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On Jordan k-Derivations of 2-Torsion Free Prime Γ_N-Rings

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Abstract. In this article, we define k-derivation and Jordan k-derivation of Γ-rings as well as different types of Γ-rings, and develop some important results relating to these concepts. In general, every Jordan k-derivation of a Γ-ring M is not a k-derivation of M. We prove that every Jordan k-derivation of a 2-torsion free prime Γ-ring (in the sense of Nobusawa) is a k-derivation.

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1. Introduction

Let M and Γ be two additive abelian groups. If there exists a mapping $(a, \alpha, b) \mapsto a\alpha b$ of $M \times \Gamma \times M \to M$ satisfying the following for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:
(a) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ and
(b) $(a\alpha b)\beta c = a\alpha(b\beta c)$,
then M is called a Γ-ring. This definition is due to Barnes [1].

If, in addition to the above, there exists a mapping $(\alpha, a, \beta) \mapsto \alpha a\beta$ of $\Gamma \times M \times \Gamma \to \Gamma$ satisfying the following for all $a, b \in M$ and $\alpha, \beta, \gamma \in \Gamma$:
(a*) $(\alpha + \beta)a\gamma = \alpha a\gamma + \beta a\gamma$, $\alpha(a + b)\beta = \alpha a\beta + \alpha b\beta$, $\alpha a(\beta + \gamma) = \alpha a\beta + \alpha a\gamma$,
(b*) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and
(c*) $a\alpha b = 0$ for all $a, b \in M$ implies $\alpha = 0$,
then M is called a Γ-ring in the sense of Nobusawa[4], or simply, a Nobusawa Γ-ring and we say that M is a Γ_N-ring. Clearly, M is a Γ_N-ring always implies that Γ is an M-ring.

Let M be a Γ-ring. Then M is said to be 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in M$. Besides, M is called a prime Γ-ring if, for all $a, b \in M$, $a\Gamma M\Gamma b = 0$ implies either $a = 0$ or $b = 0$. And, M is called semiprime if $a\Gamma M\Gamma a = 0$ with $a \in M$ implies $a = 0$. Note that every prime Γ-ring is obviously semiprime.

The notions of derivation and Jordan derivation of a Γ-ring has been introduced by M. Sapanci and A. Nakajima in [5], whereas, the concept of k-derivation of a Γ-ring has been used and developed by H. Kandamar[3]. Afterwards, the concept of Jordan generalized derivation of a Γ-ring has been developed by Y. Ceven and M. A. Ozturk in [2].

Here we introduce the concept of Jordan k-derivation of a Γ-ring as follows and then we build up a relationship between the k-derivation and Jordan k-derivation of a Γ-ring in a concrete manner.

Let M be a Γ-ring and let $d : M \to M$ and $k : \Gamma \to \Gamma$ be two additive mappings. If $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$, then d is called a derivation of M. And, for all $a, b \in M$ and $\alpha \in \Gamma$, if $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$ is satisfied, then d is called a k-derivation of M. Finally, if $d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$ holds for all $a \in M$ and $\alpha \in \Gamma$, then d is called a Jordan k-derivation of M.

From these definitions it is clear that every k-derivation of a Γ-ring M is a Jordan k-derivation of M. But, the converse statement is not true in general. Here we show that every Jordan k-derivation of a 2-torsion free prime Γ_N-ring M is a k-derivation of M. For this to happen we develop some important results as follows.

2. Main Results

Lemma 1. *Let M be a Γ_N-ring and let d be a Jordan k-derivation of M. Then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:*
(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha d(b) + b\alpha d(a)$;
(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + ak(\alpha)b\beta a + ak(\beta)b\alpha a$
$+ a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha bk(\beta)a + a\beta bk(\alpha)a + a\alpha b\beta d(a) + a\beta b\alpha d(a)$.
In particular, if M is 2-torsion free, then
(iii) $d(a\alpha b\alpha a) = d(a)\alpha b\alpha a + ak(\alpha)b\alpha a + a\alpha d(b)\alpha a + a\alpha bk(\alpha)a + a\alpha b\alpha d(a)$;
(iv) $d(a\alpha b\alpha c + c\alpha b\alpha a) = d(a)\alpha b\alpha c + d(c)\alpha b\alpha a + ak(\alpha)b\alpha c + ck(\alpha)b\alpha a$
$+ a\alpha d(b)\alpha c + c\alpha d(b)\alpha a + a\alpha bk(\alpha)c + c\alpha bk(\alpha)a + a\alpha b\alpha d(c) + c\alpha b\alpha d(a)$.
Especially, if M is 2-torsion free and if $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then
(v) $d(a\alpha b\beta a) = d(a)\alpha b\beta a + ak(\alpha)b\beta a + a\alpha d(b)\beta a + a\alpha bk(\beta)a + a\alpha b\beta d(a)$;
(vi) $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + ak(\alpha)b\beta c + ck(\alpha)b\beta a$
$+ a\alpha d(b)\beta c + c\alpha d(b)\beta a + a\alpha bk(\beta)c + c\alpha bk(\beta)a + a\alpha b\beta d(c) + c\alpha b\beta d(a)$.

Proof. Compute $d((a + b)\alpha(a + b))$ and cancel the like terms from both sides to obtain (i). Then replace $a\beta b + b\beta a$ for b in (i) to get (ii). Since M is 2-torsion free, (iii) is easily obtained by replacing α for β in (ii), and then (iv) is obtained by replacing $a + c$ for a in (iii). Again, since M is 2-torsion free and $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, (v) follows from (ii) and then finally, (vi) is obtained by replacing $a + c$ for a in (v). □

Lemma 2. *Let d be a Jordan k-derivation of a 2-torsion free Γ_N-ring M. Then for all $b \in M$ and $\beta \in \Gamma$, $k(\beta b\beta) = k(\beta)b\beta + \beta d(b)\beta + \beta bk(\beta)$.*

Proof. For all $a \in M$ and $\alpha \in \Gamma$, we have $d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$. Let $b \in M$ and $\beta \in \Gamma$. Then putting $\beta b\beta$ for α, we get $d(a\beta b\beta a) = d(a)\beta b\beta a + ak(\beta b\beta)a + a\beta b\beta d(a)$. Expanding the LHS by Lemma (1)(iii), we obtain $a(k(\beta b\beta) - k(\beta)b\beta - \beta d(b)\beta - \beta bk(\beta))a = 0$. Hence, by applying the Nobusawa condition (c*) of the definition of Γ_N-ring, we get the proof. □

Lemma 3. *If d is a Jordan k_1-derivation as well as a Jordan k_2-derivation of a 2-torsion free Γ_N-ring M, then $k_1 = k_2$.*

Proof. Obvious. □

Remark 4. If d is a Jordan k-derivation of a 2-torsion free Γ_N-ring M, then k is uniquely determined.

Definition 5. Let M be a Γ-ring. Then for $a, b \in M$ and $\alpha \in \Gamma$, we define $[a, b]_\alpha = a\alpha b - b\alpha a$.

Lemma 6. *If M is a Γ-ring, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $[a, b]_\alpha + [b, a]_\alpha = 0$;
(ii) $[a + b, c]_\alpha = [a, c]_\alpha + [b, c]_\alpha$;
(iii) $[a, b + c]_\alpha = [a, b]_\alpha + [a, c]_\alpha$;
(iv) $[a, b]_{\alpha+\beta} = [a, b]_\alpha + [a, b]_\beta$.

Proof. Obvious. □

Remark 7. Note that a Γ-ring M is commutative if and only if $[a, b]_\alpha = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 8. Let d be a Jordan k-derivation of a Γ_N-ring M. Then for $a, b \in M$ and $\alpha \in \Gamma$, we define $F_\alpha(a, b) = d(a\alpha b) - d(a)\alpha b - ak(\alpha)b - a\alpha d(b)$.

Then we have, $F_\alpha(b, a) = d(b\alpha a) - d(b)\alpha a - bk(\alpha)a - b\alpha d(a)$.

Lemma 9. *l9 If d is a Jordan k-derivation of a Γ_N-ring M, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $F_\alpha(a, b) + F_\alpha(b, a) = 0$;
(ii) $F_\alpha(a + b, c) = F_\alpha(a, c) + F_\alpha(b, c)$;
(iii) $F_\alpha(a, b + c) = F_\alpha(a, b) + F_\alpha(a, c)$;
(iv) $F_{\alpha+\beta}(a, b) = F_\alpha(a, b) + F_\beta(a, b)$.

Proof. Obvious. □

Remark 10. Note that d is a k-derivation of a Γ_N-ring M if and only if $F_\alpha(a, b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Lemma 11. *Let d be a Jordan k-derivation of a 2-torsion free Γ_N-ring M and suppose that $a, b \in M$ and $\alpha, \beta \in \Gamma$. Then*
(i) $F_\alpha(a, b)\alpha m\alpha[a, b]_\alpha + [a, b]_\alpha \alpha m\alpha F_\alpha(a, b) = 0$;
(ii) $F_\alpha(a, b)\beta m\beta[a, b]_\alpha + [a, b]_\alpha \beta m\beta F_\alpha(a, b) = 0$;
(iii) $F_\beta(a, b)\alpha m\alpha[a, b]_\beta + [a, b]_\beta \alpha m\alpha F_\beta(a, b) = 0$.

Proof. (i) Consider $G = d(a\alpha b\alpha m\alpha b\alpha a + b\alpha a\alpha m\alpha a\alpha b)$.
First, compute $G = d(a\alpha(b\alpha m\alpha b)\alpha a) + d(b\alpha(a\alpha m\alpha a)\alpha b)$ using Lemma 1 (iii) and then, $G = d((a\alpha b)\alpha m\alpha(b\alpha a) + (b\alpha a)\alpha m\alpha(a\alpha b))$ using Lemma 1 (iv). Since these two are equal, cancelling the similar terms from both sides of this equality and then rearranging them with the use of Lemma 2.5(i), we obtain the result of (i).

(ii) Considering $G = d(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$ and proceeding in the same way as in the proof of (i) by the similar arguments, we get (ii).

(iii) Interchanging α and β in (ii), we obtain (iii). □

Lemma 12. *Let M be a 2-torsion free semiprime Γ_N-ring and suppose that $a, b \in M$. If $a\Gamma m\Gamma b + b\Gamma m\Gamma a = 0$ for all $m \in M$, then $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$.*

Proof. Let m and m' be two arbitrary elements of M. Then by hypothesis, we have

$$\begin{gathered}(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b) = -(b\Gamma m\Gamma a)\Gamma m'\Gamma(a\Gamma m\Gamma b)\\ = -(b\Gamma(m\Gamma a\Gamma m')\Gamma a)\Gamma m\Gamma b = (a\Gamma(m\Gamma a\Gamma m')\Gamma b)\Gamma m\Gamma b\\ = a\Gamma m\Gamma(a\Gamma m'\Gamma b)\Gamma m\Gamma b = -a\Gamma m\Gamma(b\Gamma m'\Gamma a)\Gamma m\Gamma b\\ = -(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b).\end{gathered}$$

This implies, $2((a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)) = 0$.
Since M is 2-torsion free, $(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b) = 0$.
By the semiprimeness of M, $a\Gamma m\Gamma b = 0$ for all $m \in M$.
Hence we get, $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$ for all $m \in M$. □

Corollary 13. *If M is a 2-torsion free semiprime Γ_N-ring, then for all $a, b \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $F_\alpha(a,b)\alpha m\alpha[a,b]_\alpha = [a,b]_\alpha\alpha m\alpha F_\alpha(a,b) = 0$;
(ii) $F_\alpha(a,b)\beta m\beta[a,b]_\alpha = [a,b]_\alpha\beta m\beta F_\alpha(a,b) = 0$;
(iii) $F_\beta(a,b)\alpha m\alpha[a,b]_\beta = [a,b]_\beta\alpha m\alpha F_\beta(a,b) = 0$.

Proof. Using Lemma 12 in the result of Lemma 11, we obtain these results. □

Theorem 14. *Let M be a 2-torsion free semiprime Γ_N-ring. Then for all $a, b, c, d \in M$ and $\alpha, \beta, \gamma \in \Gamma$,*
(i) $F_\alpha(a,b)\alpha m\alpha[c,d]_\alpha = 0$;
(ii) $F_\alpha(a,b)\beta m\beta[c,d]_\alpha = 0$;
(iii) $F_\alpha(a,b)\alpha m\alpha[c,d]_\beta = 0$.

Proof. Replacing $a + c$ for a in Corollary 13 (i), we get

$$F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha + F_\alpha(c,b)\alpha m\alpha[a,b]_\alpha = 0.$$

Therefore, we get

$$\begin{gathered}F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha\alpha m\alpha F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha\\ = -F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha\alpha m\alpha F_\alpha(c,b)\alpha m\alpha[a,b]_\alpha = 0.\end{gathered}$$

Hence, by the semiprimeness of M, $F_\alpha(a,b)\alpha m\alpha[c,b]_\alpha = 0$.

Similarly, by replacing $b + d$ for b in this equality, we get

$$F_\alpha(a,b)\alpha m\alpha[c,d]_\alpha = 0.$$

Proceeding in the same way as before by the similar replacements in Corollary 13 (ii), we obtain (ii).

Finally, replacing $\alpha + \beta$ for α in (i), we get

$$F_\alpha(a,b)\alpha m\alpha[c,d]_\beta + F_\beta(a,b)\alpha m\alpha[c,d]_\alpha = 0.$$

Therefore, we have

$$F_\alpha(a,b)\alpha m\alpha[c,d]_\beta\alpha m\alpha F_\alpha(a,b)\alpha m\alpha[c,d]_\beta$$
$$= -F_\alpha(a,b)\alpha m\alpha[c,d]_\beta\alpha m\alpha F_\beta(a,b)\alpha m\alpha[c,d]_\alpha = 0.$$

Hence, by the semiprimeness of M, we get $F_\alpha(a,b)\alpha m\alpha[c,d]_\beta = 0$. □

Theorem 15. *Every Jordan k-derivation of a 2-torsion free prime Γ_N-ring M is a k-derivation of M.*

Proof. Let d be a Jordan k-derivation of a 2-torsion free prime Γ_N-ring M. Since M is prime, we get from Theorem 14 (i) that either $F_\alpha(a,b) = 0$ or, $[c,d]_\alpha = 0$ for all $a,b,c,d \in M$ and $\alpha \in \Gamma$.

If $[c,d]_\alpha \neq 0$ for all $c,d \in M$ and $\alpha \in \Gamma$. Then $F_\alpha(a,b) = 0$ for all $a,b \in M$ and $\alpha \in \Gamma$ and hence we get, d is a k-derivation of M.

But, if $[c,d]_\alpha = 0$ for all $c,d \in M$ and $\alpha \in \Gamma$, then M is commutative and, therefore, we have from Lemma 1 (i),

$$2d(a\alpha b) = 2d(a)\alpha b + 2ak(\alpha)b + 2a\alpha d(b).$$

Since M is 2-torsion free, we obtain that d is a k-derivation of M. □

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