Omega}$ from the Rayleigh distribution for $n = 5, 6, 7$ in Table 1.

Table 1

The Elements of Ω in Cubic Coefficient Estimate (CCE) from the Rayleigh Distribution for n = 5, 6, 7

$n=5$	0.049204	0.0509054	0.0545378	0.0567213
		0.0681764	0.0777652	0.0841323
			0.0926029	0.10322768
				0.1180682
$n=6$	0.357670	0.0424211	0.0454481	0.0472677
		0.564425	0.0641847	0.0693179
			0.0759378	0.0943095
				0.0956505
$n=7$	0.0306574	0.0363610	0.0389555	0.0405152
		0.0564423	0.0641847	0.0693179
			0.0759378	0.0843095
				0.0956505

Hence the coefficient of μ^* of μ and λ^* of λ in Cubic Coefficient Estimate (CCE) are respectively as

$$a_{11} = a_1;\ a_{12} = a_1 + a_2;\ a_{13} = a_1 + 2a_2 + 2a_3 \cdots a_{1n} = a_1 + (n-1)a_2 + (n-1)(n-2)a_3 + (n-1)(n-2)(n-3)a_4 \quad (5.\ 48)$$

Similarly

$$a_{21} = b_1;\ a_{22} = b_1 + b_2;\ \text{and } a_{2n} = b_1 + (n-1)b_2 + (n-1)(n-2)b_3 + (n-1)(n-2)(n-3)b_4 \quad (5.\ 49)$$

These coefficients are given in Tables 2 and 3 and variance - covariances of these estimates are given in Table 4 for n = 5, 6, 7.

Table 2

Coefficients of μ^* by Cubic Coefficients Estimate (CCE) from the Rayleigh Distribution for $5 \leq n \leq 7$

n	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
5	1.243 4932	0.2607 969	−0.058 8164	−0.119 6135	−0.325 8612		
6	1.086 9110	0.3489 273	0.0183 732	−0.078 3446	−0.1138 614	−0.262 2495	
7	0.096 0048	0.3954 130	0.0901 670	−0.039 7372	−0.0783 446	−0.109 7002	−0.21 78499

Table 3

Coefficients of λ^* by Cubic Coefficient Estimate (CCE) from the Rayleigh Distribution for $5 \leq n \leq 7$

n	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}
5	-1.097 3312	-0.116 28640	0.2276 022	0.34 3886	0.642 1268		
6	-0.955 78813	-0.228 8629	-0.109 2469	0.231 7671	0.341 117	0.5317 147	
7	0.8446 933	-0.228 8629	-0.014 704	0.156 7133	0.220 3651	0.2907 3770	0.452 9087

Table 4

Variance and Covariances of μ^* and λ^* from the Rayleigh Distribution for n = 5, 6, 7. Each value should be multiplied by λ^2

n	5	6	7
Var (μ^*)	0.086745	0.06734607	0.0548885
Var(λ^*)	0.1300687	0.1024643	0.08451043
Cov(μ^*, λ^*)	-0.0826940	-0.034785	-0.051375

Hence we conclude that the Cubic Coefficient Estimate (CCE) can replace the best linear unbiased estimate (Lloyd's) method from the efficiency point of view Cubic Coefficient Estimate (CCE) is quite simple even for small samples and involves less calculations. This method is applicable for complete samples and no viable technique is as yet available to extend it to censored data.

6. Asymptotic Properties of Cubic Coefficient Estimate (CCE)

We have from (3.28)

$$\Omega_{ij} = \frac{(i+1)(j+1)}{n^{(i+1)}}[b^0_{ij} + (n-j-1)^{(1)}b^{(1)}_{ij} + \cdots + (n-j-1)^{(i)}b^{(i)}_{ij}] \quad (6.\ 50)$$

and this becomes

$$\simeq \frac{(i+1)(j+1)}{n}b^{(i)}_{ij}$$

Let

$$\vec{B} = \frac{1}{n}\begin{bmatrix} b^{(u)}_{00} & 2b^{(0)}_{01} & 3b^{(0)}_{02} & 4b^{(0)}_{03} \\ & 4b^{(1)}_{11} & 6b^{(1)}_{12} & 28b^{(1)}_{13} \\ & & 9b^{(2)}_{22} & 12b^{(2)}_{23} \\ & & & 16b^{(3)}_{33} \end{bmatrix} \quad (6.\ 51)$$

If $|B| \neq 0$ then asymptotic coefficients $\mu^{'*}$ and $\lambda^{'*}$ of μ and λ are given by

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} \frac{\theta_0}{n} & \frac{2\theta_1}{n^{(2)}} & \frac{3\theta_2}{n^{(3)}} & \frac{4\theta_3}{n^{(4)}} \\ \frac{\{_0}{n} & \frac{2\{_1}{n^{(2)}} & \frac{3\{_2}{n^{(3)}} & \frac{4\{_3}{n^{(4)}} \end{bmatrix}$$

where

$$\begin{bmatrix} \vec{\theta'_0} \\ \cdots \\ \vec{\{'_0} \end{bmatrix} = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \\ \cdots & \cdots & \cdots & \cdots \\ \{_0 & \{_1 & \{_2 & \{_3 \end{bmatrix} = \vec{B}^{-1}[\vec{1}\vdots\vec{\alpha}] \left[\begin{bmatrix} \bar{1} \\ \cdots \\ \bar{\alpha} \end{bmatrix} B^{-1}[\vec{1}\vdots\vec{\alpha}] \right]^{-1} \quad (6.\ 52)$$

and the variances of the estimates are calculated while calculating these coefficients.

Table 5

Asymptotic Coefficients of estimate μ'^* and λ'^* from the Rayleigh distribution for $n = 5, 6, 7$ are given in Tables 5 & 6

n	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
5	1.3693 861	0.3479 291	−0.2367 639	−0.3846 929	−0.0958 579		
6	1.1411 550	0.4601 837	−0.0024 056	−0.2466 129	−0.2724 382	−0.272 4382	
7	0.9784 186	0.4920 105	0.1303 920	−0.1064 369	−0.2184 962	−0.205 7259	−0.068 1860

Table 6

n	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
5	−1.150 2775	−0.23 4223	0.340 9165	0.575 1389	0.468 4449		
6	−0.958 5644	−0.034 78609	0.092 4344	0.362 3215	0.461 8004	0.390 8711	
7	0.821 6267	0.385 4099	−0.046 5979	−0194 8093	0.338 8177	0.385 4093	0.334 6021

The asymptotic variance covariance of μ'^* and λ'^* from the Rayleigh distribution for n = 5, 6, 7 are given in Table 7. Each value may be multiplied by λ^2.

Table 7

n	5	6	7
Var (μ'^*)	0.0585286	0.0487738	0.0418061
Var(λ'^*)	0.0970063	0.0808386	0.0692902
Cov(μ'^*, λ'^*)	−0.0524101	−0.0436751	−0.0374358

Mostellet [10] has shown that linear combination of order statistics tends to normality for large samples and therefore it can be shown that these estimates also tend to normality as the sample size becomes very large.

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