VOLUME V NOVEMBER 1972

THE PUNJAB UNIVERSITY JOURNAL OF MATHEMATICS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE PUNJAB
LAHORE, PAKISTAN

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CONTENTS

GEOMETRY OF CORRELATIONS BETWEEN TWO PLANES S_2 AND S'_2.

by

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Introduction : A preliminary study of complete correlations of S_2 on S'_2 on the lines of Schubert's enumerative Geometry of conics and quadrics was published by T.H. Hirst (1, 2)†. Applications to this case are also recorded in (6). In this paper we set out to give a detailed discussion embodying subsystem of such correlations, their representations on the Semple's model Ω of complete plane correlations which form a starting point for the geometry of plane correlations with which subsequent developments are largely concerned.

The work is broadly divided into two parts. The first part deals mainly with the representations of system and subsystem of correlations on different models whereas the second part develops the theory of bases on the degeneration subvariety ∂ introducing algebraic equivalence on certain singular varieties.

PART I

§1. In what follows we will use the notations $\mathbf{x}=(x_i)$. $\mathbf{y}=(y_i)$ as the coordinate vectors of S_2 and S'_2 and $\mathbf{u}=(u_i)$, $\mathbf{v}=(v_i)$ as those for the lines of S_2 and S'_2. Equations of complete correlations are then of the form

$$y^t A x=0, \quad v^t B u=0$$

where the matrices $A=(a_{ij})$, $B=(b_{ij})$, $(i, j=0, 1, 2)$, satisfy the following set of sixteen fundamental relations :

$$\partial^{r_1 r_2}=a_{r_1 0}b_{r_2 0}+a_{r_1 1}b_{r_2 1}+a_{r_1 2}b_{r_2 2},$$

$$\partial^{r_1 r_2}=0 \text{ if } r_1\neq r_2,$$

$$\partial^{00}=\partial^{11}=\partial^{22}.$$

†The numbers in the brackets indicate numbers of papers mentioned in the references.

Received : April 2, 1971.

In the special case when S_2 is superimposed on S'_2, the coincidence complexes simply represent two distinct conics having double contact (10). When **a** is symmetric the two equations represent the two different aspects of the same conic.

§2. As follows from Semple's work his model Ω of complete correlations in the present case is manifold $\Omega \equiv \Omega_8$ in [64]. This manifold is represented on the space S_8 of coordinates a_{ij} by the cubic primals of S_8 passing simply through a Segre Variety $V_4^6 = M^{(1)}$ and also on the space of co-ordinates b_{ij} by cubic primals passing simply through a Segre Variety $W_4^6 = N^{(1)}$.

§3. **The degeneration manifolds on Ω.**

There exist three such manifolds ∂, η and $\partial\eta$. We classify them as follows :

(*i*) The manifold ∂ :
This is a seven fold on Ω lying in an ambient of a space of dimension 63. In general it represents a system of correlations for which A is of rank 2 and B of rank 1. The ∞^7 ∂-correlations mapped one to one on the points of ∂ are each of the form (P, Q; τ) where P and Q called the centres of the system of ∂-correlations are each a point of S_2 and S'_2 and τ is a homography between the pencils with vertices P and Q. Two lines are conjugate if the centre of one plane lies on either line and two points are conjugate if their joins to P and Q correspond in the homography τ. Conversely, two lines correspond in τ if each point of one is conjugate to every point of the other.

Schubert in his 'Abzah lende Geometric' (5) dealt with analogous variables (l, m; τ) where he took l, m as lines of S_3 and S'_3 (or $S_3 \equiv S'_3$) and τ as a homography between

(*i*) points of l and planes through m,

(*ii*) planes through l and planes through m, (*iii*) points of l and points of m.

Our variables (P, Q; τ) are a similar set for a pair of planes S_2 and S'_2.

(*ii*) The Manifold η.

η is dual to ∂ with A and B of ranks 1 and 2 respectively. Points of this manifold represent ∞^7 η-correlations each denoted by a symbol $(p, q; \tau)$ where p, q are fixed lines called the axes of η-correlations in S_2 and S'_2 and τ is a given homography between points of p and q. The conditions for conjugate lines and points are just dual to those for ∂ correlations.

(*iii*) The manifold $\partial\eta$.

It is a sinfold on ∂ (or η) for which the matrices A and **B** are both of ranks 1. $\partial\eta$ is the Segre product of two three folds each of order 6. Points of $\partial\eta$ represent correlations of the form $(P, p; Q, q)$ where P, p are centres and axes of the system in S_2 and Q, q are centres and axes in S'_2. Two points are conjugate in this system if the axis of one plane passes through either point and two lines are conjugate if the centre of one plane lies on either line.

§4. **The Space S_8.**

Before considering the mappings of Ω on S_8 and S'_8 we introduce the following scheme of symbols relative to the mapping of Ω on S_8.

$$M^{(1)} \quad \alpha \quad \beta \quad P \quad V_3^3(\alpha) \quad V_3^3(\beta) \quad \omega \quad \chi$$

$$M^{(2)} \quad A \quad B \quad \tau \quad V_6^3(A) \quad V_6^3(B) \quad W \quad \psi$$

Interpretations of these symbols are :

$M^{(1)}$: Rank variety with parametrization) $(a_{ij}) = (p_i q_j)$

$M^{(2)}$: Rank variety given by $|a_{ij}| = 0$ having $M^{(1)}$ as a double locus.

α, β : Sheaves of generating planes of $M^{(1)}$ parametrized by the co-ordinate vectors q_i and p_j respectively.

A, B : Sheaves of generating [5]'s of $M^{(2)}$.

P : Intersection of an α – plane with a β – plane.

τ : A contact solid of $M^{(2)}$ the intersection of an A−space with a B − space.

$V_3^3(\alpha)$, $V_3^3(\beta)$: Planar threefolds on $M^{(1)}$ generated by α−planes and β−planes respectively.

$V_6^3(A)$, $V_6^3(B)$: Generated by the spaces A and B that pass through a point P of $M^{(1)}$ Cones with the planes α and β through P as vertices.

ω : Intersection of a $V_3^3(\alpha)$ with a $V_3^3(\beta)$. It is a quadric surface.

W : Intersection of a $V_6^3(A)$ with a $V_6^3(B)$.

χ : A Veronese surface of the ∞^8 system on $M^{(1)}$ generated by collineations between α and β.

ψ : Locus of points of intersection of collinearly related spaces A and B. ψ is a five fold of order 7 on $M^{(2)}$.

In addition we introduce two more symbols :

T_4 : A tangent [4] of $M^{(1)}$ join of a pair of planes α and β.

π : A plane, vertex of Veronese envelope of primes joining collinearly related spaces A and B.

In the quadric transformation T_{22} of S_8 into the space S_8' of co-ordinates b_{ij} we have the same sort of setup as we had in S_8. In particular $M^{(1)}$ is dilated into $N^{(2)}$ a determinantal cubic primal whose equation is $|b_{ij}|=0$ and $M^{(2)}$ is contracted into $N^{(1)}$ with equations

$$\| b_{ij} \| = 0$$

Our object in this part, in the main, is to identify the types of subsystem of correlations that are represented by the above varieties in S_8, to discuss concurrently, the subsystem of correlations that have analogous representations in S'_8 and finally to axamine the models of both types of subsystem on the overall model Ω of complete correlations.

5. Mappings of Ω_8 on S_8.

The model Ω_8 of complete correlations (**a**, **b**) is mapped birationally, as we remarked, on S_8 in such a way that its prime sections are represented in S_8 by cubic primals through $M^{(1)}$. In this mapping we note the following :

(*i*) ∂ as a whole is mapped on $M^{(2)}$, each ∂—correlation (P, Q: τ) being mapped thereby on a point of $M^{(2)}$.

(*ii*) Since for η, a is of rank 1, the whole variety η is mapped on the neighbourhood of $M^{(1)}$. Every point and therefore every η—correlation $(p, q;\ \tau)$ in this mapping corresponds to the section of the neighbourhood of a point P of $M^{(1)}$ by a [5] through the tangent space T_4 to $M^{(1)}$ at P.

(*iii*) A $\partial\eta$ – correlation is mapped by the section of the neighbourhocd of P by a [5] through the tangent space T_4 aad contained in the quadric cone of $M^{(2)}$ at P.

(*iv*) The whole neighbourhood of P in S_8 is mapped in this way on a solid of the degeneration manifold η of Ω_8. Points of this solid correspond to ∞^3 homographies between the lines p and q defining the point P. The solid meets $\partial\eta$ in a quadric surface representing the whole neighbourhood of P in $M^{(2)}$.

Details of the correspondence between the homographies between p, q neighbourhoods of P and the subvarieties of the corresponding degeneration solid on η are tabulated blow.

Homographies between p and q	Ambient of the neighbourhood of P	Image variety on η	Intersection with $\partial\eta$
Free homographies	Whole space	Solid	Quadric Surface
One linear condition on the homographies	A prime through T_4	Plane	Conic
Two linear conditions	A secundum through T_4	line	Two points
Three linear conditions	A tertium through T_4	point	×

A similar table can be constructed from the specializations of the homographies between the lines representing P. In particular when a prime through T_4 specializes into a tangent [7] to $M^{(2)}$ the plane on η becomes a tangent plane to the quadric surface on $\partial\eta$ and the system of homographies then have a fixed pair of corresponding points one on p and the other on q.

Notations :—We will use the following notations for system of degenerate correlations and the corresponding subvarieties on Ω_8.

(*i*) For ∂ correlations (P, Q: τ) we use :

∂ P : as the condition symbol for P to lie on a given line.
∂Q : as the condition symbol for Q to lie on a given line.
$\partial\zeta$: as the condition symbol for the system to have a given pair of conjugate points. By § 3. (*i*) $\partial\zeta$ is then the condition that the respective joins of P,Q to a given pair of conjugate points correspond in τ.

By analogy with Schubert's development we take ∂P, ∂Q, $\partial\zeta$ to be the fundamental conditions of weight unity on the system of ∂ – correlations.

We will use the same notations for the corresponding subvarieties ∂P, ∂Q, $\partial\zeta$ on ∂.

(*ii*) For the system of η correlations (p, q ; τ) which is dual to the system of ∂–correlations we use analogous symbols ηp, ηp, $\eta\zeta$ to stand for fundamental conditions on η.

(*iii*) For $\partial\eta$ – correlations (P, q ; Q, q) we use :

$\partial\eta$ P : as the condition symbol for P to lie on a given line of S_2.
$\partial\eta$ Q : as the condition symbol for Q to lie on a given line of S'_2.
$\partial\eta$ p : as the condition symbol for p to pass through a given point of S_2.
$\partial\eta$ q : as the condition symbol for q to pass through a given point of S'_2.

§ 6. In the following we enlist a set of results identifying the different systems of correlations with the varieties of S_8 and S'_8 and examining the models of these systems on the over–all model Ω of complete plane correlations.

1. The A−space and B−space represent the system ∂P^2 and ∂Q^2 respectively. The subvarieties ∂P^2 and ∂Q^2 on Ω are represented on A and B by cubic primals passing simply through V_3^3 (α) and V_3^3 (β) respectively.

2. The contact solid τ of $M^{(2)}$ represents the system ∂P^2Q^2. The image variety ∂P^2Q^2 is a generating solid of ∂ and meets η in a quadric surface of one system of $\partial\eta$.

3. To the whole neighbourhoods of α and β planes correspond the classes ηp^2 and ηq^2. The varieties ηp^2 and ηq^2 on η meet in $\eta p^2 q^2$ representing the whole neighbourhood of point of intersection P of α and β.

4. Points of V_6^3(A) represent the system ∂P where P lies on one of the axes of the system of η-correlations, represented by the point O of $M^{(1)}$ through which the system of spaces A passes. Similarly for V_6^3(B). V_6^3(A) and V_6^3(B) meet in a five fold W of order 5 on $M^{(2)}$ representing the class ∂PQ. The varieties ∂P, ∂Q and ∂PQ are mapped on the corresponding manifolds in S_8 by their sections by cubic primals through $M^{(1)}$.

5. To the whole neighbourhood of $V_3^3(\alpha)$ there corresponds the class ηp and similarly to the whole neighbourhood of $V_3^3(\beta)$ there corresponds the class ηq of ∞^6 η-correlations. The intersection five fold ηpq of ηp with ηq is mapped on the whole neighbourhood of the intersection of $V_3^3(\alpha)$ with $V_3^3(\beta)$ which is a quadric surface.

6. Points of ψ represent the system of ∂ correlations (P, Q; τ) where P and Q correspond in a fixed collineation K between S_2 and S'_2.

7. The whole neighbourhood of a Veronese surface on $M^{(1)}$ represents a class of η-correlations (p, q; τ) where p and q correspond in a fixed collineation between S_2 and S'_2.

8. A tangent space T_4 of $M^{(1)}$ (necessarily lying on $M^{(2)}$) represents the system of ∞^4 ∂-correlations (P, Q; τ) where P and Q lie on two assigned lines of S_2 and S'_2 which correspond in τ.

9. The class of η-correlations (p, q; τ) where p, q pass through two fixed points which correspond in τ is mapped by the section of the neighbourhood of a quadric ω of $M^{(1)}$ by the tangent prime to $M^{(2)}$ along ω.

10. Points of π represent ∞^2 correlations ∂(P, Q; τ) defined by a collineation K between S_2 and S'_2 such that

(*i*) P, Q correspond by K,

(*ii*) τ is the homographic correspondence between lines through P and lines through Q which correspond by K.

The aggregate of such ∂-correlations are mapped on the manifold ∂ by a Novemic Delpezzo surface.

11. The system of η-correlations (p, q ; τ), where p and q correspond in a collineation K between S_2 and S'_2 and for such a given pair (p, q), τ is a homographic correspondence between points of p and q which correspond by K is dual to the above system. The whole system is mapped on some sorts of neighbourhood of Veronese surface. Since every pair is associated with a uniquely defined τ every point of the Veronese surface is associated with uniquely determined tangent [5] at the point corresponding to τ. The system of such η-correlations is mapped on η by a Novemic Delpezzo surface.

From the preceding results it is clear that the two rows of symbols of § 4 are so arranged that the neighbourhood of any element of the first row represents a system dual to that represented by the opposite element of the second row.

PART II

1. ∂ which is itself a nonsingular variety contains a large number of subvarieties some of which are nonsingular and some singular. We give below some results concerning singularities of some of the sub-

varieties of ∂ which are easy deductions using local parametrizations of the varieties while leave others to be established by the interested readers.

To start with we observe :

1. ∂, ∂P and ∂Q are all nonsingular.

2. $\partial\zeta$ defined by conjugacy of the point pairs A (ξ_i) and A′ (η_i) is singular. To exhibit its singularities properly we encounter below some of the subvarieties of $\partial\zeta$.

$\partial\eta\, p$, $\partial\eta q$: Two five fold systems with the axes p and q through A and A′ respectively.

$\partial\eta\, P^2$, $\partial\eta\, Q^2$: Two four fold systems with the centres at A and A′ respectively.

$\partial\eta\, pq$: A four fold system with axes through A and A′.

$\partial\eta\, P^2 q$, $\partial\eta\, pQ^2$: Two three fold subsystems of $\partial\eta\ P^2$ and $\partial\eta\ Q^2$ with axes through A′ and A respectively.

$\partial\eta\, P^2 Q^2$: A two fold system with centres at A and A′.

$\partial\zeta$ is simple at every point of $\partial\eta\, p - \partial\eta\, P^2 - \partial\eta\, pq$, $\partial\eta\, pq - \partial\eta\, P^2 q - \partial\eta\, pQ^2$ and $\partial\eta\, P^2 - \partial\eta\, P^2 q$ having simple contacts with $\partial\eta$ and ∂P at every point of these two latter varieties.

$\partial\zeta$ has double locus at every point of $\partial\eta\, P^2 q$, the tangent space being a quadric cone whose vertex is a solid and base a quadric three fold, which degenerates into a pair of 6-spaces at every point of $\partial\eta\, P^2Q^2$ embedded in $\partial\eta\, P^2 q$. It follows further that these two tangent spaces are simply the tangent spaces one to ∂P and other to ∂Q.

2. *Theory of the base on ∂.

The essence lies in representing ∂ in the three way space (**P**, **Q**, **a**) and applying repeatedly the methods of degenerate collineations. The

*From now on for varieties on ∂ and $\partial\eta$ we will omit the common factor ∂ and $\partial\eta$ and make the convention that if V_d denotes a point set variety of dimension d then $\mathbf{V}_d$ will stand for multiplicative variety in the sense used by Hodge.

following is a long list of results—a straightforward derivation of the methods of degenerate collineations (12).

1. Any d-fold ($d \geqslant 4$) on ∂ satisfies the equivalence relation

$$V_d \sim \sum_{(i)} r_i V_d^i,\ i=1, \ldots, r$$

where $r_1, \ldots, r_r$ are integers, and each V_d^i is an irreducible d-fold contained in at least one of P, Q, ζ.

2. When $d=6$, (1) gives

$$V_6 \sim r_1 P + r_2 Q + r_3 \zeta$$

The following are some self evident equivalent formulae satisfied by some of the six folds on ∂.

(*a*) $\partial\mu \sim \zeta$

(*b*) $\partial\nu \sim P+Q$ where μ, ν represent two classes of ∞^7 correlations with a pair of conjugate points and a pair of conjugate lines respectively.

3. For equivalences on P and Q each V_d satisfies the equivalence formulae

$$V_d \sim \sum r_i V_d^i,\ i=1, \ldots, r \text{ (say)}, 4 \leqslant d \leqslant 5$$

where each V_d^i is an irreducible d-fold contained in at least one of P^2, PQ, $P\zeta$ and Q^2, QP, $Q\zeta$ respectively.

4. A five fold on ζ which is not contained in $\partial\eta$ satisfies

$$V_5 - l\zeta^2 + m\zeta P + n\zeta Q$$

5. For a five on ∂ we combine (3) and (4). This leads to

$$V_5 \sim r_1 P^2 + r_2 Q^2 + r_3 \zeta^2 + r_4 PQ + r_5 P\zeta + r_6 Q\zeta$$

where the r_i's which are not all zero may be positive or negative integers. Let V_4 be an irreducible four fold which does not consist entirely of $\partial\eta$ —correlations. Then,

6. on ζ, V_4 satisfies

$$V_4 - \sum r_i V_4^i,\ i=1, \ldots, s.$$

where the r_i's which are not all zero are positive integers and each V_4^i is contained either in ζ^2 or in ζP or in ζQ.

7. $(\zeta^3, \zeta^2 P, \zeta^2 Q)$, $(\zeta^2 P, P^2\zeta, PQ\zeta)$, $(\zeta^2 Q, Q^2\zeta, PQ\zeta)$ constitute bases for four folds for algebraic equivalences on ζ^2, $P\zeta$ and $Q\zeta$ respectively.

8. The sets $(P^2Q, PQ^2, PQ\zeta)$, $(P^2\zeta, P^2Q)$, $(Q^2P, Q^2\zeta)$ form basis for four folds for virtual equivalence on PQ, P^2 and Q^2 respectively.

A base for equivalences for four folds on ∂ is obtained by combining the above. This leads to :

9. An irreducible four fold V_4 on ∂ satisfies the equivalence relation

$$V_4 \sim r_1 P^2Q + r_2 PQ^2 + r_3 P^2\zeta + r_4 Q^2\zeta + r_5 PQ\zeta + r_6 P\zeta^2 + r_7 Q\zeta^2 + r_8 \zeta^3$$

To show that $(P, \ldots, \zeta)$, $(P,^2 \ldots, Q\zeta)$ and $(P^2Q, \ldots \zeta^3)$ of (2), (5) and (9) form minimal base for varieties of respective dimension, we pick up appropriate varieties of complementary dimensions. From the study of the intersection table we claim :

10. The set of eight three folds $(P^2Q\zeta, PQ^2\zeta, P^2\zeta^2, Q^2\zeta^2, PQ\zeta^2, P\zeta^3, Q\zeta^3, \zeta^4)$ provides a minimal base for three folds on ∂.

11. For surfaces on ∂, $(P^2Q^2\zeta, P^2Q\zeta^2, PQ^2\zeta^2, P^2\zeta^3, Q^2\zeta^3, PQ\zeta^3)$ form a minimal base.

12. The set $(P^2Q^2\zeta^2, P^2Q\zeta^3, PQ^2\zeta^3)$ provides a minimal base for curves.

3. Base on $\partial\eta$

By applying to the much simpler case of $\partial\eta$ the methods used to investigate bases on ∂ the following results are obtained :

1. A base for five folds on $\partial\eta$ is generated by (P, p, Q, q)
2. A base for four folds on $\partial\eta$ is generated by $(P^2, p^2, Q^2, q^2, PQ, Pq, pQ, pq)$.
3. A base for three folds on $\partial\eta$ is generated by $(PQ^2, P^2Q, Pq^2, P^2q, pQ^2, p^2Q, pq^2, p^2q, P^2p, Q^2q)$.
4. For surfaces on $\partial\eta$ $(P^2q^2, p^2Q^2, p^2q^2, P^2pq, pQ^2q, PQ^2q, pQ^2q, P^2Q^2)$ constitute a base.

5. (P^2Q^2q, p^2Qq^2, P^2pQ^2, Pp^2Q^2) constitute a base for curves on $\partial\eta$.

We observed earlier that $\partial\eta$ is the direct product of two three folds V_3^6 and W_3^6 where V_3^6 is the aggregate of flags (4)

F: $F_o \subset F_1$ of S_2 and W_3^6 is the aggregate of flags

F′: $F'_o \subset F'_1$ of S'_2. The nonsingularity of $\partial\eta$ follows by observing that a proper collineation preserves the inclusion property.

A base for surfaces on V_3^6 consists of two surfaces

F_o: Aggregate of flags for each of which F_o lies on a line,

F_1: Aggregate of flags for each of which F_1 passes through a point.

For curves the base consists of

$F_o{}^2$, $F_1{}^2$: Aggregate of flags for each of which F_o and F_1 respectively are fixed.

A point of V_3^6 has interpretation $F_o{}^2F_1$ denoting a particular flag for which F_o and F_1 are both fixed.

With F's replaced by F′'s only we use almost identical notations and meanings for elements of bases of different dimensions on W_3^6.

It is easy to see that if the d-fold ϕ_d is a base member of $\partial\eta$, then there exist base members V_k and W_{d-k} of V_3^6 and W_3^6 such that

$$\phi_d = V_k \times W_{d-k}$$

strikingly displaying Semple's hypothesis that a base member of $V_d = V_k^{(1)} \times V_{d-k}^{(2)}$ is obtained by pairing off a base member of $V_k^{(1)}$ with that of $V_{d-k}^{(2)}$ that preserves equality of dimensions.

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BASIS IN BANACH SPACE AND ITS TOPOLOGICAL DUAL†

by

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1. Introduction. The aim of this paper is to investigate the relationship of a basis in a Banach space and its topological dual. In his inspiring paper Karlin [3] has discussed and analyzed various types of bases in Banach spaces which served as a foundation stone for future investigations in Bases problem. These results have been considerably improved by James [2], Zippin [4] and others who have presented other criteria like separability and reflexivity of Banach spaces. In this paper we prove some results by using one of the well known results of James [2] which is given in the sequel as James Lemma.

2. Preliminaries. In this section we introduce some notations and concepts of a basis in a Banach space.

If E denotes a real Banach space, then the first conjugate space E* is its topological dual and, is the space of all linear functionals f defined on E such that

$$\| f \| = \operatorname*{Sup}_{x \varepsilon E} | f(x) | \big/ \| x \| .$$

If $\{x_n\}_{n=1}^{\infty} \subset E$, then $[X_n]_{n=1}^{\infty}$ will denote the smallest linear space spanned by the elements $\{x_n\}_{n=1}^{\infty}$.

Lin sp $\{x_n\}_{n=1}^{\infty}$ (linear subspace spanned by the set) is the smallest linear subspace containing all the elements of the set ; it is not in general closed.

$\sigma(E,F)$ denotes the weakest topology on E such that all functions of F are continuous, where $F \subset E^*$.

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†Received : October 28, 1971.

Definition 1. Let E be a real Banach space. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E is said to be a basis (also called a Schauder basis) for E, if for every $x \varepsilon$ E, there exists a *unique* set of real numbers $\{a_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} a_n x_n$, *i.e.* $\| x - \sum_{n=1}^{m} a_n x_n \| \longrightarrow o$ as $m \longrightarrow \infty$.

Definition 2. The functionals $\{f_n\}_{n=1}^{\infty}$ in E* constitute a ***biorthogonal*** sequence with respect to the basis $\{x_n\}_{n=1}^{\infty}$ in E, if $f_n(x_m) = \delta_{nm}$ where δ_{nm} is the Kronecker delta symbol.

It is well-known [1] that the set of real numbers a_n defines continuous linear functionals over E. We denote them by $a_n = f_n(x) \ \forall x \varepsilon$ E.

Definition 3. A set $\Gamma \subset E^*$ is said to be regularly closed if for every f_o not in Γ, there exists an $x_o \varepsilon$ E such that $f_o(x_o) \neq o$ and $g(x_o) = o$ for all g in Γ.

James Lemma. A sequence $\{x_n\}_{n=1}^{\infty}$ $(x_n \neq o)$ forms a basis for E if and only if there exists $K > o$ such that

$$\| \sum_{i=1}^{n} a_i x_i \| \leqslant K \| \sum_{i=1}^{n+p} a_i x_i \|,$$

where n and p are any positive integers and $\{a_i\}_{i=1}^{\infty}$ is any set of real numbers.

3. Main results. Using James Lemma, we now prove our main results.

Theorem 1. Let $\{x_n\}_{n=1}^{\infty}$ be a basis for E and let $\{f_n\}_{n=1}^{\infty}$ be the corresponding biorthogonal sequence in E*. Then $\sum_{n=1}^{\infty} f(x_n) f_n$ converges for every f in $[f_n]_{n=1}^{\infty}$.

Proof. Let $f_q \varepsilon [f_n]_{n=1}^{\infty}$. By hypothesis, $\sum_{m=1}^{\infty} f_m(x)\, x_m$ converges to x, say, as $n \to \infty$.

The set $\{ \sum_{m=n}^{n'} f_m(x)\, x_m : n, n' \varepsilon Z^{+}\}$ is uniformly bounded for all x in E, $\| x \| \leqslant 1$. For, by James Lemma $\| \sum_{i=1}^{n} f_i(x)\, x_i \| \leqslant K \| x \|$ for all n. This implies that

$$\| \sum_{m=n}^{n'} f_m(x)\, x_m \| \leq 2K \| x \| \text{ for all } n, n' \tag{1}$$

Now there exists some function g ε lin sp $\{f_n\}_{n=1}^{\infty}$ such that

$$\| g - f_p \| < \varepsilon / 2K \tag{2}$$

Also we have

$$\| \sum_{n=N}^{N'} f_q(x_n) f_n \| = \| \sum_{n=N}^{N'} f_q(x_n) f_n - \sum_{n=N}^{N'} g(x_n) f_n + \sum_{n=N}^{N'} g(x_n) f_n \|$$

$$\leqslant \| \sum_{n=N}^{N'} (f_q - g)(x_n) f_n \| + \| \sum_{n=N}^{N'} g(x_n) f_n \|$$

$$\leqslant \sup_{\| x \| \leq 1} | \sum_{n=N}^{N'} (f_q - g)(x_n) f_n(x) | + \| \sum_{n=N}^{N'} g(x_n) f_n \|$$

$$< \| f_q - g \| \sup_{\| x \| < 1} \| \sum_{n=N}^{N'} f_n(x)\, x_n \| + \| \sum_{n=N}^{N'} g(x_n) f_n \|$$

For N sufficiently large the second term on the right-hand side becomes zero. In view of (1) and (2), we finally obtain

$$\| \sum_{n=N}^{N'} f_q(x_n) f_n \| \leq \varepsilon$$

which shows that the series converges.

Theorem 2. If $\sum_{n=1}^{m} f(x_n) f_n$ converges weakly in σ (E*, E)–topology to f, as $m \rightarrow \infty$ for every $f \varepsilon E^*$, where $\{f_n\}_{n=1}^{\infty} \varepsilon E^*$ is a sequence of biorthogonal functions with respect to $\{x_n\}_{n=1}^{\infty}$. Then $\{x_n\}_{n=1}^{\infty}$ forms a basis for E.

Proof. Since $\sum_{n=1}^{\infty} f(x_n) f_n(x)$ converges for all $x \varepsilon$ E, we have, in particular

$$\operatorname{Sup}_n \left| \sum_{k=1}^{n} f(x_k) f_k(x) \right| = M < \infty$$

for each $x \varepsilon$ E, and so by the uniform boundedness theorem,

$$\operatorname{Sup}_n \left\| \sum_{k=1}^{n} f(x_k) f_k \right\| = M < \infty.$$

Now the result that $\sum_{n=1}^{\infty} f_n(x)\, x_n$ converges for every $x \varepsilon [x_n]_{n=1}^{\infty}$, follows by the same method of proof as Theorem 1. We now show that $[x_n]_{n=1}^{\infty} = E$.

Let us suppose that $x \varepsilon E \setminus [x_n]_{n=1}^{\infty}$, then by the Hahn-Banach theorem, there exists a continuous linear functional $f_o \varepsilon E^*$ such that $f_o(x) = 1$ and $f_o(x_n) = o$ for all $n \geqslant 1$. By the hypothesis, $\sum_{n=1}^{m} f_o(x_n) f_n$ converges weakly to f_o, *i.e.* in the σ (E*, E)–topology. Hence

$$\sum_{n=1}^{m} f_o(x_n) f_n(x) \longrightarrow f_o(x) \text{ as } m \longrightarrow \infty.$$

But $f_o(x_n) = o$ and $f_o(x) = 1$. Consequently

$$o = \sum_{n=1}^{m} f_n(x) f_o(x_n) \longrightarrow f_o(x) = 1,$$

which is impossible. It follows that $[x_n]_{n=1}^{\infty} = E$ and therefore $\{x_n\}_{n=1}^{\infty}$ forms a basis in E.

Theorem 3. If $\{x_n\}_{n=1}^{\infty}$ is a basis for E and $[f_n]_{n=1}^{\infty} = \Gamma$ is regularly closed as a subset of E*, then $\{f_n\}_{n=1}^{\infty}$ is a basis for E*.

Proof. We have seen in theorem 1 that if $\{x_n\}_{n=1}^{\infty}$ is a basis for E, then $\sum_{n=1}^{\infty} f(x_n) f_n$ converges for every $f \in [f_n]_{n=1}^{\infty}$. For any $x \in E$, define $\hat{x}(f) = f(x)$ for all $f \in E^*$. The $\hat{x}$ is a linear map $E^* \longrightarrow K$ (real scalar field). Now it is easily seen by considering $\hat{x}_n$ that if $f = \sum_{n=1}^{\infty} b_n f_n$, then $b_n = f(x_n)$ for all n.

Hence the expression $f = \sum_{n=1}^{\infty} f(x_n) f_n$ is unique, and so $\{f_n\}_{n=1}^{\infty}$ forms a basis for $[f_n]_{n=1}^{\infty}$. Now if we can show that $[f_n]_{n=1}^{\infty} = E^*$, then $\{f_n\}_{n=1}^{\infty}$ is a basis for E*,

Assume that $\Gamma \neq E^*$, then there exists an f_o not in Γ and by hypothesis, there exists an $x_o \in E$ such that $f_o(x_o) \neq o$ and $f_n(x_o) = o$ for all $n \geqslant 1$. But $x_o = \sum_{n=1}^{\infty} f_n(x_o)\, x_n = o$ which is a contradiction. Hence

$[f_n]_{n=1}^{\infty} = \Gamma = E^*$ and therefore $\{f_n\}_{n=1}^{\infty}$ forms a basis for E^*.

Theorem 4. If $\{f_n\}_{n=1}^{\infty}$ is a basis for E^* and $\{x_n\}_{n=1}^{\infty}$ in E is a biorthogonal sequence to $\{f_n\}_{n=1}^{\infty}$, then $\{x_n\}_{n=1}^{\infty}$ is a basis for E.

Proof : Let $f \in E^*$. Then since $\{f_n\}_{n=1}^{\infty}$ forms a basis for E^*, each f in E^* can be expressed uniquely in the form $f = \sum_{n=1}^{\infty} c_n f_n$. Therefore $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$ for all $x \in E$. In particular, we have

$$\begin{aligned} f(x_m) &= \sum_{n=1}^{\infty} c_n f_n(x_m) \\ &= c_m \quad \text{for } n = m, \quad (n,m = 1,2,3\ldots.). \end{aligned}$$

Therefore $f(x) = \sum_{n=1}^{\infty} f(x_n) f_n(x)$ for all $x \in E$. Hence the series $\sum_{n=1}^{m} f(x_n) f_n$ converges weakly to f, for each $f \in E^*$.

Hence by theorem 2, $\{x_n\}_{n=1}^{\infty}$ constitutes a basis for **E**.

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TOLLER POLE ANALYSIS OF DIP STRUCTURE IN $0^- + \frac{1}{2}^+ \rightarrow 1^+ + \frac{1}{2}^+$ PROCESSES

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Abstract

It is shown that according to 0(3,1) symmetry model, the differential cross-sections for near-forward scattering in $0^- + \frac{1}{2}^+ \rightarrow 1^+ + \frac{1}{2}^+$ reactions should exhibit dip at $\alpha = 0$ and $\alpha = -1$. These predictions are not inconsistent with the recent experimental data available for $\pi^+ p \rightarrow B^+ p$.

Introduction

The explanation of dip structure in the forward and the backward differential cross-sections serves as a good criterion for the validity of any model. Many authors have tried to explain the dip structure in various reactions by using Regge theory[1, 2, 3]. Contributions have been included and cuts[4, 5, 6, 7] have also been brought in to give suitable description of dip phenemenon. But despite all these efforts, no consistent explanation of dip structure in various reactions has been given so far. Different models provide only partial explanations; a model which explains the dip structure in some backward reactions fails to do so in others. There is no Regge model which gives a unified picture of the dip structure in the scattering processes with definite spin content. In the following we shall study the dip structure in the differen-

Received : August 1, 1972.

tial cross-sections for the forward scattering processes with spin content $0^- + \frac{1}{2}^+ \rightarrow 1^+ + \frac{1}{2}^+$ on the basis of extended formalism of Toller's 0(3,1) symmetry model.[8, 9, 10] This model predicts dips at $\alpha(t)=0$ and $\alpha(t) = -1$ and these are not inconsistent with the recent measurements[11] of the differential cross-section for the process $\pi^+ p \rightarrow B^+ p$.

0(3,1) Symmetry Formalism

Toller[8] has shown that the scattering amplitudes for the elastic forward scattering which possess symmetry with respect to the homogeneous Lorentz group 0(3, 1) can be expanded in terms of unitary representation function of 0(3, 1). This formalism has been extended in an approximate way, by Delbourgo, Salam and Strathdee[12] to inelastic near-forward scattering amplitudes for the scattering processes of the type $a+b \rightarrow c+d$ having masses m_1, m_2, m_3 and m_4 and spins s_1, s_2, s_3 and s_4 respectively. According to the extended formalism, the reduced amplitude $T_{s\lambda s'\lambda'}$ can be expressed in terms of a new set of helicity non-flip amplitudes $T^{(s')}_{j'\lambda s}(s, t)$ as

$$T_{s'\lambda's\lambda} = \sum_{j'} \left| \frac{\Phi(s, t)}{s_{12}\, s_{34}} \right|^{\frac{|\Delta|}{2}} < s'\lambda' \mid |\Delta| - \Delta\, j'\lambda > T^{(s')}_{j'\lambda s}(s, t)$$

where

$$s' = s_2 + s_4, \ldots\ldots\ldots\ldots, |s_2 - s_4|$$

$$s = s_1 + s_3, \ldots\ldots\ldots\ldots, |s_1 - s_3|$$

$$\lambda' = \lambda_2 - \lambda_4,\ \lambda = \lambda_1 - \lambda_3$$

$$j' = s' + |\Delta|, \ldots\ldots\ldots\ldots, |s' - |\Delta||$$

with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ the helicities of the particles and $\left|\frac{\phi(s, t)}{s_{12}\, s_{34}}\right|$ is a kinematical factor. The amplitudes $T^{(s')}_{j'\lambda s}$ have poles. If only one pole is supposed to dominate, then $T^{(s')}_{j'\lambda s}$ can be written as

$$T^{(s')}_{j'\lambda s}(s,t)=\sum_{j_o\leqslant \min(j',s)} \tfrac{1}{2}(j_o^2-\sigma^2)\times\beta^{(s')}_{j's}(j_o,t)\times \left\{d^{j_o\sigma}_{s\lambda j'}(\xi_t)+(-1)^{s+j'}\pi_1\pi_2\pi_3\pi_4\times d^{-j_o\sigma}_{s\lambda j'}\right\} \quad \text{(A)}$$

where $d^{j_o\sigma}_{s\lambda j'}(\xi_t)$ are the representation functions of 0(3, 1) and

$$\cosh\xi_t=\frac{s-u}{\left\{2(m_1^2+m_3^2)-t\right\}^{\frac{1}{2}}\left\{2(m_2^2+m_3^2)-t\right\}^{\frac{1}{2}}}$$

To obtain the asymptotic behaviour of $T^{(s')}_{j'\lambda s}$ for high energy, the representation functions $d^{j_o\sigma}_{s\lambda j'}(\xi_t)$ are expressed in terms of $e^{j_o\sigma}_{s\lambda j'}(\xi_t)$ as

$$d^{j_o\sigma}_{s\lambda j'}(\xi_t)=e^{j_o\sigma}_{s\lambda j'}(\xi_t)+\frac{\Gamma(j'+\sigma+1)}{\Gamma(j'-\sigma+1)}\times e^{-j_o-\sigma}_{s\lambda j'}(\xi_t)\times\frac{\Gamma(s-\sigma+1)}{\Gamma(s+\sigma+1)}$$

For large ξ_t, the behaviour of $e^{j_o\sigma}_{s\lambda j'}$ is given by

$$e^{j_o\sigma}_{s\lambda j'}(\xi_t)\underset{\xi_t\to\infty}{\sim}[(2s+1)(2j'+1)]^{\frac{1}{2}}\left[\frac{\Gamma(s-\lambda+1)\Gamma(s+j_o+1)}{\Gamma(s+\lambda+1)\Gamma(s-j_o+1)}\times\frac{\Gamma(j'-\lambda+1)\Gamma(j'+j_o+1)}{\Gamma(j'+\lambda+1)\Gamma(j'-j_o+1)}\right]^{\frac{1}{2}}\times(-1)^{j'+\lambda}\times$$

$$\frac{\Gamma(\sigma+j'+1)\Gamma(-j_o-\sigma)}{\Gamma(j_o-\lambda+1)\Gamma(\sigma-\lambda+1)\Gamma(-\sigma+j'+1)}\times(\operatorname{Cosh}\xi_t)^{-(\sigma+1+j_o-\lambda)}$$

provided $j_o\geqslant\lambda$. For $j_o<\lambda$, the same formula with j_o and λ interchanged gives the asymptotic behavious of $e^{j_o\sigma}_{s\lambda j'}(\xi_t)$. The correct Regge behaviour is obtained by putting $\sigma=\alpha+1$, where α specifies the usual Regge trajectory.

Calculations and Discussion

In the 0(3,1) symmetry formalism discussed above we start with s-channel centre-of-mass helicity amplitudes

$$\langle p_3s_3\lambda_3, p_4s_4\lambda_4\mid T\mid p_1s_1\lambda_1.\ p_2s_2\lambda_2\rangle\equiv\langle\lambda_3\lambda_4\mid T\mid\lambda_1\lambda_2\rangle$$

and express them in terms of reduced amplitudes $T_{s'\lambda's\lambda}$ as follows :

$$<\lambda_3\lambda_4 \mid T \mid \lambda_1\lambda_2> = \sum_{s, s'} <s_4\lambda_4 \mid s_2\lambda_2\, s' - \lambda'>$$

$$\times T_{s'\lambda' s\lambda}(s, t) <s_3\lambda_3 s\lambda \mid s_1\lambda_1>$$

The reduced amplitudes are further expanded in terms of

$T^{(s')}_{j'\lambda s}(s, t)$ as

$$T_{s'\lambda' s\lambda}(s, t) = \sum_{j'} \mid t \mid^{\frac{|\Delta|}{2}} <s'\lambda' \mid |\Delta| - \Delta j'\lambda> \times T^{(s')}_{j'\lambda s}(s, t)$$

The amplitudes $T^{(s')}_{j'\lambda s}$ (s, t) have poles in the extended complex angular momentum σ - plane. The contribution of a typical pole to the amplitude $T^{(s')}_{j'\lambda s}$ is given by equation (A).

There are twelve helicity amplitudes for the process $0^- + \frac{1}{2}^+ \rightarrow 1^+ + \frac{1}{2}^+$, but owing to the parity, charge conjugation and time reversal invariance only six of them are independent. We choose these amplitudes as

$<0\,\frac{1}{2} \mid T \mid 0\,\frac{1}{2}>$, $<0 - \frac{1}{2} \mid T \mid 0\,\frac{1}{2}>$, $<1\,\frac{1}{2} \mid T \mid 0\,\frac{1}{2}>$,

$<1\,\frac{1}{2} \mid T \mid 0\,\frac{1}{2}>$, $<1 - \frac{1}{2} \mid T \mid 0\,\frac{1}{2}>$, $<-1 - \frac{1}{2} \mid T \mid 0\,\frac{1}{2}>$

Using the formalism discussed in the previous section and assuming that only one trajectory dominates the asymptotic behaviour of the amplitudes, we obtain the following expressions for the helicity amplitudes :—

$$(1)\ <0\,\tfrac{1}{2} \mid T \mid 0\,\tfrac{1}{2}> = \tfrac{1}{3}T_{1010} + \frac{1}{\sqrt{3}}T_{0010} \equiv \tfrac{1}{3}T^{(1)}_{101} + \frac{1}{\sqrt{3}}T^{(0)}_{001}$$

$$= -4\,\beta^{(1)}_{11}(1, t)\frac{z^{\sigma-2} + z^{-\sigma-2}}{\sigma} - \beta^{(1)}_{01}(0, t)$$

$$\times \left[\frac{\sigma}{\sigma+1}z^{\sigma-1} - \frac{\sigma}{1-\sigma}z^{-\sigma-1}\right]$$

$$(2)\ <0 - \tfrac{1}{2} \mid T \mid 0\,\tfrac{1}{2}> = \frac{\sqrt{2}}{3}\,T_{1110} = \frac{\sqrt{2}}{3} \mid \Phi' \mid^{\frac{1}{2}}\left[\frac{1}{\sqrt{3}}T^{(1)}_{112} + \frac{1}{\sqrt{2}}T^{(1)}_{111}\right]$$

where $T^{(1)}_{112}=-24\sqrt{15}\,\beta^{(1)}_{11}(0,t)\left[\frac{\sigma}{(\sigma+1)(\sigma+2)}z^{\sigma-2}-\frac{\sigma}{(1-\sigma)(2-\sigma)}z^{-\sigma-2}\right]$

$$T^{(1)}_{111}=\frac{-3}{2}\beta^{(1)}_{11}(1,t)\Big[(1-\sigma)z^{\sigma-1}-\frac{8}{\sigma}z^{\sigma-3}+(1+\sigma)z^{-\sigma-1}+\frac{8}{\sigma}z^{-\sigma-3}\Big]$$

(3) $<1\tfrac{1}{2}\mid T\mid 0\tfrac{1}{2}>=\tfrac{1}{3}T_{101-1}+\tfrac{1}{3}T_{001-1}$

$$=\tfrac{1}{3}\mid\Phi'\mid^{\frac{1}{2}}\left(\frac{1}{\sqrt{6}}T^{(1)}_{102}+\frac{1}{\sqrt{2}}T^{(1)}_{101}+\frac{1}{\sqrt{3}}T^{(1)}_{100}\right)$$

$$+\frac{1}{\sqrt{3}}\mid\Phi'\mid^{\frac{1}{2}}\left(\frac{1}{\sqrt{6}}T^{(1)}_{002}+\frac{1}{\sqrt{2}}T^{(1)}_{001}+\frac{1}{\sqrt{3}}T^{(1)}_{000}\right)$$

where

$$T^{(1)}_{102}=2\sqrt{15}\beta^{(1)}_{12}(0,t)\left[\frac{\sigma(1-\sigma)}{(1+\sigma)(2+\sigma)}z^{\sigma-2}-\frac{\sigma(\sigma+1)}{(1-\sigma)(2-\sigma)}z^{-\sigma-2}\right]$$

$$+\tfrac{1}{2}\beta^{(1)}_{12}(1,t)\Big[\frac{\sigma(1-\sigma)(2+\sigma)+(1+\sigma)(2-\sigma)}{\sigma(2+\sigma)}z^{\sigma-2}$$

$$-\frac{(2+\sigma)\,1-\sigma)-6(\sigma+1)(2-\sigma)}{\sigma(2-\sigma)}z^{-\sigma-2}\Big]$$

$$T^{(1)}_{101}=-12\,\beta^{(1)}_{11}(1,t)\,\frac{z^{\sigma-2}+z^{-\sigma-2}}{\sigma}$$

$$T^{(1)}_{100}=-\sqrt{3}\,\beta^{(1)}_{10}(0,t)\left[\frac{\sigma}{1-\alpha}z^{\sigma-1}-\frac{\sigma}{1+\sigma}z^{-\sigma-1}\right]$$

$$T^{(1)}_{002}=3\sqrt{5}\,\beta^{(1)}_{02}(0,t)\left[\frac{\sigma}{(\sigma+1)(\sigma+2)}z^{\sigma-1}-\frac{\sigma}{(1-\sigma)(2-\sigma)}z^{-\sigma-1}\right]$$

$$T^{(1)}_{001}=-\sqrt{3}\,\beta^{(1)}_{01}(0,t)\left[\frac{\sigma}{1+\sigma}z^{\sigma-1}-\frac{\sigma}{1-\sigma}z^{-\sigma-1}\right]$$

$$T^{(1)}_{000}=0$$

(4) $<-1\tfrac{1}{2}\mid T\mid 0\tfrac{1}{2}>=\tfrac{1}{3}T_{1011}+\frac{1}{\sqrt{3}}T_{0011}$

$$=\tfrac{1}{3}\,|\,\Phi'\,|^{\frac{1}{2}}\left[\frac{1}{\sqrt{6}}T^{(1)}_{102}-\frac{1}{\sqrt{2}}T^{(1)}_{101}+\frac{1}{\sqrt{3}}T^{(1)}_{100}\right]$$

$$+\frac{1}{\sqrt{3}}\,|\,\Phi'\,|^{\frac{1}{2}}\left[\frac{1}{\sqrt{6}}T^{(1)}_{002}-\frac{1}{\sqrt{2}}T^{(1)}_{001}\right.$$

$$\left.+\frac{1}{\sqrt{3}}T^{(1)}_{000}\right]$$

where all the amplitudes $T^{(s')}_{j'\lambda s}$ have been defined above.

(5) $$<1\tfrac{-1}{2}\,|\,T\,|\,0\,\tfrac{1}{2}>=\frac{\sqrt{2}}{3}T_{111-1}$$

$$=\frac{\sqrt{2}}{3}\left[\frac{1}{\sqrt{15}}T^{(1)}_{113}+\frac{1}{\sqrt{3}}T^{(1)}_{112}\right.$$

$$\left.+\sqrt{\frac{3}{5}}T^{(1)}_{111}\right]$$

where

$$T^{(1)}_{113}=120\sqrt{21}\,\beta^{(1)}_{13}(0,t)\left[\frac{\sigma^2}{(1+\sigma)(2+\sigma)(3+\sigma)}z^{\sigma-2}\right.$$

$$\left.-\frac{\sigma^2}{(1-\sigma)(2-\sigma)(3-\sigma)}z^{-\sigma-2}\right]+\frac{\sqrt{21}}{2}\beta^{(1)}_{13}(1,t)$$

$$\left[-(1-\sigma)\,z^{\sigma-1}+\frac{516\,(2-\sigma)(3-\sigma)}{\sigma(2+\sigma)(3+\sigma)}z^{\sigma-3}\right.$$

$$\left.+(1+\sigma)\,z^{-\sigma-1}+\frac{576\,(2+\sigma)(3+\sigma)}{\sigma(2-\sigma)(3-\sigma)}z^{-\sigma-3}\right]$$

and other amplitudes have already been defined above.

(6) $$<-1\tfrac{-1}{2}\,|\,T\,|\,0\,\tfrac{1}{2}>=\frac{\sqrt{2}}{3}\,T_{1111}$$

$$=\frac{\sqrt{2}}{3}\beta^{(1)}_{11}(1,t)\left[-(1-\sigma)\,z^{\sigma-1}+\frac{8}{\sigma}z^{\sigma-3}\right.$$

$$\left.+(1+\sigma)\,z^{-\sigma-1}+\frac{8}{\sigma}z^{-\sigma-3}\right]$$

In order that the differential cross section may not blow up, the various residue functions must contain σ factors which occur in the denominators of their coefficients. The corresponding amplitudes $T^{(s')}_{j'\lambda s}$ can then be written as :

$$T^{(1)}_{101} = -12C^{(1)}_{11}(1, t)\left[z^{\sigma-2} + z^{-\sigma-2}\right]$$

$$T^{(0)}_{001} = -\sqrt{3}\,C^{(1)}_{01}(0, t)\left[\sigma(1-\sigma)\,z^{\sigma-1} - \sigma(1+\sigma)\,z^{-\sigma-1}\right]$$

$$T^{(1)}_{112} = -24\sqrt{15}\,C^{(1)}_{11}(0, t)\Big[\sigma(1-\sigma)(2-\sigma)\,z^{\sigma-2} - \sigma(1+\sigma)(2+\sigma)z^{-\sigma-2}\Big]$$

$$T^{(1)}_{111} = -\frac{3}{2}C^{(1)}_{11}(1, t)\Big[\sigma(1-\sigma)z^{\sigma-1} - 8\,z^{\sigma-3} + \sigma(1+\sigma)\,z^{-\sigma-1} - 8\,z^{\sigma-3}\Big]$$

and similarly for others.

If only the leading terms are retained, we get :

$$<0\tfrac{1}{2}\,|\,T\,|\,0\tfrac{1}{2}> = C^{(1)}_{01}\,\sigma(\sigma-1)\,z^{\sigma-1}$$

$$<0\tfrac{-1}{2}\,|\,T\,|\,0\tfrac{1}{2}> = \tfrac{1}{2}\,|\,\Phi'\,|^{\frac{1}{2}}C^{(1)}_{11}(1, t)\,\sigma(\sigma-1)\,z^{\sigma-1}$$

$$<1\tfrac{1}{2}\,|\,T\,|\,0\tfrac{1}{2}> = |\,\Phi'\,|^{\frac{1}{2}}\Big[\tfrac{1}{3}C^{(1)}_{10}(0, t)\,\sigma(\sigma+1) + \sqrt{\tfrac{5}{2}}\,C^{(1)}_{02}(0, t)\,\sigma^2(\sigma-1)(\sigma-2) + \frac{1}{\sqrt{2}}C^{(1)}_{01}(0, t)\,\sigma(\sigma-1)\Big]z^{\sigma-1}$$

$$<-1\tfrac{1}{2}\,|\,T\,|\,0\tfrac{1}{2}> = |\,\Phi'\,|^{\frac{1}{2}}\Big[\tfrac{1}{3}C^{(1)}_{10}(0, t)\,\sigma(\sigma+1) + \sqrt{\tfrac{5}{2}}\,C^{(1)}_{02}(0, t)\,\sigma^2(\sigma-1)(\sigma-2) - \frac{1}{\sqrt{2}}C^{(1)}_{01}(0, t)\,\sigma(\sigma-1)\Big]z^{\sigma-1}$$

$$<1\tfrac{-1}{2} \mid T \mid 0\tfrac{1}{2}> = \Big[\tfrac{1}{8}\sqrt{\tfrac{7}{10}}\,C^{(1)}_{13}(1,t)(\sigma-3)(\sigma-2)\times (\sigma-1)\,\sigma^2(\sigma+2)(\sigma+3)+\sqrt{\tfrac{3}{10}}\,C^{(1)}_{11}(1,t)\times \sigma(\sigma-1)\Big]z^{\sigma-1}$$

$$<-1\tfrac{-1}{2} \mid T \mid 0\tfrac{1}{2}> = \tfrac{1}{2}\,C^{(1)}_{11}(1,t)\,\sigma(\sigma-1)\,z^{\sigma-1}$$

For $\sigma=1$ *i.e.* $\alpha=0$ four amplitudes become zero. However in each of the remaining two amplitudes only one term does not vanish. We therefore expect a dip at $\alpha=0$.

For $\sigma=0$ *i.e.* $\alpha=-1$, all the amplitudes disappear. We, therefore, predict a dip at $\alpha=-1$.

Now in the process $\pi^+ p \rightarrow B^+ p$ three trajectories A_1, A_2, and ω can be exchanged. It has, however, been argued that the ω-trajectory is the dominant one because the particle B^+ appears to be associated only with the particle ω[13]. This trajectory has been parametrized[14] as

$$\alpha_\omega = 0.45 + 0.9t$$

For $\alpha=0$ and $\alpha=-1$ this gives $t=-0.5$ $(GeV/c)^2$ and $t=-1.6$ $(GeV/c)^2$ respectively. Although the experimental curve is not accurate enough, there seems to be such an indication of the dips in the vicinity of the predicted values of t.

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ON CLASSIFICATION BY THE STATISTICS Z AND W

by

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1. Introduction : We consider the situation where an observation x has come from one of the two p-variate normal populations π_1, π_2 with means μ_1, μ_2 and identical covariance matrix Σ, the parameters μ_1 and μ_2 are unknown, and the two random samples drawn from π_1 and π_2 are available to aid classification of x into its relevant population. Two of the classification statistics proposed in discriminant analysis are

$$Z=\frac{N_1}{N_1+1}(x-\bar{x}_1)'\Sigma^{-1}(x-\bar{x}_1)-\frac{N_2}{N_2+1}(x-\bar{x}_2)'\Sigma^{-1}(x-\bar{x}_2)$$

$$W=[x-\tfrac{1}{2}(\bar{x}_1+\bar{x}_2)]'\Sigma^{-1}(\bar{x}_1-\bar{x}_2),$$

where $\bar{x}_1$, $\bar{x}_2$ are the sample means, and N_1 and N_2 are the sizes of samples from π_1 and π_2. The procedure of classification by the first criterion is to assign x to π_1 if $Z \leqslant 0$, and to π_2 if $Z > 0$. According to the second criterion the observation x is assigned to π_1 if $W \geqslant 0$, and to π_2 if $W < 0$.

In following any procedure we can make two kinds of errors in classification ; one is, that we decide to classify x into π_2 while it belongs to π_1 ; the other is, that we decide to classify x into π_1 while it belongs to π_2. This paper employs a large sample approach in investigating probabilities of making such errors which arise due to use of above procedures. A comparative study of these procedures is also made for various values of N_1, N_2, p, D^2 where D^2 is the Mahalanobis distance between π_1 and π_2.

2. Error Probabilities : We write Prob $(\underline{D} \mid x \in \pi_i)$ to indicate the probability of making a decision $\underline{D}$ about classification of x by a given

Received : September 1, 1972.

procedure when x comes from π_i; $i=1, 2$. Suppose, the decision $\underset{\sim}{D}$ is as arising from $Z>0$, then Prob $(Z>0 \mid x \in \pi_i)$ is the probability of classifying by Z the observation $x \in \pi_i$ into π_2. If $i=1$, this becomes an error probability since x is misclassified. The other kind of error probability due to using Z is Prob $(Z \leqslant 0 \mid x \in \pi_1)$. Similar notations will be adopted below in the case of W statistic.

Theorem 1 : If $D>0$, then

(*i*) Prob $(Z>0 \mid x \in \pi_1)$

$$=\left[1-\left(1+\frac{a_1}{N_1}+\frac{a_2}{N_2}+\frac{a_{11}}{N_1^2}+\frac{a_{22}}{N_2^2}+\frac{a_{12}}{N_1N_2}+\ldots\right)\phi(y)\right]_{y=D/2}$$

(*ii*) Prob $(Z \leqslant 0 \mid x \in \pi_2)$

$$=\left[1-\left(1+\frac{a_2}{N_1}+\frac{a_1}{N_2}+\frac{a_{22}}{N_1^2}+\frac{a_{11}}{N_2^2}+\frac{a_{12}}{N_1N_2}+\ldots\right)\phi(y)\right]_{y=D/2}$$

where

$$a_1=\tfrac{1}{2}D^{-2}(d^4+Dd^3+pd^2),$$

$$a_2=\tfrac{1}{2}D^{-2}[d^4+Dd^3-(D^2-p)d^2-D^3d],$$

$$a_{11}=\tfrac{1}{8}D^{-4}[d^8+2Dd^7+(D^2+2p+4)d^6+2(p+2)Dd^5$$
$$-(4D^2-p^2-2p)d^4-4D^3d^3-4pD^2d^2],$$

$$a_{22}=\tfrac{1}{8}D^{-4}[d^8+2Dd^7-(D^2-2p-4)d^6-2(2D^2-p-2)Dd^5$$
$$-\{D^4+2(p+4)D^2-p^2-2p\}d^4+2(D^2-p-4)D^3d^3$$
$$+(D^4+4D^2-4p)D^2d^2+4D^5d],$$

$$a_{12}=\tfrac{1}{4}D^{-4}[d^8+2Dd^7+2(p+2)d^6-2(D^2-p-2)Dd^5$$
$$-\{D^4+(p+4)D^2-p^2-2p\}d^4-(p+4)D^3d^3-2pD^2d^2],$$

and $\phi(y)$ is the cumulative distribution function of N (0, 1).

Proof: Let $F_i(y)$ be the distribution of the random variable $(Z-\mu_i) \mid \sigma_i$ when $x \in \pi_i$ $(i=1, 2)$, the parameters μ_i and σ_i being mean and variance of the asymptotic distribution of Z. It is not difficult to see that $\mu_1=-\mu_2$, $\sigma_1=\sigma_2$. Memon (1970) derives the first part of this theorem from $F_1(y)$.

To prove the second part we suppose that $\bar{F}_1(y)$ is the expression obtained by interchanging N_1 and N_2 in $F_1(y)$. As $Z \rightarrow -Z$ under this operation, we have

$$F_2(y) = \text{Prob}\left(\frac{Z-\mu_2}{\sigma_2} < y\right)$$
$$= 1 - \text{Prob}\left(\frac{-Z+\mu_2}{\sigma_2} < -y\right)$$
$$= 1 - \text{Prob}\left(\frac{-Z-\mu_1}{\sigma_1} < -y\right)$$
$$= 1 - \bar{F}_1(-y).$$

On using this result, we have

$$\text{Prob}(Z \geqslant 0 \mid x \in \pi_2) = 1 - \text{Prob}(Z < 0 \mid x \in \pi_2)$$
$$= 1 - \text{Prob}\left(\frac{Z-\mu_2}{\sigma_2} < -y\right)\Big|_{y=D/2}$$
$$= 1 - F_2(-y)\Big|_{y=D/2},$$
$$= \bar{F}_1(y)\Big|_{y=D/2}.$$

Thus if **Z** is the classification criterion, the interchange of N_1 and N_2 in error probability of one kind gives error probability of other kind.

Theorem 2 : If $D > 0$, then

(*i*) Prob $(W < 0 \mid x \in \pi_1)$

$$= \left(1 + \frac{a_1}{N_1} + \frac{a_2}{N_2} + \frac{a_{11}}{N_1^2} + \frac{a_{22}}{N_2^2} + \frac{a_{12}}{N_1 N_2} + \cdots\right)\phi(y)\Big|_{y=-\frac{D}{2}}$$

(*ii*) Prob $(W \geqslant 0 \mid x \in \pi_2)$

$$= \left(1 + \frac{a_2}{N_1} + \frac{a_1}{N_2} + \frac{a_{22}}{N_1^2} + \frac{a_{11}}{N_2^2} + \frac{a_{12}}{N_1 N_2} + \cdots\right)\phi(y)\Big|_{y=-\frac{D}{2}}$$

where

$$a_1 = \tfrac{1}{2} D^{-2} (d^4 + pd^2 + pDd),$$
$$a_2 = \tfrac{1}{2} D^{-2} [d^4 - 2Dd^3 + (D^2+p) d^2 - pDd],$$
$$a_{11} = \tfrac{1}{8} D^{-4} [d^8 + 2(p+2) d^6 + 2(p+2) Dd^5 + p(p+2) d^4 + 2p(p+2) Dd^3 + p(p+2) D^2 d^2],$$
$$a_{22} = \tfrac{1}{8} D^{-4} [5d^8 - 4Dd^7 + 2(3D^2+p) d^6 - 2(2D^2+3p+6) Dd^5 + \{D^4 + 6(p+2) D^2 + p^2 + 2p\} d^4 - 2\{(p+2) D^2 + p^2 + 2p\} \times Dd^3 + p(p+2) D^2 d^2],$$
$$a_{12} = \tfrac{1}{4} D^{-4} [d^8 - 2Dd^7 + (D^2+2p) d^6 - 2pDd^5 - p(D^2-p) d^4 + pD^3 d^3 - p^2 D^2 d^2],$$

and $\phi(y)$ is the cummulative distribution function of N (0, 1).

Proof : The proof being lengthy, we shall omit it here. However, the reader can prove this result by adopting an approach as used by Okamoto (1963), Memon (1970), and above in Theorem 1.

3. Comparison of Two Procedures : We shall now consider the two cases $N_1=N_2$ and $N_1 \neq N_2$ in comparing the classification procedures based on **Z** and **W** statistics. Since in the sense of relative desirability a better procedure should be one which minimizes the risk of misclassification of the observation x, our comparison of one procedure with the other will take into account both kinds of error probabilities.

Case $N_1=N_2$

When $N_1=N_2=N$. it is easy to observe from the above theorems that

$$\text{Prob}\,(Z>0 \mid x \,\varepsilon\, \pi_1)=\text{Prob}\,(Z\leqslant 0 \mid x \,\varepsilon\, \pi_2),$$

$$\text{Prob}\,(W<0 \mid x \,\varepsilon\, \pi_1)=\text{Prob}\,(W\geqslant 0 \mid x \,\varepsilon\, \pi_2).$$

The terms appearing with $\frac{1}{N}$ as well as $\frac{1}{N^2}$ in the asymptotic expressions of these error probabilities are found to be identical. Also since $1-\phi(-D/2)=\phi(D/2)$, we conclude

$$\text{Prob}\,(Z>0 \mid x \,\varepsilon\, \pi_1)=\text{Prob}\,(W<0 \mid x \,\varepsilon\, \pi_1),$$

$$\text{Prob}\,(Z\leqslant 0 \mid x \,\varepsilon\, \pi_2)=\text{Prob}\,(W\geqslant 0 \mid x \,\varepsilon\, \pi_2),$$

that is, the two criteria are equally good for the purpose of classification of x into its proper population when the samples drawn from π_1 and π_2 are of the same size.

Alternatively, since in this case we notice that

$$Z=\frac{N}{N+1}\,[(x-\overline{x_1})'\,\Sigma^{-1}(x-\overline{x_1})-(x-\overline{x_2})'\,\Sigma^{-1}(x-\overline{x_2})]$$

$$=-\frac{2N}{N+1}W,$$

and on which account,

$$Z>0 \Longrightarrow -\frac{2N}{N+1}\,W>0 \Longrightarrow W<0,$$

$$Z\leqslant 0 \Longrightarrow -\frac{2N}{N+1}\,W\leqslant 0 \Longrightarrow W\geqslant 0,$$

we obtain the same results as above.

Case $N_1 \neq N_2$

The problem of comparison of the two procedures is rather intricate in this case. To facilitate such a study, we have prepared a table which presents overall error probabilities evaluated at $D^2=1,2,3,4,6$; $p=1,2,3,5,10$; keeping $N_1=40$ and varying N_2 over 50,70,100. These probabilities due to the use of Z and W, where unequal, are given in the first and second rows respectively, in front of each value of p. For example, the probabilities of making the two kinds of errors in classification by Z, W when $N_1=40$, $N_2=50$, $D^2=2$, are both equal to 0.33091 at $p=5$, and 0.34383, 0.34386 at $p=10$.

The table reveals some interesting features about the classification procedures based on the statistics Z and W. The probability of making an overall error increases in case of each procedure when (*i*) the Mahalanobis distance D^2 decreases, (*ii*) the dimensionality p enlarges, (*iii*) the sample size decreases. But on comparing one with the other for the classification point of view, we notice that Z generally is slightly better than W at smaller values of D^2, larger values of p and wider differences in N_1 and N_2. This performance in favour of Z improves with increasing D^{-2}, p and or $|N_1-N_2|$. However, when the sample sizes are same, the two criteria show an equal performance whatever the values of other parameters.

Table

Overall error probabilities based on Z and W

	p	D^2 = 1	2	3	4	6
$N_1=40,\ N_2=50$	1	0.61904	0.32002	0.13580	0.04671	0.00285
	2	0.62726	0.32279	0.13679	0.04703	0.00286
	3	0.63500	0.32553	0.13778	0.04734	0.00288
	5	0.64899	0.33091	0.13975	0.04796	0.00292
	10	0.67538 0.67541	0.34383 0.34386	0.14464 0.14465	0.04954	0.00301
$N_1=40,\ N_2=70$	1	0.61879	0.31968	0.13552	0.04656	0.00283
	2	0.62593 0.62594	0.32209	0.13638	0.04683	0.00284
	3	0.63270 0.63272	0.32448	0.13725	0.04710	0.00286
	5	0.64512 0.64519	0.32918 0.32921	0.13896 0.13897	0.04765	0.00289
	10	0.66960 0.66996	0.34053 0.34066	0.14322 0.14328	0.04901 0.04905	0.00297
$N_1=40,\ N_2=100$	1	0.61861	0.31942	0.13531	0.04644	0.00281
	2	0.62494 0.62495	0.32156 0.32157	0.13608	0.04668	0.00283
	3	0.63097 0.63102	0.32368 0.32370	0.13684 0.13685	0.04693	0.00284
	5	0.64216 0.64278	0.32787 0.32793	0.13837 0.13839	0.04741	0.00287
	10	0.66493 0.66569	0.33803 0.33830	0.14217 0.14226	0.04862 0.04865	0.00294

Acknowledgement : The author is grateful to Mr. Abdul Aziz Sabir of the Department of Mathematics and Statistics, West Pakistan Agricultural University, for his assistance in preparing the probability table presented in this paper.

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A NOTE ON CONFORMALLY AND PROJECTIVE RECURRENT SPACES OF SECOND ORDER

by

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1. Introduction : This paper deals with types of Riemannian spaces whose conformal and projective tensors are recurrent with respect to second order covariant derivative. In connection with the study of such spaces we shall give a little introduction of higher order recurrent spaces.

A non-flat Riemannian manifold V_n is called a *recurrent space of order r* if the curvature tensor satisfies the condition

$$R_{hijk,\, lm\cdots n} = a_{lm\cdots n} R_{hijk}$$

where a comma indicates covariant derivative and $a_{lm\cdots n}$ is a non-zero covariant tensor of order r.

We shall denote an n-space of this kind by ${}^rK_n$, and shall call $a_{lm\cdots n}$ the *tensor of recurrence* of the space.

The definition of recurrent spaces of higher order naturally leads us towards finding analogous definitions of other well-known classes of Riemannian and non-Riemannian spaces with curvature restriction. Among them we refer here conformally and projective-recurrent spaces of order r which have the recurrence conditions as follows :

$$C_{hijk,\, lm\cdots n} = a_{lm\cdots n} C_{hijk}$$
$$P_{hijk,\, lm\cdots n} = a_{lm\cdots n} P_{hijk},$$

where C_{hijk} and P_{hijk} are conformal and projective curvature tensors.

An n-dimensional space of these types will be denoted by ${}^rCK_n$ and ${}^rPK_n$ respectively.

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Received on : July 28, 1971.

The definition of second order recurrent spaces follows from that of higher order. It has, however, been our interest to study recurrent spaces of second order. Second order Riemannian recurrent spaces were first considered by A. Lichnerowicz [2]. Later such spaces have been studied in detail by W. Roter ([4], [5]).

In the following an attempt to study conformally recurrent spaces of second order has been made.

1. Conformally Recurrent Spaces of Second Order

The concept of conformally recurrent spaces of higher order gives rise to the definition of conformally recurrent spaces of second order.

Definition 1.1. A Riemannian space V_n $(n>3)$ whose conformal curvature tensor C_{hijk} defined by

$$C_{hijk}=R_{hijk}+\frac{1}{n-2}(g_{hj}R_{ik}-g_{hk}R_{ij}+g_{ik}R_{hj}-g_{ij}R_{hk})$$
$$+\frac{R}{(n-1)(n-2)}(g_{hk}g_{ij}-g_{hj}g_{ik}) \qquad (1.1)$$

satisfies

$$C_{hijk,\,lm}=a_{lm}C_{hijk} \qquad (1.2)$$

for some non-zero tensor a_{lm}, will be called a *conformally recurrent space of second order*, and we shall denote such a space by ^2CK_n.

Based on the properties of CK_n – spaces [1] somewhat similar nature of ^2CK_n – spaces may be observed. We are able to demonstrate some theorems on ^2CK_n – spaces analogous to theorems [1] on CK_n – spaces. We shall prove the following

Theorem 1.1. If a ^2CK_n – space which is *not flat* and satisfies $R_{ij}=0$, then the space is ^2K_n, and the *scalar* $g^{rs}a_{rs}=0$ *vanishes.*

Proof : Let us assume that a ^2CK_n – space satisfies $R_{ij}=0$;
Then from (1.2) we have

$$R_{hijk,\,lm}=a_{lm}R_{hijk}.$$

Thus the space is ^2K_n.
Next, because of the Bianchi identity and the above recurrence condition, we have

$$a_{lm}R_{hijk}+a_{jm}R_{hikl}+a_{km}R_{hilj}=0 \qquad (1.3)$$

Multiplying this with g^{hl} and using $R_{ij} = 0$, we get

$$a_{lm}R^{l}{}_{ijk} = 0 \quad \text{.. (1.4)}$$

Multiplying again (1.3) with g^{lm} we get, because of (1.4),

$$\theta_o R_{hijk} = 0, \text{ where we have put } \theta_o = g^{lm}a_{lm}.$$

Since the space is assumed to be non-flat, we have $\theta_o = 0$.
This completes the proof of the theorem.

2. Einstein ^2CK_n-spaces

The following theorems hold for Einstein ^2CK_n—spaces.

Theorem 2.1 If a ^2CK_n $(n > 3)$ is an Einstein space, then either the space has constant curvature or the scalar $g^{rs}a_{rs} = \theta_o$ vanishes and the tensor of recurrence is symmetric.

Proof: Let a ^2CK_n-space be an Einstein one. Then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n} g_{ij}$$

Since $n > 3$, R is constant, and therefore

$$R_{,l} = 0, \; R_{ij,\,l} = 0.$$

Hence it follows from (1.2) that

$$R_{hijk,\,lm} = a_{lm} \left[R_{hijk} - \frac{R}{n(n-1)}(g_{hk}\, g_{ij} - g_{hj}\, g_{ik})\right] \quad \text{.. (2.1)}$$

On applying Walker's lemma 1, [6], we get, on account of (2.1),

$$b_{lm}T_{hijk} + b_{hi}\, T_{jklm} + b_{jk}\, T_{lmhi} = 0$$

where $\quad b_{lm} = a_{lm} - a_{ml}, \; T_{hijk} = R_{hijk} - \dfrac{R}{n(n-1)}\,(g_{hk}\, g_{ij} - g_{hj}\, g_{ik})$

From Walker's lemma 2, [6], it follows that

either $b_{lm} = 0$, *i.e.* $a_{lm} = a_{ml}$

or $\quad T_{hijk} = 0$ *i.e.* $R_{hijk} = \dfrac{R}{n(n-1)}\,(g_{hk}g_{ij} - g_{hj}\, g_{ik})$.

Thus an Einstein ^2CK_n $(n > 3)$ has either *constant curvature* or it has *symmetric tensor of recurrence*.

Next, as a consequence of the Bianchi identity and the relation (2.1) we obtain

$$a_{lm}\, T_{hijk} + a_{jm}\, T_{hikl} + a_{km}\, T_{hilj} = 0$$

Now assumlng the space to be *not* of constant curvature, the proof of the *vanishing of* θ_0 follows closely that of the method adopted in the foregoing theorem.

Theorem 2.2 An Einstein ^2CK_n is ^2K_n when and only when the scalar curvature vanishes.

Proof : The proof follows in a manner similar to that used in [1].

3. Ricci-symmetric ^2CK_n

Ricci-symmetric spaces are more general than Einstein spaces. A Ricci-symmetric n-space is a Riemannian n-space which satisfies $R_{ij,l}=0$.

Proceeding as in the Proof of Theorem 2.1 and remembering that $C_{ij} = g^{hk} C_{hijk} = 0$, we are able **to** give a theorem concerned with Riccl-symmetric ^2CK_n. We thus have

Theorem 3.1 If a ^2CK_n-space is Ricci-symmetric, then either the space is conformally flat or the scalar θ_0 vanishes and the tensor of recurrence is symmetric.

4. Projective-Recurrent Spaces of Second Order

Definition 4.1 A Riemannian V_n $(n > 2)$ is said to be projective-recurrent space of second order if the projective curvature tensor defined by

$$P_{hijk} = R_{hijk} - \frac{1}{n-1}(g_{hk} R_{ij} - g_{hj} R_{ik}) \quad .. \quad (4.1)$$

satisfies

$$P_{hijk,lm} = a_{lm} P_{hijk} \quad .. \quad (4.2)$$

where a_{lm} is a non-zero tensor.

We adopt our notation ^2PK_n to represent second order projective-recurrent space.

In view of somewhat similar behaviour of ^2PK_n some of its results may be brought out as those of ^2CK_n, and so we do not consider to outline proofs of those than merely give the statements. The idea of establishing some theorems for ^2PK_n-spaces is originated from the study of projective-recurrent spaces [3] as well.

5. The tensor of recurrence

We shall prove the following

Theorem 5.1 In a ^2PK_n-space the tensor of recurrence is symmetric.

Proof : From (4.2) we have

$$R_{hijk,\,lm} - \frac{1}{n-1}(g_{hk} R_{ij,\,lm} - g_{hj} R_{ik,\,lm}) = a_{lm} R_{hijk} - \frac{1}{n-1} a_{lm} (g_{hk} R_{ij} - g_{hj} R_{ik}) \quad \ldots \quad (5.1)$$

Transvecting (5.1) with g^{ij} we get

$$R_{hk,\,lm} - \frac{1}{n-1}(g_{hk} R_{,\,lm} - R_{hk,\,lm}) = \frac{n}{n-1} a_{lm} (R_{hk} - \frac{R}{n} g_{hk})$$

$$\text{or} \quad R_{hk,lm} - \frac{R_{,\,lm}}{n} g_{hk} = a_{lm} (R_{hk} - \frac{R}{n} g_{hk}) \quad \ldots \quad (5.2)$$

As usual convention we put $P_{hk} = g^{ij} P_{hijk} = \frac{n}{n-1}(R_{hk} - \frac{R}{n} g_{hk}) \quad \ldots \quad (5.3)$

It, therefore, follows from (5.2) that

$$P_{hk,lm} = a_{lm} P_{hk} \quad \ldots \quad (5.4)$$

From (5.3) we find

$$P_{hk,\,lm} - P_{hk,ml} = \frac{n}{n-1}(R_{hk,lm} - R_{hk,ml})$$

This together with (5.4) gives

$$R_{hk,\,lm} - R_{hk,\,ml} = \frac{n-1}{n} b_{lm} P_{hk} \quad \ldots \quad (5.5)$$

where $b_{lm} = a_{lm} - a_{ml}$.

We now have

$$P_{hijk,\,lm} - P_{hijk,\,ml} + P_{jklm,\,hi} - P_{jklm,\,ih} + P_{lmhi,\,jk} - P_{lmhi,\,jh}$$

$$= [R_{hijk,\,lm} - R_{hijk,\,ml} + R_{jklm,\,hi} - R_{jklm,\,ih} + R_{lmhi,\,jk} - R_{lmhi,\,kj}] - \frac{1}{n-1}[g_{hk}(R_{ij,\,lm} - R_{ij,\,ml}) - g_{hj}(R_{ik,\,lm} - R_{ik,ml})$$

$$+ g_{jm}(R_{kl,\,hi} - R_{kl,\,ih}) - g_{jl}(R_{km,\,hi} - R_{km,\,ih}) + g_{li}(R_{mh,\,jk} - R_{mh,\,kj}) - g_{lh}(R_{mi,\,jk} - R_{mi,\,kj})]$$

From Walker's lemma 1, [6], the first term of the right hand member is zero. Now by use of (4.2) and (5.5) we have

$$b_{lm}[P_{hijk} + \frac{1}{n}(g_{hk}P_{ij} - g_{hj}P_{ik})] + b_{hi}[P_{jklm} + \frac{1}{n}(g_{jm}P_{kl} - g_{jl}P_{km})] + b_{jk}[P_{lmhi} + \frac{1}{n}(g_{li}P_{mh} - g_{lh}P_{mi})]$$

$$= 0$$

This equation reduces to

$$b_{lm}T_{hijk}+b_{hi}T_{jklm}+b_{jk}T_{lmhi}=0$$

where $T_{hijk}=P_{hijk}+\frac{1}{n}(g_{hk}P_{ij}-g_{hj}P_{ik})$

$$=R_{hijk}-\frac{R}{n(n-1)}(g_{hk}g_{ij}-g_{hj}g_{ik}).$$

Obviously, $T_{hijk}=T_{jkhi}$ and since the space is not of constant curvature, T_{hijk} are not all zero.

Thus it follows from Walker's lemma 2, [6], that $b_{lm}=0$.

That is, $a_{lm}=a_{ml}$.

Next we shell state without proof the following

Theorem 6.1. If a *non-flat* ^2PK_n satisfies $R_{ij}=0$, then the spaee is ^2K_n and *the scalar* θ_0 *vanishes.*

7. Einstein ^2PK_n

The following theorems which hold for Einstein ^2PK_n are stated without proof.

Theorem 7.1. If a ^2PK_n is an Einstein space, then either the scalar θ_0 is zero or ^2PK_n reduces to the space of constant curvature.

Theorem 7.2. A necessary and sufficient condition that an Einstein ^2PK_n be ^2K_n is that the scalar curvature vanishes.

8. 2-Ricci-recurrent ^2PK_n

A ^2PK_n $(n>2)$ will be called 2-Ricci-recurrent if it satisfies

$R_{ij,lm}=\overset{*}{a}_{lm}R_{ij}$ for non zero tensors $\overset{*}{a}_{lm}$ and R_{ij}.

We shall prove the following

Lemma 8.1. If a ^2PK_n satisfies $R_{ij,lm}=\overset{*}{a}_{lm}R_{ij}$ for non-zero tensors $\overset{*}{a}_{lm}$ and R_{ij}, then $\overset{*}{a}_{lm}$ is symmetric.

Proof : Since in a ^2PK_n the tensor of recurrence is symmetric, we have

$$P_{hijk,lm}-P_{hijk,ml}=0.$$

Transvecting this w th g^{ij} we get

$$P_{hk,lm}-P_{hk,ml}=0 \tag{8.1}$$

where P_{hk} has already been defined by (5.3).
By virtue of (5.3) it follows from (8.1) that

$$R_{hk,\,lm} - R_{hk,\,ml} = 0.$$

we therefore see that

$$(\overset{*}{a}_{lm} - \overset{*}{a}_{ml})\, R_{hk} = 0.$$

Since R_{hk} is non-zero, we have

$$\overset{*}{a}_{lm} = \overset{*}{a}_{ml}.$$

Remark : The representation of the scalar $\theta_{\circ} = g^{rs} a_{rs}$ was first given by W. Roter ([4], [5]) and later used by some authors. The problem of obtaining its geometrical interpretation still remains unsolved. We leave this problem for our future research.

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