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AN INTRODUCTION TO THE STATEMENT CALCULUS

BY

S. MANZUR HUSSAIN

Introduction

In every-day life we make certain statements based on our knowledge and information. These statements may be true or untrue depending upon our own interpretations. It is also quite natural to believe that a certain statement is true in one context but untrue in another. But it is definite that a statement cannot both be true and false at the same time.

During the discussions we often make a composite statement consisting of several sub-statements using the words, ' Or ', ' & ', if and only if ', etc., as their connectives. Now it is most natural to ascertain whether the composite statement that we have made is true or false. In the mathematical world also we make similar statements and then logically study their truth or falsehood. It was C. S. Peirce (1839—1914) who was the first to discover in 1885 a mechanical device known as truth tables by means of which we can determine whether such statements are true or false.

This essay has been divided into three sections. Section 1 deals with the statement calculus and truth tables. In the next section we give an introduction of the elementary Set Theory and explain the three basic operations of union, intersection and complementation. The last section deals with the definition of Boolean Algebra based on the concepts of the Set Theory. The Boolean Algebra is after the name of G. Boole (1815—1865) who discovered this Algebra in 1847. In the end we conclude that the statement calculus is a Boolean Algebra under the operations of conjunction, disjunction and negation.

SECTION 'I'

Statement Calculus and Truth Tables

Notation :

We use the following symbols :—

$\equiv$ for equivalence

p. q, r,......for statements or assertions

(p $\equiv$ it is raining).

$\sim$ for negation

$\wedge$,, conjunction

$\vee$,, disjunction

$\longrightarrow$,, one way implication

$\longleftrightarrow$,, two way implication

Definitions :—

1. *Statement* :—It is an assertion which must be true or false but not both.

2. *Truth Value* :—The truth-fulness or falsity of a statement is called its truth value. In case the statement is true we shall represent this fact by 1 and if false, then by 0. It may be mentioned that some authors use T. & F. instead of 1 & 0.

3. *Negation* :—If p be a statement, then $\sim$ p means 'not' p e.g. p $\equiv$ it is raining, then $\sim$ p $\equiv$ it is not raining. Further if p is a true statement, then $\sim$ p will be a false statement and *vice versa.* But since there are only two possibilities for p i.e. p can either be true or false, and hence there are two possibilities for $\sim$ p. i.e. either false or true respectively. This fact can be represented by the following table.

Table No. 1 *for* $\sim$ *p*

p	$\sim$ p
0	1
1	0

4. *Conjunction* :—If p, q are statements, then 'p $\wedge$ q' is a composite statement known as the conjunction of the original statements : e -g. p $\equiv$ it is raining, q $\equiv$ the sun is shining. Then p $\wedge$ q $\equiv$ it is raining and the sun is shining. Since p & q have 2 possibilities each, therefore p $\wedge$ q will have 4 possibilities in all. We can easily see that ' p $\wedge$ q ' will be true only when both p & q are true but false otherwise. This fact can be represented by the following table :—

Table No. 2 for p $\wedge$ q

p	q	p $\wedge$ q
0	0	0
0	1	0
1	0	0
1	1	1

5. *Disjunction* :—If p & q are statements, then p $\vee$ q is a composite statement known as the disjunction of the original statement. e. g. p $\vee$ q $\equiv$ It is raining or the sun is shining. 'p $\vee$ q' has also 4 possibilities. We can easily see that 'p $\vee$ q' is false only when both of them are false but true otherwise. This fact can be represented by the following table.

Table No. 3 for p $\vee$ q

p	q	p $\vee$ q
0	0	0
0	1	1
1	0	1
1	1	1

One-way implication ($\longrightarrow$). Many statements in Mathematics are of the form ' if p then q ' or p implies q. This can also be read as ' p is sufficient for q', 'q is necessary for p', p only if q. 'p $\longrightarrow$ q' has also 4 possibilities *viz.*

(*i*) A false statement implies a false statement, (true).

(*ii*) A false statement implies a true statement, (true).

(*iii*) A true statement implies a false statement, (false).

(*iv*) A true statement implies a true statement, (true).

It is true that a false statement should ordinarily lead to a false conclusion but sometimes it happens that it implies a true statement and in that case also we consider the whole statement *viz* 'a false statement implying a true statement' a true statement. We would like to elaborate this point by an example. Let p be $1 = 2$, (false) & q be $3 = 3$ (true) $\because 1 = 2$ $\therefore 2 = 1$. Adding both sides we get $3 = 3$. Since the conclusion is true, it is, therefore, logical to consider the whole statement ($p \longrightarrow q$) as true. The statements No. (*iii*) & (*iv*) are quite straightforward and need no explanation.

The above facts can be represented by the following table.

Table No. 4 for $p \longrightarrow q$

p	q	$p \longrightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Note :—Some authors define $p \longrightarrow q$ as equivalent to $\sim p \vee q$. Let us construct the table for $\sim p \vee q$.

Table No. 5 for ($\sim p \vee q$)

p	$\sim p$	q	$\sim p \vee q$
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1

By comparison we find that the results in the column under $p \longrightarrow q$ is the same as under $\sim p \vee q$ and hence $p \longrightarrow q \equiv \sim p \vee q$.

Note :—2. $p \longrightarrow q$ is not always the same as $q \longrightarrow p$.

Example : If it is a $\triangle$ then the sum of its angles is equal to 2 rt. $\angle^s$; but the converse of the above need not be true. The above fact can also be verified by constructing table for $q \longrightarrow p$ and then comparing it with that of $p \longrightarrow q$.

Two way implication :—$p \longleftrightarrow q$. sometimes we come across statements of the type 'p if and only if q' or simply 'p iff q' then such statements involve two way implications i.e. if p then q and if q then p or symbolically $p \longleftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p)$. Necessary and sufficient condition—is an example of two-way implication. It is obvious that $p \longleftrightarrow q$ will be true only if both p & q are either false or true ; otherwise false. These facts can be represented in the form of a table :—

Table No. 6 for $p \longleftrightarrow q$

p	q	$p \longleftrightarrow p$
0	0	1
0	1	0
1	0	0
1	1	1

Note :—1. Truth table is a mechanical device by which we can judge the truth or falsehood of a composite statement.

2. If in the last column of a truth table we have all '1' s' then the composite statement is known as a tautology (universal truth) and if all o's then it is a contradiction e.g. $p \vee \sim p$ is a tautology & $p \wedge \sim p$ is a contradiction.

Table for No. 7 for '$p \vee \sim p$' and '$p \wedge \sim p$'

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
0	1	1	0
1	0	1	0

Logical Equivalence ($\equiv$) Two composite statements are said to be logically equivalent if their truth tables are identical. We have already seen that $p \longrightarrow q \equiv \sim p \vee q$' (*vide* Tables No. 4 and No. 5).

Let us now consider another example in which three statements p, q & r are involved.

Suppose we are required to prove $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

We construct their table and find that the columns under $p \vee (q \wedge r)$ & $(p \vee q) \wedge (p \vee r)$ are identical and hence the result.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	1	1	1	1	1	1

We can easily prove the following by truth tables.

1. $p \vee p \equiv p, \quad p \wedge p \equiv p$
2. $(p \vee q) \vee r \equiv p \vee (q \vee r), (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
3. $p \vee q \quad \equiv q \vee p, \quad p \wedge q \quad \equiv q \wedge p$.

4. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$; $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (already proved)
5. $p \vee 0 \equiv p$, $p \wedge 1 \equiv p$
6. $p \vee 1 \equiv 1$, $p \wedge 0 \equiv 0$
7. $p \vee \sim p \equiv 1$, $p \wedge \sim p \equiv 0$
8. $\sim \sim p \equiv p$ $\sim 1 \equiv 0$, $\sim 0 \equiv 1$.
9. $\sim (p \vee q) \equiv \sim p \wedge \sim q$, $\sim (p \wedge q) \equiv \sim p \vee \sim q$.
10. $\sim (p \wedge q) \equiv \sim (p \wedge q) \vee [\sim (q \longleftrightarrow p)]$, $(p \wedge q) = 0 \rightarrow (p \vee q) \equiv 1$

Sometimes it is easier to judge by truth tables the truth or falsehood of a composite statement which is quite complicated otherwise. For instance we are required to determine the nature of the following statement.

"It is not true that if $2 + 2 = 4$ then $3 + 3 = 5$ & $1 + 1 = 2$" Suppose p, q & r denote $2 + 2 = 4$, $3 + 3 = 5$ & $1 + 1 = 2$ respectively.

We then see that the above statement can be written as $\sim (p \rightarrow (q \wedge r)$. We now assign the values 1, 0 & 1 to p, q & r respectively. We consult the truth tables and find that

$q \wedge r \equiv 0 \wedge 1 \equiv 0$;
$p \rightarrow (q \wedge r) \equiv 1 \rightarrow 0 \equiv 0$;
& finally $\sim (p \rightarrow (q \wedge r) \equiv \sim (0) \equiv 1$.

∴ the above statement is true.

Note :—1. The judgement by truth table takes less time and space.

2. When a composite statement consists of four sub-statements or more the construction of truth table becomes more tedious and difficult.

3. There are truth tables other than those already mentioned for statements such as either p or q but not both, neither p nor q, etc. but they are less common and have not been discussed.

SECTION ' II '

Elementary Set Theory

Notation :—We shall use the following symbols

A, B, C,..............for sets.
0 for null set
1 for universal set
a, b, c,.............. for the elements of a set.

$\in$ & $\notin$ for ' belongs to' and ' does not belong to ' respectively.

iff for 'if and only if'
$<$ for inclusion
$\cup$ for union of sets
$\cap$ for intersection of sets

A′ for the complement of the set A

Definitions :—1. A ' set ' is any well defined collection of objects. e. g. : A set of all integers ; a set of all rt. angled $\triangle$s, etc.

2. Null set is the set which has no elements. It is sometimes called an empty set.

3. Set A is said to be included in Set B in case every element of A is a member of Set B i.e. if $x \in A$ then $x \in B$. A is called a subset of B. and denoted as $A \subset B$, e. g. $A = \{1, 2\}$ & $B = \{1, 2, 3, 4\}$ and $A \subset B$.

4. Universal set is the set of which all the sets under consideration are subsets. e. g. in Number System all the numbers form a universal set ; in plane geometry all the pts. form a Universal Set.

5. $A \cup B$ is a set whose elements belong to A or B or both. In the above example $A \cup B = \{1, 2, 3, 4\}$.

6. $A \cap B$ is the set whose elements belong to both A & B. In the above example $A \cap B = \{1, 2\}$.

7. A − B is the set whose elements belong to A but do not belong to B.

Suppose $A = \{1, 2, 3, 4, 5\}$

$B = \{3, 4, 5, 6, 7\}$.

$A - B = \{1, 2\}$

8. A′ is the set whose elements are members of 1 (Universal Set) but do not belong to A, i.e.

$A' = 1 - A.$

for example 1 = set of all integers.

A = set of all odd integers

Then A′ = set of all even integers.

A few basic properties of the Set :—The following relations between the sets can easily be verified.

(*i*) $A \cup B = B \cup A$, $A \cap B = B \cap A$;

(*ii*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) =$

$= (A \cap B) \cup (A \cap C)$;

(*iii*) $A \cup 0 = A$, $A \cap 1 = A$;

(*iv*) $A \cup A' = 1$, $A \cap A' = 0$.

Note :—If we examine the above relations (*i* − *iv*) we discover that these relations exhibit a duality between the operations of union and intersection. There is a pair of results in each relations. Each result in the relations (*i*) & (*ii*) is transformed into the other by interchanging $\cup$ & $\cap$. In relations (*iii*) & (*iv*) the above position will also be true provided we interchange 0 & 1 at the same time.

SECTION 'III'

Boolean Algebra.

Definition : A Boolean Algebra is a Set S of at least two elements including 0 & 1 which can be operated upon by union, intersection and complementation, such that if A & B belong to S then,

$A \cup B$, $A \cap B$ & A' belong to S ; and satisfy the relations (*i*) to (*iv*) of section II. These relations form an axiom system for Boolean Algebra. It may be pointed out that the Algebra of sets forms a model for the Boolean Algebra.

We also give another example of this algebra.

Example :—Let $S = \{1, 2, 3, 5, 6, 10, 15, 30\}$ (All the divisors of 30);

$A \cup B$ stand for *l.* c. m. of A & B ;

$A \cap B$,, ,, H. C. F. of A & B ;

A' ,, ,, $\frac{30}{A}$;

0 ,, ,, 1 ;

& 1 ,, ,, 30.

We pick up any elements from S say 6, 10, 15 & verify the relations (*i*) to (*iv*).

(*i*) $6 \cup 10 = 30 = 10 \cup 6$

Now $6 \cap 10 = 2 = 10 \cap 6$

(*ii*) $6 \cup (10 \cap 15) = 6 \cup 5 = 30$

$(6 \cup 10) \cap (6 \cup 15) = 30 \cap 30 = 30$

$\therefore \quad 6 \cup (10 \cap 15) = (6 \cup 10) \cap (6 \cup 15);$

$6 \cap (10 \cup 15) = 6 \cap 30 = 6$

$(6 \cap 10) \cup (6 \cap 15) = 2 \cup 3 = 6$

$\therefore \quad 6 \cap (10 \cup 15) = (6 \cap 10) \cup (6 \cap 15).$

(*iii*) $6 \cup 1 = 6 \qquad 6 \cap 30 = 6$

(*iv*) $6 \cup 5 = 30 \qquad 6 \cap 5 = 1.$

We see that all the axioms are satisfied.

We can similarly verify the above axioms for the other divisors and come to the conclusion that the above set S forms a Boolean Algebra.

Next we show that the statement calculus is also a Boolean Algebra under the operations of $\vee$, $\wedge$ & $\sim$;

$$x \in A \cup B \text{ iff } x \in A \text{ or } x \in B;$$
$$x \in A \cap B \text{ iff } x \in A \ \& \ x \in B;$$
$$\& \quad x \in A' \text{ iff } x \notin A$$

Let $p \equiv x \in A$, $q \equiv x \in B$ $r \equiv x \in C$.

We find that $A \cup B \equiv p \vee q$

$$A \cap B \equiv p \wedge q$$
$$A' \equiv \sim p.$$

Since $x \in 0$ is false for every x ;

$\therefore$ 0 is a universally false statement.

Again since $x \in 1$ is true for every x ;

$\therefore$ 1 is a universally true statement.

The relations (*i* to *iv*) under the above equivalence results and those known to us from section I take the following forms :—

(*i*) $p \vee q = q \vee p, \qquad p \wedge q = q \wedge p;$

(*ii*) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r),$

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(*iii*) $p \vee 0 = p, \qquad p \wedge 1 = p;$

(*iv*) $p \vee \sim p = 1 \qquad p \wedge \sim p = 0.$

MORSE THEORY AND ITS APPLICATIONS TO THE THEORY OF TOTAL ABSOLUTE CURVATURE[1]

BY

B. A. SALEEMI(2)

Morse Theory or The Calculus of Variations in the large, was developed in the papers of Marston Morse, culminating in his famous monograph 'The Calculus of Variations in the Large' in 1934. According to this Theory, the topological complexity of a space is reflected in the existence and nature of the critical points of a real-valued function defined on it. The key result used in applying this to the theory of total absolute curvature of immersed manifolds, is Sard's theorem which states : "if M_1 and M_2 are differentiable manifolds of the same dimension having a countable basis and $\phi : M_1 \rightarrow M_2$ is of class C^1, then the image of the set of singular[3] points of ϕ has measure zero in M''_2 [1][4].

I am greatly indebted to my supervisor Professor T. J. Willmore for reading the original draft of this paper and suggesting a large number of valuable improvements.

Part 1. Morse Theory :

Unless otherwise stated all our manifolds and maps will be smooth (*i.e.* of class C^{∞}). The tangent space of a manifold M at a point p will be denoted by M(p).

(1) This paper is essentially expository in nature and gives a brief survey of the theory of "Total Absolute Curvature".

(2) The author gratefully acknowledges the help of the Colombo Plan authorities for the award of a fellowship and the Panjab University, Lahore, (Pakistan authorities for granting him study-leave.

(3) The point $p \in M_1$ is called a 'singular point' of ϕ if the rank of the Jacobian of ϕ evaluated at p is less than the dimension of M_1.

(4) Numbers in brackets refer to the references at the end of this paper.

1.1 *Definition.* Let M be a manifold of dimension n, and $f : M \to R$ be a real-valued function defined on it. A point $p \in M$ is called a *critical point* of f if $df = 0$ at p. If $(x^1, x^2, \ldots, x^n)$ are local coordinates in a neighbourhood U of p, then $df = 0$ means that

$$(1) \quad \left(\frac{\partial f}{\partial x^1}\right)_p = \ldots, = \left(\frac{\partial f}{\partial x^n}\right)_p = 0.$$

The critical point p is called *non-degenerate* or *degenerate* according as the rank of the matrix $\left(\frac{\partial^2 f}{\partial x\, \partial x^j}(p)\right)$ is equal to n or less than n.

If $y = y^j(x^1, \ldots, x^n)$, $j = 1, 2, \ldots, n$, is another local co-ordinate-system about p, then

$$(2) \quad \left(\frac{\partial^2 f}{\partial y^i \partial y^j}\right)(p) = \left(\frac{\partial^2 f}{\partial x^k \partial x^l} \cdot \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial x^l}{\partial y^j}\right)(p).$$

The equation (2) shows that the non-degeneracy of a critical point is independent of the choice of the coordinate-system

1.2 If $p \in M$ is a critical point of $f : M \to R$, and $(x^1, \ldots, x^n)$ is a local coordinate-system about p, then we can talk about the *index* and *nullity* of f at p according as p is non-degenerate or degenerate. If p is non-degenerate, then the number of negative roots of the equation in λ

$$(3) \quad \left| \left(\frac{\partial^2 f}{\partial x\, \partial x^j}\right)(p) - \lambda I \right| = 0$$

is called the *index* of f at p ; and it gives all the information about the nature of the critical point p. On the other hand, if p is a degenerate critical point, then the *nullity* of f at p is defined by the equation

$$(4) \quad \text{Nullity} = n\text{-rank}\left(\left(\frac{\partial^2 f}{\partial x\, \partial x}(p)\right)\right).$$

1.3 In order to determine the index of $f : M \to R$ at a non-degenerate critical point p, one can use the following

Lemma of Morse. If p is a non-degenerate critical point of $f : M \to R$, then there exists a local co-ordinate-system $(x^1, \ldots, x^n)$ in a neighbourhood U of p such that

$$(i)\ x^1(p) = \ldots, = x^n(p) = o,$$

and $(ii)\ f(q) = f(p) - (x^1)^2 \ldots - (x^r)^2 + (x^{r+1})^2 + \ldots, + (x^n)^2$

for all $q \in U$, and r is the index of f at p. For the proof see [12].

1.4 *Examples.*

(*a*) Consider the real projective plane P^2 defined by

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$(x_1, x_2, x_3) \sim (-x_1, -x_2, -x_3),$$

where "$\sim$" is the diametrical equivalence on S^2. Let

$$f = \sum_{i,j=1}^{3} x_i x_j = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

$$= 1 + x_1x_2 + x_1x_3 + x_2x_3$$

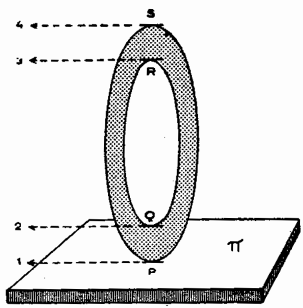
be the real-valued function on P^2. This function has all its critical points non-degenerate because

$$\left(\frac{\partial^2 f}{\partial x\, \partial x^j} \right) = +1$$

at all points.

(*b*) Let $f : R \to R$ be defined by $f(x) = x^2$. Then $df = o$ at $x = o$. Thus the origin is a critical point of f and it is non-degenerate as $\frac{df^2}{dx^2} = 2$.

(*c*) In order to see how the number of positive and negative signs in the expression for $f : M \to R$ varies with the nature of the critical points, we take as example the height function h defined on the torus T^2. Let π be the plane tangent to T^2 at p as shown in figure 1. It is clear from the figure that the height above p is minimum at p and maximum at s. We can take co-ordinates about p, say (x, y) such that



(α) $h=x^2+y^2$ at p ;
(β) $h=C_1-x^2+y^2$ at q ;
(τ) $h=C_2-x^2+y^2$ at r ;
(δ) $h=C_3-x^2-y^2$ at s.

Thus we have four non-degenerate critical points of h of indices 0, 1, 1, 2; and the number of minus signs (*i.e.* index) tells us the nature of the critical point. The situation described above can also be given an interpretation in terms of Gaussian curvature. The points of positive Gaussian curvature (elliptic points) have either index c or 2 whereas the saddle points have index 1. The critical levels 1, 2, 3, 4 refer to the points which are mapped into the same point by the Gauss normal map

$$\tilde{\nu}: T^2 \to S^2 .$$

This fact has very far-reaching implications in the theory of total absolute curvature.

From the viewpoint of homotopy, the number of minus signs appearing in the expression for h at each critical point is the dimension of the cell we must attach to go from h_i to h_j (i, j = 1, 2, 3, 4).

(*d*) The function

$f: R^2 \to R$ given by $f(x, y) = x^2y^2$ has only degenerate critical points. The set of critical points is the union of x-axis and y-axis. This set is not even a submanifold of R .

1.5 *The Morse inequalities.* Let M be a compact manifold of dimension n whose Betti numbers with respect to any field F are $\beta_0, \beta_1, \ldots, \beta_n$.

Let f : M→R be a real-valued function having only non-degenerate critical points. Let the number of critical points of index j be denoted by μ_j. Then the Morse inequalities are :

$$\mu_0 \geqslant \beta_0,$$
$$\mu_1 - \mu_0 \geqslant \beta_1 - \beta_0,$$
$$\mu_2 - \mu_1 + \mu_0 \geqslant \beta_2 - \beta_1 + \beta_0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

(5) $\mu_k - \mu_{k-1} + \ldots, \pm \mu_0 \geqslant \beta_k - \beta_{k-1} + \ldots, \pm \beta_0.$ $(1 \leqslant k \leqslant n)$

From the inequalities we deduce easily that

(6) $\mu_k \geqslant \beta_k$ for $0 \leqslant k \leqslant n$. For proofs see [12]. Also, it is known that (7) $\chi(M) = \sum_{i=0}^{n}(-1)\beta_i = \sum_{i=0}^{n}(-1)\mu_i$.

Moreover, it can be shown that if f : M→R has only non-degenerate critical points whose total number is the minimum possible number of non-degenerate critical points that a real-valued function can have on M, then $\mu_0 = \mu_n = 1$. This can be expressed by saying that such a function has only one maximum and one minimum. This fact plays a very important role in the theory of immersions with minimal total absolute curvature.

Part 2. Application

Morse Theory has become one of the most powerful tools in modern researches in Mathematics. Some results obtained in this way are :

2.1 If f : M→R has only two critical points (where M is compact), both of which are non-degenerate than M is homeomorphic to a sphere. (Reeb) [12].

2.2 A differentiable manifold can have several inequivalent differentiable structures. (Milnor) [14].

2.3 M. Morse and N.H. Kuiper generalized critical point theory to continuous functions and proved the following :

Let M be a closed n-dimensional manifold and $f : M \rightarrow R$ a continuous real-valued function. Let f have only non-degenerate critical points and let these be exactly two in number. Then M is homeomorphic to a sphere. [11] and [16].

2.4 Quite recently S. Robertson has introduced the concept of *trans-normal embeddings* of manifolds in Eculidean space and has obtained very interesting results *via* Morse Theory. For example, he proved [17] :

Any transnormal n-manifold of order 2 in E^m is diffeomorphic to the Cartesian product $V_1 \times V_2$ of differentiable manifolds V_1, V_2, where V_1 is homeomorphic to S and V_2 is homeomorphic to E^{n-j} $(0 \leqslant j \leqslant n)$.

In what follows, we shall be interested in the applications of Morse Theory to the theory of total absolute curvature of immersed manifolds.

2.5 *Definition :* Let M_1 and M_2 be two manifolds of dimensions n_1 and n_2 respectively. Let $\phi : M_1 \rightarrow M_2$ be a map. Then ϕ is called an *immersion* of M_1 into M_2 if the rank of the induced linear map on the tangent spaces is equal to n_1 at all points of M_1. If $\phi : M_1 \rightarrow M_2$ is an immersion and $\phi(p)=\phi(q) \Longleftrightarrow p = q$, then ϕ is called an *embedding*(1) of M_1 into M_2. If $n_1 = n_2$ and ϕ is an onto embedding, then ϕ is called a *diffeomorphism*, and M_1 and M_2 are said to be diffeomorphic.

2.6 Let M be a compact, connected smooth manifold and $f : M \rightarrow E^N$ be an immersion of M in euclidean space of dimension N. Let B_ν be (N—n—1)—sphere bundle(2) over M induced by f and S_0^{N-1} be the unit hypersphere in E^N with centre at the origin. Let dV, $dV \wedge ds^{N-n-1}$ and ds^{N-1} denote volume elements of M, B_ν and S_0^{N-1} respectively. Let

$$(8) \qquad \tilde{\nu} : B_\nu \rightarrow S_0^{N-1}$$

(1) Note that in the case of an embedding, the topology of M_1 must coincide with the induced topology.

(2) dimension of B_ν is equal to N-1.

be the generalized Gauss-map. Then the dual map $\vec{\nu}^*$ will bring the volume element ds^{N-1} to B_ν. Comparing these two (N—1)—forms, we have

(9) $$\vec{\nu}^*(ds^{N-1}) = G(p, v)\, dV \wedge ds^{N-n-1}.$$

We call G(p, v) the Lipschitz-Killing curvature at p. It coincides with the Gaussian curvature when $N-n=1$. The total absolute curvature at p has been defined in [2] by the following formula :

(10) $$K^*(p) = \int \left| G(p, v) \right| ds^{N-n-1}$$

where the integration is carried over the fibre of B_ν at $p \in M$. Then the total absolute curvature of M is defined by

(11) $$\tau(f, M) = \frac{1}{C_{N-1}} \int_M K^*(p)\, dv, \text{ where } C_{N-1} = \text{area of } S_o^{N-1}.$$

Let $\mathcal{J}$ be the space of all immersions of M in E^N. Then we define

(12) $$\tau = \inf_{f \in \mathcal{J}} \tau(M, f)$$

to be the *minimal total absolute curvature* of M. It is clear from the very definition of τ that it is a topological invariant of M. In many cases its value (which is always a positive integer) is given by the formula :

(13) $$\tau = \sum_{i=0}^{n} \beta i .$$

where $\beta_r, \ldots \beta_n$ are the Betti numbers of M.

On the other hand, if C_1, $C_2 \ldots$ are the homtopy classes of immersions of M into E^N, then we can introduce new topological invariants

(14) $$\tau_1 = \inf_{f \in C_1} \tau(f, M)$$
$$\tau_2 = \inf_{f \in C_2} \tau(f, M)$$

and so on.

Are $\tau_1, \tau_2, \ldots$ integers ? What is their significance ? These are still open questions.

2.7 *One-dimensional manifolds.* Even the theory of the total absolute curvature of one-dimensional manifolds is very hard and little is known. W. Fenchel first introduced the concept of total absolute curvature and proved the followings :

A_1. If C is a closed space curve, then $\tau(f, C) \geqslant 2$; and equality holds if and only if C is a plane convex curve. [6].

Fary and Milnor proved the following result about knots :

B_1. If C is a closed space curve and $\tau \leqslant 4$, then C is not knotted. [5] and [13].

Fenchel's result has been generalized to higher dimensions by Chern but the generalization of B_1 to higher dimensional knots is an unsolved problem [1].

2.8 *Two-dimensional manifolds :*

The *classification theorem* states: "a compact connected 2-dimensional manifold M is either (*i*) homeomorphic to a sphere with g handles and its Euler—Poincare characteristic χ is 2-2g,

or (*ii*) homeomorphic to a projective plane with g handles and its Euler-Poincare characteristic χ is 1-2g,

or (*iii*) homeomorphic to a Klein bottle with g handles and its Euler-Poincare characteristic χ is-2g",

This theorem shows that the sphere, the projective plane and the Klein bottle are fundamental objects of study in the field of compact 2-dimensional manifolds. If we use homology with coefficient field Z_2, then the Betti numbers of 2-dimensional compact manifolds in cases (*i*), (*ii*) and (*iii*) are (1, 2g, 1), (1, 1+2g, 1), (1, 2+2g, 1) respectively. Chern-Lashof [3] and Kuiper [8] proved the following :

A_2. If $f : M \to E^3$ is an immersion, then

(15) $\tau \geqslant 2+2g$ or $3+2g$ or $4+2g$.

Now $\chi = 2-2g$ or $1-2g$ or $-2g$. Hence this means that $\tau \geqslant 4-(2\text{-}2g)$ or $4-(1-2g)$ or $4-(-2g)$,

(16) *i.e.* $\tau \geqslant 4-\chi$.

B_2. *Definition :* An immersion $f : M—E^3$ is said to be *minimal* if

$$(17) \qquad \tau = \inf_{f \in F} \tau(f, M) = \sum_{i=0}^{2} \beta_i = 4 - \chi.$$

If $f : M \rightarrow E^3$ is minimal, then it can be shown that for almost every unit vector $z \in S_0{}^2 \subset E^3$, the linear function z.f (scalar product) : $M \rightarrow R$ has only non-degenerate criticalpoints. If μ_i (M, z.f) is the number of critical points of index i of such a function, then

$$(18) \qquad \tau = \sum_{i=0}^{2} \mu_i \text{ (M, z.f)}.$$

Now, the critical points of such a function are characterised by the equation z.df=0, *i.e.*, a critical point p is such that z is normal to the immersed manifold at f (p). If $\bar{\nu} : M \rightarrow S_0{}^2$ is the Gauss normal map which takes a point $p \in M$ into the unit vector through the origin (*i.e.* a point on $S_0{}^2$) parallel to the normal of f (M) at f (p), then the total absolute curvature of M (not necessarily minimal) is the average number of times $S_0{}^2$ is covered by $\bar{\nu}$. But, if the given immersion is minimal, then $\bar{\nu} : M \rightarrow S_0{}^2$, covers almost every point of $S_0{}^2$ an integral number of times which is equal to the minimal total absolute curvature of M. This shows the strong relationship between critical point theory and the theory of total absolute curvature.

C_2. Kuiper in [8] introduced the concepts of *topsets* and top-1-cycles and proved ;

C'_2. There are no minimal immersions for the projective plane and the Klein bottle in euclidean three-space.

In his subsequent papers [9] and [10], he made use of machinery developed in [8] and proved the following theorems :

C''_2. There exist minimal immersions for all closed(1) 2-manifolds with $\chi \leqslant -2$ in euclidean three-space E^3.

He actually constructed a minimal immersion of the projective plane with 2 handles in E^3. (in this case

$$\chi = 1 - 2g = 1 - 2.2. = -3.)$$

He conjectured that "there exists no minimal immersion in E^3 of the projective plane with one handle."

C'''_2. Let $f : M \to E^5$ be a minimal immersion of a closed 2-manifold properly into(2) E^5. Then M is a smooth projective plane, f is an embedding, and f (M) is a real Veronese surface.

An unsolved problem is the generalization of Kuiper's theorem (C'') to $P^2(C)$, $P^2(Q)$ and $P^2(IH)$.

D_2. Chern-Lashof in [3] proved the following results :

D'_2. A compact orientable surface embedded in E^3 lies on one side of the tangent plane at each point of positive Gaussian curvature if and only if the total absolute curvature is $2+2g$.

D''_2. A compact orientable surface immersed in E^3 with Gaussian curvature $\geqslant o$ is embedded and convex.

A part of the result D'_2 was generalized in the same paper to n-dimensional manifolds immersed in E^{n+1}.

2.9 *n-dimensional manifolds* :

The following are the main known results about the total absolute curvature of immersed manifolds :

A_n. (Chern-Lashof). Let M^n be a compact smooth manifold

(1) "closed" means "compact without boundary".

(2) "properly into" means "f (M) is not contained in any hyperplane of E^5".

immersed in E^N, and let β_i $(0 \leqslant i \leqslant n)$ be its ith Betti number relative to a coefficient field. Then the total absolute curvature $\tau(M^n)$ of M^n satisfies the inequality

$$(19) \qquad \tau(M^n) \geqslant \sum_{i=0}^{n} \beta_i (M^n).$$

Moreover, if the equality signs holds in (17) with the real field as the co-efficient field, then M^n has no *torsion* [3].

This is in fact a generalization of the following result of Chern-Lashof in [2] :

If $f : M^n \rightarrow E^N$ is an immersion of a closed manifold, then

$$(20) \qquad \tau(M^n) \geqslant 2.$$

A'_n. If $2 \leqslant \tau(M^n) < 3$, then M^n is homeomorphic to a sphere [2].

$A''n$. If $\tau = 2$, M^n is a convexly embedded hypersurface [2].

B_n. (Willmore-Saleemi). If $3 \leqslant \tau < 4$, then M^n is either a sphere or else it is a projective plane in the sense of Eells and Kuiper [4]. [19].

B_n. The results A_n, A'_n A''_n, B_n remain valid if the receiveing space E^{n+N} is replaced by a complete, simply-connected Riemannian manifold of non-positive curvature. [19].

C_n. (Kuiper). If $f : M^n \rightarrow E^N$ is a minimal, proper-into immersion, then $N \leqslant \frac{1}{2}n(n+3)$. [7].

C'_n. (Kuiper). Let $f_1 : M_1^{n1} \rightarrow E^{N1}$

and $f_2 : M_2^{n2} \rightarrow E^{N2}$ are immersions with total absolute curvatures τf_1 and τf_2 respectively. Then the total absolute curvature of

$$f_1 \times f_2 : M_1^{n1} \times M_2^{n2} \rightarrow E^{N1+N2}$$

is given by the formula :

$$(21) \qquad \tau_{f_1 \times f_2} = \tau_{f_1} \times \tau_{f_2}. \quad [7].$$

For its generalization and a simple proof see [19].

Almost all the above results have been obtained through the applica-tions of Morse Theory. To see how Morse Theory can be used to obtain the total absolute curvature of an immersed manifold in euclidean space we shall give a detailed proof of A_n (section 2.9).

Proof. Let $f : M^n \rightarrow E^N$ be an immersion of a closed manifold M^n in euclidean space of dimension N. Let B_ν be the bundle of $(N-n-1)$ – spheres of unit normals on M^n induced by f. Let

$$\tilde{\nu} : B_\nu \rightarrow S^{N-1}$$

be the map defined in (2.6). The total absolute curvature of M^n is the volume of the image of B_ν under $\tilde{\nu}$. The singular points of $\tilde{\nu}$ are those where the matrix $z.\, d^2f = -dz.df$, has rank less than n, that is, the points where $G(p,\nu) = 0$. The image of such points on $S_0{}^{N-1}$ is a set of measure zero (Sard's theorem). Hence for almost every unit vector $z \in S_0{}^{N-1}$, the linear function z.f restricted to M^n has only non-degenerate critical points. Hence, it follows from the Morse-inequalities that

$$(22) \qquad \sum_{k=0}^{n} \mu_k (M^n, z.f) \geqslant \sum_{i=0}^{n} \beta_i (M^n).$$

Now, the image of B_ν under $\tilde{\nu}$ is the same as the set of points $z \in S_0{}^{N-1}$ each counted a number of times equal to the number of critical points of the function z.f on M^n. It follows that

$$(23) \qquad \tau = \frac{\text{Im}\ (\nu\,(B_\nu))\ \text{on}\ S_0{}^{N-1}}{C_{N-1}} \geqslant \sum_{i=0}^{n} \beta_i (M^n).$$

Thus $$\tau \geqslant \sum_{i=0}^{n} \beta_i (M^n).$$

fI R is the field of reals and Z_p (p a prime) is a field mod p, then

(24) $\beta_i\,(M^n, R) \leqslant \beta_i\,(M^n, z_p), \quad (0 \leqslant i \leqslant n)$.

But by hypothesis, $\tau = \sum_{i=0}^{n} \beta_i\,(M^n, R)$, and by equation (23)

$\tau \geqslant \sum_{i=0}^{n} \beta_i\,(M^n, Z_p)$. Hence, this is only possible if

$\beta_i\,(M^n, R) = \beta_i\,(M^n, Z_p)$, $(0 \leqslant i \leqslant n)$, which means that M^n has no torsion.

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Address :

Panjab University, Lahore (Pakistan) and Liverpool University. Liverpool (England).

THE LAW OF MAGNETIC FIELD PRODUCTION

BY

G. T. P. TARRANT

Endless trouble is caused eventually by teaching to beginners laws which are insufficiently general and which apply only in a limited field. Thus teachers who neglect μ or k in electromagnetic theory merely because they are talking about free space and have chosen to work in systems for which μ or k is unity, have to repeat their work completely when ordinary media are under discussion.

The whole of Maxwell's displacement current idea is, in essence merely a complexity produced because the usual e.m. basic law of electromagnetism,

$$(1) \qquad \delta H = \frac{i\ \delta s\ \sin\theta}{d^2}$$

is not sufficiently general to cover all situations. One wonders, in fact, if it would not be better to start the subject from the even simpler law which exists and which is of sufficient generality to cover all situations. It is, if consistent units are used throughout,

(2) (Magnet-Motive Force) $= \int H.ds =$ {(Rate of cutting of number of lines of electrostatic induction}

$$\text{or} \quad \text{Curl } H = \frac{d\,(kX)}{dt}$$

History favours equation (1) but it was only a matter of chance that Ampere was not sure that electric currents were due to a flow of electric charges and had therefore to describe his results in terms of current.

Virtually the same discovery might have been made by an experimentalist leaning out of a train which was very close to another parallel train. If the one train was charged electrically relative to the other then the experimentalist could have detected a horizantal electric field between the two trains and could also have detected a vertical magnetic field when the two trains moved relative to each other. This experimentalist would undoubtedly have announced his discovery by saying that a movement of an electric field produces a magnetic field proportional to $\int \frac{d\,(k\mathrm{X})}{dt}$ ds. If he had previously defined X in electrostatic units but had not previously defined H he would undoubtedly have written $\mathrm{H} = \int \frac{d\,(k\mathrm{X})}{dt}$ ds and thus obtained H in electrostatic units. If the contrary had happened he would have obtained the same equation with (kX) and H being both expressed in electromagnetic units.

Several important results can be obtained quite easily from equation (2).

Amongst these are :

(*a*) If there are n charges of magnitude e per cm. length of a wire and if they are flowing so as to constitute a current i of (nev) then the number of lines of induction cutting any loop we draw round the wire will be 4 π (nev) in each second. Hence

$$\int \mathrm{H}\, ds = 4\,\pi\, i$$

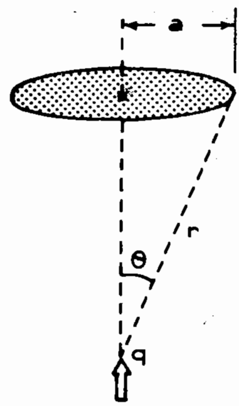
provided again that H is measured in units consistent with those of e and i. This proof seems of some interest in view of the great importance of the result and that the present proof is quite general and does not involve the doubtful conception of a unit magnetic pole.

(*b*) If the system has complete cylindrical symmetry about the wire — as will occur, for example, if two oppositely charged spheres are connected by short straight wire, than

$$\mathrm{H} = \frac{4\,\pi\, i}{2\,\pi\, r}$$

(*c*) The same formula should also be valid whenever we have a battery driving current in a closed circuit providing the circular loop is drawn close enough to one wire to ensure that the electrostatic lines of induction leaving the moving charges do not have their symmetry disturbed by the corresponding lines coming from other charges coming from other parts of the wire.

(*d*) Although law (2) cannot be derived from law (1) without the additional assumptions of Maxwell's displacement current, yet it is perfectly easy to deduce law (1) and all that follows from it form law (2).



To do this consider a charge q moving with velocity v as shown, then the number of lines of induction, N_{el}, entering the circle is

$$4\,\pi\,q\,\frac{\text{(Area of spherical cap contacting circle)}}{4\,\pi\,r^2}$$

$$= 4\,\pi\,q\,\frac{2\,\pi\,r^2\,(1 - \cos\theta)}{4\,\pi\,r^2}$$

$$\therefore\ \frac{d\,N_{el}}{dt} = 2\,\pi q\,\sin\theta\left(\frac{d\,\theta}{dt}\right) = 2\pi\,q\,\sin\theta$$

$$\times\left(\frac{\sin^2\theta}{a}\,\frac{dx}{dt}\right)\ \text{since}\ \cot\,\theta = \frac{x}{a}$$

Because the system is symmetrical about the line of velocity

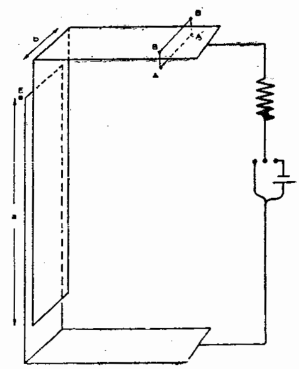
$$\frac{d\,N_{el}}{dt} = 2\pi\,a\,\frac{d\,(kX)}{dt} = 2\pi\,a\,\delta\,H$$

if we write δ H instead of H because the field is certainly very small and because we may later find the total magnetic field due to a long stream of charges similar to q flowing in the same line and thus constituting a current in a long straight wire. Then:—

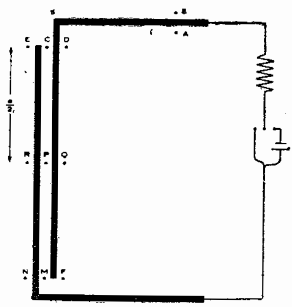
$$\delta\,H = \frac{2\pi\,q}{2\pi\,a}\ \frac{\sin^3\theta}{a}\,v = \frac{q\,v\,\sin\theta}{r^2} = \frac{i\,\delta s\,\sin\theta}{r^2}$$

It therefore follows that the statement that $H = \int \dot{D}$ ds tells us everything that the more conventional relation gives.

It does this AND MORE and is therefore MORE SUITABLE as a fundamental relation. To understand this consider the following :



Imagine two metal plates each bent at right angles as shown and placed near together in a vacuum to form a parallel plate condenser of area (ab). Assume this can be charged by a battery and discharged through a resistance. Then it is known from equation (1)—that $\int H\, ds$ round any loop surrounding the wire or the metal plate should be always $4\pi\, i$. This should be so for the loops A A′ B′B, CC′D′D, NMM′N′ and RR′Q′Q (where the primed letters denote points directly underneath the corresponding unprimed ones and near the other edge of the plate).



On the other hand $\int Hds$ round either of the loops RR′P′P or PP′Q′Q should be only $4\pi\,(\frac{1}{2}\, i)$ because these loops are drawn round only one plate of the condenser at a point half way up the overlapping portion where the current flow should be only half as great. This shows us that $\int Hds$ along PP′ must be zero.

Now we come to the crux of the matter. Consider the loop C C′P′P. The $\int Hds$ along PC and C′P′ must be zero so that $\int Hds$ round the whole loop must be $2\,\pi\, i$. But if equation (1) is really true it should be zero since no ordinary current threads the loop.

On the other hand the matter is obvious from the point of view of equation (2), for one half of the lines of induction coming from the charges passing the strip between C and D will cut the wire CC′, while over the portion PP′ the movement of the lines of induction produced by the current flow in the strip between P and Q will be exactly balanced by those coming in the opposite direction from the current flowing in the strip between R and P.

The derivation of Maxwell's equation is exceedingly simple if we employ equation (2). We have merely to consider a closed loop in space and to state that $\int Hds = \frac{d\,N_1}{dt} = \int\int (kX)\,d\,S$ if there is no conduction current or will be $\int\int \left\{k\,\dot{X}_n + 4\,\pi \left(\textit{current density}\right\}\right. d\,S$ if there is a conduction current. The application of Stokes theorem then shows immediately that curl $H = k\dot{X} + 4\,\pi$ (current density).

A final advantage comes from the adoption of the basic equation (2) is that it is equally applicable to the complete electrostatic or complete electromagnetic systems or to rationalized M.K.S. because if X is written as $\frac{e}{4\pi\,kd^2}$ then $\delta H = \frac{i\,ds\,\sin\theta}{4\pi\,d^2}$ follows immediately from equation (2) without any other changes being required.

A NEW USE FOR DIMENSIONAL ANALYSIS

BY

G. T. P. TARRANT

Mathematics Department, University of the Panjab

Mathematics deals with the consequences of (*a*) the actual laws of nature as we believe them to be, and (*b*) hypothetical laws which may, or may not, be found in the future to occur in the physical world.

Dimensional analysis is a mathematical tool which has so far been employed solely in dealing with (*a*) and has, as far as the author is aware, never been used to elucidate the consequences of purely hypothetical laws. This is probably because the mathematician gets his main pride in handling more complex operations than those of dimensional analysis while the physicist and the engineer tend to refuse to waste time on laws unless they are believed to be correct.

The problem discussed below illustrates how dimensional analysis can be applied to hypothetical laws to give results which are of genuine scientific interest. It is a problem which we can solved in two minutes by dimensional analysis but which took Darwin the major part of two years to solve by ordinary methods.

In 1911 Rutherford published his theory of the single scattering of α particles and showed that the proportion of α particles scattered at a given angle depends on (Area) $\left(\frac{nt}{r^2}\right)(\frac{1}{2}m\ V^2)^{-2}\ (\text{Ze E})^2$

Here (nt) denotes the number of scattering atoms per sq. cm. and the remaining obvious symbols are those given by Rutherford, Chadwick and Ellis, Ch 8. This theory was based on the assumption of the inverse square law of repulsion between positive charges was true even though the distances involved were of the order of 10^{-13} cm. Did the fact that this formula agreed with experiment prove that the law of force which was known to be true between, say, 1 cm and 100 cm. also applied at distances as small as 10^{-13} cm? Might it not have been possible that nearly the same formula could have been derived from some other law of repulsion? It was to answer these questions that Rutherford asked Darwin to determine what would be the proportion of α particles scattered if the law of fore was the inverse nth.

If this is the assumption, that (force) $= \frac{e_1 e_2}{k\, r^n}$, then the dimensions of $(e_1 e_2)$ must be $[\mathrm{M\ L\ T^{-2}\ k\ L^n}] = [\mathrm{M\ L^{n+1}\ T^{-2}\ k}]$ instead of the more usual $[\mathrm{M\ L^3\ T^{-2}\ k}]$.

We will now make the assumption that the fraction of the α particles scattered depends on :—

$$(\text{area}) \frac{(\text{No of scattering atoms per sq cm})}{r^2} - (\tfrac{1}{2} m\, V^2)^a\ (Ze\ E)^{b} k^c$$

Then the fact that the dimensions of a fraction are zero tells us that

$$[L^2] \frac{[L^{-2}]}{[L^2]} [ML^2\ T^{-2}]^a [M\ L^{n+1}\ T^{-2}\ k]^b [k]^c = 0$$

From this we see that :—

In M	$a + b = 0$
In L	$-2 + 2a + b(n+1) = 0$
In T	$-2a - 2b = 0$
In k	$b + c = 0$

From which we find that $b = \frac{2}{n-1} = -a = -c$, so that the formula becomes $f \propto (\text{area}) \left(\frac{nt}{r^2}\right) \left(\frac{ZeE}{\frac{1}{2}mV^2k}\right)^{\frac{2}{n-1}}$

This agrees with that found by Darwin more rigorous calculation. To compare with Chadwick's experimental results that, over a variation of V^4 of 4 to 1 the product $(f\ V^4)$ varied by less than 3 % we put the above equation into the form

$$\log (f\ V^4) - \frac{(2-n)}{(n-1)} \log (V^4) = \text{constant.}$$

This shows that the law is within 1 % of the inverse square at the most, even at these extremely short distances.

THE ACCURACY OF THE INVERSE SQUARE LAW IN ELECTROMAGENTISM

BY

G. T. P. TARRANT,

Mathematics Department, University of the Panjab.

Newton's work on gravitation made many workers in the 18th century suspect that an inverse square law would also apply to the forces between magnetic poles and between electric charges. To show the intrinsic difficulty in ascertaining the truth of these guesses by direct measurement of the forces let us assume that the force, F, varies as $\frac{1}{d^n}$. We will also assume that measurements conducted at two distances d_1 and d_2 gave forces $F_1 \pm \delta F$ and $F_2 \pm \delta F$ where F_1 and F_2 are the forces that would have been obtained in an ideal experiment with zero error and δF is the constant instrumental error in making the measurements.

Then we find $\frac{F_1 \pm \delta F}{F_2 \pm \delta F} = \left(\frac{d_2}{d_1}\right)^{n+\delta n}$ were δn is the difference between the true exponent, n, and the value, $(n+\delta n)$, that we find as a result of our errors. Hence

$$\frac{F_1\left(1 \pm \frac{\delta F}{F_1}\right)}{F_2\left(1 \pm \frac{\delta F}{F_2}\right)} = \left\{\left(\frac{d_2}{d_1}\right)^n\right\}^{1+\frac{\delta n}{n}} = \left(\frac{F_1}{F_2}\right)^{1+\frac{\delta n}{n}}$$

The limiting values of δn are then given by

$$\delta n = \pm n \frac{\log\left(1+\frac{\delta F}{F_1}\right) - \log\left(1-\frac{\delta F}{F_2}\right)}{\log\left(\frac{F_1}{F_2}\right)} = \pm \frac{n\frac{\delta F}{F_1}\left(1+\frac{F_1}{F_2}\right)}{\log\left(\frac{F_1}{F_2}\right)}$$

This has a minimum value of $3.55\, n \frac{\delta F}{F_1}$ or $7 \frac{\delta F}{F_1}$ when $\frac{F_1}{F_2} = 3.55$, or when $\frac{d_2}{d_1}=1.88$. As most students know, it is generally hard to measure deflections with an accuracy much greater than 2 % of the largest deflection encountered. We must therefore agree that Coulomb did remarkably well in proving in 1767 that the law of force for electrostatics was within 3 % of being an inverse square. This is especially so because it is unlikely that he had the benefit of the above analysis and would therefore have been almost certain to have fallen into the common trap of making d_2 much larger than the optimum.

A tremendeous step forward was made in 1773 when Cavendish devised an indirect experiment based on a theoretical argument that there should be no field inside a charged, completely closed, metal box if the law of force is an exact inverse square. This well known experiment is usually discussed today in connection with Gauss' theorem in spite of the fact that this theorem was not put forward until some 60 years later. Maxwell used the same method in 1870 and concluded that the exponent cannot differ from 2 by more than $\frac{1}{21,600}$. The most recent test, by Plimpton and Lawton (1), showed that that the law of force between electric charges at ordinary distances is an inverse square to the almost unbelievable accuracy of 1 in 10^9. This makes it to be, almost certainly, the most accurately known law in the whole of science.

The law of force between magnet poles is in a less satisfactory state largely because magnets are heavy and have poles that are ill defined and nebulous. It is, however, easy to show that if the magnet has poles, δm, at symmetrical distances L from the centre and if the law of force is the inverse nth then the fields in the 'end' on and 'broadside on' positions are

$$\frac{n2\,L\,\delta m}{r^{n+1}}\left\{1+\frac{(n+1)(n+2)}{6}\,\frac{L^2}{r^2}\right\} \text{ and } \frac{2L\,\delta m}{r^{n+1}}\left(1-\frac{n+1}{2}\,\frac{L^2}{r^2}\right)$$

respectively.

Integrating, we find for a real magnet that these two fields are

$$n \left\{ M + \frac{(n+1)(n+2)}{3r^2} \int L^3 \, dm \right\} \frac{1}{r^{n+1}} \quad \text{and} \quad \left(M - \frac{n+1}{re} \int L^3 dm \right) \frac{1}{r^{n+1}}$$

respectively.

The ratio between the two is then

$$n \left[1 + \left\{ \frac{(n+1) \ (n+2)}{3} + (n+1) \right\} \frac{\int L^3 \, dm}{M \, r^2} \right]$$

$= n \left[1 + 7 \frac{\int L^3 \, dm}{M \, r^2} \right]$ since n is near to 2. This ratio can now be written as $n(1+k/r^2)$. This means that a graph of the ratio of the fields against $1/r^2$ should be a straight line from which n could be determined with tolerable accuracy in spite of the maget having a continuous distribution of poles over its length. Unfortunately this analysis does not appear to be known and no experimental graph of this nature appears ever to have been plotted. Because of this lack of knowledge Gauss (1833) felt compelled to work at large distances from his magnet which meant that the angles of deflection of his tangent magnetometer were very small, varying from 2° 13′ 51.2″ to 0° 2′ 22.2″ of arc. The accuracy of his measurements was however of the order of 1″ so that his measurements gave the law of force between poles as the inverse square with an uncertainty of less than 1 in 2000

Compared with either of the above two laws our experimental knowledge of the law governing the production of magnetic fields by electric current elements is much less satisfactory. In the first place there is still a discussion (2) as to whether Ampere's formula for the force between two current elements $i_1 \delta s_1$ and $i \ \delta s_2$ is to be preferred to that of Biot-Savart or Grassman. Ampere's formula is expressed in terms of the angle ϵ between the two current elements and the angles β_1 and β_2 made by the two current elements with the line, r, joining them. It says that the force is in the direction of r and is

$$\delta F = \mu\, i_1\, \delta s_1\, i_2\, \delta s_2 \left(2 \cos \epsilon - 3 \cos \beta_1 \cos \beta_2\right)\frac{1}{r^2}$$

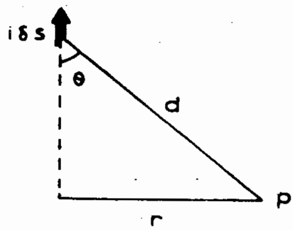
This law has the obviously attractive feature that action and reaction are equal opposite but is so less convenient mathematically than that of Biot-Savart, $\delta \bar{F} = \mu\left(i_1\, \delta \bar{s}_1\right) \times \left(i_2\, \delta \bar{s}_2\right) \times \frac{\bar{r}}{r^3}$ or $\left(i_1\, \delta \bar{s}_1\right) \times \delta \bar{B}$ where

$$\delta \bar{B} = \mu\, i_2\, \delta \bar{s}_2 \times \frac{\bar{r}}{r^3}$$

that it is hardly ever used. On the other hand the Biot-Savart law makes action and reaction between current elements to be *unequal and not even in the same direction.* This difficulty is usually ignored on the ground (*a*) that we can never isolate the contribution to B from one portion of a circuit δs_2 from that produced by the remainder so that in practice we are concerned only with $i_1\, \delta \bar{s}_1 \times \int \mu\, (i_2\, \delta \bar{s}_2) \times \frac{\bar{r}}{r^2}$ and (*b*) that the Ampere formula gives exactly the same result for the equivalent integral so that it does not matter which formula is used.

However, independent of this controversy, it is fair to ask what is the experimental or theoretical accuracy in the inverse square law involved in both equations. The answer is rather amazing. I can find no references or indications anywhere that the question has ever been considered seriously since the original work of Ampere—which could hardly have had an accuracy greater than 5 or 10 %. Undoubtedly any major departure from the inverse square would have been detected more or less accidentally. Most electrical experiments, however, do not have the accuracy to show up small departures from the law and the really accurate work with current weighers was done with apparatus of somewhat similar geometrical configuration of the coils or relies on electrical methods for measuring such quantities as the effective radii involved. The contrast between the accuracy of the law for electrostatics of 1 in 10^9 with a 10 % (or perhaps 1 %) accuracy for the law of electromagnetism justifies an examination of the problem.

The first obvious step in ascertaining the accuracy with which the exponent of r is, or can be shown to be, two is to consider the field produced by a long straight wire.



$$H=\int\frac{(ids)\sin\theta}{d^n}=\int\frac{\left(i\,\frac{(rd\theta)}{\sin^2\theta}\right)\sin\theta}{\left(\frac{r}{\sin\theta}\right)^n}$$

$$=\frac{i}{r^{n-1}}\int\sin^{n-1}\theta\,d\theta=\frac{i}{r^{n-1}}\int\left(1-\cos^2\theta\right)^{\frac{n-2}{2}}d\left(\cos\theta\right)$$

$$=\frac{i}{r^{n-1}}\int\left(1-\frac{b}{2}\cos^2\theta+\frac{\frac{b}{2}\left(\frac{b}{2}-1\right)}{2}\cos^4\theta\,..\right)d(\cos\theta)$$

if $b=n-2$

$$=\frac{2i}{r^{n-1}}$$ if the wire is infinitely long and b is very small.

Experiments at two different distances r_1 and r_2 would then give, instead of the true fields H_1 and H_2, the apparent fields, $H_1\pm\delta H$ and $H_2\pm\delta H$, so, following the argument on page 37 ,

$$\frac{H_1\pm\delta H}{H_2\pm\delta H}=\left(\frac{r_2}{r_1}\right)^{n-1+\delta n}$$

where δn is the difference between the true exponent, n, and the value that we find as a result of our errors. Hence :—

$$\frac{H_1}{H_2}\,\frac{\left(1\pm\frac{\delta H}{H_1}\right)}{\left(1\pm\frac{\delta H}{H_2}\right)}=\left\{\left(\frac{r_2}{r_1}\right)^{n-1}\right\}^{1+\frac{\delta n}{n-1}}=\left\{\frac{H_1}{H_2}\right\}^{1+\frac{\delta n}{n-1}}$$

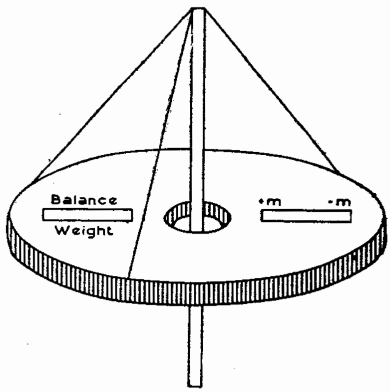
$$\therefore\quad\delta n=(n-1)\,\frac{\frac{\delta H}{H_1}+\frac{\delta H}{H_2}}{\log\left(\frac{H_1}{H_2}\right)}=\frac{\delta H}{H_1}\;\frac{\left(1+\frac{H_1}{H_2}\right)}{\log\left(\frac{H_1}{H_2}\right)}$$

which is $3.55\left(\frac{dH}{H_1}\right)$ when at its minimum value corresponding to $H_1/H_2=3.55$ or $d_2/d_1=3.55$.

As we have discussed earlier such direct measurements are not likely to allow us to determine n with an accuracy much better than 1 %.

The accuracy of simple null experiments in which the fields of one long straight wire is exactly balanced by that produced by a larger current at a larger distance are unlikely to be greater because of the dimensions of the measuring apparatus and the rapid change of field with position.

One ingenious scheme that deserves consideration is described by Maxwell(3) who, unfortunately, does not say if it was actually carried out—though one suspects that it might be the method used by Biot-Savart. In it a magnet is placed radially on a cardboard disc hanging horizantally from a long, vertical, current carrying, wire which passes near to the centre of the cardboard disc.



Then, if the field on the nearer pole +m at a distance r_1 from the wire is H_1, $H_1 = 2i/(r_1)^n$ and $H_2 = 2i/(r_2)^n$.
Hence, if the three supporting wires allow the disc to rotate about a point distant a from the wire centre the resulting moment of the forces causing the rotation will be :—

$$\delta M = m H_1 (r_1 + a) - m H_2 (r_2 + a) = 2\,im \left\{ \frac{r_1 + a}{r_1^n} - \frac{r_2 + a}{r_2^n} \right\}$$

$$= 2\,im \left(\frac{r_1 + a}{r_1^n}\right) \left\{ 1 - \frac{r_2 + a}{r_1 + a} \, \frac{r_1^n}{r_2^n} \right\}$$

So $\dfrac{\delta M}{M} = 1 - \dfrac{r_2 + a}{r_1 + a} \, \dfrac{r_1^n}{r_2^n}$ and $n \log\left(\dfrac{r_1}{r_2}\right) = \log\left(1 - \dfrac{\delta M}{M}\right) - \log\left(\dfrac{r_2 + a}{r_1 + a}\right)$

$$\therefore\ n \log\left(\frac{r_1}{r_2}\right) = -\frac{\delta M}{M} + \log\left(\frac{r_1}{r_2}\right) - \left(\frac{a}{r_2} - \frac{a}{r_1}\right)$$

$$\therefore\ n = 1 - \left(\frac{\delta M}{M} + \frac{a}{r_2} - \frac{a}{r_1}\right) \Big/ \log\left(\frac{r_1}{r_2}\right)$$

It seems unlikely that we will be able to rely on a being less than 1mm. on a radius of, say, 10 cm. Hence the accuracy is likely to be of the order of

1 in 400 whatever the accuracy in detecting the small difference in the moment of the forces.

Considerable improvement will, of course, be effected by replacing the balancing weight by an identical magnet as Maxwell actually indicated partly because this would eliminate effects produced by the earths magnetic field. If the magnets were truly identical it could probably be arranged to eliminate completely the terms dependent on a in the above expression. Even so, however, the accuracy can hardly be made very high.

A possible method that does not appear to have been considered involves the use of two pairs of Helmholtz coils. We start our discussion by noting that if one pair of coils is arranged in the Helmholtz position (for which $x=R/2$), then

$$H_1 = \frac{4\pi N_1 i R_1^2}{\left(R_1^2+x^2\right)^{\frac{n+1}{2}}} = \frac{4\pi N_1 i}{\left(\frac{5}{4}\right)^{\frac{n+1}{2}}\left(R_1^{n-1}\right)}$$

Consequently if we set up a second and larger pair of Helmholtz coils with the same current passing in opposite directions we would expect zero field at the common centre if $H_1=H_2$, *i.e.* if

$$\frac{N_1}{R_1^{n-1}} = \frac{N_2}{R_2^{n-1}}$$

However we will again have an instrumental error of $\pm\delta H$ in determining the field equality and a measuremental error in determining the radius of δR. As a result we will calculate not the true value n but $(n+\delta n)$, such that :—

$$\frac{4\pi N_1 i}{\left(\frac{4}{5}\right)^{\frac{n+\delta n+1}{2}}\left(R_1 \pm \delta R\right)^{n+\delta n-1}} = \frac{4\pi N_2 i}{\left(\frac{5}{4}\right)^{\frac{n+\delta n+1}{2}}\left(R_2 \pm \delta R\right)^{n+\delta n-1}} \pm \delta H$$

Hence, dividing by the value of H_2 which is

$$\frac{4\pi N_2 i}{\left(\frac{5}{4}\right)^{\frac{n+\delta n+1}{2}}\left(R_2 \pm \delta R\right)^{n+\delta n-1}}$$

we see that :—

$$\frac{N_1}{N_2}\left(\frac{R_2 \pm \delta R}{R_1 \pm \delta R}\right)^{n+\delta n-1} = 1 \pm \frac{\delta H}{H_2},$$

$$\text{So} \quad \left(n+\delta n-1\right) \log \frac{\left(1 \pm \frac{\delta R_2}{R_2}\right)}{\left(1 \pm \frac{\delta R}{R_1}\right)} = \log\left(1 \pm \frac{\delta H}{H_2}\right)$$

$$- \log \left\{\frac{N_1}{N_2}\left(\frac{R_2}{R_1}\right)^{n+\delta n-1}\right\},$$

$$\left(n+\delta n-1\right)\left(\pm\frac{\delta R}{R_2} \pm \frac{\delta R}{R_1}\right) = \pm \frac{\delta H}{H_2} - \delta n \log\left(\frac{R_2}{R_1}\right)$$

$$\therefore \delta n \left\{\pm \frac{\delta R}{R_2} \mp \frac{\delta R}{R_1} + \log\left(\frac{R_2}{R_1}\right)\right\} = \pm \frac{\delta H}{H_2}$$

$$+ \left(n-1\right) \left(\pm \frac{\delta R}{R_2} \mp \frac{\delta R}{R_1}\right)$$

$$\therefore \delta n = \left\{\frac{\delta H}{H_2 \log \frac{R_2}{R_1}} \pm \frac{\frac{\delta R}{R_2} - \frac{\delta R}{R_1}}{\left(\log \frac{R_2}{R_1}\right)}\right\}$$

Since with modern apparatus we should be able to detect δH of 10^{-5} gauss on a field of perhaps 10 gauss with coils having $\log (R_2/R_1)=1$, we conclude that

$$\delta n = 2 \left\{ 10^{-6} + \left(\frac{\delta R_2}{R_2} + \frac{\delta R_1}{R_1}\right)\right\}$$

The possible accuracy in the determination is thus set by the measurement of the radii which could, with ease, be 1 in 10^4.

Limitations of the same order of magnitude appear in other typical experiments so that it is very reassuring to have an argument based on the special theory of relativity which indicates that the law should be inverse square to the same accuracy as that applying with the law of electrostatics (1 in 10^9). This argument is based on a paper by Rosser(4).

The Relativistic Approach to the Problem :

Suppose an observer S′ has a charge e_v fixed at the origin of his coordinate system and observes a second charge to be moving relative to him with velocity components u_x', u_y' and u'_z. At the instant that this charge crosses the x'_u y'_u plane at the point x′, y′, the observer measures the force on this second charge e_u and finds components

$$F'_x = \frac{e_u\, e_v\, x'_u}{k\left(x_u'^2 + y_u'^2\right)^{\frac{3}{2}}}; \quad F'_y = \frac{e_u\, e_v\, y'_v}{k\left(x_u'^2 + y_u'^2\right)^{\frac{3}{2}}}; \quad F'_z = O$$

(This statement involves the assumption that the only forces involved between a stationary and a moving charge are purely electrostatic ones; but if this were not true we would surely have found some peculiarities in our treatment of the motion of an electron round a stationary nucleus when the electron velocity is enormous.)

We will next suppose there is a second observer S moving relative to the first with a velocity of v in the x direction and carrying his own instruments to measure forces and velocities. Assume there is a coordinate system which is stationary with respect to him and which is placed so that its axes are parallel to those of the first observer and so that its origin lies on the x axis of the first observer.

Then this second observer will see the charge e_v moving with a velocity of v and the second charge e_u moving with different velocity components ponents u_x, u_y, u_z . Assume that he measures the components of the force on e_u, (F_x, F_y, F_z), at the instant that he sees that charge crossing the x,y plane at the point x_u, y_u. Assume also that at that instant he notes the position of e_v on the x axis and finds it is at x_v.

Now the special theory of relativity provides us with a number of 'transformation formulae' by which we can determine what the second observer will see if we know what is seen by the first observer. These are best written in words as 'The value of x as seen by the second observer is γ times the value of $(x'+vt')$ as seen by the first observer. For conciseness we write it as $x == \gamma(x' + vt')$ where γ where stands for $\left(1-\frac{v^2}{V^2}\right)^{-\frac{1}{2}}$, where V is the velocity of light in free space. This double sign $==$ seems a useful idea because it emphasises that this is not an ordinary equation but is a transformation formula, and has to be read not as x equals $\gamma(x' \times vt')$ but as 'the value of x as seen by the second observer is γ times the value of $(x' + vt')$ as seen by the first observer'.

Then with this understanding we may write the transformation formulae with which we will be concerned as :—

$$x' == \gamma(x - vt) \quad ; \quad y' == y \quad ; \quad z' == z$$

$$u'_x == \frac{(u_x - v)\,V^2}{V^2 - v\,u_x} \; ; \; u'_y == \frac{V^2\,u_y}{\gamma\,(V^2 - v\,u_x)} \; ; \; u'_z == \frac{V^2\,u_z}{\gamma\,(V^2 - v\,u_x)}$$

$$F_x == F'_x + \frac{v}{V^2 + v\,u'_x}\left(u'_y\,F'_y + u'_z\,F'_z\right) \; ; \; F_y == \frac{V^2\,F'_y}{\gamma(V^2 + v\,u'_x)} \; ;$$

$$F_z == \frac{V^2\,F'_z}{\gamma\,(V^2 + v\,u'_x)}$$

It is important to note that all these formulae may be deduced from general considerations assuming only that true physical laws and the velocity of light in free space remain the same whatever the relative speeds of different observers. The proofs do not even mention such quantities as B,H, or E. We now start. Using the first equations we argue that :—

$$x'_u == \gamma\,(x_u - vt) \; ; 0 == \gamma\,(x_v - vt), \quad x'_u == \gamma\,(x_u - x_v)$$

Then, using the force transformation formulae, we see that :—

$$F_x == \frac{e_u\,e_v\,x'_u}{k\left(x'^2_v + y'^2_u\right)^{\frac{3}{2}}} + \frac{v}{(V^2 + v\,u'_x)}\left\{ u'_y\,\frac{e_u\,e_v\,y'_u}{k\left(x'^2_u + y^2_u\right)^{\frac{3}{2}}} + O \right\} ;$$

$$F_y == \frac{V^2}{\gamma\,(V^2 + v\,u'_x)} \quad \frac{e_u\,e_v\,y'_r}{k\left(x'^2_u + y'^2_u\right)^{\frac{3}{2}}} \quad ; \; F_z == O$$

We also see that

$$V^2 + v\,u'_x == V^2 + v\left\{\frac{(u_x - v)\,V^2}{V^2 - v\,u_x}\right\} = V^2\,\frac{(V^2 - v^2)}{(V^2 - v\,u_x)}$$

and that

$$x'^2_u + y'^2_u == \gamma^2 (x_u - x_v)^2 + y_u^2 \quad \text{and} \quad u'_y == \frac{V^2 u_y}{\gamma (V^2 - v u_x)} ;$$

$$\therefore \quad F_x = \frac{e_u e_v}{k \{ \gamma^2 (x_u - x_v)^2 + y_u^2 \}^{\frac{3}{2}}} \left\{ \gamma (x_u - x_v) + \frac{v (V^2 - v u_x)}{V^2 (V^2 - v^2)} \frac{V^2 u_y y_u}{\gamma (V^2 - v u_x)} \right\}$$

$$F_y = \frac{e_u e_v}{k \{ \gamma^2 (x_u - x_v)^2 + y_v^2 \}^{\frac{3}{2}}} \frac{V^2}{\gamma} \frac{(V^2 - v u_x) y_u}{V^2 (V^2 - u^2)} ; F_z = O$$

Note that it is now no longer necessary to use the sign == because it is obvious that the only way in which the S man can see things the same when they are both in the same system is if they are truly equal to each other.

We now simplify the above force equations by remembering that

$$\gamma^2 = \frac{1}{\left(1 - \frac{v^2}{V^2}\right)} \quad \text{and by noting that}$$

$$\gamma^2 \left(x_u - x_u\right)^2 + y_u^2 = \gamma^2 \left\{ \left(x_u - x_v\right)^2 + \left(1 - \frac{v^2}{V^2}\right) y_u^2 \right\}$$

$$= \gamma^2 \left\{ \left(x_u - x_v\right)^2 + y_u^2 - \frac{v^2}{V^2} y_u^2 \right\} = \gamma^2 \left\{ r^2 - \frac{v^2 y_u^2}{V^2} \right\}$$

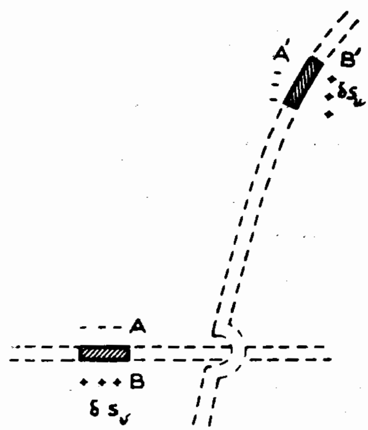
$$= \gamma^2 r^2 \left\{ 1 - \frac{v^2}{V^2} \sin^2 \theta_u \right\}$$

if θ_u is the angle between the radius vector and the x axis or the direction of movement of e_v. So the components of the force on e_v are :—

$$F_x = \frac{e_u e_v \left(x_u - x_v + \frac{v}{V^2} u_y y_u \right)}{k \gamma^2 r^3 \left\{ 1 - \frac{v^2}{V^2} \sin^2 \theta_v \right\}^{\frac{3}{2}}} ; \quad F_y = \frac{e_u e_v \left(V^2 - v u_x \right) y_u}{k \gamma^2 r^3 \left\{ 1 - \frac{v^2}{V^2} \sin^2 \theta_v \right\}^{\frac{3}{2}} V^2}$$

These equations, virtually those first obtained by Tolman(5), degenerate when the velocities are low, so that $\gamma = 1$, into the components of a huge electrostatic force plus other forces of the order of $\frac{v}{V}$ smaller. It is therefore unlikely that they can be tested as they stand. Rosser(4), however, gave a lengthy explanation showing that the force can be regarded as being produced by the electrostatic field of e_t and the magnetic induction $\propto v e_v$ The following argument seems much simpler and more directly applicable to currents in wires.

Suppose that we have two short elements of wire δs_v and δs_u containing electrons e_v and e_u which are in motion as above and equal positive charges which are at rest. Call these A, B, A′, B′ as shown. Then the total force, F*, on δs_u should be that



$$\begin{pmatrix}\text{from AA}' \text{ with}\\ \text{v \& u present}\end{pmatrix} + \begin{pmatrix}\text{from BB}' \text{ when}\\ \text{v \& u absent}\end{pmatrix} + \begin{pmatrix}\text{from AB}' \text{ when v}\\ \text{present, u absent}\end{pmatrix}$$

$$+ \begin{pmatrix}\text{from BA}' \text{ when v}\\ \text{absent, u present}\end{pmatrix}$$

Then, using the formulae given above, we find that :—

$$F_x^* = \frac{e_v\, e_u}{k r^3} \left[\left\{ \frac{(x_u - x_v) + \frac{v}{V^2}\, u_y\, y_u}{\gamma^2 \left(1 - \frac{v^2 \sin^2 \theta_v}{V^2}\right)^{\frac{3}{2}} V^2} \right\} + \left\{ \frac{x_u - x_v}{V^2} \right\} \right.$$

$$\left. - \left\{ \frac{x_u - x_v}{\gamma^2 V^2 \left(1 - \frac{v^2 \sin^2 \theta_v}{V^2}\right)^{\frac{3}{2}}} \right\} - \left\{ \frac{x_u - x_v}{V^2} \right\} \right]$$

$$F^*_y = \frac{e_v\, e_u}{kr^3} \left\{ \left\{ \frac{(V^2 - vu_x)\, y_u}{\gamma^2\, V^2 \left(1 - \frac{v^2 \sin^2 \theta_v}{V^2}\right)^{\frac{3}{2}}} + \left\{ \frac{V^2\, y_u}{V^2} \right\} \right. \right.$$

$$\left. - \left\{ \frac{V^2\, y_u}{\gamma^2\, V^2 \left(1 - \frac{v^2 \sin^2 \theta_v}{V^2}\right)^{\frac{3}{2}}} \right\} - \left\{ \frac{V^2\, y_u}{V^2} \right\} \right\}$$

$$F^*_x = \frac{e_v\, e_u}{kr^3} \qquad \frac{vu_y\, y_u}{\gamma^2\, V^2 \left(1 - \frac{v^2 \sin^2 \theta_v}{V^2}\right)^{\frac{3}{2}}}$$

and $F^*_y = - \dfrac{e_v\, e_u}{kr^3} \qquad \dfrac{vu_x\, y_u}{\gamma^2\, V^2 \left(1 - \frac{v^2 \sin^2 \theta_v}{V^2}\right)^{\frac{3}{2}}}$

So, since $\dfrac{y_u}{r} = \sin \theta_v$

$$F^*_u = \sqrt{F^{*2}_u + F^{*2}_y} = \frac{(e_v v)\,(e_u u) \sin \theta_v}{kr^2\, \gamma^2\, V^2 \left(1 - \frac{v^2 \sin \theta_v}{V^2}\right)^{\frac{3}{2}}}$$

$$= \frac{(i_v\, \delta s_v)\,(i_u\, \delta s_u) \sin \theta_v \left(1 - \frac{v^2}{V^2}\right)}{k\, r^2\, V^2 \left(1 - \frac{v^2 \sin^2 \theta_v}{V^2}\right)^{\frac{3}{2}}}$$

whilst its direction is at right angles to $(i_v\, \delta s_v)$.

This conclusion deserves some comment.

(1) It shows that, if the force is expressed as $B_v\, (e_u u \sin \theta_u)$ then the value of B_v is the standard formula for the magnetic induction from a relativistically moving charge (6). The difference is that here it is obtained without using Maxwell's equations or any other electromagnetic formulae which are based implicitly on the law we are trying to prove.

(2) At low velocities B_v becomes $\dfrac{i_v\, \delta s_v \sin \theta_v}{k\, V^2\, r^2}$ which can be written as $\dfrac{\mu\, i_v\, ds_v \sin \theta_v}{r^2}$ which is the normal Biot-Savart or Grassman expresssion.

(3) The average speed of electrons in wires carrying ordinary currents is, surprisingly enough, only of the order of 10^{-2} cm. per sec. This occurs because 1 cm. of wire of area of cross section 1 mm.^{2} contains 8.5×10^{20} atoms and a similar number of free electrons. Each electron has a charge of $1.6.\times10^{-19}$ coulombs. Hence for a current of 1 amp the electron velocity must be $\dfrac{1}{(8.5\times10^{20})\,(1.6\times10^{-19})} = \dfrac{1}{135}$ cm. per sec. For this reasons in any conceivable wire experiments the relatavistic correction terms can be forgotten.

(4) We now have a definite argument showing that if the law of force between electrostatic charges is an inverse square then the law of magnetic field production must also be an inverse square with equal accuracy or certainty.

These conclusions are valid only if it is proper to think of the fields of currents in wires as being produced by the the slow motion of the whole number of electrons contained in the wire. This seems a little doubtful since we know that currents in straight wires are propagated with the velocity of light.

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SECTION II

Problems for solution.

I. Prove that the co-efficients of all the terms except those of $a_0^p, a_1^p, \ldots\ldots, a_{p-1}^p$, in the expansion of the circulant

$$\begin{vmatrix} a_0 & \cdots\cdots & a_{p-1} \\ a_{p-1} & \cdot\cdot\; a_0 \;\cdot\cdot & a_{p-2} \\ a_1 & \cdot\;\cdot\;\cdot & a_0 \end{vmatrix}$$

of prime order p, are divisible by p.

(Proposed by S. M. Hussain
Mathematics Department,
Panjab University).

II. Prove that a necessary and sufficient condition for an immersion to be minimal in Euclidean 3-space is that $\int_{k \geqslant o} K \, ds = 4\pi$, where K is the Gaussian Curvature, of the surface.

(Proposed by B.A. Saleemi
Mathematics Department,
Panjab University).

III. Find a general formula for calculating the number of proper Ideals in J_m.

(Proposed by B.A. Saleemi
Mathematics Department,
Panjab University).

IV. If 1 cm^2 of surface emits normally an energy E in the form of an an electromagnetic wave, it produces during the short time δt of the emission a pressure of $\frac{E}{C\,\delta t}$ and thus carries a momentum of $\frac{E}{C}$. If this plane wave passes into water and then falls onto an absorber it produces a pressure of $\frac{E}{v\delta t}$ and thus carries a momentum of $\frac{E}{v.}$ What happens to the remaining momentum?

(Proposed by G. Tarrant.
Mathematics Department.
Panjab University)

V. Explain why a cyclist can ride without falling off (*a*) when he is holding on to the handlebars, (*b*) when he is not.

(Proposed by G. Tarrant
Mathematics Department,
Panjab University).

SECTION III

ON CERTAIN EXPRESSIONS INVOLVING PTH ROOTS OF UNITY

BY

S. MANZUR HUSSAIN AND A. SHAFAAT

Notation.

p = an odd prime.

$q = \frac{p-1}{2}$

w = a pth root of unity other than 1

$u_n = w^n + \frac{1}{w^n}$

m is a primitive root of p if p is of the form $(4k+1)$ and belongs to q or 2q (mod p) if p is of the form $(4k-1)$

Introduction :

1. While evaluating the determinant of 11th order in [1] which is specially related to the Partition Theory we had to find the values of expressions of the form $\Sigma\, \alpha^a \beta^b \gamma^c \delta^d \epsilon^e$, where $\alpha = w + \frac{1}{w}$, $\beta = w^3 + \frac{1}{w^3}$ $\gamma = w^2 + \frac{1}{w^2}$, $\delta = w^5 + \frac{1}{w^5}$, $\epsilon = w^4 + \frac{1}{w^4}$, Σ extends over cyclic permutations of $(\alpha, \beta, \gamma, \delta, \epsilon)$ and w stands for an 11th root of unity. The values of these expressions exhibited an interesting congruence property, namely,

$$\Sigma\, \alpha^a \beta^b \gamma^c \delta^d \epsilon^e = -2^{a+b+c+d+e-1} \pmod{11},$$

a, b, c, d, e being non-negative integers. In this paper we generalize the above expression for any p and its congruence property. We also prove some results concerning the value of this generalized expression.

We shall prove the following Theorem.

Theorem : If $\alpha_i = u_m^{\,i}$, $o \leqslant i \leqslant q$, then

$\Sigma\, \alpha_1^{a_1} \alpha^{a_2} \ldots \alpha_q^{a_q} \equiv -2^{a_1+a_2+\cdots+a_q-1}$ (mod p), where Σ extends over all cyclic permutations of $\alpha_1, \alpha_2, \ldots, \alpha_q$ and not all a_i are zero.

We begin with the following Lemma.

Lemma : $u_{n_1}, \ldots\ldots\ldots\ldots, u_{n_q}$ are permuted cyclically by every transformation $w \to w^n$, (n, p)=1, if and only if ($u_{n_1}, u_{n_2}, \ldots \ldots u_{n_q}$) is a cyclic permutation of ($u_m, u_m^{\,1}, \ldots\ldots, u_m^{\,q-1}$).

Proof : We first prove that if ($u_{n_1}, \ldots\ldots, u_{n_q}$) is a cyclic permutation of ($u_m^{\,0}, u_m^{\,1}, \ldots\ldots, u_m^{\,q-1}$) then $u_{n_1}, \ldots, u_{n_q}$ are permuted cyclically by every transformation $w \to w^n$, (n, p)=1. We note that $u_1 = u_s$, if and only if $1 \equiv \pm s$ (mod p) and that the transformation $w \to w^n$, (n, p)=1, transforms u_l into u_{ln}. We now prove that $n \equiv \pm m^i$ (mod p) where $o \leqslant i \leqslant q-1$ and (n,p)=1. If p is of the form 4k +1, then m is a primitive root of p. Hence we have $n \equiv m^j \equiv -m^{q+j}$, $o \leqslant j \leqslant p-1$ and either j or q+j−p+1 can be taken as i. If p is of the form (4k−1) and m is primitive root of p the proof is the same as before. If, however, m belongs to q (mod p) then $m^o, m^1, \ldots, m^{q-1}$ are all the quadratic residues of p. Since −1 is a non-residue of p, either n or -n is congruent to one of the integers $m^o, m^1, \ldots\ldots, m^{q-1}$. This establishes the required congruence, $n \equiv \pm m$ (mod p). Thus the transormation $w \to w^n$ is equivalent to $w \to w^{\pm m^i}$ which obviously permutes ($u_m^{\,0}, u_m^{\,1}, \ldots\ldots\ldots, u_m^{\,q-1}$) cyclically. Since ($u_{n_1}, \ldots\ldots, u_{n_q}$) is a cyclic permutation of ($u_m^{\,0}, u_m^{\,1}, \ldots, u_m^{\,q-1}$) any cyclic permutation of ($u_m^{\,0}, u_m^{\,1}, \ldots, u_m^{\,q-1}$) is a cyclic permutation of ($u_{n_1}, \ldots, u_{n_q}$). This proves that ($u_{n_1}, \ldots, u_{n_q}$) is permuted cyclically by every transformation of the form $w \to w^n$, (n, p) = 1.

Next suppose that $({}^{u}n_1, {}^{u}n_2, \ldots, {}^{u}n_q)$ is permuted cyclically by every transformation $w \to w^n$, $(n, p) = 1$. Choose n such that $n.n_1 \equiv 1 (\text{mod } p)$; then $w \to w^n$ transforms $({}^{u}n_1, \ldots, {}^{u}n_q)$ into $({}_1, {}^{u}nn_2, \ldots\ldots, {}^{u}nr_q)$. Clearly ${}^{u}{}_1 = {}^{u}n_i$ for some i, $1 \leqslant i \leqslant q$. Let ${}^{u}n_{i+1} = {}^{u}\lambda$; then by our supposition $({}^{u}\lambda n_{-1}, \ldots, {}^{u}\lambda n_{i-1}, {}^{u}\lambda, {}^{u}\lambda n_{i+1}, \ldots, {}^{u}\lambda n_q)$ is a cyclic permutation of $({}^{u}n_1, \ldots, {}^{u}n_{i-1}, {}^{u}{}_1, {}^{u}\lambda, {}^{u}n_{i+2}, \ldots, {}^{u}n_q)$ Hence ${}^{u}n_{i+2} = {}^{u}\lambda^2$, ${}^{u}n_{i+3} = {}^{u}\lambda n_{i+2} = {}^{u}\lambda^3, \ldots, {}^{u}n_q = {}^{u}\lambda^{n-i-1}$, ${}^{u}n_1 = {}^{u}\lambda^{q-i} \ldots\ldots {}^{u}n_{i-1} = {}^{u}\lambda^{q-1}$, ${}^{u}{}_1 = {}^{u}\lambda^q$. So that $({}^{u}n_1, \ldots, {}^{u}n_q) = {}^{u}\lambda^{q-i}, \ldots {}^{u}\lambda, \ldots, {}^{u}\lambda^{q-i-1})$ is a cyclic permutation of $({}^{u}\lambda, {}^{u}\lambda^2, \ldots, {}^{u}\lambda^q.)$ The relation ${}^{u}\lambda^q = {}^{u}{}_1$ shows that $\lambda^q \equiv \pm 1$ (mod p.). Hence λ belongs to q or 2q. The proof will be complete if we show that $\lambda^q \not\equiv 1$ (mod p) if p is of the form $4k+1$. Suppcse otherwise; then $-\lambda$ is a quadratic residue of p. Let n be a non-residue (mod p). Since p is of the form $4k+1, -n$ is also non-residue. Hence neither n nor -n is of the form λ (mod p) for any i, so that $({}^{u}n_1, \ldots, {}^{u}n_q) = ({}^{u}\lambda^{q-i}, \ldots, {}^{u}\lambda, \ldots, {}^{u}\lambda^{q-i-1})$ is transformed into $({}^{u}n\lambda^{q-i}, \ldots, {}^{u}n\lambda, \ldots, {}^{u}n\lambda^{q-i-1})$ by $w \to w^n$, which is not a cyclic permutation of $({}^{u}\lambda^{q-i}, \ldots, {}^{u}\lambda, \ldots, {}^{u}\lambda^{q-i-1})$ This proves the lemma.

Proof of the Theorem.

Let $f(w) = a_1^{a1} \ldots a_q^{aq}$

The relation

$$\Sigma a_1^{a1} \ldots a_q^{aq} = \sum_{r=1}^{q} f(w^r)$$

follows immediately from the Lemma. We first prove that $f(w) = n_o \times (w^o + \frac{1}{w^o}) + n_1 (w^1 + \frac{1}{w^1}) + n_2 (w^2 + \frac{1}{w^2}) + \ldots + n_q (w^q + \frac{1}{w^q})$ where $n_o, n_1, \ldots\ldots n_q$ are non-negative integers such that

$$n_o + n_1 + \ldots + n_q = 2^{a1 + \ldots + a_q - 1}$$

Now ${}^{u}\lambda \cdot {}^{u}\mu = {}^{u}(\lambda+\mu) + {}^{u}(\lambda-\mu)$,

$${}^{u}\lambda\mu\nu = {}^{u}(\lambda+\mu+\nu) + {}^{u}(\lambda+\mu-\nu) + {}^{u}(\lambda-\mu+\nu) + {}^{u}(\lambda-\mu-\nu)$$

This shows that the product of any two of the numbers can be expressed as the sum of 2^{2-1} of them and the product of three numbers among u_o,

u_1., ..., u_q as the sum of 2^{3-1} of them. By induction it follows that the product of n numbers among u_o, u_1,........, u_q is expressible as the sum of 2^{n-1} of these numbers. Hence $a_1^{a1}\ldots\ldots a_q^{aq}$ which is the product of $a_1+\ldots\ldots+a_q$ of the numbers u_o, u_1., ..., u_q can be written as the sum of $2^{a_1+\ldots+a_q-1}$ u's.

Thus $f(w) = a_1^{a_1}\ldots a_q^{a_q} = n_o\,(w^o+\frac{1}{w^o}) + n_1\,(w^1+\frac{1}{w^1}) + \ldots$

$+n_q\,(w^q+\frac{1}{w^q})$ where $n_0+n_1+\ldots+n_q = 2^{a_1+\ldots+a_q-1}$ Hence

$$\Sigma a_1^{a_1}\ldots a_q^{a_q} = \sum_{r=1}^{q} f(w^r) = \sum_{r=1}^{q} \left\{ n_0\,(w^0+\frac{1}{w^0})+n_1\,(w^1+\frac{1}{w^1})+\ldots +n_q\,(w^q+\frac{1}{w^q}) \right\}$$

$$= 2q\,n_0+n_1\,(-1)+\ldots+n_q\,(-1)$$

$$= n_0\,p-(n_0+\ldots.+n_q).$$

$$\equiv -\,2^{a_1+\cdots+a_q-1} \pmod{p}$$

This completes the proof of the theorem.

The proof also shows that, if $a_1 \geqslant 1$

$$\Sigma a_1^{a1}\ldots a_q^{aq} = n_0\,p-2^{a1+\cdots+a_q-1}$$

where $n_o=n_o\,(a_1,\ldots\ldots\ldots\ldots a_q)$ = numbers of integers of the form M : $(m^1 \pm\ldots \pm m^1) + (\pm m^2\ldots\pm m^2) + \ldots+(\pm m^q\pm\ldots\pm m^q)$ which are divisible by p, where m occurs a_i times.

It is easy to relate $n_o\,(a_1,\ldots\ldots\ldots a_q)$ to the number of solutions of a linear congruence (mod p). Let x_i be the number of times m^i occurs with a negative sign in M. Then, if

$$M = (a_1-2x_1)\,m+\ldots.+(a_q-2x_q)\,m^q$$

where $o \leqslant x_1 \geqslant a_1-1,\ o\leqslant x_2 \leqslant a_2,\ldots\ldots,\ o \leqslant x_q \leqslant a_q$ .... (*i*)

For every integer $n \not\equiv 0 \pmod{p}$, define n^{-1} as the unique solution of the congruence $n\,x \equiv 1 \pmod{p}$. Now

$$M \equiv 0 \pmod{p}$$

iff $x_1+mx_2+\ldots+m^{q-1}x_q \equiv 2^{-1}(a_1+a_2m+\ldots+m^{q-1}a_q) \pmod p$ ....(*ii*)

$n_o(a_1,\ldots\ldots, a_q)$ is, then, the number of q-tuples $(x_1,\ldots\ldots x_q)$ which satisfy the congruence (*ii*) together with restrictions in (*i*).

We now deduce some results. We shall suppose that $a_1 \geqslant 1$. It is clear that whatever is proved for a_1 is true for all non-zero a_i. When we fix our attention on a_1 alone we shall write $n_o(a_1)$ for $n_o(a_1,\ldots\ldots, a_q)$.

I. $n_o(a_1,\ldots\ldots, a_q) \leqslant a_1(a_2+1)\ldots\ldots(a_q+1)$

II. $n_o(\acute{a}_1 p) = \acute{a}_1 n_o(p) = \acute{a}_1(a_2+1)\ldots\ldots(a_q+1)$

For corresponding to any one of the $(a_2+1)\ldots\ldots(a_q+1)$ ways of choosing $x_2,\ldots\ldots x_q$ we can choose x_1 in just a_1 ways so as to make $(x_1,\ldots, x_q)$ satisfy (*i*) and (*ii*).

III. If $a_1 = s_1 p + r_1$, $s_1 \geqslant 1$, $r_1 \geqslant 1$

$n_o(s_1 p + r_1) = n_o(s_1 p) + n_o(r_1)$

The proof is similar to that of II.

It may be noted that a cyclic permutation of the arguments of n_o does not change the value of n_o. Hence by a repeated application of III the calculation of $n_o(a_1, \ldots\ldots, a_q)$ can be reduced to that of $n_o(r_1, \ldots\ldots, r_q)$ where $r_i \leqslant p-1$.

IV. $n_o(a_1) + n_o(p-a_1) = n_o(p)$, where $a_1 \leqslant p-1$.

To prove this we define $N(a_1,\ldots\ldots, a_i; l)$ to be the number of i-tuples $(x_1,\ldots\ldots,x_i)$ of integers satisfying

$$x_1+mx_2+\ldots+m^{i-1}x_i \equiv l \pmod p \ldots\ldots\ldots(iii)$$

and

$$o \leqslant x_1 \leqslant a_1,\ o \leqslant x_2 \leqslant a_2,\ \ldots\ldots,\ o \leqslant x_i \leqslant a_i \quad (iv)$$

When we fix our attention only on a_1 and l we write $N_i(a_1; l)$ for $N(a_1,\ldots\ldots, a_i; l)$. We first note that

(α) $N_i(p-1; l) = (a_2+1)\ldots\ldots(a_i+1)$

(β) $N(a_1,\ldots, a_i; l) = \sum_{n=o}^{a_1} N(a_1,\ldots, a_i; \overline{l-x_1\, m^{-1}})$

(γ) $n_o(a_1) = N_q(a_1-1; \overline{2^{-1}a_1+a_2m+\ldots+a_qm^{-1}})$

Proof of (α) is similar to that of II. (β) follows by noting that the number of i-tuples satisfying (*iii*) and (*iv*) is same as the number of $(i-1)$ tuples satisfying

$o \leqslant x_2 \leqslant a_2, \ldots\ldots, o \leqslant x_i \leqslant a_i$ and any one of the congruences

$$x_2 + mx_3 + \ldots + m^{i-2} x_i \equiv (l - o)\, m^{-1} \pmod{p}$$
$$x_2 + mx_3 + \ldots + m^{i-2} x_i \equiv (l - 1)\, m^{-1} \pmod{p}$$
$$x_2 + mx_3 + \ldots + m^{i-2} x_i \equiv (l - a_1)\, m^{-1} \pmod{p}$$

(ν) is obvious.

By (β) we have, provided $a_1 \leqslant p-2$.

$$N_i\,(p\text{-}1\,;1) = \sum_{x_1=o}^{p-1} N\,(a_2, \ldots, a_i\,;\, \overline{l\text{-}x_1}\, m^{-1}) =$$

$$= \sum_{x_1=o}^{a_1} N\,(a_2, \ldots, a_i\,;\, (\overline{l\text{-}x_1}\, m^{-1}) + \sum_{x_1=a_1+1}^{p-1} N\,(a_2, \ldots, a_i\,;\, \overline{l\text{-}x_1}\, m^{-1})$$

$$= \sum_{x_1=o}^{a_1} N\,(a_2, \ldots, a_i;\, \overline{l\text{-}x_1}\, m^{-1}) + \sum_{x_1=o}^{p\text{-}2\text{-}a_1} N\,(a_2, \ldots, a_i;\, \overline{l\text{-}x_1\text{-}a_1\text{-}1}\, m^{-1})$$

$$= \quad N_i\,(a_1\,;\, l) + N_i\,(p\text{-}2\text{-}a_1\,;\, l\text{-}a_1 - 1) \quad \ldots\ldots.. \quad (\delta)$$

Now by (α), (γ) and (δ) we have, if $a_1\text{-}1 \leqslant p\text{-}2$,

$$n_o\,(p) = N_q\,(p\text{-}1\,;\, 2^{-1}\, \overline{a_1 + a_2\, m + \ldots + a_q\, m^{q-1}})$$

$$= N_q\,(p\text{-}1\,;\, 2^{-1}.\, \overline{a_1 + ma_2 + \ldots + m^{q-1}\, a_q}) + N_q\,(p\text{-}a_1\text{-}1\,;$$

$$2^{-1}.\, \overline{a_1 + a_2 m + \ldots + a_q m^{q-1}\text{-}a_1})$$

$$= n_o\,(a_1) + N_q\,(p\text{-}a_1\text{-}1\,;\, 2^{-1}.\, \overline{a_1 + a_2\, m + \ldots + a_q\, m^{q-1} - a_1})$$

But $\quad 2^{-1}\,(a_1 + a_2 m + \ldots + a_q\, m^{q-1}) - a_1 \equiv 2^{-1}\,(p\text{-}a_1 + a_2 m + \ldots$
$\qquad + a_q\, m^{q-1}) \pmod{p}$

so that $\quad N_q\,(p\text{-}1\text{-}a_1\,;\, 2^{-1}.\,(a_1 + a_2\, m + \ldots\ldots + a_q m^{q-1}) - a_1) = n_o(p\text{-}a_1)$

This proves

$$n_o\,(p) = n_o\,(p\text{-}a_1) + n_o\,(a_1) \quad \text{for} \quad a_1 \leqslant p\text{-}1.$$

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ON THE EFFECT OF VISCOSITY ON EDGE WAVES ON A SLOPING BEACH:

BY

M. H. KAZI

Introduction

In (1), the author formulated and solved the problem of Edge Waves on a sloping beach, when a boundary layer is incorporated in the description, previously discussed by Stokes on the assumption of no viscosity. If Z-axis is taken along the length of the beach, Y-axis as vertically downwards and X-axis towards the Ocean, then for irrotational ideal flow, the formulation of surface wave problem for small amplitude motion requires that the velocity potential should satisfy : $\nabla^2\phi=0$, alongwith the boundary conditions :

$$\phi_{tt} - g\,\phi_y = 0 \quad \text{at } y=0$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x}\tan\alpha \quad \text{at} \quad y = x\tan\alpha$$

$0 \leqslant y \leqslant x \tan\alpha$, α being the slope of the beach with the horizontal. A solution of this B.V.P. is given by $\phi = e^{i\sigma t}\, e^{ikz} e^{-k(x\cos\alpha + y\sin\alpha)}$ with $\sigma^2 = gk\sin\alpha$, which is referred to as STOKES EDGE MODE. Taking the velocity vector $\vec{U}_I$ corresponding to this solution as the INTERIOR SOLUTION, the boundary layer solution obtained in (1) is

$$\vec{u}_{bl} = (-k\cos\alpha, -k\sin\alpha, ik)\,\text{Exp}\left[\, i(\sigma t + kz) - \frac{ky}{\sin\alpha} - (1+i) \times \xi\sqrt{\frac{\sigma\sin\alpha}{2\nu}}\,\right]$$

where $\xi = x - y\cot\alpha$, $y=\eta$, $z=z$.

The components of the force of friction on the sloping beach are calculated and are then distributed over the depth as components of Body Force.

The problem is then reformulated in accordance with the shallow water wave theory, taking into consideration the body force obtained from the preceding theory.

Calculation of Stresses and Components T_1 :

$$\vec{U}_{bl} = (-k\cos\alpha, -k\sin\alpha, ik)\,\phi(t, z, y)\,\mathrm{Exp}\,[-(1+i)\,\xi\beta]$$

where $\phi(t, z, y) = \mathrm{Exp}\left[i(\sigma t+kz) - \frac{ky}{\sin\alpha}\right]$

and $\beta = \sqrt{\frac{\sigma \sin\alpha}{2\nu}}$

Let T_1, T_2, T_3 denote the components of the force of friction on the points of the plane $y = x\tan\alpha$, in the directions of x, y and z-axis respectively :

$$T_i = \sigma_{ij} n_j$$

Calculations of T_1 :

$$T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$

$$n_1 = \sin\alpha$$

$$n_2 = -\cos\alpha$$

$$n_3 = 0$$

$$\therefore \quad T_1 = \sigma_{11}\sin\alpha - \sigma_{12}\cos\alpha$$

$$= \sigma_{xx}\sin\alpha - \sigma_{xy}\cos\alpha$$

$\sigma_{xy} = \nu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$: In the new co-ordinate system $\sigma_{xy}(\xi, y, z) = \nu\left(\frac{\partial u}{\partial y} - \cot\alpha\,\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi}\right)$

Since $\frac{\partial u}{\partial y}$ is negligible in comparison to $\frac{\partial v}{\partial \xi}$ and $\frac{\partial u}{\partial \xi}$, we get :

$$\sigma_{xy}(\xi, y, z) = \nu\left[-\cot\alpha\,\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi}\right]$$

$$\sigma_{xx}(\xi, y, z) = -p + 2\nu \frac{\partial u}{\partial \xi}$$

$$\therefore\ T_1 = -p\sin\alpha + 2\nu \sin\alpha \frac{\partial u}{\partial \xi} - \nu \cos\alpha \left[-\cot\alpha \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi} \right]$$

$$= -p \operatorname{Sin}\alpha + \nu \frac{\partial u}{\partial \xi} \left\{ \frac{1+\sin^2\alpha}{\sin\alpha} \right\} - \nu \cos\alpha \frac{\partial v}{\partial \xi}$$

$$u = -k\cos\alpha\, \Phi(t, z, y) \operatorname{Exp}\left[-(1+i)\beta\xi \right]$$

$$\therefore\ \frac{\partial u}{\partial \xi} = k\cos\alpha\,(1+i)\,\beta\,\Phi(t, z, y) \operatorname{Exp}\left[-(1+i)\,\xi \sqrt{\frac{\sigma \sin\alpha}{2\nu}} \right]$$

$$\frac{\partial v}{\partial \xi} = k\sin\alpha\,(1+i)\,\beta\,\Phi(t, z, y) \operatorname{Exp}\left[-(1+i)\,\xi\,\beta \right]$$

$$\therefore\ \left.\frac{\partial u}{\partial \xi}\right|_{\xi = 0} = k\cos\alpha\,(1+i)\,\beta\,\Phi(t, z, y)$$

$$\left.\frac{\partial v}{\xi}\right|_{\xi = 0} = k\sin\alpha\,(1+i)\,\beta\,\Phi(t, z, y)$$

$$\therefore\ (T_1)_{\xi = 0} = -p\sin\alpha + \nu k \left[(1+\sin^2\alpha)\cot\alpha - \sin\alpha\cos\alpha \right] (1+i)\,\beta\,\Phi(t, z, y)$$

$$\therefore\ \left. T_1 \right|_{\xi = 0} = -p\operatorname{Sin}\alpha + \sqrt{\frac{\sigma\nu \operatorname{Sin}\alpha}{2}}\; k\,(1+i)\cot\alpha \operatorname{Exp}\left[i(\sigma t + kz) - \frac{ky}{\operatorname{Sin}\alpha} \right] \qquad \ldots\ldots\ldots(1)$$

Calculation of T_2

$$T_2 = \sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z$$

$$= \sigma_{yx} \sin\alpha - \cos\alpha\, \sigma_{yy}$$

$$\sigma_{yy} = -p + 2\nu \frac{\partial v}{\partial y}$$

$$= -p + 2\nu \left\{ \frac{\partial v}{\partial \eta} - \operatorname{Cot}\alpha \frac{\partial v}{\partial \xi} \right\} = -p - 2\nu \cot\alpha \frac{\partial v}{\partial \xi}$$

Also $\sigma_{yx} = \nu \left[-\cot\alpha \dfrac{\partial u}{\partial \xi} + \dfrac{\partial v}{\partial \xi} \right]$

$$\therefore\ T_2 = \nu \left[-\cot\alpha \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi} \right] \operatorname{Sin}\alpha - \cos\alpha \left[-p - 2\nu\cot\alpha \frac{\partial v}{\partial \xi} \right]$$

$$= p\cos\alpha - \nu\cos\alpha\frac{\partial u}{\partial\xi} + \nu\frac{\partial v}{\partial\xi}\quad\frac{1+\cos^2\alpha}{\mathrm{Sin}\alpha}$$

Substituting for $\frac{\partial u}{\partial\xi}$ and $\frac{\partial v}{\partial\xi}$ we get

$$T_2\Big/_{\xi=0} = p\cos\alpha - \nu\cos^2\alpha\; k(1+i)\,\beta\,\Phi(t,z,y) + \nu(1+\cos^2\alpha)\,k(1+i)\,\beta\,\Phi(t,z,y)$$

$$= p\cos\alpha + k(1+i)\,\nu\,\beta\,\mathrm{Exp}\left[i(\sigma t+kz) - \frac{ky}{\mathrm{Sin}\alpha}\right]$$

$$\therefore T_2\Big/_{\xi=0} = p\cos\alpha + k(1+i)\sqrt{\frac{\sigma\nu\,\mathrm{Sin}\,\alpha}{2}}\;\mathrm{Exp}\left[i(\sigma t+kz - \frac{ky}{\mathrm{Sin}\alpha}\right]$$

Calculation of T_3 ....(II)

$$T_3 = \sigma_{xz}\,n_x + \sigma_{zy}\,n_y$$

$$= \sigma_{xz}\sin\alpha - \cos\alpha\;\sigma_{zy}$$

$$\sigma_{xz} = \nu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\therefore \sigma_{xz}(\xi, y, z, t) = \nu\left(\frac{\partial w}{\partial\xi} + \frac{\partial u}{\partial z}\right]$$

$$w = ik\,\Phi(t, y, z)\;\mathrm{Exp}\,[-(1+i)\,\xi\beta]$$

$$\therefore \frac{\partial w}{\partial\xi} = -ik(1+i)\,\beta\,\Phi(t, z, y)\;\mathrm{Exp}\,[-(1+i)\,\xi\,\beta]$$

$$\therefore \sigma_{xz}\Big/_{\xi=0} = -ik(1+i)\,\beta\,\nu\,\Phi(t,z,y)$$

$$\sigma_{zy} = \nu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\therefore \sigma_{zy}(\xi,y,z,t) = \nu\left[\frac{\partial w}{\partial y} - \cot\alpha\frac{\partial w}{\partial\xi} + \frac{\partial v}{\partial z}\right]$$

$$= -\nu\cot\alpha\frac{\partial w}{\partial\xi}$$

$$\therefore \sigma_{zy}\Big/_{\xi=0} = ik(1+i)\,\nu\cot\alpha\,\beta\,\Phi(t,z,y)$$

$$\therefore T_3\Big/_{\xi=0} = ik\,(1+i)\,\nu\,\Phi\,(t,z,y)\left\{-\operatorname{Sin}\alpha - \frac{\cos^2\alpha}{\operatorname{Sin}\alpha}\right\}$$

$$= -ik\,(1+i)\sqrt{\frac{\sigma\,\nu}{2\operatorname{Sin}\alpha}}\ \operatorname{Exp}\left[i\,(\sigma t+kz) - \frac{ky}{\operatorname{Sin}\alpha}\right]$$

..............(III)

TO SUMMARIZE :

$$T_1\Big/_{\xi=0} = \sqrt{\frac{\sigma\nu\sin\alpha}{2}}\,k\,(1+i)\cot\alpha\ \operatorname{Exp}\left[i\,(\sigma t+kz) - \frac{ky}{\operatorname{Sin}\alpha}\right] - p\sin\alpha$$

$$T_2\Big/_{\xi=0} = p\cos\alpha + k\,(1+i)\sqrt{\frac{\sigma\nu\operatorname{Sin}\alpha}{2}}\ \operatorname{Exp}\left[i\,(\sigma t+kz) - \frac{ky}{\operatorname{Sin}\alpha}\right]$$

$$T_3\Big/_{\xi=0} = -ik\,(1+i)\sqrt{\frac{\sigma\,\nu}{2\operatorname{Sin}\alpha}}\ \operatorname{Exp}\left[i\,(\sigma t+kz) - \frac{ky}{\operatorname{Sin}\alpha}\right]$$

DISTRIBUTION OF components of force of friction over the depth as components of body force can be accomplished by dividing the expressions for them by $x\tan\alpha$: we want to reformulate the problem in accordance with the shallow water wave theory, taking into consideration the body-force obtained from the preceding theory, neglecting the pressure terms in the above expressions. We anticipate the effect of T_2 to be negligible. We shall relate the other two components to the expressions for the interior velocity components at $\xi = 0$:

$$F_{(x)} = -\frac{\sqrt{\dfrac{\sigma\nu\operatorname{Sin}\alpha}{2}}\,k\,(1+i)\cot\alpha\ \operatorname{Exp}\left[i\,(\sigma t+kz) - \dfrac{ky}{\operatorname{Sin}\alpha}\right]}{x\tan\alpha}$$

$$u_o = k\cos\alpha\ \operatorname{Exp}\left[i\,(\sigma t+kz) - \frac{ky}{\operatorname{Sin}\alpha}\right]$$

$$\therefore\ F_{(x)} = \frac{-(1+i)}{x}\sqrt{\frac{\sigma\nu\operatorname{Sin}\alpha}{2}}\ \frac{u_o\cos\alpha}{\operatorname{Sin}^2\alpha}$$

$$= -(1+i)\,u_o\sqrt{\frac{\sigma\,\nu}{2\operatorname{Sin}\alpha}}\ \frac{\cot\alpha}{x}$$

$$= \frac{-(1+i)\, u_o \sqrt{\sigma\nu / 2\mathrm{Sin}\alpha}}{x \tan \alpha} \qquad \text{..............(IV)}$$

$$F_{(z)} = \frac{ik(1+i)}{x \tan \alpha} \sqrt{\frac{\sigma\nu}{2\mathrm{Sin}\,\alpha}}\ \mathrm{Exp}\left[i\,(\sigma t + kz) - \frac{ky}{\sin \alpha}\right]$$

$$w_o = -ik\ \mathrm{Exp}\left[i\,(\sigma t + kz) - \frac{ky}{\sin \alpha}\right]$$

$$\therefore\ F_{(z)} = \frac{-(1+i) w_o \sqrt{\dfrac{\sigma\nu}{2\mathrm{Sin}\,\alpha}}}{x \tan \alpha} \qquad \text{..............(V)}$$

$$\therefore\ F_{(x)} = -\frac{\lambda u_o}{x} \qquad \text{..............(VI)}$$

$$\text{and } F_{(z)} = -\frac{\lambda w_o}{x}$$

$$\text{where } \lambda = (1+i) \sqrt{\frac{\sigma\nu \sin \alpha}{2}}.\ \cos \alpha \qquad \text{..............(VII)}$$

'Reformulation in Linear Shallow Water Wave Theory'

We shall now formulate the problem in linear shallow water-wave theory, in which we take into consideration the effects of viscosity rather indirectly, by using $F_{(x)}$ and $F_{(z)}$, whose estimate is given by the preceding calculations, as components of body-force in the following:

The equations of linear shallow water wave theory are : (IN the same co-ordinate system)

The conservation of mass equation :

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial z}(hw) = 0 \qquad \text{..............(1)}$$

where $h = x \tan \alpha$ and $\eta \ll h$ and $u \ll 1$, $v \ll 1$

$$\frac{\partial u}{\partial t} = F_{(x)} - \frac{1}{\rho} \frac{\partial p}{\partial x} \qquad \text{..............(2)}$$

$$\frac{\partial w}{\partial t} = F_{(z)} - \frac{1}{d} \frac{\partial p}{\partial z} \qquad \text{..............(3)}$$

Assuming that the vertical acceleration is negligible

$$p = p_o + \rho g\,(y + \eta) \qquad \text{..............(4)}$$

so that (2) and (3) take the form :

$$\frac{\partial u}{\partial t} = F_{(x)} - g\,\eta_x \qquad \text{..............(5)}$$

and

$$\frac{\partial w}{\partial t} = F_{(z)} - g\,\eta_z \qquad \text{............. (6)}$$

Now we make the following assumption—we take

$F_{(x)} = -\dfrac{\lambda u}{x}$ and

$F_{(z)} = -\dfrac{\lambda w}{x}$ where λ is given by equation (VII) of the previous theory :

We then have the following formulation :

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial z}(hw) = 0$$

$$h = x \tan \alpha$$

$$\frac{\partial u}{\partial t} = -\frac{\lambda u}{x} - g\,\eta_x \qquad \text{..............(7)}$$

$$\frac{\lambda w}{\partial t} = -\frac{\lambda w}{x} - g\,\eta_z \qquad \text{..............(8)}$$

We anticipate a complex time factor $e^{i\sigma t}$ in the expressions involved so that these equations can be further simplified. Thus if we suppose

$$\left.\begin{aligned} u &= e^{i\sigma t}\,u' \\ w &= e^{i\sigma t}\,w' \end{aligned}\right\} \quad \text{etc. and}$$

then drop (′) we shall obtain :

$$\left(i\sigma + \frac{\lambda}{x}\right) u = -g\,\eta_x$$

$$\left(i\sigma + \frac{\lambda}{x}\right) w = -g\,\eta_z$$

so that we have for the boundary condition, $\dfrac{\partial \eta}{\partial n} = 0$

on $y = x \tan \alpha$ . ...(9)

Equation of conservation of mass now gives

$$i\sigma\eta + \frac{\partial}{\partial x}\left[\frac{-ghx}{\lambda+i\sigma x}\eta_x\right] + \frac{\partial}{\partial z}\left[\frac{-ghx}{\lambda+ix\sigma}\eta_z\right] = 0$$

$h = x \tan\alpha$:

$\therefore$ we end up with the following final form of the formulated problem:

$$\left[\frac{x^2}{\lambda+i\sigma x}\eta_x\right]_x + \left[\frac{x^2}{\lambda+ix\sigma}\eta_z\right]_z = \frac{i\sigma}{g}\eta\cot\alpha$$

$$\eta_n = 0 \text{ at } y = x\tan\alpha \qquad \ldots\ldots\ldots\ldots(10)$$

It has been possible to reformulate the problem according to Shallow Water-Wave theory, taking into consideration the components of body force obtained from the preceding theory. The author believes that the solution of the reformulated problem may reveal the extent to which the viscous effects are important for Edge Waves. Efforts in that direction are being continued, and it is anticipated that the effect of viscosity is to cause attenuation of waves in the Z-direction.

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Department of Mathematics,
Panjab University (New Campus), Lahore.

THE USE OF DIMENSIONAL ANALYSIS FOR CONVERTING ELECTRICAL EQUATIONS FROM UNRATIONALIZED TO RATIONALIZED FORMS

BY

G. T. P. TARRANT

Mathematics Department, University of the Panjab.

Summary

Electrical equations expressed in unrationalized units commonly differ from those in rationalized units by factors of (4π). Students often have need to convert equations which are proved in one system into the one in which they desire to work. This awkward process can be dealt with very readily by a simple extension of the theory of dimensional analysis.

INTRODUCTION

The possibility of using method of dimensions to convert unrationalized electrical equations into rationalized ones (or *vice versa*) depends essentially on understanding clearly the relation between the theory of dimensions and the theory of the names of units. Since this paper is intended to be read by students I may perhaps be pardoned for clarifying the initial simple ideas.

We can add, subtract or equate quantities only when they are expressed in the same units. A volume of 1000 cc may, or may not be equal to (1 pint + 14 fluid ounces); the statement cannot be tested without the relevant conversion factors.

In most scientific work derived units such as ccs. are preferred to arbitrary ones such as pints. This is convenient in part because the name of the unit of the new quantity can be obtained immediately by inserting the units as well as the numbers into the equation employed. Thus if the momentum of a body is written as (20 gm) $(10\ \frac{\text{cm}}{\text{sec}})$ we obtain not only the number (200) but also the unit $(\frac{\text{gm-cm}}{\text{sec}})$.

Commonly the full names of such derived units are so lengthy that we often coin shorter words such as dyne or erg instead of the full derived names $(\frac{\text{gm-cm}}{\text{sec}^2})$ of $(\frac{\text{gm-cm}^2}{\text{sec}^2})$. This habit often helps to give a better understanding of the nature of the more complicated physical quantity. For this reason work done per unit area is usually said to have units of $(\frac{\text{erg}}{\text{cm}^2})$ instead of $(\frac{\text{gm}}{\text{sec}^2})$. However for our present purpose the important thing is to develop the ideas which follows from the realization that every derived unit can, in principle, have its name expressed in terms of the names of the original arbitrary units of mass, length, time, temperature. The words 'in principle' are used because we have no actual name for the unit of dielectric constant.

The fact that the multiplication and division laws apply to the names of those parts of an equation where several factors are multiplied or divided and the fact that we can add, subtract or equate quantities only if they are in the same units allows us to made a valuable check on the reliability of equations after the numbers have been inserted. As an example, consider the expression for the pressure at a depth, h, inside a liquid of density D when the surface tension, T, is taken into account

$$P = (hDg + \frac{2T}{r})$$

$$P = (20 \text{ cm}) (1 \frac{\text{gm}}{\text{cm}^3}) (981 \frac{\text{cm}}{\text{sec}^2}) + \frac{2\left(80 \frac{\text{gm-cm}}{\text{sec}^2\text{-cm}}\right)}{(1 \text{ mm})}$$

$$= 19620 \frac{\text{gm}}{\text{cm}-\text{sec}^2} + 160 \frac{\text{gm}}{\text{mm}-\text{sec}^2}$$

The units reveal the mistake immediately and we are left to decide whether this mistake occurred in the algebraic equation or in the process of inserting the numbers.

This old technique is not driven home in teaching as much as it deserves, possibly because senior teachers are so used to using consistent units throughout that they do not appreciate how frequently mistakes are caused in elementary work by the use of mixed units. Thus the dominant interest of the advanced teacher is in checking the reliability of the algebraic equation alone.

As Focken explains in his 'Dimensional Methods and their Applications' this can be done most concisely, providing we do not use mixed units, with the aid of the convention in which (L) is written for the unit of length, etc. These symbols are called 'the dimensions' and are usually written in square brackets so that [Density] = $[ML^{-3}]$, etc. We can, therefore test the algebraic equation $P = (hDg + \frac{2T}{r})$ by writing

$$[P] = \left[(L)(ML^{-3})(LT^{-2} + \left\{\frac{MLT^{-2}}{L}\right\}L^{-1}\right] = \left[ML^{-1}T^{-2} + ML^{-1}T^{-2}\right]$$

which appears correct since we are adding quantities of the same 'dimensions'.

This is the essential reasoning behind the 'method of dimensions'. The method gives correct results because it is part of the more general analysis of the names of our units. It is short and easy but it does not normally cover problems produced by the use of mixed units or *Constants of Proportionality*. The formalism of the method of dimensions can, however, readily be extended to cover the use of mixed units if we remember that arbitrary and derived units are always connected by an equation of the type.

$$(Q_{\text{in cals}}) = k\,(Q_{\text{in joules}})$$

which is one equation involving two new quantities Q_{cal} and k. We could therefore write formally that $\left[Q_{cals}\right] = [k]\left[Q_{joules}\right]$.

This formalism involves a generalisation or extension of our previous understanding of the word 'dimensions'. If I say, 'the dimensions of heat measured in calories are those of k multiplied by those of heat measured in joules', I am using the word 'dimensions' in a more abstract and general way than has normally been done.

There are several ways of discussing dimensionally the equation $Q = S\ M\ \theta$. Amongst these are :—

(1) Measure heat in joules. Then $[M\ L^2\ T^{-2}] = [S]\ [M]\ [\theta]$ so that one new quanitity, say S, can be said to have the dimensions of $(L^2\ T^{-2}\ \theta^{-1})$. One new 'dimension' is introduced because the one defining equation contains two new quantities. S then has the units of joules per gm per °C.

(2) Measure heat in calories and regard this as a 'dimension'. Then $[S] = [Q\ M^{-1}\ \theta^{-1}]$. Two new 'dimensions' occur because the one defining equation now contains three new quantities. S then has the units of calories per gm per °C.

(3) Measure heat in joules but regard the constant of proportionality,

$$k = \left(\frac{Q\ \text{cals}}{Q\ \text{joules}}\right) \text{ as a 'dimension'. Then}$$

$$k\ Q_{\,joules}\ (\text{which in } Q_{\,cals}) = S\ M\ \theta$$

The dimensions of S will then be $[k\ L^2\ T^{-2}\ \theta^{-1}]$ and again two new 'dimensions' occur. This method is necessary if, for any reason, we insisted on measuring S in calories per gm per °C in spite of measuring Q in Joules.

This shows that the notion of regarding a constant of proportionality as a 'dimension' is, in fact, only a new way of putting an idea which has been current for half a century. If I wish to measure S in cals/gm/°C I must use either [Q] or [k] as a dimension and must make my choice according to my selection of the units in which I wish to measure Q.

Because this type of formalism complicates dimensional analysis it can be of interest only when mixed units are involved and *more than one* logical system of units is employed. This is, however, exactly the situation which occurs in the inter-conversion of rationalized and unrationalized electrical equations.

Rationalized and Unrationalized Electrical systems :

In the old electromagnetic or electrostatic systems there were really three fundamental equations introducting four new quantities, e & k, μ & m. These can be written :—

$$\text{Force} = \frac{e_1\, e_2}{k\, d^2} \;;\quad \text{Force} = \frac{m_1\, m_2}{\mu\, d^2} \;;\quad \text{Force} = mH = m\,\frac{i\, \delta s \sin\theta}{d^2}$$

Because these three new equations connected four new quantities one of them had to be taken as a dimensional primary; if the choice was k we said we were working in the complete electrostatic system and if it was μ in the complete electromagnetic system.

The new rationalized M. K.S. system embodies a completely different line of approach and tends to ignore the conception of magnetic poles. Nevertheless the system can be shown to lead to the following equations:—

$$\text{Force} = \frac{e_1\, e_2}{4\pi k\, d^2} \;;\quad \text{Force} = \frac{m_1\, m_2}{4\,\pi\,\mu\, d^2} \;;\quad \text{Force} = mH = m\,\frac{i\, \partial s \sin\theta}{4\,\pi d^2}$$

Occasionally people have followed the old e.m. or e.s. logic in their teaching but have made their results applicable in the rationalized system also. This can be done by accepting as basic the three equations :—

$$\text{Force} = \frac{e_1\, e_2}{Akd^2} \;;\quad \text{Force} = \frac{m_1\, m_2}{A\mu d^2} \;;\quad \text{Force} = mH = m\,\frac{i\, \delta s \sin\theta}{Ad^2}$$

providing A is put equal to unity for results in unrationalized systems and to $4\,\pi$ for rationalized ones.

Now from the point of view of dimensions these three equations can be regarded as introducing five new quantities,e and k, μ and m and, for the fifth, A. Two quantities must therefore be regarded as dimensional primaries A and either k or μ according as to whether we wish to work with dimensions of k or μ. Any other pair out of the five could theoretically have been selected but it is certainly a great advantage to select as dimensional primaries quantities which you are unlikely to use frequently. This is because such quantities can be ignored completely whenever they are not wanted because this cannot then affect any other dimensional factors in an equation. It is therefore important to retain A as one of the 'dimensions'.

The dimensions of e^2 can thus be regarded as being those of Akd^2 (Force). So $[e] = \left[A^{\frac{1}{2}} k^{\frac{1}{2}} L (M L T^{-2})^{\frac{1}{2}}\right] = \left[A^{\frac{1}{2}} k^{\frac{1}{2}} M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}\right]$

From this, with the help of the third equation we see immediately that the dimensions of m will be those of :—

$$\frac{(\text{Force})\ A\ d^2}{i\ \delta s} = \frac{(M\ L\ T^{-2})\ A\ (L^2)}{(A^{\frac{1}{2}}\ k^{\frac{1}{2}}\ M^{\frac{1}{2}}\ L^{\frac{2}{3}}\ T^{-2})\ L} = A^{\frac{1}{2}}\ k^{-\frac{1}{2}}\ M^{\frac{1}{2}}\ L^{\frac{1}{2}}$$

Proceeding in this way the following table of dimensions can be drawn up :

	Dimensions in terms of k	*Dimensions* in terms of μ
Electric Charge	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} k^{\frac{1}{2}} A^{\frac{1}{2}}$	$M^{\frac{1}{2}} L^{\frac{1}{2}} \mu^{-\frac{1}{2}} A^{\frac{1}{2}}$
Electric Potential	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} k^{-\frac{1}{2}} A^{-\frac{1}{2}}$	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \mu^{\frac{1}{2}} A^{-\frac{1}{2}}$
Electric Field	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} k^{-\frac{1}{2}} A^{-\frac{1}{2}}$	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2} \mu^{\frac{1}{2}} A^{-\frac{1}{2}}$
Dielectric Constant	k	$L^{-2} T^{2} \mu^{-1}$
Capacity	$L\ k\ A$	$L^{-1} T^{2} \mu^{-1} A$
Resistance	$L^{-1} T k^{-1} A^{-1}$	$L T^{-1} A^{-1} \mu$
Inductance	$L^{-1} T^{2} k^{-1} A^{-1}$	$L A^{-1} \mu$
Magnetic Field	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2} k^{\frac{1}{2}} A^{-\frac{1}{2}}$	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}} A^{-\frac{1}{2}}$
Magnetic Moment	$M^{\frac{1}{2}} L^{\frac{3}{2}} k^{-\frac{1}{2}} A^{\frac{1}{2}}$	$M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-1} \mu^{\frac{1}{2}} A^{\frac{1}{2}}$
Permeability	$L^{-2} k^{-1} T^{2}$	μ

Equations which have been derived in the A system (that is from (Force) $= \dfrac{e_1\ e_2}{A\ kd^2}$ etc.) will, in general, contain A implicitly and explicitly.

Implicitly, because A is hidden in the different terms, e, X, H, etc., and explicitly as a free power of A on one side or the other. Thus the field at the centre of a circular coil of wire carrying a current is $H = \frac{2\pi n i}{A r}$.

Such equations must balance as far as the total powers of A are concerned if we include both the explicit and the implicit powers. Thus in our example the left side contains $A^{-\frac{1}{2}}$ implicitly in H while the right contains A^{-1} explicitly and $A^{\frac{1}{2}}$ implicitly in i.

Equations which have been derived in a complete electrostatic or electromagnetic unrationalized system or in rationalized M.K.S. can be regarded as containing A only implicitly. The fact that the dimensions of A on the two sides must balance thus allows us to determine the missing explicit power of A. Another example should make this clear. Consider the non rationalized formula for the field on the axis of a finite solenoid.

$$H = \frac{2\pi n i}{x} (\cos \psi_1 - \cos \psi_2)$$

The H contains $A^{-\frac{1}{2}}$ implicitly and the i contains $A^{\frac{1}{2}}$, so that there must be an explicit A^{-1} on the right. The A containing formula is thus,

$$H = \frac{2\pi n i}{A x} (\cos \psi_1 - \cos_2)$$

But in rationalized systems $A = 4\pi$. Hence the rationalized formula must be $H = \frac{n i}{2 x} (\cos \psi_1 - \cos \psi_2)$

We will now consider an example which transforms in the opposite direction, Suppose we wish to convert the above rationalized formula into the unrationalized form. Then we note as before from the table that H and i contain implicitly $A^{-\frac{1}{2}}$ and $A^{+\frac{1}{2}}$ respectively. Hence the rationalized equation containing A must be : $H = \frac{n i}{2 x A} (\cos \psi_1 - \cos \psi_2)$ so that by merely writing $A = \frac{1}{4\pi}$ we convert to the unrationalized form of the equation.

The real value of the method is in more complicated problems where some dozen pages of argument would have to be written out afresh if one could find the formula *only* in the rationalized/unrationalized system which was *not* desired. One example of this more complicated type should suffice.

'The effective cross section of a bound electron for scattering of electromagnetic waves is known in the unrationalized electrostatic system to be

$$\sigma = \frac{8\pi e^4 p^4}{3 k^2 m^2 c^4 \left\{ (w^2 - p^2)^2 + \left(\frac{2 w^2 e^2}{3 m k c^3}\right)^2 p^2 \right\}}$$

What will it be in rationalized M.K.S.?.

In answer we note that e is the only quantity involved containing A—and that to the extent of $A^{\frac{1}{2}}$. Therefore the unrationalized equation containing A must be :—

$$\sigma = \frac{8\pi e^4 p^4}{A^2\, 3 k^2 m^2 c^4 \left\{ \left(w^2 - p^2\right)^2 + \left(\frac{2 w^2 e^2}{3 A m k c^3}\right)^2 p^2 \right\}}$$

so that the rationalized equation is :

$$\sigma = \frac{8\pi e^4 p^4}{(4\pi)^2\, 3 k^2 m^2 c^4 \left\{ (w^2 - p^2)^2 + \left(\frac{2 w^2 e^2}{12 \pi m k c^3}\right)^2 p^2 \right\}}$$

Warning. The above method of conversion can be expected to produce the correct rationalized equation only if it is supplied initially with the correct unrationalized one. Unfortunately many common expressions in the e.m. and e.s. systems are in fact incorrect because the authors have ignored μ or k merely because these happened to have unit value in the system of units in which they were working. If there is any suspicion of this it is wise to check the dimensional equations in both the k and the μ terms and to add in the necessary terms to make the dimesnsions balance.

Special problems of the Gaussian system :

Some authors have written in the Gaussian system in which electrical units are measured in e.s. and magnetic ones in e.m.u. This is possible because H_{es} = (a constant of proportionality) $H_{e \cdot m}$

Since the constant of proportionality turns out to be $3 \times 10^{10} \sqrt{\mu_{em} k_{es}}$ and since μ_{em} for free space $= 1 = k_{e.s.}$ for free space, it is usually written as c. I will, however, call it c_{ratio} or c_r to distinguish it between it and values arising in other ways which I will call $c_{ordinary}$ or c_o. Numerically the two are the same but c_o has dimensions of $L\,T^{-1}$ while c has not. Because of this the dimensions of equations written in the Gaussian system will probably fail to balance by some power of $(L\,T^{-1})$. This will occur whenever c_r is involved or whenever c_r has been cancelled with c_o during the deduction. There are two ways of dealing with the situation.

Firstly we can check dimensions as far as L and T are concerned and can assume that any failures of powers of LT^{-1} are caused by a c_r having been regarded as c_o. The equation can then be corrected and converted directly into rationalized form ignoring completely the presence of c_r. This method is rapid and easy providing we know that the original equation was correct. Its weakness is that it gives us only a partial check on the accuracy of the equation.

As an example consider problem (3) pg 566 of Ferraro's 'electromagnetic Theory' where students are asked to prove that a distance

$$d = \frac{2\,\pi\,m\,c\,v}{e\,H} \quad \text{in Gaussian units.}$$

Let us attempt to convert this to rationalized units. We start by considering the L T dimensions only and note, with the aid of the table on page 70 that the LT dimensions on the right side $= \dfrac{[L\,T^{-1}]^2}{[L^{\frac{3}{2}}\,T^{-1}]\,[L^{\frac{1}{2}}\,T^{-2}]} = [T]$ —which does not have the dimensions of a length. The L T dimensions would, however, have balanced if the c had been c_r with zero dimensions. Hence we rewrite it as $d = \dfrac{2\pi\,m\,c_r\,v}{e\,H}$. Next we remember the warning mentioned in the last section and check on k or μ. This shows us that the author had forgotten a k. (This is unimportant in Gaussian units for which k for free space is 1 but it is vital in M.K.S. where it is 8.9×10^{-16}). We therefore amend it to $d = \dfrac{2\pi m\,c_r\,v\,k}{e\,H}$. A further check is now needed that this does balance as far as the A dimension is concerned. The rationalized equation must therefore be $d = \frac{2\pi\, m\, v\, k}{e\, H}$ since the value of c_r in rationalized N.K.S. is unity.

That is the first method. The second is to construct a completely new table of dimensions using in addition to M,L,T,K, and A a new one C to to denote the dimensions of the constant of proportionality c_r. The electrostatic, or k containing, form of this table (and it is best to use this because magnetic quantities are less common than electrical ones) is exactly the same as in table (1). The only difference will be in the magnetic quantities which are given below.

		Dimensions
Magnetic field		$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2} k^{\frac{1}{2}} A^{-\frac{1}{2}} C$
Magnetic Moment		$M^{\frac{1}{2}} L^{\frac{1}{2}} k^{-\frac{1}{2}} A^{\frac{1}{2}} C^{-1}$
Permeability		$L^{-2} k^{-1} T^{2} C^{-2}$
c_r	is	C

Then all that is necessary is to check the dimensions of C (though one can do a complete check if desired). Let us return to our previous equation $d = \frac{2\pi\, m\, c\, v}{e\, H}$, H alone contains C so the equation will not balance unless $c = c_r$. The conversion then proceeds as before.

Conclusion. An extension of the theory of dimensions to cover constants of proportionality is justified by the uses it can have at present in converting equations from unrationalized to rationalised forms or in checking the accuracy of equations expressed in Gaussian units. It is, of course, also always possible that other uses may develop.

ON THE CRITICAL DETERMINANT OF AN UNBOUNDED STAR DOMAIN OF HEXAGONAL SYMMETRY

M. REHMAN

Definition.

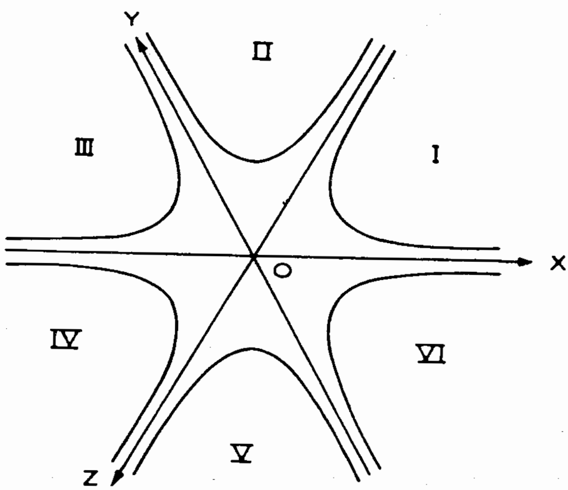
(1) Let R be any region and L a lattice with no point except O, the origin in the interior of R. Then L is said to be R-admissible.

(2) The lower bound of the determinants of all R-admissible lattices is called the critical determinant of R and is usually denoted by $\triangle$ (R).

(3) R-admissible lattices of determinant $\triangle$ (R) are called the critical lattices of R.

This paper is intended to determine the critical determinant of the following star domain S :

$$2|y|(\sqrt{3}|x| - |y|) \leqslant 1, \quad |y| \leqslant \sqrt{3}|x|$$
$$y^2 - 3x^2 \leqslant 1, \quad |y| \geqslant \sqrt{3}|x|$$



S is defined by the following equations in tri-axial co-ordinates,

$4\ ZX \leqslant 1$, in sectors I and IV

$4\ YZ \leqslant 1$, in sectors II and V

$4\ XY \leqslant 1$, in sectors III and VI

Theorem :

$$\Delta(S) = \frac{2}{\sqrt{3}}(a^2 + ab + b^2), \text{ where } 4\,ab = 1,\ b^2 = 2a^2.$$

Let S be defined by $f(x,y,z) \leqslant 1$ so that in sectors I and IV $f(x,y,z) = f_1(x,y,z) = 2\sqrt{zx}$,

in sectors II and V, $f(x,y,z) = f_2(x,y,z) = 2\sqrt{yz}$,

in sectors III and VI, $f(x,y,z) = f_3(x,y,z) = 2\sqrt{xy}$,

We apply R.P. Bambah's method in proving the theorem.

It is easy to see that $f(x,y,z)$ satisfies the following conditions :

(1) for $x \geqslant o$, $z \geqslant o$, $f(x,y,z)$ is continuous.
$f(o, o, o) = o$; for $t > o$, $f(tx, ty, tz) = tf(x,y,z)$

(2) $f(-x, -y, -z) = f(x,y,z)$.

(3) $f(x,y,z)$ is symmetric in all the variables.

(4) $f(x_1 + x_2, y_1 + y_2, z_1 + z_2) \geqslant f(x_1, y_1, z_1) + f(x_2, y_2, z_2)$
$x \geqslant o$, $z \geqslant o$

(5) for fixed x, $f(x,y,z)$ is a non-decreasing function of z.

(6) for fixed z, $f(x,y,z)$ is a non-decreasing function of x.

(7) if $o \leqslant r \leqslant 1$, then $f(1-r, -2, 1+r)$ is a non-increasing function of r.

(8) for $o \leqslant a\text{-}r \leqslant a+\frac{r}{2}$, $f(a-r, -2a+\frac{r}{2}, a+\frac{r}{2})$ is a strictly decreasing function of r.

(9) $f(o, -3, 3) < f(1, -2, 1)$.

Consider the following points (fig. 2) and define a, b by the relations.

(i) $f_1(a, -a-b, b) = f_2(a+b, -b, -a) = 2\sqrt{ab} = 1.$

(ii) $f_1(2a+b, -a-2b, b-a) = 2\sqrt{(2a+b)(b-a)} = 1,\ b > a > o.$

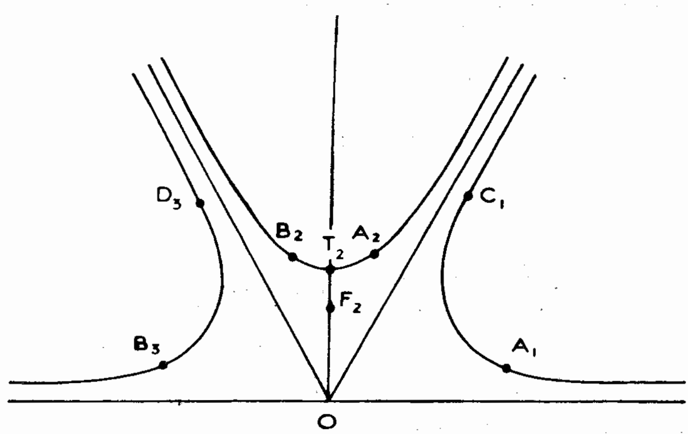
$A_1(a, -a-b, b)$, $A_2(a+b, -b, -a)$,

$B_2(a+b, -a, -b)$, $B_3(a, b, -a-b)$, $C_1(2a+b, -a-2b, b-a)$,

$D_3(2a+b, b-a, -a-2b)$

$$F_2\left(\frac{2(a^2+ab+b^2)}{2b+a}, -\frac{a^2+ab+b^2}{2b+a}, -\frac{a^2+ab+b^2}{2b+a}\right),$$

$T_2(1, -\frac{1}{2}, -\frac{1}{2})$.



It is easily seen that $\vec{OA_1} + \vec{OA_2} = \vec{OC_1}$, $\vec{OB_2} + \vec{OB_3} = \vec{OD_3}$ and F_2 is the point of intersection of $C_1 A_2$ and $D_3 B_2$.

From relations (i) and (ii) $a = \dfrac{1}{2\sqrt[4]{2}}$, $b = \dfrac{\sqrt{2}}{2\sqrt[4]{2}}$

$$2\triangle OA_1A_2 = \triangle = \frac{2}{\sqrt{3}}(a^2+b^2+ab) = \frac{3+\sqrt{2}}{2\sqrt{6}}$$

It can be easily verified that F_2 lies below T_2 or $2(a^2+b^2+ab) \leqslant 2b+a$.

$\therefore \Delta(S) \geqslant \Delta$. Also $\Delta(S) = \Delta$ if and only if L_1 (A_1, A_2), and L_2 obtained by the reflections of A_1, A_2 into the medians in their respective sectors, are S-admissible.

Lemma.

L_1 (A_1, A_2) is S-admissible, any point of L_1 has co-ordinates, $X = a\,\xi + (a+b)\,\eta$, $Y = -(a+b)\,\xi - b\,\eta$, $Z = b\,\xi - a\,\eta$ for integer values of ξ, η.

$$4\,|ZX| = 4\,|ab\,\xi^2 + \xi\eta\,\{b\,(a+b) - a^2\} - a\,(a+b)\,\eta^2|$$

$$= \left|\xi^2 + \left(1 + \frac{1}{\sqrt{2}}\right)\xi\eta - \left(1 + \frac{1}{\sqrt{2}}\right)\eta^2\right|$$

$$= \left|\left(\xi - \frac{\sqrt{2}}{2}\,\eta\right)\left(\xi + \sqrt{2}+1\,\eta\right)\right|$$

On account of hexagonal symmetry it suffices to show that $4\,|ZX| \geqslant 1$ for $\xi > 0$.

Case 1. $\eta > 0$.

$$\frac{\sqrt{2}}{2} = \frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{2+}\cdots\cdots\cdots\frac{1}{\alpha_n}$$

The convergents p_{n+1}/q_{n+1} are given by the following relations :-

$a_0 = 0$, $a_1 = 1$, $a_2 = a_3 = \dots\dots\dots = 2$

$p_{-1} = p_0 = p_1 = 1$, $p_{n+1} = a_n\,p_n + p_{n-1}$

$q_{-1} = 1$, $q_0 = 2$, $q_1 = 1$, $q_{n+1} = a_n\,q_n + q_{n-1}$

$p'_{n+1} = \alpha_n\,p_n + p_{n-1}$, $q'_{n+1} = \alpha_n\,q_n + q_{n-1}$

where α_n is the complete quotient for $n > 1$

Clearly $\alpha_n = \sqrt{2}+1$

Then, $\left|\dfrac{p_n}{q_n} - \dfrac{\sqrt{2}}{2}\right| = \dfrac{1}{q_n\,q'_{n+1}} = \dfrac{1}{q_n\,(\alpha_n q_n + q_{n-1})}$

$$= \frac{1}{q_n\left[(\sqrt{2}+1)q_n + \frac{q_n}{2} - \frac{q_{n-2}}{2}\right]}$$

$$> \frac{1}{q_n^2\,(\frac{1}{2} + \sqrt{2}+1)} \qquad \text{(i)}$$

Suppose, $\left|\frac{\xi}{\eta} - \frac{\sqrt{2}}{2}\right| < \frac{1}{\eta\,\{\xi+(\sqrt{2}+1)\eta\}}$ ....(ii)

then, $\left|\frac{\xi}{\eta} - \frac{\sqrt{2}}{2}\right| < \frac{1}{(\sqrt{2}+1)\,\eta^2}$

$\therefore \frac{\xi}{\eta} > \frac{\sqrt{}}{2} - \frac{1}{(\sqrt{2}+1)\eta^2}$

or, $\xi > \frac{\sqrt{2}}{2}\,\eta - \frac{1}{(\sqrt{2}+1)\eta}$

$> \frac{\sqrt{2}}{2}\,\eta - \frac{1}{\sqrt{2}+1}$ for $\eta > 1$ ....(iii)

If $\eta = 1$, then $\left|\xi - \frac{\sqrt{2}}{2}\right| < \frac{1}{\xi+\sqrt{2}+1}$, impossible.

In (ii) We use (iii) to find an estimate on the right. Then we have,

$$\left|\frac{\xi}{\eta} - \frac{\sqrt{2}}{2}\right| < \frac{1}{\eta^2\left\{\frac{\sqrt{2}}{2} + \sqrt{2}+1 - \frac{1}{\eta^2(\sqrt{2}+1)}\right\}}$$

$$< \frac{1}{\eta^2\left\{\frac{\sqrt{2}}{2} + \sqrt{2}+1 - \frac{1}{4(\sqrt{2}+1)}\right\}} \qquad \text{(iv)}$$

Due to (iv) we find that $\frac{\xi}{\eta}$ is a convergent of $\frac{\sqrt{2}}{2}$.

We put $\frac{p_n}{q_n} = \frac{\xi}{\eta}$. Then from (i) and (iv),

$$\frac{1}{\frac{\sqrt{2}}{2} + \sqrt{2}+1 - \frac{1}{4(\sqrt{2}+1)}} > \frac{1}{\frac{1}{2} + \sqrt{2} + 1}$$

or, $\frac{1}{2} + \frac{\sqrt{2}-1}{4} > \frac{\sqrt{2}}{2}$

or, $\sqrt{2} + 1 > 2\sqrt{2}$, impossible.

$\therefore$ Assumption (ii) is false.

Case 2. $\eta < 0$. Put $\zeta = -\eta$

Suppose, $\left|\frac{\xi}{\eta} + \sqrt{2}+1\right| < \frac{1}{|\eta|\left|\xi - \frac{\sqrt{2}}{2}\eta\right|}$

$$\text{or, } \left|\frac{\xi}{\eta} - (\sqrt{2}+1)\right| < \frac{1}{\zeta\left(\xi + \frac{\sqrt{2}}{2}\zeta\right)} \qquad \text{(i)}$$

$$\sqrt{2}+1 = 2 + \frac{1}{2+}\ \frac{1}{2+} \cdots\cdots\cdots \frac{1}{a_r,} \quad a_n = \sqrt{2}+1$$

The convergents p_{n+1}/q_{n+1} are given by the relations :-

$$p_{-1} = 1,\ p_o = 2,\ p_{n+1} = 2p_n + p_{n-1};$$

$$q_{-1} = 0,\ q_o = 1,\ q_{n+1} = 2q_n + q_{n-1}.$$

$$q'_{n+1} = (\sqrt{2}+1)\, q_n + q_{n-1},\ q_{n-1} = q_n/2 - q_{n-2}/2.$$

$$\text{Then, } \left|\frac{\xi}{\eta} - (\sqrt{2}+1)\right| = \frac{1}{q_n\, q'_{n+1}}$$

$$= \frac{1}{\zeta\left\{(\sqrt{2}+1)\,\zeta + \frac{\zeta}{2} - \frac{q_{n-2}}{2}\right\}}$$

$$> \frac{1}{\zeta^2(\sqrt{2}+1+\frac{1}{2})} \qquad \text{(ii)}$$

$$\text{From (i), } \left|\frac{\xi}{\eta} - (\sqrt{2}+1)\right| < \frac{\sqrt{2}}{\zeta^2}$$

$$\text{or, } \xi > (\sqrt{2}+1)\,\zeta - \frac{\sqrt{2}}{\zeta},\ \zeta \geqslant \zeta_o$$

$$\text{From, (i), } \left|\frac{\xi}{\zeta} - (\sqrt{2}+1)\right| < \frac{1}{\zeta^2\left(\sqrt{2}+1 - \frac{\sqrt{2}}{\zeta_o^2} + \frac{\sqrt{2}}{2}\right)} \qquad \text{(iii)}$$

$$\text{From (ii) and (iii), } \frac{1}{\sqrt{2}+1+\frac{1}{2}} < \frac{1}{\sqrt{2}+1 - \frac{\sqrt{2}}{\zeta_o^2} + \frac{\sqrt{2}}{2}}$$

$$\text{or, } \frac{\sqrt{2}}{\zeta_o^2} + \frac{1}{2} > \frac{\sqrt{2}}{2}$$

or, $\zeta_o^2 < 4 + 2\sqrt{2}$, impossibie if $\zeta_o = 3$.

$$\text{For } \eta = -1, \left|\left(\xi - \frac{\sqrt{2}}{2}\eta\right)\left(\xi + \sqrt{2}+1\,\eta\right)\right| = \phi_1(\xi)$$

$$= \left|\left(\xi + \frac{\sqrt{2}}{2}\right)(\xi - \sqrt{2} - 1)\right| > 1 \quad \text{for } \xi \geqslant 4$$

It can be verified that $\phi_1(\xi) > 1$ for $\xi = 0, 1, 2, 3,$

For $\eta = -2,$

$\phi_2 \ (\xi) = |(\xi + \sqrt{2})(\xi - 2\sqrt{2} - 2)| > 1$ for $\xi \geqslant 6.$

It can be verified that $\phi_2(\xi) > 1$ for $\xi = 0, 1, 2, 3, 4, 5.$

Hence the assumption $4 \mid Z X \mid < 1, \xi > 0$, is wrong.

$\therefore$ $L_1 (A_1, A_2)$ is $S-$admissible, and $\triangle (S) = \triangle$.

$$= \frac{2}{\sqrt{3}} (a^2 + ab + b^2) = \frac{3 + \sqrt{2}}{2\sqrt{6}} .$$

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Department of Mathematics,
Dacca University.

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