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A NOTE ON NON-COMMUTATIVE POLYNOMIAL RINGS

by

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1. Introduction

We shall suppose throughout this paper that all rings under consideration are non-associative with unity element different from zero.

Let R be a ring and $f_0, f_1, \ldots\ldots f_k$ $(k \geqslant 1)$ the $(k+1)$ abelian group endomorphisms of R subject to the restrictions :

$$f_1\ (1) = 1 \text{ and } f_i\ (1) = 0\ (i \neq 1).$$

In [4, Theorem 2.1] it was proved that there exists a polynomial ring R $[x]$ under a multiplication which satisfies

$$x\,a = \sum_{0}^{k} f_i\,(a)\ x^i \quad \ldots\ldots\ldots\ldots (1)$$

for all a ϵ R.

The question of associativity of R$[x]$ was also taken up in [4]. In this connection an attempt was made to find necessary and sufficient conditions so that R $[x]$ might reduce to the polynomial ring essentially defined first by O. ore in [1]. It was found out that, if R is an associative integral domain and f_k is a monomorphism, then R $[x]$ is associative if and only if $k=1$, f_1 is a ring homomorphism, and f_0 is an f_1 – derivation (i.e., f_0 (a b)$=f_0$ (a) $b+f_1$ (a)f_0 (b) for all a, $b \,\epsilon$ R).

Here in this short note we would like to show that this result is valid for any associative ring with unity element, not necessarily an integral domain.

2. The Theorem

First we shall record here some definitions and results of [4]. We would like to follow the notations of [2] for the sake of convenience.

If $n \geqslant 0$, $0 \leqslant r \leqslant nk$ and $a \in R$ we shall write $a_{n,r}$ for

$$\sum_{\alpha_1, \alpha_2, \ldots, \alpha_n}^{\infty} a_{f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n}},$$

where each index is sumined up from 0 to k under the condition $r = \sum_1^n \alpha i$, and $a_{f_i f_j}$ stands for $f_i\,(f_j\,(a))$. Furthermore,

$$\left.\begin{array}{l} a_{n,r} = 0 \text{ if } r > nk \quad \text{or} \quad r < 0, \\ a_{0,r} = 0 \text{ if } r \neq 0, \text{ and } a_{0,0} = a \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let $P = \sum_0^{\infty} R_i$ ($R_i = R$ for each i) be the direct sum of abelian groups R_i. Define the multiplication in P by

$$f\,g = (A_0,\ A_1,\ \ldots\ldots,\ A_{lk+m},\ 0,\ 0,\ \ldots\ldots),$$

where

$$f = (a_0,\ a_1,\ \ldots\ldots,\ a_l,\ 0,\ 0,\ \ldots\ldots)$$

$$g = (b_0,\ b_1,\ \ldots\ldots,\ b_m,\ 0,\ 0,\ \ldots\ldots)$$

are any two elements of P, and

$$A_u = \sum_{r=0}^{u} \left\{ \sum_{i=0}^{l} a_i\ (b_r)_{i,u-r} \right\}$$

Taking

$$x = (0,\ 1,\ 0,\ 0,\ \ldots\ldots\ldots)$$

we can write

$$f = \sum_0^{\infty} a_i\ x^i\ (a_i = 0 \text{ for } i > l).$$

Then (see [4. Theorem 2.1]) $P = R\,[x]$ is a ring satisfying (1) and x is left indeterminate over R. Furthermore, if R is associative, then P is associative and if only if, for $a, b \in R$,

$$\sum_{r=0}^{w} f_{w-r}(a\, b_{j,r}) = \sum_{\alpha=0}^{k} f_\alpha(a)\, b_{\alpha+j,w} \quad (j \geqslant 0,\ 0 \leqslant w \leqslant (j+1)\,k) \ldots (3)$$

$$\sum_{\alpha=0}^{k} f_\alpha(a)\, b_{\alpha+j,w} = 0 \text{ for } w > (j+1)\ k.$$

We are now ready to prove the

Theorem

Let R be an associative ring with unity element $1 \neq 0$ and f_k is a monomorphism. Then P is associative if and only if $k=1, f_1$ is a ring homomorphism, and f_0 is an f_1-derivation.

Proof :

Suppose that **P** is associative and $k \neq 1$. Then, since f_k is a monomorphism,

$$f_k(1) = 0 \implies 1 = 0,$$

which is contradiction. Hence $k=1$, and putting $j=0$, $k=1$ in (3) we obtain

$$\sum_{r=0}^{w} f_{w-r}(a\, b_{o,r}) = f_o(a)\, b_{o,w} + f_1(a)\, b_{1,w} = f_w(a\, b) \ldots\ldots (4)$$

in view of (2). Writing $w=1$, 0 respectively in (4) we get

$$f_1(a\, b) = f_o(a)\, b_{o,1} + f_1(a)\, b_{1,l}\,,$$

$$f_o(a\, b) = f_o(a)\, b_{o,o} + f_1(a)\, b_{1,o}\,.$$

Since $b_{o,1} = 0$, $b_{1,l} = f_1(b)$, $b_{1,o} = f_o(b)$ it follows that f_1 is a ring homomorphism and f_o is an f_1-derivation.

The converse is proved in [3, Theorem 1.2].

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A NOTE ON AN INTEGRAL INVOLVING BESSEL FUNCTIONS

by

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In this short note we discuss an integral operator

$$T^{\alpha}_{\nu,\lambda}(f)(x) = \int_0^{\infty} (x\ y)^{-\frac{1}{2}(\frac{1}{2}+\alpha)} W^{\alpha}_{\nu,\lambda}(4\sqrt{x\,y})\, f(y)\, dy, \tag{1}$$

where $f \in L^2(0, \infty)$ and

$$W^{\alpha}_{\nu,\lambda}(x) = x^{\frac{1}{2}} \int_0^{\infty} J_{2\nu+\alpha}(t)\, J_{2\lambda+\alpha}(x/t)\, dt/t, \tag{2}$$

J_σ is the Bessel function of the first kind.

When $\alpha=0$ (1) has been studied by Watson [6] Bhatnagar [1] and Srivastava [5]. An integral of this type has also been considered by Lowndes [4]. For the detailed study of $W^{\alpha}_{\nu,\lambda}$ see [1, 5].

The following lemma which is a special case of a theorem due to Schur [8, Theorem 319] will be needed later.

Lemma 1.

Let $f \in L^2(0, \infty)$ and let g be a function defined on $(0, \infty) \times (0, \infty)$ such that for any real number $a>0$

$$g(\alpha\, x,\ \alpha\, y) = |\, a\,|^{-1}\, g(x, y),$$

and

$$\int_0^\infty | g(x, 1) | x^{-\frac{1}{2}} dx = k < \infty.$$

Then if

$$h(x) = \int_0^\infty g(x, y) f(y) dy \tag{3}$$

$$h : L^2(0, \infty) \rightarrow L^2(0, \infty),$$

and

$$\|h\|_2 \leqslant k \|f\|_2. \tag{4}$$

We shall also use the Kober's operators of fractional integrals introduced in [3] and defined by

$$I_\nu^\alpha (f)(x) = \frac{x^{-\nu-\alpha}}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} y^\nu f(y) dy, \tag{5}$$

$$K_\nu^\alpha (f)(x) = \frac{x^\nu}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} y^{-\nu-\alpha} f(y) dy. \tag{6}$$

Lemma 2.

If $f \in L^2(0, \infty)$, $\alpha > 0$, $\nu > -\frac{1}{2}$, then I_ν^α and $K_\nu^\alpha (f)$ belong to $L^2(0, \infty)$. Also

$$m_s\left(I_\nu^\alpha (f)\right) = \frac{\Gamma(\nu - s + 1)}{\Gamma(\alpha + \nu - s + 1)} m_s(f) \tag{7}$$

and

$$m_s\left(K_\nu^\alpha (f)\right) = \frac{\Gamma(\nu + s)}{\Gamma(\alpha + \nu + s)} m_s(f), \tag{8}$$

where m_s denotes the Mellin transform.

Lemma 3.

If $\alpha > 0$, $\nu > -\frac{1}{2}$, $\lambda > -\frac{1}{2}$, then

$$x^{-\frac{1}{2}(\frac{1}{2}+\alpha)} W^{\alpha}_{\nu,\lambda} (4\sqrt{x}) \in L^2 (0, \infty).$$

Proof. We write

$$x^{-\frac{1}{2}(1+\alpha)} W^{\alpha}_{\nu,\lambda} (x)$$

$$= 2\, x^{-\frac{1}{2}\alpha} \int_0^\infty J_{2\nu+\alpha} (t)\, J_{2\lambda+\alpha} \left(\frac{4\sqrt{x}}{t}\right) dt/t$$

$$= \int_0^\infty t^{-1} (t/x)^{\frac{1}{2}\alpha} J_{2\lambda+\alpha} (4\sqrt{x}/t)\, t^{-\frac{1}{2}\alpha} J_{2\nu+\alpha} (t)\, dt.$$

Use the asymptotic formulae

$$J_\sigma (x)=0\left(x^{-\frac{1}{2}}\right), \text{ when } x \text{ is large,}$$

and

$$J_\sigma (x)=0\left(x^{\sigma}\right), \text{ when } x \text{ is small}$$

to see that $\int_0^\infty \left(t^{-\frac{1}{2}\alpha} J_{2\nu+\alpha} (t)\right)^2$ and $\int_0^\infty t^{-\frac{1}{2}} t^{\frac{1}{2}\alpha-1} J_{2\lambda+\alpha}\left(\frac{1}{\sqrt{t}}\right) dt$

are finite. The result now follows from Lemma 1.

Theorem 1.

If $\alpha > 0$, $\nu > -\frac{1}{2}$, $\lambda > -\frac{1}{2}$ and $f \in L^2 (0, \infty)$, then $T^{\alpha}_{\nu,\lambda} (f)$:

$L^2 (0, \infty) \rightarrow L^2 (0, \infty)$ and there is a finite constant such that

$$\| T^{\alpha}_{\nu,\lambda} (f) \|_2 \leqslant k \, \| f \|_2$$

Moreover

$$m_s\left(T^{\alpha}_{\nu,\lambda} (f)\right) = \frac{\Gamma (\nu+s)}{\Gamma (\nu+\alpha-s+1)} \; \frac{\Gamma (\lambda+s)}{\Gamma (\lambda+\alpha-s+1)}\, m_{(1-s)} (f) \quad (10)$$

Proof : Define

$$V(f)(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \tag{11}$$

so that

$$\| V(f) \|_2 = \| f \|_2, \quad V\{V(f)\} = f, \tag{12}$$

and

$$m_s(V(f)) = m_{(1-s)}(f). \tag{13}$$

Also set

$$s_{\alpha,\nu}(f)(x) = \int_0^\infty (xy)^{-\frac{1}{2}\alpha} J_{2\nu+\alpha}(2\sqrt{xy}) f(y)\, dy. \tag{14}$$

It is known [3] that if $\alpha > 0$, $\nu > -\frac{1}{2}$, $S_{\alpha,\nu}(f) \in L^2(0, \infty)$ and

$$m_s(S_{\alpha,\nu}(f)) = \frac{\Gamma(\nu+s)}{\Gamma(\nu+\alpha+1-s)} m_{(1-s)}(f). \tag{15}$$

Now on changing in (1) the order integration which is permissible due to Lemma 3 we see that

$$T^{\alpha}_{\nu,\lambda}(f) = S_{\alpha,\nu} V S_{\alpha,\lambda}(f). \tag{16}$$

Since $S_{\alpha,\lambda}(f) \in L^2(0, \infty)$ and $V S_{\alpha,\nu}(f) \in L^2(0, \infty)$ it immediately follows that

$$T^{\alpha}_{\nu,\lambda} : L^2(0, \infty) \to L^2(0, \infty),$$

and there is a finite constant k such that

$$\| T^{\alpha}_{\nu,\lambda}(f) \|_2 \leqslant k \| f \|_2.$$

Furthermore using (13) and (16) we obtain.

$$m_s\left(T^{\alpha}_{\nu,\lambda}(f)\right) = \frac{\Gamma(\nu+s)}{\Gamma(\nu+\alpha-s+1)} \frac{\Gamma(\lambda+s)}{\Gamma(\nu+\alpha-s+1)} m_{(1-s)}(f)$$

Theorem 2.

Let $\alpha > 0$, $\nu > -\frac{1}{2}$, $\lambda > -\frac{1}{2}$. If $f \in L^2(0, \infty)$ and $g \in L^2(0, \infty)$, then

$$\int_0^\infty f(x) T^{\alpha}_{\nu,\lambda}(g)(x)\, dx = \int_0^\infty g(t) T^{\alpha}_{\nu,\lambda} f(t)\, dt. \tag{18}$$

The result follows immediately by changing the order of integration which is permissible by Fubini's Theorem.

Theorem 3.

If $\alpha > 0$, $\nu > -\frac{1}{2}$, $\lambda > -\frac{1}{2}$ and $f \in L^2(0, \infty)$ then

$$T^{\alpha}_{\nu,\lambda} \; T^{\alpha}_{\nu,\lambda}(f) = K^{\alpha}_{\nu} \; K^{\alpha}_{\lambda} \; I^{\alpha}_{\nu} \; I^{\alpha}_{\lambda}(f) \tag{19}$$

Proof : Since $f \in L^2(0, \infty)$, $T^{\alpha}_{\nu,\lambda} \; T^{\alpha}_{\nu,\lambda}(f) \in L^2(0, \infty)$.

Applying Theorem 1 and Lemma 2, it follows that

$$m_s\left(T^{\alpha}_{\nu\,\lambda} \; T^{\alpha}_{\nu,\lambda}(f)\right)$$

$$= \frac{\Gamma(\nu+s)}{\Gamma(\nu+\alpha-s+1)} \; \frac{\Gamma(\lambda+s)}{\Gamma(\lambda+\alpha-s+1)} m_{(1-s)}\left(T^{\alpha}_{\nu\,\lambda}(f)\right)$$

$$= \frac{\Gamma(\nu+s)}{\Gamma(\nu+\alpha-s+1)} \; \frac{\Gamma(\lambda+s)}{\Gamma(\lambda+\alpha-s+1)} \; \frac{\Gamma(\nu-s+1)}{\Gamma(\nu+\alpha+s)} \; \frac{\Gamma(\lambda-s+1)}{\Gamma(\lambda+\alpha+s)} m_s(f)$$

$$= \frac{\Gamma(\nu+s)}{\Gamma(\nu+\alpha+s)} \; \frac{\Gamma(\lambda+s)}{\Gamma(\lambda+\alpha+s)} \; \frac{\Gamma(\nu-s+1)}{\Gamma(\nu+\alpha-s+1)} \; \frac{\Gamma(\lambda-s+1)}{\Gamma(\lambda+\alpha-s+1)} m_s(f)$$

$$= m_s\left(K^{\alpha}_{\nu} \; K^{\alpha}_{\lambda} \; I^{\alpha}_{\nu} \; I^{\alpha}_{\lambda}(f)\right).$$

Since both side of the last equation belong to $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ an application of inverse Millin transform proves the result. Similarly we can prove

Theorem 4.

If $\alpha>0$, $\beta>0$, $\nu>-\frac{1}{2}$, $\lambda>-\frac{1}{2}$ and $f \in L^2(0, \infty)$, then

$$T^{\alpha}_{\nu+\beta,\lambda+\beta} \; T^{\beta}_{\nu,\lambda}(f) = I^{\alpha+\beta}_{\nu} \; I^{\alpha+\beta}_{\lambda}(f) \tag{20}$$

and

$$T^{\alpha}_{\nu,\lambda} \; T^{\beta}_{\nu+\alpha,\lambda+\alpha}(f) = K^{\alpha+\beta}_{\nu} \; K^{\alpha+\beta}_{\lambda}(f) \tag{21}$$

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ASYMPTOTIC EXPANSION OF MIXED POISSON DISTRIBUTION AND THE POSTERIOR DISTRIBUTION OF POISSON PARAMETER.*

by

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Abstract :—Asymptotic expansions are derived for the mixed Poisson distribution and for the posterior distribution of the Poisson parameter, assuming that the prior distribution has continuous derivatives in its domain of variation. Expansions are also obtained for the moments of the posterior distribution. The theory is illustrated by means of a Gamma prior. It is shown that the appoximations are valid for uninformative prior density.

1. Introduction :

Let (X_1, X_2.............X_n) be a random sample from the Poisson distribution with density :

$$f(x\,u) = e^{-\mu}\,\mu^{x}/x\,!,\ x = 0, 1, 2, \ldots\ldots\ldots$$

It is known that all the information contained in the sample is given by $Y = \sum_i X_i$, which has probability density function

$$h\,(y, n/\mu) = e^{-n\mu}\,(n\,\mu)^{y}/y\,!$$

Suppose that degree of belief about μ consists of the density $g\,(\mu)$, $0 \leqslant \mu < \infty$, which has continuous derivatives in its domain of variation. In many contexts, it is important to know the behaviour of the mixed Poisson probability density function

$$k_g\ (y,\ n) = \int_0^{\infty} \frac{e^{-n\mu}(n\,\mu)^{y}}{y\,!}\,g(\mu)\,d\mu \qquad (1)$$

* Received : Oct. 1, 1971.

and the corresponding distribution function

$$K_g(y, n) = \sum^{y} k_g(x, n), \tag{2}$$

as the sample size is increased. It is also important to know how the increased sample size affects the degree of belief about μ. That is to say, what happens to the posterior density

$$h(\mu \mid y, n) = \frac{e^{-n\mu}(n\mu)^y g(\mu)}{y!\, k_g(y, n)} \tag{3}$$

when n gets large ?

One way to investigate this problem is to express (1), (2) and (3) in forms which show the effect of variation in the values of n. The present paper attempts to do this. We find asymptotic expansions of (1), (2), (3) and of the moments of (3).

2. Mixed Poisson distribution :

Let $u = n^{-1}y$, $g^{(i)}(x) = d^i g/dx^i$, and $r_i = g^{(i)}(u)/g(u)$, $i = 1, 2.......$ For any fixed value of u, we have by Taylor's theorem

$$g(\mu) = g(u)1 + \sum_{i=1}^{4} (\mu - u)^i r_i/i! + \frac{1}{120}(\mu - u)^5\, g^{(5)}(z)/g(u) \tag{4}$$

where $z = \mu + \theta(\mu - u)$, $0 < \theta < 1$.

Now, it is easy to see that

$$\int_0^\infty \frac{e^{-n\mu}(n\mu)^y}{y!} d\mu = 1/n, \int_0^\infty \frac{e^{-n\mu}(n\mu)^y u^i}{y!} d\mu = n^{-(i+1)}(y+i)\, i = n^{-(i+1)}$$

$$(y+1) \times (y+2) \dots (y+i),$$

so that

$$\int_0^\infty (\mu - u)\frac{e^{-n\mu}(n\mu)^y}{y!} d\mu = n^{-2}, \int_0^\infty (\mu - u)^2 \frac{e^{-n\mu}(n\mu)^y}{y!} d\mu = n^{-2}u + 2n^{-3},$$

$$\int_0^\infty (\mu-u)^3 \frac{e^{-n\mu}(n\mu)^y}{y!} d\mu = 5n^{-3}u + 6n^{-4}, \int_0^\infty (\mu-u)^4 \frac{e^{-n\mu}(n\mu)^y}{y!} d\mu$$

$$= 3n^{-3}u^2 + 2\,6n^{-4}u + 24n^{-5},$$

$$\int_0^\infty (\mu-u)^5 \frac{e^{-n\mu}(n\mu)^y}{y!} d\mu = 0\,(n^{-4}).$$

Using these results and (4), we obtain after some calculations

$$k_g(y, n) = n^{-1} g(u)\{1 + a_1 n^{-1} + a_2 n^{-2} + 0\,(n^{-3})\}, \tag{5}$$

where $a_1 = r_1 + \frac{1}{2}ur_2$, $a_2 = r_2 + \frac{5}{6} r_3 u + \frac{1}{8} r_4 u^2$.

Again, introducing the incomplete gamma function

$$\frac{1}{y!}\int_t^\infty e^{-z} z^y dz = \sum_{x=0}^{y} \frac{e^{-t} t^x}{x!}$$

we obtain

$$K_g(y, n) = \sum_{x=0}^{y} \int_0^\infty \frac{e^{-n\mu}(n\mu)^x}{x!} g(\mu)\, d\mu$$

$$= \int_0^\infty \left\{ \int_{n\mu}^\infty \frac{e^{-v} v^y}{y!} dv \right\} g(\mu)\, d\mu$$

$$= n\int_0^\infty \frac{e^{-n\mu}(n\mu)^y}{y!} G(\mu)\, d\mu$$

$$= G(u)\{1 + n^{-1} A_1 + n^{-2} A_2 + 0\,(n^{-3})\},$$

where $G(u) = \int_0^u g(v)\, dv$, $A_1 = \{g(u)/G(u)\}\{1 + \frac{1}{2} ur_1\}$,

and $A_2 = \{g(u) / G(u)\}\{r_1 + \frac{5}{6} ur_2 + \frac{1}{8} u^2 r_3\}$.

Obviously, $(n\, k_g(y, n) - g(u)] \to 0$, and $k_g(y, n) \to G(u)$, as $n \to \infty$. Similar results have been obtained for binomial distribution by Hald (1968).

As an example, consider gamma prior density

$g(u) = [\Gamma(s)]^{-1} w^s e^{-w\mu} \mu^{s-1} \quad u \geqslant 0, w > 0$

It is well known that the mixed Poisson distribution, in this case, has density

$$k_g(y, n) = \binom{s+y-1}{y} [w / (w+n)]^s \quad [n / w+n)]^y ,$$

which can be expanded as (5) with

$$a_1 = \tfrac{1}{2} s(s-1) u^{-1} - ws + \tfrac{1}{2} w^2 u$$

$$a_2 = \tfrac{1}{24} s(s-1)(s-2)u^{-2} - \tfrac{1}{2} ws^2 (s-1)u^{-1} + \tfrac{1}{4} w^2 s(3s+1) - \tfrac{1}{6} w^3 (3s+2) u + \tfrac{1}{8} w^4 u^2.$$

The expansions (5) and (6) held irrespective of whether the prior distribution has existing moments or not. For example, the truncated Cauchy prior

$$g(u) = (2/\Pi)(1+\mu^2)^{-2} , \quad 0 < u < \infty$$

has no existing moment, but we can find asymptotic expansion for $k_g(y, n)$ *and* $K_g(y, n)$.

The moments of the mixed Poisson distribution can be obtained readily without using (5). For the existence of these moments, it is necessary that the corresponding moments of the prior distribution should exist. For instance, denoting the moments of $k_g(y,n)$ by m_{yr} $(r=1, 2, \ldots)$, we have

$$m_{y1} = \Sigma\, y\, k_g(y, n)$$

$$= \Sigma \int_0^\infty \frac{y (n\mu)^y}{y!} e^{-n\mu} g(\mu)\, d\mu$$

$$= n \int_0^\infty \mu\, g(\mu) d\mu$$

$$m_{y2} = \Sigma\, y^2 k_g(y, n) = \Sigma\, y(y-1) k_g(y, n) + \Sigma y\, k_g(y, n)$$

$$= n^2 \int_0^\infty \mu^2 g(\mu)\, d\mu + n \int_0^\infty \mu\, g(\mu) d\mu,$$

from which it is obvious that, if the first and second moments of $g(\mu)$ exist. exist, m_{y1} and m_{y2} also exist. Further, the variance σ_y^2 of the distributions is given by

$$\sigma_y^2 = n^2 \text{ (variance of prior distribution)} + \text{n (mean of the prior distribution)}.$$

3. Posterior distribution :

Let $V=n^{\frac{1}{2}}(\mu-u)/u^{\frac{1}{2}}$. Using stirling approximation, we find that the posterior distribution of V has density

$$h(v \mid y, n) = \phi(v) \left\{ 1 + \sum_{1}^{4} R_i (nu)^{-i/2} + 0(n^{-5/2}) \right\},$$

where $\phi(v)$ is the standardized normal density

$$R_1 = (r_1 uv + \tfrac{1}{3} v^3)$$

$$R_2 = \tfrac{1}{2} u^2 v^2 r_2 + u(\tfrac{1}{3} r_1 u^4 - a_1) + (\tfrac{1}{18} v^6 - \tfrac{1}{4} v^4)$$

$$R_3 = \tfrac{1}{6} u^3 v^3 r_3 + u^2 (\tfrac{1}{6} r_2 v^5 - a_1 r_1 v) + u(\tfrac{1}{18} r_1 v^7 - \tfrac{1}{4} r_1 v^5 - \tfrac{1}{3} a_1 v^3) + (\tfrac{1}{5} v^5 - \tfrac{1}{12} v^7 + \tfrac{1}{162} - v^9)$$

$$R_4 = \tfrac{1}{24} u^4 v^4 r_4 + u^3(\tfrac{1}{18} r_3 v^6 - \tfrac{1}{2} a_1 v^2 r_2) + u^2 \tfrac{1}{36} v^8 r_2 - \tfrac{1}{8} r_2 v^6 - \tfrac{1}{3} a_1 r_1 v^4 - (a_2 - a_1^2) + u \tfrac{1}{4} a_1 v^4 - (\tfrac{1}{18} a_1 - \tfrac{1}{5} r_1) v^6 - \tfrac{1}{12} r_1 v^8 + \tfrac{1}{162} r_1 v^{10} + \tfrac{1}{1944} v^{12} - \tfrac{1}{72} v^{10} + \tfrac{47}{480} v^8 - \tfrac{1}{6} v^6$$

These expressions are quite complicated, but we seldom need them all. In fact, for moderately large values of n, it is sufficient to retain terms of the order $0(n^{-\frac{1}{2}})$ only. Further, we do not need (8) in order to evaluate moments of the posterior distribution of μ. Thus (omitting details which are quite tedious) we have

$$\begin{aligned} E(\mu \mid y, n) &= n^{-1}(y+1) k_g(y+1, n)/k_g(y, n) \\ &= n^{-1}(y+1)\{1 + n^{-1} r_1 + n^{-2}(a_1' + \tfrac{1}{2} r_2) + 0(n^{-3})\}, \end{aligned} \qquad (9)$$

$$E(\mu^2 \mid y, n)=n^{-2}(y+1)(y+2)K_g(y+2,n)/K_g(y,n)$$
$$=n^{-2}(y+1)(y+2)\{1+2r_1 n^{-1}+n^{-2}(2\underset{1}{a'}+2r_2)+O(n^{-3}) \quad (10)$$

$$E(\mu^3 \mid y, n)=n^{-3}(y+1)(y+2)(y+3)k_g(y+3, n)/k_g(y, n)$$
$$=n^{-3}(y+1)(y+2)(y+3)\{1+3r_1 n^{-1}+n^{-2}(3\underset{1}{a'}+\tfrac{9}{2}r_2)$$
$$+0(n^{-3})\}, \quad (11)$$

and in general

$$E(u^m \mid y, n)=n^{-m}(y+m)_m\{1+mr_1 n^{-1}+n^{-2}(m\underset{1}{a'}+\tfrac{1}{2}m^2 r_2)+O(n^{-3})\} \quad (12)$$

In the expressions (9) — (12), prime denotes derivative with respect to u that is,

$$\underset{1}{a'}=\underset{1}{r'}+\tfrac{1}{2}(u\underset{2}{r'}+r_1)=\tfrac{3}{2}r_2-\underset{1}{r^2}+\tfrac{1}{2}(r_3-r_1 r_2)u$$

From the forging expressions we can derive asymptotic expression for the central moments $m_{\mu p}$, $p=2, 3$........of the posterior distribution. We have, for instance,

$$m\mu_2=n^{-1}(y+1)\{n^{-1}+n^{-2}2r_1+u(r_2-r_1^2)+O(n^{-3})\},$$
$$m\mu_3=n^{-1}(y+1)\{2n^{-2}+0(n^{-3})\},$$
$$m\mu^4=n^{-1}(y+1)\{3u\,n^{-2}+0(n^{-3})\}$$

Further, it is easily verified that the skewness and kurtosis of $h(\mu \mid y,n)$ are of the orders $0(n^{-\frac{1}{2}})$ and $0(n^{-1})$, respectively. This provides an additional evidence of the asymptotic normality of $h(\mu \mid y, n)$.

To illustrate, consider again the posterior distribution when the prior distribution is gamma (7). It is immediately verified that

$$E(\mu \mid y, n)=(w+n)^{-1}(s+y)$$
$$=(u+sn^{-1})(1-n^{-1}w+n^{-2}w^2\ldots.)$$
$$=u+(s-uw)n^{-1}+(uw-s)wn^{-1}+\ldots.$$
$$=(u+1/n)[1+(s-1-uw)u^{-1}n^{-1}+\{w^2-w(s-1)u^{-1}$$
$$(s-1)u^{-2}\}n^{-2}+O(n^{-3})$$

which agrees with (9). Similarly, the second and higher moments of the posterior distribution can be identified with those given in (10) — (12).

It is evident from (8) that for a fixed u, the probability density function of V converges to $(2\Pi)^{-\frac{1}{2}} \exp - (\frac{1}{2} v^2)$, so that the posterior distribution of μ is asymptotically normal with mean u and variance $n^{-1} u$, whatever the prior distribution provided only that density has continuous derivatives. (Asymptotic normality of posterior distributions in general is considered in considerable detail by Jeffreys, 1961, and Lindley, 1968).

4. Posterior distribution with uninformative prior :

Now suppose that u has uninformative prior distribution with density $g(u) \propto 1/\mu$, which implies that $n\ \mu$ has uniform distribution over the positive real axis. (For the definition of a uniform distribution over infinite domaine. see Lindley (1965) pp. 18-19).

Evidently,

$$k_g\ (y, n) \propto 1/y,$$

which agrees with (5), because, in this situation, $a_1 = a_2 = o$, etc. Note that $k_g\ (y, n)$ is not defined for $y = o$, and that $K_g\ (y, n)$ is not defined for any value of y. This is also obvious from the fact that the prior density is not proper.

The posterior distribution of y has density

$$h(\mu \mid y, n) = n \frac{e^{-n\mu} (n\mu)^{y-1}}{(y-1)\ !}$$

so that the posterior distribution of $n\ \mu$ is $\overline{\mid (y)}.$ Confidence limits for μ can be found by using the fact that $(2n\ \mu)$ is χ^2 with $2y$ degrees of freedom.

Further,

$$E(\mu^m \mid y, n) = n^{-m}(y+m-1)_m = n^{-m}\ y(y+1)\ldots(y+m-1),$$

$m = 1, 2, \ldots\ldots$, which agrees with (12). It follows that the mean and variance of the posterior distribution with vague prior knowledge are u and $n^{-1}u$ respectively. This shows that Bayesian sestimator with quadratic loss function and uninformative prior distribution is unbiased.

The posterior distribution of $V = n^{\frac{1}{2}} (\mu - u)/u^{\frac{1}{2}}$ can be derived either directly or, alternatively, from (8) by substituting

$r_i = (-1)^i \, i\,! \,/\, u^i$, $i = 1, 2 \ldots\ldots$, and $a_1 = a_2 = o$; its density function has the same form as (8) with

$$R_1 = -v + \tfrac{1}{3} v^3, \; R_2 = v^2 - \tfrac{7}{12} v^4 + \tfrac{1}{18} v^6$$
$$R_3 = -v^3 + \tfrac{47}{60} v^5 - \tfrac{5}{36} v^7 + \tfrac{1}{162} v^9,$$
$$R_4 = v^4 - \tfrac{19}{20} v^6 + \tfrac{341}{1440} v^8 - \tfrac{13}{648} v^{10} + \tfrac{1}{1944} v^{12}.$$

5. Importance of K_g (y, n) and h (μ | y, n).

One final remark about Kg (y,n) and $h(\mu\, y,n)$ may be made. The two distributions play different roles in statistical inference. The former incorporates prior information or belief and, might, therefore serve as a tool for determining a sample size which is optimal in some sense. To take a very simple case, suppose that we want to determine a sample size n such that, with constant per unit cost of sampling, the variance of $u = n^{-1}y$ does not exceed a pre-assigned value σ_0^2. Let the mean and variance of the prior distribution be a and b^2 respectively. Recall that the variance of K_g (y, n) is $n^2 b^2 + na$, so that the unconditional variance of μ is $b^2 + a/n$. Therefore, the optimal sample size is the integer just exceeding $a/(\sigma_0^2 - b^2)$. It is evident that as n increases without bound, the variance of u approaches that of the prior distribution, which is a further evidence that K_g $(y, n) \longrightarrow$ G (u) in limit.

The posterior density h $(\mu \mid y, n)$ provides a check on one's prior knowledge in the light of the information derived from the sample. In order to find how far the prior information agrees with the sample information, one might compute

$$E_y\ E(\mu^2 \mid y, n) - E^2(\mu \mid y, n) = E_y\ (m_{\mu 2}).$$

which is a measure of concentration of $h(\mu \mid y, n)$. We have from Section 3 that this is of the order $0(n^{-1})$. Therefore, with the increase in sample size the area under $h(\mu \mid y, n)$ tends to concentrate about a single ordinate. In other words, the larger the sample, the more precise our knowledge about the true value of the parameter.

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O (3, 1) SYMMETRY MODEL AND DIP STRUCTURE IN $\pi^- p$ ELASTIC SCATTERING BETWEEN 1.7 AND 2.5 GeV/c.*

by

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Abstract :—It is shown that for small angle scattering, the dip structure in $\pi^- p$ elastic scattering between 1.7 and 2.5 GeV/c can be explained by O(3, 1) symmetry model.

1. Introduction :

Recently the differential cross-sections in $\pi^- p$ elastic scattering have been measured at incident pion laboratory momenta of 1.70, 1.88, 2.07, 2.27 and 2.5. GeV/c[1]. These measurements show dips at

$$-t=0.6-0.8,\ -t=1.6-1.9 \text{ and } -t=2.6-2.8(\text{GeV/c})^2.$$

So far no satisfactory explanation for this dip structure is available. In this article we shall show that the appearance of these dips follows as a natural consequence of O (3, 1) symmetry model.

2.O (3,1) Symmetry Model :

The helicity amplitudes for elastic forward scattering of strongly interacting particles possess symmetry with respect to homogeneous Lorentz group O (3,1). Toller[2] has shown that these amplitudes can be expanded in terms of principal series of unitary representation functions of O(3,1). This scheme has been extended approximately by Delbourgo,

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Salam and Strathdee[3] to inelastic non-forward scattering processes. In this scheme the reduced amplitudes $T_{S'\lambda' S\lambda}$ (s, t) for the scattering process $1+2\rightarrow 3+4$ of particles 1, 2, 3, 4 having unequal masses and spins S_1, S_2, S_3, S_4, are expressed in terms of a set of helicity non-flip amplitudes $T^{(S')}_{J'\lambda S}$ (s, t) as

$$T_{S'\lambda' S\lambda}(s, t) = \Sigma_{J'} \left| \frac{\phi\ (s, t)}{s_{12}\ \ s_{34}} \right|^{|\Delta|/2} < S'\lambda' \mid \Delta \mid \Delta\ J'\lambda > T^{(S')}_{J'\lambda S}(s,t), \quad (1)$$

where $\lambda = \lambda_1 - \lambda_3$, $\lambda' = \lambda_2 - \lambda_4$, $\Delta = \lambda' - \lambda$,

$S = \mid S_1 - S_3 \mid, \ldots, S_1 + S_3$, $S' = \mid S_2 - S_4 \mid, \ldots, S_2 + S_4$,

$J' = \Big| S' - \mid \Delta \mid \Big|, \ldots, S' + \mid \Delta \mid$ and $\left| \frac{\phi\ (s,\ t)}{s_{12}\ \ s_{34}} \right|^{|\Delta|/2}$ is a kinematical factor. The amplitudes $T^{(S')}_{J'\lambda\ S}$ are then written as

$$T^{(S')}_{J'\lambda S} = \sum_{\text{poles}\ j_\circ} \sum_{\leqslant \min(J',S)} \tfrac{1}{2} (j_0^2 - \sigma^2)\ \beta^{(S')}_{J'S}\ (j_\circ, t) \Big[d^{\,j_\circ\ \sigma}_{S\lambda J'}\ (\xi_t) + (-1)^{S+J'}\ \pi_1\, \pi_2\, \pi_3\, \pi_4\ \ d^{\,-j_\circ \sigma}_{S\lambda J'}\ (\xi_t) \Big]$$

where π_1, π_2, π_3, π_4 are the intrinsic parities of the particles and $d^{\,j_\circ\ \sigma}_{S\lambda J'}$ (ξ_t) are the unitary representation functions of O(3,1) with

$$\cosh\ \xi_t = \frac{s - u}{\{2\,(m_1^2 + m_3^2) - t\ \}^{\frac{1}{2}}\, \{2\,(m_2^2\ + m_4^2) - t\ \}^{\frac{1}{2}}} \quad (3)$$

In order to get the high energy asymptotic expressions for amplitudes the functions $d^{\,j_\circ\ \sigma}_{S\lambda S'}$ can be written as

$$d^{\,j_\circ\ \sigma}_{S\lambda J'} = e^{\,j_\circ\ \sigma}_{S\lambda J'} + \frac{\Gamma\,(J' + \sigma + 1)}{\Gamma\,(J' - \sigma + 1)}\ e^{\,-j_\circ \sigma}_{S\lambda J'}\ \frac{\Gamma\,(S - \sigma + 1)}{\Gamma\,(S + \sigma + 1)} \quad (4)$$

where the functions $e^{j_\circ\ \sigma}_{S\lambda S'}$ have the asymptotic behaviour

$$e^{j_\circ\ \sigma}_{S\lambda S'}(\xi_t) \underset{\xi_t \to \infty}{\to} [(2S+1)\ (2S'+1)]^{\frac{1}{2}} \left[\frac{\Gamma\ (S-\lambda+1)}{\Gamma\ (S+\lambda+1)}\ \frac{\Gamma\ (S+j_\circ+1)}{\Gamma\ (S-j_\circ+1)}\right.$$

$$\left.\frac{\Gamma\ (S'-\lambda+1)}{\Gamma\ (S+\lambda+1)}\ \frac{\Gamma\ (S'+j_\circ+1)}{\Gamma\ (S'-j_\circ+1)}\right]^{\frac{1}{2}} (-1)^{S'+\lambda}$$

$$\times \frac{\Gamma\ (\sigma+S'+1)}{\Gamma\ (j_\circ-\lambda+1)}\ \frac{\Gamma\ (-j_\circ-\sigma)}{\Gamma\ (\sigma-\lambda+1)\ \Gamma\ (-\sigma+S'+1)} (z)^{-(\sigma+1+j_\circ-\lambda)}$$

$$,\ z=\cosh \xi_t$$

provided $j_\circ \geqslant \lambda$. If $j_\circ < \lambda$ the corresponding asymtotic expression is obtained from (5) by interchanging $j_\circ$ and λ.

3. Calculation and Discussion:

The number of independent helicity amplitudes for $\pi^- p$ elastic scattering is two. If only a single pole contribution is considered to be significant, then asymptotic expressions for relevant $d^{j_\circ\ \sigma}_{S\lambda J'}$ are given by

$$d^{0\sigma}_{000} \to \frac{1}{\sigma} (z^{\sigma-1} - z^{-\sigma-1}), \tag{6}$$

$$d^{0\sigma}_{001} \to -\sqrt{3}\ \frac{\sigma+1}{\sigma(\sigma-1)}\ z^{-\sigma-1} - \sqrt{3}\ \frac{1}{\sigma}\ z^{\sigma-1}, \tag{7}$$

$$d^{0\sigma}_{111} \to \frac{12}{\sigma\ (1-\sigma)} z^{-\sigma-2} + \frac{12}{\sigma\ (1+\sigma)} z^{\sigma-2} \tag{8}$$

$$d^{1\sigma}_{111} \to \frac{3}{1-\sigma} z^{-\sigma-1} + \frac{24}{\sigma(1-\sigma^2)} z^{\sigma-3} \tag{9}$$

The corresponding expressions for $d^{j_\circ, -\sigma}_{S\lambda S'}$ are obtained by changing σ to $-\sigma$ in the above expressions (6)–(9).

The helicity amplitudes for $\pi^- p$ elastic scattering are given by

$$<0\ \tfrac{1}{2}\ |\ T\ |\ 0\tfrac{1}{2}> \ = \ \beta^{(0)}_{00}(0,\ t)\ \sigma\ (z^{-\sigma-1} - z^{\sigma-1})$$

$$- \beta_{10}^{(0)}(0,t)\left[\frac{\sigma^2}{1-\sigma} z^{-\sigma-1} - \frac{\sigma}{\sigma+1} z^{\sigma-1}\right] \qquad (10)$$

$$\langle 0\text{-}\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} \rangle = 12\sqrt{\tfrac{2}{3}}\,\beta_{11}^{(0)}(0,t)\left[\frac{\sigma}{\sigma-1} z^{-\sigma-2} + \frac{\sigma}{\sigma+1} z^{\sigma-2}\right]$$

$$+ \sqrt{\tfrac{3}{2}}\,\beta_{11}^{(0)}(1,t)\left[(1+\sigma) z^{-\sigma-1} + (1-\sigma) z^{\sigma-1}\right]$$

$$+12\sqrt{\tfrac{2}{3}}\,\beta_{11}^{(0)}(1,t)\frac{1}{\sigma}\left[z^{\sigma-1} - z^{-\sigma-3}\right] \qquad (11)$$

A look at equations (10) and (11) shows that some terms blow up at $\sigma = 1$, $\sigma = -1$ and $\sigma = 0$, making the differential cross-section infinite. In order to avoid this, the corresponding residue functions must contain factors $\sigma-1$, $\sigma+1$ and σ. The above amplitudes then become

$$\langle 0\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} \rangle = \beta_{00}^{(0)}(0,t)(\alpha+1)[z^{-\alpha-2} - z^{\alpha}]$$

$$-(\alpha+1)\, C_{10}^{(0)}(0,t)\,[(\alpha+1)(\alpha+2) z^{-\alpha-2} + \alpha z^{\alpha}] \qquad (12)$$

$$\langle 0\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} \rangle = 1\ \sqrt{\tfrac{2}{3}}\, C_{11}^{(0)}(0,t)(\alpha+1)[(\alpha+2) z^{-\alpha-3} + \alpha z^{\alpha-1}]$$

$$+ \sqrt{\tfrac{3}{2}}\, \beta_{11}^{(0)}(1,t)\,[(\alpha+2) z^{-\alpha-2} - \alpha z^{\alpha}$$

$$+12\sqrt{\tfrac{2}{3}}\, C_{11}^{(0)}(1,t)\,[\,(z^{\alpha-2} - z^{-\alpha-4}] \qquad (13)$$

where $\alpha = \sigma - 1$.

For $\alpha = -1$, the helicity amplitudes containing only the leading terms are

$$\langle 0\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} \rangle = \beta_{00}^{(0)}(0,t)(\alpha+1)\,[z^{-\alpha-2} - z^{\alpha}]$$

$$- C_{10}^{(0)}(0,t)(\alpha+1)[\,(\alpha+1)(\alpha+2) z^{-\alpha-2} + \alpha z^{\alpha}]$$

$$\langle 0\text{-}\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} \rangle = \sqrt{\tfrac{3}{2}}\, \beta_{11}^{(0)}(1,t)\,[(\alpha+2) z^{-\alpha-2} - \alpha z^{\alpha}]$$

which reduce to

$$\langle 0\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} \rangle = 0$$

$$\langle 0\text{-}\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} \rangle = \sqrt{6}\, \beta_{11}^{(0)}(1,t)\, z^{-1}$$

As all the terms except one disappear, we expect a dip at $\alpha = -1$. The exchanged trajectory in $\pi^- p$ elastic scattering is ρ which is parametrised as

$$\alpha = 0.57 + 0.96\, t\ [4].$$

For $\alpha = -1$, this gives a dip at $t = -1.64$ in good agreement with experiment[1].

For $\alpha = 0$, the helicity amplitudes with leading terms are

$$< 0\tfrac{1}{2} \mid T \mid C\tfrac{1}{2} > = -\beta^{(0)}_{00}(0, t)\,(\alpha + 1)\, z^{\alpha} - C^{(0)}_{10}(0, t)\,\alpha(\alpha + 1)\, z^{\alpha}.$$

$$< 0 -\tfrac{1}{2} \mid T \mid C\tfrac{1}{2} > = -\sqrt{\tfrac{3}{2}}\, \beta^{(0)}_{11}(1, t)\,\alpha\, z^{\alpha}$$

which reduce to

$$< 0\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} > = -\beta^{(0)}_{00}(0, t)$$

$$< 0 -\tfrac{1}{2} \mid T \mid 0\tfrac{1}{2} > = 0.$$

Again as all the terms except one disappear, we expect a dip at $\alpha = 0$, *i.e.* $t = -0.6$. This dip has also been observed[1].

As mentioned in Section 2, the O (3, 1) symmetry formalism is valid only for small angle scattering. Therefore, we do not expect an explanation of the dip at $t = -2.8$ from this formalism.

It may be mentioned that the dips which have been observed at $t = -0.6$ and $t = -1.7$ disappear at very high energies. The simple Toller pole theory does not give any explanation of this phenomenon. Perhaps cut contributions play a significant role at very high energy in filling up the gap and making the dips in the energy range 1.7 to 2.5. GeV/c disappear.

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A PROPERTY OF SPEARMAN'S RANK CORRELATION COEFFICIENT

by

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Let 1, 2, ..., n be the natural ranking N of a group of objects $A_1, A_2, \ldots\ldots, A_n$ according to some quality. Suppose that $x_1, x_2, \ldots\ldots x_n$, is the ranking J given by a judge to $A_1, A_2, \ldots\ldots, A_n$. Denote e_{ij} as the difference between the ith and jth ranks. Kendall (1938) proposes

$\tau = \sum_{i>j=1}^{n} a_{ij}\ (J) / \sum_{i>j=1}^{n} a_{ij}\ (N)$ as the measure of correlation between N and J where a_{ij} is taken -1 or $+1$ as $a_{ij} < 0$ or > 0 in a ranking. We shall show below that if a_{ij} instead of ± 1 is taken as such, the quantity

$\sum_{i>j=1}^{n} a_{ij}\ (J) / \sum_{i>j=1}^{n} a_{ij}\ (N)$ turns to be Spearman's rank correlation coefficient ρ.

Since $a_{ij}\ (J) = x_i - x_j$, it is seen that

$$\sum_{i>j=1}^{n} a_{ij}\ (J) = \sum_{i=1}^{n} (2i-1)\ x_i - n \sum_{i=1}^{n} x_i$$

$$= \frac{n^3 - n}{6} - \sum_{i=1}^{n} (i - x_i)^2.$$

In case of natural ranking, we obtain

$$\sum_{i>j=1}^{n} c_{ij}\ (N) = \frac{n^3 - n}{6}.$$

Now the proof is easy to complete.

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The proposed alternate expression of Spearman's rank correlation coefficient is useful for its computation as well as in seeking more knowledge about rank correlation.

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A NOTE ON LIPSCHITZ SPACES

by

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We use E_n to denote n-dimensional Euclidean space and for each $x=(x_1, x_2, \ldots, x_n) \in E_n$, we write $|x| = \left(x_1^2 + x_2^2 + \ldots x_n^2\right)^{\frac{1}{2}}$. The space $L^p(E_n)$ is denoted by L^p. As usual the norm in L^p is given by

$$\|f\|_p = \int_{E_n} (|f(x)|^p dx)^{1/p} \quad 1 \leqslant p < \infty,$$

$$\|f\|_\infty = \inf [\sigma : m(\{x : |f(x)| > \sigma\}) = 0],$$

where m represent Lebesgue measure on E_n.

The Fourier transform of a function f in L^p is denoted by $\hat{f}$ and the convolution is designated by $f * g$.

For some real number $\alpha > 0$, $b > 0$, let $G^\alpha(., b)$ be the Function on E_n given by

$$G^\alpha(x, b) = \frac{\pi^{-\frac{1}{2}n} 2^{-\alpha} b^{-\alpha}}{\Gamma(\frac{1}{2}\alpha)} \int_0^\infty a^{\frac{1}{2}(n-\alpha)-1} e^{-|x|^2 a - \frac{b^{-2}}{4a}} da.$$

It has the following properties.

(*i*) $G^\alpha(bx, b) = b^{-n} G^\alpha(x, 1)$;

(*ii*) $G^\alpha(., b)$ is everywhere positive and an integrable function on E_n;

(*iii*) $[G^\alpha(., b)]^\wedge(x) = (2\pi)^{-\frac{1}{2}n} b^{-\alpha} (b^{-2} + |x|^2)^{-\frac{1}{2}\alpha}$;

(*iv*) $\|G^{\alpha}(.,b)\|_1=1$,

(*v*) $G^{\alpha+\beta}(.,b)=G^{\alpha}(.,b)_{*}\,G^{\beta}(.,b)$ $\alpha>0$, $\beta>0$.

For any $f \in L^p$, $1\leqslant p\leqslant\infty$, and for any $\alpha>0$, $b>0$, the Bessel potential $J^{\alpha}(f)(.,b)$ of order α of f is the covolution $G^{\alpha}(.,b) * f$.

It is well known that if $f \in L^p$, $1\leqslant p\leqslant\infty$,

(*vi*) $\|J^{\alpha}(f)\|_p \leqslant \|f\|_p$.

Also if $1<p<\infty$

(*vii*) $\|J^{\alpha}(f)-f\|_p\to 0$ as $\alpha\to 0+$,

and

(*viii*) $\|J^{\alpha}(f)(.,b)-f\|_p\to 0$ as $b\to 0+$.

Proof : Using Minkowski's integral inequality and the property (*i*) we get

$$\|J^{\alpha}(f)(x,b)-f(x)\|_p\leqslant\int_{E_n} G^{\alpha}(t,1)\,\|f(x-bt)-f(x)\|_p\,dt$$

$$\to 0 \text{ as } b\to 0+.$$

If $f(x,h)$ is measurable in x and h ($x, h \in E_n$), define

$$\|f(x,h)\|_{pq}=\{\int_{E_n}\|f(x,h)\|_p^q\ h^{-n}\,dh\}^{1/q},\ 1<q<\infty,$$

$$\|f(x,h)\|_p=\sup_{|h|>0}\|f(x,h)\|_p.$$

For any $\alpha>0$, the Lipschitz space $\Lambda(\alpha,p,q)$ is the set of function in L^p for which the norm

$$\|f\|_{\alpha,p,q}=\|h^{-\alpha}\{f(x+h)-f(x)\}\|_{pq}+\|f\|_p$$

is finite.

For a detailed discussion of the Lipschitz space see [1, 3, 4].

We use k to denote a positive constant depending on the parameters involved. These constants are not necessarily the same on any two occurances.

Lemma 1.

If $0 < \alpha < 1$, then

$$G^{\alpha}(x, b) \leqslant k\, b^{j-\alpha} \, | x |^{(\alpha-n)-j}, \; j=0, 1, 2, \ldots .$$

Proof. In the property (i) use the fact

$$e^{-b^{-2}/4a} \leqslant k\, b^{j}\, \alpha^{\frac{1}{2}j}, \; j=0, 1, \ldots ,$$

to obtain

$$G^{\alpha}(x, b) \leqslant k\, b^{j-\alpha} \int_{0}^{\infty} a^{\frac{1}{2}(n-\alpha+j)-1} \, e^{-\;|x|^{2}a} \, d\,a$$

$$= k\, b^{j-\alpha} \, | x |^{(\alpha-n-j)}.$$

Lemma 2.

If $\alpha > 0$, then

$$\left| \frac{\partial^{2}}{\partial b^{2}} G^{\alpha}(x,b) \right| \leqslant k\,(1 + |x|\, b^{-1} + |x|^{2}\, b^{-2})\, b^{-2}\, G^{\alpha}(x, b).$$

Proof. To prove the result differentiate twice an alternative integral representation.

$$G^{\alpha}(x,b) = \frac{\pi^{-\frac{1}{2}(n+1)} \; b^{-\alpha} \; e^{-b^{-1}|x|}}{2^{\frac{1}{2}(n+\alpha-1)}\, \Gamma(\frac{1}{2}\alpha)\, \Gamma\left(\frac{n-\alpha}{2} - 1\right)}$$

$$\int_{0}^{\infty} e^{-|x|t} (b^{-1}\, t + \tfrac{1}{2} t^{2})^{\frac{1}{2}(n-\alpha-1)} dt,$$

and use the inequality $b^{-1}\, t \leqslant b^{-1}\, t + \frac{1}{2}\, t^{2}$, $t > 0$.

Lemma 3.

Let ψ be a non-negative function defined over $(0, \infty)$ and for $\alpha > 0$, $p \geqslant 1$, let $\psi(s)$ be given by either as

$$\psi(s) = \int_{0}^{s} \psi(u)\, du, \; \alpha < 0$$

or as

$$\psi(s)=\int_0^\infty \psi(u)\, du,\ \alpha > 0,$$

then

$$\left[\int_0^\infty \left\{s^\alpha \psi(s)\right\}^p \frac{ds}{s}\right]^{1/p} \leqslant |\alpha|^{-1}\left[\int_0^\infty \left\{s^{\alpha+1} \psi(s)\right\}^p \frac{ds}{s}\right]^{1/p}.$$

For proof see [2, page 239].

Theorem

If $0 > \alpha < 1$, $1 \leqslant p,\ q \leqslant \infty$, then there is a finite constant k such that

$$\|f\|_p + \| b^{2-\alpha} \frac{\partial^2}{\partial b^2} J^\alpha(f)(.,b)\|_{pq} \leqslant \|f\|_{\alpha,p,q}. \tag{1}$$

Proof. Set

$$A = \|h^{-\alpha} f(x+h) - f(x)\|_{pq}, \tag{2}$$

$$B = \| r^{-\alpha} \phi_r(x)\|_{pq}, \tag{3}$$

where for $r > 0$

$$\phi_r(x) = \frac{1}{\sigma_n}\int_{\Sigma_n} \{f(x+r\omega) - f(x)\}\, d\omega, \tag{4}$$

ω is the measurement on the surface of the unit sphere Σ_n on E_n with $\omega(\Sigma_n) = \sigma_n$; and

$$C = \| b^{2-\alpha} \frac{\partial^2}{\partial b^2} J^\alpha(f)(.,b)\|_{pq} \tag{5}$$

Since

$$\int_{E_n} \frac{\partial^2}{\partial b^2} G^\alpha(x,b)\, dx = 0,$$

$$\frac{\partial^2}{\partial b^2} J^\alpha(f)(x,b) = \int_{E_n} \{f(x-y) - f(x)\} \frac{\partial^2}{\partial b^2} G^\alpha(y,b)\, dy$$

$$= \int_0^\infty \phi_r(x)\, r^{n-1} \frac{\partial^2}{\partial b^2} g(r,b)\, dr,$$

where $\phi_r(x)$ is given in (4) and $g(r,b) = G^{\alpha}(x, b)$, $|x| = r$. Using Minkowski's integral inequality and lemma 2 we obtain.

$$\left\| \frac{\partial^2}{\partial b^2} J^{\alpha}(f)(x,b) \right\|_p \leqslant \int_0^{\infty} \| \phi_r(x) \|_p \, r^{n-1} \left| \frac{\partial^2}{\partial b^2} g(r,b) \right| dr$$

$$\leqslant k \, \{ b^{-2} \int_0^{\infty} \|\phi_r(x)\|_p \, r^{n-1} \, g(r,b) \, dr$$

$$+ \, b^{-3} \int_0^{\infty} \|\phi_r(x)\|_p \, r^{n} \, g(r,b) \, dr$$

$$+ \, b^{-4} \int_0^{\infty} \|\phi_r(x)\|_p \, r^{n+1} \, g(r,b) \, dr \}.$$

Also, on using the estimates given in Lemma 1, it follows that for $q = \infty$

$$\left\| \frac{\partial^2}{\partial b^2} J^{\alpha}(f)(x,b) \right\|_p$$

$$\leqslant k \sup_{r>0} r^{-\alpha} \|\phi_r(x)\|_p \left(b^{-\alpha-2} \int_0^{\infty} r^{2\alpha-1} dr + b^{-\alpha} \int_b^{\infty} r^{2\alpha-3} \, dr \right)$$

$$= k \, b^{\alpha-2} \, B,$$

so that for $q = \infty$

$$C = \sup_{b>0} \left(b^{2-\alpha} \left\| \frac{\partial^2}{\partial b^2} J^{\alpha}(f)(x,b) \right\|_p \right) \leqslant k B, \tag{6}$$

Suppose now $1 < q < \infty$. Then again using Minkowski's inequality, Lemmas 1 and 3 we have

$$C \leqslant k \left[\int_0^{\infty} \left\{ \int_0^{\infty} \|\phi_r(x)\|_p \, r^{n-1} \frac{\partial^2}{\partial b^2} g(r,b) \, dr \right\}^q \frac{db}{b^{1+q(\alpha-2)}} \right]^{1/q}$$

$$\leqslant k\Big[\int_0^\infty\Big(b^{-2\alpha}\int_0^b\|\phi_r(x)\|_p r^{\alpha-1}\,dr\Big)^q\frac{db}{b}\Big\}^{1/q}$$

$$+\,k\Big[\int_0^\infty\Big(b^{2(1-\alpha)}\int_h^\infty\|\phi_r(x)\|_p\, r^{\alpha-3}\,dr\Big)^q\frac{db}{b}\Big]^{1/q}$$

$$\leqslant k\Big[\int_0^\infty\Big(b^{-\alpha}\,\|\phi_r(x)\|_p\Big)^q\frac{db}{b}\Big]^{1/q}=k\ \mathrm{B}. \tag{7}$$

It is also known [3] that

$$\mathrm{B}\leqslant k\ \mathrm{A}. \tag{8}$$

Hence combining (6), (7) and (8) we prove the result.

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SOME THEOREMS ON GENERALIZED BATEMAN K-FUNCTION*

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SUMMARY:

Bateman K-function is a special kind of confluent Hypergeometric function. In 1953 Chakravarti gave the *first* generalization of this function by adding one more parameter. In the present article the author generalizes Bateman K-function by introducing two parameters and believes that further generalization is no more possible. Besides the generalization of the results of Chakravarti the author obtains many interesting results and the properties of this new function. Its relation with important special functions have been worked out and the Integrals involving this generalized K-function have been evaluated and expressed as a finite series of the terms involving Hypergeometric, parabolic cylinder and the well known Meijer G-functions which are of interest to pure and applied mathematicians.

1. Introduction :

The Bateman K-function was first introduced by Bateman [1] in the form

$$(1.1)\quad K_n(x)=\frac{2}{\pi}\int_0^{\pi/2} \text{Cos}\,(x\tan\theta-n\,\theta)\,d\,\theta$$

and then generalized by Chakravarti [3] in the form

$$(1.2)\quad K_n^u(x)=\frac{2}{\pi}\int_0^{\pi/2} 2^u\,\text{Cos}^u\,\theta\;\text{Cos}\,(x\tan\theta-n\,\theta)\,d\,\theta,\; u>-1$$

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The object of this paper is to generalize further this Bateman K-function in the form

(1.3) $K_n^{u,v}(x) = \frac{2}{\pi} \int_0^{\pi/2} \sin^u \theta \, \text{Cos}^v \theta \, \text{Cos}\,(x \tan\theta - n\,\theta)\, d\,\theta$

and to make a systematic study of this new function. We will in general take n, u, v to be positive integers. The following symbols and notations will be used.

The classical Laplace transform of $f\,(t)$ as defined by

(1.4) $F(p) = p \int_0^{\infty} e^{-pt} f\,(t)\; d\,t \qquad R\,(p) > 0$

is symbolically denoted by $F\,(p) = f\,(t)$

The symbol D_a^n (f) denotes the n^{th} differential coefficient of the function $f\,(x)$ at the point $x=a$.

The expressions $F\,(\mp\, x)$ and $F\,(\pm\, x)$ are defined as follows

(1.5) $F\,(\pm\, x) = F\,(x) + F\,(-x)$

$F\,(\mp\, x) = F\,(x) - F\,(-x)$

Many interesting special functions have been used in the study of the generalized Bateman K-functions. Corresponding references have been given, wherever they occur, by writing (for definition see [ ]).

2. Basic Properties, Recurrence relations and the Differential equations Satisfied by Generalised Bateman K-function :

In this section the recurrence relations and the differential equation satisfied by $K_n^{u,v}$ is obtained. It will also be shown that by giving special values to the parameters and the variable x the function $K_n^{u,v}(x)$ can be expressed in terms of the Hypergeometric function ${}_1F_2\left[\begin{matrix}\alpha \\ \beta\; r\end{matrix}; x\right]$ and ${}_3F_2\left[\begin{matrix}a,b,c \\ \alpha,\beta\end{matrix}; x\right]$ (for definition see [7], pp 182).

Theorem 1 :

If n, u and v be positive integers and $K_n^{u,v}(x)$ is defined by (1.3) then

$$(2.1)\quad K_n^{u+1,v}x = \sum_{r=0}^{[u/2]} \frac{(-1)^{\frac{u-1}{2}}}{2^{u+v+1}} \left[K^v_{n+u+2r\mp 1}(x) - K_{n-u+2r\pm 1}(x) \right]$$

where

$$K^n_{l\pm m}(x) = K^n_{l+m}(x) + K^n_{l-m}(x)$$

If we take $v = 0$ in (2.1) we then obtain

$$K_n^{u+1,0}(x) = \sum_{r=0}^{[u/2]} \frac{(-1)^{\frac{u-1}{2}}}{2^{u+1}} \left[K_{n+u-2r\mp 1}(x) - K_{n-u+2r\pm}(x) \right]$$

Theorem 2 :

If $K_n^{u,v}(x)$ is defined by (1.3) then it satisfies the following recurrence relations.

$$(2.2)\quad K_n^{u,v+1}(x) = \tfrac{1}{2}\left[K_{n+1}^{u,v}(x) + K_{n-1}^{u,v}(x) \right]$$

$$(2.3)\quad D_x\, K_{n-1}^{u,v}(x) + D_x\, K_{n+1}^{u,v}(x) = K_{n-1}^{u,v}(x) - K_{n+1}^{u,v}(x)$$

$$(2.4)\quad K_n^{u,v}(x) = \tfrac{1}{2}\left[D_x\, K_{n+1}^{u-2,v+1}(x) - D_x K_{n-1}^{u-2,v+1}(x). \right]$$

$$(2.5)\quad D_x\, K_n^{u,v+1}(x) = \tfrac{1}{2}\left[K_{n-1}^{u,v}(x) - K_{n+1}^{u,v}(x) \right]$$

$$(2.6)\quad K_n^{u,v}(x) = K_n^{u,v-2}(x) - K_n^{u+2,v-2}(x)$$

$$= K_n^{u-2,v}(x) - K_n^{u-2,v+2}(x)$$

Theorem 3 :

If $K_n^{u,v}(x)$ is defined by (1.3) the

$$(2.7)\quad K_n^{2u,v}(x) = \sum_{r=0}^{u} (-1)^r \binom{u}{r} 2^{-v-2r} K_n^{v+2r}(x)$$

$$(2.8)\quad K_n^{0,u}(x) = 2^{-v-1}\left[K_{n-1}^{v-1}(x) - K_{n+1}^{v-1}(x)\right]$$

$$(2.9)\quad K_0^{0,u}(x) = \frac{2}{\sqrt{\pi}\,\Gamma(v/2+1} \left(\frac{x}{2}\right)^{\frac{v+1}{2}} K_{\frac{v+1}{2}}(x)$$

Theorem 4 :

If $K_n^{u,v}(x)$ is defined by (1.3) and $R\,(n-u-v)>0$

$$(2.10)\quad K_n^{u,v}(0) = \frac{1}{\pi}\ \frac{\Gamma\left(\frac{u}{2}+1\right)\Gamma\left(\frac{v+n}{2}+\frac{1}{2}\right)}{\Gamma\left(1+\frac{u+v+n}{2}\right)} \cdot {}_3F_2\left(\begin{matrix}\frac{1-n}{2}, \frac{-n}{2}, \frac{u+1}{2} : 1\\ \frac{1}{2}, \frac{1-v-n}{2}\end{matrix}\right)$$

Theorem 5 :

If n is a positive integer and $K_n^{u,v}(x)$ is defined by (1.3) then

$$(2.11)\quad K_n^{u,v}(x) = \sum_{r=0}^{[n/2]} \left[\binom{n}{2r}\{A_r(x)+B_r(x)\} - \binom{n}{2r+1}\{D_x A_r(x) + D_x B_r(x)\}\right]$$

$$(2.12)\quad K_n^{u,v}(-x) = \sum_{r=0}^{[n/2]} \left[\binom{n}{2r}\{A_r(x)+B_r(x)\} - \binom{n}{2r+1}\{D_x A_r(x) + D_x B_r(x)\}\right]$$

where

$$A_r(x)=\frac{(-1)^r}{\pi}\ \frac{\Gamma\left(\frac{u+1}{2}+r\right)\Gamma\left(\frac{n+v+1}{2}-r\right)}{\Gamma\left(1+\frac{u+v+n}{2}\right)}\ {}_1F_2\left(\begin{matrix}\frac{u+1}{2}+r\\ \frac{1-v-n}{2}+r,\ \frac{1}{2}\end{matrix};\ \frac{x^2}{4}\right)$$

$$B_r(x)=\frac{(-1)^r}{\pi}\ \frac{\Gamma\left(r-\frac{n+v}{2}-1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{v+n}{2}-r\right)}\cdot\left(\frac{x}{2}\right)^{1+v+n-2r}$$

$$\times\ {}_1F_2\left(\begin{matrix}\frac{u+v+n}{2}+1\\ \frac{v+n}{2}+1-r,\ \frac{v+n}{2}-r+\frac{3}{2}\end{matrix};\ \frac{1}{4}x^2\right)$$

Proof of Theorems 1-5 :

To prove theorem 1 use the expansion of $\text{Sin}^u\,\theta$ and $\text{Sin}^n\,\theta$ in the definition of $K_n^{u,v}(x)$. To prove theorems 2 and 3 use the definition of $K_n^{u,v}(x)$ and proceed. The results follow after a little computation. To prove theorem 4 put $x=0$ in (1.3) and use the expansion of Cos $n\theta$. After a little simplification one obtains

$$\text{(2.13)}\quad K_n^{u,v}(0)=\frac{1}{\pi}\sum_{r=0}^{[n/2]}\frac{(-)^r\,\Gamma(n+1)}{\Gamma(2r+1)\,\Gamma(n-2r+1)}$$

$$\times\frac{\Gamma\left(\frac{u+1}{2}+r\right)\Gamma\left(\frac{v+n+1}{2}-r\right)}{\Gamma\left(\frac{u+v+n}{2}+1\right)}$$

On using the various properties of Gamma function and the definition of ${}_3F_2\left(\begin{matrix}\alpha_1\ \alpha_2\ \alpha_3\\ \beta_1\ \beta_2\end{matrix};\ 1\right)$ the equation (2.13) reduces to (2.10) To prove theorem 5 expand Cos $(x\tan\theta-n\theta)$ in (1.3), use the expansion of Cos $n\theta$ and Sin $n\theta$ and then evaluate each integral with the help of the result of Erdelyi [6, pp 14 (29)]. Calculations being lengthy but straightforward.

Theorem 6 :

The generalized Bateman K-function $K_n^{u,v}(x)$ as defined by (1.3) satisfies the following differential equation.

(2.14) $X D_x^3 K_n^{u,v}(x) + (1-v) D_x^2 K_n^{u,v}(x) + (n-x) D_x K_n^{u,v}(x)$

$-(1+u) K_n^{u,v}(x) = 0$

Proof : Let us consider the equation

(2.15) $\int_0^{\pi/2} D_\theta \left[\text{Sin}^u \theta . \text{Cos}^{v+2}\theta . \ \text{Sin}(x \tan\theta - n\theta) \right] d\theta = 0$

From (2.15) we can write

(2.16) $\int_0^{\pi/2} \Big[u \ \text{Sin}^{u-1}\theta \ \text{Cos}^{v+3}\theta \ \text{Sin}(x\tan\theta - n\theta)$

$-(v+2) \text{Sin}^{u+1}\theta \ \text{Cos}^{v+1}\theta \ \text{Cos}(x\tan\theta - n\theta) - n \ \text{Sin}^u\theta \ \text{Cos}^{v+2}\theta$

$\times \text{Cos}(x\tan\theta - n\theta) + \times \text{Sin}^u\theta \ \text{Cos}^v\theta \ \text{Cos}(x\tan\theta - n\theta) \Big] d\theta$

Integration and the use of (1.3) and (2.2) yields

(2.17) $(v-n+2) \ K_{n-1}^{u,v+1}(x) - (v+n+2) K_{n+1}^{u,v+1}(x) + 2x \ K_n^{u,v}(x)$

$-u\left[K_{n-1}^{v-2,v+3}(x) - K_{n+1}^{u-2,v+3}(x) \right] = 0$

On using (2.6) in (2.17) and then simplifying with the help of (2.2) we obtain

(2.18) $\frac{1}{2}(v+u-n+2)\left[K_{n-1}^{u,v}(x) + K_n^{u,v}(x) \right] - \frac{1}{2}(v+u+n+2)$

$\times \left[K_n^{u,v}(x) + K_{n+2}^{u,v}(x) \right] + 2\times \ K_n^{u \ v}(x) - u$

$\left[K_{n-1}^{u-2,v+1}(x) - K_{n+1}^{u-2,v+1}(x) \right] = 0$

On differentiating (2.18) and using (2.3) and (2.4) we find

$$(2.19)\quad (v+u-n+2)\,K_{n-2}^{u,v}(x)+(v+u+n+2)\;K_{n+2}^{u,v}(x)$$

$$+2(u-v)K_{n}^{u,v}(x)+4\times D_x\;K_{n}^{u,v}\;(x)=0$$

Now from (2.18) and (2.19) we can see

$$(2.20)\quad (v+u-n+2)\,K_{n-2}^{u,v}(x)+(u-n-v+2\times)\,K_{n}^{u,v}\,(x)$$

$$+2xD_xK_{n}^{u,v}\,(x)+u\Big[K_{n+1}^{u-2,v+1}(x)-K_{n-1}^{u-2,v+1}(x)\Big]=0$$

If we differentiate (2.20) and use (2.6) we then find

$$(2.21)\quad 2(x-v)D_x\,K_{n}^{u,v}\,(x)+2\times D_x^2\,K_{n}^{u,v}\,(x)+2(1+u)\,K_{n}^{u,v}\,(x)$$

$$=(n-2-u-v)\Big[K_{n-2}^{u,v}(x)-K_{n}^{u,v}(x)\Big]$$

On substituting the value of $K_{n-2}^{u,v}\,(x)$ from (2.20) in (2.21) we have

$$(2.22)\quad 2\times D_x^2\,K_{n}^{u,v}(x)-2v\,D_x\,K_{n}^{u,v}\,(x)+2(n-x)\,K_{n}^{u,v}\,(x)$$

$$=u\Big[K_{n+1}^{u-2,v+1}(x)-K_{n-1}^{u-2,v+1}(x)\Big]$$

The result now follows on differentiating (2.22) and then using (2.4).

Theorem 7 :

If $K_{n}^{u,v}\,(x)$ is defined by (1.3) then we have the following integral and differential formulas.

$$(2.23)\text{a}\quad \int_0^{\infty} e^{-x}\,K_{n}^{u,v}(x+a)d\,x\;=K_{n-1}^{u,v}\,(a)$$

$$(2.23)\text{b}\quad \int_a^{\infty} e^{-x}\,K_{n}^{u,v}(x)\;d\,x\;=e^{-a}\,K_{n-1}^{u,v}\,(a)$$

$$(2.23)\text{c}\quad D_a^n\Big[e^{-a}\,K_{0}^{u,v+n}\,(a)\;=(-1)^n\,e^{-a}\,K_{n}^{u,v}\,(a)$$

Proof: To prove (2.23)a substitute the value of $K_n^{u,v}(x+a)$ from (1.3) in the integral of (2.23)a and evaluate the double integral so obtained from the results of Erdelyi [6, pp 72 (1) & pp 14 (1)]. The result (2.23)b follows from (2.23) a and (2.23)c follows from (2.23)b.

Theorem 8.

Let $K_n^{u,v}(x)$ be defined by (1.3) then

$$\text{(2.24)} \quad D_x K_n^{u,v+2}(x) = K_{n-1}^{u,v-1}(x) - K_n^{u,v}(x)$$

Proof : It is not very hard to verify that

$$\int_0^\infty e^{-y}\left[K_n^{u,v}(x+y) + K_n^{u,v}(x-y)\right] dy = 2\,K_n^{u,v+2}(x)$$

$$\int_0^\infty e^{-y}\left[K_n^{u,v}(x+y) - K_n^{u,v}(x-y)\right] dy = 2\,D_x K_n^{u,v+2}(x)$$

On adding the above equations and using (2.23)a the result (2.24) follows at once.

Theorem 9.

If $K_n^{u,v}(x)$ be defined by (1.3) then

$$\text{(2.25)} \quad K_0^{u,v+n}(a) = \frac{1}{\pi}\left[\frac{\Gamma\frac{(1+u)}{2}\,\Gamma\frac{(1+v+n)}{2}}{\Gamma\left(1+\frac{u+v+n}{2}\right)} \cdot {}_1F_2\left(\begin{matrix} u+1 \\ \frac{1-v-n}{2},\ \frac{1}{2}\end{matrix};\frac{a^2}{4}\right)\right.$$

$$\left. + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\frac{(-1-v-n)}{2}}{\Gamma\left(\frac{v+n}{2}+1\right)}\left(\frac{a}{2}\right)^{v+n+1} \cdot {}_1F_2\left(\begin{matrix}\frac{u+v+n}{2}+1 \\ \frac{v+n}{2}+1,\ \frac{v+n+3}{2}\end{matrix};\frac{a^2}{4}\right)\right]$$

Proof: The result (2.25) can be obtained from (1.3) and the result of Erdelyi [6, pp 14 (29)].

Theorem 10.

If n is not an add integer and $K_n^{u,v}(x)$ be defined by (1.3) then

$$(2.26)\quad K_n(a) = \frac{2}{\sqrt{\pi}} e^{a} \frac{(-1)^n}{\Gamma(\frac{n}{2}+1)} D_a^n \left[e^{-a} \left(\frac{a}{2}\right)^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(a) \right]$$

where $K_v(z)$ is a Modified Bessel function of the second kind. (For definition see [10], pp 372).

Proof :

Taking $u=0$, $v=0$ in (2.23)c, substituting $K_0^{o,n}(a)$ from (2.25) and then using the following well known result [8]

$${}_0F_1(v+1\ ; Z^2/4) = (Z/2)^{-v}\,\Gamma(v+1)\,I_v(Z)$$

we obtain after a little simplification

$$\Gamma(\frac{n}{2}+1)\, K_n(a) = \frac{e^{a}\sqrt{\pi}\,(-1)^n}{\sin\frac{\pi}{2}(n+1)} D_a^n$$

$$\left[e^{-a}\left(\frac{a}{2}\right)^{\frac{n+1}{2}} \left\{ I_{-\frac{n+1}{2}}(a) - I_{\frac{n+1}{2}}(a) \right\} \right]$$

provided n is not an odd integer. Now on using the result ([6], pp 371)

$$I_{-v}(Z) - I_v(Z) = \frac{2}{\pi}(\sin v\pi)\, K_v(Z)$$

in the above equation the result (2.26) follows atonce. $I_v(z)$ being the Modified Bessel function of the first kind. (For definition see [10] pp 372).

3. Laplace Transform and the Generating Function of the Generalized Bateman K-Function :

Theorem 11.

The classical Laplace transform of $K_n^{u,n}(x)$ as defined by (1.3) is given by

$$(3.1)\qquad 2(-1)^m\, p^{2m+1}\frac{(1-p)^{n-l-m-1}}{(1+p)^{n+l+m+1}} = K_{2n}^{2m,\,2l}(x)$$

provided that R $(p)>0$, $(n-l-m-1)>0$. m and $n-1$ being positive integers.

Proof: From (1.3) we can write

$$(3.2)\qquad K_{2n}^{2m,\,2l}(x)=\frac{2}{\pi}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\int_0^{\pi/2}$$

$$\operatorname{Cos}^{2l+2r}(2\,\theta)\operatorname{Cos}(x\tan\theta-n\,\theta)\,d\theta$$

Now using (1.2) in (3.2) we obtain

$$(3.3)\qquad K_{2n}^{2m,\,2l}(x) = \sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{1}{2^{2l+2r}}K_{2n}^{2l+2r}$$

provided $2l+2r>-1$.

It is well known that [3]

$$(3.4)\qquad p.2^{2l+2r+1}\,(1-p)^{n-l-r-1}\,(1+p)^{-n-l-r-1} = K_{2n}^{2m+2l}(x)$$

By virtue of (3.3) and (3.4) one can easily verify

$$K_{2n}^{2m,\,2l}(x)=2\sum_{r=0}^{m}(-1)^r\binom{m}{r}p(1-p)^{n-l-r-1}\,(1+p)^{-n-l-1}$$

$$=2\,p\frac{(1-p)^{n-l-1}}{(1+p)^{n+l+1}}\sum_{r=0}^{m}\binom{m}{r}(-1)^r\,(1-p^2)^{-r}$$

$$=2(-1)^m\,(1-p)^{n-l-m-1}\,(1+p)^{-n-l-m-1}\,p^{2m+1}$$

This completes the proof.

Theorem 12.

If $K_n^{u,v}(x)$ is defined by (1.3) and $\phi_2(a, b, c,; z; \alpha, \beta, r)$ denotes the generalized Hypergeometric series function (for definition see [6], pp. 385) then the generating function of $K_n^{u,v}(x)$ is given by

$$(3.5)\quad \sum_{n=0}^{\infty} (-1)^n x^n K_2^{2m,2l}(t)=\frac{2(-1)^m t^{2l+1}}{(1-x)\Gamma(2l+2)}\phi_2\Big[-l-m-1,-l-m,-1;\;(2l+1);\; t,-t,-\frac{1+x}{1-x}t\Big]$$

provided that $|x|<1$,

Proof: On multiplying (3.1) by $(-1)^n x^n$ and summing from 0 to ∞ we find.

$$(3.6)\quad \sum_{n=0}^{\infty} (-1)^n x^n K_{2n}^{2m,2l}(t)=\frac{2(-1)^m p^{2m+1}}{(1-p^2)^{l+m+1}}\sum_{n=0}^{\infty}(-1)^n x^n\left[\frac{1-p}{1+p}\right]^n$$

$$\text{Since } \sum_{n=0}^{\infty}(-1)^n x^n\left(\frac{1-p}{1+p}\right)^n=\frac{1+p}{1-x}\left[+1\frac{1}{p}\left(\frac{1+x}{1-x}\right)\right]$$

hence (3.6) reduces to

$$(3.7)\quad \sum_{n=0}^{\infty}(-1)^n x^n K_{2n}^{2m,2l}(x)$$

$$=p.\frac{2(-1)^m}{1-x}p^{-2l-2}\left(1-\frac{1}{p}\right)^{-l-m-1}\left(1+\frac{1}{p}\right)^{-l-m}\left[1+\frac{1}{p}\left(\frac{1+x}{1-x}\right)\right]^{-1}$$

Now using the result of Erdelyi [6, pp 222 (5)] we obtain

$$(3.8)\quad t^{2l+1}\phi_2\left[-l-m-1,-l-m,-1;(2l+1);b,-t,-\frac{1+x}{1-x}t\right]$$

$$=p\Gamma(2l+2)p^{-2l-2}\left(1-\frac{1}{p}\right)^{-l-m-1}\left(1+\frac{1}{p}\right)^{-l-m}\left(1+\frac{1}{p}\frac{1+x}{1-x}\right)^{-1}$$

On comparing (3.7) and (3.8) the result (3.1) follows atonce.

4. Connections of Generalized Dateman K-Function with the other Special Functions.

In this section $K_n^{u,v}(x)$ has been expressed in terms of whittaker Confluent Hypergeometric function $M_{k,\mu}(x)$ and also as *finite* series of the terms involving parabolic cylinderical function $D_v(x)$ and generalized Laguerre function L_n^{α} respectively. The infirite series involving the sum $K_n^{u,v}(x) + K_n^{u,v}(-x)$ have been expressed in terms of the Hypergeometric function ${}_1F_1\binom{a}{b;x}$ ${}_1F_2\binom{a}{b,c;x}$, the Modified Bessel function of the second kind $K_v(x)$ and the Weber function $E_l(x)$. For the definition of the special function $M_{k,\mu}(x)$, $D_v(x)$, $L_n^{\alpha}(x)$ and $E_i(x)$ see Wittakar and Watson ([10], pp 337 & 347) and Erdelyi ([7] pp 369 & 372) From now and onwards we will write $K_n^{u,v}(\pm x)$ for the sum $K_n^{u,v}(x)+K_n^{u,v}(-x)$. This follows from notation (1.5).

Theorem 13.

The generalized Bateman K-function $K_n^{u,v}(x)$ as defined by (1,3) is related to the Whittaker confluent Hypergeometric function $M_{k,\mu}(x)$ according to the following identities.

$$(4.1)\quad K_{2n}^{2m,2l}(x) = \frac{2^{-l-m}(-1)^{n-l-1}}{\Gamma(2l+2m+2)} \cdot D_x^{2m} \left[X^{l+m} M_{n,l+m+\frac{1}{2}}(2x)\right]$$

where m is a positive integer and $1 \pm m \geqslant -1$

$$(4.2)\quad K_{2n}^{2m\,2l}(x) = (-1)^{n-l}\left(\frac{x}{2}\right)^{l+\frac{1}{2}} \times \sum_{r=0}^{m} \binom{m}{r}\left(\frac{x}{2}\right)^{r} \frac{1}{\Gamma(2l+2r+1)} \left[M_{n+\frac{1}{2},\,l+r}(2x) - M_{n-\frac{1}{2},\,l+r}(2x)\right]$$

Proof : From (3.1) it is easy to see that

$$(4.3)\quad K_{2n}^{2m,2l}(x) = p\,2\,(-1)^{2-l-1}p^{-2l-2}\left(1-\frac{1}{p}\right)^{n-l-m-1}\left(1+\frac{1}{p}\right)^{-n-l-m-1}$$

where $R(p)>0$, m and $(n-1)$ are positive integers and $R(n-1-m-1)>0$

Also by virtue of the result of Erdelyi [6, pp 215 (10)] we may write

$$(4.4)\quad x^{l+m}\, M_{n,\, l+m+\frac{1}{2}}(2x) = p\, 2^{l+m+1}\, \Gamma(2l+2m+2)$$
$$.\, p^{-2l-2m-2}\left(1-\frac{1}{p}\right)^{n-l-m-1}\left(1+\frac{1}{p}\right)^{-n-l-m-1}$$

where R $(p)>0$ and R$(1+m)>-1$.

Now by virtue a well known property of Laplace transform [6, pp 129 (8)] and (4.4) we have

$$(4.5)\qquad D_x^{2m}\left[x^{l+m}\, M_{n,l+m+\frac{1}{2}}(2x)\right]$$
$$= p\left[2^{l+m+1}\,\Gamma(2l+2m+2)p^{-2l-2}\left(1-\frac{1}{p}\right)^{n-l-m-1}\left(1+\frac{1}{p}\right)^{-n-l-m-1}\right.$$

provided $(1-m)>-1$.

On comparing (4.3) and (4.5) we obtain the result (4.1) under the conditions mentioned in the theorem.

Remark :

From (4.2) it is obvious that

$$K_{2n}^{2m,2l}(x) = O(x^{2l+\frac{1}{2}}) \text{ as } x \to 0$$

Also one can verify that (3.1) can be simplified to

$$(4.5)\quad K_{2n}^{2m,2l}(x) = \sum_{r=0}^{m}\left[x^r\binom{m}{r}\left\{p\,(1-p)^{n-l-r-1}\right.\right.$$
$$\left.\left..(1+p)^{-n-l-r} - p\,(1-p)^{n-l-r}(1+p)^{-n-l-r-1}\right\}\right]$$

Now from the result of Erdelyi [6, pp 215 (10)] one can deduce

$$(4.7)\qquad x^{l+r-\frac{1}{2}}\, M_{n-\frac{1}{2},\, l+r}(2x)$$
$$= p(-1)^{n-l-r-1}\, 2^{l+r+\frac{1}{2}}\,\Gamma(2l+2r+1)(1-p)^{n-l-r-1}$$
$$(1+p)^{-n-l-r}$$

and

$$(4.8)\qquad x^{l+r-\frac{1}{2}}\, M_{n+\frac{1}{2},l+r}(2x)$$

$$-p(-1)^{n-l-r}2^{l+r+\frac{1}{2}}\,\Gamma(2l+2r+1)(1-p)^{n-l-r}(1+p)^{-n-l-r-1}$$

On using (4.7) and (4.8) in (4.6) the result (4.2) is obtained without any difficulty.

Theorem 14.

If $K_n^{m,l}(x)$ is defined by (1.3) then

$$(4.9)\quad e^{-\frac{1}{2}x}K_{2n}^{2m,2l}(\tfrac{1}{2}x)=\sum_{r=0}^{m}(-1)^{n-l-1}\binom{m}{r}\frac{2^{-2l-2r}}{\Gamma(n+l+r+1)}$$

$$x\left[(n+l+r)\,D_x^{n-l-r-1}\left(x^{n+l+r-1}e^{-x}\right)+D_x^{n-l-r}\left(x^{n+l+r}e^{-x}\right)\right]$$

Proof : From (4.6) it is not very hard to verify that

$$(4.10)\quad e^{-\frac{1}{2}x}K_{2n}^{2m,2l}\left(\frac{x}{2}\right)=\sum_{r=0}^{m}(-1)^{n-l-1}\binom{m}{r}2^{-2l-2r}$$

$$\cdot\left[p^{n-l-r}(1+p)^{-n-l-r}+p^{n-l-r+1}(1+p)^{-n-l-r-1}\right]$$

On using the result of Erdelyi [6, pp 144]

$$(4.11)\quad D_x^m\left[x^a e^{-x}\right]=p\Gamma(a+1)\,p^m(1+p)^{-a-1},\ R(p)>-1$$

and the equation (4.10) the result (4.9) follows after a littles computation.

Remark : From the equation (4.9) it is a easy to see that

$$K_{2n}^{2m,2l}(x)=O\left(e^{-\frac{1}{2}x}.\,x^{m+n+l}\right)\text{as } x\to\infty$$

Theorem 15

The generalized Bateman K-function $K_n^{uv}(x)$ as defined by (1.3) is related to the Laguerre function $L_n^{\alpha}(x)$ according to the following identities.

$$(4.12)\quad e^{\frac{1}{2}x}K_{2n}^{2m,2l}(\tfrac{1}{2}x)=\sum_{r=0}^{m}(-1)^{n-l-1}\binom{m}{r}\left(\frac{x}{2}\right)^{2l+2r}$$

$$\cdot\frac{\Gamma(n-l-r)}{\Gamma(n+l+r+1)}\left[(n+l+r)\,L_{n-l-r-1}^{2l+2r}(x)+(n-r-l)\,L_{n-l-r}^{2l+2r}(x)\right]$$

$$(4.13) \qquad K_{2n}^{2m,2l}(x)=2(-1)^{n-l-1}\frac{\Gamma(n-l-m)}{\Gamma(n+l+m+1)}$$

$$\cdot\, D_x^{2m}\left[e^{-x}\; x^{2l+2m+1}\, L_{n-l-m-1}^{2l+2m+1}(2\;x)\right]$$

provided $1\pm m>-1$.

Proof : If we multiply (4.9) by $e^{\frac{1}{2}x}$ and use Rodrigues formula $e^{x}\, D_x^{n}\left[x^{n+a}\; e^{-x}\right]=n\,!\; x^{a}\; L_n^{a}\;(x)$ we obtain (4.12)

Again from the definitions of $M_{k,u}(x)$ and $L_n^{a}\;(x)$ i.e.

$$(4.14) \quad M_{k,\mu}(x)=x^{\mu+\frac{1}{2}}e^{-\frac{1}{2}x}\;{}_1F_1\left(\begin{matrix}\mu-k+v_2\\ 2\mu+1\end{matrix};x\right)$$

$$(4.15) \quad L_n^{a}(x)=\frac{\Gamma(a+n+1)}{\Gamma(a+1)\Gamma(n+1)}\cdot\;{}_1F_1\left(\begin{matrix}-n\\ 1+a\end{matrix};x\right)$$

we obtain

$$(4.16) \quad M_{k,\;\mu}(x)=\frac{\Gamma(2\mu+1)\,\Gamma(k-\mu+\frac{1}{2})}{\Gamma(k+\mu+\frac{1}{2})}.\;x^{\mu+\frac{1}{2}}.\;e^{-\frac{1}{2}x}\;L_{k-\mu-\frac{1}{2}}^{2\mu}\;(x)$$

On substituting the value of $M_{n,\,l+m+\frac{1}{2}}\,(2\,x)$ from (4.16) in (4.1) we obtain (4.13).

Theorem 16

The generalized Bateman K-function $K_n^{u,v}\;(x)$ as defined by (1.3) is related to the parabolic cylinder function $D_v\,(x)$ as follows

$$(4.17)\quad K_{2n}^{2m,2l}\left(\frac{x}{2}\right)=\sum_{r=0}^{m}\frac{(-1)^r\binom{m}{r}}{2^{n-l-r+\frac{1}{2}}}\cdot\frac{x^{2l+2r-\frac{1}{2}}}{\Gamma(2l+2r-\frac{1}{2})}\cdot\frac{\Gamma(n-l-r)}{\Gamma(n+l+r+1)}$$

$$\cdot\left[(n+r+l)\sum_{j=0}^{n-l-r-1}\frac{(-1)^j\,2^{j+1}\,\Gamma(2l+2r+j-\frac{1}{2})}{\Gamma(j+1)\Gamma(n-l-r-j)}\cdot D_{2(n-l-r-j-\frac{1}{2})}(\sqrt{2}\;x)\right.$$

$$\left.+\,(n-r-l)\sum_{j=0}^{n-l-r}\frac{(-1)^j\,2^j\,\Gamma(2l+2r+j-\frac{1}{2})}{\Gamma(j+1)\,\Gamma(n-l-r-j+1)}\cdot D_{2(n-l-r-j+\frac{1}{2})}(\sqrt{2}\;x)\right]$$

provided m is a positive integer and $m \leqslant n-l-1$.

Proof : We know that [8, pp 209]

$$(4.18) \quad L_n^a(x) = \sum_{j=0}^{n} \frac{\Gamma(a-b+j)}{\Gamma(a-b)\Gamma(j+1)} L_{n-j}^b(x)$$

From (4.18) it can be shown

$$(4.19) \quad L_{n-l-r-1}^{2l+2r}(x) = \frac{(-1)^{n-l-r-1} x^{-\frac{1}{2}} e^{\frac{1}{2}x} 2^{-n+l+r+\frac{1}{2}}}{\Gamma(2l+2r-\frac{1}{2})}$$

$$\times \sum_{j=0}^{n-l-r-1} \frac{(-1)^j 2^j \Gamma(2l+2r+j-\frac{1}{2})}{\Gamma(j+1)\Gamma(n-l-r-j)} D_{2(n-l-r-j-\frac{1}{2})}(\sqrt{2}\, x)$$

$$(4.20) \quad L_{n-l-r}^{2l+2r}(x) = \frac{(-1)^{n-l-r} x^{-\frac{1}{2}} e^{\frac{1}{2}x} 2^{-n+l+r-\frac{1}{2}}}{\Gamma(2l+2r-\frac{1}{2})}$$

$$\cdot \sum_{j=0}^{n-l-r} \frac{(-)^j 2^j \Gamma(2l+2r+j-\frac{1}{2})}{\Gamma(j+1)\Gamma(n-l-r-j+1)} D_{2(n-l-r-j+\frac{1}{2})}(\sqrt{2}\, x)$$

Use of these results in (4.12) yields (4.17).

Theorem 17 :

The sum of the series involving generalized Bateman K-function $K_n^{u,v}(x)$ as defined by (1.3) can be expressed in terms of one of the special functions, ${}_1F_2\left(\begin{matrix}\alpha \\ \beta, r\end{matrix}; x\right)$, ${}_1F_1\left(\begin{matrix}\alpha \\ \beta\end{matrix} : x\right)$ and $E_1(x)$ as follows.

$$(4.21) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} K_{2n+1}^{u,n}(\pm x) = \tfrac{1}{2}\, \beta\left(\frac{u+1}{2}, \frac{v+1}{2}\right)$$

$$\cdot\ {}_1F_2\left(\begin{matrix}\frac{u+1}{2} \\ \frac{1-v}{2}, \frac{1}{2}\end{matrix}; \frac{x^2}{4}\right) + \frac{\sqrt{\pi}\, 2^{-v-2}}{\Gamma(1+\frac{v}{2})} . x^{v+1}\, \Gamma\left(\frac{-1}{2}\ \frac{-v}{2}\right)$$

$${}_1F_2\left(\begin{matrix}\frac{u+v}{2}\ 1+ \\ \frac{1+v}{2}, \frac{v+3}{2}\end{matrix}; \frac{x^2}{4}\right)$$

where v is an odd positive integer.

(4.22) $\sum_{j=0}^{l} \binom{l}{j} K_{n-l+j+1}^{2m, n-l-2m-1}(+)$

$$= \frac{2^{l+1}(-)^m t^{n-2m} e^{-t}}{\Gamma(n-2m+1)} \cdot {}_1F_1\left(\begin{matrix}-2m \\ n+1-2m\end{matrix}; t\right)$$

provided $R(n\text{-}2m+1)>0$

(4.23) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} K_{2n}^{+2,o}(\mp n) = \frac{2}{\pi}[e^{-x} + x\,E_i(-x) - 1]$

Proof : Using the definition (1.3) of $K_n^{u,v}(x)$ we have

(4.24) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} K_{2n+1}^{u,v}(\pm x)$

$$= \frac{4}{\pi} \int_0^{\pi/2} \text{Sin}^u\theta\ \text{Cos}^v\theta\ \text{Cos}(x \text{ ton } \theta) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{Cos}(2n+1)\theta\right] d\theta$$

Since the series under the integral sign has the sum $\frac{\pi}{4}$ (see [4], pp 98) hence

(4.25) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} K_{2n+1}^{u,v}(\pm x) = \int_0^{\pi/2} \text{Sin}^u\theta\ \text{Cos}^v\theta\ \text{Cos}(x\tan\theta)\, d\theta$

Now on evaluating this integral with the help of the result [5, pp 172] the result (4.21) follows atonce.

To prove (4.22) consider the result of Erdelyi [6, pp 215 (10)] in the form

(4.26) $\frac{t^{\frac{n}{2}-m-\frac{1}{2}}\, e^{-\frac{t}{2}}}{\Gamma(n-2m+1)} M_{\frac{n}{2}+m+\frac{1}{2}, \frac{n}{2}-m}(t) = p^{2m+1}(1+p)^{-n-1}$

Also it is not very hard to verify that

(4.27) $p^{2u+1}(1+p)^{-n-1} = 2^{-l} \sum_{j=0}^{l} \binom{l}{j} p^{2u+1}(1-p)^j (1+p)^{l-n-j-1}$

Now if we take the inverse Laplace transform of (3.1) and use (4.26) and (4.27) we then obtain

$$(4.28) \quad \sum_{j=0}^{l} \binom{l}{j} K_{n-l+j+1}^{2m,\, n-l-2m-1}(t)$$

$$= \frac{2^{l+1}(-1)^m}{\Gamma(n-2m+1)} \cdot t^{\frac{n}{2}-m-\frac{1}{2}} \cdot e^{-\frac{1}{2}t} \cdot M_{\frac{n}{2}+m+\frac{1}{2},\, \frac{n}{2}-m}(t)$$

On using the result [6, pp 386]

$$M_{k,\,\mu}(x) = x^{\mu+1} e^{-\frac{1}{2}x} {}_1F_1\left(\begin{matrix}\frac{1}{2}+\mu+k\\ 2\mu+1\end{matrix}; x\right)$$

in (4.28) the result (4.22) follows under the conditions stated in the theorem.

In order to prove (4.23) abserve that

$$(4.29) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} K_{2n}^{u,v}(\mp x)$$

$$= \frac{4}{\pi} \int_0^{\pi/2} \theta \, \text{Sin}^u \theta \, \text{Cos}^u \theta \, \text{Sin}(x \tan\theta) \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{2n\,\theta} \text{Sin } 2n\,\theta \right] d\theta$$

Now if we substitute $-\frac{1}{2}$ form the sum of the d series under the integral sign [4, pp 102] and take $u=2$, $v=0$ then (4.29) reduces to

$$(4.30) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} K_{2n}^{-2,0}(\mp x) = -\frac{2}{\pi} \int_0^{\pi/2} t^{-2} (\tan^{-1} t) \, \text{Sin } x\,t \, dt$$

Evaluating the integral on the right with the result of Ditkin [5, pp 278] the result (4.23) follows after a little simplification.

Theorem 18 :

The series involving the sum $K_n^{u,v}(\pm x)$ can be expressed in terms of the Modified Bessel function $K_v(x)$ as follows.

If

$$\alpha(m,n) = \frac{m(m-1)\ldots(m-n+1)}{(m+1)\ldots\ldots(m+n)}; \; \beta(m,n) = \frac{2.4\ldots\quad 2m}{3.5\ldots(2m+1)}$$

Then

(4.31) $$\sum_{n=0}^{m} \alpha(m, n)\, K_{2n}^{u,v}(\pm x)$$

$$= \frac{2}{\sqrt{\pi}} \left(\frac{x}{2}\right)^{\frac{v+1}{2}} \left[\beta(m, n) \cdot \frac{(x/2)^m}{\Gamma\left(\frac{v}{2}+m+2\right)} K_{\frac{v+1}{2}+m}(x) - \frac{1}{\Gamma\left(\frac{v}{2}+1\right)} K_{\frac{v+1}{2}}(x)\right]$$

(4.32) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} K_{2n+1}^{u,v}(\pm x)$$

$$= \frac{(-1)^u \sqrt{\pi}}{2^{u+v+\frac{1}{2}} \cdot \Gamma(u+v+1)} D_x^{2u} \left[x^{u+v+\frac{1}{2}} K_{u+v+\frac{1}{2}}(x)\right]$$

Proof : It is well known that

(4.33) $$2 \sum_{n=0}^{\infty} \alpha(m, n) \operatorname{Cos} 2n\,\theta = \beta(m, n) \operatorname{Cos}^{2m} \theta - 1.$$

where $\alpha(m, n)$ and $\beta(m, n)$ are defined above.

From the definition (1.3) of $K_n^{u,v}(x)$ and (4.33) we have

(4.34) $$\sum_{n=0}^{\infty} \alpha(m, n)\, K_{2n}^{u,v}(\pm x)$$

$$= \frac{4}{\pi} \int_0^{\pi/2} \operatorname{Sin}^u \theta \operatorname{Cos}^u \theta \operatorname{Cos}(x \tan \theta) \left[\sum_{n=0}^{m} \alpha(m, n) \operatorname{Cos} 2n\,\theta\right] d\theta$$

$$= \beta(m, n)\, K_0^{u,v+2m} - K_0^{u,v}(x)$$

Now setting $u=0$ and using (2.9) in (4.34) the result (4.(31) follows after a little simplification.

To prove (4.32) observe that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} K_{2n+1}^{u,v} (\pm x)$$

$$= \frac{4}{\pi} \int_0^{\pi/2} \mathrm{Sin}^{2u} \theta \, \mathrm{Cos}^{2v} \theta \, \mathrm{Cos}\,(x \tan \theta) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \mathrm{Cos}\,(2n+1)\theta \right] d\theta$$

On substituting $\frac{\pi}{4}$ for the sum of the series (see [4] and using the result [5, pp 172] the result (4.32) follows without any difficulty.

5. Integrals Involving Generalized Bateman K-Function :

If f (x, t) is a known function of p and t then

$$\phi(t) \; def \int_0^{\infty} f(x, t) \; K_n^{u,v}(x) \; dx$$

is called the integral transform of $K_n^{u,v}(x)$ and $f(x, t)$ is called the *kernel* of the transform. In this section a few integral transforms of $K_n^{u,v}(x)$ have been obtained by choosing kernels in different forms. The Laplace and Mellin transform of $K_n^{u,v}(x)$ have been obtained and expressed as a finite series of Hypergeometric functions. It has been shown that taking the kernel $f(x, t) = e^{-x - x^2/2t} \cdot D(2x/\sqrt{2t})$ the transform of $K_n^{u,v}(x)$ is a finite series of the terms involving G-function $G_{\gamma,\delta}^{\alpha,\beta}\left(x \middle| \begin{matrix} a_1 \ldots a_\gamma \\ b_1 \ldots b_\delta \end{matrix}\right)$ (For the definition of G-function see [7], pp 206) Kernels involving exponential and the Bessel function of the first kind $J_v(x)$ have also been considered. (For the definition of $J_v(x)$ see [10] pp 355). It has also been shown that by giving special values to the parameters the product $K_a^{b,c}(x) \, K_\alpha^{\beta,\gamma}(x)$ can be integrated and expressed as a finite series of the generalized Bateman K-functions.

Theorem 19 :

If $K_n^{u,v}(x)$ is defined by (1.3) then

(5.1) $$\int_0^\infty \left[K_{2n}^{0,v}(x)\right]^2 dx = \frac{\Gamma(v+2)}{\Gamma(v+\frac{3}{2})}(\pi)^{-\frac{3}{2}} - \frac{1}{2}\sum_{r=0}^{n-1}\left[K_{2r+1}^{0,v+1}(x)\right]^2$$

Proof : From (2.3) we have

(5.2) $$D_x K_n^{u,v}(x) + D_x K_{n+2}^{u,v}(x) = K_n^{u,v}(x) - K_{n+2}^{u,v}(x)$$

On multiplying both sides by $K_n^{u,v}(x)$ and then by $K_{n+2}^{u,v}(x)$ we obtain after addition

$$\tfrac{1}{2} D_x\left[K_n^{u,v}(x) + K_{n+2}^{u,v}(x)\right]^2 = \left[K_n^{u,v}(x)\right]^2 - \left[K_{n+2}^{u,v}(x)\right]^2$$

On repeating this process it is easy to see that

(5.3) $$\tfrac{1}{2} D_x\left\{\sum_{r=0}^{n-1}\left[K_{2r}^{u,v}(x) + K_{2r+2}^{u,v}\right]^2\right\} = \left[K_0^{u,v}(x)\right]^2 - \left[K_{2n}^{u,v}(x)\right]^2$$

Since $K_{2n}^{0,v}(x) \to 0$ as $x \to \infty$ hence the integration of (5.3) with $u=0$ gives

(5.4) $$\int_0^\infty\left[K_{2n}^{0,v}(x)\right]^2 dx = \int_0^\infty\left[K_0^{0,v}(x)\right]^2 dx - \tfrac{1}{2}\sum_{r=0}^{n-1}\left[K_{2r}^{0,v}(0) + K_{2r+2}^{0,v}(x)\right]^2$$

Now evaluating the integral on the right with the help of (2.9) and the result of Erdelyi [6, pp 127 (1)] and simplifying with the help of (2.2) the result (5.1) follows after a little computation.

Theorem 20 :

If $K_n^{u,v}(x)$ is defined by (1.3) then

(5.5) $$\int_0^\infty K_{2n}^{0,v}(x)\, K_{2n-1}^{0,v-2}(x)\, dx$$

$$=\frac{\Gamma\,(\nu+2)}{\Gamma\,(\nu+\frac{3}{2})}(\pi)^{-\frac{3}{2}}-\tfrac{1}{2}\Big[K_{2n}^{0,v}(o)\Big]^2-\sum_{r=0}^{n-1}\Big[K_{2r+1}^{0,v+1}\,(x)\Big]^2.$$

Proof: It is not very hard to verify that

$$\int_0^\infty e^{-y}\Big[K_n^{u,v}\,(x+y)+K_n^{u,v}\,(x-y)\Big]dy=2\,K_n^{u,v+2}\,(x)$$

$$\int_0^\infty e^{-y}\Big[K_n^{u,v}\,(x+y)-K_n^{u,v}\,(x-y)\Big]d\,y=2\,D_x\,K_n^{u,\,v+2}\,(x)$$

On adding the above equations and using (2.23) a we abtain the recurrence relation

(5.6) $\qquad D_x\,K_n^{u,v}\;(x)=K_{n-1}^{u,v-2}\,(x)-K_n^{u,v}\,(x)$

Now if we multiply (5.6) by $K_n^{u,v}\,(x)$, take $u=0$, replace n by $2n$ and use theorem 19 we obtain (5.5.) after a little simplification.

Theorem 21 :

If $K_{2n}^{u,v}\,(x)$ is defined by (1.3) then its Laplace transform is given by

(5.7) $$\int_0^\infty e^{-px}\,K_{2n}^{u,v}\,(x)\,d\,x\equiv\sum_{r=0}^{n-1}\frac{(1-p)^{n-r-1}}{(1+p)^{n-r}}\Big[K_{2r}^{u,v}(0)+K_{2r+2}^{u,v}\,(0)\Big]$$

$$=\Big[\frac{1-p}{1+p}\Big]^n\frac{1}{\pi}\;\frac{\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{\nu+3}{2}\right)}{\Gamma\left(\frac{u+\nu}{2}+2\right)}\cdot\;{}_2F_1\left[\begin{matrix}1,\;\frac{u+1}{2}\\ \frac{u+\nu}{2}+2\end{matrix};1-p^{-2}\right]$$

where $|\arg p^{-2}|<\pi$, $R\,(p)>1$ and $K_n^{u,v}\,(o)$ is given by (2.12).

Proof : Let us denote

(5.8) $\qquad L_n^{u,v}\;(p)=p\int_0^\infty e^{-px}\,K_n^{u,v}\,(x)$

Using the definition (5.8) and the result (2.3) it can be shown that

$$(5.9) \qquad L^{u,v}_{2n-2}(p) - L^{u,v}_{2n}(p)$$

$$= -\left[K^{u,v}_{2n-2}(o) + K^{u,v}_{2n}(o)\right] + p\int_0^\infty e^{-px}\left(K^{u,v}_{2n-2}(x) + K^{u,v}_{2n}(x)\right)dx$$

From this it follows

$$(5.10) \qquad L^{u,v}_{2n}(p) = \frac{1-p}{1+p} L^{u,v}_{2n-2}(p) + \frac{p}{1+p}\left[K^{u,v}_{2n}(o) + K^{u,v}_{2n-2}(o)\right]$$

Using the formula (5.10) successively we obtain

$$(5.11) \qquad L^{u,v}_{2n}(p) = \left(\frac{1-p}{1+p}\right)^n L^{u,v}_o(p)$$

$$+ \; p \sum_{r=0}^{n-1} \frac{(1-p)^{n-r-1}}{(1+p)^{n-r}}\left[K^{u,v}_{2r}(o) + K^{u,v}_{2r+2}(o)\right]$$

Finally

$$(5.11) \qquad L^{u,v}_o(p) = \frac{2}{\pi}\int_0^\infty e^{-px}\int_0^{\pi/2} \mathrm{Sin}^u\,\theta\ \mathrm{Cos}^v\,\theta\ \mathrm{Cos}\,(x\tan\theta)\,d\theta\,dx$$

$$= \frac{2}{\pi}\int_0^\infty e^{-px}\int_0^\infty \frac{t^u\ \mathrm{Cos}\,(x\,t)}{(1+t^2)^{\frac{u+v}{2}+1}}\,dt\ dx$$

On changing the order of integration, which is valid by de la Vallee Poussion theorem [2, pp 504], we obtain

$$L^{u,v}_0(p) = \frac{2}{\pi}\int_0^\infty \frac{t^u}{(1+t^2)^{\frac{u+v}{2}+1}}\int_0^\infty e^{-px}\,\mathrm{Cos}\,(x\,t)\,dx\ dt$$

which may be written in the form

$$(5.12) \qquad L^{u,v}_0(p) = \frac{2p}{\pi}\int_0^\infty \frac{1}{p^2+t^2}\,\frac{t^u}{(1+t^2)^{\frac{u+v}{2}+1}}\,d\,r$$

Now evaluating the integral in (5.12) with the help of the result of Erdelyi [6, pp 310 (23)] we obtain

$$(5.13) \qquad L_0^{u,v}(p) = \frac{1}{\pi} \frac{\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{u+3}{2}\right)}{\Gamma\left(\frac{u+v}{2}+2\right)} {}_2F_1\left(\begin{matrix} 1, \frac{u+1}{2} \\ \frac{u+v}{2}+2 \end{matrix}; 1-p^{-2}\right)$$

provided that $|\arg p^{-2}| < \pi$ and $0 < R\left(\frac{u+1}{2}\right) < R\left(\frac{u+v}{2}+2\right)$

On substituting the expression for $L_0^{u,v}(p)$ from (5.13) in (5.11) the result (5.7) follows atonce.

Theorem 22 :

If $K_n^{u,v}(x)$ is defined by (1.3) then the Laplace transform of $K_{2n+1}^{u,v}(x)$ is given by

$$(5.14) \qquad \int_0^\infty e^{-px} K_{2n-1}^{u,v}(x)\,dx = \sum_{r=0}^{n-1} \frac{(1-p)^{n-r-1}}{(1+p)^{n-r}} \left[K_{2r+1}^{u,v}(o) + K_{2r+3}^{u,v}(o)\right]$$

$$+ B\left(\frac{u+1}{2}, \frac{u}{2}+2\right) \cdot {}_2F_1\left(\begin{matrix} \frac{u+v+3}{2}, \frac{u+1}{2} \\ \frac{u+v+5}{2} \end{matrix}; 1-p^{-2}\right)$$

$$+ \frac{(1-p)^n}{(1+p)^n} B\left(\frac{u+3}{2}, \frac{v}{2}+1\right) \cdot {}_2F_1\left[\begin{matrix} \frac{u+v+3}{2}, \frac{u+3}{2} \\ \frac{u+v+5}{2} \end{matrix}; 1-p^{-2}\right]$$

where $R(p) > 1$, $|\arg p^{-2}| < \pi$ and $K_n^{u,v}(o)$ is defined by (2.12)

Proof : The proof is exactly similar to the proof of theorem 21.

Theorem 23 :

If $K_n^{u,v}(x)$ is defined by (1.3) then the Mellin transform of this function is given by

(5.15) $$\int_0^\infty x^{s-1}\, K_n^{u,v}(x)\, dx = \frac{\Gamma(s)}{\pi}\, \frac{\Gamma\left(\frac{u}{2}+1-\frac{s}{2}\right)\Gamma\left(\frac{v+n+1+s}{2}\right)}{\Gamma\left(\frac{u+v+n}{2}+1\right)}$$

$$\times \left\{ \left(\operatorname{Cos}\frac{\pi}{2}s\right) . {}_3F_2\left[\begin{matrix} \frac{1-n}{2}, \frac{-n}{2}, \frac{u+1-s}{2} \\ \frac{1}{2}, \frac{1-v-n-s}{2} \end{matrix} ; 1\right]\right.$$

$$\left. + n\left(\operatorname{Sin}\frac{\pi}{2}s\right).\ {}_3F_2\left[\begin{matrix} \frac{1-n}{2}, \frac{1-n}{2}, 1+\frac{u-s}{2} \\ \frac{3}{2}, 1-\frac{u+n+s}{2} \end{matrix} ; 1\right]\right\}$$

provided that $(n-u-v)>0$ and $|s|<1$

Proof : Observe that

$$\int_0^\infty x^{s-1}\, K_n^{u,v}(x)\, d\, x$$

$$= \frac{2}{\pi}\int_0^\infty x^{s-1}\int_0^{\frac{\pi}{2}} \operatorname{Sin}^u\theta\, \operatorname{Cos}^u\theta\, \operatorname{Cos}(x\tan\theta - n\,\theta)\, d\theta\, d\,x$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \operatorname{Sin}^u\theta\, \operatorname{Cos}^u\theta \int_0^\infty x^{s-1}\Big[\operatorname{Cos}(x\tan\theta)\ \operatorname{Cos} n\,\theta + \operatorname{Sin}(x\tan\theta)$$

$$\operatorname{Sin} n\,\theta\Big]d\,x,\ d\,\theta$$

Now evaluating the inner integral with the help of the results of Erdelyi [6, pp 317 (10) and pp 319 (21)] we obtain after a little simplification

(5.16) $$\int_0^\infty x^{s-1}\, K_n^{u,v}(x)$$

$$= \frac{2}{\pi}\Gamma(s)\left\{\left(\text{Cos}\,\frac{\pi}{2}\right)\int_0^\infty \text{Sin}^{u-s}\,\theta,\ \text{Cos}^{v+s}\,\theta.\ \text{Cos}\,n\,\theta\,d\,\theta\right.$$

$$\left.+\left(\text{Sin}\,\frac{\pi}{2}\,s\right)\int_0^{\pi/2} \text{Sin}^{u-s}\,\theta\ \text{Cos}^{v+s}\,\theta\ \text{Sin}\,n\,\theta\,d\,\theta\right\}$$

By theorem 5 we know that if R $(n-u-v)>0$ then

(5.17) $$\frac{2}{\pi}\int_0^{\pi/2} \text{Sin}^u\,\theta\ \text{Cos}^u\,\theta\ \text{Con}\ n\,\theta\ d\theta$$

$$=\frac{1}{\pi}\ \frac{\Gamma\left(\frac{u}{2}+1\right)\Gamma\left(\frac{v+n}{2}+\frac{1}{2}\right)}{\Gamma\left(1+\frac{u+v+n}{2}\right)}.\ {}_3F_2\left[\begin{matrix}\frac{1-n}{2},\ \frac{-n}{2},\ \frac{u+1}{2} \\ \frac{1}{2},\ \frac{1-v-n}{2}\end{matrix};\ 1\right]$$

It can be easily shown that if R $(n-u-v+1)>0$ then

(5.18) $$\frac{2}{\pi}\int_0^{\pi/2} \text{Sin}^u\,\theta\ \text{Co}^u\,\theta\ \text{Sin}\ n\,\theta\ d\,\theta$$

$$=-\frac{n}{\pi}\ \frac{\Gamma\left(\frac{v+n}{2}\right)\Gamma\left(1+\frac{u}{2}\right)}{\Gamma\left(1+\frac{u+v+n}{2}\right)}.\ {}_3F_2\left[\begin{matrix}\frac{1-n}{2},\ \frac{1-n}{2},\ 1+\frac{u}{2} \\ \frac{3}{2},\ 1-\frac{v-n}{2}\end{matrix};\ 1\right]$$

Using (5.17) and (5.18) in (5.16) the result (5.15) is established under the conditions stated in the theorem.

Theorem 24 :

If $K_n^{u,v}(x)$ is defined by (1.3) then

(5.19) $$\int_0^\infty e^{-t+\frac{1}{2}yt}\ K_{2n}^{2m,\,2l}\left(\tfrac{1}{2}\,y\,t\right)d\,t$$

$$=(-1)^m\,(2-y)^{2m}\,(y-1)^{n-l-m-1}\ y^{2l}\ 2^{-2l-2m}$$

provided that $(n-m-l-1)\geq 0$, $n-l$ is a positive integer and $R(y)<2$.

Proof : From (4.6) it is not very hard to deduce that

(5.20) $$e^{-t+\frac{1}{2}yt}\, K_{2n}^{2m,2l}\left(\tfrac{1}{2}yt\right)$$

$$= p \sum_{r=0}^{m} (-1)^r \binom{m}{r}\left(\frac{y}{2}\right)^{2l+2r}\left[\frac{(y-p-1)^{n-l-r-1}}{(p+1)^{n+l+r}} - \frac{(y-p-1)^{n-l-r}}{(p+1)^{n+l+r+1}}\right]$$

where $(n-1)$ and m are positive integers and $R\,(n-l-m-1)>0$.

The result (5.19) is clearly a consequence of (5.20.)

Particulars Cases.

If we take $y=1$ and $y=2$ we then obtain

Corr 24.1 $$\int_0^\infty e^{-t/2}\, K_{2n}^{2m,2l}\left(\tfrac{1}{2}t\right)=0$$

corr 24.2 $$\int_0^\infty K_{2n}^{2m,2l}(t)\, dt=0$$

Also if we set $m=0$ and $y=x^2$ in (5.19) and differentiate both sides with respect to x and then use the Rodrigues formula for the Legendre polynomial (for definition and Rodrigues formlua see [10, pp 302]) we then have,

corr 24.3 $$\int_0^\infty D_x^{n-l}\left[e^{-t+\frac{1}{2}tx^2}\, K_{2n}^{o,2l}\left(\tfrac{1}{2}x^2\right)\right]x^{-4l}\, dx$$

$$=2^{n-l-1}\,\Gamma(n-l)\left[D_x P_{n-l-1}(x) - 2\, P_{n-l}(x)\right]$$

provided that $0<x<1$ and $R\,(1) > -1$.

Theorem 25 :

If $K_n^{u,v}(x)$ is defined by (1.3) and $D_v\,(x)$ denotes the parabolic cylinderical function then for $t>0$ we have

$$(5.21)\quad \int_0^\infty e^{-\frac{x}{2}-\frac{x^2}{8t}} D_{2w-1}\left(\frac{x}{\sqrt{2t}}\right) K_{2n}^{2m,2l}(\tfrac{1}{2}x)\,dx$$

$$= \frac{(-1)^{n-l-1}}{\sqrt{\pi}} 2^{n-l+w-3/2}\, t^{\frac{l+w}{2}} \sum_{r=0}^{m} \binom{m}{r} \left(\frac{\sqrt{t}}{2}\right)^r \frac{1}{\Gamma((n+l+r+1)}$$

$$\cdot\left[(n+l+r)\, G_{2,3}^{2,2}\left(t \,\middle|\, \begin{matrix} \frac{3-n-w}{2}, \frac{2-n-w}{2} \\ \frac{l+r+w+1}{2}, \frac{l+r-w+2}{2}, \frac{w-r-l}{2} \end{matrix}\right)\right.$$

$$\left. + 2\, G_{2,3}^{2,2}\left(t \,\middle|\, \begin{matrix} \frac{2-w-n}{2}, \frac{1-n-w}{2} \\ \frac{l+r-w+1}{2}, \frac{l+r-w+2}{2}, \frac{w-r-l}{2} \end{matrix}\right)\right]$$

Proof: From (4.6) we have

$$(5.22)\quad e^{-\frac{1}{2}t} K_{2n}^{2m,2l}(\tfrac{1}{2}t)$$

$$= p \sum_{r=0}^{m} (-1)^{n-l-1}\, 2^{-2l-2r} \binom{m}{r} \left[\frac{p^{n-l-r-1}}{(1+p)^{n+l+r}} + \frac{p^{n-l-r}}{(1+p)^{n+l+r+1}}\right]$$

Now by virtue of the result of Erdelyi [6, pp 133 (35)] and (5.22) we obtain

$$(5.23)\quad \sqrt{\frac{2}{\pi}}(2t)^{-w} \int_0^\infty e^{-\frac{1}{2}x-\frac{x^2}{8t}} D_{2w-1}\left(\frac{x}{\sqrt{2t}}\right) K_{2n}^{2m,2l}(\tfrac{1}{2}x)\,dx$$

$$= p^w \sum_{r=0}^{m} \frac{(1-)^{n-l-1}}{2^{2l+2r}} \binom{m}{r} \left[\frac{p^{\frac{1}{2}(n-l-r-1)}}{(1+\sqrt{p})^{n+l+r}} + \frac{p^{\frac{1}{2}(n-l-r)}}{(1+\sqrt{p})^{n+l+r+1}}\right]$$

If we denote the Mellin transform of $f(x)$ by $F(s)$ defined by

$$F(s) = \int_0^\infty x^{s-1} f(x)\,dx$$

and represent this equation by the symbolic notation $F(s) \overset{m}{\underset{s}{-}} f(t)$ then we have the following well known result (see [9], pp 351)

Lemma :

If $f(t) \overset{m}{\underset{s}{-}} F(s)$ and $g(t) \overset{m}{\underset{s}{-}} G(s)$ then

$$(5.24)\qquad \int_0^\infty f(at)\, g(bt)\, dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, G(1-s)\, a^{-s}\, b^{-1+s}\, ds$$

Now with the help of the results of Erdelyi [6, pp. 336 (1) and pp. 312 (1)] and (5.24) it is easy to verify

$$(5.25)\qquad \int_0^\infty x^{2v-1} e^{-x-x^2/8t} D_{2w-1}\left(\frac{x}{\sqrt{2t}}\right) dx$$

$$= \frac{2^{2v+w-1}}{\sqrt{\pi}} \cdot t^{\frac{v+w}{2}} \cdot G_{2,3}^{2,2}\left(t \left| \begin{matrix} 1-\frac{v+w}{2}, \frac{1}{2}-\frac{v+w}{2} \\ \frac{v-w}{2}, \frac{1}{2}+\frac{v-w}{2}, \frac{1}{2}-\frac{v-w}{2} \end{matrix} \right. \right)$$

Again from the results of Erdelyi [6, pp. 238 (1) and pp. 133 (35)] it is obvious that

$$(5.26)\qquad p^{w+\frac{1}{2}} (1+\sqrt{p})^{-2v}$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2t)^{-w-\frac{1}{2}} \frac{1}{\Gamma(2v)} \int_0^\infty e^{-\frac{x^2}{8t}} D_{2w}\left(\frac{x}{\sqrt{2t}}\right) x^{2v-1} e^{-x}\, dx$$

On combining (5.25) and (5.26) we obtain a well known result

$$(5.27)\qquad p^{w+\frac{1}{2}} (1+\sqrt{p})^{-2v}$$

$$= \frac{2^{2v-1}}{\pi\, \Gamma(2v)} \cdot t^{\frac{v-w-1}{2}} \cdot G_{2,3}^{2,2}\left(t \left| \begin{matrix} 1-\frac{v+w}{2}, \frac{1}{2}-\frac{v+w}{2} \\ \frac{v-w}{2}, \frac{1}{2}+\frac{v-w}{2}, \frac{1}{2}-\frac{v-w}{2} \end{matrix} \right. \right)$$

No from (5.27) it is easy to see that

$$(5.28) \qquad \sum_{r=0}^{m} \frac{(-1)^{n-l-1}}{2^{2l+2r}} \binom{m}{r} \left[\frac{p^{w+\frac{1}{2}(n-l-r-1)}}{(1+\sqrt{p})^{n+l+r}} + \frac{p^{w+\frac{1}{2}(n-l-r)}}{(1+\sqrt{\ })^{n+l+r+1}} \right]$$

$$= \sum_{r=0}^{m} \frac{(-1)^{n-l-1}}{\pi} \frac{2^{n-l-r-1}}{\Gamma(n+l+r+1)} \binom{m}{r} t^{\frac{1}{2}(l+r-w)}$$

$$\cdot \left[(n+l+r)\, G_{2,3}^{2,2} \left(t \,\middle|\, \begin{matrix} \frac{3}{2}-\frac{n+w}{2}, 1-\frac{n+w}{2} \\ \frac{l+r-w}{2}, \frac{l+r-w}{2}+1, -\frac{l+r+w}{2} \end{matrix} \right) \right.$$

$$\left. + 2\, G_{2,3}^{2,2} \left(i \,\middle|\, \begin{matrix} 1-\frac{n+w}{2}, \frac{1}{2}-\frac{n+w}{2} \\ \frac{l+r-w}{2}+\frac{1}{2}, \frac{l+r-w}{2}+1, -\frac{l+r+w}{2} \end{matrix} \right) \right]$$

On combining (5.28) with (5.23) we obtain (5.21) without any difficulty.

Particular Cases :

If we set $w=\frac{1}{2}$ in (5.23) and use the result

$$p^{\frac{m}{2}} = \left(1 - \frac{1}{1+\sqrt{p}}\right)^{m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1+\sqrt{p})^{-j-m}$$

we then obtain, after a little simplification, an interesting result

$$(5.29) \qquad \int_0^\infty e^{-\frac{1}{2}x - \frac{x^2}{8t}} K_{2n}^{2m,2l}\left(\tfrac{1}{2}x\right) dx$$

$$= \sum_{r=0}^{m} \frac{(-1)^{n-l-1}}{2^{2l+2r}} \cdot \binom{m}{r} \left[\sum_{j=0}^{n-l-r-1} (-1)^j \binom{n-l-r-1}{j} p^{\frac{1}{2}} (1+\sqrt{p})^{-2l-2r-1-j} \right.$$

$$\left. + \sum_{j=0}^{n-l-r} (-1)^j \binom{n-l-r}{j} p^{\frac{1}{2}} (1+\sqrt{p})^{-2l-2r-1-j} \right]$$

Again if we use the result

$$p^{\frac{1}{2}}(1+\sqrt{p})^{-m} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(2\,t)^{\frac{m-1}{2}}\; e^{\frac{1}{2}t}\, D_{-m}(\sqrt{2\,t})$$

in (5.29) we obtain another interesting result.

$$(5.30)\qquad \int_0^\infty e^{-\frac{1}{2}x-\frac{x^2}{8\,t}}\, K_{2n}^{2m,\,2l}\left(\tfrac{1}{2}\,x\right)\, d\,x$$

$$= 2\left(\frac{t}{2}\right)^{l+\frac{1}{2}}(-1)^{n-l-1}\; e^{\frac{1}{2}t}\sum_{r=0}^{m}\left(\frac{t}{2}\right)^r\binom{m}{r}$$

$$\cdot\left[\sum_{j=0}^{n-l-r-1}(-1)^j\binom{n-l-r-1}{j}(\sqrt{2\,t})^j\, D_{-j-2l-2r-1}(\sqrt{2\,t})\right.$$

$$\left.+\sum_{j=0}^{n-l-r}(-1)^j\binom{n-l-r}{j}(\sqrt{2\,t})^j\, D_{-j-2l-2r-1}(\sqrt{2\,t})\right]$$

provided $t>0$

Theorem 26 :

If $K_n^{u,v}(x)$ is defined by (1.3) and $J_v(x)$ denotes the Bessel function of the first kind (for definition see [10] pp. 101) then

$$(5.31)\qquad \int_0^\infty x^{-w-\frac{1}{2}}\, J_{2w+1}(2\sqrt{x\,t})\, K_{2n}^{2m,2l}(x)\, d\,x$$

$$=(-1)^{n-2l+w-1}\; t^{-w-\frac{1}{2}}\, K_{2n}^{2l-2w,2w+2m}(t)$$

Proof: Let $f(t) = p\,F(p)$ then from the result of Erdelyi [6, pp. 132 (32)] we have

$$(5.32)\qquad t^w\int_0^\infty x^{-w}\, J_{2w}(2\sqrt{x\,t})\, f(x)\, d\,x = p^{-2w}\, g\left(\frac{1}{p}\right)$$

Now using (3.1) in (5.32) the result (5.31) follows after a little simplification.

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