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Stability Analysis of Predator-Prey Population Model with Time Delay and Constant Rate of Harvesting

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Abstract. This paper studies the effect of time delay and harvesting on the dynamics of the predator - prey model with a time delay in the growth rate of the prey equation. The predator and prey are then harvested with constant rates. The constant rates may drive the model to one, two, or none positive equilibrium points. When there exist two positive equilibrium points, one of them is possibly stable. In the case of the constant rates are quite small and the equilibrium point is not stable, an asymptotically stable limit cycle occurs. The result showed that the time delay can induce instability of the stable equilibrium point, Hopf bifurcation and stability switches.

Key Words: Predator-prey, Limit cycle, Time delay, Harvesting rate, Hopf bifurcation.

1. Introduction

The Lotka-Volterra model is one of the earliest predator-prey models to be based on sound mathematical principles. It forms the basis of many models used today in the analysis of population dynamics and is one of the most popular models in mathematical ecology. In both the analysis and experiment, the predator and prey can coexist by reducing the frequency of contact between them, Luckinbill [13]. In the context of predator-prey interaction, some studies that treat population can be extended by considering harvesting, stocking, diffusion, and time delay. In the model with harvesting, some studies relate the population to the economic problems. The time delay is considered into the population dynamics when the rate of change of the population is not only a function of the present population but also depends on the past population.
One predator-one prey system in Hogart et al. [10] where both the predator and prey are harvested with constant yield has been considered and the stability at maximum sustainable yield is established. Martin and Ruan [14] have analyzed generalized Gause predator-prey models where the prey is harvested with constant rate while Kar [12] considered the

predator-prey model with the predator harvested and suggested that it is ideal to study the combined harvesting of predator and prey population models. The effect of constant rate of harvesting has been studied by Holmberg [11] and the results showed that the constant catch quota can lead to both oscillations and chaos and an increased risk for over exploitation. While the effects on population size and yield of different levels of harvesting of a predator in a predator-prey system have been explored by Matsuda and Abrams [15] and showed that the predator may increase in population size with increasing £shing effort.

Brauer and Soudack [3] have analyzed the global behavior of a predator-prey system under constant rate predator harvesting. They showed how to classify the possibilities and determine the region of stability. They found that if the equilibrium point is asymptotically stable, which is determined by a local linearization, then every solution whose initial value is in some neighborhood of the stable equilibrium point tends to it as the time approaches in£nity. There exists an asymptotically stable limit cycle when the constant rate is small and the equilibrium point is unstable. A predator-prey model with Holling type using harvesting efforts as control has been presented by Srinivasu et al. [17] and showed that with harvesting, it is possible to break the cyclic behavior of the system and introduces a globally stable limit cycle in the system.

The effect of constant rate of harvesting on the dynamics of predator-prey systems has been investigated by many authors, see, for example, Brauer and Soudack [2, 4], Myerscough et al. [16], Dai and Tang [7], Xiao and Ruan [18]. Some interesting dynamical behaviors have been observed such as the stability of the equilibria, existence of Hopf bifurcation and limit cycles. It is also observed that in some cases, before a catastrophic harvest rate is reached the effect of harvesting is to stabilize the equilibrium point of the population system. In this paper we present a deterministic and continuous model for predator - prey population based on Lotka - Volterra model which is extended by incorporating time delay and constant rates of harvesting of both populations. The objective of this paper is to study the combined effects of harvesting and time delay on the dynamics of predator-prey model.

2. The Predator - Prey Population Model

We consider a predator - prey model based on Lotka - Volterra model with one predator and one prey populations. The model for the rate of change of prey population (x) and predator population (y) is

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{K}) - \alpha xy \\ \frac{dy}{dt} &= -cy + \beta xy. \end{aligned} \tag{2.1}$$

The model includes parameter K, the carrying capacity, for the prey population in the absence of the predator. The parameter r is the intrinsic growth rate of the prey, c is the mortality rate if the predator without prey, α measures the rate of consumption of prey by the predator, β measures the conversion of prey consumed into the predator reproduction rate. All the parameters are assumed to be positive.

The equilibrium points of model (2.1) are $(0,0)$, $(K,0)$ and $E^* = (x^*, y^*) = (\frac{c}{b}, \frac{r(K\beta-c)}{\alpha\beta K})$. In order to get a positive equilibrium point we assume that $K\beta - c > 0$. The Jacobian matrix of model (2.1) takes the form

$$J = \begin{pmatrix} r - \frac{2rx}{K} - \alpha y & -\alpha x \\ \beta y & -c + bx \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix J at the equilibrium point E^* is $f(\lambda) = \lambda^2 + \frac{cr}{\beta K}\lambda + \frac{cr}{\beta K}(\beta K - c)$ and the eigenvalues have negative real parts. It means that the equilibrium points E^* is locally asymptotically stable. Furthermore, since $K\beta - c > 0$ then the equilibrium point E^* is also globally asymptotically stable, see Ho and Ou [9].

3. The Predator-Prey Model with Time Delay and Constant Rate of Harvesting

We consider the predator and prey populations of model (2.1) where both populations are subjected to a constant rate of harvesting. Before we go to the model with time delay, we need to analyze the stability of the equilibrium point of the model without time delay. The model without time delay is

$$\begin{aligned} \frac{dx}{dt} &= x(r - bx - \alpha y) - H_x \\ \frac{dy}{dt} &= y(-c + \beta x) - H_y, \end{aligned} \tag{3.1}$$

where r, $b = \frac{r}{K}$, α, c, β, H_x, H_y are positive constants. The constants H_x and H_y denote the rate of harvesting for the populations x and y respectively.

By setting $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ then we have the relations

$$x(r - bx - \alpha y) = H_x \tag{3.2}$$

$$y(-c + \beta x) = H_y. \tag{3.3}$$

From (3.2) we have $y = \frac{rx - bx^2 - H_x}{\alpha x}$ which follows that $r^2 - 4bH_x$ should be positive in order to get the equilibrium point in the positive quadrant. Hence we have to assume that $H_x < \frac{r^2}{4b}$. Since H_y is positive, then from (3.3) we should assume that $x > \frac{c}{\beta}$.

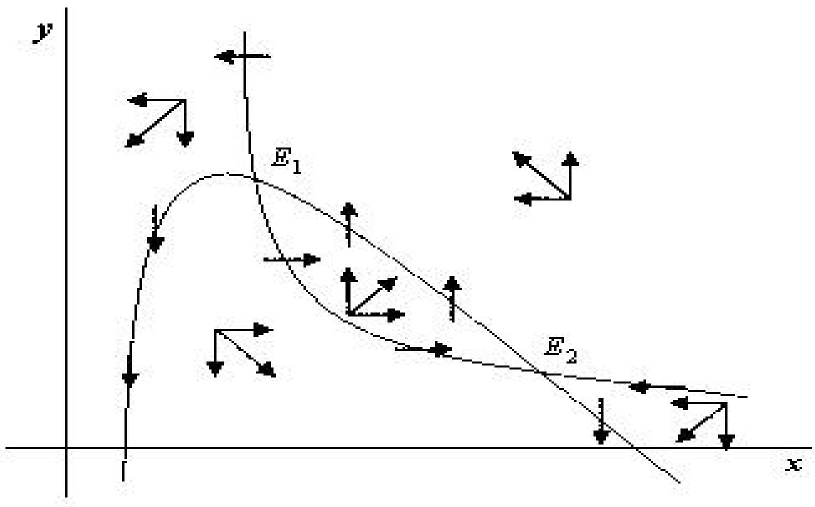

FIGURE 1. Phase plane and directions of the trajectories

From the phase plane, we know that it is possible to get two, one or no equilibrium points, Figure 1. There are two positive equilibrium points of the model when

$b\beta x^3 - (r\beta + bc)x^2 + (H_x\beta + H_y\alpha + rc)x - H_x c < 0$, for some positive $x > \frac{c}{\beta}$. Let the two equilibrium points be $E_1 = (x_1, y_1)$ and $E_2 = (x_2, y_2)$. The equilibrium point E_1 is possible to be asymptotically stable, while the equilibrium point E_2 is not stable, it is a saddle point.

To analyze the stability of the equilibrium point E_1 we linearize the model around the equilibrium point E_1 . The Jacobian matrix of the model is

$$J = \begin{pmatrix} r - 2bx - \alpha y & -\alpha x \\ \beta y & -c + \beta x \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix at this point is

$$\lambda^2 - (P + S)\lambda + PS + QR = 0, \tag{3.4}$$

where

$$\begin{aligned} P &= r - 2bx_1 - \alpha y_1, \\ Q &= \alpha x_1, \\ R &= \beta y_1, \quad \text{and} \\ S &= -c + \beta x_1. \end{aligned}$$

Then the equilibrium point E_1 is asymptotically stable when $PS + QR > 0$ and $P + S < 0$.

Example 1. *Consider model (3.1) with parameters* $r = 1$, $b = 0.01$, $\alpha = 1$, $c = 0.3$, $\beta = 0.05$, $H_x = 0.01$, *and* $H_y = 0.02$. *The equilibrium points of the model in the positive quadrant are* $E_1 = (6.42819, 0.93416)$ *and* $E_2 = (99.56243, 0.00428)$. *The eigenvalues associated with the equilibrium point* E_1 *are* $-0.02066 \pm 0.54633i$ *and the eigenvalues associated with equilibrium point* E_2 *are* -0.99177 *and* 4.67437. *This reveals that the equilibrium point* E_1 *is asymptotically stable while the equilibrium point* E_2 *is a saddle point and unstable.*

Example 2. *Consider again model (3.1) with parameters* $r = 1$, $b = 0.01$, $\alpha = 1$, $c = 0.3$, $\beta = 0.05$, $H_x = 0.1$, *and* $H_y = 0.2$. *There are two equilibrium points of the model in the positive quadrant, they are,* $E_1 = (10.51819, 0.88531)$ *and* $E_2 = (95.42203, 0.04473)$. *The equilibrium point* E_1 *has eigenvalues* $-0.06512 \pm 0.66313i$ *and the equilibrium point* E_2 *has eigenvalues* -0.91354 *and* 4.43147. *This means that both equilibrium points are not stable.*

From Examples 1 and 2 we know that the equilibrium E_1 may be a stable or an unstable equilibrium point. It depends on the values of the parameters and the level of constant rate of harvesting. Apparently, the equilibrium point E_1 tends to the equilibrium point E^* when the harvesting function H_x and H_y approach zero. If the equilibrium point E^* for the non-harvesting model is asymptotically stable, then the eigenvalues of the Jacobian matrix of the linearized system have negative real parts. Since the eigenvalues are continuous in H_x and H_y, the equilibrium point E_1 is asymptotically stable for suf£ciently small $H_x > 0$ and $H_y > 0$. On the other hand, when the equilibrium point E_1 is unstable, there exists an asymptotically stable limit cycle. Theory of perturbation of periodic solutions, Coddington and Levinson [5], shows that there is an asymptotically stable limit cycle for small $H_x > 0$ and $H_y > 0$. Thus, the qualitative behavior of the system for $H_x = 0$ and $H_y = 0$ carries over to small $H_x > 0$ and $H_y > 0$.

Now we consider the predator - prey population model with time delay and constant rate of harvesting. Both predator and prey populations are subjected to constant rate of harvesting. The model is

$$\begin{aligned}\frac{dx(t)}{dt} &= rx(t) - bx(t)x(t-\tau) - \alpha x(t)y(t) - H_x,\\ \frac{dy(t)}{dt} &= -cy(t) + \beta x(t)y(t) - H_y.\end{aligned} \tag{3.5}$$

A predator-prey model with time delays in the growth rate of the predator population and the prey harvested with constant rate has been analyzed by Martin and Ruan [14]. They showed that the time delays can induce instability, oscillations via Hopf bifurcation and switching stability.

To linearize the model about the equilibrium point E_1 of model (3.5), let $u(t) = x(t) - x_1$ and $v(t) = y(t) - y_1$. We then obtain the linearized model

$$\begin{aligned}\dot{u}(t) &= (r - bx_1 - \alpha y_1)u(t) - bx_1 u(t-\tau) - \alpha x_1 v(t)\\ \dot{v}(t) &= \beta y_1 u(t) + (-c + \beta x_1)v(t).\end{aligned} \tag{3.6}$$

From the linearized model we obtain the characteristic equation

$$\Delta(\lambda,\tau) = \lambda^2 + a_1\lambda e^{-\lambda\tau} - a_2\lambda - a_3 e^{-\lambda\tau} + a_4, \tag{3.7}$$

where

$$\begin{aligned}a_1 &= bx_1\\ a_2 &= r - c - bx_1 + \beta x_1 - \alpha y_1\\ a_3 &= -bcx_1 + b\beta x_1^2, \quad \text{and}\\ a_4 &= -rc + a\beta x_1 + bcx_1 + b\beta x_1^2 - \alpha c y_1.\end{aligned}$$

For $\tau = 0$, the characteristic equation (3.7) becomes $\lambda^2 + (a_1 - a_2)\lambda - a_3 - a_4 = 0$. This characteristic equation is the same with the characteristic equation (3.7). The eigenvalues of the characteristic equation are either real and negative or complex conjugate with negative real parts if and only if

$$a_1 - a_2 > 0 \quad \text{and} \quad -a_3 + a_4 > 0. \tag{3.8}$$

Hence, in the absence of time delay, the equilibrium point E_1 is locally asymptotically stable if and only if both conditions $a_1 - a_2 > 0$ and $-a_3 + a_4 > 0$ are satis£ed.

Now for $\tau \neq 0$, if $\lambda = i\omega$, $\omega > 0$, is a root for the characteristic equation (3.7), then we have

$$\omega^2 + a_1 i\omega\cos(\omega\tau) + a_1\omega\sin(\omega\tau) - a_2 i\omega - a_3\cos(\omega\tau) + a_3 i\sin(\omega\tau) + a_4 = 0.$$

Separating the real and imaginary parts, we get

$$\begin{aligned}-\omega^2 + a_4 + a_1\omega\sin(\omega\tau) - a_3\cos(\omega\tau) &= 0\\ -a_2\omega + a_1\omega\cos(\omega\tau) + a_3\sin(\omega\tau) &= 0,\end{aligned}$$

or equivalently

$$\begin{aligned}-\omega^2 + a_4 &= -a_1\omega\sin(\omega\tau) + a_3\cos(\omega\tau)\\ a_2\omega &= a_1\omega\cos(\omega\tau) + a_3\sin(\omega\tau).\end{aligned} \tag{3.9}$$

Squaring both sides gives

$$\omega^4 - 2a_4\omega^2 + a_4^2 = a_1^2\omega^2 \sin^2(\omega\tau) - 2a_1a_3\omega \sin(\omega\tau)\cos(\omega\tau) + a_3^2 \cos^2(\omega\tau)$$
$$a_2^2\omega^2 = a_1^2\omega^2 \cos^2(\omega\tau) + 2a_1a_3\omega \sin(\omega\tau)\cos(\omega\tau) + a_3^2 \sin^2(\omega\tau).$$

Adding both equations and regrouping by powers of ω, we obtain the following fourth degree polynomial

$$\omega^4 - (a_1^2 + 2a_4 - a_2^2)\omega^2 + a_4^2 - a_3^2 = 0. \tag{3.10}$$

Then we obtain

$$\omega_\pm^2 = \frac{1}{2}\{(a_1^2 + 2a_4 - a_2^2) \pm \sqrt{(a_1^2 + 2a_4 - a_2^2)^2 - 4(a_4^2 - a_3^2)}\}. \tag{3.11}$$

From the equation (3.11), it follows that if

$$a_2^2 - 2a_4 - a_1^2 > 0 \quad \text{and} \quad a_4^2 - a_3^2 > 0, \tag{3.12}$$

then the equation (3.10) does not have any real solutions.

To £nd the necessary and suf£cient conditions for nonexistence of time delay induced instability, we now use the following theorem.

Theorem 3. *(Kar,* [12]*). A set of necessary and suf£cient conditions for an equilibrium point* (x_*, y_*) *to be asymptotically stable for all* $\tau \geq 0$ *is*

(1) *The real parts of all the roots of* $\Delta(\lambda, 0) = 0$ *are negative,*

(2) *For all real* ω *and* $\tau \geq 0$, $\Delta(i\omega, \tau) \neq 0$, *where* $i = \sqrt{-1}$.

Theorem 4. *If conditions (3.8), (3.12) and Theorem 3 are satis£ed, then the equilibrium point* E_1 *is locally asymptotically stable for all* $\tau \geq 0$.

Again, if

$$\begin{aligned} a_4^2 &- a_3^2 > 0, a_2^2 - 2a_4 - a_1^2 < 0, \quad \text{and} \\ (a_2^2 &- 2a_4 - a_1^2)^2 > 4(a_4^2 - a_3^2), \end{aligned} \tag{3.13}$$

hold, then there are two positive solutions of $\omega_\pm^2$. Substituting $\omega_\pm^2$ into equation (3.9) and solving for τ, we obtain

$$\tau_k^\pm = \frac{1}{\omega_\pm} \arctan\{\frac{\omega_\pm(a_1\omega_\pm^2 - a_1a_4 + a_2a_3)}{a_1a_2\omega_\pm^2 + a_3(a_4 - \omega_\pm^2)}\} + \frac{2k\pi}{\omega_\pm}, k = 0, 1, 2, \cdots. \tag{3.14}$$

Differentiating equation (3.7) with respect to τ, we obtain

$$(2\lambda - a_2 + a_1e^{-\lambda\tau} - \tau(a_1\lambda - a_3e^{-\lambda\tau}))\tfrac{d\lambda}{d\tau} = \lambda(a_1\lambda - a_3)e^{-\lambda\tau},$$

therefore

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda - a_2}{\lambda(a_1\lambda - a_3)e^{-\lambda\tau}} + \frac{a_1}{\lambda(a_1\lambda - a_3)} - \frac{\tau}{\lambda}.$$

From equation (3.7), we have $e^{-\lambda\tau} = \frac{-(\lambda^2 - a_2\lambda + a_4)}{(a_1\lambda - a_3)}$. Then we obtain

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda - a_2}{-\lambda(\lambda^2 - a_2\lambda - a_4)} + \frac{a_1}{\lambda(a_1\lambda - a_3)} - \frac{\tau}{\lambda}.$$

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda - a_2}{-\lambda(\lambda^2 - a_2\lambda - a_4)} + \frac{a_1}{\lambda(a_1\lambda - a_3)} - \frac{\tau}{\lambda}.$$

Thus,

$$\begin{aligned}
\text{sign}\frac{d(\text{Re}\lambda)}{d\tau}_{\lambda=i\omega} &= \text{sign}\{\text{Re}(\frac{d\lambda}{d\tau})^{-1}\}_{\lambda=i\omega} \\
&= \text{sign}\{\text{Re}(\frac{2\lambda - a_2}{-\lambda(\lambda^2 - a_2\lambda + a_4)})_{\lambda=i\omega} + \text{Re}(\frac{a_1}{\lambda(a_1\lambda - a_3)})_{\lambda=i\omega} \\
&+ \text{Re}(\frac{-\tau}{\lambda})_{\lambda=i\omega}\} \\
&= \text{sign}\{\text{Re}(\frac{2i\omega - a_2}{-i\omega(-\omega^2 - a_2 i\omega + a_4)}) + \text{Re}(\frac{a_1}{i\omega(a_1 i\omega - a_3)}) \\
&+ \text{Re}(\frac{-\tau}{i\omega})\} \\
&= \text{sign}\{\frac{a_2^2 + 2\omega^2 - 2a_4}{a_2^2\omega^2 + (-\omega^2 + a_4)^2} - \frac{a_1^2}{a_1^2\omega^2 + a_3^2}\}.
\end{aligned}$$

From equation (3.10), we know that

$$a_1^2\omega^2 + a_3^2 = \omega^4 + (a_2^2 - 2a_4)\omega^2 + a_4^2 = a_2^2\omega^2 + (-\omega^2 + a_4)^2,$$

then we obtain

$$\begin{aligned}
\text{sign}\{\frac{d(\text{Re}\lambda)}{d\tau}\}_{\lambda=i\omega} &= \text{sign}\{\frac{a_2^2 + 2\omega^2 - 2a_4}{a_2^2\omega^2 + (-\omega^2 + a_4)^2} - \frac{a_1^2}{a_2^2\omega^2 + (-\omega^2 + a_4)^2}\} \\
&= \text{sign}\{2\omega^2 - (a_1^2 + 2a_4 - a_2^2)\}. \qquad (3.15)
\end{aligned}$$

Theorem 5. *Let $\tau_k^{\pm}$ be de£ned by equation (3.14). If the conditions (3.7) and (3.13) are satis£ed, then the equilibrium point E_1 is stable when $\tau \in [0, \tau_0^+) \bigcup (\tau_0^-, \tau_1^+) \bigcup \cdots \bigcup (\tau_{m-1}^-, \tau_m^+)$ and unstable when $\tau \in [\tau_0^+, \tau_0^-) \bigcup (\tau_1^+, \tau_1^-) \bigcup \cdots \bigcup (\tau_{m-1}^+, \tau_{m-1}^-)$, for some positive integer m. Therefore there are bifurcations at the equilibrium point E_1 when $\tau = \tau_k^{\pm}$, $k = 0, 1, 2, \cdots$.*

Proof. Since the conditions (3.8) and (3.13) are satis£ed, then to prove the theorem we need only to verify the transversality conditions, see Cushing [6],

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=\tau_k^+} > 0 \quad \text{and} \quad \frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=\tau_k^-} < 0,$$

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=i\tau_k^+} > 0 \text{ and } \frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=i\tau_k^-} < 0.$$

From (3.15) and (3.11), it follows that

$$\begin{aligned}\text{sign}\{\frac{d(\text{Re}\lambda)}{d\tau}\}_{\lambda=i\omega_+} &= \text{sign}\{2\omega_+^2-(a_1^2+2a_4-a_2^2)\}\\ &= \text{sign}\{\sqrt{(a_1^2+2a_4-a_2^2)^2-4(a_4^2-a3^2)}\},\end{aligned}$$

therefore,

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\omega=\omega_+,\tau=\tau_k^+}>0,$$

$$\tfrac{d(\text{Re}\lambda)}{d\tau}|_{\omega=\omega_+,\tau=\tau_k^+}>0.$$

Again,

$$\begin{aligned}\text{sign}\{\frac{d(\text{Re}\lambda)}{d\tau}\}_{\lambda=i\omega_-} &= \text{sign}\{2\omega_-^2-(a_1^2+2a_4-a_2^2)\}\\ &= \text{sign}\{-\sqrt{(a_1^2+2a_4-a_2^2)^2-4(a_4^2-a3^2)}\},\end{aligned}$$

therefore,

$$\frac{d(\text{Re}\lambda)}{d\tau}|_{\omega=\omega_-,\tau=\tau_k^-}<0.$$

Hence, the transversality conditions are satis£ed. This completes the proof. □

Example 6. *Consider model (3.5) with parameters* $r = 3.5$*,* $b = 0.04$*,* $\alpha = 1$*,* $c = 0.3$*,* $\beta = 0.05$*,* $H_x = 0.02$*, and* $H_y = 0.01$*. The equilibrium point of the model is* $E_1 = (6.06146, 3.25424)$*. For* $\tau = 0$*, the Jacobian matrix of the model associated with the equilibrium point has eigenvalues* $-0.11804 \pm 0.98570i$*. This means that the equilibrium point of the model without time delay is stable. The conditions (3.8) and (3.13) are satis£ed. Some trajectories of* $x(t)$ *and* $y(t)$ *with various time delays are given in Figures 2, 3, and 4.*

From Figures 2a and 2b with time delay $\tau = 1.2$, the equilibrium point $(6.06146, 3.25424)$ is stable. Figures 3a and 3b with time delay $\tau = 1.53$ show that the equilibrium point $(6.06146, 3.25424)$ is unstable. The £rst critical value of time delay is $\tau = \tau_0^+ = 1.37941$. When $\tau < 1.37941$, the equilibrium point $(6.06146, 3.25424)$ is asymptotically stable; when $\tau = 1.37941$ the equilibrium point $(6.06146, 3.25424)$ loses its stability; and when $\tau > 1.37941$ but less then the second critical value of time delay, the equilibrium point $(6.06146, 3.25424)$ becomes unstable and there is a bifurcating periodic solution, see Figure 4. Following Theorem 5 we have

$$\begin{array}{rlr} \tau_0^+ = 1.37941, & & \tau_0^- = 5.39314,\\ \tau_1^+ = 6.98104, & & \tau_1^- = 12.53884,\\ \tau_2^+ = 12.58266, & & \tau_2^- = 19.68453,\\ \tau_3^+ = 18.68453, & \text{and} & \tau_3^- = 26.83023. \end{array}$$

Then we have 2 stability switches from stability to instability and to stability.

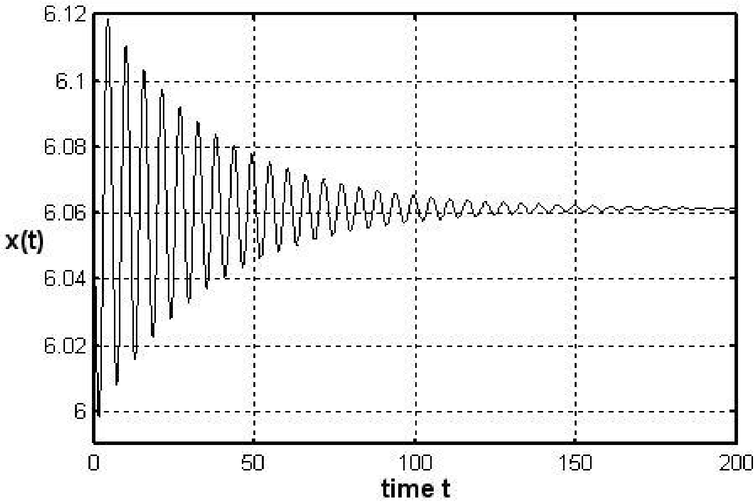

FIGURE 2a. Trajectory of prey with $x(0) = 6.0715$ and $\tau = 1.2$

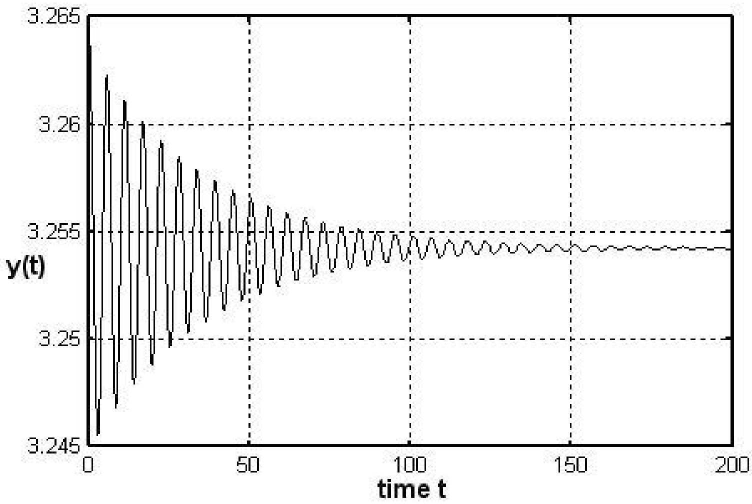

FIGURE 2b. Trajectory of predator with $y(0) = 3.2642$ and $\tau = 1.2$

4. DISCUSSION

In the analysis of the positive equilibrium point of model (3.1), it is quite dif£cult to determine the value of the equilibrium points analytically. We just state that there exists either one, or two, or none positive equilibrium points by inspection the phase plane of the model. In the case of two positive equilibrium points occur, one of the equilibrium point is possibly stable and the other is a saddle point. In this paper, we just analyze the case of there exist two positive equilibrium points and focus on analyzing the effect of the time delay on the stable equilibrium point. Actually we may also try to analyze the effect of the time delay on the stability of the unstable equilibrium point.

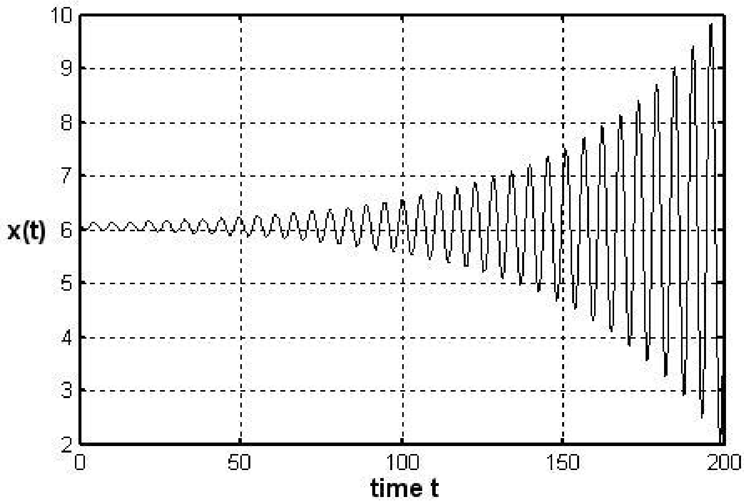

FIGURE 3a. Trajectory of prey with $x(0) = 6.0715$ and $\tau = 1.53$

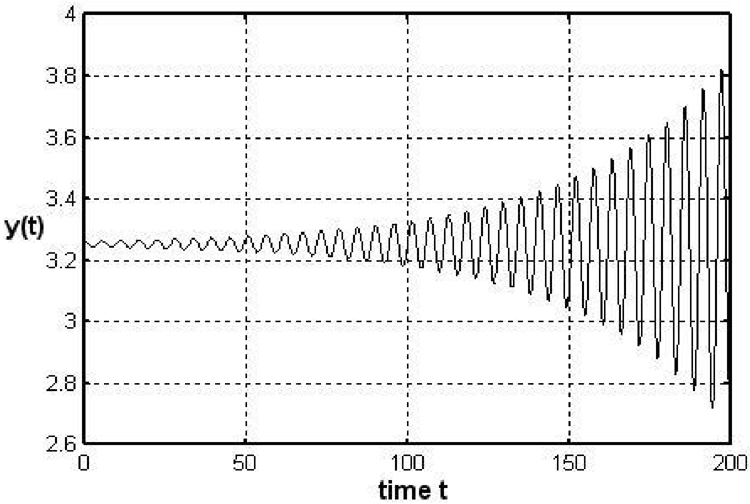

FIGURE 3b. Trajectory of predator with $y(0) = 3.2642$ and $\tau = 1.53$

There is still a lot of work to do in the predator-prey models with time delay and harvesting. For example, it would be interesting to consider time delay and harvesting in generalized Gause-type predator-prey model and in some another generalized predator-prey models as in Martin and Ruan [14]. It would also be interesting to study the Wangersky-Cunningham model with some delays in both the predator and prey model as in the Bartlett's model, see Bartlett [1] and Hasting [8].

REFERENCES

[1] M.S. Bartlett, *On theoretical models for competitive and predatory biological systems*, Biometrika, **44** (1957), 27-42.

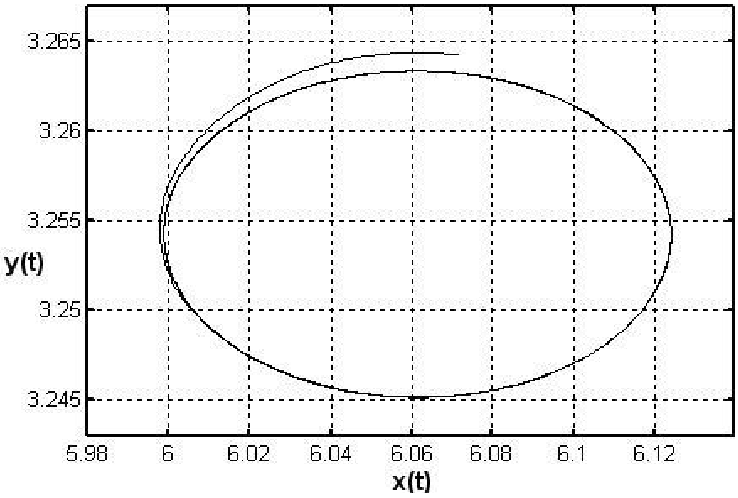

FIGURE 4. Trajectory of $(x(t), y(t))$, with $x(0) = 6.0715$, $y(0) = 3.2642$ and $\tau = 1.37941$

[2] F. Brauer and A.C. Soudack, *Stability regions and transition phenomena for harvested predator-prey systems*, Journal Math. Biology, **7** (1979), 319-337.

[3] F. Brauer and A.C. Soudack, *Stability regions in predator-prey systems with constant-rate prey harvesting*, Journal Math. Biology, **8** (1979), 55-71.

[4] F. Brauer and A.C. Soudack, *Coexistence properties of some predator-prey systems under constant rate harvesting and stocking*, Journal Math. Biology, **12** (1981), 101-114.

[5] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw - Hill, New York, (1955).

[6] J.M. Cushing, *Integrodifferential equations and delay models in population dynamics*. Heidelberg: Springer-Verlag, (1977).

[7] G. Dai and M. Tang, *Coexistence region and global dynamics of a harvested predator-prey system, SIAM J. Appl. Math*. **58** (1998), 193-210.

[8] A. Hasting, *Delays in recruitment at different trophic levels: effects on stability*, J. Math. Biol. **21** (1984), 35-44.

[9] Ho, C.P. and Ou, Y.L. . *In¤uence of time delay on local stability for a predator-prey system*, Journal of Tunghai Science, **4** (2002), 47-62.

[10] Hogarth, W.L., Norbury, J., Cunning, I. and Sommers, K., *Stability of a predator-prey model with harvesting*, Ecological Modelling, **62** (1992), 83-106, ibid. **4** (2002), 47-62.

[11] J. Holmberg, *Socio-ecological principles and indicators for sustainability*, PhD Thesis, Goteborg University, Sweden, (1995).

[12] T.K. Kar, *Selective harvesting in a prey-predator £shery with time delay*, Mathematical and Computer Modelling, **38** (2003), 449-458.

[13] L.S. Luckinbill, *Coexistence in laboratory populations of paramecium aurelia and its predator didinium nasutum*, Journal of Ecology, **54**, 6, (1973), 1320-1327.

[14] A. Martin and S. Ruan, *Predator-prey models with time delay and prey havesting*, J. Math. Biol, **43** (2001), 247-267.

[15] H. Matsuda and P.A. Abrams, *Effects of predators-prey interaction and adaptive change on sustainable yield*, Can. J. Fish. Aquat. Sci./J. Can. Sci. Halieut. Aquat., **61**, 2, (2004), 175-184.

[16] M.R. Myerscough, B.F. Gray, W.L. Hogarth, J. Norbury . *An analysis of an ordinary differential equation model for a two-species predator-prey system with harvesting and stocking*, Journal Math. Biology, **30** (1992), 389-411.

[17] P.D. Srinivasu, S. Ismail and C.R. Naidu, *Global dynamics and controllability of a harvested prey-predator system*, J. Biological Systems, **9**, 1, (2001), 67-79.

[18] D. Xiao and S. Ruan, *Bogdanov-Takens bifurcations in predator-prey systems with constant rate of harvesting*, Fields Institute Communications, **21** (1999), 493-506.