VOLUME III APRIL 1970

THE PUNJAB UNIVERSITY JOURNAL OF MATHEMATICS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE PUNJAB
LAHORE, PAKISTAN

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OF
MATHEMATICS

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CONTENTS

NUMERICAL ITERATION OF SIMULTANEOUS SEMI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS OVER A NON-RECTANGULAR REGION*

by

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INTRODUCTION

Point-wise numerical iteration by over-relaxation of finite difference analogs of the Dirichlet Boundary Value Problem [1]* over rectangular regions was presented by Young [2]. Greenspan [3] extended this method to the Nuemann Boundary Value Problem over rectangular regions and the Dirichlet Boundary Value Problem over a region with curved boundary. He further reported that the solution to Nuemann Problem over a region with curved boundary did not converge. Forsythe and Wason [4] pointed out that this difficulty was mainly due to the finite-difference approximation of the normal derivative on the curved boundary. It was later shown by the author, [5], that if appropriate care was exercised in this finite-difference approximation and in the over-relaxation process, then the method will be feasible for regions with curved boundaries, not only for Nuemann Problem, but even for the Mixed Boundary Value Problem. This point is further elaborated later in the paper. In all the works cited above, attention was focused only on a single second-order partial differential equation with prescribed boundary conditions.

In some problems of interest in Fluid Mechanics, a higher order partial differential equation may arise quite naturally. For instance, the problem of slow flow of a fluid with uniform density and viscosity gives rise to a fourth-order partial differential equation in terms of a stream function [1]. However, an alternate formulation of the same problem in terms of two variables, the stream function and the vorticity function,

* This research was conducted by Hydronautics, Inc., Laurel, Md., and was supported by the office of Saline Water, U.S. Department of Interior under contract No. 14-01-001-1246.

will result in two simultaneous, second-order partial differential equations. Generally, the later formulation will result in coupled boundary conditions. This paper concerns the numerical interation by over-relaxation of such a system of partial differential equations.

STATEMENT OF PROBLEM

In search for high-performance desalination units, one of the suggestions [6] to enhance heat transfer during evaporation of saline waters is to use rivulet flows on vertical metal plates lined with low surface-tension materials such as teflon. It is presumed that water will form rivulets of circular arc shape running down the plate, instead of a continuous film and hence high conduction rates will occur at the rivulet corners. Since salt accumulation on the rivulet free surface at any section will vary along this surface, a surface-tension gradient will set in. This will tend to drive the liquid on the free-surface toward the rivulet corner, which, in turn, will create a swirling motion within the rivulet.

With proper simplifications, the swirling motion within this surface-tension driven rivulet is expressed by the following mathematical problem. The details of the derivation are deliberately omitted here and the interested reader is refered to [6].

1. Differential Equations :

$$\frac{\partial^2 \zeta}{\partial x^2}+\frac{\partial^2 \zeta}{\partial y^2}=\zeta \tag{1}$$

$$\frac{\partial^2 \zeta}{\partial x^2}+\frac{\partial^2 \zeta}{\partial y^2}+\left(R_s \frac{\partial \psi}{\partial y}\right)\frac{\partial \zeta}{\partial x}-\left(R_s \frac{\partial \psi}{\partial x}\right)\frac{\partial \zeta}{\partial y}=0 \tag{2}$$

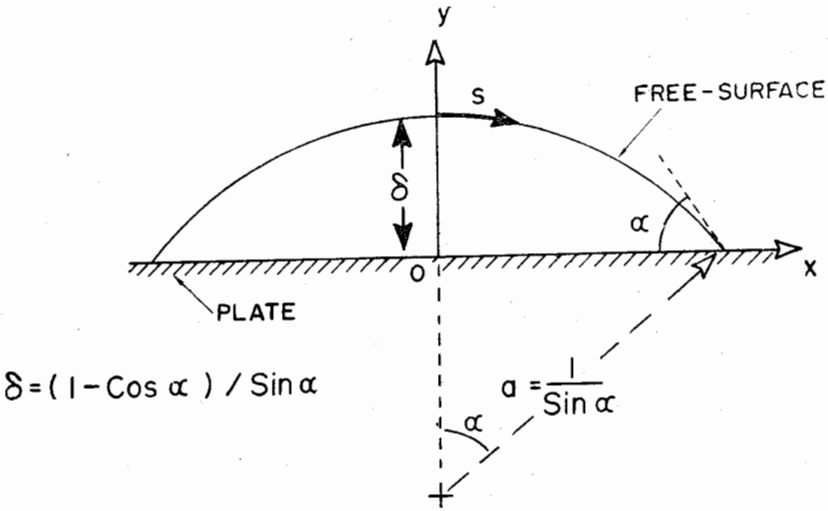
Here, $\psi=\psi(x, y)$=stream function

$\zeta=\zeta(x, y)$=vorticity function

R_s=Reynolds number, a constant parameter, characteristic of the flow

x, y=length co-ordinates along and normal to the plate respectively, in rivulet cross-section.

2. Boundary Conditions :



(Fig. 1)

$$\psi = 0 \text{ on all boundaries} \tag{3}$$

$$\zeta = 0 \text{ for } x = 0 \textit{ i.e.} \text{ at the line of rivulet symmetry} \tag{4}$$

$$\zeta = \frac{\partial^2 \psi}{\partial y^2} \text{ for } y = 0, \textit{ i.e.} \text{ along the plate surface} \tag{5}$$

and $$\zeta = \frac{\partial T}{\partial s} + \frac{2}{a} \frac{\partial \psi}{\partial n} \text{ along the rivulet free-surface} \tag{6}$$

Here, n and s are normal and tangential to the free-surface respectively, and a is the radius of the curvature of that surface.

3. Equation of Free-Surface :

$$x^2 + y^2 + 2y \left(\frac{\text{Cos } \alpha}{\text{Sin } \alpha} \right) = 1 \tag{7}$$

Here α is the contact angle for the rivulet, a property of the lining material.

4. Input Function :

The term $\frac{\partial T}{\partial s}$ in Eq. 6 is the contribution of the known surface-

tension gradient and hence itself known. It may be pointed out that Eq. 6 is the direct consequence of the balance of surface-tension and shear force at the free-surface.

It should be noted that the differential equations in Eq. 1 and 2 are coupled with Eq. 2 being semi-linear. The boundary conditions on ψ are well-defined, but those on ζ are ψ–dependent *i.e.* coupled.

Without getting involved with the physics of the problem, its mathematical content could be summarized in the following question :

> Given a function $T(x, y)$ described on the curve represented by Eq. 7 (α known), Eq. 1 and 2 are to be solved simultaneously with boundary constraints as in Eq. 3, 4, 5, and 6 to determine functions $\psi(x, y)$ and $\zeta(x, y)$ throughout the region bounded by the x-axis, the y-axis and the curve in Eq. 7.

Even for the simplest possible distribution of $T(x,y)$, the above mathematical problem is hopelessly complicated to yield to an analytical solution.

The problem, although phrased here quite arbitrarily, is a real physical one and hence warrants the search for at least a numerical answer.

SYSTEM OF DIFFERENCE EQUATIONS

Consider a semi-linear partial differential equation as :

$$A(x, y)\frac{\partial^2 u}{\partial x^2}+C(x, y)\frac{\partial^2 u}{\partial y^2}+D(x, y)\frac{\partial u}{\partial x}+E(x, y)\frac{\partial u}{\partial y}+F(x, y)\,u=G(x, y) \tag{8}$$

For a known ζ–distribution Eq. 1 is a special case of Eq. 8 and so is Eq. 2 for a known distribution of ψ. Thus, Eq. 8 represents both Eq. 1 and 2, with u being ψ and ζ respectively. Hence, for the purpose of generation of a finite-difference analog, Eq. 8 is considered as defined over a region R bounded by a boundary B. If R is spanned by a square grid of mesh size h, then at any interior point (x_0, y_0) the difference equation corresponding to Eq. 8 is : [7]

$$\alpha_0 u(x_0, y_0)=\alpha_1 u(x_0+s_1h, y_0)+\alpha_2 u(x_0, y_0+s_2h)+\alpha_3 u(x_0-s_3h, y_0)+\alpha_4 u(x_0, y_0-s_4h)-t(x_0, y_0) \tag{9}$$

Where

$$\begin{aligned}
\alpha_1 &= (2A\ (x_0, y_0) + hs_3\ D\ (x_0, y_0))/s_1(s_1+s_3)\\
\alpha_2 &= (2C\ (x_0, y_0) + hs_4\ E\ (x_0, y_0))/s_2(s_2+s_4)\\
\alpha_3 &= (2A\ (x_0, y_0) - hs_1\ D\ (x_0, y_0))/s_3(s_1+s_3)\\
\alpha_4 &= (2C\ (x_0, y_0) - hs_2\ E\ (x_0, y_0))/s_4(s_2+s_4)\\
\alpha_0 &= \alpha_1+\alpha_2+\alpha_3+\alpha_4 - F\ (x_0, y_0)\ h^2
\end{aligned} \tag{10}$$

and

$$t = G\ (x_0, y_0)\ h^2.$$

Here

$$s_1 = \frac{x_1 - x_0}{h},\ s_3 = \frac{x_0 - x_3}{h} \tag{11}$$

$$s_2 = \frac{y_2 - y_0}{h},\ s_4 = \frac{y_0 - y_4}{h}$$

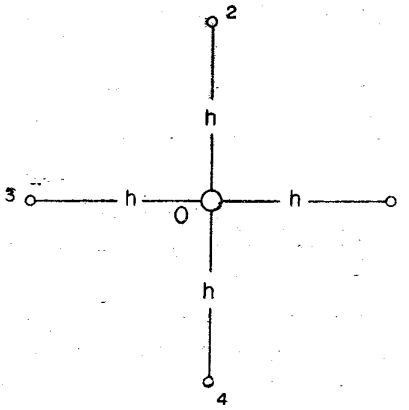


Fig. 2)

Referring to Figure 2 the s_i $(i=1,2,3,4)$ indicate the relative position of the neighbours of a point in the square grid. Hence these may be called the neighbourhood scale fractions. If the region R is a rectangle, then a square grid may be so chosen, that the boundary points coincide with the nodes of the grid. In that case, all s_i are equal to 1 for each point interior to R. Otherwise, in general

$$s_i \leq 1, \text{ for } i=1, 2, 3, 4.$$

Eq. 9 represents the required difference equations corresponding to the

co-efficients of the given differential equation in Eq. 8. A proper choice of h will give α_i such that

$$\alpha_0 \geqq \alpha_1+\alpha_2+\alpha_3+\alpha_4 \tag{12}$$

Eq. 9 for node in the grid will give a system of simultaneous equations as :

$$\sum_{i=1}^{N} a_{ij}\, u_i = b_j \tag{13}$$

where N is the total number of nodes at which the unknown values of the function are to be determined and b_j are the known values contributed by the boundary conditions. For the Dirichlet problem, N is the number of the interior points of R, whereas for the mixed boundary value problem, N is the number of the interior points plus the points on the boundary on which the normal derivative is specified. The condition required in Eq. 12 guarantees that the coefficient matrix (a_{ij}) in Eq. 13 is diagonally dominant.

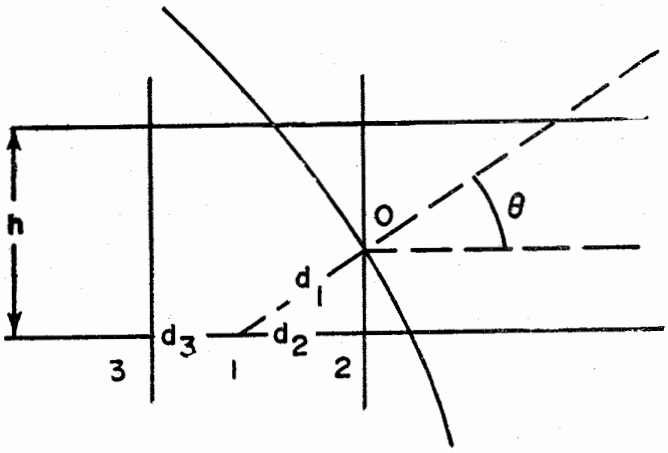
ITERATIVE METHOD

Among the Point Iterative Methods for the solution of a system of linear difference equation (Equation 13), Young's method of successive over-relaxation has proved to be the [4] most successful. The convergence of a solution by this method requires the coefficient matrix (a_{ij}) to have a special property, which Young [2, 7] has called property—A. For the Dirichlet problem in a rectangular region, with the boundary passing through the nodes of a square mesh, the co-efficient matrix of the difference system does possess that required property. Hence, for that problem the convergence of the itreration process is guaranteed [2].

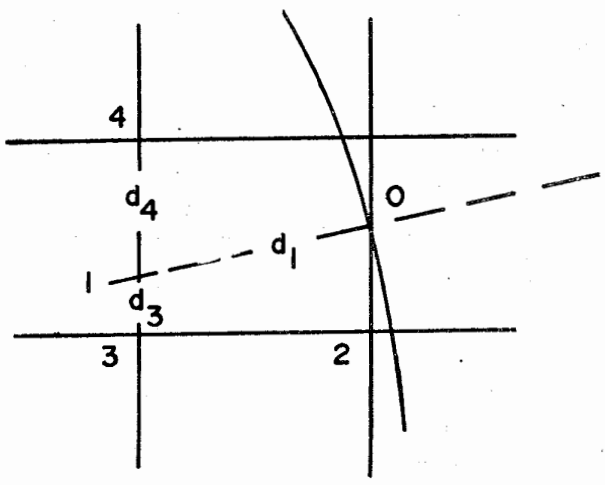
The property—A of Young is further relaxed [3, 4] and it is shown that such a method of iteration will converge even for a Dirichlet problem for a region bounded by a curved boundary. It is observed that the essential features of the co-efficient matrix are that it is : (1) irreducible, (2) symmetric, (3) positive definite and (4) diagonally dominant.

When the normal derivative is specified on a boundary which is parallel to the grid lines of the mesh, the extension of the method for the Dirichlet problem is rather simple, since the mesh points are on the boundary and the

normal to the boundary is along one of the grid lines. However, when the specified normal derivative is to be satisfied on a curved boundary, then the numerical method adopted for approximating the normal derivative affects the coefficient matrix quite adversely as shown in the next paragraph.



(*a*)



(*b*) (Fig. 3)

APPROXIMATION FOR NORMAL DERIVATIVE ON A CURVED BOUNDARY

Referring to Figure 3*a*, the most obvious way to approximate the normal derivative is to express it in terms of its components along the co-ordinate axes as :

$$\left.\frac{\partial u}{\partial n}\right|_{\text{B.P.}} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \tag{14}$$

It is shown that the co-efficient matrix for the difference equations employing Eq. 14 loses both symmetry and diagonal dominance [3,4]. However, if a first order approximation to the normal derivative is used as :

$$\left.\frac{\partial u}{\partial n}\right|_{\text{B.P.}} = \frac{u_1 - u_0}{d_1} \tag{15}$$

Where

$$u_1 = u_2\frac{d_3}{h} + u_3\frac{d_2}{h} \tag{16}$$

(d_1, d_2, d_3 are shown in Figure 3*a*) then the co-efficient matrix is at least diagonally dominant, although not symmetric. It is to be noted that in Eq. 16 d_2 and d_3 are both to be less than h. If the normal to the boundary at a point is as shown in the Figure 3*b*, then Eq. 16 is to be replaced by

$$u_1 = u_4\frac{d_3}{h} + u_3\frac{d_4}{h} \tag{17}$$

(d_3, d_4 are shown in Figure 3b) thus guaranteeing the diagonal dominance of tne co-efficient matrix.

PROCEDURAL REMARK

Since all the conditions required of the co-efficient matrix for the convergence of the iterative solution are only sufficient and not necessary, there is no reason to assume that the convergence of the iterative method may not be possible even if any one of the properties is not present in the coefficient matrix. Based on this conjecture, Greenspan conducted numerical experiments on a Laplace equation with normal derivatives specified over a circular boundary and found that the iteration did not

converge [3]. However, a procedural modification of the iteration scheme has been reported [5] to converge and was utilized here. This procedure and the mechanics of grid generation on digital computer is described in the Appendix at the end.

METHOD OF ITERATION

For a given $T(x, y)$ on the free-surface of the rivulet, the differential equations and the boundary conditions in Eq. 1-5 are discretised according to Eq. 9 and 15. The iterative procedure is started by first assuming an arbitrary distribution of ψ and ζ in the region. For this ζ, the finite-difference analog of Eq. 1 is solved at each point to re-estimate ψ, conforming to the boundary conditions in Eq. 2. With this re-estimated $\psi-$ distribution, the co-efficients of Eq. 2 are evaluated at each point, and boundary conditions on ζ are estimated according to Eq. 5 and 6. The given distribution of T on free-surface is also employed in Eq. 6. Having established these, the finite-difference analog to Eq. 2 is solved at each point in the region to give a second estimate of ζ distribution. Hopefully, this second estimate of ζ will conform to the differential equation and the boundary conditions better than the first one assumed arbitrarily. If this process is repeated again and over again, solutions to the original problem should be approached.

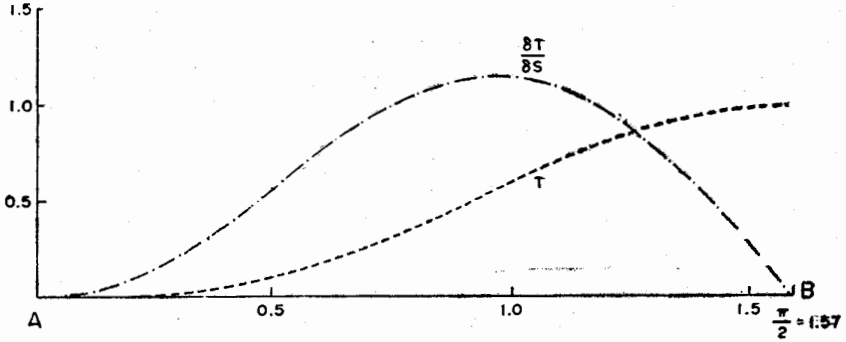
COMPUTATIONAL RESULTS

Figure 4 shows a distribution of T on the free-surface given by $\alpha=90°$ in Eq. 7. The abscissa of this plot is the arc length of the free-surface. Here, for computational experiment, T is assumed as

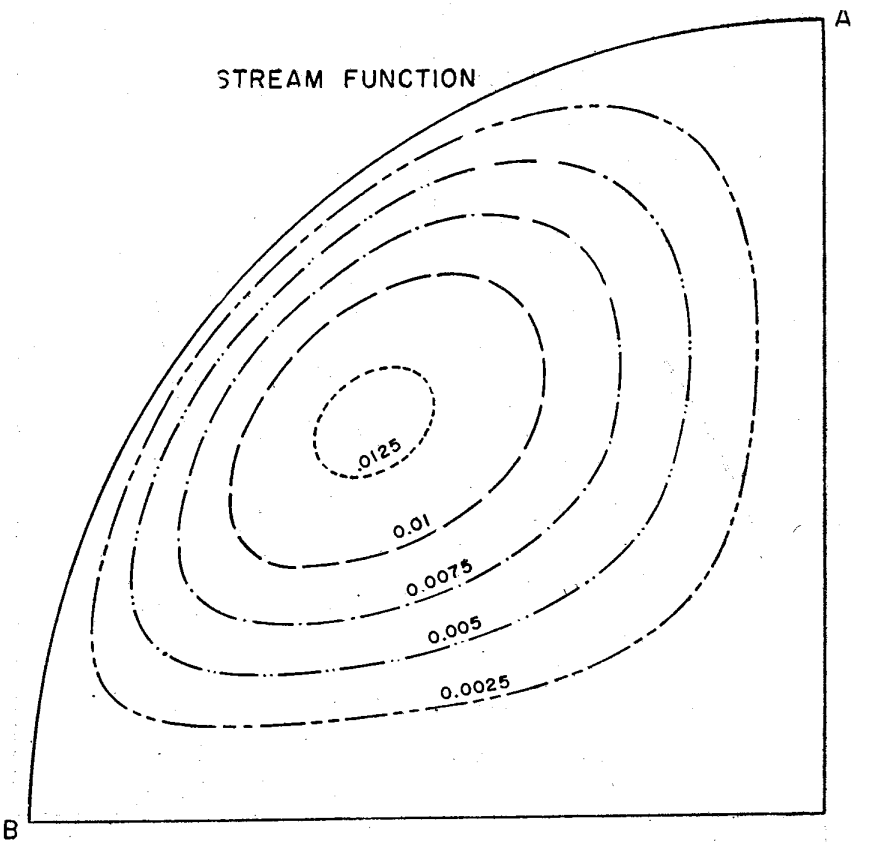
$$T=\mathrm{Cos}^3\,(\pi/2\,(1-s)) \tag{18}$$

For this assumed distribution of $\frac{\partial T}{\partial s}$, and $R_s=50$, the above iteration scheme was carried out on an IBM 1130 digital computer. The iterations were repeated for about 100 times. The final distributions of ψ and ζ are shown in Fig. 5 and 6, respectively. The distributions seem reasonable intuitively, and the results of heat transfer obtained by this

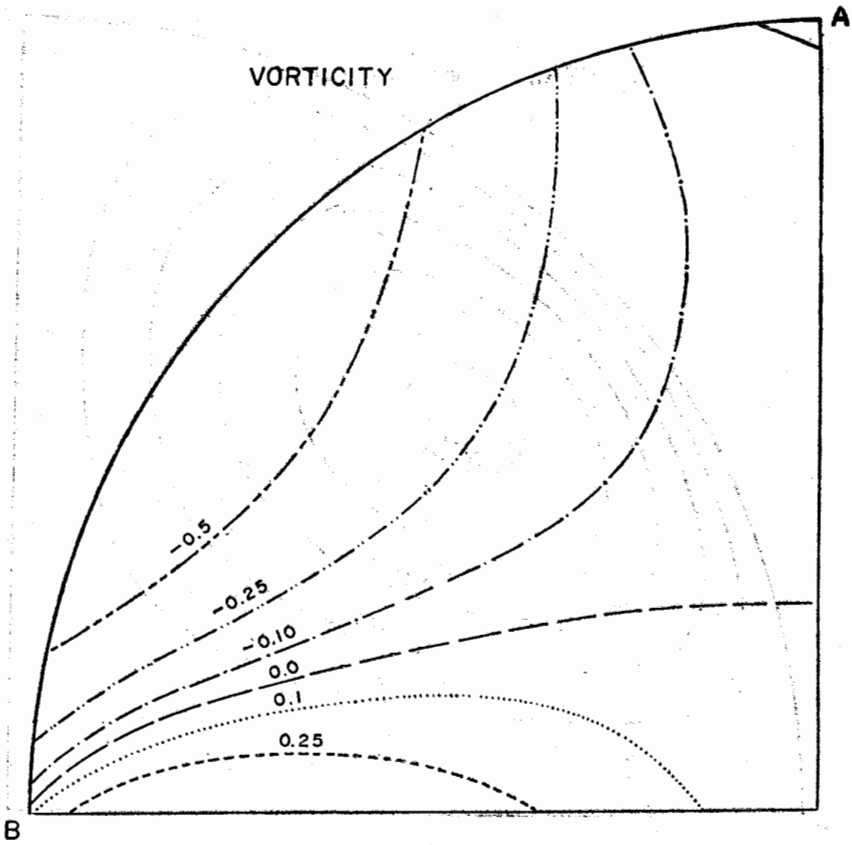
same procedure are being reported separately elsewhere [8]. Here it is intended to emphasize that even if this is only a numerical experimentation on a digital computer, it does open a way to handle the simultaneous semi-linear partial differential equations with coupled boundary conditions which, otherwise, will not, in general, be tractable. The numerical experimentation has shown a definite trend of convergence of iterative scheme; however, a proof of convergence is lacking and needs further research and exploration.



(Fig. 4)



(Fig. 5)



(Fig. 6)

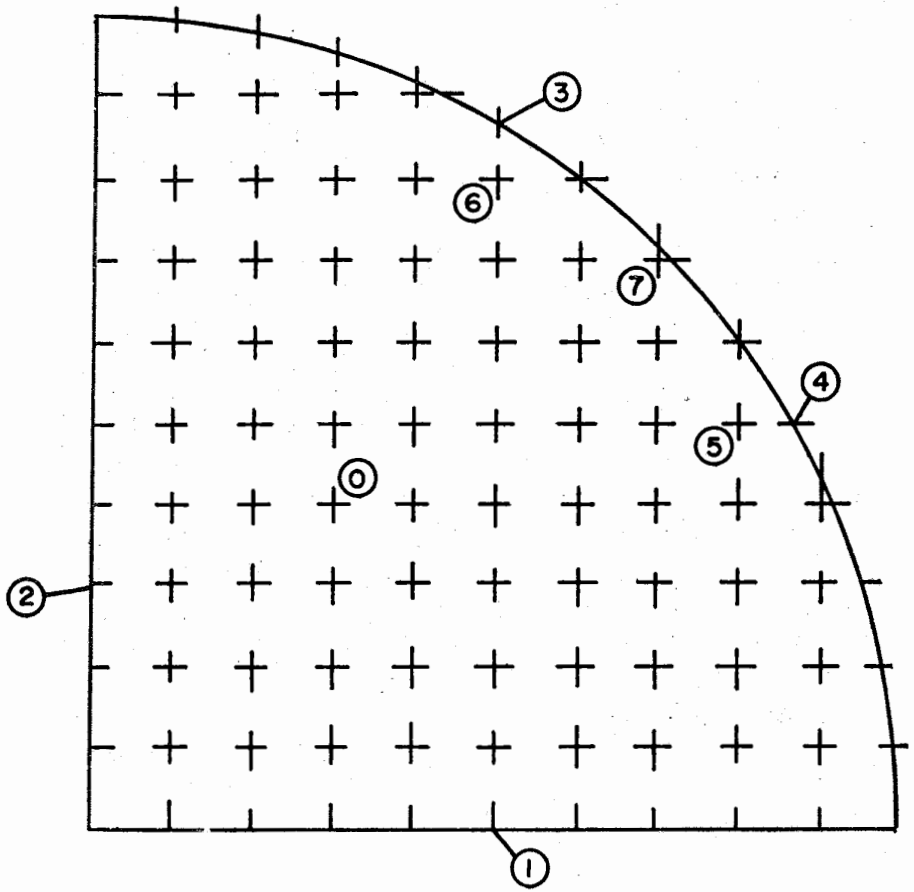
APPENDIX I

GRID GENERATION

In order to achieve a point-by-point iteration on the difference equations in Eq. 9, the co-efficients α_i for respective points will be needed. These depend on the co-efficients of the differential equation Eq. 8 and the scale factor as shown in Eq. 10. For the interior points, S_i's are all unity, but for points near the boundary, these should be estimated as for Eq. 11.

This could, in principle, be done by laying out a grid graphically, and picking off the co-ordinates for the various points manually. However, in practice, this is rather crude and cumbersome, if not impossible, especially if the grid size is to be decreased for increased accuracy, and various regions with different α's are to be investigated. In view of this, an automatic computer routine was developed which picked off all the internal and boundary points of the rectangular grid network imposed on the aeutal region, labeling the different points as shown in Fig. 7. The points are classified as : (1) Points on the x-axis ; (2) Points on the y-axis ; (3) Points on the curved boundary and vertical grid lines ; (4) Points on the curved boundary and horizontal grid lines; (5) (6) and (7) are points interior but adjacent to the curved boundary with a neighbouring point as 3 or 4, or 3 and 4, respectively. The points interior to the region, with all its neighbours also interior, are labeled as 0.

These choices are required in order to estimate the normal derivatives on the curved boundary using Eq. 14 or 15, and generating automatically the appropriate scale factors for finite difference equation, Eq. 9.



(Fig. 7)

APPENDIX II

Finite-difference equations of the form in Eq. 13 are obtained for each interior point only. Even if the boundary condition involves the specification of the normal derivative at the curved boundary, the function values at the interior points are relaxed to conform to the differential equation by solving Eq. 13. Having found the new function values at the interior points, the values at the curved boundary are changed so as to remain conformed to the boundary conditions.

Essentially, this amounts to iterating or relaxing first a Dirichlet type problem and then, after the interation, forcing the values of the function at the boundary points to conform to the prescribed conditions, if the normal derivative is specified. Hence, mathematically, the value of the function at the boundary points changes from iteration-step to iteration-step. Putting it another way, for the Nuemann or Mixed Boundary Value Problem, Eq. 13 is actually replaced by

$$\sum_{i=1}^{N} a_{ij}\, u_i^{(n)} = b_j^{(n-1)} \tag{18}$$

where N is always the number of interior points and n is iteration-step number. For more detail, the reader is referred to [5].

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SYMMETRIES AND DYNAMICS IN HIGH ENERGY PHYSICS

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The purpose of this review is to indicate the kind of mathematics used in high energy physics and to survey the subject as it enters the seventies. Some of the original work referred to has been carried out by the author in collaboration with M. Rafique, University of the Punjab, Lahore.

1. Introduction:

In ordinary non-relativistic quantum mechanics the elastic scattering problem

$$a+b \rightarrow a+b \tag{1}$$

is studied using the one-particle Schrodinger wave equation appropriate to the motion of a relative to b (with h equal to Planck's constant divided by 2π)

$$-\frac{h^2}{2\mu}\nabla^2\psi+\mathrm{V}(\mathbf{r})\,\psi=\mathrm{E}\,\psi$$

where μ is the reduced mass of a and b, $\mathrm{V}(\mathbf{r})$ is a potentil fucntion, E the energy, ψ a probability amplitude. One seeks solutions of the form

$$\psi(\mathbf{r})=e^{i\mathbf{k}.\mathbf{r}}+f(\theta,\phi)\,\frac{e^{ikr}}{r} \tag{3}$$

the first term representing a plane wave propagating in the direction of the vector $\mathbf{k}$ and the second an outgoing spherical wave (there is an overall time dependence $\exp(-iEt/h)$). The physics of the problem is contained essentially in the amplitude f whose squared modulus $|f|^2$ is the so-callaed *differential scattering cross-section* which measures the ratio of particles emerging at θ, ϕ (considering a specific direction and looking at the spherical wave) to particles in the incident beam (the plane wave). The problem is in principle solvable being mathermatically well defined — given $\mathrm{V}(\mathbf{r})$

solve equation (2) to find $f(\theta, \phi)$ the amplitude describing the result of a counting process.

In relativistic high energy physics the problem is somewhat ill defined and often one knows fairly little about the amplitude T (the analogue of f) in detailed terms. We treat T as a function of serveral variables

$$s,\ t,\ u\ ;\ \mathrm{I},\ \mathrm{B},\ \mathrm{Y} \qquad (4)$$

whose meaning I propose to explain. The semi-colon in (4) divides the world of high energy physicists into two camps peopled by persons whose activities are significantly different and whose inter-relationship is generally unproductive. The variables I, B, Y refer to *internal quantum numbers i.e.* to the detailed nature and properties of the "elementary" or "fundamental" objects which are involved in the scattering process. This kind of physics is called *symmetries*. The variables s, t, u correspond to the scattering itslef (energies, momenta, etc) for given values of I, B, Y. This kind of physics is called *dynamics*.

Perhaps one should say that "elementary" is a misleading term to apply in particle physics. Very few objects (such as the proton) are stable and most (such as the neutron decaying via $n \rightarrow p + e + \nu$) decay spontaneously into other objects. The division of objects into "elementary' and "composite" is quite arbitrary depending on an arbitrary statement of what one means by a short decay time. Short lived objects are usually thought of as non-elementary. Rather than make a distinction of such blatant artificiality a common philosophy is to regard no object as fundamental and every object as being composed of other objects (inculding perhaps itself) — *bootstrap* philosophy

In dynamics one accepts as given some set of basic objects so in this sense dynamics treats every object as fundamental.

2. Symmetries :

The quantum number B is probably the easiest to explain roughly to the layman. It stands for *baryon number i.e.* the number of particles of a family of objects (to which belong the familiar objects kown as proton and neutron) whose number is conserved in some sense in processes of the type considered.

The properties of proton p and neutron n are very similar and there is evidence from scattring theory that forces $p-p$, $p-n$, $n-n$ are charge independnet. It is convenient to treat p and n as two different *charge states* of the same particle the *nucleon* (Q$=+1$ for p, Q$=0$ for n).

If one introduces operators $a_p\dagger$, $a_n\dagger$ which create respectively a proton and a neutron (the operators a_p, a_n destroy the corresponding particles) one may construct four operators

$$a_p\dagger a_p,\ a_p\dagger a_n,\ a_n\dagger a_p,\ a_n\dagger a_n \tag{5}$$

which leave unaltered the baryon number. The commutation relations for these operators are

$$a_k\dagger a_l + a_l a_k\dagger = \delta_{kl}$$
$$a_k a_l + a_l a_k = 0.$$

Convenient combinations of the operators (5) are

$$\mathrm{B} = a\dagger_p a_p + a_n\dagger a_n \tag{6}$$

the baryon number itself and

$$\begin{aligned} \tau_+ &= a_p\dagger a_n \\ \tau_- &= a_n\dagger a_p \\ \tau_0 &= \tfrac{1}{2}(a_p\dagger a_p - a_n\dagger a_n) = \mathrm{Q} - \tfrac{1}{2}\mathrm{B} \end{aligned} \tag{7}$$

where τ_+ changes a neutron into a proton and τ_- does the reverse ; Q is the electiric charge.

A possilbe representation of these operators is provided by the matrices

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad \tau_0 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

operating in the space spanned by vectors

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

representing respectively the proton and the neutron. The charge operator for example is represented by

$$\mathrm{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The commutation relations for the operators (7) follow from those for the a's and are familiar to everyone with an elementary knowledge of angular mometum theory. They are

$$\begin{aligned} [\tau_0, \tau_+] &= \tau_+ \\ [\tau_0, \tau_-] &= -\tau_- \\ [\tau_+, \tau_-] &= 2\tau_0 \end{aligned} \tag{8}$$

identical to those for angular momentum operator

$$J_{\pm} = J_x \pm i J_y, \quad J_0 = J_3. \tag{9}$$

The relations (8) define a Lie algebra isomorphic with the usual angular momentum algebra.

By analogy with angular momentum theory there exists an operator I^2 (analogous to $J^2 = J_x^2 + J_y^2 + J_z^2$) having eigenvalues of the form $I(I+1)$ and $p-n$ states can be classified into multiplets — sets of $2I+1$ states with eigenvalues I_0 of τ_0 varying from $-I$ to I. In the two nucleon system one finds two *isotopic* multiplets as listed in the table (10)

State	I_0	I	
$p_1\, p_2$	1	1	
$\frac{1}{\sqrt{2}}(p_1 n_2 + n_1 p_2)$	0	1	triplet
$n_1 n_2$	−1	1	
$\frac{1}{\sqrt{2}}(p_1 n_2 - n_1 p_2)$	0	0	singlet

(10)

The operator I^2 commutes with the elements $\tau_{\pm}$, τ_0 of the Lie algebra and is in fact the only *Casimir operator* of the algebra which means that it is a rank one algebra (a Casimir operator is one other than the unit operator which commutes with every element and the number of such operators defines the rank of the Lie algebra). Just as J_+, J_3 can be thought of as generators of the group 0(3), the operators τ_+, τ_0 can be thought of as

generators of a group SU (2) a unimodular unitary group of transformations in a two dimensional isotopic spin space (spanned by vectors representing p and n).

Other baryons are observed and their properties have been catalogued. For example the reactions

$$\pi^- + p \rightarrow \Sigma^- + k^+ \tag{11}$$

and

$$\pi^\circ + n \rightarrow \Sigma^+ + k^- \tag{12}$$

though seemingly possible on the basis of considerations of charge, baryon number, energy, etc. are not both observed despite their obvious similarity. The operation of this selection rule is conveniently expressed by introducing a new quantum number S (the *strangeness*) whose conservation is used to account for the observations. In the example the strangeness balance is

$$\begin{aligned} 0+0 &= -1+1 \\ 0+0 &\neq -1-1 \end{aligned} \tag{13}$$

Existent in nature is a chargeless particle $\wedge$ of strangeness -1 whose properties are similar to those of the neutron (strangeness 0) and it might be convenient to treat n and $\wedge$ as equivalent in a strangeness space as differen states of the same particle. This has been done and a U-spin formalism identical to the above described isotopic spin formalism has been developed and the same rank one Lie algebra and group SU(2) are involved.

If one combines the $p-n$ equivalence with $\wedge - n$ equivalence one arrives at a formalism with a basic $p-n-\wedge$ triplet (three different states of a single entity the *Sakaton*) out of which one may build more complex systems e.g. two Sakaton systems. The underlying group is SU(3) which contiains of course SU(2) sub-groups and the multiplet structure of composite systems reduces to a study of group representations.

The Sakata model does not agree with physics in that the permitted composite structures of the model do not match at all well the observations of high energy physics. However, an identical model—the *quark model*—based on a fictious (or perhaps simply unobserved in the present experimental region) triplet of objects called *quarks* in which $p, n, \wedge$ etc. appear as composite entitties. Not only does this SU(3) model provide a pattern

fitting observation but also it has proved its predictive value (Ω^- discovery).

The quantum numbers B, I, Y (Y is the hypercharge B+S) describe symmetries of the basic entities. Let us now turn from symmetries to dynamics and take B, I, Y as given.

3. Dynamics :

In $s-t-u$ physics it is actually often convenient to discard the selection rules altogether and artificially reimpose them at the end of the calculation. In any event in a description such as this it is simplest to deal with chargeless, spinless, particles specified completely by energy-momentum four-vectors. Also as I do not wish to have to define the term antiparticle I shall assume that no antiparticles exist or simply that all my particles (they have no charge) are their own antiparticles.

The process

$$a+b \rightarrow c+d \tag{14}$$

may be described by four energy-momentum four-vectors p_i, $i=1, 2, 3, 4$, formally meaured ingoing. There appear to be seven invariants which one can construct from four four-vectors with vanishing sum (energy-momentum conservation)

$$p_i^2=m_i^2, \ i=1, 2, 3, 4 \tag{15}$$

the particle masses and

$$\begin{aligned} s&=(p_1+p_2)^2=(p_3+p_4)^2 \\ t&=(p_1+p_3)^2=(p_2+p_4)^2 \\ u&=(p_1+p_4)=(p_2+p_3)^2 \end{aligned} \tag{16}$$

but actually

$$s+t+u=\sum_{i=1}^{4} m_i^2 \tag{17}$$

so that there are two independent invariant dynamical variables s, t (or s, u or t, u) to be considered.

4. Crossing

In this simple theory the description is completely symmetrical and indeed could equally well represent a process (depending literally on one's point of view)

$$a+d \to b+c \tag{18}$$

in which the variable u plays the role of the squared energy previously played by s.

The *crossing theorem* (discovered in perturbation theory) and apparently a general principle of physics states that a single analytic (with some qualifications) function T (s, t, u) describes all three processes

$$\begin{aligned} a+b &\to c+d \\ a+c &\to b+d \\ a+d &\to b+c \end{aligned} \tag{19}$$

or more precisely that the three amplitudes which a priori describe each *channel* separately are in fact continuations of one another.

Mathematically the analytic structure of collision amplitudes is a very important study which finds expression in so-called *dispersion relations* (integral representations). These relations provide a set of dynamical equations which constrain but do not completely determine an amplitude and experimental data have been satisfactorily described using dispersion relations. Moreover, they have proved to have predictive value ($\pi-\pi$ resonance) and lead to predictions about relationships between particle masses which complement symmetry theories (sum-rules).

5. Regge Behaviour[1] :

In potential theory the equation determining the poles of the *partial wave amplitudes* $a_l(s)$ defined by

$$f(s, t)=\sum_l (2l+1)\, a_l(s)\, \mathrm{P}_l(\cos\theta) \tag{20}$$

where t is simply related to the scattering angle θ, coincides with that defining the bound state energy levels. This leads to the notion of *Regge poles* in high energy physics. One constructs an amplitude $a(l, s)$ defined for complex l which coincides with $a_l(s)$ when $l=0, 1, 2, \ldots$. Among the singularities of $a(l,s)$ are poles corresponding to bound states, resonances, etc. in the system considered. The trajectory in the complex l-plane of such a pole as s varies is given by

$$l=\alpha(s) \tag{21}$$

and is called a *Regge* (pole) *trajectory*. At various points on such a trajectory occur points which can be associated with bound states, resonances, etc. This clearly puts all objects (bound states, resonances, etc.) on the same trajectory into families and reinforces the idea that no object is elementary — no elementary particles, only Regge poles. For example the N and N*** (pion-nucleon resonance) lie on the same trajectory—the nucleon was conventionally an elementary particle while clearly the third nucleon resonance was composite. This dynamical scheme of particle classification is not of great value because many Regge trajectories pass through but one point which can be assigned to an observed object—other members of the same Regge family presumeably appearing only at energies above the present experimental range. A detailed review of Regge theory in low energy potential scattering and its extension to particle calssifications has been given by Rafique, M.Sc. thesis, Wales, 1965.

In potential theory by suitable contour deformation the Sommerfield-Waston transformed partial wave series can be manipulated to exhibit the large t behaviour (properties of Legendre polynomials) in a form dominated by a single Regge pole. Translated into high energy physics one attributes high energy behaviour in the t-channel to the exchange of objects *Reggions* related to poles in the s-channel corresponding to resonant or bound state behaviour.

$$f(s,t) \underset{t\to\infty}{\sim} g(s)\,(-t)^{\alpha(s)}$$

Not only does this prove important in discussions of convergence of dispersion relations but has been raised almost to the rank of a fundamental principle—the *principle of duality*.

A model combining the requirements of crossing and Regge behaviour has been invented by Veneziano[2] and has received much successful attention from theorists.

Now neither crossing nor Regge behaviour have any rigorous basis but both have weighty support in perturbation theory from the study of infinite classes of terms.

6. Difficulties :

The terms of the perturbation series are in the form of multiple intergrals and the problem consists of finding

(*a*) the points of no-analyticity of each term

(*b*) classifying the termwise results with a view to making assertions about the sum function.

The programme (*a*) is in a reasonable state but the same can only be said of (*b*) asymptotically. This may surprise the reader because dispersion relations (and the Mandelstam conjecture) have been very successful and are NOT firmly based even in perturbation theory—they are the product of intuition aquired in fourth order perturbation theory ! Asymptotic behaviour based on intuition aquired in non-relativistic theory seems dubious but is indeed very well founded in the high energy perturbation model.

We (the author in collaboration with Rafique) have tried to do something about singularity classification by using Plucker's equations of algebraic geometry[3,4]. The points of possible singularity for each term lie on curves whose implicit equations were first written down by Landau (5, 6). Only one infinite family of *Landau curves* (the so-called ladder diagram curves) have been classified (Regge and Barucchi (7)) and our studies so far indicate that ladder diagrams are by no means typical. The main feature not exhibited in Regge and Barucchi's analysis is the occurrence of cusped Landau curves. Computer drawings[4] indicate empirically the complexity of the problem and the probability of one being unable to perform the eliminations necessary to obtain from Landau's equations the explicit equation of the curve (which, conveniently, one hopes to write in terms of a single parameter using automorphic functions). This is so because a theorem of Salmon[8] states that the genus of a curve in general exceeds by unity the number of independent circuits which can be made on the curve. The higher the genus the more complicated and more unfamiliar are the automorphic functions needed for the parametrisation.

7. Conclusion.

It is always unwise to attempt predictions in any subject and high energy physics is no exception. The experimental situation must have more than usual impact on theorists when the new higher energy accelerators (such as the European 300 GeV machine) come into operation. Who can tell which methods or which "fundamentls" may be discarded ?

Surely, despite past failures, theorists will continue to seek a marriage of the successful SU(3) symmetry schemes with schemes for Lorentz space ?

Despite the difficulties hinted at in section (6) will not theorists persevere with that seems to me to be their only model formally satisfying the axioms of quantum field theory—the perturbation series ? New avenues of approach such as that of de Alfaro et al.[9] using differential equations instead of integrals may lead to success. Personally I favour attempts to build upon the sort of expertise gained in single term analyses (typified by the work of Rafique, Ph.D. thesis, Wales, 1967) to study infinite sets of terms with a view to finding Landau singularities for suitable sum functions. As for asymptotic behaviour generalisation from termwise behaviour to that of infinite sums has largely been achieved and this in itself is some guarantee that perturbation theory and the Regge theory of high energies will continue to excite interest in the seventies.

The Work of Olive and Negrine (in progress in Cambridge, England) suggests that Landau cruve analysis and asymptotic behaviour are intimately connected and this too encourages a belief that perturbative methods developed in the sixties will have relevance in the seventies.

A definitive account of the subject is given in ref. (10).

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ON THE MOBIUS INVERSION FORMULA

by

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A galance at the Mobius inversion formula suggests that one has to do with some sort of inversion, but apparently none of the books on number theory points out explicitly what this inversion in reality is. The purpose of this note is just to do this.

The formula can be looked upon as an illustration of a very simple and basic fact of group theory. All we assume on part of the reader is some familiarity with the group concept and the elementary divisiblity properties of natural numbers, in particular the fact ("the fundamental theorem of arithmetic") that every natural number >1 can be expressed uniquely, upto the order of the factors, as a product of powers of finitely many prime numbers.

The Mobius inversion formula is concerned with arithmetical functions. By an arithmetical function f we shall mean a function defined on the set, N, of all natural numbers, with values in the real, or even complex, numbers, such that $f(1)=1$. (More abstractly, we could allow f to have values in any commutative ring with identity element, 1.) Some examples of arithmetical functions are :

(1) The divisor function $\tau(n)$;

$\tau(n)$=The number of (natural) divisors of n, including 1 and n.

(2) The φ function (or, totient function) of Euler ;

$\varphi(n)$=The number of all positive integers $m \leqslant n$, which have no common divisor with n, except 1.

(3) The function ε defined by $\varepsilon(n)=1$ for every $n \in \mathrm{N}$.

(4) The function i defined by

$$i(n)=\begin{cases} 1, \text{ if } n=1, \\ 0, \text{ if } n>1. \end{cases}$$

Consider the set, A, of all arithmetical functions $f : \mathrm{N} \to \mathrm{D}$, where D is the set of all integers, or of all rational, or real, or complex numbers (or, for that matter, any commutative ring with identity, 1).

For any $f, g \in A$ let us define a function $f*g$ by the equation

$$(1) \qquad (f*g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right) \text{ for each } n \in N,$$

where the summation extends over all the natural divisors of n ; ($d \mid n$ is read "d divides n"). Obviously $f*g \in A$, for any $f, g \in A$. Thus the set A is closed under the operation* definied by (1). (This operation is an arithmetical analogue of that of convolution in Analysis).

Of great importance in number theory are the multiplicative arithmetical functions. $f \in A$ is called multiplicative, if for every pair of relatively prime or as we shall say, coprime natural numbers m, n, we have

$$(2) \qquad f(mn) = f(m) f(n) \quad \text{for all } m, n \in N \text{ with } (m, n,) = 1$$

(m, n are called coprime, if their greatest common divisor, (m, n), is 1.)

The functions ε and i defined above are both multiplicative; so are τ and φ. A proof that φ is multiplicative can be found in any book on elementary number theory, *e.g.* [1], Theorem 60, page 53. We shall be interested also in the set, M, of all multiplicative arithmetical functions $f : N \to D$, that is, in the subset

$$M = \{f \in A : f \text{ is multiplicative}\}$$

of A. We assert that M, too is closed under the operation *; that is, $f*g \in M$ for all $f, g \in M$. Indeed, for any pair of coprime numbers m, n, we have

$$\begin{aligned}
(f*g)(m)\,(f*g)(n) &= \left(\sum_{d \mid m} f(d)\, g\left(\frac{m}{d}\right)\right)\left(\sum_{k \mid n} f(k)\, g\left(\frac{n}{k}\right)\right) \\
&= \sum_{d \mid m} \sum_{k \mid n} f(d)\, f(k)\, g\left(\frac{m}{d}\right) g\left(\frac{n}{k}\right) \\
&= \sum_{d \mid m} \sum_{k \mid n} f(dk)\, g\left(\frac{mn}{dk}\right) \\
&= \sum_{l \mid mn} f(l)\, g\left(\frac{mn}{l}\right) \\
&= (f*g)(mn).
\end{aligned}$$

The last step deserves a word of explanation. Since $(m,n)=1$, $l=dk$ runs through the divisors of mn, as d and k run through the divisors of

m and n, respectively. Moreover, then $(d,k)=1$ and $\left(\frac{m}{d}, \frac{n}{k}\right)=1$. Thus,

$$\sum_{l \mid mn} = \sum_{d \mid m} \sum_{k \mid n} .$$

Hence,

$$(f*g)(mn)=(f*g)(m).\ (f*g)(n)$$

for all $m,n \in \mathrm{N}$ with $(m,n)=1$.

We may now enunciate the

Theorem. A is an abelian group under the operation *, and M is a subgroup of A.

Proof. It has been shown that A, as well as M, is closed under *. Let us prove first that * is a commutative operation, that is, $f*g=g*f$, for all $f, g \in \mathrm{A}$. This is clear if we observe that with d, also $\frac{n}{d}$ runs through all the divisors of n, so that $\sum_{d \mid n} = \sum_{\frac{n}{d} \mid n}$

Hence in (1) we may replace d by $\frac{n}{d}$ in the right hand side to obtain

$$\sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{\frac{n}{d} \mid n} f\left(\frac{n}{d}\right) g(d) = \sum_{d \mid n} g(d)\, f\left(\frac{n}{d}\right) = (g*f)(n)$$

Thus, $(f*g)(n)=(g*f)(n)$, for each $n \in \mathrm{N}$. This means, $f*g=g*f$.

As for the associativity, we calculate ($f, g, h \in \mathrm{A}$),

$$\begin{aligned}((f*g)*h)(n) &= \sum_{d \mid n} (f*g)(d)\, h\left(\frac{n}{d}\right) \\ &= \sum_{d \mid n} \left(\sum_{k \mid d} f(k)\, g\left(\frac{d}{k}\right)\right) h\left(\frac{n}{d}\right) \\ &= \sum_{d \mid n} \sum_{k \mid d} f(k)\, g\left(\frac{d}{k}\right) h\left(\frac{n}{d}\right)\end{aligned}$$

On the other hand,

$$\begin{aligned}(f*(g*h))(n) &= ((g*h)*f)(n) \\ &= \sum_{d \mid n} (g*h)(d)\, f\left(\frac{n}{d}\right)\end{aligned}$$

$$= \sum_{d \mid n} f\left(\frac{n}{d}\right)\left(\sum_{k \mid d} g\left(\frac{d}{k}\right) h(k)\right)$$

$$= \sum_{d \mid n} \sum_{k \mid d} f\left(\frac{n}{d}\right) g\left(\frac{d}{k}\right) h(k).$$

Now, $k \mid d$ and $d \mid n$, if and only if $\frac{n}{d} \mid \frac{n}{k}$ and $\frac{n}{k} \mid n$. Hence in the last summation we may replace k by $\frac{n}{d}$ and d by $\frac{n}{k}$ without altering the sum ($\frac{d}{k}$ remains thereby unchanged); this shows that $((f*g)*h)(n)$ is equal to $(f*(g*h))(n)$, for all $n \in \mathbf{N}$.

The element of A, which serves as the identity with respect to $*$, is the function i defined earlier ; for,

$$(i*f)(n) = \sum_{d \mid n} \angle(d) f\left(\frac{n}{d}\right) = f(n),$$

since $i(d)=0$, as soon as $d>1$.

The existence of an inverse element (and its uniqueness) to each $f \in A$ is easy to see. We have to exhibit, for every $f \in A$, an element $h \in A$ such that $f*h=i$. But, this requirement automatically furnishes us with the function, h, sought, because putting $n=1,2,3,..$ in the equation

$$(f*h)(n) = \sum f\left(\frac{n}{d}\right) h(d) = i(n),$$

we can successively calculate $h(1)=1$, $h(2)$, $h(3)$.....and thus effectively determine h.

With this, we have completely proved that A is an abelian group under $*$. To show that M is a subgroup of A, that means, M is itself a group under $*$, it remains to verify that for any $f \in M$, its inverse, h, also belongs to M. We prove $h(mn)=h(m)\,h(n)$, for all m, $n \in \mathbf{N}$ with $(m, n,)=1$, by induction on mn. The assertion is trivially ture, when $mn=1$. Assume now $mn>1$, and that $h(lk)=h(l)\,h(k)$ holds for all coprime pairs l, k with $1k<mn$. Then

$$(\#) \qquad (f*h)(mn) = h(mn) + \sum_{d \mid mn} f(d)\, h\left(\frac{mn}{d}\right),$$

where Σ extends over all $d>1$, which divide mn. As remarked earlier, $\sum_{d \mid mn} = \sum_{l \mid m} \sum_{k \mid n}$; using the induction hypothesis and the fact that is multiplicative, we see

$$\sum_{d \mid mn} f(d)\, h\left(\frac{mn}{d}\right) = \left(\sum_{l \mid m} f(1)\, h\left(\frac{m}{l}\right)\right)\left(\sum_{k \mid n} f(k)\, h\left(\frac{n}{k}\right)\right) - h(m)\, h(n)$$

$$= (f*h)(m)\,(f*g)(n) - h(m)\,h(n).$$

Hence the equation (#) becomes

$$(f*h)(mn) = (f*h)(m)\,(f*h)(n) + h(mn) - h(m)\,h(n)\ ;$$

but $f*h=i$ is multiplicative, hence we must have $h(mn)=h(m)\,h(n)$. This establishes the assertion, and the theorem is completely proved.

We now want to compute the inverse of the element $\varepsilon \in \mathrm{M}$, where $\varepsilon(n)=1$, for all $n \in \mathrm{N}$. We assert that its inverse, μ, is the function characterized by the following equations :

$$(3)\qquad \mu(n) = \begin{cases} 1, & \text{if } n=1, \\ 0, & \text{if } p^2 \mid n, \text{ where } p \text{ is some prime number,} \\ (-1)^k & \text{if } n=p_1 \ldots p_k, \text{ where the } p_i\text{'s are all distinct prime numbers } (k \geqslant 1). \end{cases}$$

By virtue of the fundamental theorem of arithmetic, the equations (3) do really determine a unique arithmetical function, μ.

Now, obviously $\mu(1)=\varepsilon(1)\ \mu(1)=(\varepsilon*\mu)(1)=i(1)=1$. Let p be any prime number ; then $0=i(p)=(\varepsilon*\mu)(p)=1+\mu(p)$. Hence, $\mu(p)=-1$. Further, $0=i(p^2)=(\varepsilon*\mu)(p^2)=1+\mu(p^2)$, whence $\mu(p^2)=0$ follows. If $k \geqslant 2$ is any integer, then

$$0=i(p^k)=1+\mu(p)+\ldots+\mu(p^{k-1})+\mu(p^k)\ ;$$

hence,, by induction on k, we see that

$\mu(p^k)=0$, for all $k \geqslant 2$, and any prime number p.

μ, being the inverse of a multiplicative arithmetical function is itself multiplicative. Hence, if $p_1, \ldots, p_k$ are all distinct prime numbers $(k \geqslant 1)$, then

$$\mu(p_1 \ldots p_k) = \prod_{i=1}^{k} \mu(p_i) = (-1)^k.$$

Lastly, if $n \in \mathrm{N}$ is divisible by the square of a prime number p, then there

is an integer $k \geqslant 2$ and a natural number m not divisible by p, such that $n=mp^k$. Then $\mu(n)=\mu(m)\ \mu(p^k)=0$, since $(p, m)=1$ and $\mu(p^k)=0$.

Thus we have determined the inverse, μ, of the element $\varepsilon \in \mathbf{M}$. We are now ready to obtain the theorem known as the

MOBIUS INVERSION FORMULA.

Two arithmetical functions f and g are connected by the equations

$g(n)=\sum_{d \mid n} f(d)$, for each $n \in \mathrm{N}$, if and only if they are related by the equations

$f(n)=\sum_{d \mid n} g(d)\ \mu\left(\frac{n}{d}\right)$, for all $n \in \mathrm{N}$, where μ is the Mobius function defined by (3).

Moreover, in tnis situation, g is multiplicative if and only if f is mul-multiplicative.

There remains nothing to prove but only to interpret the above enunciation and assertion in our framework. In our terms it means :

For two elements $f, g, \in \mathrm{A}$, the relation $g=f^*\varepsilon$ holds, if and only if the relation $f=g^*\mu$ holds. This is, however, completely clear. If $g=f^*\varepsilon$, then $g^*\mu=(f^*\varepsilon)\ ^*\mu=f^*\ (\varepsilon^*\mu)$, by the associative law,

$$=f^*i$$
$$=f.$$

The converse is similar. Since $\varepsilon, \mu \in \mathbf{M}$, if $g=f^*\varepsilon$, then g is multiplicative, when f is so ; conversely if $f=g^*\mu$ and g is multiplicative, then f is multiplicative too.

Naturally we have proved much more than would be needed to esta-establish just the Mobius Inversion Formula. But the little extra effort involved in our treatment is more than well paid by the insight it gives. Not only does it reveal the inversion formula as an illustration of a simple, purely group theoretic phenomenon, but it also serves to introduce in a natural way (being the inverse, with respect to a certain group law, of a very trivial and natural element) the Mobius function μ, while in the customary way it is introudced off-hand without any motivation.

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AN INTEGRAL EQUATION INVOLVING A HYPERGEOMETRIC FUNCTION

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Introduction : Recently E.R. Love ((3)) solved some integral equations involving hypergeometric functions by means of fractional integration. This followed A. Erdelyi's investigations ((1)) into the solutions of an integral eqution which contains the Legendre function as kernel. Srivastava ((5)) also used fractional integration to discuss certain integral equations with polynomial kernels. Here we shall employ the fractional integral operators to solve the integral equation

$$\int_a^x (x-t)^{\lambda-\frac{1}{2}} \mathrm{M}_v{}^{\lambda}\left(\frac{x}{t}\right) f(t)\, dt = g(x),\ (0<a<x<\infty), \qquad (1.1)$$

where $\mathrm{M}_v{}^{\lambda}(z)$ is a function derived from certain polynomials which are special cases of Jacobi polynomials, $\lambda > -\frac{1}{2}$, $\mathrm{M}_v{}^{\lambda}\left(\frac{x}{t}\right)$ is defined for $\left|\frac{x}{t}-1\right|<2$, g is a given function and f is to be determined.

Fractional Integration : Let C_0 be the class of those continuous functions on the interval (a, b), open at a, where $0 < a < b < \infty$, which are integrable at a, and C_n where n is a positive integer, be the class of all those functions which are n-times continuously differentiable on (a, b) and which satisfy $f^{(k)}(a+)=0$, $k=0, 1, 2, \ldots\ldots, n-1$, while f^n is integrable at a.

Let I be the operator of integration defined by

$$\mathrm{If} = \int_a^x f(t)\, dt,\ (a < x < b < \infty) \qquad (2.1)$$

so that the operation of $k-$ times repeated integration is expressed as

$$\mathrm{I}^k f=\int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)\,dt,\ k=1, 2, \ldots, \tag{2.2}$$

and set

$$\mathrm{I}^0 f=f(x),\ \mathrm{I}^{-k} f=f^{(k)},\ k=1, 2, \ldots. \tag{2.3}$$

The most important properties may be summarised as follows. For $f \in \mathrm{C}_k$, $k=0, 1, 2, \ldots$, we have

$$f(x)=\mathrm{I}^k f^{(k)}(x). \tag{2.4}$$

If $f \in \mathrm{C}_j$, $j=1, 2, \ldots$, and k is an integer (positive, negative or zero) for which $j+k \geqslant 0$, then $\mathrm{I}^k f$ exists and belongs to C_{j+k}; if l is a further non-negtive integer which does not exceed $j+k$, then

$$\left(\frac{d}{dx}\right)^l \mathrm{I}^k f(x)=\mathrm{I}^{k-l} f(x) \tag{2.5}$$

exists and belongs to C_{j+k-l}.

We shall now extend these ad other results to non-negative values of the index. For $\alpha > 0$, we follow Riemann and Liouville in defining integration of order α as

$$\mathrm{I}^\alpha f(x)=\int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt,\ (0 \leqslant a < x < b < \infty). \tag{2.6}$$

Many authors have proved the existence almost everywhere of (2.6) for integrable functions. Under heavier restrictions upon f *i.e.* $f \in \mathrm{C}_0$, $\mathrm{I}^\alpha f$ exists and belongs to C_0.

For $\alpha > 0$, $\beta > 0$, $f \in \mathrm{C}_0$

$$\mathrm{I}^\alpha \mathrm{I}^\beta f=\mathrm{I}^\beta \mathrm{I}^\alpha f=\mathrm{I}^{\alpha+\beta} f \tag{2.7}$$

This can be proved by interchanging the order of integration in the repeated integral indicated on the left hand side of (2.7).

We now define $\mathrm{I}^\alpha f$ for $\alpha < 0$ as inverse operation to $\mathrm{I}^{-\alpha}$, i.e. define $g=\mathrm{I}^\alpha f$ for $\alpha < 0$ and $f \in \mathrm{C}_0$ to be the solution in C_0, if it exists, of the integral equation $f=\mathrm{I}^{-\alpha} g$. Hence, for any real α, the statement that

$I^{\alpha}f$ exists implies that f and $I^{\alpha}f$ both belong to C_{o}. With this extension (2.7) holds for all real α and β (positive, negative and zero).

A sufficient condition for the existence of $I^{\alpha}f$, where $\alpha < 0$, is that $f \in C_{k}$ for some $k \geqslant -\alpha$; and

$$I^{\alpha}f(x)=I^{\alpha+k}f^{k}(x). \tag{2.8}$$

Moreover, if j is a non-negative integer not exceeding $k+[\alpha]$, where $[\alpha]$ is the integral part of α then

$$\left(\frac{d}{dx}\right)^{j} I^{\alpha}f(x)=I^{\alpha-j}f(x)=I^{\alpha-j+k}f^{(k)}(x). \tag{2.9}$$

For $\alpha > 0$, let us denote C_{α} the class of functions representable in the form $I^{\alpha}f$ with $f \in C_{o}$. This definition gives the class C_{n} of functions when $\alpha=n$. If $\alpha \geqslant 0$, $\beta \geqslant 0$ and $f \in C_{\alpha}$ then $I^{\beta}f \in C_{\alpha+\beta}$. If $0 < \beta < \alpha$ and $f \in C_{\alpha}$, then $f \in C_{\beta}$, and it follows that for $0 < \beta < \alpha$, $C_{\alpha} \subset C_{\beta}$. If $\alpha=-n$, n is a positive integer, then $I^{\alpha}f$ exists and belongs to C_{o} if and only if $f \in C_{n}$. If $\alpha=-n+\vartheta$, where n is a positive integer and $0 < \vartheta < 1$, then $f \in C_{n-1}$ is necessary and $f \in C_{n}$ is sufficient condition for the existence of $I^{\alpha}f$ in C_{o}, while a condition that is both necessary and sufficient is that $I^{\vartheta}f \in C_{n}$.

A new Function : We begin this section by defining polynomials $M_{n}(x)$ by

$$\sum_{n=0}^{\infty} M_{n}(x)\, t^{n}=(1-t)\,(1-2\,x\,t+t^{2})^{-1} \tag{3.1}$$

These polynomials are special Jacobi polynomials. In detail

$$M_{n}(x)=\frac{n\,!}{(\frac{1}{2})_{n}} P_{n}^{(-\frac{1}{2},\,\frac{1}{2})}(x), \tag{3.2}$$

where $P_{n}^{(\alpha,\,\beta)}$ are the Jacobi polynomials. To prove (3.2) we use the following relation for Jacobi polynomials (((4)), p. 256)

$$\sum_{n=0}^{\infty} \frac{(a+b+1)_n}{(1+a)_n} P_n^{(a,\,b)}(x)\, t^n$$

$$=(1-t)^{-1-a-b}\, {}_2F_1\left(\tfrac{1}{2}(1+a+b),\ \tfrac{1}{2}(2+a+b);\ 1+a;\ \frac{2t(x-1)}{(1-t)^2}\right), \quad (3.3)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.

$$\sum_{n=0}^{\infty} M_n(x)\, t^n=(1-t)(1-2xt+t^2)^{-1}$$

$$=(1-t)^{-1}\left(1-\frac{2t(x-1)}{(1-t)^2}\right)^{-1}$$

$$=(1-t)^{-1}\, {}_2F_1\left(\tfrac{1}{2}, 1; \tfrac{1}{2}; \frac{2t(x-1)}{(1-t)^2}\right)$$

and with $a=-\frac{1}{2}$, $b=\frac{1}{2}$, we obtain from (3.3).

By equating the coefficients in the last equation we prove (3.2). This relation shows that many formulas on Jacobi polynomials can be converted into results on M_n-polynomials. These polynomials can also be introduced by the relation,

$$M_n(x)={}_2F_1\left(-n,\ n+1;\ \tfrac{1}{2};\ \frac{1-x}{2}\right), \quad (3.4)$$

and

$$M_n(\cos\theta)=\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}}. \quad (3.5)$$

(3.5) follows from (3.4) and 2.8 (11) of Erdelyi ((2)).

The Tchebicheff polynomials $T_n(x)$ and $U_n(x)$ are often defined by the relations ((4)).

They satisfy the equations

$$T_n(x)=U_n(x)-x\,U_{n-1}(x),\ (n \geqslant 1), \quad (3.6)$$

$$(1-x^2)\,U_n(x)=x\,T_n(x)-T_{n+2}(x). \quad (3.7)$$

The generating functions for $T_n(x)$ and $U_n(x)$ are

$$(1-2xt+t^2)^{-1}=\sum_{n=0}^{\infty} U_n(x)\, t^n \quad (3.8)$$

$$(1-xt)(1-2xt+t^2)^{-1}=\sum_{n=0}^{\infty} T_n(x)\, t_n. \quad (3.9)$$

Using (3.8) and (3.9) we observe that

$$\sum_{n=0}^{\infty} [T_n(x)+(x-1)\,U_n(x)]\,t^n = x\sum_{n=0}^{\infty} M_n(x)\,t^n.$$

Comparing the coefficients in this equation we obtain

$$x\,M_n(x)=T_n(x)+(x-1)\,U_n(x).$$

Using (3.7) we have

$$M_n(x)=U_n(x)-U_{n-1}(x),\ (n \geqslant 1). \tag{3.10}$$

Similarly, making use of (3.7), we get

$$(x^2-1)\,M_n(x)=T_{n+2}(x)-(1+x)\,T_{n+1}(x)+x\,T_n(x). \tag{3.11}$$

We now define two functions by the formulas

$$M_v(x)={}_2F_1\left(-v,\ v+1;\ \tfrac{1}{2};\ \tfrac{1}{2}(1-x)\right), \tag{3.12}$$

and

$$M_v^{\lambda}(x)={}_2F_1\left(-v,\ v+1;\ \lambda+\tfrac{1}{2};\ \tfrac{1}{2}(1-x)\right). \tag{3.13}$$

Various relations involving the functions (3.12) and (3.13) can be found by actual computation, and we shall discuss these in an other paper.

The integral equation. We shall now investigate the solution of the integral equation.

$$\int_a^x (x-t)^{\lambda-\frac{1}{2}}\ M_v^{\lambda}\left(\frac{x}{t}\right) f(t)\,dt=g(x), \tag{4.1}$$

where $0<a<x<b<\infty$, $\lambda>-\frac{1}{2}$, $M_v^{\lambda}\left(\frac{x}{t}\right)$ is defined for $\left|\frac{x}{t}-1\right|<2$, g is the given function and f is to be determined.

We write

$$M_v^{\lambda}(x)={}_2F_1\left(-v,\ v+1;\ \lambda+\tfrac{1}{2};\ \tfrac{1}{2}(1-x)\right)$$

$$=\frac{\Gamma(\lambda+\frac{1}{2})}{2^v\,\Gamma(v+1)\,\Gamma(\lambda-v-\frac{1}{2})}$$

$$\int_0^1 v^{\lambda-v-3/2}(1-v)^v\,(x+1-v\,(x-1))^v\,dv, \tag{4.2}$$

We shall now establish a result on the fractional integral of $M_v^{\lambda}(x)$. In these lines $\alpha > 0$, t and x vary over the interval (a, b), H is Heaviside's unit function defined by

$$H(u)=1, \text{ if } u > 0, \ =0 \text{ if } u < 0.$$

Consider, for $\alpha > 0$, $c > 0$, and for each t, the function of x

$$I^{\alpha}\left\{\frac{H(x-t)}{\Gamma(c)}(x^2-t^2)^{c-1}\right\}=\frac{H(x-t)}{\Gamma(\alpha)\,\Gamma(c)}$$

$$\int_t^x (x-u)^{\alpha-1}(u^2-t^2)^{c-1}\,du.$$

Write $u=x-(x-t)\,v$ and obtain

$$I^{\alpha}\left\{\frac{H(x-t)}{\Gamma(c)}(x^2-t^2)^{c-1}\right\}$$

$$=\frac{(x-t)^{\alpha+c-1}}{\Gamma(\alpha)\,\Gamma(c)}\int_0^1 v^{\alpha-1}(1-v)^{c-1}(x+t-(x-t)\,v)^{c-1}\,dv.$$

On account of (3.2), we thus have

$$I^{\alpha}\left\{\frac{H(x-t)}{\Gamma(c)}(x^2-t^2)^{c-1}\right\}$$

$$=H(x-t)\,\frac{(2t)^{c-1}}{\Gamma(\alpha+c)}(x-t)^{\alpha+c-1}M_{c-1}^{\alpha+c-\frac{1}{2}}\left(\frac{x}{t}\right). \qquad (4.3)$$

In (4.1), if $\lambda+\frac{1}{2} > \nu+1 > 0$, then the conditions for (4.3) are satisfied, thus its application to (4.1) yields

$$g(x)=\int_a^x I^{\lambda-\nu-\frac{1}{2}}\left\{\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\,H(x-t)\,(2t)^{-\nu}(x^2-t^2)^{\nu}\right\}f(t)\,dt. \qquad (4.4)$$

If it is permitted to interchange the operation $I^{\lambda-\nu-\frac{1}{2}}$ with the integral with respect to t, we use the operator

$$I_2^{\alpha} f(x)=\frac{2}{\Gamma(\alpha)}\int_a^x (x^2-t^2)^{\alpha-1}f(t)\,t\,dt$$

to obtain

$$g(x)=\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\nu+1)}\, I^{\lambda-\nu-\frac{1}{2}}\int_a^x (x^2-t^2)^{\nu}\,(2t)^{-\nu} f(t)\, H(x-t)\, dt \qquad (4.5)$$

$$= \Gamma(\lambda+\tfrac{1}{2})\, I^{\lambda-\nu-\frac{1}{2}}\, I_2{}^{\nu+1}\left\{(2x)^{-\nu-1} f(x)\right\} \qquad (4.6)$$

which leads to the solution

$$f(x)=\frac{(2x)^{\nu+1}}{\Gamma(\lambda+\frac{1}{2})}\, I_2{}^{-\nu-1}\; I^{\nu+\frac{1}{2}-\lambda}\; g(x). \qquad (4.7)$$

Necessary and Sufficient Conditions : To find necessary and sufficient conditions for the existence of the solution f, in C_o, of the integral equation (4.1), we use the following result ((1)). For every f in C_o

$$I_n{}^{\alpha} f = I^{\alpha} h,$$

where h belongs to C_o.

We shall now show that the necessary and sufficient condition for the existence of solution f in C_o is that g belongs to $C_{\lambda+\frac{1}{2}}$

Indeed, if $f \in C_o$ and $\nu \leqslant -1$, then

$$I_2{}^{\nu+1}\ \{(2x)^{-\nu-1} f(x)\} = I^{\nu+1} h(x),$$

Where h is in C_o and from (4.6), we see that

$$g(x)=\Gamma(\lambda+\tfrac{1}{2})\, I^{\lambda-\nu-\frac{1}{2}}\, I^{\nu+1} h(x)=\Gamma(\lambda+\tfrac{1}{2})\, I^{\lambda+\frac{1}{2}} h(x).$$

Thus $g \in C_{\lambda+\frac{1}{2}}$ is the necessary condition.

On the other hand, if $g(x)=I^{\lambda+\frac{1}{2}} h(x)$, for $h \in C_o$,

$$I^{\nu-\lambda+\frac{1}{2}} g(x)=I^{\nu-\lambda+\frac{1}{2}} I^{\lambda+\frac{1}{2}} h(x)=I^{\nu+1} h(x)=I_2{}^{\nu+1} k(x),$$

where $k \in C_o$.

Thus $$f(x)=\frac{(2x)^{\nu+1}}{\Gamma(\nu+\frac{1}{2})}\, I_2{}^{-\nu-1}\; I_2{}^{\nu+1}\, k(x),$$

so that $$f(x)=\frac{(2x)^{\nu+1}}{\Gamma(\lambda+\frac{1}{2})}\, k(x)$$

and this belongs to C_o. Hence $g \in C_{\lambda+\frac{1}{2}}$ is necessary and sufficient condition for the existence of the solution in C_o.

Explicit Solutions : The explicit form of the solution (4.7) is different according as v, $\lambda-v-\frac{1}{2}$ or both of these numbers are or are not non-negative integers. First assume $\lambda-v-\frac{1}{2}=m$, where m is a positive integer, then

$$\mathrm{I}^{v-\lambda+\frac{1}{2}}\ g(x)=g^{(m)}(x). \tag{6.1}$$

If, in addition, also $v=n$, n is an integer, then (4.7) may be written as

$$f(x)=\frac{(2x)^{n+1}}{\Gamma(n+m+1)}\left(\frac{d}{dx^2}\right)^{n+1} g^{(m)}(x),\ \lambda=n+m,\ v=n. \tag{6.2}$$

If v is not an integer and $v<n$, then

$$\mathrm{I}_2{}^{-v-1}=\mathrm{I}_2{}^{-n-1}\ \mathrm{I}_2{}^{n-v},$$

and we get

$$f(x)=\frac{2\,(2x)^{v+1}}{\Gamma(n-v)\ \Gamma(m+v+1)}\left(\frac{d}{dx^2}\right)^{n+1}\int_a^x (x^2-t^2)^{n-v-1} g^{(m)}(t)\,dt \tag{6.3}$$

for $\lambda=m+v+\frac{1}{2}$.

Now if $\lambda-v-\frac{1}{2}$ is not an integer and $\lambda-v-\frac{1}{2}<m$, then $\mathrm{I}^{v-\lambda+\frac{1}{2}}$ $=\mathrm{I}^{v-\lambda+\frac{1}{2}+m}\ \mathrm{I}^{-m}$ and accordingly

$$\mathrm{I}^{v-\lambda+\frac{1}{2}}g(x)=\frac{1}{\Gamma(v-\lambda+m+\frac{1}{2})}\int_a^x (x-t)^{v-\lambda+m-\frac{1}{2}}\ g^{(m)}(t)\,dt \tag{6.4}$$

If, in this case $\lambda=n$ is an integer then (4.7) may be transcribed

$$f(x)=\frac{(2x)^{n+1}}{\Gamma(v-\lambda+m+\frac{1}{2})\ \Gamma(\lambda+\frac{1}{2})}\left(\frac{d}{dx^2}\right)^{n+1}\int_a^x (x-t)^{n-\lambda+m-\frac{1}{2}}\ g^{(m)}(t)\,dt. \tag{6.5}$$

for $v=n$, $\lambda<n+m+\frac{1}{2}$.

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ELEMENTARY PARTICLES AND TOPOLOGICAL CONCEPTS

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A new approach involving the use of topological notions in Physics has recently been suggested by Campbell[1] and Antoine[2].

This short note contains a brief discussion regarding the use of topological concepts for the characterization of elementary particles.

Let a be a physical system and S_a be the set of all physical states of a. At the time $t=0$, let s be the state of the system. The other states of the system can be obtained from s with the help of an operator U_t in the sense that $U_{t_1}(s)$ represents the state of the system at $t=t_1$. It follows that the set $\{U_t\}$ is a semi-group under the successive application of operator as multiplication[3]. We shall denote this semi-group by A. Let {S} be the set of all physical systems and let {A} be the set of all semi-groups corresponding to the members of {S}. We shall define an equivalence relation over {S} to characterize elementary particles.

Let C be a category[4] whose objects are the members of {A} and whose multiplicative system M is the set of morphisms of A's. Let C′ be a sub-category of C whose objects are members of {A} and whose multiplicative system M′ of morphisms is the subset of M consisting of those mappings which are iso-morphisms. With the help of these morphisms we can introduce a relation over the set {A} as follows:

Let $A_1, A_2 \in \{A\}$. Then A_1 is said to be related to A_2, abbreviated as "$A_1 \sim A_2$", if there exists an $h \in M'$ such that $h: A_1 \to A_2$ is a

morphism between A_1 and A_2. That this relation is an equivalence relation over the set {A} can be seen as follows :

(*i*) Every $A \in \{A\}$ is equivalent to itself under the identity morphism (whose existence is guaranteed by the definition of a category), *i.e.* $A \sim A$ for every $A \in \{A\}$.

(*ii*) If h: $A_1 \to A_2$ is a morphism then h^{-1}: $A_2 \to A_1$ is also a morphism and hence $A_1 \sim A_2$ implies $A_2 \sim A_1$. That is the relation is symmetric.

(*iii*) If $A_1 \sim A_2$ and $A_2 \sim A_3$, then there exist morphisms $h, g \in M'$ such that h: $A_1 \to A_2$ and g: $A_2 \to A_3$. Since $gh \in M'$ by the definition of a category and gh: $A_1 \to A_3$, we conclude that $A_1 \sim A_2$ and $A_2 \sim A_3$ together imply $A_1 \sim A_3$.

This substantiates our claim that the aforesaid relation is an equivalence relation over the set {A}.

Now let $a_1, a_2 \in \{S\}$. Then we say that a_1 is equivalent to a_2 if and only if $A_1 \sim A_2$, where A_1 and A_2 are the semi-groups associated with a_1 and a_2 respectively. One readily verifies that this is an equivalence relation over the set {S}. Thus we have induced an equivalence relation over {A} as described above. Then by the well-known property of an equivalence relation it follows that {S} can be partitioned into mutually exclusive and collectively exhaustive classes of physical systems, where each equivalence class consists of physical systems equivalent to each other. An equivalence class shall be called elementary if each of its members is a single particle system. The elementary classes so obtained will represent elementary particles. This provides us with a procedure for characterizing elementary particles.

It may, however, be pointed out that if we consider that category C with its multiplicative system M (consisting of all morphisms), then the following difficulty arises in regard to the definition of an equivalence relation over {A}: in case of a homomorphism (not isomorphism) many elements of its domain may be mapped into a single element of its codomain and hence the invertibility required for establishing the symmetry (*ii*) of the relation (as defined above) is not possible. By

reinterpreting the concept of invertibility regarding morphisms, it may be possible to overcome this apparently difficult situation. We hope to discuss this in some later article.

An alternative approach to solve the above problem could be to introduce an equivalence relation over {S} without resorting to the set {A}. To accomplish this we may proceed as follows. Two systems a_1 and a_2 are equivalent if and only if a_1 and a_2 have the same number of particles. It can be verified that it is an equivalence relation over {S}. Let e_1 be the equivalence class of singleton systems (a system is called singleton system if it has just one particle) of {S}. Let $\{A_1\}$ be the set of semigroups associated with the systems in e_1. Let C_1 be a category whose objects are members of $\{A_1\}$ and whose multiplicative system is the set of isomorphisms of A_1's. We can partition e_1 into equivalence classes by inducing an equivalence relation over it as described above. These equivalence classes will correspond to elementary particles and hence the consequential characterization.

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ON THE FUNCTION C (n)=q^e_k(n) − q^o_k(n) AND SOME INEQUALITIES IN PARTITION THEORY

By

S. MANZUR HUSSAIN

Definitions. $p(n)$=Number of unrestricted partitions of n

$q^r_k(n)$=Number partitions of n into just r distinct parts $\leqslant k$.

$q^r_k(0)=1$

$q^o_k(n)$=Number of partitions of n into an odd Number of distinct parts $\leqslant k$.

$q^o_k(0)=0$

$q^e_k(n)$=Number of partitions of n into even Number of parts $\leqslant k$.

$q^e_k(0)=1$

$$c_k(n)=c(n)=\sum_{r=1}^{\infty}(-1)^r\, q^r_k(n).$$

$c(0)=1$.

1. The following properties of $q^r_k(n)$ & $c(n)$ are obvious from definitions.

$$q^r_k(n)=0, \qquad k<r. \tag{1.1}$$

$$q^r_k(n)=0, \qquad n>\frac{r(r+1)}{2} \tag{1.2}$$

$$q^r_k(n)=0, \qquad n>\frac{r(2k-r+1)}{2} \text{ and } k\geqslant r\geqslant 1. \tag{1.3}$$

$$q^r_k(n)=1, \qquad n=\frac{r(r+1)}{2} \text{ or } \frac{r(2k-r+1)}{2} \text{ \& } k\geqslant r\geqslant 1. \tag{1.4}$$

$$c(n)=(-1)^r, \qquad n=\frac{r(r+1)}{2} \text{ or } \frac{r(2k-r+1)}{2} \text{ \& } k\geqslant r\geqslant 1. \tag{1.5}$$

$$c(n)=0, \qquad n>\frac{r(2k-r+1)}{2} \ \&$$

$$k \geqslant r \geqslant 1. \tag{1.6}$$

2. We have found a few more interesting properties of $c(n)$ which we state in the form of the following theorem :

Theorem 1. (*a*) $\sum_{n=0}^{\infty} c(n)=0.$

(*b*) $c(s)=+c(n-s), \qquad k=\text{even},$

$=-c(n-s), \qquad k=\text{odd},$

$$\text{and } 0 \leqslant s \leqslant n=\frac{k(k+1)}{2}$$

(*c*) $c_{k+1}(r)=c_k(r), \qquad 0 \leqslant r \leqslant k$

$$c_{k+1}(r)=c_k(r)-c_k \quad (r-\overline{k+1}),$$

$$k+1 \leqslant r \leqslant \frac{k(k+1)}{2},$$

$$c_{k+1}(r)=-c_k\,(r-\overline{k+1}),$$

$$\frac{k(k+1)}{2}+1 \leqslant r \leqslant \frac{(k+1)(k+2)}{2}.$$

(*d*) $c_{k+1}(r)=0, \qquad k+2 \equiv 0 \pmod 4,$

$=0, \pmod 2 \quad k+1 \equiv 0 \pmod 4,$

$$\& \ r=\frac{(k+1)(k+2)}{4}.$$

(*e*) $\sum_{n=1}^{\frac{k(k+1)}{2}} n\,c(n)=0, \qquad k \geqslant 2$

(*f*) $\sum_{n=0}^{\frac{k(k+1)}{2}} c(2n)=0, \qquad k(k+1)\equiv 0 \pmod 4,$

$\sum_{n=0}^{\frac{k(k+1)-2}{4}} c(2n)=0, \quad k(k+1)\equiv 2 \pmod 4.$

In order to prove the above properties we first show that $c(n)$ is the generating function of $\prod_{n=1}^{k}(1-x^n)$. In other words we prove the following :—

Lemma :— If $\phi_k(x)=\prod_{1}^{k}(1-x^n)$

$$\text{then } \phi_k(x)=\sum_{n=0}^{\infty} c(n)\,x^n$$

Consider the identity

$$\begin{aligned}\prod_{n=1}^{k}(1-x^n)=1&-(x+x^2+..x^k)+(x.x^2+x.x^3+..+x^{k-1}.x^k)\\&-(x.x^2.x^3+x.x^3.x^4+..+x^{k-2}.x^k)\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+(-1)^r\,[(x.x^2..x^r)+(x.x^3..x^{r-1})+..+(x^{k-r+1}.x^{k-r+2}..x^k)]\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+(-1)^k(x.x^2..x^k),\end{aligned}\qquad (2.1)$$

where the general term on the right hand side of (2.1) is the sum of ${}^{k}c_{r}$ terms of the type $x^{a_1}.x^{a_2}..x^{a_r}$ and where $a_i \leqslant k$ are all distinct.

We can easily see that the above identity can also be written as

$$\phi_k(x)=1+\sum_{r=1}^{k}(-1)^r\sum_{s=\frac{r(r+1)}{2}}^{\frac{r(2k-r+1)}{2}} c_s\,x^s. \qquad (2.2)$$

We now have to determine the coefficient c_s of x^s. Let us suppose that $a_1+a_2+..+a_r=s$. This set of r distinct integers contributes just one to the coefficient c_s. The same is the case with each such set of integers a_i taken r at a time whose total sum $=s$. Thus the total contribution to c_s=the total no. of such sets and this by definition is equal to $q_k^r(s)$. Therefore, we have $c_s=q_k^r(s)$. Consequently (2.2) can be re-written as

$$\phi_k(x)=1+\sum_{r=1}^{k}(-1)^r\sum_{s=\frac{r(r+1)}{2}}^{\frac{r(2k-r+1)}{2}} q^r{}_k(s)\,x^s. \tag{2.3}$$

$$=1-\sum_{s=1}^{k} q^1_k(s)\,x^s+\sum_{s=3}^{2k-1} q^r_k(s)\,x^s+\ldots.$$

$$+(-1)^r\sum_{s=\frac{r(r+1)}{2}}^{\frac{r(2k-r+1)}{2}} q^r_k(s)\,x^s+\ldots.$$

$$+(-1)^k.x^{\frac{k(k+1)}{2}}$$

Re-arranging the terms on the right hand side of the above identity in ascending powers of x and using the properties from (1.1) to (1.6) we have finally the required form, namely :

$$\phi_k(x)=\sum_{n=0}^{\frac{k(k+1)}{2}} c(n)x^n. \tag{2.3}$$

$$=\sum_{n=0}^{\infty} {}^*c(n)\,x^n$$

Incidentally we have also proved that

$$\sum_{s=\frac{r(r+1)}{2}}^{\frac{r(2k-r+1)}{2}} q^r_k(s)={}^kc_r \tag{2.4}$$

We are now in in a position to prove the theorem :

${}^*c(n)=q^e_k(n)-q^o_k(n)$, *vide* [1].

Proof of (*a*) Since $\phi_k(1)=0$,

$$\text{therefore } \sum_{n=0}^{\infty} c(n)=0$$

(*b*) Since $\phi_k(x)=\sum_{s=0}^{\frac{k(k+1)}{2}} c(s)\, x^{-s}$

and $x^{\frac{k(k+1)}{2}} \cdot \phi_k(x^{-1})=\phi_k(x)$, $k=$even ;

$=-\phi_k(x)$, $k=$odd.

After substituting the values of $\phi_k(x)$ and $\phi_k(x^{-1})$ in terms of finite series on comparing the coefficients of x^s, we have that required results.

(*c*) We know that $\sum_{n=0}^{\frac{k(k+1)}{2}} c_k(n)\, x^n\ (1-x^{k+1})$

$$= \sum_{n=0}^{\frac{(k+1)(k+2)}{2}} c_{k+1}(n)\, x^n.$$

By equating coefficients of equal powers of x on both the sides of the above identity we have the relations in (*c*). These relations may be called recurrence formulae for $c(n)$

(*d*) From the recurrence formula we know that

$$c_{k+1}(r)=c_k(r)-c_k(r-(k+1)), \qquad k+1 \leqslant r \leqslant \frac{k(k+1)}{2}$$

There are two cases to be considered.

(1) Let $k=$even, $\qquad c_k(r)=c_k(r-(k+1))$

provided $r=\frac{(k+1)(k+2)}{4}$.

Since r is an integer, we place further restriction on k and say that $k+2\equiv 0 \pmod 4$.

Hence under the restrictions mentioned above, we have

$$c_{k+1}(r)=0.$$

(2) Let k=odd. $\qquad c_k(r)=-c_k(r-(k+1))$

provided $r=\dfrac{(k+1)(k+2)}{4}$

As above we place further restriction on k and say that $k+1\equiv 0 \pmod 4$ and consequently we have

$$c_{k+1}(r)\equiv 0 \pmod 2.$$

(*e*) Let $k \geqslant 2$

Since $\phi'^{*}_{k}(x)=\sum_{n=1}^{\frac{k(k+1)}{2}} n\,c(n)\,x^{n-1}$ and $\phi'_k(1)=0$

therefore $\sum_{n=1}^{\infty} n\,c(n)=0$

We can easily verify that the generalised form of the property is

$$\sum_{n=m-1}^{\infty} [(n-m+2)(n-m+3)\ldots(n)]\;c(n)=0 \qquad (2.5)$$

when $k \geqslant m$.

(*f*) Since $\phi_k(x)+\phi_k(-x)=2\sum_{n=0}^{\frac{k(k+1)}{4}} c(2n)\,x^{2n}$, $k(k+1)\equiv 0 \pmod 4$,

$$=2\sum_{n=0}^{\frac{k(k+1)-2}{4}} c(2n)\,x^{2n},$$

$k(k+1)\equiv 2 \pmod 4$.

---

* For instance $q(5,0)=2$ *i.e.* ..... and ...

Now corresponding to the above partition of $q(5,0)$, we have the partitions ..... and .... of $q(6,0)$ respectively in addition to two more partitions of $q(6,0)$ *viz* ... and ..

By substituting $x=1$, we have the required property.

4. In this section we prove some inequalities involving the function $p(n)$ in theorem 2.

Theorem 2: (*a*) $p(n)+p(n-2) \geqslant 2p(n-1)$, $\quad n \geqslant 2$.

(*b*) $p(n) \geqslant p(n-k)+p(k)$ $\quad n > k \geqslant 1$.

(*c*) $\sum_{r=0}^{\frac{k(k+1)}{2}} c(r)\, q(n-r,\ k) > 0$, $n \geqslant \frac{(k+1)(k+2)}{2}$,

and $q(n, k)=p(n)-p(n-\overline{k+1})$.

(*a*) We give a combinatorial proof.

Let $q(n, k)=p(n)-p(n-\overline{k+1})$.

We can easily see that generating function of $q(n, 0)$ is given by the expression $\frac{1}{\prod_{r=2}^{\infty}(1-x^r)}$. In other words $q(n, 0)$=No. of partitions of n into parts $\geqslant 2$. Now $q(n, 0) \geqslant q(n-1, 0)$; because for each partition* π enumerated by $q(n-1, 0)$ there exist a partition enumerated by $q(n, 0)$ obtained by adding one to the largest part of π.

(*b*) From (*a*) we know that

$p(n)-p(n-1) \geqslant p(n-1)-p(n-2)$, $\quad n \geqslant 2$.

or $p(m)-p(m-1) \geqslant p(k)-p(k-1)$, $\quad m \geqslant k \geqslant 1$

or $p(m)+p(k-1) \geqslant p(k)+p(m-1)$.

If $k=2$ and $m=n-1$,

then $p(n-1)+p(1) \geqslant p(2)+p(n-2)$.

If $k=3$ and $m=n-2$

then $p(n-2)+p(2) \geqslant p(3)+p(n-3)$ and so on.

We have ultimately

$p(n-1)+p(1) \geqslant p(k)+p(n-k)$.

But $p(n) \geqslant p(n-1)+p(1)$.

Hence $p(n) \geqslant p(k)+p(n-k)$.

We have also a very simple combinatorial proof of the above inequality.

Let $p(n-k)=m$ and $p(k)=n$.

Pick up one of the partitions of $(n-k)$ and one of k.

Combine the partitions in any manner. This combined partition will be a partition of n. The total no. of such combination will be $m\,n$. But $mn \geqslant m+n$ $(m \,\&\, n \geqslant 2)$.

Hence $p(n) \geqslant p(n-k)+p(k)$.

The generalized form of the above inequality will be

$$p(n) \geqslant p(k_1)+p(k_2)+\ldots\ldots+p(k_n)$$

when $n=k_1+k_2+\ldots\ldots+k_n$.

(*c*) Since $$\frac{1}{\prod_{r=1}^{\infty}(1-x^r)} = \sum_{n=0}^{\infty} p(n)x^n.$$

Therefore $$\frac{x^{k+1}}{\prod_{r=1}^{\infty}(1-x^r)} = x^{k+1}.\sum_{n=0}^{\infty} p(n)x^n$$

$$= \sum_{n=k+1}^{\infty} p(n-k+1)x^n.$$

From the above equations we can easily see that

$$\frac{(1-x^{k+1})}{\prod_{r=1}^{\infty}(1-x^r)} = \sum_{n=0}^{k} p(n)x^n + \sum_{n=k+1}^{\infty} q(n,k)x^n.$$

or $$\frac{1}{\prod_{r=1}^{k}(1-x^r)\prod_{r=k+2}^{\infty}(1-x^r)} = \sum_{n=0}^{k} p(n)x^n + \sum_{n=k+1}^{\infty} q(n,k)x^n.$$

or $$\frac{1}{\prod_{r=k+2}^{\infty}(1-x^r)} = \sum_{n=0}^{\frac{k(k+1)}{2}} c(n)x^n \left\{ \sum_{n=0}^{k} p(n)x^n + \sum_{n=k+1}^{\infty} q(n,k)x^n \right\}$$

$$= \text{terms upto } x^{\frac{k(k+3)}{2}} +$$

$$\sum_{n=\frac{(k+1)(k+2)}{2}}^{\infty} \left\{ \sum_{r=0}^{\frac{k(k+1)}{2}} c(r).\, q\,(n-r), k) \right\} x^n.$$

Since the coefficients of x^s, where $s \geqslant k+2$ in the expansion of the left hand side are always positive ; therefore

$$\sum_{r=0}^{\frac{k(k+1)}{2}} c(r).\, q(n-r,\, k) > 0, \text{ Whenever } n \geqslant \frac{(k+1)(k+2)}{2} \tag{4.1}$$

Incidentally we have also prove that

$$\sum_{r=0}^{k} c(r)\, p(k-r)=0, \tag{4.2}$$

which is obtainable by equating coefficients of x^k on both the sides of the above identity.

We now consider a few cases of the inequality in (4.1.).

Ist Case. If $k=1$, then

$$p(n)+p(n-3) > p(n-1)+p(n-2), \qquad n \geqslant 3 \tag{4.3}$$

2nd Case. If $k=2$, then

$$p(n)+p(n-4)+p(n-5) > p(n-1)+p(n-2)+p(n-6), \quad n \geqslant 6 \tag{4.4}$$

3rd Case. If $k=3$, then

$$p(n)+2p(n-5)+p(n-10) > p(n-1)+p(n-2)+p(n-8)+p(n-9), \quad n \geqslant 10, \tag{4.5}$$

and so on.

On a careful study of the above of inequalities we observe that

(*i*) On each side an equal number of terms are involved

(*ii*) the sums of the integers whose partitions are being added are the same, and

(*iii*) these integers belong to the block of $\left[\frac{(k+1)(k+2)}{2}+1\right]$ consecutive integers.

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DISTRIBUTION OF THE CLASSIFICATION STATISTIC Z WHEN COVARIANCE MATRIX IS KNOWN

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1. INTRODUCTION :

Consider the situation where the vector observation x is known to belong to one of the two p-variate normal populations $\pi_1 : N(\mu_1, \Sigma)$ and $\pi_2 : N(\mu_2, \Sigma)$, the positive definite matrix Σ being known. The samples $x_{11}, x_{12}, \ldots, x_{1N_1}$ and $x_{21}, \ldots, x_{2N_2}$ drawn respectively from π_1 and π_2 are independent of each other. The parameters μ_1 and μ_2, where $\mu_1 \neq \mu_2$, are unknown, and so they are estimated from samples by the sample means

$$\bar{x}_1 = (1/N_1) \sum_{i=1}^{N_1} x_{1i} \qquad \bar{x}_2 = (1/N_2) \sum_{i=1}^{N_2} x_{2i}.$$

Then

$$Z = \frac{N_1}{N_1+1} (x-\bar{x}_1)' \Sigma^{-1} (x-\bar{x}_1) - \frac{N_2}{N_2+1} (x-\bar{x}_2)' \Sigma^{-1} (x-\bar{x}_2)$$

is a criterion proposed by Kudo (1959) and John (1960, 1963) in discriminant analysis for classifying the observation x into its correct population, the procedure according to which is to assign x to the population π_1 if $Z \leqslant 0$ and to the population π_2 if $Z > 0$. Memon (1968) studies this classification criterion when Σ is unspecified. This paper obtains, for the case when Σ is fully known, an asymptotic expansion of the distribution of Z with respect to N_1^{-1} and N_2^{-1} by the studentization method of Hartley, as also used in Han (1969). Expressions for probabilities of misclassification which arise due to the use of this criterion are also derived therefrom for the purpose of numerical computations.

2. THE MAIN RESULTS :

Let $F(y)$ be the distribution function of the random variable $(Z-\mu)/\sigma$ under the assumption that x comes from the normal population π_1. The parameters $\mu=-D^2$ and $\sigma^2=4D^2$ are mean and variance of the asymptotic distribution of Z given that $x \in \pi_1$ where D^2 is the Mahalanobis distance between π_1 and π_2.

2.1. *Case* $N_1 \neq N_2$.

Theorem. If $D > 0$, then an asymptotic expression of $F(y)$ is given by

$$
\begin{aligned}
&[1+N_1^{-1}(G_1+\tfrac{1}{2}A_1)+N_2^{-1}(G_2+\tfrac{1}{2}B_1)\\
&\quad+N_1^{-2}\{G_{11}+\tfrac{1}{2}(A_1G_1+A_2)+\tfrac{1}{8}D_1\}+N_2^{-2}\{G_{22}\\
&\quad+\tfrac{1}{2}(B_1G_2+B_3)+\tfrac{1}{8}D_2\}\\
&\quad+(N_1N_2)^{-1}\{G_{12}+\tfrac{1}{2}(A_1G_2+A_3)+\tfrac{1}{2}(B_1G_1+B_2)\\
&\quad+\tfrac{1}{4}D_3\}+\ldots\ldots]\,\Phi(y),
\end{aligned}
$$

where $d=d/dy$, $\Phi(y)$ is the *cdf* of N (0, 1) and

$$
\begin{aligned}
A_1&=D^{-2}[d^4+p(d^2-Dd)],\\
A_2&=D^{-3}(d-D)[2d^4+pd^2-pDd],\\
A_3&=-D^{-3}[2d^5+2Dd^4+pd^3]\\
B_1&=D^{-2}(d+D)[d^3+Dd^2+pd],\\
B_2&=D^{-3}[2d^5+2Dd^4+pd^3],\\
B_3&=-D^{-3}(d+D)^2[2d^3+2Dd^2+pd],\\
D_1&=D^{-4}[\{d^4+pd^2-pDd\}^2+4d^6-4Dd^5+2p(d^2-Dd)^2],\\
D_2&=D^{-4}(d+D)^2[\{d^3+Dd^2+pd\}^2+4d^4+4Dd^3+2pd^2],\\
D_3&=D^{-4}[(d+D)(d^4+pd^2-pDd)(d^3+Dd^2+pd)\\
&\qquad+4d^6+4Dd^5+2pd^4],\\
G_1&=\tfrac{1}{2}D^{-1}(d^3+pd),\\
G_2&=-\tfrac{1}{2}D^{-1}[d^3+2Dd^2+(D^2+p)d],\\
G_{11}&=\tfrac{1}{8}D^{-2}[(d^3+pd)^2+4d^4-4Dd^3+2pd^2-4pDd],\\
G_{12}&=-\tfrac{1}{4}D^{-2}[(d^3+pd)\{d^3+2Dd^2+(D^2+p)d\}\\
&\qquad+4d^4+4Dd^3+2pd^2],\\
G_{22}&=\tfrac{1}{8}D^{-2}[\{d^3+2Dd^2+(D^2+p)d\}^2\\
&\qquad+4d^4+12Dd^3+2(6D^2+p)d^2+4(D^2+p)Dd].
\end{aligned}
$$

Proof. We can express the statistic **Z** in the form

$$b\,[\{x-\bar{x}_1+\alpha\,(\bar{x}_2-\bar{x}_1)\}'\,\Sigma^{-1}\,\{x-\bar{x}_1+\alpha\,(\bar{x}_2-\bar{x}_1)\}-\alpha\,(\alpha+1)\,Q],$$

where

$$b=(N_1-N_2)/(N_1+1)\,(N_2+1),$$
$$\alpha=N_2\,(N_1+1)/(N_1-N_2),$$
$$Q=(\bar{x}_1-\bar{x}_2)'\,\Sigma^{-1}\,(\bar{x}_1-\bar{x}_2).$$

By using this expression it can be seen that the conditional characteristic function of Z when $\bar{x}_1$, $\bar{x}_2$ are fixed and the observation x comes from π_1, is given by

$$\exp\,[-\theta\,b\alpha\,(\alpha+1)Q+\sum_{j=1}^{p}\frac{\theta\,b\,v_j^2}{1-2\theta b}-\frac{p}{2}\log\,(1-2\theta b)],$$

where

$\theta=it$,

$v=C\,[\mu_1-\bar{x}_1+\alpha\,(\bar{x}_2-\bar{x}_1)]$,

C is a $p\times p$ nonsingular matrix such that $C\,\Sigma\,C'=I$,

$Q=(\bar{x}_1-\bar{x}_2)'\Sigma^{-1}\,(\bar{x}_1-\bar{x}_2)$.

Due to invariance of the distribution of the Z statistic under any nonsingular linear transformation, we may suppose without any loss of generality that

$$\mu_1=0,\ \mu_2=\mu_0,\ \Sigma=I,$$

where μ_0' is $1\times p$ vector $(D, 0, .., 0)$. These substitutions now simplify the conditional characteristic function of **Z** to

$$\exp\,[-\theta b\alpha\,(\alpha+1)\,Q+\sum_{j=1}^{p}\frac{\theta b v_j^2}{1-2\theta b}-\frac{p}{2}\log\,(1-2\theta b)]$$

where

$$v=[-(\alpha+1)\,\bar{x}_1+\alpha\bar{x}_2],$$
$$Q=(\bar{x}_1-\bar{x}_2)'\,(\bar{x}_1-\bar{x}_2).$$

The characteristic function of $(Z-\mu)/\sigma$ under the assumption that $x \in \pi_1$ is

$$\phi(\theta)=E\,[\exp.\,\{\tfrac{1}{2}\,\theta D^{-1}\,(Z+D^2)\}\mid \pi_1]=E_{\bar{x}_1,\,\bar{x}_2}\,[\psi(\bar{x}_1,\,\bar{x}_2)]$$

where $E_{\bar{x}_1,\,\bar{x}_2}$ indicates expectation w. r. t. the joint distribution of $(\bar{x}_1,\,\bar{x}_2)$,

$$\psi\,(\bar{x}_1,\,\bar{x}_2)=\exp\,[A\,(\bar{x}_1,\,\bar{x}_2)],$$

and

$$A\,(\bar{x}_1,\,\bar{x}_2)=\frac{\theta D}{2}-\frac{\theta b\alpha\,(\alpha+1)}{2D}\,Q+\frac{1}{2D}\sum_{j=1}^{p}\frac{\theta b\,v_{2j}^2}{1-\frac{\theta b}{D}}$$

$$-\frac{p}{2}\log\left(1-\frac{\theta b}{D}\right).$$

Now since $\psi\,(\bar{x}_1,\,\bar{x}_2)$ is analytic about the point $(\bar{x}_1,\,\bar{x}_2)=(0,\,\mu_\circ)$, so in a Taylor expansion about an origin $(0,\,\mu_\circ)$

$$\exp\,[A\,(\bar{x}_1,\,\bar{x}_2)]$$

$$=[\{\sum_{i=1}^{p}\bar{x}_{1i}\,\frac{\partial}{\partial\mu_{1i}}+\sum_{i=1}^{p}\,(\bar{x}_{2i}-\mu_{\circ i})\frac{\partial}{\partial\mu_{2i}}\}\,\psi\,(\mu_1,\,\mu_2)]_\circ,$$

where $]_\circ$ is the value at the point $(0,\,\mu_\circ)$. So

$$\phi\,(\theta)=E_{\bar{x}_1,\,\bar{x}_2}\,[e^{A(\bar{x}_1,\,\bar{x}_2)}]$$

$$=\bigcirc\,e^{A(\mu_1,\,\mu_2)}]_\circ \tag{3.1}$$

where

$$\bigcirc=1+\frac{1}{2N_1}\sum_r\frac{\partial^2}{\partial\mu^2{}_{1r}}+\frac{1}{2N_2}\sum_r\frac{\partial^2}{\partial\mu^2{}_{2r}}+\frac{1}{8N_1^2}\sum_{r,\,s}\frac{\partial^4}{\partial\mu^2{}_{1r}\partial\mu^2{}_{1s}}$$

$$+\frac{1}{8N_2^2}\sum_{r,\,s}\frac{\partial^4}{\partial\mu^2{}_{2r}\partial\mu^2{}_{2s}}+\frac{1}{4N_1N_2}\sum_{r,\,s}\frac{\partial^4}{\partial\mu^2{}_{1r}\partial\mu^2{}_{2s}} \tag{3.2}$$

where each subscript runs over the range $1,\,..,\,p$.

We consider now

$$\phi(\theta) = \bigcirc e^{A(\mu_1, \mu_2)}]_{\circ} . \tag{3.3}$$

where A (μ_1, μ_2) comes to

$$\frac{\theta D}{2} - \frac{\theta b\alpha(\alpha+1)}{2D} Q + \frac{1}{2D} \sum_{j=1}^{p} \frac{v^2 j}{1-\frac{\theta b}{D}} - \frac{p}{2} \log\left(1-\frac{\theta b}{D}\right),$$

and

$$v = [-(\alpha+1)\mu_1 + \alpha\mu_2],$$
$$Q = (\mu_1 - \mu_2)'(\mu_1 - \mu_2).$$

At the point $\mu_1 = 0$, $\mu_2 = \mu_{\circ}$ it can be shown that

$$A(0, \mu_{\circ}, I) = \left(1 + \frac{G_1}{N_1} + \frac{G_2}{N_2} + \frac{G_{11}}{N^2_1} + \frac{G_{22}}{N^2_2} + \frac{G_{12}}{N_1 N_2} + \ldots\right) e^{\theta^2/2}. \tag{3.4}$$

The constants appearing in the expansion of $\phi(\theta)$ are functions of θ and are same as defined in the statement of the theorem for $\theta = -d$.

The coefficient of $\frac{1}{2N_1}$ in (3.2) is

$$\sum_r \frac{\partial^2 e^A}{\partial \mu^2_{1r}}\Big]_{\circ} = \sum_r \left[\frac{\partial^2 A}{\partial \mu^2_{1r}} + \left(\frac{\partial A}{\partial \mu_{1r}}\right)^2\right]_{\circ} e^{A_{\circ}}.$$

Since

$$\frac{\partial A}{\partial \mu_{1r}}\Big]_{\circ} = -\frac{\theta^2 b^2 \alpha(\alpha+1)}{D\left(1-\frac{\theta b}{D}\right)} \delta_{1r},$$

$$\frac{\partial^2 A}{\partial \mu^2_{1r}}\Big]_{\circ} = \frac{\theta b(\alpha+1)}{D\left(1-\frac{\theta b}{D}\right)}\left(1+\frac{\theta b\alpha}{D}\right),$$

so on simplification the above comes to

$$\left(A_1 + \frac{A_1 G_1 + A_2}{N_1} + \frac{A_1 G_2 + A_3}{N_2} + \ldots\ldots\right) e^{\theta^2/2}. \tag{3.5}$$

The coefficient of $\frac{1}{2N_2}$ in (3.2) obtained similarly is

$$\sum_r \frac{\partial^2 e^A}{\partial \mu^2_{2r}}\Big]_{\circ} = \left(B_1 + \frac{B_1 G_1 + B_2}{N_1} + \frac{B_1 G_2 + B_3}{N_2} + \ldots\right) e^{\theta^2/2} \tag{3.6}$$

We can show

$$\frac{\partial^2 A}{\partial \mu_{ir}\partial \mu_{is}}=\frac{\partial^3 A}{\partial \mu_{ir}\partial^2 \mu_{is}}=\frac{\partial^3 A}{\partial \mu^2_{ir}\partial \mu_{is}}=\frac{\partial^4 A}{\partial \mu^2_{ir}\partial \mu^2_{is}}=0,$$

$$\frac{\partial^3 A}{\partial \mu^3_{ir}}=\frac{\partial^4 A}{\partial \mu^4_{ir}}=0 \qquad \text{for } r\neq s \text{ and } i=1, 2$$

Using these we have

$$\sum_{r,s}\frac{\partial^4 e^A}{\partial \mu^2_{ir}\partial \mu^2_{is}}\Big]_0=\left[\sum_r\left\{\left(\frac{\partial A}{\partial \mu_{ir}}\right)^2+\frac{\partial^2 A}{\partial \mu^2_{ir}}\right\}\right]^2$$

$$+\sum_r\left\{4\left(\frac{\partial A}{\partial \mu_{ir}}\right)^2\frac{\partial^2 A}{\partial \mu^2_{ir}}+2\left(\frac{\partial^2 A}{\partial \mu^2_{ir}}\right)^2\right\}\Big]_0 e^{A_0},$$

which on simplification gives the coefficient of $\frac{1}{8N_i^2}$

$$(3.7) \qquad =D_1 e^{\frac{\theta^2}{2}} \qquad \text{if } i=1$$

$$(3.8) \qquad =D_2 e^{\frac{\theta^2}{2}} \qquad \text{if } i=2.$$

On using

$$\frac{\partial^3 A}{\partial \mu_{1r}\partial \mu^2_{2s}}=\frac{\partial^3 A}{\partial \mu^2_{1r}\partial \mu_{2s}}=\frac{\partial^4 A}{\partial \mu^2_{1r}\partial \mu^2_{2s}}=0 \text{ for } r\neq s$$

we have

$$\sum_{r,s}\frac{\partial^4 e^A}{\partial \mu^2_{1r}\partial \mu^2_{2s}}\Big]_0=\left[\sum_r\left\{\left(\frac{\partial^2 A}{\partial \mu^2_{1r}}\right)^2+\frac{\partial^2 A}{\partial \mu^2_{1r}}\right\}\right]\left[\sum_s\left\{\left(\frac{\partial A}{\partial \mu_{2s}}\right)^2+\frac{\partial^2 A}{\partial \mu^2_{2s}}\right\}\right]$$

$$+\sum_{r,s}\left\{4\frac{\partial A}{\partial \mu_{1r}}\frac{\partial A}{\partial \mu_{2s}}\frac{\partial^2 A}{\partial \mu_{1r}\partial \mu_{2s}}+2\left(\frac{\partial^2 A}{\partial \mu_{1r}\partial \mu_{2s}}\right)^2\right\}\Big]_0 e^{A_0},$$

and since $\frac{\partial^2 A}{\partial \mu_{1r}\partial \mu_{2s}}\Big]_0=-\frac{\theta^2 b^2 \alpha\,(\alpha+1)}{D^2\left(1-\frac{\theta b}{D}\right)}\delta_{rs}$

we obtain the coefficient of $\frac{1}{4N_1N_2}$

$$(3.9) \qquad \sum_{r,s}\frac{\partial^4 e^A}{\partial \mu^2_{1r}\partial \mu^2_{2s}}=D_3 e^{\frac{\theta^2}{2}}.$$

The characteristic function $\phi(\theta)$ of the random variable $(Z-\mu)/\sigma$ on the assumption that $x \in \pi_1$ is obtained from 3.2 to 3.9.

The technique for inversion of a characteristic function of the form $(-\theta)^r \phi(\theta)$ is well-known (see Cramer's result, 1946, pp. 225) and was used by various authors. According to it, the cumulative distribution function corresponding to $(-\theta)^r \phi(\theta)$ is given by $d^r F(y)$. It is now easy to see that the cumulative distribution function is expressible in an asymptotic expansion as given in the statement of the theorem.

Corollary 3. 1. If Z is the classification statistic used in discriminating an observation x, then the probability of misclassifying the observation into the population π_2 when it comes in fact from the population π_1, is given by

$$\left[1-\left(1+\frac{a_1}{N_1}+\frac{a_2}{N_2}+\frac{a_{11}}{N_1^2}+\frac{a_{22}}{N_2^2}+\frac{a_{12}}{N_1N_2}+\cdots\right)\Phi(y)\right]_{y=D/2}$$

where

$$a_1=\frac{1}{2}D^{-2}(d^4+Dd^3+pd^2),$$

$$a_2=\frac{1}{2}D^{-2}[d^4+Dd^3-(D^2-p)d^2-D^3d],$$

$$a_{11}=\frac{1}{8}D^{-4}[d^8+2Dd^7+(D^2+2p+4)d^6+2(p+2)Dd^5$$
$$-(4D^2-p^2-2p)d^4-4D^3d^3-4pD^2d^2],$$

$$a_{22}=\frac{1}{8}D^{-4}[d^8+2Dd^7-(D^2-2p-4)d^6-2(2D^2-p-2)Dd^5$$
$$-\{D^4+2(p+4)D^2-p^2-2p\}d^4+2(D^2-p-4)D^3d^3$$
$$+(D^4+4D^2-4p)D^2d^2+4D^5d],$$

$$a_{12}=\frac{1}{4}D^{-4}[d^8+2Dd^7+2(p+2)d^6-2(D^2-p-2)Dd^5$$
$$-\{D^4+(p+4)D^2-p^2-2p\}d^4-(p+4)D^3d^3-2pD^2d^2],$$

and $\Phi(y)$ is the cumulative normal distribution function of N(0, 1).

Proof. Since the probability of misclassifying the observation into π_2 when it comes in fact from π_1 is

$$1-\text{Prob}\{Z<0 \mid \pi_1\}=1-F(y)\Big]_{y=D/2},$$

the proof follows immediately from consideration of the previous theorem.

2.2. *Case* $N_1=N_2$. When the sample sizes are equal it is discovered that the conditional characteristic function of $(Z-\mu)/\sigma$ given that $x \in \pi_1$ is not derivable from the conditional characteristic function found in the proof of previous theorem. We can, however, establish the following result on similar lines. The probability of misclassifying this observation x into π_2 when it comes in fact from π_1 is then easy to derive therefrom.

$$F(y)=\left(1+\frac{a_1}{2N}+\frac{a_2}{4N^2}+\cdots\right)\Phi(y),$$

where

$$a_1=D^{-2}[2d^4+2Dd^3-(D^2-2p)d^2-D^3d]$$

$$a_2=\frac{1}{2}D^{-4}[4d^8+8Dd^7+8(p+2)d^6-8(D^2-p-2)Dd^5$$
$$-\{3D^4+4(p+5)D^2-4p^2-8p\}d^4$$
$$+2(D^2-2p-10)D^3d^3+(D^4+4D^2-12p)D^2d^2+4D^5d]$$

and $\phi(y)$ is the cumulative distribution function of $N(0, 1)$.

The fact that the characteristic function obtained in the case $N_1=N_2$ is not derivable from that when $N_1 \neq N_2$, would then also be true for their respective cumulative distribution functions. But when we substitute $N_1=N_2=N$ into the asymptotic expansion of $F(y)$ for the case $N_1 \neq N_2$, it is interesting to find out that the resulting expression becomes identical with the one obtained in this section, so that up to order N^{-2} the previous theorem as well as corollary can be regarded as holding true in general.

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