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DEPARTMENT OF MATHEMATICS
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CONTENTS

A SET OF n-DUAL INTEGRAL EQUATIONS*

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Dual integral equations have their origin in the mixed boundary value problems arising in potential theory. Weber [8] was perhaps the first to consider dual integral equations but the systematic treatment for the solution of such equations was initiated by Titchmarsh [5]. In the current decade Erdogan and Bahar [2], Westmann [9], Szefer [4] and a few other authors have considered simultaneous dual integral equations. In this paper, a solution for a given set of n-dual integral equations is found in terms of a Fredholm's integral equation of the second kind. The matrix notation and the following discontinuous integral [7] are used

$$(1)\quad \int_0^\infty J_\nu(\alpha t)\, J_{\nu-1}(\alpha x)\, d\alpha = \begin{cases} 0 & t < x \\ \dfrac{x^{\nu-1}}{t^\nu} & t > x \end{cases}$$

provided $\operatorname{Re}\ \nu > 0$.

The method is a direct extension of Szefer's solution [4]. It is primarily based on the method used by Copson [1] and later on extended by Lowengrub and Sneddon [3].

1. **Transformation :** Consider the set of n-dual integral equations with Bessel function kernels

$$(2a)\quad \int_0^\infty D(\alpha)\, \phi(\alpha)\, J_\nu(\alpha t)\, d\alpha = f(t) \qquad 0 < t < 1$$

$$(2b)\quad \int_0^\infty E(\alpha)\, \phi(\alpha)\, J_\nu(\alpha t)\, d\alpha = g(t) \qquad t > 1$$

where $l(x)$ is an unknown vector. Substituting the value of $\chi(\alpha)$ from (6) in (5*a*), we get

$$\int_0^\infty t^\nu \int_1^\infty l(x) J_{\nu-1}(\alpha x)\ J_\nu(\alpha t)\ dx\ d\alpha \qquad 0 < t < 1$$

Inverting the order of integration, we have

$$(7) \qquad \int_1^\infty t^\nu l(x)\, dx \int_0^\infty J_\nu(\alpha t)\, J_{\nu-1}(\alpha x)\ d\alpha \qquad 0 < t < 1$$

Since $x > 1$, we observe that $t < x$ always; hence applying the discontinuous intergral (1), the integral (7) vanishes identically and hence the equations (5*a*) are statisfied.

Now substitute (6) in (5*b*), then we have

$$(8) \qquad \int_0^\infty L(\alpha)\,[t^\nu \int_1^\infty l(x)\, J_{\nu-1}(\alpha x)\ dx]\ J_\nu(\alpha t)\ d\alpha = g(t) \qquad t > 1$$

Assuming for the matrix $L(\alpha)$

(9) $\quad L(\alpha) = I + V(\alpha)$

where I is the unit matrix and

$$(10) \qquad V(\alpha) = \begin{Bmatrix} V_{11}(\alpha), L_{12}(\alpha), \ldots\ldots, L_{1n}(\alpha) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_{n1}(\alpha), L_{n2}(\alpha), \ldots\ldots, V_{nn}(\alpha) \end{Bmatrix}; \ V_{rr}(\alpha) = L_{rr}(\alpha) - 1$$

we obtain form (8) that

$$(11) \qquad g(t) = \int_0^\infty I\,[t^\nu \int_1^\infty l(x)\, J_\nu(\alpha t)\, J_{\nu-1}(\alpha x)\ dx]\ d\alpha$$

$$+ \int_0^\infty V(\alpha)\,[t^\nu \int_1^\infty l(x)\, J_\nu(\alpha t)\ J_{\nu-1}(\alpha x)\ dx]\ d\alpha$$

Inverting the order of integration in (11), we get

$$g(t)=\int_1^\infty t^\nu l(x)\,dx\int_0^\infty J_\nu(\alpha t)\,J_{\nu-1}(\alpha x)\,d\alpha$$

$$+\int_1^\infty\left[\int_0^\infty V(\alpha)\,J_\nu(\alpha t)\,J_{\nu-1}(\alpha x)\,dx\right]t^\nu\,l(x)\,dx.$$

Again applying the discontinuous integral (1), we obtain

$$(12)\qquad g(t)=\int_1^t x^{\nu-1}\,l(x)\,dx+t^\nu\int_1^\infty\left\{\int_0^\infty V(\alpha)\,J_\nu(\alpha\,t)\,J_{\nu-1}(\alpha x)\,dx\right\}l(x)\,dx$$

Putting

$$(13)\qquad \int_1^t x^{\nu-1}\,l(x)\,dx=Q(t)$$

and assuming that

$$(14)\qquad \lim_{t\to\infty} Q'(t)=0$$

we have

$$(15)\qquad t^{\nu-1}\,l(t)=Q(t).$$

Substituting (13) and (15) in (12) we get

$$(16)\qquad g(t)=Q(t)+t^\nu\int_1^\infty\left\{\int_0^\infty V(\alpha)\,J_\nu(\alpha t)\,J_{\nu-1}(\alpha x)\,dx\right\}x^{1-\nu}\,Q'(x)\,dx$$

Putting

$$(17)\qquad \int_0^\infty V(\alpha)\,J_\nu(\alpha t)\,J_{\nu-1}(\alpha x)\,d\alpha=N(x,t)$$

equations (16) may be written as

$$(18)\qquad g(t)=Q(t)+t^\nu\int_1^\infty N(x,t)\,x^{1-\nu}\,Q'(x)\,dx.$$

Integrating by parts and making use of condition (14), we get

$$(19) \qquad g(t)=Q(t)-t^{\nu}\int_{1}^{\infty}\left[N(x,t)\,x^{1-\nu}\right]'\,Q(x)\,dx$$

Taking

$$(20) \qquad [N(x,t)\,x^{1-\nu}]'=K(x,t)$$

we finally obtain

$$(21) \qquad Q(t)=g(t)+t^{\nu}\int_{1}^{\infty}K(x,t)\,Q(x)\,dx$$

This is a set of Fredholm's integral eqations of the second kind, which can be reduced to a single equation of that type. Hence the problem is formally solved.

Notice that for $n=1$, the problem reduces to Tranter's case [6].

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ON GENERATING SETS OF INTEGERS

By

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The content of this article arose as a special case of general investigation of the following problem :

Let $I_N = \{1, 2, 3, \ldots, N\}$ and let $G_m = \{g_1, g_2, \ldots, g_m\}$

be a set of positive distinct integers. If $n \in I_N$ and $n = \epsilon_1 g_1 + \epsilon_2 g_2 + \ldots + \epsilon_m g_m$, where $\epsilon_i \in E = \{-1, 0, 1\}$ then we say that n admits an *E-representation* with respect to G_m.

What is the minimum value of m for which every member of I_N admits an E- representation ?

§ 1. **Set of Independent Integers.**

Here we prove an interesting result based on the following definition.

Definition 1. A set G_m of positive distinct integers is called a set of independent integers if none of its members admits an E-representation with respect to the set of its remaining members.

That sets of independent integers exist in abundance is obvious.

Theorem 1. The number of positive integers (distinct or coincident) which admit E-representations with respect to the set G_m of independent integers is equal to

$$\tfrac{1}{2}(3^m - 1).$$

Proof. Let $D_r = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be one of the combinations of r elements of G_m. Consider the expression $\sum_{i=1}^{r} \delta_{j_i} \lambda_{j_i}$, where each δ_{j_i} is either $+1$ or -1 and each j_i from 1 to r. The total number of integers (positive as well as negative) which can be obtained by considering all possible permutations of the set $\{1, 2, \ldots, r\}$ is clearly equal to $\sum_{i=0}^{r} {}^rC_i$. It follows from the symmetry of the expression

$\sum_{i=1}^{r} \delta j_i \lambda j_i$ that half of the integer so obtained will be positive.

Hence the total number of positive integers M obtainable from all the combination of r members of G_m is given by

$$M = \frac{1}{2} \sum_{r=1}^{m} {}^mC_r \sum_{i=0}^{r} {}^rC_i \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Since $\sum_{i=0}^{r} {}^rC_i = 2^r$, the equation (1) can be written as

$$M = \frac{1}{2} \sum_{r=1}^{m} 2^r . {}^mC_r \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

It is known that $3^m = 1 + \sum_{r=1}^{m} 2^r \, {}^mC_r$, therefore the equation (2) can be cast in form

$$M = \frac{1}{2}(3^m - 1), \text{ which completes the proof.}$$

§ 2. Generating Sets.

Our solution of the problem posed in the beginning of this article utilizes the concept of a generating set whose definition runs as follows :

Definition 2. A well-ordered set $G_m = \{g_1, g_2, \ldots, g_m\}$ of distinct positive integers is called a generating set of I_N if it meets the following requirements :

(*i*) $\sum_{i=1}^{m} g_i = N$, *i.e.* g_i's constitute a partition of N.

(*ii*) Every $n \in I_N$ admits E-representation with respect to G_m.

Note 1. Obviously for all N, I_N does not possess a generating set. For instance I_5 has no generating set.

Note 2. If G_m is a generating set of I_N such that its members constitute a set of independent integers, then it is easy to verify that

$$N \leqslant \tfrac{1}{2}(3^m - 1) \qquad \text{[compare Theorem 1]}$$

Theorem 2. Let G_m be a generating set of I_N and let $n \in I_N$. If $n \leqslant g_t$, then

$$n = \sum_{i=1}^{t} (1-\epsilon_i)\, g_i.$$

Proof. Let

$$N - n = \sum_{i=1}^{m} \epsilon_i\, g_i \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

Since $N = \sum_{i=1}^{m} g_i$, it follows that

$$n = \sum_{i=1}^{m} (1-\epsilon_i)\, g_i$$

$$= \sum_{i=1}^{t} (1-\epsilon_i)\, g_i + \sum_{i=t+1}^{m} (1-\epsilon_i)\, g_i, \qquad \ldots \quad (4)$$

where each $(1-\epsilon_i)$ is either 0, 1 or 2.

Let $s > t$ and $(1-\epsilon_s) \neq 0$. Then $n \geqslant g_s$ which contradicts the fact that G_m is well-ordered.

Hence we conclude that

$$(1-\epsilon_i) = 0 \text{ for } i = t+1,\ t+2,\ \ldots,\ m.$$

Therefore

$$n = \sum_{i=1}^{t} (1-\epsilon_i)\, g_i \qquad \ldots \qquad \ldots \qquad \ldots \quad (5)$$

Corollary. The first member of every generating set is unity.

The proof of this assertion follows from equation (5) for $n=1$ $t=1$.

Following result characterizes the generating sets.

Theorem 3. Every well-ordered set $G_m = \{1, g_2, g_3, \ldots, g_m\}$ with distinct members is a generating set if and only if

$$g_t \leqslant 2 \sum_{i=1}^{t-1} g_i + 1 \text{ for } 2 \leqslant t \leqslant m.$$

Proof. Let G_m be a generating set of I_N.

Suppose that $g_t \leqslant 2 \sum_{i=1}^{t-1} g_i + 1$, does not hold for $t = s$.

This implies that $g_s > 2 \sum_{i=1}^{s-1} g_i + 1$

$$= 2M + 1, \text{ where } M = \sum_{i=1}^{s-1} g_i.$$

It follows that

$$g_s - \sum_{i=1}^{s-1} g_i > M + 1$$

which implies that $M+1$ does not admit an E-representation with respect to G_s.

Let $g_s = 2M + 1 + K, \qquad K > 0$

Suppose $M+1$ admits an E-representation with respect to G_{s+1}. Since $g_{s+1} > g_s$, we may write

$$g_{s+1} = g_s + L, \qquad L > 0$$

consider the equation

$$(M+1) + g_{s+1} = \sum_{i=1}^{s-1} \epsilon_i g_i + \epsilon_s g_s + \epsilon_{s+1} g_{s+1} \qquad \ldots \quad (6)$$

Since equation (6) is not valid for any of the admissible value of ϵ_{s+1}, it follows that $(M+1) + g_{s+1}$ has no E-representation with respect to G_{s+1}.

We may, therefore, conclude that in general

$(M+1) + \sum_{i=1}^{\lambda} g_{s+i}$ does not have an E-representation with respect to $G_{s+\lambda}$, $1 \leqslant \lambda \leqslant m-s$. This implies that G_m is not a generating set. This contradiction proves that

$$g_t \leqslant 2 \sum_{i=1}^{t-1} g_i+1,\ 2 \leqslant t \leqslant m.$$

Conversely we suppose that $G_m=\{1, g_2, \ldots, g_m\}$ is a well-ordered set of distinct elements whose members satisfy

$$g_t \leqslant 2 \sum_{i=1}^{t-1} g_i+1 \text{ for } 2 \leqslant t \leqslant m.$$

Clearly the sets $\{1, 2\}$ and $\{1, 3\}$ which correspond to $t=2$ are generating sets. Suppose that this result holds for $t=n$. Now we shall establish that the required result is valid for $t=n+1$.

Since $g_{n+1} \leqslant 2 \sum_{i=1}^{n} g_i+1$, we may write $g_{n+1}=2N+1+K$, .. (7)

where $N= \sum_{i=1}^{n} g_i$ and K is fixed integer lying between 0 and N.

We now claim that G_{n+1} generates I_{3N+1-K}

Since G_n is a generating set of I_N, it follows that

$$N-J= \sum_{i=1}^{n} \epsilon_i g_i, \text{ where } 0 \leqslant J \leqslant N \quad .. \quad (8)$$

From (7) and (8) we get

$$g_{n+i} - \sum_{i=1}^{n} \epsilon_i g_i=N+J-K+1 \quad .. \quad .. \quad (9)$$

and $$g_{n+1} + \sum_{i=1}^{n} \epsilon_i g_i=3N-J-K+1 \quad .. \quad ... \quad (10)$$

Giving different values to J in (9) and (10), we obtain the set

$$\{N-K+1, N-K+2, \ldots, 3N-K+1\}$$

Hence G_{n+1} generates the set I_{3N-K+1}. By mathematical induction it follows that G_m is a generating set of I_m, where $M= \sum_{i=1}^{m} g_i$

Let $G_m = \{1, g_2, g_3, \ldots, g_m\}$ be a generating set and let $G_n = \{1, g_2, \ldots, g_n\}$ be one of its subsets. Then such a subset will be called a *primary subset of* G_m.

Theorem 4. Every primary subset of a generating set is also a generating set.

Proof. Let G_n be a primary subset of a generating set G_m. Since G_m is generating set, it follows from theorem 3 that

$$g_t \leqslant 2 \sum_{i=1}^{t-1} g_i + 1 \text{ for } 2 \leqslant t \leqslant m.$$

For $n \leqslant m$, we have

$$g_t \leqslant 2 \sum_{i=1}^{t-1} g_i + 1 \text{ for } 2 \leqslant t \leqslant n,$$

Hence it follows from the converse of theorem 3 that G_n is a generating set.

ON THE NUMBER OF GENERATING SETS

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1. Introduction

The concept of a generating set of $I_N = \{1, 2, \ldots, N\}$ was introduced in [1] and the following result was established.

Theorem. Every well-ordered set $G_m = \{1, g_2, g_3, \ldots, g_m\}$, $g_i \neq g_j$ for $i \neq j$, is a generating set of I_N, where

$$N = 1 + \sum_{i=2}^{m} g_i, \text{ if and only if}$$

$$g_t \leqslant 2\left(\sum_{i=1}^{t-1} g_i\right) + 1, \qquad 2 \leqslant t \leqslant m. \tag{1}$$

If G_m is a generating set of I_N, then m will be called the *order* of the generating set G_m. In the note we have proved a result which gives the exact number of generating sets of I_N of order m.

2. Number of Generating Sets

Let N be a given positive integer. Then the total number of generating sets of I_N of order m will be denoted by $\wedge(N, m)$.

Theorem 1.

$$g'_{m-1} = \left\{ \frac{N - 1 - \sum_{i=1}^{m-2} g_i}{2} \right\}$$

$$\wedge(N, m) = \sum_{g_2=2}^{3} \; \sum_{g_3=g_2+1}^{2\sum_{i=1}^{2} g_i + 1} \cdots\cdots \sum_{g_{m-2}=g_{m-3}+1}^{2\sum_{i=1}^{m-3} g_i + 1} \; \sum_{g_{m-1}=g_{m-2}+1}^{g_{m-1}=2\sum_{i=1}^{m-2} g_i + 1} \delta_{g'_{m-1}}^{g_{m-1}},$$

$$g'_{m-1} = \left\{ \frac{N + 1 - 3\sum_{i=1}^{m-2} g_i}{3} \right\}$$

Where [ ] is the bracket function and δ is the Kronecker delta.

Proof: Let G_m be a generating set of I_N of order m. Then given $g_1, g_2, \ldots, g_{m-1}$, the last member g_m is uniquely determined and is given by

$$g_m = N - \sum_{i=1}^{m-1} g_i \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Assuming that the first $m-2$ members of G_m are given, then g_{m-1} and g_m will take all values satisfying the relations :

$$\left.\begin{array}{l} g_{m-1}+g_m = N - \sum_{i=1}^{m-2} g_i \\ \text{and } g_{m-1} \geqslant g_{m-2}+1;\ g_m \geqslant g_{m-1}+1 \end{array}\right\} \quad \ldots \quad \ldots \quad (3)$$

From (1) we have

$$g_m \leqslant 2 \sum_{i=1}^{m-1} g_i + 1$$

which together with equation (2) gives

$$g_{m-1} \geqslant \left[\frac{N+1-3 \sum_{i=1}^{m-2} g_i}{3} \right] \quad \ldots \quad \ldots \quad (4)$$

From well-orderedness of G_m, we get

$$g_{m-1}+1 \leqslant g_m = N - \sum_{i=1}^{m-1} g_i,$$

which easily leads to

$$g_{m-1} \leqslant \left[\frac{N-1-\sum_{i=1}^{m-2} g_i}{2} \right] \quad \ldots \quad \ldots \quad (5)$$

Hence the allowable values of g_{m-1} satisfy the following relations

$$g_{m-2}+1 \leqslant g_{m-1} \leqslant 2 \sum_{i=1}^{m-2} g_i + 1 \quad \ldots \quad \ldots \quad (6)$$

$$\text{and} \left[\frac{N+1-3\sum_{i=1}^{m-2} g_i}{3}\right] \leqslant g_{m-1} \leqslant \left[\frac{N-1-\sum_{i=1}^{m-2} g_i}{2}\right] \quad \text{.. (7)}$$

Since (6) and (7) hold simultaneously and $g_1, g_2, \ldots, g_{m-2}$ are fixed, it follows that the number of generating sets corresponding to the variations of g_{m-1} and g_m is given by

$$\sum_{g_{m-1}=g_{m-2}+1}^{g_{m-1}=2\sum_{i=1}^{m-2} g_i+1} \delta_{g'_{m-1}}^{g_{m-1}} \qquad \text{.. .. (8)}$$

$$g'_{m-1} = \left[\frac{N-1-\sum_{i=1}^{m-2} g_i}{2}\right], \qquad g'_{m-1} = \left[\frac{N+1-3\sum_{i=1}^{m-2} g_i}{3}\right]$$

It is clear that the expression (8) is a function of $g_1, g_2, \ldots, g_{m-2}$. Since the remaining g_i 's satisfy

$$g_{m-j-1}+1 \leqslant g_{m-j} \leqslant 2\sum_{i=1}^{m-j-1} g_i+1, \qquad 2 \leqslant j \leqslant m-2,$$

we conclude that the total number of generating sets $\wedge(N, m)$ is given by

$$\wedge(N,m) = \sum_{g_2=2}^{3} \; \sum_{g_3=g_2+1}^{2\sum_{i=1}^{2} g_i+1} \cdots\cdots \sum_{g_{m-2}=g_{m-3}+1}^{2\sum_{i=1}^{m-3} g_i+1} \; \sum_{g_{m-1}=g_{m-2}+1}^{g_{m-1}=2\sum_{i=1}^{m-2} g_i+1} \delta_{g'_{m-1}}^{g_{m-1}}$$

$$g'_{m-1} = \left[\frac{N-1-\sum_{i=1}^{m-2} g_i}{2}\right], \qquad g'_{m-1} = \left[\frac{N+1-3\sum_{i=1}^{m-2} g_i}{3}\right]$$

Which is the required result. .. (9)

Illustration :

To facilitate the understanding of the above result we write down the full expression for $m=5$.

$$\Lambda(N,5)=\sum_{g_2=2}^{g_2=3}\;\sum_{g_3=g_2+1}^{2\sum\limits_{i=1}^{2} g_i+1}\;\sum_{\substack{g_4=g_3+1\\ g'_4=\left[\frac{N+1-3\sum\limits_{i=1}^{3} g_i}{3}\right]}}^{\substack{g'_4=\left[\frac{N-1-\sum\limits_{i=1}^{3} g_i}{2}\right]\\ g_4=2\sum\limits_{i=1}^{3} g_i+1}} \delta_{g'_4}^{g_4}$$

$$=\sum_{g_3=3}^{7}\;\sum_{\substack{g_4=g_3+1\\ g'_4=\left[\frac{N-8-3g_3}{3}\right]}}^{\substack{g'_4=\left[\frac{N-4-g_3}{2}\right]\\ g_4=2g_3+7}} \delta_{g'_4}^{g_4}+\sum_{g_3=4}^{9}\;\sum_{\substack{g_4=g_3+1\\ g'_4=\left[\frac{N-11-3g_3}{3}\right]}}^{\substack{g'_4=\left[\frac{N-5-g_3}{2}\right]\\ g_4=2g_3+9}} \delta_{g'_4}^{g_4}$$

$$=\sum_{\substack{g_4=4\\ g'_4=\left[\frac{N-17}{3}\right]}}^{\substack{g'_4=\left[\frac{N-7}{3}\right]\\ g_4=13}} \delta_{g'_4}^{g_4}+\sum_{\substack{g_4=5\\ g'_4=\left[\frac{N-20}{3}\right]}}^{\substack{g'_4=\left[\frac{N-8}{2}\right]\\ g_4=15}} \delta_{g'_4}^{g_4}+\sum_{\substack{g_4=6\\ g'_4=\left[\frac{N-23}{3}\right]}}^{\substack{g'_4=\left[\frac{N-9}{2}\right]\\ g_4=17}} \delta_{g'_4}^{g_4}$$

$$+\sum_{\substack{g_4=7\\ g'_4=\left[\frac{N-26}{3}\right]}}^{\substack{g'_4=\left[\frac{N-10}{2}\right]\\ g_4=19}} \delta_{g'_4}^{g_4}+\sum_{\substack{g_4=8\\ g'_4=\left[\frac{N-29}{3}\right]}}^{\substack{g'_4=\left[\frac{N-11}{2}\right]\\ g_4=21}} \delta_{g'_4}^{g_4}+\sum_{\substack{g_4=5\\ g'_4=\left[\frac{N-23}{3}\right]}}^{\substack{g'_4=\left[\frac{N-9}{2}\right]\\ g_4=17}} \delta_{g'_4}^{g_4}$$

$$\sum_{\substack{g_4=6 \\ g'_4=\left[\frac{N-26}{3}\right]}}^{\substack{g'_4=\left[\frac{N-10}{2}\right] \\ g_4=19}} \delta_{g'_4}^{g_4} + \sum_{\substack{g_4=7 \\ g'_4=\left[\frac{N-29}{3}\right]}}^{\substack{g'_4=\left[\frac{N-11}{2}\right] \\ g_4=21}} \delta_{g'_4}^{g_4} + \sum_{\substack{g_4=8 \\ g'_4=\left[\frac{N-32}{3}\right]}}^{\substack{g'_4=\left[\frac{N-12}{2}\right] \\ g_4=23}} \delta_{g'_4}^{g_4}$$

$$+ \sum_{\substack{g_4=9 \\ g'_4=\left[\frac{N-35}{3}\right]}}^{\substack{g'_4=\left[\frac{N-13}{2}\right] \\ g_4=25}} \delta_{g'_4}^{g_4} + \sum_{\substack{g_4=10 \\ g'_4=\left[\frac{N-38}{3}\right]}}^{\substack{g'_4=\left[\frac{N-14}{2}\right] \\ g_4=27}} \delta_{g'_4}^{g_4}$$

It is interesting to note that

(*i*) $\wedge\ (15, 5) = 1$ and $\wedge\ (N, 5) = 0$ for $N \leqslant 14$.

(*ii*) $\wedge\ (121, 5) = 1$ and $\wedge\ (N, 5) = 0$ for $N \geqslant 122$.

3. Restricted Generating Sets

To investigate the behaviour of $\wedge\ (N, m)$ when some restrictions are imposed on the member of G_m, we define the following :

Definition 1. A generating set G_m of I_N is called an *even generating set* if each g_i, $i \neq 1$ is even.

Definition 2. A generating set G_m of I_N is called on *odd generating set* if each g_i is odd.

Let $\wedge^e(N, m)$ and $\wedge^o(N, m)$ stand for the number of even and odd generating sets of I_N respectively.

Theorem 2.

$$\wedge^e(N, m) = \wedge\left(\frac{N-1}{2}, m-1\right).$$

Proof :

Let $G_m=\{1, g_2, g_3, .., g_m\}$ be an even generating set of I_N. Then

$$g_i=2\,h_i, \qquad 2\leqslant i\leqslant m \qquad \text{.. .. (10}$$

$$\rightarrow \quad N=2\sum_{i=2}^{m} h_i+1$$

$$\rightarrow \quad \sum_{i=2}^{m} h_i=\frac{N-1}{2}. \qquad \text{.. .. .. (11)}$$

Firstly we shall establish that the set

$\{h_2, h_3, .., h_m\}$ is a generating set of $I_{\frac{N-1}{2}}$.

obviously $h_2=1$: Since G_m is a generating set, we have

$$g_t\leqslant 2\sum_{i=1}^{t-1} g_i+1$$

$$\rightarrow \quad 2h_t\leqslant 4\sum_{i=2}^{t-1} h_i+3$$

$$\rightarrow \quad h_t\leqslant 2\sum_{i=2}^{t-1} h_i+1+\tfrac{1}{2}.$$

Utilizing the bracket function, we get

$$h_t\leqslant 2\sum_{i=1}^{t-1} h_i+1$$

which proves that the set $\{h_2, h_3, .., h_m\}$ is a generating set of $I_{\frac{N-1}{2}}$.

It follows from equation (10) that there is a one-one correspondence between the even generating sets of I_N of order m and the generating sets of $I_{\frac{N-1}{2}}$ of order $(m-1)$. Hence we conclude that

$$\wedge^e(N, m)=\wedge\left(\frac{N-1}{2}, m-1\right), \qquad \text{.. .. (12)}$$

Theorem 3.

$$\wedge^{o}(N, m) = \wedge\left(\frac{m+N}{2}, m\right),$$

where $m+N$ is an even integer.

Proof :

Let G_m be an odd generating set of I_N . Then each

$$g_i = 2h_i - 1 \qquad 1 \leqslant i \leqslant m \qquad \ldots \qquad \ldots \quad (13)$$

$$\rightarrow \quad \sum_{i=1}^{m} h_i = \frac{m+N}{2}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (14)$$

Using similar arguments to Theorem 2, it can easily be proved that the set $\{h_1, h_2, \ldots, h_m\}$ is a generating set of $I_{\frac{m+N}{2}}$. Furthermore it follows from equation (13) that there is a one-one correspondenee between the odd generating sets of I_N of order m and the generating sets of $I_{\frac{m+N}{2}}$ of order m. Hence it follows that

$$\wedge^{o}(N, m) = \wedge\left(\frac{m+N}{2}, m\right) \qquad \ldots \qquad \ldots \quad (15)$$

which proves our assertion.

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AN ELEMENTARY APPROACH TO PARTITION THEORY

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1. Notations :

For the sake of precision and conciseness we shall adopt the following notations :

I $P(N)$ =number of all the partition of N.

II $P(N, m)$ =number of partitions of N into m parts.

III $P^d(N, m)$ =number of partitions of N into m distinct parts.

IV $P^{<r, s>}(N, m)$=number of partitions of N into m parts of the form $(ra_i - s)$, where a_i's are positive integers.

V $P^o(N, m)$ =number of partitions of N into m odd parts.

VI $P^e(N, m)$ =number of partitions of N into m even parts.

VII $P^{d \,\&\, o}(N, m)$ =number of partitions of N into m distinct and odd parts.

VIII $P^{d \,\&\, e}(N, m)$ =number cf partitions of N into m distinct and even parts.

IX $P^{d\&<r,s>}(N,m)$=number of partitions of N into m distinct parts of the form $(ra_i—s)$.

2. Relation between unrestricted and restricted Partitions.

In this section we have established a relation between unrestricted partitions and partitions into parts of the from $(ra_i - s)$.

Let N be a positive integer and let $A_m=\{a_1, a_2, \ldots, a_m\}$ be a well-ordered set of positive integers. Clearly $P(N, m)$ is equal to the number of solutions of the diophantine equation.

$$a_1 + a_2 + \ldots + a_m = N$$

or $$\sum_{i=1}^{m} a_i = N$$

Now let $B_m = \{ b_1, b_2, \ldots, b_m \}$ be another well-ordered set of positive integers and let.

$F : B_m \rightarrow A_m$ be an order preserving bijective map defined by

$$a_i = rb_i - s \tag{2}$$

were $r \geqslant 1$, $s \geqslant o$ and $rb_i \geqslant s$. Then we have the following

Theorem 1.

$$P^{<r, s>}(N, m) = P\left(\frac{N + ms}{r}, m\right)$$

where $\frac{N + ms}{r}$ is some integer

Proof : We divide the proof of this theorem into two parts

(I) We first put $s = o$ and prove

$$P^{<r, o>}(N, m) = P\left(\frac{N}{r}, m\right)$$

Substituting $s = o$ in (2) and using (1) we get

$$b_1 + b_2 + b_3 + \ldots + b_m = \frac{N}{r} \tag{3}$$

But by hypothesis we have

$$a_1 + a_2 + a_3 + \ldots + a_m = N \tag{4}$$

Since the number of solutions of the diophantines (3) and (4) are equal and number of solutions of (4) equals

$P^{<r, o>}(N, m)$ we conclude that

$$P^{<r, o>}(N, m) = P\left(\frac{N}{r}, m\right) \tag{5}$$

(II) Now putting $r = 1$ in (2) and using (1) we get

$$b_1 + b_2 + b_3 + \ldots + b_m = N + ms \tag{6}$$

were $a_1 + a_2 + a_3 + \ldots + a_m = N$ (7)

Equality of number of solutions of (6) and (7) leads to the result

$$P^{<1,\, s>}(N, m) = P(N + ms, m) \tag{8}$$

combining (5) and (8) we obtain

$$P^{<r,\, s>}(N, m) = P\left(\frac{N + ms}{r}, m\right) \tag{9}$$

which is the required result.

Note : $P^{<r,\, s>}(N, m) = O$ if $(N+ms)$ is not divisible by r.

Corollary 1.

$$P(N, m) = P^{<1,\, O>}(N, m)$$

Corollary 2.

$$P^{e}(N, m) = P^{<2,\, O>}(N, m)$$

$$= P\left(\frac{N}{2}, m\right) \text{ where N is even} \tag{10}$$

Corollary 3.

$$P^{o}(N, m) = P^{<2,\, 1>}(N, m)$$

$$= P\left(\frac{N+m}{2}, m\right) \text{ where } N+m \text{ is even} \tag{11}$$

Note.—It is obvious from (2) that if b_i's are distinct, then a_i's are also distinct and hence with similar arguments we get

$$P^{d\ \&\ <r,\, s>}(N, m) = P^{d}\left(\frac{N+ms}{r}; m\right) \tag{12}$$

3. Unrestricted Partitions.

In this section we have established a result which gives an exact formula for P (N, m). We have given three independent proofs of the following.

Theorem 2.

$$P(N, m) = \sum_{a_1=1}^{\left[\frac{N}{m}\right]} \sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]} \cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N - \sum_{i=1}^{m-3} a_i}{3}\right]} \left[\frac{N+2-\sum_{i=1}^{m-3} a_i - 3a_{m-2}}{2}\right]$$

Proof 1. Since A_m is well-ordered, it follows that

$$a_t \geqslant a_{t-1} \qquad \text{for } 2 \leqslant t \leqslant m \tag{13}$$

Now $$\sum_{i=1}^{m} a_i = N$$

$$a_m = N - \sum_{i=1}^{m-1} a_i \tag{14}$$

Hence (13) and (14) give

$$\left[\frac{N - \sum_{i=1}^{m-2} a_i}{2} \right] \geqslant a_{m-1} \geqslant a_{m-2} \tag{15}$$

Similarly

$$\left[\frac{N - \sum_{i=1}^{m-3} a_i}{3} \right] \geqslant a_{m-2} \geqslant a_{m-3},$$

and hence, in general

$$\left[\frac{N - \sum_{i=1}^{m-r-1} a_i}{r+1} \right] \geqslant a_{m-r} \geqslant a_{m-r-1} \tag{16}$$

If $a_1, a_2, \ldots, a_{m-2}$ are assigned fixed values then it is clear that to each allowable value of a_{m-1}, there corresponds a unique partition of N. Equivalently the number of partitions of N for fixed values of $a_1, a_2, \ldots, a_{m-2}$, equals the number of allowable values of a_{m-1} satisfying condition (15). Hence for fixed values of $a_1, a_2, \ldots, a_{m-2}$, the number of partitions of N is given by :

$$\left[\frac{N - \sum_{i=1}^{m-2} a_i}{2} \right] - a_{m-2} + 1.$$

$$= \left[\frac{N + 2 - \sum_{i=1}^{m-3} a_i - 3a_{m-2}}{2} \right]. \tag{17}$$

Similarly for fixed values of $a_1, a_2, \ldots, a_{m-3}$, the number of partitions corresponding to the admissible values of a_{m-2} and a_{m-1} is

$$\sum_{a_{m-2}=m-3}^{\left[\frac{N-\sum_{i=1}^{m-3} a_i}{3}\right]} \left[\frac{N+2-\sum_{i=1}^{m-3} a_i-3a_{m-2}}{2}\right].$$

Hence the total number of partitions of N into m parts corresponding to the permissible values of all a_i's is given by

$$P(N, m)=\sum_{a_1=1}^{\left[\frac{N}{m}\right]} \sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]} \cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N-\sum_{i=1}^{m-3} a_i}{3}\right]} \left[\frac{N+2-\sum_{i=1}^{m-3} a_i-3a_{m-2}}{2}\right] \quad (18)$$

which is the required result for $m \geqslant 3$.

Our second proof uses the following lemma.

Lemma : $P(N, 2)=\left[\frac{N}{2}\right].$

Proof : Clearly P (N, 2) is equal to the number of solutions of the diophantine equation

$$a_1+a_2=N \quad (19)$$

since $a_2 \geqslant a_1 > 0$, it follows that

$$\left[\frac{N}{2}\right] \geqslant a_1 \geqslant 1.$$

which implies that $P(N, 2)=\left[\frac{N}{2}\right].$

This conforms to the result given in [3].

Proof II:

We know from [2] that

$$P(n, k)=P(n-1, k-1)+P(n-k, k),$$

or in general

$$P(n, k)=P(n-1, k-1)+P(n-1-k, k-1)+\dots \left[\frac{n}{k}\right] \text{ terms.}$$

According to our notation, the last equation can be cast in the form

$$P(N, m)=P(N-1, m-1)+P(N-1-m, m-1)+P(N-1-2m, m-1) + \dots \left[\frac{N}{m}\right] \text{ terms} \qquad (20)$$

or

$$P(N, m)=\sum_{a_1=1}^{\left[\frac{N}{m}\right]} P(N+(m-1)-ma_1, m-1) \qquad (21)$$

which determines the allowable values of a_1.

To determine a_2, we apply the above recursion formula to $P(N+(m-1)-ma_1, m-1)$ to obtain

$$P(N+(m-1)-ma_1, m-1)=\sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]} P(N+(m-2) -(m-1)\, a_2-ma_1, m-2).$$

To see this we note from the recursion formula (20) that there will be $\left[\frac{N+(m-1)-ma_1}{(m-1)}\right]$ terms in the expression for $P(N+(m-1)-ma_1, m-1)$. Hence our a_2 will satisfy the relation:

$$\left[\frac{N+(m-1)-ma_1}{m-1}\right] \geqslant a_2 \geqslant 1. \qquad (22)$$

In order to allow repetitions and to introduce well orderedness in the sets representing the required partitions, we may add (a_1-1) to both the bounds of a_2 without altering the recursive character of $P(N+(m-1)-ma_1, m-1)$.

Therefore from (22) we have

$$\left[\frac{N+(m-1)-ma_1}{m-1}\right]+(a_1-1)\geqslant a_2\geqslant 1+a_1-1$$

$$\left[\frac{N-a_1}{m-1}\right]\geqslant a_2\geqslant a_1.$$

Thus

$$P(N+(m-1)-ma_1,\ m-1)$$

$$=\sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]} P(N+(m-2)-(m-1)a_2-a_1,\ m-2). \qquad (23)$$

Similarly

$$P\ (N+(m-2)-(m-1)a_2-a_1,\ m-2)$$

$$=\sum_{a_3=a_2}^{\left[\frac{N-(a_1+a_2)}{m-2}\right]} P(N+(m-3)-(m-2)a_3-\sum_{i=1}^{2} a_i,\ m-3). \quad (24)$$

Substitution from (23) and (24) in (21) leads to

$$P\ (N,\ m)=\sum_{a_1=1}^{\left[\frac{N}{m}\right]}\ \sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]}\ \sum_{a_3=a_2}^{\left(\frac{N-\sum_{i=1}^{2} a_i}{m-2}\right)}$$

$$P\ (N+m-3-(m-2)\ a_3-\sum_{i=1}^{2} a_i,\ m-3) \qquad (26)$$

Repeated application of the above procedure, in general, gives the following

$$P\ (N,\ m)=\sum_{a_1=1}^{\left[\frac{N}{m}\right]}\ \sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]}\ \cdots \sum_{a_r=a_{r-1}}^{\left(\frac{N-\sum_{i=1}^{r-1} a_i}{m-r+1}\right)} P\ (N+(m-r)$$

$$-(m-r+1)a_r-\sum_{i=1}^{r-1} a_i,\ m-r). \qquad (26)$$

putting $r=m-2$, equation (26) yields the following

$$P(N, m)= \sum_{a_1=1}^{\left[\frac{N}{m}\right]} \sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]} \cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N-\sum_{i=1}^{m-3} a_i}{3}\right]}$$

$$P(N+2-3a_{m-2}-\sum_{i=1}^{m-1} a_i, 2). \qquad (27)$$

Hence by the preceding Lemma equation (27) gives

$$P(N,m)=\sum_{a_1=1}^{\left[\frac{N}{m}\right]} \sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]} \cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N-\sum_{i=1}^{m-3} a_i}{3}\right]} \left[\frac{N+2-\sum_{i=1}^{m-3} a_i-3a_{m-2}}{2}\right]$$

which completes the proof of theorm 2.

Proof III

It is clear that the number of solutions of the diophantine equation

$$a_1 + a_2 + a_3 + \ldots + a_m = N \qquad (28)$$

is $P(N, m)$

Then given a_1, the number of solutions of

$$a_2 + a_3 + \ldots + a_m = N - a_1 \qquad (29)$$

is $P(N-a_1, m-1)$. Since a_i 's are positive and are well-ordered it follows that

$$1 \leqslant a_1 \leqslant \left[\frac{N}{m}\right].$$

Equation (29) is equivalent to $\left[\frac{N}{m}\right]$ equations which correspond to different values of a_1. Obviously the sum of the number of solutions of all these equations will be equal to the number of solutions of the equation (28)

$$P(N, m) = \sum_{a_1=1}^{\left[\frac{N}{m}\right]} P(N-a_1, m-1). \tag{31}$$

This provides us with a recursion formula which is, of course, different from the recursion formula given in [1].

The recursive character of this formula gives

$$P\left(N-\sum_{i=1}^{2} a_i,\ m-2\right) = \sum_{a_3=a_2}^{\left[\frac{N-\sum_{i=1}^{2} a_i}{m-2}\right]} P\left(N-\sum_{i=1}^{3} ai, m-3\right).$$

and hence in general

$$P\left(N-\sum_{i=1}^{r} a_i, m-r\right) = \sum_{a_r+1=a_r}^{\left[\frac{N-\sum_{i=1}^{r} a_i}{m-r}\right]} P\left(N-\sum_{i=1}^{r+1} a_i, m-r-1\right). \tag{32}$$

r iterations of (31) yield

$$P(N, m) = \sum_{a_1=1}^{\left[\frac{N}{m}\right]} \sum_{a_2=a_1}^{\left[\frac{N-a_1}{m-1}\right]} \cdots \sum_{a_r=a_{r-1}}^{\left[\frac{N-\sum_{i=1}^{r-1} a_i}{m-r+1}\right]} P\left(N-\sum_{i=1}^{r} a_i, m-r\right). \tag{33}$$

For $r=m-2$, expression (32) reduces to

$$P\left(N-\sum_{i=1}^{m-2} a_i,\ 2\right)=\sum_{a_{m-1}=a_{m-2}}^{\left[\frac{N-\sum_{i=1}^{m-2} a_i}{2}\right]} P\left(N-\sum_{i=1}^{m-1} a_i,\ 1\right).$$

$$=\sum_{a_{m-1}=a_{m-2}}^{\left[\frac{N-\sum_{i=1}^{m-2} a_i}{2}\right]} 1. = \left[\frac{N+2-\sum_{i=1}^{m-3} a_i-3a_{m-2}}{2}\right]$$

Hence for $r=m-2$, expression (33) becomes

$$P(N,m)=\sum_{a_1=1}^{\left[\frac{N}{m}\right]}\ \sum_{a_2=a_1}^{\left[\frac{N-1}{m-1}\right]} \cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N-\sum_{i=3}^{m-3} a_i}{3}\right]} \left[\frac{N+2-\sum_{i=1}^{m-1} a_i-3a_{m-2}}{2}\right]$$

which establishes the theorem.

Corollary 1. $P(N)=\sum_{m=3}^{N} P(N,m)+\left[\frac{N}{2}\right]+1.$

The Corollary 1 may be used to compute the exact number of unrestricted partitions of a given integer N.

Corollary 2.

In Theorem 1 we established

$$P^{<r,\, s>}(N,\, m) = P\left(\frac{N+ms}{r},\, m\right)$$

Substituting $\left(\frac{N+ms}{r},\, m\right)$ for N in theorem 2 we get

$$P^{<r,\, s>}(N,m) = \sum_{a_1=1}^{\left[\frac{N+ms}{rm}\right]} \sum_{a_2=a_1}^{\left[\frac{N+ms-r_{a1}}{r(m-1)}\right]} \cdots$$

$$\cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N+ms-r\sum_{i=1}^{m-3} a_i}{3r}\right]} \left[\frac{N+ms+r(2-\sum_{i=1}^{m-3} a_i-3a_{m-2}}{2r}\right] \quad (35)$$

As special cases of corollary we have

Corollary 3.

$$P^{e}(N,\, m) = \sum_{a_1=1}^{\left[\frac{N}{2m}\right]} \sum_{a_2=a1}^{\left[\frac{N-2a_1}{2(m-1)}\right]} \cdots$$

$$\cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N-2\sum_{i=1}^{m-3} a_i}{6}\right]} \left[\frac{N+4-2\sum_{r=1}^{m-3} a_i-6a_{m-2}}{4}\right] \quad (36)$$

Corollary 4.

$$P^{O}(N, m) = \sum_{a_1=1}^{\left[\frac{N+m}{2m}\right]} \sum_{a_2=a_1}^{\left[\frac{N+m-2a_1}{2(m-1)}\right]} \cdots$$

$$\cdots \sum_{a_{m-2}=a_{m-3}}^{\left[\frac{N+m-2\sum_{i=1}^{m-3} a_i}{6}\right]} \left[\frac{N+m+4-2\sum_{i=1}^{m-3} a_i-6a_{m-2}}{4}\right] \quad (37)$$

4. Distinct Partitions

In the following we shall be concerned with the number of partitions of N into m distinct parts and prove the following :

Theorem 3.

$$P^{d}(N, m) = \sum_{a_1=1}^{\left[\frac{2N-m(m-1)}{2m}\right]} \sum_{a_2=a_1+1}^{\left[\frac{2N-(m-1)(m-2)-2a_1}{2(m-1)}\right]} \cdots$$

$$\cdots \sum_{a_{m-2}=a_{m-3}+1}^{\left[\frac{2N-6-2\sum_{i=1}^{m-3} a_i}{6}\right]} \left[\frac{2N-2-2\sum_{i=1}^{m-3} a_i-6a_{m-2}}{4}\right].$$

where $a_i=O$ for $i \leqslant O$ and also $m \geqslant 2$.

Proof : Let $A_m=\{a_1, a_2, \ldots, a_m\}$ be a well-ordered set of positive distinct intergers such that

$$\sum_{i=1}^{m} a_i = N \tag{38}$$

Since a_i 's are distinct, we have

$$a_t \geqslant a_{t-1} + 1 \text{ for } 2 \leq t \leqslant m \tag{39}$$

Writing (38) in the form

$$a_m = N - \sum_{i=1}^{m-1} a_i$$

and using (39) for $t=m$, we get

$$\left[\frac{N-1-\sum_{i=1}^{m-2} a_i}{2}\right] \geqslant a_{m-1} \geqslant a_{m-2}+1. \tag{40}$$

Similarly

$$\left[\frac{N-3-\sum_{i=1}^{m-3} a_i}{3}\right] \geqslant a_{m-2} \geqslant a_{m-3}+1,$$

and hence, in general,

$$\left[\frac{N-\frac{r(r+1)}{2}-\sum_{i-1}^{m-r-1} a_i}{r+1}\right] \geqslant a_{m-r} \geqslant a_{m-r-1}+1. \tag{41}$$

It follows from arguments similar to those given in the proof 1 of Theorem 2, that the number of distinct partitions of N for fixed values of $a_1, a_2, \ldots, a_{m-2}$, is equal to

$$\left[\frac{N-1-\sum_{i=1}^{m-3} a_i - 3a_{m-2}}{2}\right] \tag{42}$$

Since $\left[\frac{2N-r(r+1)-2\sum_{i=1}^{m-r-1} a_i}{2(r+1)}\right] \geqslant a_{m-r} \geqslant a_{m-r-1}+1$, it follows that the total number of partitions of N into m distinct parts is given by

$$P^d(N, m) = \sum_{a_1=1}^{\left[\frac{2N-m(m-1)}{2m}\right]} \; \sum_{a_2=a_1+1}^{\left[\frac{2N-(m-1)(m-2)-2a_1}{2(m-1)}\right]} \cdots$$

$$\cdots \sum_{a_{m-2}=a_{m-3}+1}^{\left[\frac{2N-6-2\sum_{i=1}^{m-3} a_i}{6}\right]} \left[\frac{2N-2-2\sum_{i=1}^{m-3} a_i - 6a_{m-2}}{4}\right] \qquad (43)$$

which is the required result.

Corollary 1.

(*i*) $P^d(N, 2) = \left[\frac{N-1}{2}\right]$

(*ii*) $P^d(N, 2) = O$ for $N < \frac{m(m+1)}{2}$.

Both these results are easily deducible from (43).

Note : Corollary 1 clearly shows that the diophantine equation $\sum_{i=1}^{m} a_i = N < \frac{m(m+1)}{2}$ has no distinct solution in positive integers.

Deductions

From (12) we have

$$P^{d\,\&\,<r,\,s>}(N, m) = P^d\left(\frac{N+ms}{r}, m\right)$$

Substituting $\left(\frac{N+ms}{r}\right)$ for N in (43) we get

Corollary 2.

$$P^{d \,\&\, <r,\, s>}(N, m) = \sum_{a_1=1}^{\left[\frac{2(N+ms) - rm(m-1)}{2rm}\right]} \; \sum_{a_2=a_1+1}^{\left[\frac{2(N+ms) - r(m-1)(m-2) - 2ra_1}{2r(m-1)}\right]} \cdots \sum_{a_{m-2}=a_{m-3}+1}^{\left\{\frac{2(N+ms) - 6r - 2r\sum_{i=1}^{m-3} a_i}{6r}\right\}} \left\{\frac{2(N+ms) - 2r - 2r\sum_{i=1}^{m-3} a_i - 6ra_{m-2}}{4r}\right\} \quad (44)$$

Corollary 3.

$$P^{d \,\&\, e}(N, m) = \sum_{a_1=1}^{\left[\frac{N - m(m-1)}{2m}\right]} \; \sum_{a_2=a_1+1}^{\left[\frac{N - (m-1)(m-2) - 2a_1}{2(m-1)}\right]} \cdots\cdots \sum_{a_{m-2}=a_{m-3}+1}^{\left\{\frac{N - 6 - 2\sum_{i=1}^{m-3} a_i}{6}\right\}} \left\{\frac{N - 2 - 2\sum_{i=1}^{m-3} a_i - 6a_{m-2}}{4}\right\} \quad (45)$$

Corollary 4.

$$P^{d\&O}(N) = \sum_{a_1=1}^{\left[\frac{N+m-m(m-1)}{2m}\right]} \sum_{a_2=a_1+1}^{\left[\frac{N+m-(m-1)(m-2)-2a_1}{2(m-1)}\right]} \cdots \sum_{a_{m-2}=a_{m-3}+1}^{\left[\frac{N+m-6-2\sum_{i=1}^{m-3} a_i}{6}\right]} \left[\frac{N+m-2-2\sum_{i=1}^{m-3} a_i - 6a_{m-2}}{4}\right]. \qquad (46)$$

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UNSTEADY EXPANSIONS OF A RELAXING GAS*

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Abstract :—The structure of a centered rarefaction fan generated by impulsively withdrawing a piston from a vibrationally relaxing gas is examined. Equations of motion are solved in the neighbourhood of the origin of the piston path in the time-distance plane. Expansions are used in series in powers of the radial distance from the origin, with coefficients depending on the transverse coordinate. The zero and the first order solutions are given, and it is shown that the solutions are continuous and uniformly valid throughout the flow field, that is, the tail of the fan does not develop into a shock in the near-frozen region.

1. Introduction.

Let us consider a medium of a single diatomic gas in which the rotational mode is in equilibrium with the translational mode and the only non-equilibrium effect which is present is that due to the vibrational relaxation. A centered rarefaction wave or 'fan' bounded by a 'head' and a 'tail' is generated by withdrawing a piston with velocity U_p ($>$O) from such a vibrationally relaxing gas. In space x and time t coordinates system the initial position of the piston is taken to be the origin. The gas is at equilibrium at rest for $t <$ O and occupies the whole region $x \leqslant$ O. For $t \geqslant$ O, there are three distinct regions O, r and g Fig. 1.

In region o is the original undisturbed gas at given temperature, T_{oO} and pressure, P_{oO}. Region r is the rarefaction fan in which the gas is rapidly expanding. The extent of the fan depends on the conditions in region o and the piston velocity U_p. g is the region between the tail of the fan and the piston, which we call the 'post-tail region'.

* The paper is based on the work reported in author's Ph.D. Thesis submitted to Manchester University in 1967.

The relaxation process is described by the linear equation

$$\frac{D\sigma}{Dt} = \rho\,\Phi\,(\bar{\sigma} - \sigma) \tag{1}$$

where 'D/Dt' is the subtantive derivative following a particle path, σ is the vibrational energy, $\bar{\sigma}$ its local equilibium value, ρ, the density and $\Phi = \Phi$ (T), a function of the translational temperatrue T only, is the relaxation frequency. The quantity $[\rho\,\Phi]^{-1}$ has the dimension of time and has been termed the relaxation time, τ by many authors.

On the basis of the assumption that $D\sigma/Dt$ remains finite, it follows that as $\tau \to O$, $\sigma \to \bar{\sigma}$, leading to the establishment of a local equilibrium as a fluid element moves through the flow field. On the other hand, when $\tau \to \infty$, $D\sigma/Dt \to O$ or σ remains constant along a fluid element irrespective of the value of $(\bar{\sigma} - \sigma)$, and the flow approaches the so called frozen limit.

It can be shown [*e.g.* Broer (1958)] that in flows with finite non-zero relaxation time the characteristics of the equations of motion are those determined by the frozen flow and any disturbance propagates with the local frozen speed of sound a_f defined by $a_f = \left(\frac{\partial p}{\partial s}\right)^{\frac{1}{2}}_{s,\,\sigma}$,

p being the pressure and s being the entropy. Only when the relaxation time is zero, the propagation speed is the equilibrium speed of sound

$$a_e = \left(\frac{\partial p}{\partial \rho}\right)^{\frac{1}{2}}_{s,\ \sigma = \bar{\sigma}}.$$

But the development of the flow as τ decreases from infinity to zero is not immediately obvious. As a result most of the works are confined in predicting how this transition takes place along the wave head, and hence these analyses are concerned with the propagation of weak disturbances in an undisturbed region like o [Chu (1957) ; Jones (1963)], assuming τ constant. Wood and Parker (1958) have studied this problem in some greater detail showing that the flow tends to become isentropic at large times due to the decrease of the gradients through the wave ; they have also pointed out that the more quickly relaxing the gas is, the repid is the decay. These analyses do not throw much light on the flow pattern in the neighbourhood of the tail.

From the characteristic solution of the present problem, Johannesen (1965) suggested that a de-excitation shock develops at the tail near the origin. Clarke (1964) considered a similar problem in which the piston is not withdrawn impulsively but it is withdrawn with continuous monotonically increasing velocity. He found that no shock wave formation takes place in the expansion region.

In the corresponding two-dimensional problem, Glass and Tokano (1963) studied the structure of a Prandtl-Meyer fan in a non-equilibrium flow of dissociated oxygen around a corner. They found from their characteristic solution that a recombination shock wave is generated at the tail of the fan. The shock wave is strongest at the corner and decays with increasing distance. Glass and Tokano's analysis, however, contradicts their assumption that the vibrational excitation is in equilibrium with the translational and the rotational modes and is always at its ground state. They have taken the vibrational mode to be frozen at the corner, while it is in equilibrium with the translational mode elsewhere. This means that the vibrational energy is transferred to the translational mode at an enormous rate. Again they have used irrelevant boundary conditions. The same problem for an ideal, dissociating gas was discussed before by Cleaver (1959) and (Appleton 1960), but neither of them discovered the existence of such a recombination shock at the tail.

In their experimental investigations with shock heated gases, Holbeche and Woodley (1966) found that a 'breakway' of the observed pressure variation from the computed profile occurs at the tail ; but this happens when the flow is already near equilibrium. They do not think that this is a de-excitation shock associated with the vibrational relaxation. Recently Blythe (1969) has discussed the possibility of shock formation both within and down stream of the fan.

3. Equations of Motion

In formulating the equations of motion we have used the heat sink analogy of Johannesen (1961) which gives the correspondence between a real gas of variable specific heats and his $\alpha-$gas of constant specific heats to which heat is added at a rate equal to that at which energy is released by the vibrational mode of the real gas. P, ρ, u and T, having their usual meaning in gas dynamics, are the same for both the real gas and the $\alpha-$gas as these depend only on the translational mode which is in equilibrium. The sound speed a of the $\alpha-$gas is identical with the frozen sound speed of the real gas, but the entropy s is a property only of the $\alpha-$gas : it is related to P and ρ by the equation

$$p = c\rho^{\gamma} \exp \{(s-s^*)/c_v\}, \quad (2)$$

where γ and c_v are respectively the ratio of the specific heats of the $\alpha-$gas and its specific heat at constant volume. Consequently these are constants. s^* is some reference value of s and C is a constant.

Following this '$-D\sigma/Dt$' is a heat addition term and the equations of motion are

$$\frac{\partial \rho}{\partial} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = O \quad (3)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = O \quad (4)$$

$$\frac{Ds}{Dt} = -\frac{1}{T}\frac{D\sigma}{Dt} \quad (5)$$

$$\frac{D\sigma}{Dt} = \rho\, \Phi\, (\sigma - \bar{\sigma}) \quad (6)$$

$$p = \rho\, \mathcal{R}\, T \quad (7)$$

where $\mathcal{R}$ is the gas constant.

The harmonic oscillator expression for $\bar{\sigma}$ is

$$\bar{\sigma}(T) = \frac{\mathcal{R}\, \Theta_v}{[exp\,(\Theta_v/T) - 1]} \quad (8)$$

Φ is given by [see Phinney (1964)]

$$\Phi/\Phi_{\Theta_v} = (T/\Theta_v).\ \exp\ (-B\ \{(T/\Theta_v)^{-\frac{1}{3}} - 1)\ \} \qquad (9')$$

where Φ_{Θ_v} is the value of Φ at $T = \Theta_v$, Θ_v being the characteristic temperature of vibration of the gas, and

$$B = 3.211 \times \log_e 10 = 7.394 \qquad (10)$$

Following Johannesen et al (1967), we take $U = \sqrt{\mathcal{R}\Theta_v}$ as a characteristic velocity, $L = [\mathcal{R}\,\Theta_v]^{3/2} / (P_{oO}\,\Phi_{\Theta_v})$ as a reference length define the nondimensional variables each denoted by a circumflex by the following equations :

$$X = L\hat{x}\ ,\ t = \frac{L}{U}\hat{t}$$

$$P = P_{oO}\,\hat{P}\ ,\ \rho = \frac{P_{oO}}{U^2}\,\hat{\rho}\ ,\ T = \Theta_v\,\hat{T}$$

$$s = \mathcal{R}\,\hat{s}\ ,\ \sigma = \mathcal{R}\,\Theta_v\,\hat{\sigma}\ ,\ \Phi = \Phi_{\Theta_v}\,\hat{\Phi}\ ,\ c_v = \mathcal{R}\,\hat{c}_v \qquad (A)$$

$$u = U\hat{u}\ ,\quad a = U\hat{a}$$

The double suffix 'oO' represents the constant values of the variables in region o.

From equations (7′), (8′), (9′), we get

$$\hat{P} = \hat{\rho}\,\hat{T} \qquad (7)$$

$$\hat{\sigma}\,(\hat{T}) = \frac{1}{[\exp\,(1/\hat{T})\ -1]} \qquad (8)$$

$$\hat{\Phi}\,(\hat{T}) = \hat{T}.\ \exp.\ \{-B\,(\hat{T}^{-1/3} - 1)\ \} \qquad (9)$$

without loss of generality we may assume that

$$S^* = S_{oO} = O \qquad (11)$$

which is clearly compatible with the relations (A).

Noting that $a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$ and $\hat{c}_v\,(\gamma - 1) = 1$, we deduce from (2), (11) and (A), the following relations :

$$\hat{P} = \left(\frac{\hat{a}}{\hat{a}_0}\right)^{\frac{2\gamma}{\gamma - 1}} .\ \exp\ (-\hat{s}) \qquad (12)$$

$$\hat{\rho}=\hat{\rho}_{oO}\,(\hat{a}/\hat{a}_{oO})^{\frac{2}{\gamma-1}}.\ \exp.(-\hat{s})$$

Henceforth we shall omit the use of circumflexes and all variables will refer to nondimensional variables.

Introducing (A) in (5) and (6) we see that along the particle paths, C_o, given by $(dx/dt)=u$, we have

$$\frac{Ds}{Dt}=-\frac{1}{T}\frac{D\sigma}{Dt} \tag{13a}$$

and

$$\frac{D\sigma}{Dt}=\rho\,\Phi\,(\bar{\sigma}-\sigma) \tag{13b}$$

Again substituting relations (A) and (12), equations (3) and (4) become respectively

$$\frac{2}{\gamma-1}\left(\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}\right)+a\frac{\partial u}{\partial x}-a\left(\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}\right)=0.$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{2a}{\gamma-1}.\frac{\partial a}{\partial x}-a.\frac{a}{\gamma}.\frac{\partial s}{\partial x}=0.$$

By adding and subtracting these, we see that there are two sets of characteristics C and C_- such that

$$\left(\frac{du}{dt}\right)_{\pm}\pm\frac{2}{\gamma-1}\left(\frac{da}{dt}\right)_{\pm}\mp\frac{1}{\gamma}\left(\frac{ds}{dt}\right)_{\pm}=\mp\frac{\gamma-1}{a}.\frac{D\sigma}{Dt} \tag{13c, d}$$*

$$\text{along } C_{\pm}:\ \left(\frac{dt}{dt}\right)_{\pm}=u\pm a$$

where $\left(\frac{d}{dt}\right)_{\pm}=\frac{\partial}{\partial t}+(u\pm a)\frac{\partial}{\partial x}.$

These show that any disturbance in the flow field propagate with the speed a, thus confirming aur earlier comment that for non-zero finite sound speed is that determined by the frozen flow.

The fan is formed by the family of the C_- characteristics all originating from the origin. Thus the head which moves into the undisturbed gas at rest in region o, has equation

$$x=-a_{oO}t \tag{14}$$

The head is therefore a straight line for all values of t

* The upper signs correspond to equation (13c), while the lower signs correspond to equation (13d).

4. Solutions by the Method of Series Expansions.

To describe the state of motion in the neighbourhood of the origin, O, we introduce new independent variables R and θ defined by

$$x = R\cos\theta$$

$$t = R\sin\theta$$

so that R and θ are the polar coordinates in the $x-t$ plane with the positive direction of the X-axis as the initial line.

Following the solution of a corresponding flow problem for the case of a non-relaxing gas, we assume series expansions for u, a, s and σ of the form

$$u_k = \sum_{n=o}^{\infty} u_{kn}(\theta)\, R^n$$

$$a_k \sum_{n=o}^{\infty} a_{kn}(\theta)\, R^n \qquad (15)$$

$$s_k = \sum_{n=e}^{\infty} s_{kn}(\theta)\, R^n$$

$$\sigma_k = \sum_{n=o}^{\infty} \sigma_{kn}(\theta)\, R^n$$

where the suffix k has been used to denote the values of the respective variables in region k ; so $k=o$, r or g.

T_k, ρ_k, p_k are then obtained by iterating these upon relations $T_k = a^2{}_k/\gamma$ and (12). Hence.

$$T_{ko} = a^2_{ko}/\gamma;\; T_{kl} = 2a_{ko}\, a_{kl}/\gamma \qquad (16)$$

$$\rho_{kO} = \rho_{oO}\,(a_{kO}/a_{oO})^{\frac{2}{(\gamma-1)}} e^{-s_{kO}};\; \rho_{k1} = \rho_{kO}\left(\frac{2}{\gamma-1}\, a_{kl}/a_{kO} - s_{kl}\right) \qquad (17)$$

$$p_{kO} = (a_{kO}/a_{oO})^{\frac{2\gamma}{(\gamma-1)}} e^{-s_{kO}};\; p_{kl} = p_{kO}\left(\frac{2\gamma}{\gamma-1}\, a_{kl}/a_{kO} - s_{kl}\right) \qquad (18)$$

$$\Phi_{kO}=T_{kO}\, e^{-B\,(T_{kO}^{-1/3}-1)}\ ;\ \Phi_{k1}=(1+BT_{kO}^{-1/3}/3).\,(\Phi_{kO}/T_{kO}).\,T_{k1} \tag{19}$$

$$\bar{\sigma}_{kO}=[e^{1/T_{kO}}-1]^{-1}\ ;\ \bar{\sigma}_{k1}=(\bar{\sigma}_{kO}/T_{kO})^{2}\,.\,e^{1/T_{kO}}.\,T_{k1} \tag{20}$$

Also letting

$$F_k=\rho_k\,\Phi_k\,(\bar{\sigma}_k-\sigma_k)\quad , \tag{21}$$

we have

$$F_{kO} = \rho_{kO}\,\Phi_{kO}\,(\bar{\sigma}_{kO} - \sigma_{kO}) \tag{22}$$

$$F_{k1} = (\rho_{kO}\,\Phi_{k1} + \rho_{k1}\,\Phi_{kO})\,(\bar{\sigma}_{kO}-\sigma_{kO}) + \rho_{kO}\,\Phi_{kO}\,(\bar{\sigma}_{k1}-\sigma_{k1})$$

4.1. Equations in (R, θ) Coordinates

Recasting equations (13*a*) – (13*d*) in (R, O) coordinates, we have respectively,

$$[\cos\theta-u_k\sin\theta]\frac{\partial s_k}{\partial\theta}+R\,[\sin\theta+u_k\cos\theta]\frac{\partial s_k}{\partial R}=-\frac{F_k}{T_k}R \tag{23a}$$

$$[\cos\partial-u_k\sin\theta]\frac{\partial\sigma_k}{\partial\theta}+R\,[\sin\theta+u_k\cos\theta]\frac{\partial\sigma_k}{\partial R}=F_k\,R \tag{23b}$$

$$[\cos\theta-(u_k\pm a_k)\sin\theta]\left[\frac{\partial}{\partial\theta}\left(u_k\pm\frac{2a_k}{\gamma-1}\right)\mp\frac{a_k}{\gamma}\ \frac{\partial s_k}{\partial\theta}\right]$$
$$+R\left[\sin\theta+(u_k\pm a_k)\cos\theta\right]\left[\frac{\partial}{\partial R}\left(u_k\pm\frac{2a_k}{\gamma-1}\right)\mp\frac{a_k}{\gamma}\ \frac{\partial s_k}{\partial R}\right]$$
$$=\mp\frac{(\gamma-1)\,F_k}{a_k}\,R \tag{23c, d}$$

Substituting (15) in equations (23*a*)—(23*d*) and equating the coefficients of like powers of R from both sides, there result the following systems of equations :

Zero Order Equations

$$[\cos\theta-u_{kO}\sin\theta]\,\sigma'_{kO}=O \tag{24a}$$

$$[\cos\theta-u_{kO}\sin\theta]\,s'_{kO}=O \tag{24b}$$

$$\left[\cos\theta-(u_{kO}\pm a_{kO})\sin\theta\right]\left[u'_{kO}\pm\frac{2}{\gamma-1}\,a'_{kO}\mp\frac{a_{kO}}{\gamma}s'_{kO}\right]=O, \tag{24c, d}$$

where primes denote differentiation with respect to θ, that is $/\equiv\frac{d}{d\theta}$: $k=$ *o*, *r* or *g*.

First Order Equations

$$[\cos\theta - u_{kO}\sin\theta]\sigma'_{k1} + [\sin\theta + u_{kO}\cos\theta]\,\sigma_{k1} - u_{k1}\sin\theta\,\sigma'_{kO} = F_{kO} \tag{25a}$$

$$[\cos\theta - u_{kO}\sin\theta]s'_{k1} + [\sin\theta + u_{kO}\cos\theta]\,s_{k1} - u_{k1}\sin\theta\,s'_{kO} = F_{kO}/T_{kO} \tag{25b}$$

$$\Big[\cos\theta - (u_{kO} \pm a_{kO})\sin\theta\Big]\Big[u'_{k1} \pm \frac{2a'_{k1}}{\gamma-1} \mp \frac{a_{kO}}{\gamma}s'_{k1} \mp \frac{a_{k1}}{\gamma}s'_{kO}\Big]$$
$$-(u_{k1} \pm a_{k1})\sin\theta\Big[u'_{kO} \pm \frac{2}{\gamma-1}a'_{kO} \mp \frac{a_{kO}}{\gamma}s'_{kO}\Big]$$
$$+[\sin\theta + (u_{kO} \pm a_{kO})\cos\theta]\Big[u_{k1} \pm \frac{2a_{k1}}{\gamma-1} \mp \frac{a_{kO}}{\gamma}s_{k1}\Big] = \mp\left(\frac{\gamma-1}{a_{kO}}\right)F_{kO} \tag{25c, d}$$

$k = o, r$ or g.

4.2. Solutions in region o.

Since in this region the gas remains undisturbed for all $t \geqslant O$, u_{oO}, a_{oO}, σ_{oO}, s_{oO} are constants and are given by $u_{oO} = s_{oO} = O$, $a_{oO} = (\gamma T_{oO})^{\frac{1}{2}}$, $\sigma_{oO} = [\exp(1/T_{oO}) - 1]^{-1}$ (26a)

and

$$u_{on} = a_{on} = \sigma_{on} = s_{on} = O \text{ for all } n \geqslant 1. \tag{26b}$$

4.3. Boundaries and Boundary Conditions for effecting solutions in regions *r* and *g*.

The head of the fan is a straight line and is recognised by

$$\theta = \theta_H = \cot^{-1}(-a_{oO}) = \pi - \cot^{-1}(a_{oO}) \tag{27}$$

independently of R [from (14)].

The flow quantities are continuous across the head. Hence from (26a) and (26b),

$$u_{rO}(\theta_H) = s_{rO}(\theta_H) = O,\ a_{rO}(\theta_H) = a_{oO} = (\gamma T_{oO})^{\frac{1}{2}},\ \sigma_{rO}(\theta_H) = [\exp(1/T_{oO}) - 1]^{-1} \tag{28a}$$

and

$$u_{rn}(\theta_H) = a_{rn}(\theta_H) = \sigma(\theta_H) = s_{rn}(\theta_H) = O;\ n \geqslant 1 \tag{28b}$$

The piston path is also a straight line and has equation

$$\theta=\theta_{P}=\cot^{-1}(u_{P}) \tag{29}$$

Thus

$$u_{go}\ (\theta_{P})=u_{P}, \tag{30}$$

$$u_{gn}\ (\theta_{P})=0, \text{ for } n \geqslant 1.$$

The tail, that is, the last minus characteristic of the fan is not a straight line. Let us assume that the tail is given by

$$\theta=\theta_{T}=\delta_{o}+\delta_{1}R+\ .\ .\ .\ .\ .$$

Then since the slope of the tail is $\left(\frac{dx}{dt}\right)=u_r\ (\theta_T,R)\ a_r\ (\theta_T, R)$, θ_T will satisfy the equation

$$[\{u_r(\theta_T, R)-a_r(\theta_T, R)\}\cot\theta_T+1]\,R\frac{d\theta_T}{dR}=\cot\theta_T-\{u_r(\theta_T, R)-a_r(\theta_T,R)\} \tag{32}$$

Substituting (15) and (31) in (32) and making use of Taylor series expansions, we obtain,

$$\delta_0=\cot^{-1}[u_{rO}(\delta_0)-a_{rO}(\delta_0)] \tag{33a}$$

$$\delta_1=-[u_{r1}[\delta_0)-a_{r1}(\delta_0)]\sin^2\delta_0 \tag{33b}$$

Noting that the quantity

$$\zeta_k=u_k+\frac{2a_k}{\gamma-1},\ k=o,\ r \text{ or } g \tag{34}$$

remains constant and continuous throughout the flow field in the corresponding flow problem for a non-relaxing gas. We assume that ζ, σ and s will remain continuous across the tail at least upto the first order. That is,

$$\zeta_r\ (\theta_T,\ R)=\zeta_g\ (\theta_T,\ R)\ ;\ \sigma_r\ (\theta_T,\ R)=\sigma_g\ (\theta_T,\ R);\ S_r(\theta_T,\ R)=S_g\ (\theta_T,\ R)$$

or iterating (15) and (31) on these,

$$\zeta_{r0}\ (\delta_o)=\zeta_{g0}\ (\delta_o)\ ;\ \sigma_{rO}\ (\delta_o)=\sigma_{g0}\ (\delta_o)\ ;\ S_{rO}\ (\delta_o)=S_{g0}\ (\delta_o) \qquad \ldots(35a)$$

$$\zeta_{r1}\ (\delta_o)+\zeta'_{r0}\ (\delta_o)\ \delta_1=\zeta_{g1}\ (\delta_o)+\zeta'_{gO}\ (\delta_o).\ \delta_1\ ;\ \sigma_{r1}\ (\delta_o)+\sigma'_{ro}\ (\delta_o)\ \delta_1=\sigma_{g1}\ (\delta_o)+\sigma'_{go}\ (\delta_o).\ \delta_1;$$

$$S_{r1}\ (\delta_o)+S'_{ro}\ (\delta_o).\ \delta_1=S_{g1}\ (\delta_o)+S'_{g0}\ (\delta_o).\ \delta_1 \tag{35b}$$

The solutions obtained on the basis of conditions (35) will be uniformly valid provided the tail of the fan and the first C_- characteristic of region g are concident lines to a first order of approximations. This is true if

$$u_r(\theta_T, R) - a_r(\theta_T, R) = u_g(\theta_T, R) - a_g(\theta_T, R)$$

upto first order : that is, if

$$u_{rO}(\delta_o) - a_{rO}(\delta_o) = u_{gO}(\delta_o) - a_{gO}(\delta_o) \tag{36a}$$

$$u_{r1}(\delta_o) - a_{r1}(\delta_o) + [u'_{ro}(\delta_o) - a'_{ro}(\delta_o)].\ \delta_1 = [u_{g1}(\delta_o) - a_{g1}(\delta_o)] + [u'_{go}(\delta_o) - a'_{go}(\delta_o)].\ \delta_1 \tag{36b}$$

4.4 Solutions of the Zero Order Equation.

Since $(\cos\theta - u_{kO}\sin\theta)$ is non-zero, except at the piston, solutions of equations (24*a*) and (24*b*), satisfying the conditions (28*a*) and (35*a*) are

$$\sigma_{rO} = \sigma_{gO} = \text{constant} = [\exp.(1/T_{oO}) - 1]^{-1} \tag{37}$$

$$s_{rO} = s_{gO} = \text{constant} = s_{oO} = O \tag{38}$$

Equations (24*c*) and (24*d*) now reduce respectively to

$$[\cos\theta - (u_{kO} + a_{kO})\sin\theta]\left[u'_{kO} + \frac{2}{\gamma - 1}a'_{kO}\right] = O \tag{24c}$$

$$[\cos\theta - (u_{kO} - a_{kO})\sin\theta]\left[u'_{kO} - \frac{2}{\gamma - 1}a'_{kO}\right] = O;\ k\ r \text{ or } g. \tag{24d}$$

Their solutions appropriate to the region r which is a centered rarefaction fan formed by the minus characteristics are

$$u_{rO} + \frac{2a_{rO}}{\gamma - 1} = \zeta_{rO} = \text{constant} = \frac{2a_{oO}}{\gamma - 1} \qquad \text{[from (28a)]}$$

and $u_{rO} - a_{rO} = \cot\theta$;

that is,

$$u_{rO} = \frac{2}{\gamma - 1}(a_{oO} + \cot\theta) \tag{39}$$

$$a_{rO} = \frac{2}{\gamma - 1}\left(a_{oO} - \frac{\gamma - 1}{2}\cot\theta\right) \tag{40}$$

solutions of (24*c*) and (24*d*) appropriate to region *g* subject to the conditions (30) and (35*a*), namely, $u_{gO}(\theta_p) = u_p$

and $\zeta_{gO}(\xi_{rO}) = \zeta_{ro}(\xi_o) = \dfrac{2a_{oO}}{\gamma - 1}$ are

$$u_{gO} = \text{constant} = u_p \tag{41}$$

$$a_{gO} = \text{constant} = a_{oO} - \frac{\gamma - 1}{2} u_p \tag{42}$$

To make the solutions uniformly valid, we have to take

$u_{rO}(\xi_o) = \text{up}$ and $a_{rO}(\xi_o) = a_{gO}$

whence from (33*a*) and (42),

$$\xi_o = \cot^{-1}[u_p - a_{gO}] = \cot^{-1}\left[\frac{\gamma - 1}{2} u_p - a_{oO}\right] \tag{43}$$

The exclusion of cavitation that is, the exclusion of the situation in which $a_{gO} \leqslant O$ leads to the restriction

$$u_p < \frac{2a_{oO}}{\gamma - 1} \quad \text{[from (42)]} \tag{44}$$

Thus the zero order solutions are just the same as those for a centered rarefaction wave in a non-relaxing gas of constant heat capacity ratio γ. The flow field is, therefore, frozen at the origin.

4.5 Solutions of the First Order Equations

Using the zero order solutions, the solutions of equations (25*a*), (25*b*), (25*c*) in region *r* satisfying the boundary conditions (28*b*) can be given respectively as

$$\sigma_{r1} = a_{r_O}^{2N} \sin\theta . \; X_1(\theta) \tag{44}$$

$$s_{r1} = a_{rO}^{2N} \sin\theta \; Y_1(\theta) \tag{45}$$

$$\zeta_{r1} = u_{r1} + \frac{2a_{r1}}{\gamma - 1} = \frac{a_{rO}}{\gamma} s_{r1} + a_{rO}^{N} \sin\theta . \; Z_1(\theta) \tag{46}$$

where

$$2N = (\gamma + 1)/(\gamma - 1), \tag{47}$$

$$X_1(\theta) = -\int_{\theta_H}^{\theta} a_{rO}^{-(2N+1)} F_{rO}(\theta) \operatorname{cosec}^2\theta \, d\theta \tag{44a}$$

$$Y_1(\theta) = \gamma \int_{\theta_H}^{\theta} a_{rO}^{-(2N+3)} F_{rO}(\theta) \operatorname{cosec}^2 \theta \, d\theta \qquad (45a)$$

$$Z_1(\theta) = -\int_{\theta_H}^{\theta} a_{rO}^{-N}\left[\left(\frac{1}{2N\gamma}\right) a_{rO}^{2N} Y_1(\theta) - \frac{\gamma-1}{2} a_{rO}^{-2} F_{rO}\right] \operatorname{cosec}^2 \theta \, d\theta \qquad (46a)$$

Again using the zero order solutions, (25d) reduces to

$$\frac{\gamma+5}{3\gamma-1} u_{r1} - \frac{2a_{p1}}{\gamma-1} = \frac{\gamma+1}{3\gamma-1}\left(a_{rO}\, s_{r1}/\gamma - \frac{\gamma-1}{a_{rO}} F_{rO} \sin\theta\right) = h_{r1} \text{ (say)}$$

whence from (46) and (48),

$$u_{r1} = \frac{3\gamma-1}{4(\gamma+1)}[\zeta_{r1} + h_{r1}] \qquad (49)$$

$$a_{r1} = \frac{(\gamma-1)(3\gamma-1)}{8(\gamma+1)}\left[\frac{\gamma+5}{3\gamma-1}\zeta_{r1} - h_{r1}\right] \qquad (50)$$

In region g, the zero order solutions have constant values Hence the solutions of equations (25a)–(25d) can be given as

$$\sigma_{g1} = F_{gO} \sin\theta \, [1 + A_1(\cot\theta - u_p)] \qquad (51)$$

$$s_{g1} = -(F_{gO}/T_{gO}) \sin\theta \, [1 + A_2(\cot\theta - u_p)] \qquad (52)$$

$$\zeta_{g1} = u_{g1} + \frac{2a_{g1}}{\gamma-1}$$

$$= \frac{a_{gO}}{\gamma} s_{O1} - (\gamma-1)\frac{F_{gO}}{a_{gO}} \sin\theta \, [1 + A_3\{\cot\theta - (u_p + a_{gO})\}] \qquad (53)$$

$$u_{g1} - \frac{2a_{g1}}{\gamma-1} = -\frac{a_{gO}}{\gamma} s_{g1} + (\gamma-1)\frac{F_{gO}}{a_{gO}} \sin\theta \, [1 + A_4\{\cot\theta - (u_p - a_{gO})\}] \qquad (54)$$

where A_1, A_2, A_3, A_4 are constants of integration.

From (52), (53), (54) we have

$$u_{g1} = -\frac{\gamma-1}{2} \cdot \frac{F_{gO}}{a_{gO}}[(A_3 - A_4)(\cot\theta - u_p) - (A_3 + A_4) a_{gO}] \sin\theta \qquad (55)$$

$$a_{g1} = \frac{\gamma-1}{2}\left[\frac{a_{gO}}{\gamma}\left(-\frac{F_{gO}}{T_{gO}}\right)\right] \cdot \{1 + A_2(\cot\theta - u_p)\}$$

$$-(\gamma-1)\frac{F_{gO}}{a_{gO}}\left\{1 + \frac{A_3 + A_4}{2}(\cot\theta - u_p) - \frac{A_3 - A_4}{2} a_{gO}\right\}\Bigg] \sin\theta \qquad (56)$$

Since σ_{kO}, s_{kO}, ζ_{kO} ($k=r$ or g) are constants throughout the flow field, the conditions (35*b*) reduce to

$$\sigma_{g1}(\xi_o)=\sigma_{r1}(\xi_o),\ s_{g1}(\xi_o)=s_{r1}(\xi_o),\ \zeta_{g1}(\xi_o)=\zeta_{r1}(\xi_o) \qquad (35b)$$

Hence, using cot $\xi_o=u_p-a_{gO}$, these give

$$A_1=[1-\sigma_{r1}(\xi_o)/(F_{gO}\sin\xi_o)]/a_{gO} \qquad (57a)$$

$$A_2=[1+T_{gO}\,s_{r1}(\xi_0)/F_{gO}\sin\xi_o)]/a_{gO} \qquad (57b)$$

$$A_3=[1+\{\zeta_{r1}(\xi_o)-\frac{a_{gO}}{\gamma}s_{g1}(\xi_o)\}.\ a_{gO}/\{(\gamma-1)F_{gO}\sin\xi_o\}]/(2a_{gO}) \qquad (57c)$$

where

$$s_{g1}(\xi_o)=-\frac{F_{gO}}{T_{gO}}[1-A_2\,a_{gO}]\sin\xi_o \qquad (58)$$

Again on the piston path we have u_{gn} (θ_P $=O$, $n\geqslant 1$.

Therefore, from (55),

$$A_4=-A_3 \qquad (57d)$$

Thus A_1, A_2, A_3, A_4 are determined from the relations (57) and the first order solutions are specified in region *g*.

Making use of the zero order solutions and the expression for ξ_1, we see from (36*b*) that the solutions will be uniformly valid to a first order of approximation, provided we have,

$$u_{g1}(\xi_o)-a_{g1}(\xi_o)=2[u_{r1}(\xi_o)-a_{r1}(\xi_o] \qquad (59)$$

which can easily be seen to be satisfied from the following relations :

$$u_{r1}(\xi_o)+\frac{2}{\gamma-1}a_{r1}(\xi_o)=u_{g1}(\xi_o)+\frac{2}{\gamma-1}a_{g1}(\xi_o) \qquad (60a)$$

$$\frac{\gamma-5}{3\gamma-1}u_{r1}(\xi_o)-\frac{2}{\gamma-1}a_{r1}(\xi_o)=\frac{\gamma+5}{3\gamma-1}\left[u_{g1}(\xi_o)-\frac{2}{\gamma-1}a_{g1}(\xi_o)\right] \qquad (60b)$$

[obtained from (48) and (54)]

5. Results and Discussions

The solutions given above are analytic except that there are three integrals X_1, Y_1, Z_1 which are evaluated numerically. The presence of the factors $e^{-B(T_{ro}^{-1/3}-1)}$ and $\left[\left(e^{1/T_{ro}}-1\right)^{-1}-\sigma_{ro}\right]$ in F_{rO} in the integrand of each of these integrands makes it impossible to give analytic expressions for them.

We obtained solutions for various values of u_p and T_{oO}. The first order terms for entropy and sound speed, that is s_{k1} and a_{k1} ($k=r$ or g) are positive throughout, except at the head of the fan where these are identically zero. Hence temperature, pressure and density at points near the origin have higher values than their counterparts in the non-relaxing simple wave. This is owing to the transfer of energy from the vibrational to the translational mode corresponding to the negative σ_{k1} everywhere. These effects are maximum to the line $\theta = \frac{\pi}{2}$.

It is seen that for a given T_{oO} and hence for a fixed position of the head, the solutions have identical values at common points in the fan for all u_p and the higher the values of u_p is, the more to the right is the tail of the fan. Hence the behaviour of the tail does not happen to be the same for all cases. For example, in cases (*i*) $u_p=4$, $T_{oO}=2$; (*ii*) $u_p=2$, $T_{oO}=1$, (*iii*) $u_p=4$, $T_{oO}=1$, it is seen that $\delta_1 < O$, that is, the tail is initially accelarated, while in cases (*iv*) $u_p=1, T_{oO}=1$; (*v*) $u_p=1$, $T_{oO}=2$, (*vi*) $u_p=1$, $T_{oO}=0.5$, $\delta_1 > O$, showing that the tail is initially decelerated due to the relaxation effects.

The solutions are uniformly valid throughout the whole flow field and the tail remains to be a C_-—Characteristic, that is, the tail does not develop into a shock at least to a first order of approximation.

In order to provide a means of justifying the merits of the present analysis, the problem was also solved by the method of characteristics following the techniques of Johannesen, Bird and Zienkiewicz (1967). For the purpose of comparison of the numerical results given by the two methods, a number of points on individual particle paths as determined by the methods of characteristics were chosen and the values of different flow variables, namely, u, p, T and σ were then computed by the method of series expansions. It is seen that the agreement between the two sets of solutions are remarkably close. Although the series solutions are, strictly speaking, valid for $R < 1$, it can be seen that these solutions are in excellent agreement with the characteristic solutions for larger values of R, even for $R \simeq 30$ for $u_p = 1$ and $T_o O = 0.5$ (see Table I).

TABLE I

COMPARISON WITH CHARACTERISTIC SOLUTIONS

$\hat{u}p = 1$

$\hat{T}oO = 0.5$

Methods	x	t	u	p	T	—
Series	0.147690	0.215063	0.124897	0.808839	0.470605	0.156474
Charac.			0.124896	0.808838	0.470604	0.156474
Series	0.121819	0.314944	0.374894	0.518421	0.414507	0.156226
Charac.			0.374848	0.518421	0.414507	0.156227
Series	0.0911157	0.384772	0.499878	0.410453	0.387795	0.156073
Charac.			0.499878	0.410452	0.387795	0.156075
Series	0.370858	0.586646	0.750003	0.250991	0.337032	0.155784
Charac.			0.75002	0.250990	0.337031	0.155788
Series	0.335450	0.923412	1.00018	0.147881	0.289813	0.155563
Charac.			1.00018	0.147880	0.289811	0.155568
Series	1.83482	2.42251	1.00018	0.148104	0.290116	0.154939
Charac.			1.00017	0.148101	0.290112	0.154947
Series	5.83316	6.42018	1.00018	0.148699	0.290924	0.153275
Charac.			1.00016	0.148679	0.290908	0.153291
Series	12.3306	12.9166	1.00018	0.149665	0.292238	0.1505710
Charac.			1.00015	0.149588	0.292181	0.150610
Series	21.3272	21.9119	1.00018	0.151002	0.294057	0.148625
Charac.			1.00013	0.150787	0.293904	0.146922

The series solutions are analytic, and for smaller values of R, these will therefore given more accurate results than those given by the method of charaeteristics. Hence the method of series expansions establishes that the solutions are continuous near the origin and the tail does not develop into a shock at least to a first order of approximation.

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GEOMETRICAL INTERPRETATION OF LAGRANGE MULTIPLIERS IN MECHANICS

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1. INTRODUCTION :

It has been asserted by Gaskill and Arenstein[1,2] that Lagrange multipliers can be given a geometrical interpretation. To substantiate their viewpoint, they have made use of the concepts of differentiable manifolds and their tangent bundles. Both of their articles (almost identical in content) contain conceptual errors some of which are listed below :

(*a*) No distinction has been made between the generalized co-ordinates and the configuration space co-ordinates of a physical system. For example, the constraints on a system could be regarded as hypersurfaces (surfaces when the configuration space is the Euclidean space of dimension 3) in the configuration space but they are not, in general, so in the space of generalized co-ordinates.

(*b*) Concepts of differentiable manifold[3] and tangent bundle[4] have not been used in their proper sense. For instance, if M is a smooth manifold of dimension n, then its tangent bundle, usually denoted by $T(M)$, is also a differentiable manifold of dimension $2n$. Moreover, if $(p, X) \in T(M)$, then $(x_1, x_2, \ldots, x_n; \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$ can be taken as the co-ordinates of (p, X) via the coordinates $(x_1, x_2, \ldots, x_n)$ in a neighbourhood of p in M, where '.' denotes differentiation with respect to some appropriate parameter t. Hence if the position of a physical system at time $t = t_0$ is specified by $(x_1, x_2, \ldots, x_n)$, then $\dot{x}_1 \big|_{t=t_0}, \dot{x}_2 \big|_{t=t_0}, \ldots, \dot{x}_n \big|_{t=t_0}$ will be the components of its velocity ; and in this sense, $T(M)$ may be regarded as the phase space of the system. It may, however, be mentioned that in Refs. 1, 2, the tangent bundle of a manifold is confused with a vector space which neither conforms to its standard use nor is justifiable.

(*c*) The assumption that the intersection of two or more hypersurfaces is also a hypersurface is, in general, not valid. For example, if

$$S_r=\{(x_1, x_2, \ldots, x_n): \varphi_r (x_1, x_2, \ldots, x_n)=0\}, r=1, 2, \ldots, k,$$

are $k<n$ hypersufaces in the Euclidean space E_n, then

$$S=\bigcap_{r=1}^{k} S_r=\{(x_1, x_2, \ldots, x_n):$$

$$\varphi_1 (x_1, x_2, \ldots, x_n)=0,\ \varphi_2 (x_1, x_2, \ldots, x_n)=0,\ \varphi_n (x_1, x_2, \ldots, x_n)=0\}$$

will also be a hyper-surface of codimension k if

(*i*) $S=\bigcap_{r=1}^{k} S_r$ is non-empty ;

(*ii*) the rank of the Jacobian matrix

$$\begin{bmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \cdots\cdots & \frac{\partial\varphi_1}{\partial x_n} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_2} & \cdots\cdots & \frac{\partial\varphi_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\varphi_k}{\partial x_1} & \frac{\partial\varphi_k}{\partial x_2} & \cdots\cdots & \frac{\partial\varphi_k}{\partial x_n} \end{bmatrix}$$

is k at all points of S. The requirement (*ii*) implies that the vectors $\nabla\varphi_1, \nabla\varphi_2, \ldots\ldots, \nabla\varphi_k$ are linearly independent. No such requirements are mentioned in the articles cited above.

(*d*) The concept of an 'eigenvalue' is not in consonance with its accepted definition.

(*e*) No geometrical interpretation of Lagrange multipliers has been given except that they appear as multipliers of vectors in forming a linear combination.

Because of the aforementioned points, many of the arguments in Refs. 1 and 2, are either ambiguous or fallacious. The purpose of this article is two-fold: (*i*) To give geometrical interpretation of the virtual work; (*ii*) To explain the geometrical significance of Lagrange multipliers. To achieve this, we shall use the following :

Normal Bundle. Let S be a hypersurface in E_n of codimension k. Let $T_p(S)$ be its tangent space at a point $p \in S$. The orthogonal complement $N_p(S)$ of $T_p(S)$ with respect to E_n at p is of dimension k which is the vector space of normals to S at p. Set

$$N(S)=\{(p, v) : p \in S \text{ and } v \in N_p(S)\}.$$

Then N(S) can be given a differentiable structure and hence can be converted into a differentiable manifold. The differentiable manifold $N(S)$ is of dimension n and is called the normal bundle of S in E_n.

2. GEOMETRICAL INTERPRETATION OF VIRTUAL WORK AND LAGRANGE MULTIPLIERS

Consider a physical system consisting of m particles. In case the system is unconstrained, its configuration can be specified by $3m=n$, say, coordinates $x_1, x_2, .., x_n$. Let there be $k < n$, holonomic constraints on the system given by

$$S_r=\{(x_1, x_2, .., x_n) : \varphi_r(x_1, x_2, .., x_n, t)=0\}, \; r=1, 2, .., k. \tag{1}$$

These constraints shall be consistent if

(*i*) $S = \bigcap_{r=1}^{k} S_r$ is non-empty ;

(*ii*) the matrix

$$\begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots\cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \cdots\cdots & \frac{\partial \varphi_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \varphi_k}{\partial x_1} & \frac{\partial \varphi_k}{\partial x_2} & \cdots\cdots & \frac{\partial \varphi_k}{\partial x_n} \end{bmatrix} \tag{2}$$

has rank k at all points of S, where S represents the surface

$$\varphi(x_1, x_2, .., x_n, t)=0.$$

In view of these consistency conditons, S is a differentiable manifold which may be called the state-manifold[5] of the system. Let $T(S_1)$, $T(S_2)$, ..., $T(S_k)$ be the tangent bundles of the surfaces $S_1, S_2, .., S_k$ respectively. Then the tangent bundle $T(S)$ of S is given by :

$$T(S)=T(S_1) \cap T(S_2) \cap \ldots \cap T(S_k). \quad (3)$$

Hence $T(S)$ may be regarded as the phase space of the consrtained system. Moreover, if $N(S_r)$, $r=1, 2...., k$ are the normal bundles of S_r, $r=1, 2, ..., k$ and $N(S)$ is the normal bundle of S in E_n, then

$$N(S_r) \subseteq N(S), \; r=1, 2, .., k. \quad (4)$$

Let $p \in S$ and let $\nabla\varphi_1, \nabla\varphi_2, ..., \nabla\varphi_k$ be the gradient vectors (which are, of course, along the normals to $S_1, S_2, ..., S_k$ at p) of the constraining surfaces at p. Then, by consistency condition (*ii*), the vecors $\nabla\varphi_1, \nabla\varphi_2, ..., \nabla\varphi_k$ are linearly independent and hence span the normal vector space to S at p. Since $\nabla\varphi$ is along the normal to S, it follows that

$$\nabla\varphi=\mu_1 \nabla\varphi_1+\mu_2 \nabla\varphi_2+......+\mu_k \nabla\varphi_k, \quad (5)$$

where $\mu_1, \mu_2, ..., \mu_k$ are parameters which vary from point to point on S. It follows from the definition of $N(S)$ that $(p, \nabla\varphi) \in N(S)$.

Let us suppose that $\mathbf{F}$ be the resultant applied force on the system. Then the holonomically constrained system will be in static equilibrium if and only if the resultant force on the system is zero. But the only forces acting on the system are $\mathbf{F}$ and the reaction of the surface S. Since at the point of equilibrium, the reaction of S is directed along the normal to S at that point, it follows that the reaction is equal to $\Lambda \nabla\varphi$. Hence for a static equilibrium of the system

$$\mathbf{F}+\Lambda \nabla\varphi=\mathbf{0}. \quad (6)$$

It is evident from equation (6) that $\mathbf{F}$ is also along the normal to S whose sense is opposite to that of $\Lambda \nabla\varphi$.

Let the system be given instantaneous infinitesimal displacement $\delta \mathbf{R}$ consistent with the constraints. Then $\delta \mathbf{R}$ lies in the tangent vector space of S. Since **F** is along the normal to S at the point of equilibrium, it follows

$$\mathbf{F} \cdot \delta \mathbf{R} = 0. \tag{7}$$

That is, the virtual work is zero. This affords us a geometrical description of the virtual work which appears more logical than the hitherto known explanations.

Combining equations (5) and (6), we get

$$\mathbf{F} = -\Lambda \nabla \varphi = -\Lambda [\mu_1 \nabla \varphi_1 + \mu_2 \nabla \varphi_2 + \ldots\ldots + \mu_k \nabla \varphi_k]$$

$$= \lambda_1 \nabla \varphi_1 + \lambda_2 \nabla \varphi_2 + \ldots\ldots + \lambda_k \nabla \varphi_k, \tag{8}$$

where

$$\lambda_r = -\Lambda \mu_r, \; r = 1, 2, \ldots\ldots, k.$$

If the system is partially conserved, then we may write

$$\mathbf{F} = -\nabla V + \mathbf{F}_{nc}. \tag{9}$$

Hence it follows from the Lagrange formulation[6] that

$$\sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} \right] \hat{e}_i - \mathbf{F}_{nc} - \sum_{r=1}^{k} \lambda_r \nabla \varphi_r = 0, \tag{10}$$

which shows that $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ are the Lagrange multipliers. However, it follows from equation (5) that if $q_1, q_2, \ldots\ldots, q_{n-k}$ are the coordinates of p on S, then $q_1, q_2, \ldots, q_{n-k}; \lambda_1, \lambda_2, \ldots, \lambda_k$ are the coordinates of $(p, \nabla \varphi)$ in $X(S)$. Thus Lagrange multipliers together with the coordinates on S, can be used to coordinatize the points of $N(S)$. Therefore, geometrically Lagrange multipliers could be regarded as the coordinates of those points (p, v) of $N(S)$ for which $v = \nabla \varphi$. Note that this interpretation is very different from the one asserted in Refs. 1, 2.

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ABSOLUTE CONVEXITY IN THE FREE NILPOTENT GROUP OF CLASS 3 ON 2 GENERATORS

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1. Introduction :

If a group G has a full order (for the sake of convenience we shall simply say "order") relation, $\leqslant$, such that for each $a, b, c \in$ G if $a \leqslant b$ then both $ac \leqslant$ bc and $ca \leqslant cb$, we say that G is an ordered group and that $\leqslant$ is an order on G. If G is ordered by $\leqslant$ and C is a non-empty subset of G such that when $a, b \in$ C, then all x ($\in$ G) between a and b are also in C, we say that C is convex in G with respeect to $\leqslant$ or simply that C is convex. Furthermore if C is convex in G in every order on G then we say that C is absolutely convex in G.

Vinogradov [5] has proved that the centre of the free nilpotent group of class 2 on 2 generators is absolutely convex. Moreover Teh [4] has shown that the set of all orders on an abelian group of finite rank can be constructed in terms of the natural order on the set of real numbers in the following way.

Suppose that A is a torsion free abelian group (note that if A is not torsion free then the set of all orders on A is the empty set) generated by $a_1, a_2 \ldots, a_r$. The set of all orders on A may be described as :

An element $x = a_1^{p1}\, a_2^{p2} \ldots\ldots a_r^{pr}$ (p_i are integers) of A is positive if, and only if,

$$\lambda_{11} p_1 + \lambda_{12} p_2 + \ldots\ldots + \lambda_{1r} p_r > 0,$$

Or $\lambda_{11} p_1 + \ldots\ldots\ldots + \lambda_{1r} p_r = 0$ but $\lambda_{21} p_1 + \ldots\ldots + \lambda_{2r} p_r > 0,$

Or ....

Or $\lambda_{11} p_1 + \ldots + \lambda_{1r} p_r = \ldots, = \lambda_{r-1,1} p_1 + \ldots + \lambda_{r-1,r} p_r = 0$
but $\lambda_{r1} p_1 + \ldots + \lambda_{rr} p_r > 0$;

where $\geqslant$ is the natural order on the set of real numbers and λ_{ij} $(i, j=1, 2 \ldots\ldots, r)$ are some real numbers subject to the condition that

$$\Delta = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \cdots\cdots & \lambda_{1r} \\ \lambda_{21} & \lambda_{22} & \cdots\cdots & \lambda_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{r1} & \lambda_{r2} & & \lambda_{rr} \end{vmatrix} \neq 0.$$

In this paper we will show that both centres of the free nilpotent group of class 3 on 2 generators are absolutely convex. Moreover we will construct the set of all orders on this group.

II. Results.

It is well known (See Neumann [3], that all free groups are ordered groups. We now consider the free nilpotent group F of class 3 generated by a, b such that $[b, a]=c$, $[c, a]=d$, $(c, b)=e$. By definition d, e, commute with all a, b, c, d, e. Thus by straightforward calculation we have

$$a^{-n} \quad c^{m} \quad a^{n} = c^{m} \quad d^{mn},$$

$$b^{-n} \quad c^{m} \quad b^{n} = c^{m} \quad e^{mn},$$

and $a^{-n} \; b^{m} \; a^{n} = b^{m} \; c^{mn} \, d^{\frac{1}{2}mn(n-1)} \; e^{\frac{1}{2}nm(m-1)}$
for all integral values of m, n. Now by the repeated application of these formulae every element f of F can be expressed uniquely as

$$f = a^{p_1} \; b^{p_2} \; c^{p_3} \; d^{p_4} \; e^{p_5},$$

where p_i are integers.

Denoting the members of the upper central series of F by Z_1 and Z_2 we have

$$Z_1 = < d, e > \quad \text{and} \quad Z_2 = < c, d, e >.$$

The following results will be needed in the sequel.

If x is an element of a group G then $\bar{I}(x)$ will denote the intersection of all the normal isolated subgroups of G containing the element x. It may be pointed out that a nilpotent group can be ordered if, and only if, it is torsion free.

Result 1 (*Kibriya* [3]) A subgroup C of an ordered nilpotent group G is absolutely convex in G if, and only if C is a member of the upper central series of G, say $C = Z_i$ for some positive integer i, and $\bar{I}(x) \supseteq C$ for all $x \in Z_{i+1} \setminus Z_i$.

Result 2 (*Kibriya* [1]) If the rank of the centre of an ordered nilpotent group is 1 then the centre is absolutely convex.

Result 3 (*Kibriya* [2]). If H is an absolutely convex normal subgroup of an ordered group G and K is a subgroup of G such that $K \supseteq H$, then K is absolutely convex in G if, and only if, K/H is absolutely convex in G/H.

Theorem 1. If F is the free nilpotent group of class 3 generated by a, b then Z_1 and Z_2 are both absolutely convex in F.

Proof : Any element x of $Z_2 \setminus Z_1$ can be expressed as

$$x = c^l \, d^m \, e^n$$

where l, m, n are integers. Now consider $\bar{I}(x)$.

By definitiin x and $g^{-1} x g$ (for all $g \in F$) are elements of $\bar{I}(x)$. Thus $a^{-1} x a = a^{-1} [c^l d^m e^n) a = cc^l d^m e^n] d^l = xd^l \in \bar{I}(x)$; in other words $d^l \in \bar{I}(x)$, so that $d \in \bar{I}(x)$.

Similarly $b^{-1} x a = x e^l \in \bar{I}(x)$, consequently $e \in \bar{I}(x)$.

Thus $\bar{I}(x) \supseteq \langle d, e \rangle = Z_1$. Hence by Result 1, Z_1 is absolutely convex in F.

Clearly the rank of the factor group Z_2/Z_1 is 1 and it is the centre of the torsion free nilpotent group F/Z_1, so that by Result 2, Z_2/Z_1 is absolutely convex in F/Z_1. Hence by Result 3, Z_2 must be absolutely convex in F.

Theorem 2. If F is the free nilpotent group of class 3 on 2 generators then the set of all orders on F may be described in terms of the natural order on the set of all real numbers.

Proof : Suppose that F is the free nilpotent group of class 3 on 2 generators and Z_1, Z_2 are the members of its upper central series.

By definition Z_1, Z_2/Z_1 and F/Z_2 are abelian torsion free groups of finite rank, thus by Teh [4] the set of all orders on them can be described. Moreover Z_2 is absolutely convex in F, therefore every order on F induces an order on F/Z_2 and conversely every order on F/Z_2 can be refined to an order on F. Similarly since Z_1 is absolutely convex in F and therefore in Z_2, every order on Z_2 induces an order on Z_2/Z_1 and conversely. Hence the set of all orders on F can be described as :

An element f of F is positive if, and only if, $f \in F \setminus Z_2$ and $f Z_2$ is positive in some order on F/Z_2 or $f \in Z_2 \setminus Z_1$ and $f Z_1$ is positive in some order on Z_2/Z_1 or $f \in Z_1$ and f is positive in an order on Z_1.

Corollary. If F is the free nilpotent group of class 2 on 2 generators then the set of all orders on F may be described in terms of natural order on the set of all real numbers.

THE PROOF IS OBVIOUS.

21*st March*, 1971.

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A NOTE ON TWO GENERATOR FINITE GROUPS WITH TWO RELATIONS

By

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In this note we give a presentation of a class of finite groups with two generators and two defining relations, that is, we prove the following :—

Theorem : The group

$$G(n, r) = gp\{a, b;\ a^n = 1,\ (a^{-1} b)^r = (ba^{-1})^{r+1}\} \quad (1)$$

is finite :

First we observe that the relation $(a^{-1} b)^r = (ba^{-1})^{r+1}$ implies that a and b have the same order :

$$(a^{-1} b)^r = b\,(a^{-1} b)^r a^{-1}$$

or $\quad (a^{-1} b)^r\ a\,(a^{-1} b)^{-r} = b$

Also from the same relation we have

$$a^{-1}(ba^{-1})^r\ a = (ba^{-1})^{r+1}$$

If $\quad ba^{-1} = x$, then

$$a^{-1} x^r a = x^{r+1} \quad .. \quad (2)$$

Taking rth power of both sides of (2)

$$a^{-1} x^{r^2} a = x^{r(r+1)}$$

That is

$$a^{-2} x^{r^2} a^2 = a^{-1} x^{r(r+1)} a = (a^{-1} x^r a)^{r+1}$$

$$= (x^{r+1})^{r+1} = x^{(r+1)^2}$$

Suppose that

$$a^{-k} x^{r^k} a^k = x^{(r+1)^k} \quad .. \quad (3)$$

is true.

Taking rth power of both sides of (3)

$$a^{-k} x^{r^{k+1}} a^k = x^{r(r+1)^k}$$

That is

$$a^{-k-1}\, x^{r^{k+1}}\, a^{k+1} = (a^{-1}\, x^{r}\, a)^{(r+1)^k}$$

$$= (x^{r+1})^{(r+1)^k} = x^{(r+1)^{k+1}}$$

Hence by Mathematical induction

$$a^{-n}\, x^{r^n}\, a^{n} = x^{(r+1)^n} \quad \text{for all positive integers } n.$$

Thus if $a^n=1$, then the order m of x is a divisor of $(r+1)^n-r^n$

Now both r, $(r+1)$ are coprime to $(r+1)^n-r^n$ and so x^r and $x^{(r+1)}$ genrate the same cyclic group as x.

Consider now the group

$$G=gp\{a, x\ ;\ a^n=x^m=1,\ a^{-1}x^r a=x^{r+1}\}$$

As r, $(r+1)^n-r^n$ are coprime, there exists an integer k such that $x^{kr}=x$. Then $a^{-1}\, x\, a=a^{-1}\, x^{kr}\, a=a^{-1}\, x^r\, a \ldots a^{-1}\, x^r\, a$ (k times).

$$= x^{(r+1)k} = x^{rk+k} = x^{k+1}$$

However since $(x^{k+1})^r=x^{kr+r}=x^{r+1}$, $gp\{x^{k+1}\}=gp\{x\}$ and therefore G is an extension of $gp\{x\}$ of order $m \mid (r+1)^n-r^n$ by $gp\{a\}$ of order n.

But as $x=ba^{-1}$, G contains together with a also the element b and so $G=G(n, r)$. Hence the theorem.

We note that when $n=2$, then for different values for r we get all the dihedral groups of order $2m$, m odd.

ON A SUM FUNCTION OF FUNCTIONS OF PARTITIONS (II)

By

S. MANZUR HUSSAIN

and

M. H. KAZI

1. Introduction

In [1] we proved some congruences mod p involving functions $P_r(n)$, where

$$P_1(n) = \sum_{k=0}^{n} P(k)\, P(n-k),$$

$$P_r(n) = \sum_{k=0}^{n} P(k)\, P_{r-1}(n-k),\ r \geqslant 2,$$

$$P_r(o)=1,$$

and $P(n)$ are unrestricted partitions of n. In [2] we introduced the sum function $S(n)=S(n,p)=\sum_{r=1}^{p} P_r(n)$ and proved a few congruence properties of $S(n)$ mod p. In this paper we use extended definition of the sum function :

$$S(n)=S(n, lp+s) = \sum_{r=1}^{lp+s} P_r(n),\ l > 1,\ s \geqslant 0$$

and prove some of its congruence properties mod p.

2. We first prove the following

Lemma : $P_{lp+s}(n) \equiv \sum_{k=0}^{r} P(k)\, P_{(l-1)p+s}(n-kp) \pmod p$

where $l \geqslant 1$ and $r = \left[\frac{n}{p}\right]$ (2.1)

Since the generating function for $P_r(n)$ is $\frac{1}{\prod^{\infty}(1-x^n)^{r+1}}$, it follows that

$$P_{lp+s} = \sum_{k=0}^{n} P_{p-1}(k)\, P_{(l-1)p+s}(n-k) \tag{2.2}$$

we proved in [1] that

$P_{p-1}(n) \equiv 0 \pmod p$, if p, does not divide n,

$\equiv P(m) \pmod p$, if $n = mp$.

Using these congruence relations in (2.2), we obtain the lemma.

We now prove the following

Theorem 1.

$$\sum_{k=1}^{mp+r} P(k)\, S(mp+r-k) = \sum_{k=1}^{m} P(k)\, P_{(l-1)p+s+1}(\overline{m-k}\,p+r) + P_{(l-1)p+s+1}(mp+r) - P_1(mp+r) \pmod p.$$

where $m \geqslant 1$ and $0 \leqslant r \leqslant p-1$.

Proof : By definition

$$S(mp+r) = \sum_{k=0}^{mp+r} P(k)\,[P(mp+r-k) + S(mp+r-k) - P_{lp+s}(mp+r-k)]$$

$$0 = \sum_{k=0}^{mp+r} P(k)\, P(mp+r-k) + \sum_{k=1}^{mp+r} P(k)\, S(mp+r-k)$$

$$-\Big[\sum_{k=0}^{r} + \sum_{k=r+1}^{p+r} + \sum_{k=p+r+1}^{2p+r} + \cdots\cdots \sum_{(j-1)p+r+1}^{jp+r} + \cdots\cdots$$

$$\cdots + \sum_{(m-2)p+r+1}^{(m-1)p+r} + \sum_{(m-1)p+r+1}^{mp+r} (P(k)\, P_{lp+3}(mp+r-k))\Big]$$

Using the lemma we obtain

$$0 \equiv P_1(mp+r) + \sum_{k=1}^{mp+r} P(k)\, S(mp+r-k) - \Big[\sum_{k=0}^{r} P(k) \Big(\sum_{k_1=0}^{m} P(k_1)$$

$$P_{(l-1)p+s}(mp+r-k-pk_1) + \sum_{k=r+1}^{p+r} P(k) \Big(\sum_{k_1=0}^{m-1} P(k_1)$$

$$P_{(l-1)p+s} \quad (mp+r-k-pk_1)\,)$$

$$+ \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

$$+ \sum_{(j-1)p+r+1}^{jp+r} P(k) \sum_{k_1=0}^{m-j} P(k_1)\; P_{(l-1)p+s}\; (mp+r-k-pk_1)$$

$$+ \; .. \; .. \; .. \; .. \; .. + \sum_{(m-2)p+r+1}^{(m-1)p+r} P(k)\, P_{(l-1)p+s} \quad (mp+r-k)$$

$$+ \sum_{(m-1)p+r+1}^{mp+r} P(k)\, P_{(l-1)p+s}\, (mp+r-k)\,]$$

(mod p). On simplification, we get

$$\sum_{k=1}^{mp+r} P(k)\, S(mp+r-k) \equiv \sum_{k=1}^{m} P(k)\, P_{(l-1)p+s+1}\, (\overline{m-k}\; p+r)$$

$$+\; P_{(l-1)p+s+1} \quad (mp+r)$$

$$-\; P_1\,(mp+r) \quad (\text{mod } p)$$

It may be remarked that in the special case when $l=1$ and $s=0$, we recover

$$\sum_{k=1}^{mp+r} P(k)\, S(mp+r-k) \equiv \sum_{k=1}^{m} P(k)\, P_1\, (\overline{m-k}\; p+r) \quad (\text{mod } p)$$

as proved in [2].

Theorem II.

(*i*) $S(n,\, l\,p+s) \equiv \sum_{r=1}^{s} P_r\,(n)\; (\text{mod } p)$ when $0 \leqslant n \leqslant p-2$

(*ii*) $S(p-1,\, lp+s) \equiv l + \sum_{r=1}^{s} P_r\,(p-1) \quad (\text{mod } p)$

(*iii*) $S(p,\, l\,p+s) \equiv s\,l + \sum_{r=1}^{s} P_r\,(p) \quad (\text{mod } p)$

(*iv*) $S(p+1,\, lp+s) \equiv l/2\,(s^2+3s+4) + \sum_{r=1}^{s} P_r\,(p+1) \quad (\text{mod } p)$

Proof of (i) & (ii)

When $\left[\frac{n}{p}\right] = 0$, the lemma gives

$$P_{lp+s}(n) \equiv P_{(l-1)p+s}(n) \pmod p$$
$$\equiv P_s(n) \pmod p$$

and so $S(n, lp+s) \equiv l(P_1+P_2+\ldots P_p)+\sum_{r=1}^{s} P_r \pmod p$

$$\equiv l\,S(n,p)+\sum_{r=1}^{s} P_r \pmod p \qquad (2.3)$$

We proved in (2) that

$$S(n,p) \equiv 0 \pmod p \text{ when } 0 \leqslant n \leqslant p-1 \qquad (2.4)$$

and $S(p-1, p) \equiv 1 \pmod p$ (2.5)

Using (2.5) and (2.4) in (2.3), we obtain congruences (*i*) and (*ii*)

Proof of (iii) and (iv) :

When $\left[\frac{n}{p}\right]=1$, the lemma gives

$$P_{lp+s}(n) \equiv P_{(l-1)p+s}(n)+P_{(l-1)p+s}(n-p) \pmod p \qquad (2.6)$$

when $n=p$, we get

$$P_{lp+s}(p)\equiv P_{(l-1)p+s}(p)+1 \pmod p$$
$$\equiv P_s(p)+l \pmod p$$

$$S(p, lp+s)\equiv l(P_1+\ldots+P_p)+p\frac{(l-1)\,l}{2}+\sum_{r=1}^{s} P_r+s\,l \pmod p$$

$$\equiv lS(p,p)+\sum_{r=1}^{s} P_r+sl \pmod p$$

But $S(p,p)=0$, as proved in [2]

Hence $S(p, lp+s)\equiv s\,l+\sum_{r=1}^{s} P_r(p) \pmod p$.

When $n=p+1$, (2.6) gives

$$P_{lp+s}(p+l)\equiv P_{(l-1)p+s}(p+1)+P_{(l-1)p+s}(1) \pmod p$$

$$\equiv P_{(l-1)\,p+s}\,(p+1)+(s+1) \pmod p$$

$$\equiv P_s(p+1)+l(s+1) \pmod p$$

and so

$$S(p+1, lp+s) \equiv P_1(p+1)+\ldots+P_p(p+1)$$
$$+\{P_1(p+1)+1.2\}+\{P_2(p+1)+1.3\}$$
$$+\{P_p(p+1)+1(P+1)\}$$
$$+\{P_1(p+1)+2.2\}+\{P_2(p+1)+2.3\}+\ldots+\{P_p(p+1)+2(p+1)\}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\{P_1(p+1)+(l-1)2\}+\{P_2(p+1)+(l-1)3\}+\ldots+\{P_p(p+1)+(l-1)(p+1)\}$$
$$+\{P_1(p+1)+l.2\}+\{P_2(p+1)+l.3\}+\ldots+\{P_s(p+1)+l.(s+1)\} \pmod p$$

$$\equiv lS(p+1,p)+\sum_{r=1}^{s} P_r(p+1)+l\frac{(s+3).s}{2} \pmod p \qquad (2.7)$$

When $l=1$, $s=0$, $r=2$ and $m=1$, theorem I gives

$$\sum_{k=1}^{p+2} P(k)S(p+2-k, p)\equiv P(1)\ P_1(2) \pmod p.$$

Hence $P(1)\ S(p+1,p)+P(2)\ S(p,p)+P(3)\ S(p-1,p)$

$$+\sum_{k=4}^{p+2} P(k)\ S(p+2-k, p) \equiv 5 \pmod p.$$

Since $S(p,p)\equiv 0 \pmod p$

$S(p-1,p)\equiv 1 \pmod p$ and $S(n,p)\equiv 0 \pmod p$, when $0 \leqslant n \leqslant p-2$, it follows that

$$S(p+1,p) \equiv 2 \pmod p \qquad (2.8)$$

From (2.7) and (2.8) we obtain

$$S(p+1, lp+s)\equiv\frac{l}{2}\{s^2+3s+4\}+\sum_{r=1}^{s} P_r(p+1) \pmod p.$$

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