VOLUME II APRIL 1969 NUMBER 2

THE PUNJAB UNIVERSITY JOURNAL OF MATHEMATICS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE PUNJAB
LAHORE

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ON A PACKING INEQUALITY FOR PLANE SECTORIAL NORM DISTANCES

Dedicated to Arnold E. Ross

by

HANS ZASSENHAUS

In the geometry of numbers there occur distance functions like

$$N_n(P, Q)=\left|\sqrt[n]{\prod_{j=1}^{n}(y_i-x_i)}\right|, (P=x_1,\dots x_n), Q=(y_1,\dots y_n),$$

$$N_{2,m}(P, Q)=\left|\sqrt[m]{\prod_{1\leqslant j\leqslant m}\left(x\cos\frac{j\pi}{m}+y\sin\frac{j\pi}{m}\right)}\right| \left(\overrightarrow{PQ}=\{x, y\}\right)$$

$$N^*_{2,m}(P, Q)=|\sqrt{u_m v_m}|, (u_m=r\cos\varphi, v_m=r\sin\varphi, 0\leqslant\varphi<\frac{\pi}{2}, r\geqslant 0,$$

$$\overrightarrow{PQ}=\left\{r\cos\left(\frac{2\varphi}{m}+\frac{j\pi}{m}\right), r\sin\left(\frac{2\varphi}{m}+\frac{j\pi}{m}\right)\right\},$$

$$0\leqslant j<m, j\in Z)$$

which have the following properties:

(1) $N(P, Q,)$ is a real valued non-negative function of the pairs of points P, Q of the euclidean n-dimensional space E_n,

(2) (translation invariance):

$N(P, Q)=N(R, S)$ if $\overrightarrow{PQ}=\overrightarrow{RS}$,

(3) (homogeneity):

$N(O, P')=\lambda N(O, P)$ if $\overrightarrow{OP'}=\lambda\overrightarrow{OP}$, $\lambda\geqslant O$,

(4) (central symmetry):

$N(O, P')=N(O, P)$ if $\overrightarrow{OP'}=-\overrightarrow{OP}$,

(5) the E_n is the non-overlapping union of finitely many closed cones $E_1, \ldots, E_p$ emanating from the origin such that each of them contains interior points and that, moreover, for any parallelogram $OP_1P_2P_3$ with one vertex at the origin O and the other three vertices contained in the same sector E_j there holds the anti-triangle inequality

$$N(O, P_2) \geqslant N(O, P_1) + N(O, P_3).$$

The pointsets $C_1, \ldots, C_p$ are called the *sectors* of the *sectorial norm distance* N which is defined by the properties (1)–(5). Or, equivalently, the 'gauge body' of all points of N-distance less than 1 from the origin O is an open region S that is starred and centrally symmetric about O such that $E_n - S$ is the non-overlapping union of finitely many closed convex pointsets $S_1, \ldots, S_p$ such that for every point P of S_i also the ray away from O belongs to S_i and the pointset C_i formed by all rays from O to points of S_i contains inner points.

A *sectorial triangle* is defined as a triangle with vertices P, Q, R such that the two points X, Y defined by setting

$$\overrightarrow{OX} = \overrightarrow{PQ}, \ \overrightarrow{OY} = \overrightarrow{QR}$$

belong to the same sector. Hence there holds the anti-triangle inequality.

(6) $N(P, Q) + N(Q, R) \leqslant N(P, R)$.

Let N be a plane sectorial norm distance. A pointset M is said to be *N-admissible*, if the N-distance of any two distinct points of M is at least 1.

Denote by $t(N)$ the greatest lower bound of the areas of the N-admissible non-sectorial triangles. For example for $N = N_2$ we have $t(N) = \frac{1}{2}\sqrt{5}$, the area of the N_2-admissible triangle OQR where $Q = (1, 1)$, $R = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$.

We consider a finite N-admissible pointset M and a Jordan-polygon $\Pi = P_1 .. P_b P_1$ with vertices in M such that every point of M that is not on the boundary of Π lies in the interior.

Theorem : The area $A(\Pi)$ of Π, the N-circumference

$$C(\Pi) = \sum_{i=1}^{b} N(P_i, P_{i+1}), \quad (P_{b+1} = P_1)$$

and the number of points N in M satisfies the inequality

$$(7) \qquad \frac{A(\Pi)}{2t(N)} + \frac{1}{2} C(\Pi) + 1 \geqslant |M|$$

in the event that $t(N)$ is positive. This inequality has been proved by an ingenious argument in the never published thesis of Norman E. Smith [3] for the special case $N = N_2$. The theorem represents a generalization which permits to establish $\frac{1}{2t(N)}$ as an upper bound for the irregular packing density with respect to plane sectorial norm distance. The bound is sharp in Norman E. Smith's case and coincides with the inverse of the critical mesh. M. Rahman [2] gives two other cases in which $2t(N)$ is less than the mesh of the critical lattice. It is known in one case [1] but not in the other one that the irregular packing density cannot be greater than the regular packing density.

Proof of the theorem: If (7) would be wrong then there would be a counter example M, Π with minimum value of $b + |M|$.

If there are three points P_i, P_{i+1}, P of M such that P is in the interior of Π and $P_i P P_{i+1}$ is a sectorial triangle contained in Π for which

$$(8) \qquad N(P_i, P_{i+1}) \geqslant N(P_i, P) + N(P, P_{i+1})$$

then for all points X of M belonging to the triangle $P_i X P_{i+1}$, but not to the straight segment $P_i P_{i+1}$ the triangle $P_i X P_{i+1}$ is sectorial again. Among the points X let Y be a point closest to the line $P_i P_{i+1}$. It follows that

$$N(P_i, P_{i+1}) \geqslant N(P_i, Y) + N(Y, P_{i+1}),$$

that there is no point of M contained in triangle $P_i\,YP_{i+1}$ other than the 3 vertices and that the triangle $P_i\,Y\,P_{i+1}$ belongs to M. Upon replacement of the Jordan polygon Π by

$$\Pi' = P_1\,P_2 \ldots P_i\,Y\,P_{i+1} \ldots P_b\,P_1$$

the number of boundary points is increased by 1 such that all points of M again either are vertices of Π' or they belong to the interior of Π'.

Hence

$$(9) \qquad \frac{A(\Pi')}{2t(N)} + \frac{1}{2}\,C(\Pi') + 1 \geqslant |M|$$

Let us note that

$$A(\Pi) = A(\Pi') + A(P_i\,Y\,P_{i+1}) > A(\Pi')$$
$$C(\Pi) = N(P_i, P_{i+1}) - N(P_i, Y) - N(Y, P_{i+1}) \geqslant 0$$

hence (9) implies (7), a contradiction. It follows that there is no sectorial triangle $P_i\,PP_{i+1}$ contained in Π for which (8) is satisfied such that P is inner point of M. If, however, $P = P_j$, where $i+1 < j \leqslant b$ and where no point of M other than the vertices belongs to triangle $P_i\,P\,P_{i+1}$, then either $b=3$,

$$\frac{A(\Pi)}{2t(N)} + \frac{1}{2}\,C(\Pi) + 1 \geqslant 0 + \frac{1}{2}\cdot 4 + 1 = 3 = |M|$$

or $b > 3$. $j = i+2$,

$$\frac{A(\Pi')}{2t(N)} + \frac{1}{2}\,C(\Pi') + 1 \geqslant |M| - 1$$
$$(\Pi' = P_1\,P_2 \ldots P_{i+2} \ldots P_b\,P_1)$$

$$A(\Pi) \geqslant A(\Pi'),$$
$$C(\Pi) - C(\Pi') = N(P_i, P_{i+1}) - N(P_i, P) + N(P, P_{i+1})$$
$$\geqslant 2\,N(P, P_{i+1}) \geqslant 2$$

hence again (7),

or $b > 3$, $j > i+2$,

$$(10) \quad \frac{A(\Pi'')}{2t(N)} + \frac{1}{2}\,C(\Pi'') + 1 \geqslant |M_1|$$
$$(\Pi'' = P_{i+1} \ldots P_j\,P_{i+1})$$

$$(11)\quad \frac{A(\Pi''')}{2t(N)} + \frac{1}{2} C(\Pi''') + 1 \geqslant |M_2|$$

$$(\Pi''' = P_j \ldots P_b\, P_1 \ldots P_i\, P_j)$$

$$M_1 \cap M_2 = \{P\}$$
$$M_1 \cup M_2 = M$$
$$A(\Pi) \geqslant A(\Pi'') + A(\Pi''')$$
$$C(\Pi) - C(\Pi'') - C(\Pi''') = N(P_i, P_{i+1}) - N(P_i, P) - N(P, P_{i+1}) \geqslant 0$$

hence upon addition of (10), (11) and subtraction of 1 on both sides once again (7) is obtained.

Thus it follows that there is no point P of M other than P_i, P_{i+1} for which the triangle $P_i\, P\, P_{i+1}$ is contained in Π and sectorial subject to (8).

If it is possible to find a triangulation of Π into non-sectorial triangles using only the points of M as vertices then by Euler's formula the number λ of these triangles satisfies :

$$|M| + \tau + 1 = \frac{3}{2}\tau + \frac{b}{2} + 2$$

so that

$$\tau = 2|M| - b - 2$$
$$A(\Pi) \geqslant 2t(N)$$
$$\frac{A(\Pi)}{2t(N)} \geqslant |M| - \frac{b}{2} - 1$$
$$\frac{1}{2} C(\Pi) \geqslant \frac{b}{2}$$
$$1 \geqslant 1$$

hence upon addition of the last three inequalities again (7).

The theorem therefore will follow from the following Lemma: Given a Jordan polygon $\Pi = P_1 P_2 \ldots P_b P_1$ and a finite pointset M containing $P_1, P_2, \ldots, P_b$ such that every point of M distinct from $P_1, \ldots, P_b$ belongs to the interior of Π. For no point P of M for which $P \neq P_i$, $P \neq P_{i+1}$, the triangle $P_i\, P\, P_{i+1}$ belongs to Π and is sectorial such that (8) is satisfied. Then there is a triangulation of π into non-sectorial triangles using precisely the points of M.

Proof : If the lemma would be wrong then there would be a counter example with minimum value of $|M|$.

For any point chain $X_0 X_1 \ldots X_l$ the broken l-path $X_0 X_1 \ldots X_l$ is defined as the chain of straight segments $X_0 X_1, \ldots, X_{l-1} X_l$. It is said to be an l, M, π, S-path (or sectorial l, M, π-path, or sectorial M, π-path) from X_0 to X_l if

(1) each of the $l+1$ points $X_0, X_1, \ldots, X_l$ belongs to M

(2) each vector $\overrightarrow{X_i X_{i+1}}$ $(0 \leqslant i < l)$ is non zero and belongs to the same sector.

It follows that each vector $\overrightarrow{X_i X_j}$ $(0 \leqslant i < j \leqslant l)$ is non-zero and belongs to S.

A *refinement* of the l, M, π, S-path is defined as an $(l+1)$, M, π, S-path $X_0 X_1 \ldots X_i X X_{i+1} \ldots X_l$.

A sectorial M, π-path is said to be *maximal* if there is no refinement.

If there is a sectorial M, π-path from P to Q then after a finite number of refinements a maximal M, π-path from P to Q will be obtained.

If the vertices $P_i, P_{i+1}, \ldots, P_{i+l}$ of π satisfy the condition that the vectors $\overrightarrow{P_i P_{i+1}}, \ldots, \overrightarrow{P_{i+l-1} P_{i+l}}$ belong to the same sector then the broken path $P_i P_{i+1}$ is a maximal sectorial l, M, π-path from P_i to P_{i+l}†. Such a sectorial l, M, π-path is said to be a *boundary* M, π-path.

Any subpath $X_i, \ldots X_j$ (where $1 \leqslant i < j \leqslant l$) of the sectorial l, M, π-path $X_0 X_1 \ldots X_l$ is a sectorial M, π-path. If the sectorial M, π-path $X_0 X_1 \ldots X_l$ is maximal then every subpath is also maximal sectorial.

† We set $P_{b+1} = P_1, P_{b+2} = P_2, \ldots, P_{2b-1} = P_{b-1}$.

If there is a maximal sectorial M, π-path from one vertex of π to another one that is not a boundary M, π-path then there is a subpath

$$P_i\ X_1\ X_2 \ldots X_{l-1}\ P_j \text{ (where } 1 \leqslant i < j < i+b;\ i \leqslant b)$$

for which none of the points $X_1, \ldots X_{l-1}$ is a vertex of π.

It follows that the two Jordan polygons

$$\pi_1 = P_i X_1 X_2 \ldots X_{l-1} P_j P_{j+1} \ldots P_{i+b-1} P_i,$$
$$\pi_2 = P_i X_1 X_2 \ldots X_{l-1} P_j P_{j-1} \ldots P_i$$

and the corresponding intersections $M_h = M \cap \pi_h$ $(h=1, 2)$ both satisfy the assumptions of the lemma such that there is a triangulation of π_h into non-sectorial triangles. The two triangulations together constitute a triangulation of π into non-sectorial triangles

From now on we make the additional assumption that every maximal sectorial M, π-path leading from one vertex of π to another one is a boundary sectorial M, π-path.

If $b>3$ then, after suitable numbering, there will be an index i such that $2<i<b$ and that the straight segment $P_1\ P_i$ belongs to π and that P_1, P_i are the only points of the straight segment $P_1\ P_i$ that are on the boundary of π and that there is the maximal sectorial M, π, S-path $P_1\ P_2 \ldots P_i$ and i is as large as possible.

If there is a vertex of M in the Jordan polygon $\pi' = P_1\ P_2 \ldots P_i\ P_1$, say the vertex P_j (where $i<j \leqslant b$), then there are vertices P_i', P_j', of π' such that $1<i'<i<j' \leqslant b$ such that the straight segment $P_i'\ P_j'$ belongs to π', but that it has only the points P_i', P_j' with the boundary of π' in common. It follows that either $P_i'\ P_i'_{+1} \ldots P_j'$ or $P_i'\ P_i'_{-1} \ldots P_1\ P_b\ P_{b-1} \ldots P_j'$ form a maximal sectorial path from P_i' to P_j'. In the first event the vector $\overrightarrow{P_i P_j'}$ would be in S and also in the sector that is opposite to S. In the second event the vector $P_1\ P_j'$ would belong to S and also to the sector that is opposite to S. In any case there arises a contradication. It follows that the vertices P_1, P_2, $\ldots P_i$ are the only vertices of π belonging to π'.

Thus we have shown that none of the straight segments $P_{i'}\ P_{j'}$ (where $1<i'<i<j'\leqslant b$) belongs to Π. All vertices $P_1, P_2, \ldots P_i$ of Π must lie on the same side of the straight line $P_1 P_i$. This is because the broken path $P_1 P_2 \ldots P_i$ is sectorial.

Let the ray $P_1 \hat{X}$ move to the other side of ray $P_1 P_i$ such that another vertex P_j of Π is met for the first time in such a way that the straight segment $P_i\ P_j$ belongs to Π and the intersection of this straight segment with the boundary of Π consists only of the endpoints. Hence $i<j\leqslant b$. From the argument given above it follows that none of the straight segments $P_{i'}\ P_j$ $(1<i'<i)$ belongs to Π. Moreover there are vertices $P_{i'}$, $P_{i''}$ (where $1<i'<i''\leqslant i$) such that the line $P_{i'}\ P_{i''}$ intersects line $P_1 P_j$ in a point X lying between P_1 and P_j and the ray XP_j is between the ray from X that is opposite ray $XP_{j'}$ and the ray XY with Y being defined by the equation $\overrightarrow{XY} = \overrightarrow{P_1 P_i}$. Since both vectors $\overrightarrow{XY}$, $\overrightarrow{P_{i'} X}$ belong to C it follows that the vector $\overrightarrow{XP_j}$ belongs to S, hence also the vector $\overrightarrow{P_1 P_j}$ belongs to S. Hence $j=b$ because of the maximal property of i. But in this event there is the sectorial triangle $P_1\ P_i\ P_b$ which is a contradiction.

There remains to discuss the case $b=3$. If $|\ M\ | = 3$ then Π itself is a non-sectorial triangle. Therefore $|\ M\ | > 3$.

There is a point P of M in the interior of the triangle $P_1\ P_2\ P_3 = \Pi$ such that neither the vector pair $\overrightarrow{P_1 P_3}$, $\overrightarrow{P_3 P}$ nor the vector pair $\overrightarrow{PP_3}$, $\overrightarrow{P_3 P_2}$ are in the same sector.

There is a sectorial τ, M, Π-path $P_1 Q_1 \ldots Q_{\tau-1} P$ connecting P_1, P with maximum value of τ and there is a sectorial m, M, Π-path $PR_1 \ldots R_{m-1} P_2$ connecting $P_1\ P_2$ with maximum value of m. Because of the property of P mentioned above all of the points $Q_1, \ldots Q_{\tau-1}$,

$R_1, \ldots . R_{m-1}$ are in the interior of π. Let us choose P in such a way that $t+m$ is minimum. It follows that the path $P_2 Q_1, \ldots . Q_{\tau-1} PR_1 \ldots .$ $R_{m-1} P_2$ is a Jordan path.

Hence the two Jordan polygons

$$\pi' = P_1 Q_1 \ldots . Q_{\tau-1} PR_1 \ldots . R_{m-1} P_2 P_3 P_1$$
$$\pi'' = P_1 Q_1 \ldots . Q_{\tau-1} PR_1 \ldots . R_{m-1} P_2 P_1$$

respectively satisfy the assumption of the lemma with respect to the pointsets

$$M' = \pi' \cap M, \; M'' = \pi'' \cap M$$

respectively.

As was shown already there are triangulations of π', π'' into non-sectorial triangles using precisely the pointsets $\pi' \cap M$, $\pi'' \cap M$ respectively as vertices. The two triangulations together provide a triangulation of π into non-sectorial triangles using precisely M as vertices. Hence the lemma.

The Ohio State University, September 1966.

BIBLIOGRAPHY

1. Hughes, Sr. M. Rosanna. On a Packing Problem with Hexagonal Symmetry, Dissertation University of Notre Dame, June 1964, p. 1-52.

2. Rahman M. On the Minimum Areas of certain N-admissible Triangles in Unbounded Star-Bodies of Symmetry, submitted for Publication in the Proceedings of the Pakistan Philosophical Society, September 1966, 5 p.

3. Smith, Norman E. On a Packing Problem of Statistical Number Geometry, Ph. D. Thesis, McGill University, 1951.

4. Zassenhaus, Hans. Modern Developments in the Geometry of Numbers, Bull. Am. Math. Society, Sept. 1961, 67 (No. 5), p 427-428.

IMPACT OF HOMOLOGICAL ALGEBRA ON THE THEORY OF ABELIAN GROUPS

by

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The object of this article is to describe the important role which Homological Algebra has played in the development of the theory of abelian groups. Homological algebra is a branch of mathematics which has emerged mainly during the past fifteen years from algebraic topology by abstracting most of the powerful algebraic techniques from their topological setting. In view of the fact that homological algebra is a highly specialized subject I feel called upon to detail some of its basic concepts. Because of the limitation of space it will not be possible for me to describe in detail the latest developments in the field of abelian groups. I will therefore have to content myself with broadly indicating how homological algebra has influenced the abelian group theory.

Let me begin with some definitions.

1. Group Homomorphisms

Definitions. A homomorphism ϕ: A$\rightarrow$B from the abelian group A to the abelian group B is a mapping satisfying $(a+a')\ \phi=a\ \phi+a'\ \phi$. The *kernel* of ϕ is the subgroup $\phi^{-1}(0)$ of A, *i.e.*, the subgroup consisting of elements which are mapped on to zero under the mapping ϕ. The subgroup Aϕ of B is the *image* of ϕ, the factor group A/ϕ^{-1} (0) is the *co-image*, and the factor group B/A ϕ the *co-kernel* of ϕ. The homomorphism ϕ is a *monomorphism if* its kernel is zero, and *epimorphism* if its co-kernel is zero, and an *isomorphism* if it is both. If A′ is a subgroup of A, then the monomorphism i : A′$\rightarrow$A defined by $a'\ i=a'$, $a' \in$ A′, is called the *inclusion map* or *injection,* and the epimorphism $p: A \rightarrow A/A'$ which maps each element of A onto its coset is called the *projection.*

Definition. A sequence of abelian groups and homomorphisms

$$\ldots\ldots \rightarrow A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \longrightarrow \ldots\ldots \qquad (1)$$

is *exact* at A_n if the image of the homomorphism ϕ_{n+1} is the kernel of the homomorphism ϕ_n. The sequence is *exact* if it is exact at A_n for each *n.*

We note that the squance $O \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A/A' \rightarrow O$ is exact. Thus we express the fact that A′ is subgroup of A and A″ the quotient group A/A′ by saying that the sequence $O \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow O$...... (2) is exact.

2. Tensor Product, Torsion Product, Hom. and Ext.

Definition. The *tensor product* $A \otimes B$ of two abelian groups A, B is the abelian group generated by elements $a \otimes b$, $a \in A$, $b \in B$, with the following relations :

$$(a_1+a_2) \otimes b = a_1 \otimes b + a_2 \otimes b,$$

$$a \otimes (b_1+b_2) = a \otimes b_1 + a \otimes b_2.$$

Definition. Let $O \rightarrow R \xrightarrow{\lambda} F \xrightarrow{\mu} A \rightarrow O$ be a presentation of an abelian group A (*i.e.* A is expressed as the quotient of a free abelian group F). Then it can be shown that the kernel of the homomorphism $\bar{\lambda}: R \otimes B \rightarrow F \otimes B$, where B is any abelian group, is independent, upto isomorphism, of the choice of μ (see [11]) and is called the *torsion product* of A and B and is denoted by Tor (A, B) or A $*$ B.

Definition. Let A, B be two abelian groups. Then the set of all homomorphisms A→B form an abelian group under the composition defined by $a(\phi+\varphi) = a\phi + a\varphi$, $a \in A$, ϕ, φ: A→B.
This group is usually written as Hom (A,B) or A $\pitchfork$ B (the latter notation is due to E.C. Zeeman).

Definition. Let $O \rightarrow R \xrightarrow{\lambda} F \xrightarrow{\mu} A \rightarrow O$ be a presentation of A.

Then it can be proved that the co-kernel of the homomorphism $\dot{\lambda} : F \otimes B \rightarrow R \otimes B$ is independent, upto isomorphism, of the choice of μ (see [11]) and is called the *group of extensions* of B by A and is written as Ext. (A, B) (or A † B).

3. Projective and Injective groups[1].

Definition. An abelian group P is *projective* if, given any epimorphism $\phi : A \rightarrow B$ and any homomorphism $\beta : P \rightarrow B$, there is a homomorphism $\alpha : P \rightarrow A$ such that $\alpha \phi = \beta$.

Definition. A group I is *injective* if, given any monomorphism ϕ: $B \rightarrow A$ and any homomorphism $\beta : B \rightarrow I$, there is a homomorphism α: $A \rightarrow I$ such that $\phi \alpha = \beta$.

It can be proved that an abelian group is projective if and only if it is free and injective if and only if it is divisible.

4. Categories and Functors

Definition. A set E of elements $\{\gamma\}$ is called a multiplicative system if, for some pairs $\gamma_1, \gamma_2 \in E$ a product $\gamma_1 \gamma_2 \in E$ is defined.

An element $e \in E$ is called an *identity* if $\gamma_1 e = \gamma_1$ and $e \gamma_2 = \gamma_2$ whenever $\gamma_1 e$ and $e \gamma_2$ are defined. The multiplicative system is called an *abstract category* if the following axioms are satisfied :

(*i*) the triple product $(\gamma_1 \gamma_2) \gamma_3$ is defined if and only if $\gamma_1 (\gamma_2 \gamma_3)$ is defined. When either is defined the associative law $(\gamma_1 \gamma_2) \gamma_3 = \gamma_1 (\gamma_2 \gamma_3)$ holds. This triple product will be written as as $\gamma_1 \gamma_2 \gamma_3$;

(*ii*) the triple product $\gamma_1 \gamma_2 \gamma_3$ is defined whenever both products $\gamma_1 \gamma_2$ and $\gamma_2 \gamma_3$ are defined ;

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1. By a group we shall always mean an abelian group.

(*iii*) for each $\gamma \in E$ there exist identities e_1, e_2, $\in E$ such that $e_1\gamma$ and $\gamma\, e_2$ are defined.

Definition: A *cotegory E* consists of a collection $\{C\}$ of elements called objects and a collection $\{\gamma\}$ of elements called mappings. The objects are in $1-1$ correspondence $C \rightarrow 1_C$ with the set of identities of the abstract category. Thus to each mapping γ there correspond unique objects C_1 and C_2 such that $\overset{1}{C_1}\gamma$ and $\overset{1}{C_2}\gamma$ an defined. The objects are called the *domain* and the *codomain* (or *range*) of γ respectively. We write $\gamma : C_1 \rightarrow C_2$. We now give some examples of categories :

(*i*) the category of topological spaces and continuous maps ;

(*ii*) the category of abelian groups and homomorphisms ;

(*iii*) the category of vector spaces over a field and linear transformations.

Definition. Let *E* and *D* be two categories and let T be a function which maps the objects of *E* to the objects of *D* and maps of *E* to maps of *D*. Then T is called a *covariant functor* if the following axioms are satisfied:

(*i*) if $\gamma : C_1 \rightarrow C_2$, then $\gamma\, T : C_1\, T \rightarrow C_2\, T$,

(*ii*) $\overset{1}{C}{}^{T} = 1_{CT}$

(*iii*) if $\gamma_1\, \gamma_2$ is defined, then $(\gamma_1\, \gamma_2)\, T = (\gamma_1\, T)\, (\gamma_2\, T)$.

The map T is called a *contravariant functor* if these axioms are replaced by

(*i*) if $\gamma : C_1 \rightarrow C_2$, then $\gamma\, T : C_2\, T \rightarrow C_1\, T$,

(*ii*) $\overset{1}{C}{}^{T} = 1_{CT}$

(*iii*) if $\gamma_1\, \gamma_2$ is defined, then $(\gamma_1\, \gamma_2)\, T = (\gamma_2\, T)\, (\gamma_1\, T)$.

Tensor product is an example of a covariant functor and Hom. that of a contravariant functor (For details of categories and functors see [1]).

5. Direct and Inverse Limits.

Definition. A relation $\alpha \leq \beta$ in a set M is called a *quasi-order* if it is reflexive and transitive. A *directed set* M is a quasi-ordered set such that for each pair $\alpha, \beta \in M$, $\exists$ a $\gamma \in M$ for which $\alpha < \gamma$ and $\beta < \gamma$.

Definition. A direct system of sets $\{X, \pi\}$ over a directed set M is a function which attaches to each $\alpha \in M$ a set X^α and, to each pair α, β such that $\alpha < \beta$ in M, a map

$$\pi_\alpha^\beta : X^\alpha \to X^\beta$$

such that, for each $\alpha \in M$

$$\pi_\alpha^\alpha = \text{identity},$$

and for $\alpha < \beta \leq \gamma$ in M,

$$\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma.$$

An *inverse system* of sets $\{X, \pi\}$ over a directed set M is a function which attaches to each $\alpha \in M$ a set X_α, and to each pair α, β such that $\alpha < \beta$ in M, a map

$$\pi_\alpha^\beta : X_\beta \to X_\alpha$$

such that

$$\pi_\alpha^\alpha = \text{identity}, \ \alpha \in M,$$

$$\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma, \ \alpha < \beta < \gamma \text{ in M}.$$

Definition. Let $\{G, \pi\}$ be a direct system over the directed set M where each G^α is an abelian group and each π_α^β is a homomorphism. Let $\Sigma \oplus G$ denote the direct sum of the groups of $\{G, \pi\}$.

For each $\alpha < \beta$ in M and each $g^\alpha \in G^\alpha$ the element

$$g^\alpha \pi_\alpha^\beta - g^\alpha$$

of $\Sigma \oplus G$ is called a relation. Let Q be the subgroup of $\Sigma \oplus G$ generated by all relations. The *direct limit* $\{G, \pi\}$ is the factor group

$$G^{\infty} = \Sigma \oplus G/Q$$

The natural map $\Sigma \oplus G \to G^{\infty}$ defines homomorphisms $\pi_{\alpha} : G^{\alpha} \to G^{\infty}$ called *projections.*

We can similarly define the *inverse limit* of an inverse system of groups. (For details see [1]).

6. Exact Sequences

It is well known in homological algebra that if the sequence

$$0 \to A' \to A \to A'' \to 0 \qquad \ldots \quad \ldots \quad \ldots \qquad (1)$$

is exact, then for any abelian group B the following sequences

$$0 \to A' * B \to A * B \to A'' * B \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0 \quad (2),$$

$$0 \to A'' \pitchfork B \to A \pitchfork B \to A' \pitchfork B \to A'' \dagger B \to A \dagger B \to A' \dagger B \to 0, \quad (3),$$

$$0 \to B \pitchfork A' \to B \pitchfork A \to B \pitchfork A'' \to B \dagger A' \to B \dagger A \to B \dagger A'' \to 0 \quad (4),$$

are exact (see [11].)

The exact sequences led to the study of other exact sequences in the theory of abelian groups. We call the exact sequence $A' \xrightarrow{\alpha} A \to A''$ *pure exact* if $A'\alpha$ is a pure subgroup of A (A$'$ is called a *pure subgroup* of A if, for each integer n, $n \mid a$ in A, $a \in A'$, implies that $n \mid a$ in A'). We can similarly define the group of *pure extentions* Pext (A, B) (We shall denote it by $A \varphi B$), Harrison [9] showed that if the sequence (1) is pure exact and B is any group then the following sequences

$$0 \to A'' \pitchfork B \to A \pitchfork B \to A' \pitchfork B \to A'' \varphi B \to A \varphi B \to A' \varphi B \to 0 \quad (5),$$

$$0 \to B \pitchfork A' \to B \pitchfork A \to B \pitchfork A'' \to B \varphi A' \to B \varphi A \to B \varphi A'' \to 0 \quad (6),$$

are again exact.

Fuchs [4] showed that if the sequence (1) is pure exact then the sequences

$$0 \rightarrow A' \otimes B \rightarrow A \otimes B \rightarrow A'' \otimes B \rightarrow 0 \quad \ldots \quad (7),$$

$$0 \rightarrow A' * B \rightarrow A * B \rightarrow A'' * B \rightarrow 0 \quad \ldots \quad (8),$$

$$0 \rightarrow B \pitchfork A' \rightarrow B \pitchfork A \rightarrow B \pitchfork A'' \quad \ldots \quad (9),$$

$$0 \rightarrow A'' \pitchfork B \rightarrow A \pitchfork B \rightarrow A' \pitchfork B \quad \ldots \quad (10),$$

$$B \dagger A' \rightarrow B \dagger A \rightarrow B \dagger A'' \rightarrow 0 \quad \ldots \quad (11),$$

$$A'' \dagger B \rightarrow A \dagger B \rightarrow A' \dagger B \rightarrow 0 \quad \ldots \quad (12),$$

are pure exact.

I have generalized these concepts (see [25]) and studied P-pure exact sequences and the group of P-pure extensions, where P is a family of primes. In the same paper I proved the theorem, stated below, which includes the case of pure exact sequences (7) and (8) as its special cases.

Theorem : If the sequence $0 \rightarrow A' \xrightarrow{\lambda} A \xrightarrow{\mu} A'' \rightarrow 0$ is pure exact and T is an additive covariant functor, commuting with the formation of direct limits, then the sequence

$$0 \rightarrow T(A') \xrightarrow{\lambda T} T(A) \xrightarrow{\mu T} T(A'') \rightarrow O \quad \ldots \quad (13)$$

is also pure exact.

For further generalizations we refer to the paper of Fuchs [8]

7. Duality

Duality plays a significant part in homological algebra. One may have already observed duality in the definitions given above, for example, image and co-image, kernel and cokernal, domain and co-domain, projective and injective, direct limits and inverse limits. Hom and Ext. are dual, in a sense, to tensor product and torsion product. Duality has also led to many new concepts in the theory of abelian groups.

7.1 Algebraically compact groups. The concept of an algebraically compact group which has a role, in a sense, dual to that of a direct sum of cyclic groups was introduced by Kaplansky in his book ([13] 'Infinite Abelian Groups'). A systematic theory of algebraically compact abelian groups is now available (see [4], [5], [7]).

It is indeed one of the remarkable results of the abelian group theory that any one of the following properties characterizes an algebraically compact group A.

(*i*) A is a direct summand of a group that admits a compact topology.

(*ii*) A is a direct summand of a complete direct sum (or direct product) of finite cyclic and quasi-cyclic groups.

(*iii*) A is a direct summand of every group that contain A as a pure subgroup.

(*iv*) A is pure injective (cf [16]).

(*v*) A is of the form $D \oplus B$ where D is a divisible group and B is Hausdorff and complete in its n-adic topology (see [16], [12])

(*vi*) A is of the form $A = D \oplus \sum_p {}^* A_p$, where D is a divisible group and A_p, for each prime p, is a p-adic module that is Hausdorff and complete in its p-adic topology.

(*vii*) If a system of linear equations over A has the property that every finite subsystem is solvable in A, the whole system has a solution in A.

7.2 Co-torsion groups. The theory of another important class of groups, called *co-torsion groups*, which are in a way, dual to torsion groups, was evolved by Harrison [9]. Co-torsion groups are characterized by the fact that their all extensions by torsion-free groups split. For example $A \dagger B$ is co-torsion. They have the following striking properties (see [5], [6]).

(*i*) a homomorphic image of a co-torsion group is co-torsion;

(*ii*) a subgroup H of a co-torsion group G is co-torsion if the factor group G/H is reduced ;

(*iii*) an extension of a co-torsion group by a co-torsion group is co-torsion ;

(*iv*) a complete direct sum of a family of groups is co-torsion if and only if every member is co-torsion ;

(*v*) a reduced co-torsion group G is algebraically compact if and only if its first Ulm subgroup $G^1 = \bigcap_n nG$ vanishes ;

(*vi*) Ulm subgroups of co-torsion groups are co-torsion and Ulm factors of co-torsion groups are algebraically compact;

(*vii*) a torsion or torsion-free co-torsion group is algebraically compact.

7.3. Cogenerators, Cocyclic and finitely cogenerated groups.

Let me add one more illustration of how duality gave rise to a new concept. It is well-known what a cyclic group is but perhaps it is not as well-known what a cocylic group is. This concept was first introduced by Maranda [16] . Let me explain it is some detail. We observe that a cyclic group is characterized by the property :

Let G be any group and ϕ any homomorphism from G to A, $\exists$ an element $a \in$ A such that if $a \in$ Im ϕ then ϕ is an epimorphism, a being called a generator of A.

In a heuristically dual sense, we call a group *cocyclic* if it has the following property :

Let G be any group and ϕ any homomorphism from A to G, $\exists$ an element $a \in$ A such that if a does not belong to Ker ϕ then ϕ is a monomorphism, a will be called a *co-generator* of A.

Thus a group A is cocyclic if $\exists\ a \in$ A, $a \neq 0$ such that every non-trivial subgroup of A contains a. Such groups are known to be of the type $Z_{p^k}(k \leqslant \infty)$, p a prime. We note that a finite cyclic p-group is self-dual.

Parallel to the theory of generaters of a group I have developed the theory of *cogenerators* of a group and introduced *finitely cogenerated* groups.

Let G be a group and let S be a subset of its non-zero elements such that every non-trivial subgroup of G intersects S non-vacuously, then we say that G is co-generated by S or S is a set of *co-generators* of G. If S is finite, then we call G a *finitely cogenerated* group.

We know that a finitely generated group is a direct sum of a finite number of cyclic groups and that its subgroups satisfy the maximum condition. We thus expect a finitely cogenerated group to be a direct product of a finite number of cocyclicgroups and expect its subgroups to satisfy the minimum condition. That this is true is shown by a theorem which we shall only state here.

Theorem. The following statements are equivalent :

(*i*) G is a finitely cogenerated group;

(*ii*) G is an essential extension of a finite group ;

(*iii*) G is a torsion group of finite rank ;

(*iv*) G is a direct product of a finite number of cocyclic groups ;

(*v*) the subgroups of G satisfy the minimum condition.

8. Structure of Tensor Product, Torsion Product, Hom and Ext.

Homological algebra has also contributed to the structural aspect of the theory of abelian groups. The structure of the tensor product of two p-groups and that of a p-group and a torsion-free group were first given by Fuchs in his famous book [2] (Abelian groups). It is shown there that if A and B are p-groups and G a torsion free group then $A \otimes B \cong U \otimes V$ where U and V are basic subgroups of A and B respectively, and that $A \otimes G \cong \sum_{\gamma} \oplus A$ where γ is the rank of G / pG.

U is called a *basic subgroup* of a p-group A if

(*i*) U is a direct sum of cyclic groups;

(*ii*) U is pure in A;

(*iii*) the factor group A / U is divisible.

Thus if A and B are torsion groups A⊗B is a direct sum of cyclic groups.

The structure of the torsion subgroup of A⊗B where A and B are any abelian groups, was described by Fuchs in [4] (see also [24]). Let A_t denote the torsion subgroup of a group A and A_f the torsion-free group A / A_t. Then the structure of the torsion subgroup $(A \otimes B)_t$ of A⊗B is given by

$$(A \otimes B)_t \cong \sum_p \oplus [U_p \otimes V_p \oplus \sum_{\gamma_b} \otimes A_p \oplus \sum_{\gamma_a} \oplus B_p],$$

where U_p, V_p are basic subgroups of the p-components A_p, B_p of A_t, B_t respectively and γ_a, γ_b denote the ranks of $A_f/p A_f$ and $B_f/p B_f$ respectively.

In fact, the structure of A ⊗ B when one of the groups is torsion is comyletely known (*cf*, [6], [24]). The structure of the tensor product of two torsion-free groups is not yet known.

For the structure of the torsion product A ∗ B we refer to Nunke's paper [19] (On the structure of Tor).

The structure of Hom (A, B) is not yet known in the general case when A, B are any groups. However, it is known in many particular cases. Fuchs has proved (see [3]) that the algebraic structure of the character group of a discrete abelian group A depends only on certain cardinal invariants of A. Pierce [20] has shown that Hom (A, B) can be completely described if A is a torsion group.

For the structure theory of the group of extensions, which is rather elaborate, we refer to [9], [10], [15], [17], [18], [27].

9. Generators of A $*$ B (Tor (A, B)).

We observe that most of the important functors in homological algebra have the remarkable property that they can be described by a canonical system of generators and relations. The first striking example in this direction is Eilenberg-Maclane's description of the functor Tor (A, B) (see [0]). They define T (A, B) to be the abelian group generated by the elements $(a, b)_h$ where $a \in A$, $b \in B$, and h is an integar such that $h\,a=o$, $h\,b=o$; these elements are subject to the relations :

(i) $(a+a', b)_h=(a, b)_h+(a', b)_h$, $h\,a=o$, $h\,a'=o$, $h\,b=o$;

(ii) $(a, b+b')_h=(a, b)_h+(a, b')_h$, $h\,a=o$, $h\,b=o$, $h\,b'=o$;

(iii) $(ka, \; b)_h=(a, b)_{kh}$, $k\,h\,a=o$, $h\,b=o$, k being an integer;

(iv) $(a, \; kb)_h=(a, b)_{kh}$, $h\,a=o$, $k\,h\,b=o$, k being an integer.

Then it can be shown that T (A, B) $\cong$ A $*$ B (defined as above). This description led to the study of new functors. For example, I have studied functors S^n_{γ} $(A_1, A_2, \ldots, A_n)$ (see [22]) which include Tor as a special case. I have also introduced functors $L^n_{\gamma, 1}$ $(A_1, A_2, \ldots, A_n)$ and discussed their properties (*c.f.* [23]).

Moreover, the isomorphism T (A, B) $\cong$ A $*$ B has also led to some problems in a different direction. We have defined A $*$ B as the kernel of the homomorphism $\bar{\lambda}$: R $\otimes$ B$\rightarrow$F $\otimes$ B, where $O\rightarrow R\xrightarrow{\lambda} F\xrightarrow{\mu} A\rightarrow O$ is a presentation of A. We note that the element $(a, b)_h$ of T (A, B) corresponds in this isomorphism to an element $r \otimes b$ (we shall call such an element a monomial) where $\gamma\,\lambda=h\,f, f \in F, f\,\mu=a$. We thus observe that the kernel of the homomorphism $\bar{\lambda}$: R$\otimes$B$\rightarrow$F$\otimes$B is generated by monomials. This observation gave rise to the following question; given monomorphisms ϕ: A'$\rightarrow$A, ψ : B'$\rightarrow$B, is the kernel of the homomorphism $\phi \otimes \psi$: A' $\otimes$ B'$\rightarrow$A $\otimes$ B generated by monomials? The question was answered by me in the negative by giving two counter-examples (see [21]).

Let me reproduce one of them here.

Let $A = Z(a) \oplus Z_{16}(a_1)$, $B = Z(b) \oplus Z_{16}(b_1)$,
$A' = Z(a') \oplus Z_4(a_1')$, $B' = Z(b') \oplus Z_4(b_1')$,

where $Z(a)$ denotes the infinite cyclic groups generated by a and $Z_{16}(a_1)$ the cyclic group of order 16 generated by a_1. The inclusions

$A' \subseteq A$, $B' \subseteq B$ are given by

$$a' = 4a + 2a_1,$$
$$b' = 4b + 2b_1,$$
$$a_1' = 4a_1,$$
$$b_1' = 4b_1.$$

One can easily verify that $a' \otimes b_1' + a_1' \otimes b'$ is an element of the kernel but it is not a linear combination of the monomials present in the kernel. In particular $a' \otimes b_1'$, $a_1' \otimes b'$ are not in the kernel.

Now the following questions, which are so simple in form, still remain to be settled.

(*i*) What is a necessary and sufficient condition that the kernel of $\phi \otimes \psi : A' \otimes B' \to A \otimes B$ may be generated by monomials?

(*ii*) What is a necessary and sufficient condition that $\phi \otimes \psi$: $A' \otimes B' \to A \otimes B$ may be a monomorphism?

10. Some Isomorphisms

We have two important isomorphisms in homological algebra of abelian groups (which can be derived from the Künneth formulae):

(*i*) $A \otimes (B * C) \oplus A * (B \otimes C) \cong (A \otimes B) * C \oplus (A * B) \otimes C$,

(*ii*) $(A \otimes B) \dagger C \oplus (A * B) \pitchfork C \cong A \pitchfork (B \dagger C) \oplus A \dagger (B \pitchfork C)$.

Many significant results have been obtained by applying these isomorphisms. But personally, I have been interested in the structure of the groups themselves that appear on each side of (*i*) and (*ii*) and thus examining these isomorphisms. I have shown the structure of each side of (*i*) [24], proving that

$$A \otimes (B * C) \oplus A * (B \otimes C)$$
$$\cong \sum_{p} \oplus [U_p \otimes (V_p * W_p) \oplus U_p * (V_p \otimes W_p) \oplus \sum_{\alpha_p} \oplus (V_p \otimes W_p)$$
$$\oplus \sum_{\beta_p} \oplus (W_p \otimes U_p) \oplus \sum_{\gamma_p} \oplus (U_p \otimes V_p) \oplus \sum_{r_p(A_f)} \oplus (B_p * C_p)$$
$$\oplus \sum_{r_p(B_f)} \oplus (C_p * A_p) \oplus \sum_{r_p(C_f)} \oplus (A_p * B_p)],$$
$$(A \otimes B) * C \oplus (A * B) \otimes C$$
$$\cong \sum_{p} \oplus [(U_p \otimes V_p) * W_p \oplus (U_p * V_p) \otimes W_p \oplus \sum_{\alpha_p} \oplus (V_p \otimes W_p)$$
$$\oplus \sum_{\beta_p} \oplus (W_p \otimes U_p) \oplus \sum_{\gamma_p} (U_p \otimes V_p) \oplus \sum_{r_p(A_f)} \oplus (B_p * C_p)$$
$$\oplus \sum_{r_p(B_f)} \oplus (C_p * A_p) \oplus \sum_{r_p(C_f)} \oplus (A_p * B_p)],$$

where U_p, V_p, W_p are basic subgroups of A_p, B_p, C_p, the p-components of the torsion subgroups A_t, B_t, C_t of A, B, C, respectively, $A_p/U_p \cong \sum_{\alpha_p} \oplus Z_{p^\infty}$, $B_p/V_p \cong \sum_{\beta_p} \oplus Z_{p^\infty}$, $C_p/W_p \cong \sum_{\gamma_p} \oplus Z_{p^\infty}$, and $r_p(A_f)$ = the rank of $A_f/p\,A_f$.

The structure of each side of (*ii*) is not yet known. I can give the structure in a significant special case when C is algebraically compact (see [26]). The difficulty in the general case when A, B, C are any groups is perhaps because of the fact that Hom, Ext and inverse limits are not strictly dual to tensor product, torsion product and direct limits. For example, every abelian group can be expressed as the direct limit of its finitely-generated subgroups (A finitely generated group is a direct sum of cyclic groups). No such result is known for inverse limits. I believe the best result available is: the inverse limit of reduced algebraically compact groups is again algebraically compact (C*f*. [7]). I still hope that it may be possible to find the structure of both the sides of (*ii*). Perhaps some relevant concepts of *co-pure* quotient gtoups and *co-basic* quotient groups or some modified inverse limits may have to be found. In this connection I would also like to point out that Maclane [14] introduced the functor Trip (A, B, C) which is isomorphic to either side of (*i*). Some one may be able to define a suitable functor co-trip (A, B, C) which would be isomorphic to each side of (*ii*).

I have necessarily omitted even to mention some very important results. But even if I have been able to suggest the significance of the impact of homological methods I should be satisfied.

REFERENCES

[0] Eilenberg, S., and S. Maclane: On the groups H (π, *n*) II, Ann, Math. 60 (1954), 49-139.

[1] Eilenberg, S. and N. Steenrod, Founditions of Algebraic Topology (Princeten, 1952).

[2] Fuchs, L. Abelian Groups, Publishing House of the Hungarian Academy of Sciences Budapest, 1958.

[3] Fuchs, L. On Character Groups of Discrete Abelian Groups, Acta Math. Acad. Sci. Hung. 10 (1959), 133-140.

[4] Fuchs, L. Notes on Abelian groups I, Annales Univ. Sci. Budapest, 2 (1959), 9-23.

[5] Fuchs, L. Notes on Abelian Groups II, Acta Math., Acad. Sci., Hung. 11 (1960), 117-125.

[6] Fuchs, L. Recent Results and Problems on Abelian Groups, Topics in Abelian groups, Chicago (1963).

[7] Fuchs, L. On Algebraically Compact Abelian Groups, Jour. Nat. Sci. and Math 3 (1963), 73-82.

[8] Fuchs, L. Some Generalizations of the Exact Sequences, Proceedings of the Colloq, on Abellian Groups, Budopest (1964).

[9] Harrison, D. K. Infinite Abelian groups and Homological Methods, Ann. Math. 69 (1959), 366-391.

[10] Harrison, D. K. On the Structure of Ext. Topics in Abelian groups, Chicago (1963).

[11] Hilton, P. J., and S. Wylie. Homology theory, Cambridge University Press (1960).

[12] Hulanicki, A. On Algebraically compact groups, Bull. Acad. Pol. Sci. 10 (1962), 71-75.

[13] Kaplansky, I. Infinite Abelian Groups (Ann. Arbor, 1954).

[14] Maclane, S. Triple Torsion Products and Multiple Künneth Formulas, Math. Ann. 140 (1969), 51-64.

[15] Maclane, S. Group Extensions by Primary Abelian Groups, Trans. Amer. Math. Soc. 95 (1960), 1-16.

[16] Maranda, J. M. On Pure Subgroups of Abelian groups, Archiv. d. Math. 11 (1960), 1-13.

[17] Nunke, R. J. Modules of Extensions over Dedekind Rings, Ill. J. Math. 3 (1959), 222-241.

[18] Nunke, R. J. On the extensions of a Torsion Module, Pac. J. Math. 10 (1960), 597-606.

[19] Nunke, R. J. On the structure of Tor, Proceedings of the Colloquium on Abelian Groups, Budapest (1964).

[20] Pierce. R. S. Homomorphisms of Primary Abelian Groups, Topics in Abelian Groups, Chicago (1963).

[21] Yahya, S. M. Kernel of the Homomorphism $A'\otimes B' \rightarrow A\otimes B$ Jour. Nat. Sci. and Math. 3 (1963), 139-145.

[22] Yahya, S. M. Functors $S_r^n(A_1, \ldots, A_n)$, Jour, Nat. Sci. and Math. 3 (1963), 41-56.

[23] Yahya, S. M. Functors $L_{r_1,}^n$ $(A_1, \ldots, A_n)$, Jour. Nat. Sci. and Math. 4 (1964), 133-141.

[24] Yahya, S. M. Structure of Maclane's group Trip (A_1, A_2, A_3) and its Generalizations, Jour. Nat. Sci. and Math. 4 (1964) 143-161.

[25] Yahya, S. M. P-pure Exact Sequences and the Group of P-pure Extensions, Anuales Univ. Sci. Budapest, 5 (1962), 179-191.

[26] Yahya, S. M. On isomorphism $(A \otimes B) \dagger C \oplus (A * B) ⋔ C \cong A ⋔ (B \dagger C) \oplus A \dagger (B ⋔ C)$, To appear in Jour. Nat. Sci. and Math.

[27] Zuravskii, V. S. On the Group of Abelian Extensions of Abelian Groups (Russian), Dokl. Akad. Nauk SSSR, 134 (1960), 29-32.

NOTE ON REDUCTION PROCEDURES

by

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1. Introduction

Reduction procedures such as those of Brown (1) and Melrose (2) relate the single loop Feynman diagrams of various collision amplitudes and, unless it can be proved that single loop diagrams are in some way typical of the perturbation series to which they belong, such procedures remain of rather academic interest. On the other hand if a reduction procedure which relates Feynman diagrams of different orders in the perturbation series for a given amplitude could be given then the analytic properties of the amplitude might be obtainable from the study of one or several "basic diagrams". Patashinski *et al* (3) have given such a procedure : suppose that z is a point of a Landau curve γ for a Feynman diagram of arbitrary complication whose internal masses m_i are given : then there exists a reduced diagram with the same configuration of external lines and no internal vertex whose internal masses are $M_j(m_i, z)$ such that the point z lies on the Landau curve Γ for the reduced diagram. This is a very weak result equivalent to saying that a complex electrical network of resistors can be replaced by a simpler one of the same effective resistance by inserting suitable resistors between the external terminals and eliminating all internal ones.

This reduction procedure has proved useful only when the M_j are independent of z.

2. Wigwam Singularities

The present authors (4) have studied the Wigwam diagram in order to investigate some properties of Landau curves previously discussed by one of us (5). The Wigwam diagram is the vertex graph specified by

$$p_1=q_2+q_4-q_5,\ p_2=q_1-q_2,\ p_3=q_3-q_4,\ 0=q_1+q_3-q_5 \tag{1}$$

where p_i denote the four-momentum of an external line ($p_i^2=z_i$) while q_i that of an internal line of mass m_i. The Landau curve $W(z_1, z_2, z_3)$ is a pair of quadric surfaces and this suggests that the Wigwam diagram might be simply related to Triangle diagrams specified by

$$P_1=Q_3-Q_2,\ P_2=Q_1-Q_3,\ P_3=Q_2-Q_1 \tag{2}$$

in an obvious notation. One might hope to find a constant set of M_j values on each quadric. If we fix z_2 and z_3 we obtain four points of γ which we denote by $z_{1,k}$, $k=1,2,3,4$. If we apply the reduction procedure to find masses M_j $(m_i, z_{1,k}, z_2, z_3)$ for Γ^1 we find that the M_j depend on both z_2 and z_3. Further if we compute the points of Γ which correspond to the masses M_j and the given values of z_2 and z_3 we obtain two values of z_1 namely $z_{1,k}$ where $z_1{}^1{}_{,k}$ does not coincide with any point z_{1t}, $t=1,2,3,4$. We thus verify the truth of the theorem of Patashinski *et al* and convince ourselves that it is totally irrelevant to the problem of finding "basic diagrams".

3. Result

If the value of z_3 is fixed then the Wigwam Landau curve is a pair of conics which coincide with the Landau curves belonging to Triangle graphs having the same value of z_3 and with masses

$$M_1=m_1 \tag{3}$$

$$M_3=m_2 \tag{4}$$

$$M_2=\sqrt{\left\{m_4^2+m_5^2-2c(m_1+a)+2J(z_3)\sqrt{(c^2-m_4^2)}\sqrt{(a^2-m_3^2)}\right\}} \tag{5}$$

where $a=(m_5^2-m_1^2-m_3^2)/(2m_1)$ (6)

and c satisfies

$$m_3^2\, c^2+c\; a\lambda+\tfrac{1}{4}\lambda^2+a^2m_4^2-m_3^2m_4^2=0 \tag{7}$$

with $\lambda=z_3-m_3^2-m_4^2$,

where J (z_3) has the value $+1$ for some values of z_3 and value -1 for others. The masses M_1, M_2, M_3 are different from those given by the procedure of Patashinski *et al.*

4. Conclusion

There is therefore a suggestion that the location of Triangle singularities is relevant to the location of Wigwam singularities. Possibly the Triangle graph is a "basic diagram" for locating vertex singularties : perhaps by means of suitable sums (or integrales) over the masses M_i the whole perturbation series (or a significant part of it) can be generated ?

References

1. Brown, L. M. (1961). Nuovo Cimento, **22**, 178.
2. Melrose, D. B. (1965). Nuovo Cimento, **40**, 181.
3. Patashinki, A. Z., Rudik, A. P., Sudakov, V. V. (1961). J.E.T.P., **40**, 298.
4. Cunningham, J., Rafique, M. (1967). Nuclear Physics, **B1**, 21.
5. Cunningham, J. (1964). Rev. Mod. Phys., **36**, 833.

TOTAL ABSOLUTE CURVATURE OF $M_1 \# M_2$ (1)

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1. Introduction

Let M be a compact, connected, C^{∞}-manifold of dimension n. Let

$$f: M \longrightarrow E^{n+N}, \ (N \geqslant 1)$$

be a C^{∞}-immersion of M into euclidean space E^{n+N} of dimension $n+N$. Let B_{ν} be the bundle of unit normals on M induced by f and S_o^{n+N-1} be the sphere of unit vectors in E^{n+N}. Let

$$\widetilde{V}: B_{\nu} \longrightarrow S_o^{n+N-1}$$

be the canonical map given by

$$\widetilde{V}[p, \nu(p)] = E[\nu(p)]$$

where E is the end-point map which translates unit normal vector $\nu(p)$ to the origin and identifies its end-point with a point on S_o^{n+N-1}. Let dV, $d\sigma_{N-1}$ and $d\Sigma_{n+N-1}$ denote the volume elements of M, the fibre S^{N-1} of B_{ν}, and S_o^{n+N-1} respectively. Then we have

(1) $(\widetilde{V})^*(d\Sigma_{n+N-1}) = G(p, \nu(p))\, dV \wedge d\sigma_{N-1}$.

The scalar factor $G(p, \nu(p))$ is known as the

Lipschitz-Killing curvature of M at p in the direction of $\nu(p)$ [4]. We note that $G(p, \nu(p)) = 0$ where the rank of $\widetilde{V} < n+N-1$.

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(1) This paper is a part of the author's doctoral dissertation submitted to the University of Liverpool in January, 1966.

(2) The author is greatly indebted to his supervisor Professor T. J. Willmore for introducing him to modern Differential Geometry.

Following S. S. Chern [1] we define the *total absolute curvature of M at p* by

$$(2)\quad K^*(p)=\int_{S^{N-1}} |\, G\,(p, \nu(p))\,|\, d\sigma_{N-1}$$

where S^{N-1} is the fibre at $p \in M$. The total absolute curvature τ_f of M is then given by

$$(3)\quad \tau_f=\frac{1}{C_{n+N-1}}\int_M K^*(p)\, dV$$

where $C_{n+N-1}=\dfrac{2\pi^{\frac{n+N}{2}}}{\Gamma\left(\frac{n+N}{2}\right)}$ is the area of the sphere $S_\circ^{n+N-1}$.

2. Connected sum $M_1 \# M_2$

Let M_1, M_2 be two compact, connected $\overset{\infty}{C}$-manifolds of the same dimension n. Then the connected sum $M_1 \# M_2$ is a manifold obtained by removing an n-cell from each of the two, and then piecing the two manifolds together along the resulting boundaries.

A $\overset{\infty}{C}$-structure can be constructed on $M_1 \# M_2$ from the $\overset{\infty}{C}$-structures of M_1 and M_2 [7].

Let $1, \alpha_1, \ldots \alpha_{n-1}, 1$ and $1, \beta_1, \beta_2, \ldots \beta_{n-1}, 1$ be the Betti numbers of M_1 and M_2 respectively. Then it follows from Myer-Vietoris theorem [3] that the Betti numbers of $M_1 \# M_2$ are $1, \alpha_1+\beta_1, \ldots, \alpha_{n-1}+\beta_{n-1}, 1$. Moreover, if χ is the Euler-Poincare characteristic operator, then

$$(4)\quad \chi(M_1 \# M_2)=\chi(M_1)+\chi(M_2)-2.$$

Let C(M) denote the minimum number of critical points a non-degenerate, real-valued, $\overset{\infty}{C}$-function can have on M. An immersion

$$f: M \rightarrow E^{n+N}$$

is called a *minimal immersion* if

(5) $\tau_f = C(M)$.

It follows from Morse-inequalities [6] that

(6) $\tau_f = C(M) \geqslant \sum_{i=0}^{n} \gamma_i \, (M)$

where γ_i is the ith Betti number of M.

Let τ denote the total absolute curvature.

Then it follows that

(7) $\tau(M_1) \geqslant \sum_{i=0}^{n} \alpha_i$,

(8) $\tau(M_2) \geqslant \sum_{i=0}^{n} \beta_i$,

and

(9) $\tau \, (M_1 \# M_2) \geqslant \sum_{i=0}^{n} \alpha_i + \sum_{i=0}^{n} \beta_i - 2$.

In the special case of $n=2$, we have

(10) $\tau \, (M) \geqslant \gamma_0 + \gamma_1 + \gamma_2 = 2 + \gamma_1$.

Also

(11) $\chi(M) = \gamma_0 - \gamma_1 + \gamma_2 = 2 - \gamma_1$.

Hence we have

(12) $\tau(M) \geqslant 4 - \chi(M)$.

Definition : An immersion of a 2-*manifold M is said to be minimal if it gives.*

(13) $\tau \, (M) = 4 - \chi(M)$.

Let τ_1, τ_2 be the minimal total absolute curvatures of M_1 and M_2 respectively. Then we have

$$\tau_1 = 4 - \chi(M_1)$$

and $$\tau_2 = 4 - \chi(M_2).$$

Hence it follows from (13) that

$$\tau\,(M_1\#M_2) \geqslant 4-\chi(M_1\#M_2)$$
$$=4-\chi(M_1)-\chi(M_2)+2$$
$$=6-(4-\tau_1)-(4-\tau_2).$$

Hence

(14) $\tau\,(M_1\#M_2) \geqslant \tau_1+\tau_2-2.$

2·1. Special cases

By means of formula (14), we can calculate the total absolute curvatures of various special manifolds as shown below :—

(*a*) Let $M_1=S^n=M_2$. Then
$\tau\,(S^n\#S^n) \geqslant 2+2-2=2.$

In general
$\tau\,(S^n\#\ldots\ldots\#S^n\ (m \text{ times})) \geqslant 2m-2\,(m-1)=2.$

(*b*) Let $M_1=S^2$ and $M_2=$2-dimensional torus T^2.

Then

$$\tau\,(S^2\#T^2) \geqslant 2+4-2=4.$$

In general, the total absolute curvature of a 2-sphere S^2 with m handles is given by :

$$\tau\,(S^2\#(T^2\#\ldots\ldots\#T^2)) \geqslant 2+4m-2m=2\,(m+1).$$

(*c*) Let $M_1=T^2$ and $M_2=T^2\#T^2\ldots\ldots m$ times. Then
$\tau\,(T^2\#\ldots\ldots(m+1) \text{ times}) \geqslant 4\,(m+1)-2m=2\,(m+2).$

(*d*) The Betti numbers of the 3-dimensional torus $S^1\times S^1\times S^1$ and the manifold $S^1\times S^2\#S^1\times S^2\#S^1\times S^2$ are the same, namely, (1, 3, 3, 1). It follows from [8] that

$$\tau\,(S^1\times S^1\times S^1)=\tau(S).\quad \tau(S).\quad \tau(S)=2.2.2=8.$$

Also from (14) we have

$$\tau(S^1\times S^2\#S^1\times S^2\#S^1\times S^2) \geqslant \tau(S^1\times S^2)+\tau(S^1\times S^2)+\tau(S^1\times S^2)-4$$
$$=4+4+4-4=8.$$

3. Convex extensions

Let $f:\ M\to E^{n+N}$ be a $\overset{\infty}{C}$-immersion where $M=$(single point). Then it is easy to see that $\tau_f=1$. From this we conclude that the minimum decrease in the total absolute curvature of a manifold, when an n-cell is removed, is 1. On the other hand, if $g' : M - D^n \rightarrow E^{n+N}$ is a $\overset{\infty}{C}$-immersion, then the minimum increase in the total absolute curvature of g' is 1 when g' is extended to $g : M \rightarrow E^{n+N}$. *Such an extension of g' is called a convex extension.*

Let M_1 and M_2 be compact, connected, $\overset{\infty}{C}$-manifolds of the same dimension n and $M_1 \# M_2$ be their connected sum. Let f: $M_1 \# M_2 \rightarrow E^{n+N}$ be a $\overset{\infty}{C}$-immersion of $M_1 \# M_2$ into E^{n+N}. Let $f \mid M_1 - C^n$ and $f \mid M_2 - D^n$ be denoted by f'_1 and f'_2 respectively. Then we can prove the following :

Theorem (3.1). *Let τ_1 and τ_2 be the minimal total absolute curvatures of M_1 and M_2 respectively. Let $f : M_1 \# M_2 \rightarrow E^{n+N}$ be a minimal immersion such that f'_1, f'_2 admit convex extensions. Then*

(15) $\tau_F (M_1 \# M_2) \geqslant \tau_1 + \tau_2 - 2.$

for all $\overset{\infty}{C}$-immersions F: $M_1 \# M_2 \rightarrow E^{n+N}$.

Proof. Let $B_\nu (M_1 \# M_2)$ denote the bundle of unit normals induced on $M_1 \# M_2$ by f. Then, from (2) and (3), we have

$$(16) \quad \tau_f = \frac{1}{C_{n+N-1}} \int_{B_\nu(M_1 \# M_2)} \left| \left(\overline{V}\right)^* \left(d\Sigma_{n+N-1}\right) \right|$$

$$\geqslant \frac{1}{C_{n+N-1}} \int_{B_\nu(M_1 - D^n)} \left| \left(\overline{V}\right)^* \left(d\Sigma_{n+N-1}\right) \right|$$

$$+ \frac{1}{C_{n+N-1}} \int_{B_\nu(M_2 - D^n)} \left| \left(\overline{V}\right)^* \left(d\Sigma_{n+N-1}\right) \right| .$$

or $\tau f \geqslant \tau f_1' + \tau f_2'$.

Let $f_1 : M_1 \longrightarrow E^{n+N}$

and $f_2 : M_2 \longrightarrow E^{n+N}$

be the convex extensions of f_1' and f_2' respectively.

Then, from (16), we have

$$(17) \quad \tau_f \geqslant \tau_{f_1'} + \tau_{f_2'} = (\tau_{f_1} - 1) + (\tau_{f_2} - 2)$$
$$= \tau_{f_1} + \tau_{f_2} - 2 \geqslant \tau_1 + \tau_2 - 2.$$

Since f is minimal, we have

$$\tau_F \geqslant \tau_f$$

for all $F : M_1 \# M_2 \longrightarrow E^{n+N}$. Hence, from (17), we have

$$\tau_F \geqslant \tau_1 + \tau_2 - 2.$$

We hope to consider the following problem in our future research work.

Problem. In the above notation, what is a necessary and sufficient condition that f_1' and f_2' admit convex extensions ?

References

1. Chern, S. S. (1955). La g'eometric des sousvariete's d'un espace euclidean a plusieurs dimensions. L' Enseignement Mathematique, 40.
2. Chern, S. S. and Lashof, R. K. (1957) (1958). On the total curvature of immersed manifolds. I. Amer. Jr. of Maths. LXXIX. II. Mich. Math. Jr. 5.
3. Hilton, P. and Wiley, S. (1960). Homology Theory, Oxford University Press.
4. Fenchel, W. (1940). On the total curvature of riemannian manifolds. Jr. London Math. SOC. 15.
5. Kuiper, N. H. (1960). On surfaces in E^3. Bull. SOC. Math. de Belg. XII.
6. Milnor, J. W. (1963). Morse Theory Annals of Mathematics studies (51), Princeton University Press, Princeton, N. J.
7. Seifert, H. (1931). Konstruction dreidimensionaler geschlossener Raume. Ber. Verh. Sachs. Akad. Wiss. Leipzig.
8. Willmore, T. J. and Saleemi, B. A. (1966). The total absolute curvature of immersed manifolds. Jr. London Math. SOC. 41.

A PROBABILITY INEQUALITY

by

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In this paper we present a somewhat isolated result of Probability Theory which illustrates the power of simple results like Hölders inequality.

Let X_i, $i=1, \ldots, n$ be random variables and $S_i=\sum_{j=1}^{i} X_j$ the sequence of partial sums. Take t real and denote mathematical expectation by E. Write v_i^t for $E(|X_i|^t)$ when this exists and define the random variable $S^t=\sum_{i=1}^{n} X_i |S_i|^t$. We prove the following:-

(1) *Theorem: Suppose* v_i^{t+1} *exists,* $i=1, \ldots, n$. *Then* S^t *has expectation and*

$$E(S^t) \leqslant \frac{(n+1)^t}{(t+1)} \sum_{i=1}^{n} v_i^{t+1}$$

if $t \geqslant 1$.

To establish this result we prove a lemma which is of independent interest.

(2) *Lemma: Let* $x_i, i=1, \ldots, n$ *be a sequence of real numbers and put* $s_i=\sum_{j=1}^{i} x_j$, $i=1, \ldots, n$. *Then*

$$s^t = \sum_{i=1}^{n} x_i |s_i|^t \leqslant \frac{(n+1)^t}{(t+1)} \sum_{i=1}^{n} |x_i|^{t+1}$$

if $t \geqslant 1$.

Proof: Take $t \geqslant 1$. When $n=1$, (2) is true because $2^t \geqslant t+1$. Suppose (2) is true for $n=m$. Then by assumption

$$(3)\ s^t = \sum_{i=1}^{m} x_i \mid s_i \mid^t + x_{m+1} \mid s_{m+1} \mid^t$$

$$\leqslant \frac{(m+1)^t}{(t+1)} \sum_{i=1}^{m} \mid x_i \mid^{t+1} + x_{m+1} \mid s_{m+1} \mid^t.$$

By Hölders inequality ([1], p 19),

$$(4)\ \mid s_{m+1} \mid^t \leqslant (m+1)^{t-1} \sum_{i=1}^{m+1} \mid x_i \mid^t$$

which, together with (3), shows that

$$(5)\ s^t \leqslant \frac{(m+1)^t}{(t+1)} \sum_{i=1}^{m} \mid x_i \mid^{t+1} + (m+1)^{t-1} \left[\mid x_{m+1} \mid^{t+1} + \sum_{i=1}^{m} \mid x_{m+1} \mid x_i \mid^t \right].$$

It is well known, ([1], p 15), that for a, $b \geqslant 0$, $u > 1$, $v = u/(u-1)$,

$$\frac{1}{a^u} \frac{1}{b^v} \leqslant \frac{a}{u} + \frac{b}{v}.$$

Put $a = \mid x_i \mid^{t+1}$, $b = \mid x_{m+1} \mid^{t+1}$, $u = 1 + 1/t$ to see that for $i = 1, \ldots, n$,

$$(6)\ \mid x_i \mid^t \mid x_{m+1} \mid \leqslant \frac{t}{t+1} \mid x_i \mid^{t+1} + \frac{1}{t+1} \mid x_{m+1} \mid^{t+1}.$$

Using (6) in (5) shows, after simplification that

$$(7)\ s^t \leqslant [(m+1)^t + t(m+1)^{t-1}] \left[\frac{1}{t+1} \sum_{i=1}^{m+1} \mid x_i \mid^{t+1} \right].$$

Since $(m+1)^t + t(m+1)^{t-1} \leqslant (m+2)^t$ for $t \geqslant 1$ by the binomial theorem, (7) shows that (2) is true for $n = m+1$, and thus for all integer n, which completes the proof.

(2) can be false if $t < 1$ as can be seen by taking $n = 1$, $x_1 \geqslant 0$, $0 < t < 1$, because $2^t < t+1$.

The theorem is proved from (2) by noticing that

$$(8)\ s^t \leqslant \frac{(n+1)^t}{t+1} \sum_{i=1}^{n} \mid X_i \mid^{t+1}$$

at all points of the relevant probability space for which $| X_i |^{t+1}$ is real, whereas (8) holds automatically when $| X_i |^{t+1}$ is infinite. Therefore expectations can be taken in (8) without reversing the inequality.

(1) need not hold for $t<1$ as is illustrated by taking $n=1$, $0<t<1$ and letting X_1 be rectangularly distributed in the unit interval.

(9) *Remark* : (1) is a useful inequality. Take $| X_i | \leqslant 1$, $i=1,\ldots .n$, $t \geqslant 1$. Then $E(S^t) \leqslant \sum_{i=1}^{n} i^t$. Applying (2) shows

$$(10)\quad E(S^t) \leqslant \frac{(n+1)^t}{t+1} \sum_{i=1}^{n} 1 = \frac{n(n+1)^t}{(t+1)}$$

whereas by simply comparing the sum with $\int_1^{n+1} x^t \, dx$

$$(11)\quad E(S\ ) \leqslant ((n+1)^{\ +1} - 1)/(t+1)$$

which is not as good an estimate of E(S) as (10).

Finally we note that (2) is analogous to an inequality of Bellman [2] for integrals of periodic functions which is given in discrete form in [1], *p* 184. This work has application to numerical integration of ordinary differential equations.

References

1. Beckenbach, E.F. and Bellman, R. (1961). *Inequalities*. Springer-Verlag, Berlin.
2. Bellman, R. (1939). A note on periodic functions and their derivatives. *J. London Math. Soc.*, **14**, 140-142.

VALUES OF SYMMETRIC FUNCTIONS

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§ 1: The sum of the $\binom{n}{j}$ products of $(x_1, \ldots, x_n)$ taken j at a time without repetition of any x_i in a product is called the elementary symmetric function of degree j in the arguments $x_1, x_2, \ldots, x_n$ and is often denoted by S_j, the arguments being tacit. In general, a symmetric function is a function of several variables which remains unaltered when any two of the variables are interchanged. In the theory of equations, one learns some properties of symmetric functions, but their arithmetical properties do not seem to have been investigated. Recently Birch [1] while discussing the sums of the d^{th} powers in p-adic field points out that symmetric functions are useful for the determination of sums of d^{th} powers in any field. In his paper he proves.

Theorem 1: Given a set x of integers of p-adic field K we can find a set v consisting of at most d^{16d^2} integers such that

$$S_i(x) = S_i(v) \text{ for } i = 1, 2, \ldots, d.$$

Combining this result with the fact that there exist polynomials F_j with rational integer co-efficients such that $t_j = F_j(S_1, \ldots, S_j)$ identically, he deduces that every element of K expressible as a sum of d^{th} powers can be expressed as a sum of at most d^{16d^2} such d^{th} powers. He also proves theorem 1 when K is a finite field of p^f elements, with v containing $\frac{1}{2}(5^d - 1)$ elements. But in none of the fields has he identified the set of possible values for the first d symmetric functions. Our object is to identify this set. We also wish to find how many variables may be necessary in the set x for $S_1(x), \ldots, S_d(x)$ to take a given d-tuple of values in this set.

Let R be any field ; let R^d be the set of d-tuples of R and let R^{∞} be the set of infinite sequences of elements of R with only finitely many non-zero terms. By taking the d-tuples of R^d as the first d terms of sequences of R^{∞} we have natural maps.

$$R^{\infty} \longrightarrow \ldots . \longrightarrow R^d \longrightarrow R^{d-1} \longrightarrow \ldots . \longrightarrow R^1.$$

We give R^{∞} an additive group structure by defining for two sequences $\{a_i\}$, $\{b_j\}$ of R^{∞},

$$\{a_i\} \oplus \{b_j\} = \{c_k\}$$

with $c_k = a_k + \sum_{j=1}^{k-1} a_j b_{k-j} + b_k$. Since the power series with only finitely many non-zero terms are really polynomials this corresponds with formal multiplication of the power series

$$\left(1+\sum_{i=1}^{\infty} a_i x^i\right)\left(1+\sum_{j=1}^{\infty} b_j x^j\right)=\left(1+\sum_{k=1}^{\infty} c_k x^k\right)$$

If $x \in R^n$ and j is any positive integer, then, as usual, let $S_j(x)$ denote the elementary symmetric function of weight j in x. Write $S(x)=(S_0, S_1, \ldots .)$. Here, it is convenient to take $S_0=1$, so that if x, v are two sets of elements, $S(x) \oplus S(v) = S(x, v)$. Thus if Σ denotes the set of all sequences of R which can occur as values for the symmetric functions of a set x of elements of R, then Σ is an additive sub-semi-group of R^{∞}. Write $\Sigma(d)$ for the set of values taken by $[S_1(x), \ldots ., S_d(x)]$; there are obvious homomorphisms

$$\Sigma \longrightarrow \Sigma(d) \longrightarrow \Sigma(d-1) \longrightarrow \ldots \ldots$$

We wish to identify $\Sigma(d)$; and if possible we wish to show that for each element $(\sigma_1, \ldots . \sigma_d) \in \Sigma(d)$ we can find a relatively small set of elements x with

$$S_i(x) = \sigma_i \text{ for } i=1, 2, \ldots ., d.$$

We will have made a good start on our problem if we can exhibit a system of generators for Σ.

§ 2: Presently we consider symmetric functions over a finite field $K=k_{pf}$ with p^f elements. Our main object is to identify the set $\Sigma(d)$

of values taken by the first d symmetric functions ; we also wish to find how many variables $x_1, \ldots x_n$ may be necessary for $S_1(x), \ldots S_d(x)$ to take a given d-tuple of values in $\Sigma(d)$.

Our identification is in two parts. In section 2.1 we solve the rather easier problem of identifying the values taken by sums of powers. Having done this, in the section 2.2 we apply the results to the identification of symmetric functions; the main extra difficulty occurs when we consider $S_j(x)$ for j divisible by p. We obtain rather good estimates for the number of variables necessary in terms of p, d and f; unfortunately we have not been able to find a good estimate depending on d alone. We cannot even improve appreciably in the $\frac{1}{4}(5^d - 1)$ given by Birch [1]. In fact, the task of determining the number of variables necessary is not so easy as it seems. For example in the congruence field mod 5, $S_1(x) \equiv S_2(x) \equiv 0 \pmod 5$ and $S_3(x)$ not congruent to 0 (mod 5) cannot be obtained with less than six elements in x.

§ 2.1: If x is a set of variables, we have written $t_j(x)$ for the sums of the j^{th} powers of the x's. Let $k = k_p$ be the prime field with p elements, so that K is the extension of k of degree f. The Galois group of K over k is cyclic of order f, generated by the Frobonius automorphism $\omega : x \rightarrow x^p$.

Theorem 2.11.

Ler $b = (b_1, \ldots b_d)$ be a set of d elements of K satisfying $b_m = b_n^p$ whenever $m \equiv n\,p \pmod{p^f - 1}$ .. (1)

then we can find a set of x of at most

$$\min\left(f\,d\,(p-1),\ (p^f - 1)(p-1)\right)$$

elements of K such that

$$t_n(x) = b_n \text{ for } n = 1, 2, \ldots, d \qquad .. (2)$$

Conversely, if x is any set of elements of K, then

$$t_m(x) = (t_n(x))^p \text{ whenever } m \equiv n\,p \pmod{p^f - 1} \qquad .. (3)$$

Proof : We prove the last bit first.
Suppose $m \equiv n\,p \pmod{p^f - 1}$

Then $t_n = \Sigma\, x_s^n$, so

$$(t_n(x))^p = (\Sigma\, x_s^n)^p = \Sigma\, x_s^{np} = \Sigma\, x_s^m = t_m(x)$$

by the binomial theorem and since $x^{p^f} = x$ for all $x \in K$.

Now for the main part of the theorem; suppose that $b_1, \ldots . b_d$ satisfy (1). Write Q short for $p^f - 1$, and let $x_1, \ldots . x_Q$ be an enumeration of the non-zero elements of K. Consider the set of linear equations.

$$\sum_{s=1}^{Q} r_s\, x_s^m = b_m \text{ for } m = 1, \ldots . d; \qquad (4)$$

we have $b_m = b_n$ for $m \equiv n \pmod{p^f - 1}$, and

$$\det(x_s^m) = \Pi\,(x_s - x_t) \neq 0,$$

$$m, s = 1, \ldots . Q$$

so the equations are certainly consistent, so we can solve (4) for $r_1, \ldots . r_Q$ in K. Unfortunately, this is not good enough; we want (4) to be soluble with $r_1, \ldots, r_Q$ in the prime field k; that is, we want $r_1, \ldots, r_Q$ to be left invariant by G_K/k.

So consider instead the larger set of equations

$$\sum_{s=1}^{Q} r_s\, x_s^{mp^t} = b_m^{p^t} \text{ for } m = 1, \ldots, d$$

$$t = 0, \ldots, f-1; \qquad \ldots (5)$$

By (1), no more than Q of the equations are different and they remain consistent, so they are certainly soluble for $r_1, \ldots, r_Q$ in K; and since the set of equations (5) is left invariant by G_K/k, we can find $r_1, \ldots, r_Q$ in k.

So each of $r_1, \ldots . r_Q$ is represented by an integer between 0 and $p-1$; take x as the set of elements of K consisting of x_1 repeated g_1 times, $\ldots, x_Q$ repeated g_Q times; then

$$t_m(x) = \Sigma g_s\, x_s^m = b_m \text{ for } m = 1, \ldots, d$$

as required. We note that the set x contains at most $(p-1)\,Q=(p-1)(p^f-1)$ elements.

If $f\,d<p^f-1$, there are only $f\,d$ equations in the set (5); even those need not be independent. Suppose there are F independent equations in the system (5), say

$$\sum_{s=1}^{Q} r_x\, x_s{}^{m}k = b'_k \text{ for } k=1, \ldots, F.$$

Then we can select $y_1, \ldots, y_F$ from $x_1, \ldots, x_Q$ so that

$$\det_{k,j=1,\ldots,F} y_j^{m_k} \neq 0 \,;$$

and then we can solve

$$\sum_{j=1}^{F} r_j\, y_j^{m}k = b'_k \text{ for } k=1, \ldots, F$$

with $r_1, \ldots, r_F$ in k. We can thus find a set x of at most $(p-1)\,F<(p-1)\,f\,d$ elements of K with $t_m\,(x)=b_m$ for $m=1, 2, \ldots, d$.

This concludes the proof of the theorem.

Our theorem, of course, enables us to identify the values of a single sum of m^{th} powers. Given m, let r bs the least positive integer such that $mp^r=m \pmod{p^f-1}$; so that p^r is the least power of p such that $p^f-1 \mid m\,(p^r-1)$. Then by our theorem we can find a set x of elements of K such $t_m\,(x)=b_m$ if and only if $b_m=b_m^{p^r}$. In general $f=r$, and this is no restriction; but, for example, an element of k_q is sum of 4^{th} powers if and only if it lies in the prime sub field k_3. In particular, if m is equal to a prime number q, then from our theorem we can find $t_q\,(x)=b_q$ if and only if $b_q=b_q^{p^r}$ where p^r is the least power of p such that $(p^r-1) \mid q\,(p^r-1)$. If q is not of the form $\dfrac{p^f-1}{p^r-1}$ then $f=r$ and $b_q=b_q^{p^f}$ for all elements of K. Therefore every element of K can be expressed as sum of q^{th} powers. But if q is a prime of the form $\frac{p^f-1}{p^r-1}$ then only those elements can be expressed a sum of q^{th} powers which satisfy the equation $x^{p^r}=x$ and hence the q^{th} powers form a subfield of p^r elements. We deduce.

Lemma 2.1.

Suppose q is a prime. Then every element of K is expressible as sum of q^{th} powers of elements of K unless $q=\frac{p^f-1}{p^r-1}$ for some divisor r of f in which special case the q^{th} powers form a subfield of p^r elements.

Hence from our theorem we have deduced a result of Tornhein [2], and theorem 1 of Bateman and Stemmler [3]. From our theorem we can also deduce the following result of Bateman and Stemmler [3, Theorem, 3]. Let K be the set of integers of an algebraic number field and suppose that q is prime. If q is expressible in the form p^f-1/p^d-1, where p is prime and f and d are positive integers, and p has in K a prime ideal factor of degree f, then some element of K is not a sum of q^{th} powers. The reason is that if P is a prime ideal in K of degree f which divides p, then the finite field with NP elements falls under the exceptional case of our lemma 2.1. Thus by lemma 2.1 not all residue classes module P contain sums of q^{th} powers. Therefore, the set of elements of K expressible as sum of q^{th} powers is properly contained in K. In tnis case, Bateman and stemmler found it easy to prove the converse, namely that if every element of K is not expressible as sum of q^{th} powers, then either q is ramified or q is expressible in the form p^f-1/p^d-1 where p is a prime and f and d are positive integers, and p has in K a prime ideal factor of degree f. Of course, we can hardly hope to obtain a good estimate for the number of variables for representation by a single sum of powers by means of a theory designed for simultaneous respresentation by several sum of powers.

Before we pass on, we remark that if $d=p^f-1=Q$, then the equations (4) must be left invariant by $G_{K/k}$ if they are to be soluble with r_1, ...,

$r_d \in k$; and the determinant det x_s^m of the coefficients on the

$$m, s=1,\ldots d.$$

left hand side is non-zero, so there is one to one correspondence between d-tuples $(r_1, \ldots. r_d)$ of elements of k^d and possible values $b_1, \ldots. b_d$ for $t_1, \ldots, t_d$, In particular, there is one particular d-tuple $(b_1^*, b_2^*, \ldots\ b_d^*)$ for which $r_1=r_2 \ldots =r_d=-1$; and then to solve $t_m(x)=b_m^*$ $m=1, \ldots, d$ the set x must contain each of $x_1, \ldots, x_d$ repeated at least $(p-1)$ times, accordingly our estimate $(p-1)(p^f-1)$ is the best possible. For instance, we ean only solve $t_m(x)=0$, $m=1, \ldots, 7$, $t_8(x)=1$ by taking x as the non-zero elements of k_9 each repeated at least twice.

§ 2.2. In this present section we want to identify $\Sigma(d)$ when $R=K$, the field with p^f elements.

Write $\Sigma^{(p)}$ for the subgroup of Σ consisting of sequences $\{\sigma_i\}$ with $\sigma_i=0$ for $(i, p)=1$. More generally for $r \geqslant 1$, let $\Sigma^{(p^r)}$ be the subgroup of Σ consisting of sequences $\{\sigma_i\}$ with $\sigma_i=0$ unless $i \equiv 0 \pmod{p^r}$; in particular $\Sigma^{(1)}$ is 1.

The following is well known [see for example (4)]

Lemma 2.21. (Newton's Formula)

$$t_K - t_{K-1} S_1 + t_{K-2} S_2 + \ldots + (-1)^K K s_K = 0$$

identically.

Thus there are polynomials with rational integer co-efficients such that $t_j = F_j(S_1, \ldots, S_j)$ identically. Conversely if $j<p$, there are polynomials F_j whose co-efficients are units mod p such that

$$S_j = F_j'(t_1, \ldots, t_j) \text{ identically.}$$

Suppose $d<p$ and that $(a_1, \ldots, a_d)$ is any d-tuple of K. Write $b_i=F_i(a_1, \ldots, a_i)$ for $i=1, 2, \ldots, d$. By theorem 2.11 we can determine a set x of atmost $f\,d\,(p-1)$ elements of K such that $t_i(x)=b_i$

for $i=1, 2, \ldots, d$. The restriction $b_m = b_n^p$ whenever $m \equiv np$ (mod p^f-1) is not relevant if $d<p$. Thus we can determine a set such that $t_i(x)=F_i(a_1, \ldots, a_i)$ for $i=1, 2, \ldots, d$; which implies that we can find a set x such that $S_i(x)=a_i$ for $i=1, 2, \ldots, d$. We deduce

Lemma 2.22. If $d<p$, then given d-tuples $(a_1, \ldots, a_d)$ of K, we can find a set x of atmost $f\,d\,(p-1)$ elements of K, such that $S_i(x)=a_i$ for $i=1, \ldots, d$.

Thus if $d<p$, $\Sigma(d)=K^d$. We see that $S_1, S_2, \ldots, S_{r-1}$ are independent.

In what follows we suppose that $d \geqslant p$.

Suppose that $S_i(x)=0$ whenever $i \leqslant p^f-1$, $(i, p)=1$; then from Lemma 2.21, it follows that the first p^f-1 sum of powers of the same set x also vanish. Cansider the set of equations linear in $r_1, \ldots, r_{p^f-1}$

$$\sum_{s=1}^{(p^f-1)=Q} r_s x_s^m = 0 \text{ for } m=1, 2, \ldots, Q$$

then

$$\det(x_s^m) = \Pi (x_s - x_t) \neq 0$$

$$m, s = 1, \ldots, (p^f-1)$$

hence the equations have no non-trivial solution for the r's. Hence each $r_s \equiv 0 \pmod p$ and consequently the sum of m^{th} powers of the set x consisting of each x_s repeated r_s times vanishes for all m. From Newton's formula, it follows that

$$S_m(x)=0 \text{ whenever } (m, p)=1. \quad \text{We deduce}$$

Lemma 2.23. If $S_i(x)=0$ whenever $i \leqslant p^f-1$ and $(i, p)=1$ then $S_m(x)=0$ whenever $(m, p)=1$.

Lemma 2.24.

$$\left(\sum_{i=0}^{N} a_i Z^i\right)^p = \sum_{i=0}^{N} a_i^p Z^{ip} \text{ for } a_0, \ldots, a_N \in K$$

Suppose then that $\sum_{i=0}^{N} a_i Z^i=0$ is an equation with roots $x_1, \ldots, x_N$; then the equation $\sum_{i=0}^{N} a_i^p Z^{ip}=0$ has roots $x_1, x_2, \ldots, x_N$ each repeated p times. If $\sigma \in \Sigma$ so that $S_i(x)=\sigma_i$ for each i, then the $x's$ are the roots of the equation $\Sigma(-1)^i \sigma_i f^i=0$. Denote the set consisting of elements of x repeated p times by $x^{(p)}$; then the equation with roots $x^{(p)}$ is

$$\Sigma(-1)^{ip} \sigma_i^p Z^{ip}=0, \text{ so } S_{ip}(x^{(p)})=\sigma_i^p \text{ for } i=1, 2, \ldots, p.$$

So whenever $\underline{\sigma} \in \Sigma$ there is a sequence $\underline{\hat{\sigma}} \in \Sigma^{(p)}$ with $\hat{\sigma}_i=0$ for $(i,p)=1$

and $\hat{\sigma}_{jp}=\sigma_j{}^p$

and if $\underline{\sigma}$ are the symmetric functions of a set of F elements of K, then $\underline{\hat{\sigma}}$ are the symmetric functions of the same set repeated p times.

Conversely, if $\underline{\hat{\sigma}} \in \Sigma^{(p)}$, then $\Sigma(-1)^i \hat{\sigma}_i Z^i$ is a p^{th} power, so its roots are p times repeated; so $\underline{\hat{\sigma}}$ can occur only as the symmetric functions of a set of elements p times repeated.

In a similar way, the sequences of $\Sigma^{(p^r)}$ are obtained by repeating p^r times. We deduce

Lemma 2.25. Suppose that $S_i(x)=\sigma_i$ for $i=1, 2, \ldots$.
Write $x^{(p^r)}$ for the set of elements of x each repeated p^r times. Then

$$S_i(x^{(p^r)})=0 \text{ if } p^r \text{ does not divide } i,$$
$$\text{and } S_{jp^r}(x^{(p^r)})=\sigma_j^{p^r}$$

Conversely, if $\underline{\hat{\sigma}} \in \Sigma^{(p^r)}$ and $S_i(y)=\hat{\sigma}_i$ for some set y then y must consist of p^r-fold repeats.

We prove the following for later application,

Lemma 2.26. If $S_i(x)=0$ whenever $i<p^{f+r}$ and p^{r+1} does not divide i, then x consists of elements repeated p^{r+1} times.

Proof. We prove the lemma by induction on r. When $r=0$, the lemma is certainly true by lemma 2 23. We suppose that it is true for $(r-1)$. Now if $S_i(x)=0$ whenever $i<p^{f+r}$ and i is not divisible by p^{r+1}, the case $r=0$ implies that x consists of p-fold repeats. Suppose that x is y repeated p times. Then $S_i(y)=0$ whenever $i<p^{f+\gamma-1}$ and p^r does not divide i and since the lemma is true for $(r-1)$, y consists of p^r-fold repeats. Thus the set x, such that $S_i(x)=0$ whenever $i<p^{f+r}$ and i is not divisible by p^{r+1} consists of p^{r+1}-fold repeats; this completes the induction.

Definition :

The elements $S_1, \ldots.$ generate Σ if each element S of Σ can be written uniquely as a sum of multiples of $S_1, \ldots,$ $S=\sum_{i=1}^{\infty} \lambda_i S_i$.

We work out the structure of Σ first in the rather simpler case when $K=k$, a congruence field.

For $m=1, 2, \ldots.$ write Σ_m for the subset of Σ consisting of possible vectors of symmetric functions $S(x)$ with $S_1(x)=S_2(x)=\ldots.=S_{m-1}(x)=0$; $\Sigma_1=\Sigma$ Then $\Sigma_m/\Sigma_{(m+1)}$ is in $1-1$ correspondence with the possible values for $S_m(x)$ when $S_1(x)=\ldots.=S_{m-1}(x)=0$; Write S_m for this set. In order to find a set of generators for Σ, it is enough to describe $S_m \cong \Sigma_m/\Sigma_{m+1}$ for $m>1$.

If $S_i(x)=\sigma$ is soluble for any value σ in k, then it is soluble for all values of σ in k ; thus S_i either consists simply of the zero vector or it is in $1-1$ correspondence with k. Further by lemma 2 23, S_i is empty unless $i<p$ or p/i by Lemma 2.26, S_i is empty unless $i=qp^r$ with $(q, p)=1, q<p$; and in this case S_i is in $1-1$ correspondence with k.

So a set of generators of Σ is

$$(S^{(1)}, S^{(2)}, \ldots., S^{(p)}, S^{(2p)}, \ldots.)$$

where whenever $i=qp^r$ with $q<p$, $S^{(i)}$ has $S_j^{(i)}=0$ for $j<i$, $S_i^{(i)}\neq 0$. This determines the structure of Σ completely.

Let $a=(a_1, \ldots, a_d)$ be an element of $\Sigma(d)$, that is, there exists a set x such that $S_i(x)=a_i$ for $i=1, \ldots, d$. Suppose that $p^r \leqslant d \leqslant p^{r+1}$. From Lemma 2.22 it follows that $S_1, S_2, \ldots S_{p-1}$ and hence S_{p^r}, $S_{2p^r}, \ldots S_{p^{r+1}-p^r}$ for $r=1, \ldots$ are independent. The number of elements necessary in the set y for the symmetric functions S^e, $S_{2p^e}\ldots$, $S_{p^{e+1}-p^e}$ to have a given setof values is at most $p^e(p-1)^2$. Thus the number of elements necessary in the set x is at most

$$(p-1)^2+p(p-1)^2+\ldots+p^{r+1}(p-1)^2=(p^{r+2}-1)(p-1)\leqslant(p^2d-1)(p-1)$$

For the more complicated case $K=k_{pf}$, we have to proceed more carefully.

If $m=qp^r$ with $(q,p)=1$, we write Σ_m for the subset of Σ consisting of possible vectors of symmetric functions $S(x)$ with

$$S_{ip^j}(x)=0 \text{ if } j<r,\ (i,p)=1$$

$$S_{ip^r}(x)=0 \text{ if } i<q,\ (i,p)=1;$$

then $\Sigma_1=\Sigma$ Also $\Sigma_{qp^r}=\Sigma_{p^r(p^f-1)}=\Sigma_{p^{r+1}}$ whenever $q\geqslant p^f-1$, by Lemma 2.26.

Write $\Sigma_{qp^r}/\Sigma_{(q+1)p^r}=S_{qp^r}$ for short, then we see $\Sigma\cong\prod_{r=0}^{\infty}\prod_{q=1}^{p^f-1}S_{qp^r}$. To identify Σ; it is enough to identify S_{qp^r} whenever $p<p^f$, $(q,p)=1$. The set S_{qp^r} is in $1-1$ correspondence with the set of values of σ such that

$$S_{ip^j}(x)=0 \text{ if } j<r\ (i,p)=1$$

$$S_{ip^r}(x)=0 \text{ for } i<q$$

$$S_{qp^r}(x)=\sigma \text{ are soluble}$$

Now $S_i(y)=0$ for $i=1,\ldots,q-1$ and $s_q(y)=\sigma_1$ implies that $t_i(y)=0$ for $i=1,\ldots,q-1$ and $t_q(y)=(-1)^{q-1}q\,\sigma_1$ and vice versa.

If p^{δ} is the least power of p such that $p^f - 1 \mid q(p^{\delta} - 1)$ then $t_i\ (y) = (-1)^{q-1}\, q\sigma_1$ is soluble if and only if $\sigma_1 = \sigma_1{}^{p^{\delta}}$. Thus a set y such that $S_i(y) = 0$ for $i = 1, 2, \ldots, q-1$ and $s_q\ (y) = \sigma_1$ exists only for those σ_1 with $\sigma_1 = \sigma_1{}^{p^{\delta}}$ where p^{δ} is the least power of p such that $p^f - 1 \mid q(p^{\delta} - 1)$. Thus by Lemma 2.26, there exists a set $y^{(p^r)} = x$ such that

$$(2.2)\quad \begin{cases} S_{ip^j}\,(x) = 0 \text{ if } j < r\ (i, p) = 0 \\ S_{ip^r}\,(x) = 0 \text{ for } i < q \\ S_{qp^r}\,(x) = \sigma_1{}^{p^r} = \sigma \text{ are soluble.} \end{cases}$$

if and only if $\sigma = \sigma^{p^{\delta}}$, where p is the least power of p such that $p^f - 1 \mid q\ (p^{\delta} - 1)$.

For each q, let $C(q)$ be the set of $\sigma \in K$ such that $\sigma = \sigma^{p^{\delta}}$ Let $A(q)$ be a set of generators of $C(q)$. Then for each of these generators and for each $r \geqslant 0$ we can find x such that (2.2) is satisfied. We thus get a set $\hat{A}\ (pq^r)$ of generators of Σ_{qp^r} module $\Sigma_{(q+1)p^r}$. Then taking $\hat{A}\ (qp^r)$ when $(q, p) = 1$ and $q \leqslant p^f - 1$ we get a set of generators for Σ. This determines the structure of Σ completely.

Furthermore if $a = (a_1, \ldots, a_d)$ is an element of $\Sigma(d)$, then there exists a set x such that $S_i\ (x) = a_i$ for $i = 1, \ldots, d$; and if $p^r \leqslant d \leqslant p^{r+1}$, then x consists of at most

$$f\,[(p-1)^2 + p(p-1)^2 + \ldots + p^{r+1}\,(p-1)^2] \leqslant f(p^2 d - 1)\,(p-1) \text{ elements.}$$

§ 3. From now on K will be a p-adic field with ring of integers θ and prime ideal $p = (\pi)$. The rational prime above is p, the ramification index is e so that $(\pi)^e = p$ and the residue class field $\theta/p = k$ has p^f elements. We denote the set of n tuples of any set E by E^n.

If the residue class field $\theta/p = k$ is infinite, then every element of K can be expressed as a sum of d^{th} powers. By Newton's formula, the identification of $\Sigma(d)$ in this case is trivial.

Thus we may suppose that k is finite.

In section 3.1 we have dealt with the case $d<p^f$; the results are mainly quoted from [1]. In section 3.2 we try the problem when $d<p^f$ but we are unable to solve it, however we have slightly improved the number of veriable given in [1]. Section 3.3. deals with the case when K is a rational p-adic field. By analogy with the work of Ramanujam [5] one would expect that the unramified case might be easier; we are unable to give any good results. By the improvement of our method it may be possible to lessen the restrictions on the values taken by the first d symmetric functions.

The identification of $\Sigma(d)$ when K is a p-adic field and $d\geqslant p^f$ seems to be distinctly hard. Even in the rational p-adic case we find it difficult to determine $\Sigma(d)$. We can show that if

$$S_1(x)\equiv S_2(x)\equiv S_3(x)\equiv 0 \tag{4}$$

then $S_4(x)\equiv 0$ (2); but this is the only case in which we know of any congruence restrictions on the values of symmetric functions that is not implied by the finite field theory.

Our estimate of number of variables necessary is in terms of d, p and f; one would wish to find it in terms of d alone; but we are unable to improve materially the d^{16a^2} given by Birch [1].

§ 3.1. Write $D=f(pd-1)(p-1)$ for short. We shall identify the set $\Sigma(d)$ for $d<p^f$ and given an estimate of the number of variables $x_1, \ldots, x_n$ which may be necessary for $S_1(x), \ldots, S_d(x)$ to take a given set of d-tuples of values in $\Sigma(d)$.

We start with a known version of Hensel's lemma (lemma 3 of [1]) whose proof we reproduce for completeness.

Lemma 3.11. Let $r\geqslant 1$. Suppose that $a\epsilon\theta^d$ $y\epsilon\theta^D$, $z^{(r)}\epsilon\theta^d$ are such that

(3.1) $z_i^{(r)}$ not congruent to $z_j^{(r)}$ (mod π) for $i\neq j$

and (3.2) $S_K(y, z^{(r)}) \equiv a_K \pmod{\pi^r}$ for $K=1, 2, \ldots, d$;

then we can find $z^{(r+1)} \in \theta^d$ such that

$$(3.3)\quad z^{(r+1)} \equiv z^{(r)} \pmod{\pi^r}$$

$$\text{and } (3.4)\quad S_K(y, z^{(r+1)}) \equiv a_K \pmod{\pi^{r+1}} \text{ for } K=1, 2, \ldots, d.$$

Proof:—The congruence (3.3) is equivalent to $z^{(r+1)} = z^{(r)} + \pi^r t$ where $t \epsilon \theta^d$, so it is enough to show that we can find t such that

$$S_K(y, z^{(r)} + \pi^r t) \equiv a_K \pmod{\pi^{(r+1)}} \text{ for } K=1, 2, \ldots, d.$$

$$\text{But } S_K(y, z^r + \pi^r t) \equiv S_K(y, z^r) + \pi^{r} \sum_{i=1}^{d} (\partial S_j/\partial z_k)\ (\pi^{2r})$$

So since $r \geqslant 1$, it is enough to solve the linear congruence.

$$(3.5)\quad \sum_{j=1}^{d} t_j\, [\partial S_K/\partial z_j] = \pi^{-r}\, [a_K - s_K(y, z^{(r)})]\ (\pi)$$

The determinant formed by the coefficients $\partial s_K/\partial z_j$ is of vandermonde type; it has value $\pm \Pi\, (z_s - z_t)$; and so does not vanish mod π by (3.1); so (4.5) is certainly soluble.

This completes the proof of lemma 3.11.

Now if $d < p^f$, then we can choose $z^{(1)} \epsilon \theta^d$ so that $z_i^{(1)}$ not congruent to $z_j^{(1)} \pmod{\pi}$ for $i \neq j$.

Let $(a_1, \ldots, a_d)$ by any set of values taken by the first d symmetric functions in the residue class field $k_p f$, and suppose that $b_i \equiv a_i \pmod{\pi}$ for $i=1, \ldots, d$. Then by the results of § 2. we can find a set $y \epsilon \theta^D$ so that

$$S_j(y, z) \equiv b_j \pmod{\pi} \text{ for } j=1, \ldots, d.$$

Now we apply lemma 3.11, for each $r \geqslant 1$ we find

$$z^{(r)} \equiv z^{(1)} \pmod{\pi^r} \text{ so that}$$

$$S_j(y, z^{(j)}) \equiv b_j \pmod{\pi^r}$$

Finally we let $r \to \infty$. By the compactness of θ the sequence $\{z^{(r)}\}$ has a limit point, call it z, and then

$$S_j(y, z) = b_j \text{ for } j = 1, 2, \ldots, d.$$

Thus the sequence of values taken by the first d symmetric functions are just those whose reductions module p are sequences of values taken by the symmetric functions in the residue class field k_{p^f}; and since the strncture of the latter is known, we know $\Sigma(d)$ completely for $d < p^f$ when K is a p-adic field.

Furthermore if $a = (a_1, \ldots, a_d)$ is an element of $\Sigma(d)$, then there exists a set x of at most $d + f\ (p^2d - 1)(p - 1)$ elements such that $s_i(x) = a_i$ for $i = 1, 2, \ldots, d$.

§ 3.2. In this section we consider the case when $d \geqslant p^f$. Birch [1] has proved that if $a \in \theta^l$ and $y \in \theta^l$ satisfy

$$S_i(y) \equiv a_i\ (\pi^{4l}) \text{ for } i = 1, 2, \ldots, d.$$

then by a Hensel's lemma type argument we can find a set

$$v \in \theta^{m+d} \text{ such that}$$

$$S_i(v) = a_i \text{ for } i = 1, 2, \ldots, d.$$

Thus the identification of $\Sigma(d)$ when $d \geqslant p^f$ reduces to the identification of the sets z such that

$$S_i(z) = a_i\ (\pi^{4l}) \text{ for } i = 1, 2, \ldots, d.$$

Unfortunately, we have not been able to identify these sets.

Birch has also proved that there is a set z such that $S_i(z) = a_i\ (\pi^{4l})$ for $i = 1, \ldots, d$ consisting of at the most $p^{4d^2 f} - p^{4lf} + 1$ elements. We can improve this number of variables slightly. For instance: if z consists of a repeated $p^{\tau+s}$ times, then $S_j(z) \equiv 0\ (p^\tau)$ for $j = 1, 2, \ldots, p^s$. Thus one gains nothing by repeating elements more than $p^{\tau+s}$ times; hence we obtain all possible sequences of residue classes modulo p^τ for $s_1(z), \ldots, s_{p^s}(z)$ by taking at most $p^{\tau+s+\tau sf}$ variables in z. Hence p-adically we need no more than p^{8df} variables.

Thus if $a=(a_1, \ldots, a_d)$ is an element of $\Sigma(d)$, then there exists a set of at most p^{8df} elements such that $s_i(x)=a_i$ for $i=1, 2, \ldots, d$.

By examining Birch's argument more closely, we can do even better; since Hensel type arguments may be applied to $S_i(y)=a_i\ (\pi^R)$ with R notably less than $4d$.

§ 3.3. In the rational p-adic case, we can do a little better. In this section, $(p)=(\pi)$, and the residue class field consists of p elements. So the residue classes modulo a power p^R have representatives which are rational integers.

We have already dealt with the case $d<p$ in section 3.1, so we may suppose $d\geqslant p$. For convenience in arithmetic, we take d as a power of p, $d=p^S$.

We prove the following two lemmas whose idea goes back to Birch [1].

Lemma 3.31.

Let $a \in \Theta^d$, $z \in \Theta^d$. Let the power of p dividing $\prod_{i=1}^{j-1}(z_i - z_j)$ be $p^{\nu(j)}$ for each $j=1, 2, \ldots, d$ and suppose that

$$\prod_{i=1}^{j-1}(z_k-z_i)\equiv 0 \pmod{p^{\nu(j)}} \text{ for } 2\leq j\leq K\leq d$$

and $a_i\equiv 0\ (p^{\nu(d)})$ for each $i\leq d$;

then we can find a set $y \in \Theta$ such that

$S_i(y, z)=a_i$ for $i=1, \ldots, d$.

Proof:—By Newton's formula

$S_i(y, z)=a_i$ for $i=1, \ldots, d$

if and only if

$t_i(y, z)=b_i$; where b_i is a polynomial in $S_1, S_2, \ldots, S_i$ with positive rational integer co-efficients and $b_i\equiv 0\ (p^{\nu(d)})$ for each $i\leq d$.

Let $b_i = \lambda_i p^{\nu(d)}$

Consider the set of equations

$$\sum_{m=1}^{d} \gamma_m (z_m)^i = \lambda_i p^{\nu(d)} \text{ for } i=1, 2, \ldots, d.$$

These have solutions

$$\gamma_m = p^{\nu(d)} \left(\frac{\underset{i,\, m=1,\, \ldots,\, d}{\det} \; z_m^i \text{ but } \lambda_i \text{ in place of } (z_m)^i}{\underset{i,\, m=1,\, \ldots,\, d}{\det} \; z_m^i} \right)$$

Obviously, the power of p in r_m is at least p^0; so we can find r_m as p-adic integer. So given any power p^R we can find rational integers $r_1, \ldots, r_d$ each at least 1 so that

$$\sum_{m=1}^{d} \gamma_m (z_m)^i \equiv b_i \; (p^R) \text{ for } 1, 2, \ldots, d.$$

Take a set $y^{(r)}$ consisting of z_1 repeated (r_1-1) times, ...; z_d repeated $r_d - 1$ times, then

$$t_i (y^{(r)}, z) \equiv b_i \; (p^R) \text{ for } i=1, 2, \ldots, d.$$

If R is large enough, a form of Hensel's lemma is applicable: we can find sets $y^{(1)}, y^{(2)}, \ldots$ so that

$$y^{(s+1)} = y^{(s)} \pmod{p^s}$$

and $t_i \; (y, z) \equiv b_i \; (p^{R+s-1})$ for $i=1, 2, \ldots, d$
$s = 1, 2, \ldots$

Let $s \to \infty$ then by the compactness of θ, the sequence $\{y^s\}$ has a limit call it y, and then

$$t_i (y, z) = b_i \text{ for } i=1, \ldots, d.$$

and

$$S_i (y, z) = a_i \text{ for } i=1, \ldots, d.$$

Lemma 3.32. (See lemma 7 of [1])

We can find a sequence $\{z_j\}$ of elements of θ such that, whenever $2 \leq j \leq K$, $\prod_{i=1}^{j-1} (z_K - z_i)$ is divisible by at least as high a power of p as $\prod_{i=1}^{j-1} (z_j - z_i)$ and $\prod_{i=1}^{j-1} (z_j - z_i)$ is not divisible by $p^{j/p-1}$.

Proof:—We simply take $z_i = i$ for $i=1, \ldots, d$, and the lemma follows immediately.

Thus combining lemma 3.31 and lemma 3.32, we get

Lemma 3.33 : Let $a \in \theta^d$ satisfy

$$a_i \equiv 0 \pmod{p^{d/p-1}} \text{ for } i \leq d$$

then we can find a set $z \in \theta$ such that

$$S_i(z) = a_i \text{ for } i=1, \ldots, d.$$

We deduce

Theorem 3.31.

Suppose that d is positive power of p. Let $a \in \theta^d$ be such that for some set $x \in \theta$

$$S_i(x) = a_i \pmod{p^{d/p-1}} \text{ for } i=1, 2, \ldots, d;$$

then we can find a set z of elements of θ such that

$$S_i(z) = a_i \text{ for } i=1, 2, \ldots, d.$$

In particular, we can solve $S_i(z) = a_i$ whenever $a_i \equiv 0 \pmod{p^{d/p-1}}$ for $i=1, \ldots, d$.

This is a weak result, but no more seems to be obvious. Note that if $d < p-1$, then the theorem asserts that for every $a \in \theta^d$ we can find z such that $S_i(z) = a_i$ for $i=1, 2, \ldots, d$. This is not a particular case of the theorem, since we have assumed implicityly that $d \geqslant p$; but the result is true by Section 3.1.

§ 4. We see that the problem of identification of $\Sigma(d)$ does not turn out to be so easy as we thought. The paper solves the problem it set out to solve, but one would like to find explicityly the relation between the first d symmetric functions. It is not too difficult to do so with the help of our theorem 2.11 and Newton's formula, but it is distinctly messy.

By analogy with the finite field one would desire to identify the set Σ when R is a ring of rational integers, there one has already trouble over the reals, if $S_i(x)=a_i$, $i=1, 2, \ldots, d$, then $t_i(x)=b_i$ for some b_i given in terms of a_i's by Newton Formula. Clearly $b_i \geqslant 0$ whenever i is even, and this implies inequalities between the symmetric functions; for instance $b_2 \geqslant 0$ implies $S_1^2 \geqslant 2S_2$. When R is the ring of rational integers, the congruence conditions interact with these inequalities and one has the additional trouble that any non-zero positive integer is at least 1.

References

1. B. J. Birch. Waring's promblem for p-adic number fields. Acra Arithmetica IX (1964) pp. 169-176.
2. L. Tornheim. Sum of nth powers in fields of prime characteristic; Duke Math. J, Vol. 4 (1938) pp. 399-362.
3. P. J. Bateman and R. M. Stemmler. Waring's problem for algebraic number field, and primes of the form $(p^r-1)/(p^d-1)$, Illinois J. Math. 6 (1962). pp. 142-156.
4. O. Perron. Algebra 1, 3rd Edition, Berlin, 1961.
5. C. P. Ramanujam. Sums of mth powers in p-adic rings, Mathematics 10 (1963) 137-146.

SOME PROPERTIES OF PERMUTATIONAL PRODUCTS OF GROUPS[1]

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1. Introduction : The concept of permutational products of groups was introduced by B. H. Neumann in [1]. This group theoretic construction is based on a method given by him in his famous essay [3] for the embeddability of an amalgam with a single group amalgamated, in a permutation group. Use of this construction was made to answer various questions about the embedding theory of group amalgams (cf. [1], [2], [3]). The present article is divided into four sections. In section 2, the definition of permutational product of groups together with some other concepts related to it is given. A fundamental lemma which forms the basis of the theory developed latter and called hereafter Neumann's lemma is also given in this chapter.

In section 3, we examine the structure of a permutational product of an amalgam in which the amalgamated subgroup possesses in both the constituents transversals that it centralises and show that for this kind of amalgam the permutational product corresponding to transversals centralised by the amalgamated subgroup belongs to the least variety[3] containing both the constituents.

Some analogies between the generalised free product and permutational product of groups with an amalgamated subgroup are discussed

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1. This article forms a part of a thesis submitted to the Australian National University for Master's degree.

2. The author gratefully acknowledges finincial support from A.N.U. and grant of study leave by the Punjab University. Professors B.H. Neumann and Hanna Neumann are thanked for their guidance, constant encouragement and advice.

3. A variety is a class of groups closed under the operations of taking subgroups, epimorphic images and cartesian products (cf. [8], [12]).

in section 4. A result similar to theorem 1.1 [3] is proved for permutational products of groups.

Following B. H. Neumann, let F* denote one of the following properties of groups: being locally finite (LF), of finite exponent (FE), or periodic (P). It is known that a soluble amalgam, that is, an amalgam of soluble groups or an amalgam of groups having the property F* need not be embeddable in a soluble or F* group respectively, (cf. [1], [2]). However under some sufficient conditions on the constituents, B. H. Neumann has shown it to be possible. These results are given in section 5.

§ 2.1. We begin by defining the notion of an amalgam[1] and its embeddability. An amalgam $\underline{\underline{A}}$ of (for convenience only) the groups A and B with a common subgroup H is an 'incomplete group' whose elements are those of A and B with the elements of H thought of as identified in the two groups. The product of two elements of $\underline{\underline{A}}$ is defined if and only if they both belong to A or both belong to B, and its value is as in that group. If there is a group G containing A and B as subgroups such that in G the intersection of A and B is the prescribed group H, then we speak of an 'embedding' of the amalgam $\underline{\underline{A}}$=am (A, B; H) in G. A and B are called 'constituents' of $\underline{\underline{A}}$ and H the 'amalgamated subgroup'.

By a transversal of a subgroup H of a group A we shall mean a set $S \subseteq A$ such that every element of A is uniquely representable in the form

$$a = sh,\ s \in S,\ h \in H$$

We say that H centralises S if the members of S and H commute.

2.2. Permutational product of Groups. We now come to the definition of a permutational product (cf. [1]). Let $\underline{\underline{A}}$=am (A, B; H)

1. This term was first introduced by Baer [10]. A deeper account of results concerning group amalgams and their embuddability can be found in [3], [4], [5] and [11].

be an amalgam of the groups A and B. We choose transversals S of H in A and T of H in B. Form the set product $K=S\times T\times H$. The elements of K are ordered triplets (s, t, h), $s \in S$, $t \in T$, $h \in H$. For each $a \in A$, we define a mapping $\rho(a) : K \to K$ by

$$(s, t, h)^{\rho(a)} = (s', t, h')$$

where $s' \in S$, $h' \in H$ are determined by the equation $sha = s''h'$

Similarly for b in B we define a mapping $\rho(b)$: $K \to K$ by

$$(s, t, h)^{\rho(b)} = (s, t'', h'')$$

where

$$thb = t''\, h''.$$

It is easy to verify that for $a=b \in H$ no ambiguity arises in the definition of ρ. Moreover, the mapping $\rho : A \to \rho(A)$ is a homomorphism; for if a, a' are two elements of A, then

$$(s, t, h)^{\rho(a)\,\rho(a')} = (s', t, h')^{\rho(a')}$$
$$= (s'', t, h'')$$

where $sha = s'h'$, $s'h'a' = s''h''$ so that $shaa' = s''h''$ which means that

$$(s, t, h)^{\rho(aa')} = (s'', h'').$$

Thus $\rho(a)\,\rho(a') = \rho(aa')$. The prcof for $\rho(b)\,\rho(b') = \rho(bb')$ is similar. It follows, therefore, that $\rho(A) = \{\rho(a);\ a \in A\}$ and $\rho(B) = \{\rho(b);\ b \in B\}$ are groups. However the homomorphism $\rho : A \to \rho(A)$ turns out to be an isomorphism, for if $\rho(a) = i_K$, the identity mapping of K, then

$$(s, t, h)^{\rho(a)} = (s, t, h)$$

for all $(s, t, h) \in K$ means that $sha = sh$ for all $s \in S$, $h \in H$ and therefore $a = 1$.

The above remarks show that the mappings $\rho(a)$, $\rho(b)$; $a \in A$, $b \in B$ are in fact permutations of K. Furthermore, the intersection of $\rho(A)$ and $\rho(B)$ is $\rho(H)$, because if $\rho(a) \in (B)$ then $\rho(a)$ leaves the first component of each triplet (s, t, h) fixed and so

$$(s, t, h)^{\rho(a)} = (s, t, ha),$$

therefore $ha \in$ **H**, that is $a \in$ H.

The permutation group P of K generated by $\rho(A)$ and $\rho(B)$ contains isomorphic copies of A and B with $\rho(A) \cap \rho(B) = \rho(H)$ isomorphic to H and, therefore, embeds the amalgam $\underline{\underline{A}}$. P *is called a permutational product of* $\underline{\underline{A}} = am(A, B; H)$. We use here the indefinite article because P depends not only on $\underline{\underline{A}}$ but also on the choice of transversals S, T of H in A, B respectively (for details see [1]). By $P(\underline{\underline{A}}; S, T)$ we shall denote the permutational product of $\underline{\underline{A}}$ corresponding to the transversals S, T of H in A and B respectively.

2.3. Next we define the free product of groups as follows: Let $\{G_\alpha\}$, be a family of groups indexed by a set I of finite or infinite cardinality. A group G, which we shall write as $\pi^*_{\alpha \in I} G_\alpha$, is said to be the free product of G_α, ($\alpha \in$ I) if

(*i*) the subgroups G_α generate G, that is, if every element $g \neq 1$ of G is expressible as a product of a finite number of elements from the G_α:

$$g = g_{\alpha_1} g_{\alpha_2} \cdots g_{\alpha_n};\ g_{\alpha_i} \in G_{\alpha_i}\quad i = 1, 2, \ldots, n \qquad \text{(A)}$$

where

$$g_{\alpha_i} \neq 1,\ \alpha_i \neq \alpha_j \text{ for } j = i \pm 1; \text{ and}$$

(*ii*) the expression (A) is unique for every $g \neq 1$ in G.

If the set I is finite, we shall use the notation

$$G = G_1 * G_2 * \ldots * G_n.$$

The subgroups G_α of G are called the 'free factors' of G while the expression (A) is called the 'normal form' of an element g of G.

2.4. As stated above, a change in the transversals of the amalgamated subgroup greatly alters the nature and character of the permutational product of an amalgam $\underline{\underline{A}}$. However, under certain conditions on the amalgam, the isomorphism type of permutational product of $\underline{\underline{A}}$ is unaltered.

The following result which plays a key role in what is given later, mentions one such condition.

2.41. Neumann's Lemma[1] : Given two groups A and B with an amalgamated subgroup H, let B be a transversal of H in A which is centralised by H, then the isomorphism type of the permutational product P ($\underline{\underline{A}}$; S, T) is independent of the change of transversals T in the other constituent, *i.e.* in B.

Proof: Let T and T′ be two distinct transversals of H in B and P ($\underline{\underline{A}}$; S, T), P′ ($\underline{\underline{A}}$; S, T′) permutational products of A, B corresponding to the transversals S, T and S, T′ of H in A and B respectively. We define a one – one mapping φ from $K = S \times T \times H$ to $K' = S \times T' \times H$ in the following manner:

If $(s, t, h) \in K$, then

$$(s, t, h)^{\varphi} = (s, t', h')$$

where $(s, t', h') \in K'$ and $th = t'h'$.

Let $a \in A$, then, since φ^{-1} exists,

$$\begin{aligned}(s, t', h')^{\varphi^{-1}\rho(a)\varphi} &= (s, t, h)^{\rho(a)\varphi} \\ &= (s_1, t, h_1)^{\varphi} \\ &= (s_1, t', h_2 h_1)\end{aligned}$$

where $sha = s_1 h_1$, $th = t'h'$, $th_1 = t'h_2 h_1$ (1)

Also $(s, t', h')^{\rho'(a)} = (s_2, t', h_2')$

where $sh'a = s_2 h_2'$ (2)

Now from $th = t'h'$, we have $t' = thh'^{-1}$ and putting it in $t = t'h_2$, we get $t = thh'^{-1} h_2$

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1. For an original version of this result see theorem 2 [1].

so that $hh'^{-1}h_2=1$ *i.e.* $h'=h_2h$.

Therefore

$$sh'a=sh_2ha=h_2sha, \text{ by assumption that } [s,h]=1^{1} \text{ for all } h\in H,\ s\in S.$$
$$=h_2s_1h_1 \quad \text{from (1),}$$
$$=s_1h_2h_1 \quad \text{by assumption}$$
$$=s_2h_2' \quad \text{from (2).}$$

Therefore $s_1=s_2$, $h_2h_1=h_2'$, and $\varphi^{-1}\ \rho(a)\ \varphi=\rho'(a)$ for all $a\in A$. For $b\in B$, we have

$$(s,t',h')^{\varphi^{-1}\rho(b)\varphi}=(s,t,h)^{\rho(b)\varphi}$$
$$=(s,t_1,h_1)^{\varphi}$$
$$=(s,t_1',h_1')$$

and

$$(s,t',h')^{\rho'(b)}=(s,t_2'\ h_2)$$

where in the first case

$$thb=t_1h_1=t_1'h_1'$$

and from the second equation,

$$t'h'b=t_2'h_2'.$$

Since $th=t'h'$, therefore,

$$thb=t'h'b=t_1'h_1'=t_2'h_2'$$

Hence $t_1'=t_2'$, $h_1'=h_2'$ so that $\varphi^{-1}\ \rho(b)\ \varphi=\rho'(b)$ for all $b\in B$, *i.e.* $\varphi^{-1}\ P\varphi=P'$ and therefore P and P' are isomorphic.

This gives us

2.42. Corollary: If H is a direct factor in A, then the isomorphism type of the permutational product is independent of the change of transversals in B.

Proof: This is immediate: choose a complementary direct factor as a transversal.

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1. $[s, h]$ denotes the commutator $s^{-1}h^{-1}sh$.

2.43. Corollary. If H possesses in both A and B, at least one transversal which it centralises, then all permutational products of the amalgam formed with transversals of which at least one centralises H, are isomorphic.

Proof: Let S_1 and T_1 be transversals of H in A and B respectively which are centralised by H, and S and T any arbitrary transversals. Then by lemma 2.31.

$$P(\underline{\underline{A}}; S, T_1) \cong P(\underline{\underline{A}}; S_1, T_1) \cong P(\underline{\underline{A}}; S_1, T)$$

as required.

One may, quite naturally expect that when the amalgamated subgroup has transversals which it centralises in both the contituents, there is only one isomorphism type of permutational product. However, the following example shows that this is hoping too much.

2.44. Example: The groups A and B are taken as isomorphic to the dihedral group of order 12 which can be considered as the direct product of the dihedral group of order 6 by a cyclic group of order 2.

Thus

$$A = gp\,\{a, b, c;\ a^3 = b^2 = c^2 = (ab)^2 = [a, c] = [b, c] = 1\}$$
$$B = gp\,\{a', b', d;\ a'^3 = b'^2 = d^2 = (a'\ b')^2 = [a', d] = [b', d] = 1\}$$

We take H as

$$H = gp\,\{g, h;\ g^3 = h^2 = (gh)^2 = 1,\ g = a = a',\ h = b = b'\}$$

The transversals $S = (1, c)$, $T = (1, d)$ are centralised by H.

The permutational product $P(\underline{\underline{A}}; S, T)$ is the direct product of the Four group by a group isomorphic to H and so has order 24. However, if we choose the transversals as $S' = (h, c)$, $T' = (g, d)$, the permutational product $P'(\underline{\underline{A}}; S', T')$ turns out to be of order 72. P and P′ are obviously non-isomorphic.

As already mentioned, when H is central in both A and B and so centralises all its transversals in the constituents, then the isomorphism type of the permutational product is unique. In fact it is then the generalised direct product of A and B amalgamating H (cf. [1]). The examples given by B. H. Neumann [1] also show how drastic the effect of a change in the transversals can be if the amalgamated subgroup is not central in both the constituents. It may therefore be asked whether in all other cases, excepting the one above (*i.e.* of H being central in both A and B) permutational products of A and B always depend on the choice of transversals of the amalgamated subgroup. This, however, is not the case, as the following example constructed by B. H Neumann in a different context (cf. [2]) shows:

2.45. Example: Let H be the restricted direct product of an infinite number of cyclic groups of order 2, that is

$$H = gp\{h_0, h_1, h_2, \ldots ; h_i^2 = [h_i, h_j] = 1, \; j, i = 0, 1, 2, \ldots\}$$

Let α, β be the automorphisms of H defined by

$$h_{2i}^{\alpha} = h_{2i+1}, \; h_{2i+1}^{\alpha} = h_{2i}, \; (i = 0, 1, 2, \ldots)$$

and

$$h_0^{\beta} = h_0, \; h_{2i+1}^{\beta} = h_{2i+2}, \; h_{2i+2}^{\beta} = h_{2i+1}, \; (i = 0, 1, 2, \ldots)$$

We now extend H by cyclic groups $C_1 = gp\ \{a; a^2 = 1\}$ and $C_2 = gp\ \{b; b^2 = 1\}$ corresponding to these automorphisms to get two groups A and B respectively. Thus

$$A = gp\ \{a, H; a^2 = h_i^2 = [h_i, h_j] = 1, h_{2i}^{a} = h_{2i+1}, h_{2\cdot+1}^{a} = h_{2i}\}$$

$$B = gp\ \{b, H; b^2 = h_i^2 = [h_i, h_j] = 1, h_0^b = h_0, h_{2i+1}^b = h_{2i+2}, h_{2i+2}^b = h_{2i+1}\}$$

Let P be a permutational product of A and B amalgamating H, then in P, ab is an element of infinite order, because for any non-zero integer n:

$$h_0^{(ab)^n} = h_0^{ab.(ab)^{n-1}} = h_1^{b(ab)^{n-1}} = h_2^{(ab)^{n-1}} = \ldots = h_{2n} \neq h_0$$

If F is the free product of A and B amalgamating H then, by the definition of the free product, there is a homomorphism of F onto P. To show that F and P are isomorphic, it is, therefore, enough to prove that it is impossible to add an additional relation in F different from those already implied by the relations of A and B, without making any of the groups collapse.

Now it follows from the general theory of free products with one amalgamated subgroup that a general element of P can be written uniquely in the form

$$r = ha^{\varepsilon_1}\, bab \;\ldots\; ab^{\varepsilon_2}$$

$\varepsilon_i = 0$ or 1, $i = 1, 2$ and $h \in$ H. Hence a relation $r = 1$ gives

$$h = b^{\varepsilon_2}\, abab \;\ldots\; a^{\varepsilon_1}$$

If the right hand side is equal to 1, then this is a relation in H; hence we may assume it to be different from 1. Then ε_1, ε_2 cannot simultaneously be 1 or 0, for the right hand side in such a situation becomes $(ba)^{m+1}$ or $(ab)^m$ for some integer m according as $\varepsilon_1 = \varepsilon_2 = 1$ or $\varepsilon_1 = \varepsilon_2 = 0$. Therefore, because ab and ba are of infinite order in any group embedding the amalgam, they are of infinite order in P whereas h is of order 2. Thus either ε_1 or ε_2 is zero. Without any loss of generality, suppose that $\varepsilon_2 = 0$, then $\varepsilon_1 = 1$ and

$$h = ababa \;\ldots\; ba.$$

The right hand side has an odd number of factors, each of order 2, hence it is a conjugate of the central factor, 'a' in our case, by a power of ba:

$$h = a^{(ba)^k}$$

for some integer k. But H $= gp\,\{h_i\}$ is normal in each of the constituents A and B, hence transforming h by $(ba)^{-k}$ gives $a \in$ H, which is impossible because this leads to the collapse of A. Therefore no proper homomorphic image of F embeds the amalgam of A and B. The kernel of this homomorphism being trivial, an isomorphism between F and P is established. As the free product of an amalgam is unique to within isomorphism this amalgam possesses only one permutational product but for isomorphisms.

Thus the case of the amalgamated subgroup being central in one of the constituents is, by no means, the only one for which we get a unique permutational product of an amalgam of two groups.

The above example also proves another interesting fact; that in some cases the free product of two groups with amalgamation may coincide with their permutational product.

However, the free product of two groups with trivial amalgamation can never coincide with their permutational product because the permutational product of such an amalgam degenerates into their direct product and since by a theorem of Baer and Levi [7], a group which is decomposable into the free product of its subgroups cannot be decomposed into their direct product, we conclude that in such a case (amalgam with trivial amalgamation), the permutational product of A, B is always different from their free product.

Finally we mention that there exist amalgams of two groups such that their permutational product is different from their free product, the amalgamated subgroup is central in none of the constituents and still we have only one isomorphism type of permutational product. This is shown by the following example:

2.45. Example : Let

$$A = gp\ \{a, b;\ a^3 = b^2 = (ab)^2 = 1\}$$
$$B = gp\ \{c, d;\ c^3 = d^2 = (cd)^2 = 1\}$$
and
$$H = gp\ \{h;\ h^2 = 1,\ h = a = c\}$$

All the permutational products of A and B are different from their free product (this is due to the fact that a permutational product of a proper amalgam of the finite groups being subgroups of the permutation group on a finite set $K = S \times T \times H$ where S and T are coset representatives of H in A and B, is finite whereas their free product is always infinite). Moreover, the amalgamated subgroup is central in none of the constituents. But the only different looking permutational products of this amalgam corresponding to distinct transversals, are given by :

$$P_1 = gp\,\{a,\ b,\ c;\ b^2 = c^2 = (bc)^6 = 1,\ (bc)^2 = a\}$$

$$P_2 = gp\,\{a',\ b',\ c';\ a'^3 = b'^2 = c'^2 = (b'c')^2 = (c'a')^2 = (a'b')^2 = 1\}$$

P_1 can also be generated by b and c alone. The mapping

$$b \longrightarrow b'a',\ c \longrightarrow c'$$

is an isomorphism between P_1 and P_2.

§ 3. We now examine the structure of the permutational product P $(\underline{\underline{A}}$; S, T) of $\underline{\underline{A}}$ corresponding to the transversals S and T which are centralised by H. It will be shown that this particular permutational product possesses the properties of the generalised direct product of groups. In fact, we shall prove that this permutational product is itself the generalised direct product of some groups isomorphic to subgroups of the constituents in $\underline{\underline{A}}$.

Let us denote by C_A (S) the centraliser of S in A, then we have:

3.1. Lemma: Let $H \subseteq C_A(S) \cap C_B(T)$, then in the permutational product P $(\underline{\underline{A}}$; S, T) of $\underline{\underline{A}}$, $\rho(S)$ and $\rho(T)$ commute elementwise

Proof: Let $(s_1, t_1, h) \in K = S \times T \times H$, then for any $s \in S$ and $t \in T$, we have,

$$(s_1, t_1, h_1)^{\rho(s)\,\rho(t)} = (s_2, t_1, h_2)^{\rho(t)} = (s_2, t_2, h_3)$$

where $s_1 h_1 s = s_2 h_2,\ t_1 h_2 t = t_2 h_3$ (1)

Also $(s_1, t_1, h_1)^{\rho(t)\,\rho(s)} = (s_1, t'_2, h'_2)^{\rho(s)} = (s'_2, t'_2, h'_3)$

where $s_1 h'_2 s = s'_2 h'_3,\ t_1 h_1 t = t'_2 h'_2$ (2)

From (1), and (2), we have,

$$s_1 s = s_2 h_2 h_1^{-1} = s'_2 h'_2 h'_2{}^{-1} \ i.e.\ s_2 = s'_2,\ h_2 h_1^{-1} = h'_3 h'_2{}^{-1} \quad (3)$$

and $t_1 t = t_2 h_2^{-1} h_3 = t'_2 h_1^{-1} h'_2$

$$t_2 = t'_2,\ h_2^{-1} h_3 = h_1^{-1} h'_2 \quad (4)$$

From (3) and (4), $h_3 = h_2 h_1^{-1} h'_2 = h'_3$. Therefore $\rho(s)\,\rho(t) = \rho(t)\,\rho(s)$.

This being true for all $(s_1, t_1, h_1) \in$ K, and $s_1 \in$ S, $t_1 \in$ T, we have

$$[\rho(S), \rho(T)] = 1.$$

3.2. Theorem: Let H possess in A and B transversals S and T respectively which it centralises, then the permutational product $\underline{\underline{P}}$(A; S, T) of $\underline{\underline{A}} = am$ (A, B: H) can be represented as the generalised direct product of any of the following three sets of groups.

(*i*) The groups K, L, and ρ(H) where
K=*gp* {ρ(S)}, L=*gp* {ρ(T)}; or

(*ii*) ρ(A) and L; or

(*iii*) ρ(B) and K.

Proof: To prove (*i*) we first note that since ρ(S) and ρ(T) commute elementwise (lemma 3.1), so do also the groups K=*gp* {ρ(S)} and L=*gp* {ρ(T)}. Let K ∩ L=R, then R is central in K because R is a subgroup of L which centralises K; so also it is in T because then we consider it as a subgroup of K, and L and K commute elementwise.

Moreover, R being a subgroup of both K and L and so also of ρ(A) and ρ(B), R is contained in ρ(H). Furthermore, R is central in ρ(H) because it is a subgroup of K and L which are centralised by ρ(H). Since K, L and ρ(H) contain R as a central subgroup and P is generated by these groups it is their generalised direct product amalgamating R.

We now proceed to the proof of (*ii*). The group P which is generated by ρ(S), ρ(T), ρ(H), can also be generated by ρ(A) and ρ(T) *i.e.* by ρ(A) and L only, where L=*gp* {ρ(T)}. Let now ρ(A) ∩ L=R_1. Since R_1 is a subgroup of ρ(A) and L, R_1 is contained in ρ(H) (the meet of A and B being only H). As a subgroup of ρ(H), R_1, L commute elementwise because H centralises T and so also L. Also R_1 is central in ρ(H) because R_1 is in L. Since ρ(A) is generated by ρ(S) and ρ(H), and L centralises ρ(S), therefore R_1 centralises ρ(S), that is R_1 is central also in $\rho(A)$. P being generated by $\rho(A)$ and L with their meet central in both, is their generalised direct product amalgamating R_1.

The proof of (*iii*) is exactly the same as that of (*ii*)

The theorem is now completely proved.

The following theorem gives the nature of subgroups generated by elements of A and B contained in the centraliser of H in A and B respectively, in a permutational product P of A and B amalgamating H.

3.3. **Theorem:** Let $\{a_i\}$ and $\{b_j\}$ be two sets, finite or infinite, of elements of A and B respectively such that each $a_i \in C_A(H)$, $b_i \in C_B(H)$ and H abelian, then in any permutational product P of A and B, $A_m = gp\{\rho(a_1), \ldots, \rho(a_m)\}$, $B_n = gp\{\rho(b_1), \ldots, \rho(b_n)\}$ generate their generalized direct product.

Proof: Let $(s, t, h) \in K = S \times T \times H$, then we have to show that $\rho(a_i)\,\rho(b_j) = \rho(b_j)\,\rho(a_i)$ for $i = 1, 2, \ldots, m$ $j = 1, 2, \ldots, n$.

Now $(s, t, h)^{\rho(a_i)\,\rho(b_j)} = (s', t, h')^{\rho(b_j)} = (s', t', h'')$

where $sha_i = sa_ih = s'h'$ and $th'b_j = tb_jh' = t'h''$ (1)

Also $(s, t, h)^{\rho(b_j)\cdot\rho(a_i)} = (s, t_1, h_1)^{\rho(a_i)} = (s_1, t_1, h_2)$

where $sh_1a_i = sa_ih_1 = s_1h_2$, and $thb_j = tb_jh = t_1h_1$ (2)

From (1) and (2), we have,

$$sa_i = s_1h_2h_1^{-1} = s'h'h^{-1} \text{ and } tb_j = t_1h_1h^{-1} = t'h''h'^{-1}$$

Therefore $s_1 = s'$, $h_2h_1^{-1} = h'h^{-1}$, $t_1 = t'$, $h_1h^{-1} = h''h'^{-1}$

that is, $h_2 = h'h^{-1}\,h_1 = h_1h^{-1}\,h' = h''$ (because H is abelian).

Thus $\rho(a_i)\,\rho(b_j) = \rho(b_j)\,\rho(a_i)$ and consequently, A_m and B_n generate their generalised direct product in P.

Remark 1. If $a_i = s_i^*h_i^*$, $b_j = t_j^*h_j^*$, then for all $h \in H$,

$$1 = [a_i, h] = [s_i^*h_i^*, h] = [s_i^*, h]^{h_i^*}\,[h_i^*, h] = [s_i^*, h]^{h_i^*}$$

gives, $[s_i^*, h] = 1$

Therefore $s_i^* \in C_A(H)$ for all $i=1, 2, \ldots, m$

similarly: $t_j^* \in C_B(H)$ for all $j=1, 2, \ldots, n$

Thus if $S^*=\{s^*, s^* \in C_A(H)\}$, $T^*=\{t^* \in C_B(H)\}$ then since $[\rho(s^*), \rho(t^*)]=1$ for all $s^* \in S^*$, $t^* \in T^*$, if $K^*=gp\{\rho(S^*)\}$, $L^*=gp\{\rho(T^*)\}$, then K^* and L^* generate their direct product in any permutational product of A and B amalgamating an abelian subgroup H.

Remark 2. The condition on H about its being abelian Is necessary. Example 2.44 would suffice to show this.

§ 4. Although the concept and nature of the generalised free product with amalgamations is entirely different from that of a permutational product of a given family of groups still there are some results which exhibit certain analogies in the behaviour of free products and permutational products. For example, it is known that if $\{G_\alpha\}$ and $\{G'_\alpha\}$, (α belonging to an indexed set I) are two families of groups each having a common subgroup H and H′ respectively, and a system of homomorphisms φ_α of G_α onto G'_α where any two φ_α and φ_β agree on H, is given, and further if F and F′ are the free products of $\{G_\alpha\}$ and $\{G'_\alpha\}$ amalgamating H and H′ respectively then there exists a homomorphism φ of F onto F′ which extends all the φ_α. The proof of the above theorem is due to Hanna Neumann (cf. for example [3] Theorem 1.1) We prove a corresponding result giving a relationship between the permutational products of two families of groups G_α and G'_α amalgamating H and H′ respectively. We have, of course, to choose the transversals in a particular manner. It is sufficient to show this for the permutational products of only two groups because the proof for more than two groups is not at all different.

Let A, B and A′, B′, be the given groups having common subgroups H and H′ respectively. Let S and S′ be coset representatives of H and

H' in A and A' respectively. Let φ_A be a homomorphism of A onto A'. If S' is the set of distinct elements in the image $S\varphi_A$ of S under this mapping, then we say that S and S' are "equivalent transversals" of H and H' in A and A' respectively. We similarly choose a pair of equivalent transversals T and T' of H and H' in B and B' respectively corresponding to a homomorphism φ_B: $B \to B'$ which coincides with φ_A on H.

We now prove the following:

4.1. Theorem: Let $\varphi_A: A \to A'$, $\varphi_B: B \to B'$ be homomorphisms of A onto A' and of B onto B' such that $\varphi_A \mid H = \varphi_B \mid H$, and further $A = SH$, $B = TH$. If a pair of transversals S', T' of H' in A', B' equivalent to S and T respectively, is chosen then there exists a homomorphism φ of the permutational product $P(\underline{\underline{A}}; S, T)$ of $\underline{\underline{A}} = am\ (A, B; H)$ onto $P'(\underline{\underline{A}}'; S', T')$ of $\underline{\underline{A}}' = am\ (A', B'; H')$, which extends both φ_A and φ_B.

Proof: First we show that in the permutational products $P(\underline{\underline{A}}; S, T)$ and $P'(\underline{\underline{A}}'; S', T')$, the mappings

$$\rho(a) \to \rho'(a') \text{ and } \rho(b) \to \rho'(b')$$

where $a \xrightarrow{\varphi_A} a'$, $b \xrightarrow{\varphi_B} b'$, $a \in A$, $b \in B$, $a' \in A'$, $b' \in B'$, are homomorphisms of $\rho(A)$ onto $\rho'(A')$ and of $\rho(B)$ onto $\rho'(B')$ respectively.

Denote by ρ^{-1} the inverse of the isomorphic mapping $a \to \rho(a)$ of A onto $\rho(A)$ for $a \in A$. Then if

$$a_1 \xrightarrow{\varphi_A} a'_1,\ a_2 \xrightarrow{\varphi_A} a'_2$$

we have $\rho(a_1)\, \rho^{-1}\, \varphi_A\, \rho' = (a_1)\, \varphi_A\, \rho' = (a'_1)\, \rho' = \rho'\ (a'_1)$

$$\rho(a_2)\, \rho^{-1}\, \varphi_A\, \rho' = \rho'(a'_2)$$

and $\rho(a_1)\, \rho(a_2)\, \rho^{-1}\, \varphi_A\, \rho' = \rho(a_1 a_1)\, \rho_A^{-1}\, \rho'$

$$= (a_1 a_2)\, \varphi_A\, \rho'$$

$$= (a'_1 a'_2)\, \rho'$$

$$=\rho'(a'_1a'_2)$$
$$=\rho'(a'_1).\ \rho'(a'_2).$$

Thus $\rho(a)\rightarrow\rho'(a')$ is a homomorphism of $\rho(A)$ onto $\rho'(A')$, $a\in A$, $a'\in A'$. Similarly $\rho(b)\rightarrow\rho'(b')$ gives a homomorphism of $\rho(B)$ onto $\rho'(B')$.

To prove that P′ is a homomorphic image of P, we have to show that any law which holds in P also holds in P′. Let $(s, t, h)\in K=S\times T\times H$ and let w be a word in elements from $\rho(A)$ and $\rho(\mathbf{B})$, that is

$$w=\rho(a_1)^{\delta_1}\rho(b_1)\ \ldots.\ \rho(a_n)\ \rho(b_n)^{\delta_2}$$
$$=\rho(a_1{}^{\delta_1})\ \rho(b_1)\ \ldots.\ \rho(a_n)\ \rho(b_n{}^{\delta_2})$$

where δ_1 and δ_2 are 0 or 1, and $a_i\in A$, $b_j\in B$. Then

$$(s,\ t,\ h)^w=(s,\ t,\ h)^{\rho(a_1^{\delta_1})\ \rho(b_1)\ldots.\rho(a_n)\ \rho(b_n^{\delta_2})}$$

$$=(s_1,\ t_1,\ h_1^*)^{\rho(a_2)\ldots.\rho(a_n)\rho(b_n^{\delta_2})}$$

. . . .

. . . .

$$=(s_n,\ t_n,\ h_n^*).$$

Here $sha_1^{\delta_1}=s_1h_1$ and $th_1b_1=t_1h_1{}^*$

$$s_1h_1^*a_2=s_2h_2\qquad t_1h_2b_2=t_2h_2^*$$

. . . .

. . . .

$$s_{n-1}h_{n-1}^*\ a_n=s_nh_n,\quad t_{n-1}h_nb_n=t_nh_n^*.$$

Also if the elemehts s', t' of S′, T′ correspond to the elements s, t of S, T under the homomorphisms φ_A, φ_B respectively and $a'_i=a_i\varphi_A$, $b'_i=b_i\varphi_B$, then

$$(s', t', h')^{\rho'(a'^{\delta_1}_1)\,\rho'(b'_1)\ldots\ \rho'(a'_n)\,\cdot\rho'(b'^{\delta_2}_n)}$$

$$=(s\varphi_A,\ t\varphi_B,\ h\varphi_A)^{\rho'(a_1^{\delta_1}\varphi_A)\ \rho'(b_1\varphi_B)\ldots\rho'(a_n\varphi_A)\ \rho'(b_n^{\delta_2}\varphi_B)}$$

$$=(s_1\varphi_A,\ t_1\varphi_B,\ h_1^*\varphi_A)^{\rho'(a_2\varphi_A)\ \rho'(b_2\varphi_B)\ldots\rho'(a_n\varphi_A)\ \rho'(b_n^{\delta_2}\varphi_B)}$$

....

....

$$=(s_n\varphi_A,\ t_n\varphi_B,\ h_n^*\ \varphi_A)$$

because $s\varphi_A\ h\varphi_A\ a_1^{\delta_1}\varphi_A=(sha_1^{\delta_1})\varphi_A=(s_1h_1)\varphi_A=s_1\varphi_A h_1\varphi_A$

and $s_i\varphi_A h_i^*\varphi_A a_{i+1}\varphi_A=(s_ih_i^*a_{i+1})\varphi_A=(s_{i+1}\ h_{i+1})\ \varphi_A=s_{i+1}\varphi_A h_{i+1}\varphi_A$.

Similarly $t\varphi_B h_1\varphi_B b_1\varphi_B=(th_1b_1)\varphi_B=(t_1h_1^*)\ \varphi_B=t_1\varphi_B h_1^*\varphi_B$.

and $t_i\varphi_B\ h_{i+1}\varphi_B b_{i+1}\varphi_B=(t_ih_{i+1}b_{i+1})\varphi_B=(t_{i+1}h^*_{i+1})\ \varphi_B=t_{i+1}\varphi_B h^*_{i+1}\varphi_B$

Thus if

$$\gamma\ (\ldots,\ \rho(a),\ \rho(b),\ \ldots)=1$$

is a relation in $P\ (\underline{A};\ S,\ T)$, then it is also a relation in $P'\ (\underline{A}';\ S',\ T')$,

By van Dyck's theorem, there exists a homomorphism φ of P onto P′ which extends both φ_A and φ_B. This complete the proof of the theorem.

If we are given an amalgam of two groups A and B amalgamating H and if

$$A=A_1\times A_2\times\ldots\times A_n,\ B=B_1\times B_2\times\ldots\times B_n$$

in such a way that

$$A\cap B=H=H_1\times H_2\times\ldots\times H_n$$

where $H_i=A_i\cap B_i$ for all $i=1,\ 2,\ \ldots,\ n$, one will expect the permutational product of A and B to be the direct product of the permutational products of the amalgams *am* $(A_i,\ B_i;\ H_i)$ provided the transversals are chosen in the natural way. The following theorem for which we suppose the groups to have, without any loss of generality, only two factors, confirms this guess.

4.2. Theorem : Let $P_1(\underline{\underline{A}}_1; S_1, T_1)$ and $P_2(\underline{\underline{A}}; S_2, T_2)$ be permutational products of A_1, B_1, amalgamating H_1 and of A_2, B_2 amalgamating H_2. Let further, $A=A_1\times A_2$, $B=B_1\times B_2$ such that $H=H_1\times H_2$. Then the permutational products $P(\underline{\underline{A}}; S, T)$ of A and B amalgamating H is the direct product of P_1 and P_2 if $S=S_1\times S_2$ and $T=T_1\times T_2$.

Proof. We first prove that in P, the group P'_1 generated by $\rho'(A_1)$, $\rho'(B_1)$ is a direct factor of P. For this we have to show that $\rho'(A_1)$, $\rho'(B_2)$ and $P'(A_2)$, $P'(B_1)$ commute elementwise.

Since the transversals of H in A and B are taken as $S=S_1\times S_2$, $T=T_1\times T_2$, therefore, every $(s, t, h)\in K=S\times T\times H$, can be written as:

$$(s_2s_1,\ t_1t_2,\ h_1h_2)$$

where $s_i \in S_i$, $t_i \in T_i$, $h_i \in H_i$, $i=1, 2$.
Let $a_1 \in A_1$, $b_2 \in B_2$, then

$$(s_2s_1,\ t_1t_2,\ h_1h_2)^{\rho'(a_1)\ \rho'(b_2)}=(s_2s'_1,\ t_1t_2,\ h_2h'_1)^{\rho'(b_2)}$$
$$=(s_2s'_1,\ t_1t'_2,\ h'_2h'_1)$$

where $s_2s_1h_2h_1a_1=s_2s_1h_1a_1h_2=s_2s'_1h_2h'_1=s_2s'_1h'_1h_2$
i.e. $s_1h_1a_1=s'_1h'_1$
and $t_1t_2h_2h'_1b_2=t_1t_2h_2b_2h'_1=t_1t'_2h'_2h'_1$
hence $t_2h_2b_2=t'_2h'_2$
However, also $(s_2s_1,\ t_1t_2,\ h_2h_1)^{\rho'(b_2)\ \rho'(a_1)}=(s_2s'_1,\ t_1t'_2,\ h'_2h'_1)$.

Therefore, $[\rho(a_1), \rho(b_2)]=1$ and consequently in P, $\rho'(A_1)$, $\rho'(B_2)$ commute elementwise. By symmetry $\rho'(A_2)$ and $\rho'(B_1)$ also commute elementwise. Hence if

$$P'_1=gp\ \{\rho(A_1), \rho(B_1)\} \text{ and } P'_2=gp\ \{\rho'(A_2), \rho'(B_2)\}$$

then P'_1 and P'_2 commute element by element. Also, since A_1, A_2; B_1, B_2; A_1, B_2; A_2, B_1 all interesect trivially therefore $P'_1 \cap P'_2=\{1\}$. Since P is generated by P'_1 and P'_2 together, P'_1 and so also P'_2 is a direct factor in P. Thus $P=P'_1\times P'_2$. We now show that $P'_i\cong P_i$, where P_i is the permutational product of A_i, B_i amalgamating H_i

corresponding to the transversals S_i, T_i of H_i in A_i and B_i respectively, $(i=1, 2)$.

We take a word

$$w=\rho(a_{11})\ \rho(b_{11})\ldots\rho(a_{1n})\ \rho(b_{1n}),\ a_{1i}\in A_1,\ b_{1i}\in B_1, \text{ in } P_1.$$

Then if $(s_1, t_1, h_1)\in K_1=S_1\times T_1\times H_1$, we have

$$\begin{aligned}(s_1, t_1, h_1)^{\rho(a_{11})\ \rho(b_{11})\ldots\rho(a_{1n})\ \rho(b_{1n})}\\ &=(s_2, t_1, h'_2)^{\rho(b_{11})\ldots\rho(a_{1n})\ \rho(b_{1n})}\\ &=(s_2, t_2, h_2)^{\rho(a_{12})\ \rho(b_{12})\ldots\rho(a_{1n})\ \rho(b_{1n})}.\\ &\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\\ &=(s_{n+1}, t_{n+1}, h_{n+1})\end{aligned}$$

where

$$\begin{aligned} &s_1h_1a_{11}=s_2h'_2 &&\text{and} && t_1h'_2b_{11}=t_2h_2\\ &s_2h_2a_{12}=s_3h'_3 &&,, && t_2h'_3b_{12}=t_3h_3\ .\\ &\ldots\ldots\ldots\ldots && && \ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots && && \ldots\ldots\ldots\ldots\\ &s_nh_na_{1n}=s_{n+1}\ h'_{n+1}, && && t_nh'_{n+1}\ b_{1n}=t_{n+1}\ h_{n+1}\end{aligned}$$

Also if $(s_2s_1, t_2t_1, h_2h_1)\in K=S\times T\times H$, then

$$\begin{aligned}(s_2s_1, t_2t_1, h_2h_1)^{\rho'(a_{11})\ \rho'(b_{11})\ldots\rho'(a_{1n})\ \rho'(b_{1n})}\\ &=(s_2s_{n+1}, t_2t_{n+1}, h_2h_{n+1}).\end{aligned}$$

Therefore if $w=1$ is a relation in P_1, it is also a relation in P'_1, but the converse also holds. Since the relations of P_1 and P'_1 are in one to one correspondence therefore they are isomorphic. Similarly $P_2\cong P'_2$. Thus

$$P=P_1\times P_2$$

as required.

That the choice of transversals of H in A and B in this particular way is necessary, is shown by the following example.

4.3. Example : Let A_1 and B_1 be symmetric groups of degree three, that is

$$A_i = gp\ \{a_i,\ c_i;\ a_i^3 = c_i^2 = (a_i c_i)^2 = 1\}$$

$$B_i = gp\ \{b_i,\ c_i;\ b_i^3 = c_i^2 = (b_i c_i)^2 = 1\}$$

and $$A = A_1 \times A_2,\ B = B_1 \times B_2 \text{ and } H = H_1 \times H_2$$

where $H_i = gp\ \{c_i;\ c_i^2 = 1\}$, Let P_i be the permutational product of A_i, B_i amalgamating H_i corresponding to the transversals $S_i = \{c_i,\ a_i,\ a_i^2\}$ and $T_i = \{1,\ b_i,\ b_i^2\}$. Then each of the $P'_i s' i = 1, 2$, has order 162 (cf. B. H. Neumann [1]) and therefore the order of $P_1 \times P_2$ is $162 \times 162 = 26244$. However, the permutational product P of A and B corresponding to the transversals $S = \{1,\ a_1,\ a_1^2\} \times \{1,\ a_2,\ a_2^2\}$ and $T = \{1,\ b_1,\ b_1^2\} \times \{1,\ b_2,\ b_2^2\}$, being an extension of an elementary abelian group of exponent 3 and order 81 by the four group is of order 324. P is, therefore, not isomorphic to $P_1 \times P_2$.

$P(\underline{\underline{A}};\ S,\ T)$ is, of course, isomorphic to $P'_1 \times P'_2$ where

$P'_i = P'_i\ (\underline{\underline{A}}_i;\ S'_i,\ T'_i)$ with $S'_i = \{1,\ a_i,\ a_i^2\}$, $T'_i = \{1,\ b_i,\ b_i^2\}$.

§ 5. Let P be a property satisfied by the groups of a certain amalgam $\underline{\underline{A}}$, (*e.g*, the property of being finite, soluble, etc.). As is shown in [1] and [2] an amalgam with a property P may not always be embeddable in a group having the same property. Sufficient conditions of one kind or another on the amalgam are, therefore essential. A condition which is fairly close to the hypothesis that the amalgamated subgroup be central in both the constituents is the existence of transversals, one in each of the constituents, which are centralised by the amalgamated subgroup.

We have seen in theorem 3.2 that when the amalgamated subgroup H has, in the constituents A and B, transversals S and T respectively which it centralises, the permutational product $P(\underline{\underline{A}};\ S,\ T)$ of A and B amalgamating H is the generalised direct product of K, L and $\rho(H)$

where K$=gp$ $\{\rho(S)\}$, L$=gp$ $\{\rho(T)\}$, amalgamating K $\cap$ L$=$R. But then in such a case P ($\underline{\underline{A}}$; S, T) belongs to the least variety containing both A and B, so that if A and B are in a variety $\underline{\underline{V}}$, we have :

5.1. Theorem: Let $\underline{\underline{A}}=am$ (A, B; H) be an amalgam of two groups belonging to a variety $\underline{\underline{V}}$, then $\underline{\underline{A}}$ is embeddable in a group beloning to $\underline{\underline{V}}$ provided that H possesses transversals S and T which it centralises in both A and B.

B. H. Neumann (cf. [1]) has shown that an amalgam of two soluble groups is embeddable in a soluble group if the amalgamated subgroup is central in one of the constituents. The above remark slightly varies this result. However, going further, we prove that, as suggested by Neumann's lemma 2.41, the condition that the amalgamated subgroup is central in one of the constituents can be replaced by the requirement that it possesses in one of the constituents a transversal which it centralises.

Some more results concerning the embeddability of a soluble amalgam (that is, an amalgam of soluble groups.) in a soluble group using a different sufficient condition will also be obtained.

We first repeat some of the definitions in [1]. Given two soluble groups A and B with a common subgroup H, by S and T we shall denote arbitrary but then fixed transversals of H in A and B respectively. By B^S, we mean the set of all functions on S with values in **B**. This is turned into a group by defining the multiplication of any two functions $f, g \in B^S$ as

$$fg(s)=f(s)\,g(s) \text{ for all } s \in S$$

Definition: A mapping γ of the set K of all triplets (s, t, h) $s \in S$, $t \in T$, $h \in H$, into itself is a quasi-multiplication (or more precisely a quasi **B**$-$S multiplication) if there is a function f on S to **B** such that

$$(s, t, h)^{\gamma}=(s, t', h')$$

with
$$t'h'=thf(s).$$

The mapping γ associated with f is denoted by $\gamma(f)$ The set of all such functions are known to form a group Γ isomorphic to B^S (cf. lemma 5.1 [1]).

To see that the results in [1] and [2] hold under the weaker condition, namely the existence of a transversal centralised by the amalgamated subgroup in one of the constituents, it is enough to show that the fundamental Lemma 5.2 [1] holds.

We restate the lemma making use of the new hypothesis.

5.2. Lemma (Compare with Lemma 5.2 [1]). Let H possess in one of the constituents, say A, a transversal S which it centralises. Then $\rho(A)$ normalises Γ. More precisely, for $a \in A$, $\gamma=\gamma(f) \in \Gamma$ there is an element $\gamma'=\gamma(f')$ of Γ such that

$$\rho^{-1}(a)\ \gamma(f)\ \rho(a)=\gamma(f')$$

for $f' \in B^S$ and ρ a permutation of $K=S\times T\times H$.

Proof: The proof is essentially the same as in [1] except that here we just use the condition that H is contained in the centraliser of one of its transversals say S in a constituent A.

We compute

$$\gamma'=\rho(a)^{-1}\ \gamma(f)\ \rho(a)=\rho(a^{-1})\ \gamma(f)\ \rho(a)$$

for $a \in A$, $\gamma(f) \in \Gamma$, $f \in B^S$. Let $(s, t, h) \in K=S\times T\times H$, then

$$(s, t, h)^{\rho(a)^{-1}\ \gamma(f)\ \rho(a)}=(s_1, t, h_1)^{\gamma(f)\ \rho(a)}$$
$$=(s_1, t_1, h_2)^{\rho(a)}$$
$$=(s_2, t_1, h_3)$$

where $sha^{-1}=s_1h_1,\ th_1\,f(s_1)=t_1h_2,\ s_1h_2a=s_2h_3$ (*i*)

Also $(s, t, h)^{\gamma(f')}=(s, t', h')$

with $thf'(s)=t'h'$ (*ii*)

We have to show that $s=s_2$, $t'=t_1$, $h'=h_3$ to prove that $\gamma(f')=\rho(a^{-1})\ \gamma(f)\ \rho(a)$.

Now from (*i*) we have

$$sh=s_1h_1a=s_1h_1h_2^{-1}s_1^{-1}s_2h_3=s_2h_1h_2^{-1}h_3$$

($\because$ $[s_1, h_1 h_2^{-1}]=1$ $[s_2, h_1 h_2^{-1}]=1$), Therefore $s=s_2$ and $h=h_1 h_2^{-1} h_3$, that is

$$h_3=h_2 h_1^{-1} h \qquad (i')$$

Also $t_1 h_3=t_1 h_2 h_1^{-1} h$ from (i')

$=t h_1 f(s_1) h_1^{-1} h$ from (i)

$=t h h^{-1} h_1 f(s_1)\ h_1^{-1} h$

$=t h (h_1^{-1} h)^{-1} f(s_1) h_1^{-1} h.$

Thus $t_1 h_3 = t h c$

where $c=(h_1^{-1}h)^{-1} f(s_1)\ h_1^{-1} h$

which depends only s_1 and $h_1^{-1}h$. If we write

$$(sha^{-1})^{\sigma}=(sa^{-1})^{\sigma}=s_1,\ (sha)^{-\sigma+1}=h_1$$

we see that s_1 is independent of h. Also from $sha^{-1}=s_1 h_1$ that is, $h_1^{-1}\ hsa^{-1}=s_1$, we have,

$$h_1^{-1} h=s_1 a s^{-1}=(sa^{-1})^{\sigma}\ as^{-1}$$

which depends only on a and not on h. Thus if we define the elements f, g, f' of B^S by

(1) $f_1(s)=f(s_1)=f[(sa^{-1})^{\sigma}]$

(2) $g(s)=h_1^{-1} h=(sa^{-1})^{\sigma}\ as^{-1}$

then $f'(s)=(h_1^{-1}h)^{-1} f(s_1)\ h_1^{-1}\ h.$

$=(g(s))^{-1} f_1(s)\ g(s)$

for all $s \in S$ and we have

$$f'=g^{-1} f_1\ g$$

and $\gamma'=\gamma(f_1)=\gamma(g^{-1} f_1\ g)$

where $f_1 \in B^S$, $g \in H^S$. This completes the proof of the lemma.

This gives us

5.21. Corollary (Compare with corollary 5.2. [1]). If $H \subseteq C_A\ (S)$, then

$$[\rho(A), \Gamma] \subseteq \Gamma$$

Here [K, L] means the group generated by all commutators $[k, l]$ $k \in K$, $l \in L$. The proof of the above corollary follows from the fact that $\rho(S)$ normalises Γ.

5.22. Corollary (Compare with corollary 5.3 [1]). If $H \subseteq C_A(S)$, then

$$[\rho(A), \Gamma'] \subseteq \Gamma'$$

where Γ' denotes the derived group of Γ.

5.23. Corollary (Compare with corollary 5.3. [1]. If $H \subseteq C_A(S)$, then

$$[\rho(A), \rho(B)] \subseteq \Gamma'$$

Consequently, we have,

5.3. Theorem (Compare with theorem 5.4) [1]. If, in one of the constituents, say A, the amalgamated subgroup H possesses a transversal S which is centralised by H, and if further, A and B are soluble of length l and m respectively, then the permutational product $P(\underline{\underline{A}}; S, T)$ of A and B is soluble of length n where n satisfies the relation

$$n \leq l + m - 1.$$

We further remark, without going into the details, that the results proved in [2] based on lemma 5.2 [1] still hold under this weaker assumption.

Let F* denote one of the following properties of a group; being locally finite (LF), of finite exponent (FE), or being periodic (P). We discuss here the embeddability of a soluble or F* amalgam in a soluble or F* group respectively, making use of a sufficient condition of somewhat different nature. The following lemma plays a key role in the discussion that follows.

5.4. Lemma : Let the groups A and B be extensions of a normal subgroup S of A by H and of a normal subgroup T of B by H respectively. Then S and T serve as transversals and the permutational product $P(\underline{\underline{A}}; S, T)$ of the amalgam $\underline{\underline{A}} = am\,(A, B; H)$ belongs to the least variety containing both A and B.

Proof: We first show that in P, $\rho(S)$ and $\rho(T)$ commute elementwise. Let $(s_1, t_1, h_1) \in K = S \times T \times H$, then for $s \in S$, $t \in T$, we have

$$(s_1, t_1, h_1)^{\rho(s)\,\rho(t)} = (s_1 s', t_1, h_1)^{\rho(t)} = (s_1 s', t_1 t', h_1)$$

and $\quad (s_1, t_1, h_1)^{\rho(t)\,\rho(s)} = (s_1 s', t_1 t', h_1)$

where $\quad s_1 h_1 s = s_1 s' h_1,\ t_1 h_1 t = t_1 t' h_1$

in both cases. Therefore $[\rho(s), \rho(t)] = 1$ for all $s \in S$, $t \in T$.

Before going further into the details of the proof of the above lemma we remark that for a slightly more general situation when $S \cap T = Z \neq \{1\}$ is central in both S and T and S_1, T_1, given by $S_1 Z = S$, $T_1 Z = T$ are taken as transversals, $\rho(S_1)$ and $\rho(T_1)$ still commute elementwise in the permutational product $P(\underline{\underline{A}}; S_1, T_1)$ of the amalgam $\underline{\underline{A}} = am\,(A, B: \{Z, H\})$.

Since P is generated by $\rho(S)$, $\rho(T)$ and $\rho(H)$ and moreover $\rho(H)$ normalises both $\rho(S)$ and $\rho(T)$ and hence also $\rho(S) \times \rho(T)$, P is an extension of $\rho(S) \times \rho(T)$ by $\rho(H)$.

Next we look at the amalgam of the groups A and B rather differently. We regard these groups as generated by S, H, and T, H_1 respectively and suppose that there is a fixed isomorphism between H and H_1 so that the amalgam of A and B consists of quintuplets $(A, B, H, H_1, \varphi;\ H\varphi = H_1)$. We take the direct product G of A and B. Since $A = SH$, $B = TH_1$ and S, T are normal subgroups of A and B respectively, $S \times T$ is normal in G. Take the 'diagonal'

$$H' = \{(h, h_1) = (h, h\varphi);\ h \in H,\ h_1 \in H_1,\ h_1 = h\varphi\}$$

of the direct product $H \times H_1$ in G. H' is clearly isomorphic to H. Also the groups $A' = \{(sh, h\varphi);\ s \in S,\ h \in H\}$ and $B' = \{(h, th\varphi);\ h \in H,\ t \in T\}$ are isomorphic to A and B respectively under the isomorphisms $a = sh \rightarrow a' = (sh, h\varphi)$, $b = th\varphi \rightarrow b' = (h, th\varphi)$ and since $(sh, h\varphi) = (h', th'\varphi)$ implies $sh = h'$, $h\varphi = th'\varphi$ which give $s = 1 = t$, $h = h'$, the intersection of A' and B' is precisely H'.

The groups A' and B' can also be taken as generated by $S' = \{(s, 1);\ s \in S\} \cong S$ and H' and by $T' = \{(1, t);\ t \in T\} \cong T$ and H'

respectively. However, since

$$h'^{-1}s'h' = (h^{-1}, h^{-1}\varphi)(s, 1)(h, h\varphi)$$
$$= (s^h, 1),$$

and $$h'^{-1}t'h' = (1, t^{h\varphi}).$$

$h' \in H'$, $s' \in S'$, $t' \in T'$; H′ induces the same automorphisms in S′ and T′ an H and H_1 do in S and T respectively. The group P′ generated by $S' \times T'$ and H′ in $A \times B$, therefore, contains isomorphic copies A′, B′ of A and B, intersect in a common subgroup H′ isomorphic to H and H_1 and is an extension of $S' \times T'$ by H′ corresponding to the above automorphisms.

As shown above P also is an extension of $\rho(S) \times \rho(T) \cong S' \times T'$ by $\rho(H) = H'$. Further, these two extensions correspond to the 'same' groups of automorphisms as induced by H in S and T and are, therefore 'equivalent'. (cf. Kurosh, [14]).

Thus P′ is isomorphic to P. Since P is a subgroup of $A \times B$ and belongs to the least variety containing both A and B, P $(\underline{\underline{A}}; S, T)$ also has this property. This completes the proof of the lemma.

As a consequence of the above remarks, we have :

5.41. Corollary : A soluble or nilpotent amalgam of two groups A and B which are extensions of their normal subgroups S and T respectively by a group H, is embeddable in a soluble or nilpotent group.

5.42. Corollary : If the groups A and B of lemma 5.4 have the property F*, then their amalgam is embeddable in an F* group.

5.43. Corollary : If the groups A and B of lemma 5.4 are p-groups for the same p, that is, very element has order a power of p, then their amalgam is embeddable in a p-group.

In the case of finite groups this is a very special case of a result of Graham Higman. [13].

REFERENCES

1. Neumann, B.H; Permutational products of groups. J. Austral. Math. Soc. Vol. I (1960), pp. 299-310.

2. Neumann, B.H; On amalgams of periodic groups. Proc. Roy. Soc. (A) 255 (1960) pp. 477-489.

3. Neumann, B,H; An essay on free products of groups with amalgamations. Phil. Trans. Roy. Soc. London, (A) 246 (1954) pp. 503-554.

4. Neumann, H; Generalised free products of groups with amalgamated subgroups, I. Am. J. Math. 70 (1948) pp. 590-625.

5. Neumann, H: Generalised free products of groups with amalgamated subgroups, II. Am. J. Math. 71 (1949) pp. 491-540.

6. Schreier, Otto: Der Untergruppen der freien Gruppen. Abh. Math. Sem. Univ. Hamburg, 5, (1927) pp. 161-183).

7. Baer, R and Levi, F: Free Produkte und ihre untergruppen, Comp. Math., Vol 3 (1936) pp. 391-398.

8. Neumann, B.H: On topics in infinite groups. Lecture notes, Tata Institute of Fundamental Research, Bombay, 1960.

9. Neumann, B. H: and Neumann, Hanna: A remark on generalised free products J. Lond. Math. Soc. 25 (1950) 202-204.

10. Baer, R: Free sums of groups and their generalisations; An analysis of the associative law. Amer. J. Math. 81 (1949) pp. 706-742.

11. Neumann, B.H, and Neumann, H: A contribution to the embedding theory of group amalgams. Proc. Lond. Math. Soc. (3). 3 (1953) pp. 245-256.

12. Neumann, H. Varieties of groups: Lecture notes. Manchester Coll. Sc. Tech. 1962-1963.

13. Higman, Graham: Amalgams of p-groups, J. of Algebra I; (1964) 301-305.

14. Kurosh, A.G: The Theory of Groups, 2nd ed. Translated from Russian by K. A. Hirsch, two volumes, Chelsea Publishing Co. N. Y., 1955.

15. Majeed, A; Permutational products of groups and embedding theory of group amalgams: M.A. Thesis, Australian National University, 1966.

ON A SUM FUNCTION OF FUNCTIONS OF PARTITIONS

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1. Introduction :—Some congruences involving functions $P_r\ (n)$ were proved with respect to mod. p in [1]; where

$$P_1\ (n)=\sum_{k=0}^{n} P(k)\ P(n-k),$$

$P_r(n)=\sum_{k=0}^{n} P(k)\ P_{r-1}\ (n-k)$, and $P(n)$ are unrestricted partitions of n. In this paper we introduce the function $S(n)=S(n,\ p)=\sum_{r=1}^{p} P_r\ (n)$ & prove a few congruence properties of $S(n)$ with respect to mod. p.

2. Theorem I :

$$\sum_{k=1}^{mp+r} P(k)\ S(mp+r-k)\equiv\sum_{k=1}^{m} P(k)\ P_1(\overline{m-kp}+r)\ (\text{mod. } p),$$

where $m\geq 1$ & $0\leq r\leq p-1$

We prove the following lemma:—

Lemma: $P_p\ (n)\equiv\sum_{k=0}^{r} P(k)\ P(n-pk)$ mod. $p)$;

where $rp\leq n<(r+1)\ p$.

Proof:—$P_p\ (n)=\sum_{k=0}^{n} P(k)\ P_{p-1}\ (n-k)=\sum_{k=0}^{n} P_{p-1}\ (k)\ P(n-k)$

$$=\sum_{k=0}^{p-1} P_{p-1}(k)\ P(n-k)+P_{p-1}(p)\ P(n-p)$$

$$+\sum_{k=p+1}^{2p-1} P_{p-1}\ (k)\ P(n-k)+P_{p-1}\ (2p)\ P(n-2p)+\ldots.$$

$$+\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$+\sum_{k=mp+1}^{(m+1)p-1} P_{p-1}(k)\, P(n-k)+P_{p-1}(mp)\, P(n-\overline{m+1}\,p)$$

$$+\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$+\sum_{k=\overline{r-1}\,p+1}^{rp-1} P_{p-1}(k)\, P(n-k)+P_{p-1}(rp)\, P(n-rp)$$

$$+\sum_{rp+1}^{n} P_{p-1}(k)\, P(n-k)$$

$$=P_{p-1}(0)\, P(n)+\sum_{m=0}^{r}\ \sum_{k=mp+1}^{\overline{m+1}\,p-1} P_{p-1}(k)\, P(n-k)$$

$$+\sum_{m=0}^{r-1} P_{p-1}(\overline{m+1}\,p)\, P(n-(m+1)\,p)$$

$$+\sum_{rp+1}^{n} P_{p-1}(k)\, P(n-k). \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \quad (2.1)$$

We have already proved in [1] that

$$P_{p-1}(n)\equiv 0 \pmod{p}, \text{ if } p \text{ does not divide } n.$$
$$\equiv P(m) \pmod{p} \text{ if } n=mp.$$

using these relations (2.1) we obtain

$$P_p(n)\equiv \sum_{k=0}^{r} P(k)\, P(n-pk) \pmod{p} \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \quad (2.2)$$

Proof of the Theorem :

By definition

$$S(mp+r)=\sum_{k=0}^{mp+r} P(k)\, [P(mp+r-k)+S(mp+r-k)-P_p(mp+r-k)];$$

$$0=\sum_{k=0}^{mp+r} P(k)\, [P(mp+r-k)]+\sum_{k=1}^{mp+r} P(k)\, S(mp+r-k)$$

$$-\Bigg[\sum_{k=0}^{r} P(k)\, P_p(mp+r-k)+\sum_{k=r+1}^{p+r} P(k)\, P_p(mp+r-k)+\ldots.$$

$$+\sum_{k=sp+r+1}^{(s+1)p+r} P(k)\, P_p(mp+r-k)+\ldots+\sum_{k=\overline{m-1}\,p+r+1}^{mp+r} P(k)\, P_p(mp+r-k)\Bigg];$$

using (2.2) we obtain

$$0 \equiv \sum_{k=0}^{mp+r} P(k)\, P(mp+r-k) + \sum_{k=1}^{mp+r} P(k)\, S(mp+r-k)$$

$$- \Bigg[\sum_{k=0}^{r} P(k) \sum_{k_1=0}^{m} P(k_1)\, P(mp+r-k-pk_1)$$

$$+ \sum_{k=r+1}^{p+r} P(k) \sum_{k1=0}^{m-1} P(k_1)\, P(mp+r-k-pk_1) + \quad \ldots \ldots \ldots \ldots \ldots \ldots$$

$$+ \sum_{k=sp+r+1}^{(s+1)p+r} P(k) \sum_{k_1=0}^{m-s-1} P(k_1)\, P(mp+r-k-pk_1) + \ldots.$$

$$+ \sum_{(m-1)p+r+1}^{mp+r} P(k)\, P(mp+r-k) \Bigg] \pmod{p}.$$

On simplification, we get

$$\sum_{k=1}^{mp+r} P(k)\, S(mp+r-k) \equiv P(m)\, P_1(r) + P(m-1)\, P_1(p+r) + P(m-2)\, P_1(2p+r)$$

$$+ \ldots. + P(k)\, P_1(\overline{m-k}p+r) + \ldots.$$

$$+ P(1)\, P_1(\overline{m-1}p+r) \pmod{p}$$

$$\equiv \sum_{k=1}^{m} P(k)\, P_1(\overline{m-k}p+r) \pmod{p}. \qquad \ldots \ldots \quad (2.3)$$

$$\text{Cor: } \sum_{k=1}^{5m+t} P(k)\, S(5m+t-k) \equiv 0 \pmod{5}; \qquad \ldots \ldots \ldots \ldots \quad (2.4)$$

when $t=2$, 3 & 4.

In [1] we proved that $P_{p-4}\ (pn+t) \equiv 0 \pmod{p}$ when t is a non-residue of $\frac{r(r+1)}{2} \equiv t \pmod{p}$ and $P_{p-4}\ (pn+p-b) \equiv 0 \pmod{p}$ when b is a least positive residue of $8b \equiv 1 \pmod{p}$.

When we substitute $p=5$ in (2.3) & use the above-mentioned results we obtain (2.4).

3. Theorem 2.

(*a*) $S(n) \equiv 0 \pmod{p}$ $\qquad 0 \leq n \leq p-2$

(*b*) $S(p-1) \equiv 1 \pmod{p}$

(*c*) $S(p) \equiv 0 \pmod{p}$

We prove the following lemma :—

Lemma 2. $\sum_{k=1}^{n} P(k)\ S(n-k) \equiv 0 \pmod{p}\ 1 \leq n \leq p-1$

Proof : $S(n) = \sum_{0}^{n} P(k)\ [P(n-k) + S(n-k) - P_p(n-k)]$

$$= P_1(n) + P(o)\ [S(n) - P_p(n)] + \sum_{1}^{n} P(k)\ [S(n-k) - P_p(n-1)]$$

$$\text{or} \quad \sum_{k=1}^{n} P(k)\ S(n-k) = \sum_{k=1}^{n} P(k)\ P_p(n-k) + P_p(n) - P_1(n)$$

$$\equiv \sum_{k=0}^{n} P(k)\ P(n-k) - P_1(n) \pmod{p}$$

(using Lemma 1)

$$\equiv 0 \pmod{p} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.1)$$

Proof of (*a*)

Since $S(o) \equiv 0 \pmod{p}$
& $S(1) \equiv 0 \pmod{p}$,

the proof follows from (3.1) by induction.

Proof of (*b*) & (*c*)

From Theorem 1, we obtain

$$\sum_{k=1}^{p+r} P(k)\ S(p+r-k) \equiv P(k)\ P_1(r) \pmod{p}.$$

When $r=0$, we have

$$\sum_{k=1}^{p} P(k)\ S(p-k) \equiv P(1)\ P_1(0) \pmod{p}.$$

or $S(p-1) \equiv 1 \pmod{p}$.
(using (*a*))

When $r=1$ we have

$$\sum_{k=1}^{p+1} P(k)\ S(p+1-k) \equiv P(1)\ P_1(1) \pmod{p}$$

or $P(1)\ S(p) + P(2)\ S(p-1) \equiv 2 \pmod{p}$
(using (*a*))

or $S(p) \equiv 0 \pmod{p}$ (using (*b*))

In the end it may be mentioned that it would be interesting to investigate further congruence properties of $S(n)$.

REFERENCE

1. S. Manzur Hussain & M. H. Kazi 'On Some Congruences Involving Functions of Partitions', J. of Sc. Research, University of the Punjab Vol. I. No. 2.

CONTENTS