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CONTENTS

REDUCIBLE AND IRREDUCIBLE 2-GENERATOR SUBGROUPS OF SL(2, C)

BY

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1. Introduction :

Let $a, b \in$ SL (2, C) with tr $a=\alpha$, tr $b=\beta$, tr $ab=\gamma$. In this paper we give a necessary and sufficient condition on the trace values of a, b and $a\,b$ for the group $< a, b > \subseteq$ SL (2, C) to be irreducible. Also if, for, any triplet α, β, γ of real or complex numbers $F_{\alpha, \beta, \gamma}$ denotes a subgroup of SL (2, C) which is generated by a pair of matrices a, b with tr $a=\alpha$, tr $b=\beta$, tr $ab=\gamma$ then we shall show that, provided $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0$, $F_{\alpha, \beta, \gamma}$ is an irreducible subgroup of SL (2, C) determined upto conjugacy.

2. Notations and definitions :

Throughout this paper GL $(n$, F) will denote the general linear group of degree n over a field F. The collection of those matrices in GL $(n$, F) whose determinant is 1, the identity element of F, is a subgroup of GL $(n$, F) called the special linear group of degree n and is denoted by SL $(n$, F). For any a $\in$ GL $(n$, F) tr a denotes the sum of diagonal elements of a. Also I represents the identity matrix.

If V is a vector space of dimension n over a field F then the set GL_n (V) of all invertible linear transformations of V to V is isomorphic to GL $(n$, F). Let G be a subgroup of GL_n (V). A subspace W of V is said to be invariant under G if x W $\subseteq$ W for all $x \in$ G. G is said to be irreducible if any and only if the only subspaces of V which are invariant under G are the null space {0} and V itself. Otherwise G is said to be reducible. For various properties of reducible and irreducible groups and conditions characterising these groups, see Dixon's books [1] and [2] and Wehrfritz's book [4].

3. Two generator irreducible sub-groups :

This section contains proofs of results about two generator irreducible sub-groups of SL (2,C) mentioned in the introduction above.

3.1 Theorem :

Let $a, b \in SL(2,C)$ with tr $a=\alpha$, tr $b=\beta$, tr $ab=\gamma$. Then $G = \langle a, b \rangle$ is irreducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0$$

Proof :

Suppose that $\langle a, b\rangle$ is irreducible. Then $a \neq I$, $b \neq I$. By a similarity transformation, both a and b can be brought into the

from $$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \mu'_2 \\ \mu'_3 & \beta-\mu_1 \end{pmatrix} \tag{1}$$

where $\lambda + \lambda^{-1} = \alpha = \text{tr}\, a \neq \pm 2$, $\mu_1(\beta - \mu_1) - \mu'_2\mu'_3 = 1$, $\beta = \text{tr}\ b$, $\mu'_2 \neq 0$, $\mu'_3 \neq 0$ because of irreducibility : or

$$\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \beta-\mu_1 \end{pmatrix} \tag{2}$$

when tr $a=\pm 2$. In the latter case $\mu_2 \neq 0$ because of irreducibility of $\langle a, b\rangle$. In the case of (1), we conjugate the matrices by

$$c = \begin{pmatrix} \delta & 0 \\ 0 & \delta' \end{pmatrix}$$

and obtain

$$a' = c\ a\ c^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

$$b' = c\ b\ c^{-1} = \begin{pmatrix} \mu_1 & \delta\delta'^{-1}\mu'_2 \\ \delta^{-1}\delta'\mu'_3 & \beta-\mu_1 \end{pmatrix}$$

Since $\mu'_3 \neq 0$, we choose δ, δ' such that $\mu'_3 = \delta\delta'^{-1}$. Then

$$a' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ b' = \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta-\mu_1 \end{pmatrix} \tag{3}$$

where $\mu_2 = \mu'_2\ \mu'_3 = -1 + \mu_1(\beta - \mu_1)$.

In case (2), that is when $\alpha = \pm 2$, we conjugate both matrices in (2) by

$$c = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$$

so that

$$a'' = c\,a\,c^{-1} = \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$b'' = c\,b\,c^{-1} = \begin{pmatrix} \mu_1 - \delta\mu_2 & \mu_2 \\ \delta\mu_1 + \mu_3 - \delta^2\mu_2 - \delta(\beta - \mu_1) & \delta\mu_2 + \beta - \mu_1 \end{pmatrix}$$

Take $\delta = \mu_1 / \mu_2$, $\mu_2 \neq 0$. Then

$$a'' = \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ b'' = \begin{pmatrix} 0 & \mu_2 \\ -1/\mu_2 & \beta \end{pmatrix},\ \mu_2 \neq 0 \tag{4}$$

Now

$$a'\,b' = \begin{pmatrix} \lambda\mu_1 & \lambda\mu_2 \\ \lambda^{-1} & \lambda^{-1}(\beta - \mu_1) \end{pmatrix}$$

$$a''\,b'' = \pm\begin{pmatrix} 0 & \mu_2 \\ -1/\mu_2 & \beta + \mu_2 \end{pmatrix}$$

in case (3) and (4) respectively. Since tr $a'b' =$ tr $a\,b$ and tr $a''\,b'' =$ tr $a\,b$ in the respective cases and tr $a\,b = \gamma$, we have :

$$\mu_1 = (\gamma - \beta/\lambda)/(\lambda - \lambda^{-1}) \quad \mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4)/(\alpha^2 - 4), \tag{5}$$

$\alpha \neq \pm 2$ in the latter equation in case (3) and

$$\mu_2 = \gamma' - \beta, \qquad \gamma' = \pm\gamma \tag{6}$$

in case of (4). Hence a, b can be transformed by a suitable conjugation into the forms

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix},\ \lambda + \lambda^{-1} = \alpha \neq \pm 2$$

with μ_1, μ_2 being given by (5); and into the form :

$$\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & \gamma' - \beta \\ -1/(\gamma' - \beta) & \beta \end{pmatrix},\ \gamma' = \pm\gamma$$

when $\alpha = \pm 2$. By the irreducibility of $<a, b>$

$$\mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4) / (\alpha^2 - 4) \neq 0, \alpha \neq \pm 2,$$

that is

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0 \qquad (7)$$

which conforms to the condition

$$\gamma' - \beta \neq 0, \quad \gamma' = \pm\gamma \qquad (8)$$

for the case (2) when $\alpha = \pm 2$.

Conversely, if conditions (7) and (8) are satisfied then $\mu_2 \neq 0$ in both cases so $<a, b>$ is irreducible. Hence $<a, b>$ is irreducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0.$$

The following corollary is immediate.

3.2 **Corollary :**

Let $a, b \in SL(2, C)$ with tr $a = \alpha$, tr $b = \beta$, tr $ab = \gamma$ Then $<a, b>$ is reducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = 0$$

3.3 **Corollary :**

Suppose that $<a, b>$ is a subgroup of SL (2, C) and let λ λ^{-1} ; μ, μ^{-1} be the characteristic roots of a, b respectively. Then $<a, b>$ is reducible if and only if

$$\text{tr } ab = \lambda/\mu + \mu/\lambda \text{ or } \lambda\mu + \lambda^{-1}\mu^{-1}$$

Proof :

By corollary 3.2 $<a, b>$ is reducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = 0, \alpha = \text{tr } a, \beta = \text{tr } b, \gamma = \text{tr } ab.$$

Now $\alpha = \lambda + \lambda^{-1}$, $\beta = \mu + \mu^{-1}$. Putting the values of α, β in terms of the characteristic roots of a and b and solving the quadratic equation in γ given above we get

$$\gamma = [\alpha\beta \pm \{(\alpha^2 - 4)(\beta^2 - 4)\}^{\frac{1}{2}}] / 2$$

$$= \lambda/\mu + \mu/\lambda \text{ or } \lambda\mu + \lambda^{-1}\mu^{-1}.$$

3.4 **Corollary** :

If α, β, γ satisfy the inequality (*) of theorem 3.1 then there exist $a, b \in SL(2, C)$ such that tr $a=\alpha$, tr $b=\beta$, tr $a\,b=\gamma$ and $< a, b >$ is an irreducible subgroup. Moreover, a and b are completely determined upto conjugacy, that is if tr $a'=\alpha$, tr $b'=\beta$ and tr $a'\,b'=\gamma$ then there exists $c \in GL(2, C)$ such that $a'=c\,a\,c^{-1}$, $b'=c\,b\,c^{-1}$.

P oof :

From equations (3) and (4) of theorem 3.1 we see that any two matrices $a, b \in SL(2, C)$ can be brought, by conjugation into the form

$$a_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad b_1 = \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix}$$

when tr $a = \alpha \neq \pm 2$ $\mu_1 = (\gamma - \beta / \lambda) / (\lambda - \lambda^{-1})$

$\mu_2 = -(\alpha^2+\beta^2+\gamma^2-\alpha\beta\gamma-4) / (\alpha^2-4) \neq 0$; or into the form

$$a_1 = \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & \gamma'-\beta \\ -1/(\gamma'-\beta) & \beta \end{pmatrix} \quad \gamma' = \pm\gamma$$

when tr $a_1 = \pm 2$. If a' and b' are any two other elements in SL (2, C) with tr $a'=\alpha=$tr a, tr $b'=\beta=$tr b, tr $a'\,b'=\gamma=$ tr $a\,b$, then a', b' are conjugate to a_1, b_1 respectively. Henee these exist $c_1, c_1 \in GL(2, C)$ such that

$$a_1=c_1\,a\,c_1^{-1}, \quad b_1=c_1\,b\,c_1^{-1}$$

and

$$a_1=c_2\,a'\,c_2^{-1}, \quad b_1=c_2\,b'\,c_2^{-1}$$

and so

$$a' = c\,a\,c^{-1}, \quad b' = c\,b\,c^{-1}$$

where $c = c_2^{-1}\,c_1$.

4. This section deals with the simplification of generators of irreducible groups discussed in section 3.

4.1 **Theorem :**

Let $G = <a, b>$ be an irreducible subgroup of SL (2,C.)

(*i*) If tr $a = \alpha = \pm 2$, tr $b = \beta = \pm 2$, tr $a\,b = \gamma$, then a, b

can be put, by a similarity transformation, simultaneously into the from

$$\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & \gamma'-2 \\ 0 & 1 \end{pmatrix}, \quad \gamma'=\pm\gamma$$

respectively, the signs being taken appropriately.

(*ii*) If the trace of at least one of the a and b, say of a is different from ± 2, then a, b can be put, by a similarity transformation, simultaneously into the form :

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta-\mu_1 \end{pmatrix} \tag{2}$$

respectively, where $\lambda+\lambda^{-1}=\alpha=$ tr a, $\beta=$ tr b, $\gamma=$tr $a\,b$

$\mu_1=(\gamma-\beta/\lambda)/(\lambda-\lambda^{-1})$, $\mu_2=-(\alpha^2+\beta^2+\gamma^2-\alpha\beta\gamma-4/(\alpha^2-4)$, $\alpha\neq 2$.

Proof :

We have already shown if $<a, b>$ is irreducible, a and b can be put in the from

$$a=\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, b=\begin{pmatrix} 0 & \gamma'-\beta \\ -1/(\gamma'-\beta) & \beta \end{pmatrix} \tag{1'}$$

where tr $a=\pm 2$, the signs being taken appropriately, and

$$a=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, b=\begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta-\mu_1 \end{pmatrix} \tag{2'}$$

otherwise; λ, β, γ, μ_1, μ_2 are as given above.

In (1′), if $\beta=\pm 2$, then we conjugate b by

$$c=\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$$

and obtain

$$c\,b\,c^{-1}=\begin{pmatrix} -\delta\,(\gamma'\mp 2) & \gamma'\mp 2 \\ -\delta^2\,(\gamma'\mp 2)\mp 2\delta-1/(\gamma'\mp 2) & -\delta(\gamma'\mp 2) \end{pmatrix}$$

Choose $\delta=\mp 1\,/\,(\gamma'\mp 2)$. Then

$$c\,b\,c^{-1}=\begin{pmatrix} \pm 1 & \gamma'\mp 2 \\ 0 & \pm 1 \end{pmatrix}, \quad c\,a\,c^{-1}=a$$

If, however, tr $a=\alpha\neq\pm 2$ then, as shown before, we have

$$a=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad b=\begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta-\mu_1 \end{pmatrix}$$

for all β where λ, β, μ_1, μ_2 are as given in the theorem

4.2 **Theorem :**

Let $G = <a, b>$ be an irreducible subgroup of $SL(2, C)$, λ a characteristic root of a and μ a characteristic root of b. Then a and b can be brought, by conjugation simultaneously into the from

$$\begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}, \quad \xi \neq 0, \eta \neq 0$$

Proof :

We can suppose that the trace of at least one of the a and b is different from ± 2 for otherwise the fact that a and b can be put in the form mentioned in the statement of the theorem is part (i) of theorem 3.5. Without any loss of generality we can assume that $\text{tr}\, a = \alpha \neq \pm 2$. Then, by theorem 3.5, a and b can be brought into the form.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix}$$

respectively, $\lambda + \lambda^{-1} = \alpha = \text{tr}\, a$, $\beta = \text{tr}\, b$, $\gamma = \text{tr}\, a\, b$,
$\mu_1 = (\gamma - \beta/\lambda)/(\lambda - \lambda^{-1})$, $\mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4)/(\alpha^2 - 4)$.
To prove the theorem we require an invertible c such that

$$c^{-1}\, a\, c = \begin{pmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{pmatrix}, \quad c^{-1}\, b\, c = \begin{pmatrix} \mu & * \\ 0 & \mu^{-1} \end{pmatrix}$$

Take

$$c = \begin{pmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_4 \end{pmatrix}$$

Then

$$a\, c = c \begin{pmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} * & \gamma_3 \lambda^{-1} \\ * & \gamma_4 \lambda^{-1} \end{pmatrix}$$

$$b\, c = c \begin{pmatrix} \mu & * \\ 0 & \mu^{-1} \end{pmatrix} = \begin{pmatrix} \mu\gamma_1 & * \\ \mu\gamma_2 & * \end{pmatrix}$$

Thus if we choose $\begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix}$ as eigenvector for λ^{-1} for a and $\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ as eigenvector for μ for b, then $c^{-1}\, a\, c$ and $c^{-1}\, b\, c$ are in the required from. Here, of course, the eigenvetors

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \text{ and } \begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix}$$

are linearly independent. (Otherwise, they would span a 1-diemsnional space invariant under $<a, b>$ contrary to the irreducibility of $<a, b>$).

4.3 **Corollary :**

Suppose that $<a, b>$ is an irreducible subgroup of SL (2, R) with tr $a=\alpha \geqslant 2$, tr $b=\beta \geqslant 2$. Then a and b can be brought simultaneously into the form

$$\begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}, \quad \xi \neq 0, \quad \eta \neq 0,$$

where $\lambda \geqslant 1$, $\mu \geqslant 1$.

Proof :

Just invert the matrices if necessary.

4.4 **Theorem :**

Suppose that

$$a=\begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \quad b=\begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

then $<a, b>$ is reducible if and only if either

$\xi\eta=0$ or $\xi\eta=-(\lambda-\lambda^{-1})\ (\mu-\mu^{-1})$.

Proof :

Here tr $a\, b = \xi\, \eta+\lambda\, \mu + \lambda^{-1}\, \mu^{-1}$. Hence the condition

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\, \beta\, \gamma - 4 = 0$$

of reducibility of $<a, b>$, after substituting

$\lambda+\lambda^{-1}$ for α, $\mu+\mu^{-1}$ for β, $\xi\eta+\lambda\mu+\lambda^{-1}\mu^{-1}$ for γ becomes

$$\xi\, \eta\, [\xi\, \eta + (\lambda + \lambda^{-1})\, (\mu - \mu^{-1})] = 0$$

i.e. $\xi\eta=0$ or $\xi\eta=-(\lambda-\lambda^{-1})\, (\mu-\mu^{-1})$.

4.5 **Corollary :**

Let

$$a=\begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix},\ b=\begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

If $\xi\eta=0$ or $\xi\eta=-(\lambda-\lambda^{-1})(\mu-\mu^{-1})$, then $<a, b>$ is not free on two generators.

Proof :

Here $<a, b>$ is reducible.

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A NOTE ON CLOSE-TO-CONVEX FUNCTIONS

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1. A function g: with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ that is regular in γ: ($\gamma = |z| < 1$) is called convex if

$$\text{Re} \frac{(zg'(z))'}{g'(z)} > 0, \quad z \in \gamma$$

A function f: with $(f)(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is called close-to-convex in γ if there exists a convex function g such that

$$\text{Re} \frac{f'(z)}{g'(z)} > 0, \quad z \in \gamma$$

Let C_r denote the closed curve which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$ and let $L_r(f)$ denote the length of C_r. Duren [2] has proved that if f is close-to-convex,

$$L_r(f) \leqslant L_r(K),$$

where K is the Koebe function.

We prove the following generalization of this result.

Theorem :

Let f be a close-to-convex function. Then for $\lambda > 1$,

$$L_r(f)^{\lambda} \leqslant L_r(K)^{\lambda}$$

Proof :

By definition

$$\frac{f'(z)}{g'(z)} = h(z), \text{ Re } h(z) > 0.$$

The function h can be represented as a stieltjes integral

$$\int_0^{2\pi} \frac{1+ze^{is}}{1-ze^{is}}\,dv(s), \qquad \int_0^{2\pi} dv(s) = 1,$$

where $v(s)$ is non-decreasing.
Further if g is a normalized convex function, there is a non-decreasing $\mu(t)$ such that [2]

$$g'(z) = \exp\left(\int_0^{2\pi} \log\,(1-z\;e^{it})^{-2}\,d\,\mu(t), \int_0^{2\pi} d\,\mu(t) = 1.\right.$$

Now, with $z = re^{i\theta}$, let

$$L_r\,(f)^{\lambda} = \int_0^{2\pi} |\,zf'(z)\,|^{\lambda}\,d\theta = r^{\lambda} \int_0^{2\pi} |\,f'(z)\,|^{\lambda} d\theta$$

$$= r^{\lambda} \int_0^{2\pi} |\,g'(z)\,h\,(z)\,|^{\lambda}\,d\theta$$

$$= r^{\lambda} \int_0^{2\pi} \left(\left| \int_0^{2\pi} \frac{1+ze^{is}}{1-ze^{is}}\,dv\,(s) \right\|^{\lambda} \exp \int_0^{2\pi} \log(1-ze^{it})^{-2} d\mu t \right|^{\lambda} \Bigg) d\theta$$

Using the exponential form of the arithmetic-geometric inequality we have

$$L_r\,(f)^{\lambda} < \int_0^{2\pi} \left\{ \left(\int_0^{2\pi} \left| \frac{1+ze^{is}}{1-ze^{is}} \right| dv(s) \right)^{\lambda} \int_0^{2\pi} \frac{d\mu(t)}{|\,1-ze^{it}\,|\,2\lambda} \right\} d\theta$$

To simplify the integral $\left(\int_0^{2\pi} \left| \frac{1+ze^{is}}{1-ze^{is}} \right| dv\,(s)\right)^{\lambda}$, we apply the

Holder's inequality with $p = \lambda > 1$ and $\frac{1}{q} = 1 - \frac{1}{p}$ and obtain

$$\left(\int_0^{2\pi} \left| \frac{1+ze^{is}}{1-ze^{is}} \right| dv(s)\right)^{\lambda} \leq \left(\int_0^{2\pi} \left| \frac{1+ze^{is}}{1-ze^{is}} \right|^{\lambda} dv(\ )\right)\left(\int_0^{2\pi} dv(s)\right)^{p/q}$$

$$= \int_0^{2\pi} \left| \frac{1+ze^{is}}{1-ze^{is}} \right|^{\lambda} dv(s)$$

Hence

$$L_r^{\lambda}(f) \leq \int_0^{2\pi} \left[\int_0^{2\pi} \left| \frac{1+ze^{is}}{1-ze^{is}} \right|^{\lambda} dv(s) \int_0^{2\pi} \frac{d\mu(t)}{|1-ze^{it}|^{2\lambda}}\right] d\theta$$

$$= \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|(1+re^{i(\theta+s)}|^{\lambda}}{|1-re^{i(\theta+s)}|^{\lambda} \, |1-re^{i(\theta+t)}|^{2\lambda}} \, d\mu(t)\, dv(s)\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2\pi} I(r,s,t)\, dv(s)\, d\mu(t),$$

where

$$I(r,s,t,) = \int_{-\pi}^{\pi} \frac{|1+re^{i(\theta+s)}|^{\lambda}}{|1-re^{i(\theta+s)}|^{\lambda}} \frac{d\theta}{|1-re^{i(\theta+t)}|^{2\lambda}}$$

we need the following.

Lemma [1]

If $F(\theta)$ and $G(\theta)$ are non-negative integrable functions and $\overset{*}{F}(\theta)$, $\overset{*}{G}(\theta)$ are their respective symmetrically decreasing re-arrangements as defined in [3, p 278], then

$$\int_{-\pi}^{\pi} F(\theta)\, G(\theta)\, d\theta \leqslant \int_{-\pi}^{\pi} \overset{*}{F}(\theta)\, \overset{*}{G}(\theta)\, d\theta$$

Applying this lemma, we have

$$I\,(r, s, t,) \leqslant I\,(r, o, o)$$

Thus

$$L_r\,(f)^{\lambda} = r^{\lambda} \int_{0}^{2\pi} |\, f'(z)|^{\lambda}\, d\theta$$

$$\leqslant \int_{0}^{2\pi} \frac{\left|1 + re^{i\theta}\right|^{\lambda}}{\left|1 - re^{i\theta}\right|^{3\lambda}}\, d\theta$$

$$= L_r\,(K)^{\lambda}$$

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SIMULTANEOUS DUAL INTEGRAL EQUATIONS WITH TRIGONOMETRIC KERNELS

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Dual integral equations arise in the solution of mixed boundary value problems. Szefer (1) and a few other authors have solved simultaneous dual integral equations with Bessel function $J_\nu(t)$ as kernel where Re $\nu > 0$. Here, we shall consider the following simultaneous dual integral equations.

$$(1a) \quad \int_0^\infty D(\alpha)\, \phi(\alpha) \sin \alpha t \; d\alpha = f(t) \qquad 0 < t < 1$$

$$(1b) \quad \int_0^\infty E(\alpha)\, \phi(\alpha) \sin \alpha t \; d\alpha = g(t) \qquad t > 1$$

where $D(\alpha) = \|D_{rs}\|$, $E(\alpha) = \|E_{rs}\|$ are non-singular $n \times n$ matrices of known functions ; $f(t)$, g (t) are columns of known functions (with n coordinates). $\phi(\alpha)$—a column of unknown functions (with n coordinates). The solution is obtained in terms of a Fredholm's integral equation of the second kind.

We require two lemmas which are generalized transformations of the well known solution of Abel's integral equation (cf. [3, p. 229]) under conditions which are sufficient for our purposes.

Lemma 1.

If $h(x)$ is continuously differentiable in the interval $[1, \infty)$ and $0 \leqslant \mu < 1$, then the solution of

$$h(x) = \int_x^\infty \frac{g(t)}{(t^2 - x^2)^\mu}\, dt \qquad (x > 1)$$

is given by

$$g(t) = \frac{-2 \sin \mu \pi}{\pi} \frac{d}{dt} \int_t^{\infty} \frac{x\, h(x)}{(x^2 - t^2)^{1-\mu}} dx$$

Lemma 2.

If $h(x)$ and $h'(x)$ are continuous in $a \leqslant x \leqslant b$ and $0 < \mu < 1$, then the solution of the integral equation

$$h(x) = \int_a^x \frac{g(t)}{(x^2 - t^2)^{\mu}} dt \qquad a < x < b$$

is given by

$$g(t) = \frac{2 \sin \mu \pi}{\pi} \frac{d}{dt} \int_a^t \frac{x h(x)}{(t^2 - x^2)^{1-\mu}} dx$$

Solution

We shall solve these integral equations in two steps. First, we shall put $f(t) = 0$ and then $g(t) = 0$. The solution of the set (1) is then obtained by just adding the two solutions.

Step I.

$$(2a) \qquad \int_0^{\infty} D(\alpha)\ \phi(\alpha)\ \sin\ \alpha t\ \ d\alpha = 0 \qquad 0 < t < 1$$

$$(2b) \qquad \int_0^{\infty} E(\alpha)\ \phi(\alpha)\ \sin\ \alpha t\ \ d\alpha = g(t) \qquad t > 1$$

Let us put $D(\alpha)\, \phi(\alpha) = \chi(\alpha)$, so that equations (2) reduce to

$$(3a) \qquad \int_0^{\infty} \chi(\alpha)\ \sin\ \alpha t\ \ d\alpha = 0 \qquad 0 < t < 1$$

$$(3b) \qquad \int_0^{\infty} L(\alpha)\ \chi(\alpha)\ \sin\ \alpha t\ \ d\alpha = g(t) \qquad t > 1.$$

where $L(\alpha) = E(\alpha)\, D^{-1}(\alpha)$.

We now consider the solution in the form

$$(4) \quad \chi(\alpha) = \int_1^\infty l(x)\, J_0(\alpha x)\, dx$$

where $l(x)$ is an unknown vector. Substituting (4) in the L.H.S. of (3*a*), we have

$$\int_0^\infty \left[\int_1^\infty l(x)\, J_0(\alpha x)\, dx \right] \sin \alpha t \quad d\alpha = 0$$

which gives on inversion of the order of integration

$$(5) \qquad \int_1^\infty l(x)\, dx \int_0^\infty J_0(\alpha x) \quad \sin \quad \alpha t \quad d\alpha = 0.$$

From the Weber-Schafheitlin integral

$$(6) \qquad \int_0^\infty J_0(\alpha x) \quad \sin \quad (\alpha t)\, (d\alpha) = \begin{cases} (t^2 - x^2)^{-\frac{1}{2}} & t > x \\ 0 & t < x \end{cases}$$

Since in (3 *a*), $t < 1$, we have $t < x$ and the integral vanishes identically

On substituting (4) in (3 *b*), we obtain

$$(7) \qquad \int_0^\infty L(\alpha) \{ \int_1^\infty l(x)\, J_0(\alpha x)\, dx \} \sin \alpha t \quad d\alpha = g(t) \quad t > 1$$

Let us put

$$(8) \quad \alpha L(\alpha) = I + V(\alpha)$$

where I is the unit matrix and

$$\begin{bmatrix} V_{11}(\alpha) & \alpha L_{12}(\alpha) \ldots\ldots\ldots & \alpha L_{ln}(\alpha) \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \alpha L_{nl}(\alpha) & \alpha L_{n2}(\alpha) & V_{nn}(\alpha) \end{bmatrix} : V_{rr}(\alpha) = \alpha L_{rr}(\alpha) - 1$$

then we have

$$(9) \quad g(t) = \int_0^\infty \frac{1}{\alpha} \{I + V(\alpha)\} \int_1^\infty l(x)\, J_0(\alpha x)\, dx\} \sin \alpha t\, d\alpha \quad t > 1$$

$$(10) \quad \therefore g(t) = \int_0^\infty \frac{1}{\alpha} \{ \int_1^\infty l(x)\, J_0(\alpha x)\, dx\} \sin \alpha t \quad d\alpha$$

$$+\int_0^\infty \frac{1}{\alpha} V(\alpha) \{ \int_1^\infty l(x) J_o(\alpha x)(dx) \} \sin \alpha t \, d\alpha \quad t > 1.$$

Inverting the order of integration, we have

$$(11) \quad g(t) = \int_1^\infty l(x)\, dx \int_0^\infty \frac{1}{\alpha} J_0(\alpha x) \sin \alpha t \, d\alpha$$

$$+\int_1^\infty l(x)\, dx \int_0^\infty \frac{1}{\alpha} V(\alpha) J_0(\alpha x) \sin \alpha t \, d\alpha \quad t > 1$$

Differentiating both sides with respect to 't' and making use the of fact (cf. [2, p. 405])

$$(12) \quad \int_0^\infty J_o(\alpha x) \cos \alpha t \, d\alpha = \begin{cases} (x^2 - t^2)^{-\frac{1}{2}} & t < x \\ 0 & t > x \end{cases}$$

we have

$$(13) \quad g'(t) = \int_t^\infty \frac{l(x)}{(x^2 - t^2)^{\frac{1}{2}}} dx + \int_1^\infty l(x)\, dx \int_0^\infty V(\alpha) J_0(\alpha x) \cos \alpha t \, d\alpha$$

Making use of the integral representation for $J_o(\alpha x)$

$$(14) \quad J_o(\alpha x) = \frac{2}{\pi} \int_x^\infty \frac{\sin y\alpha}{(y^2 - x^2)^{\frac{1}{2}}} dy$$

we get

$$(15) \quad g'(t) = \int_t^\infty \frac{l(x)}{(x^2 - t^2)^{\frac{1}{2}}} dx + \int_1^\infty l(x)\, dx \int_0^\infty$$

$$V(\alpha) \frac{2}{\pi} \int_x^\infty \frac{\sin y\,\alpha}{(y^2 - x^2)^{\frac{1}{2}}} dy \cos \alpha t \, d\alpha$$

Inverting the order of integration

$$(16)\quad g'(t) = \int_t^{\infty} \frac{l(x)}{(x^2-t^2)^{\frac{1}{2}}}\,dx + \frac{2}{\pi}\int_1^{\infty} l(x)\,dx \int_x^{\infty} \frac{dy}{(y^2-x^2)^{\frac{1}{2}}} \int_0^{\infty} V(\alpha)\sin y\alpha \cos \alpha t\, d\alpha$$

Let us substitute

$$(17)\quad \bar{V}(y, t) = \int_0^{\infty} V(\alpha)\sin y\alpha \cos \alpha t\, d\alpha \text{ and } g'(t) = G(t)$$

then we have

$$(18)\quad G(t) = \int_t^{\infty} \frac{l(x)}{(x^2-t^2)^{\frac{1}{2}}}\,dx + \frac{2}{\pi}\int_1^{\infty} l(x)\left\{\int_1^{\infty} \frac{\bar{V}(y,t)}{(y^2-x^2)^{\frac{1}{2}}}\,dy\right\} dx$$

If we put

$$(19)\quad \int_t^{\infty} \frac{l(x)}{(x^2-t^2)^{\frac{1}{2}}}\,dx = m(t), \text{ where } m(t) \text{ is a vector}$$

and apply Lemma 1, we have

$$(20)\quad l(x) = -\frac{2}{\pi}\frac{d}{dx}\int_x^{\infty} \frac{t\, m(t)}{(t^2-x^2)^{\frac{1}{2}}}\,dt$$

Substituting (19) and (20) in equation (18), we get

$$(21)\quad G(t) = m(t) - \frac{4}{\pi^2}\int_1^{\infty}\left\{\frac{d}{dx}\int_x^{\infty}\frac{z\, m(z)}{(z^2-x^2)^{\frac{1}{2}}}\,dz \int_x^{\infty}\frac{\bar{V}(y,t)}{(y^2-x^2)^{\frac{1}{2}}}\,dy\right\} dx$$

Putting

$$(22)\quad \int_x^{\infty} \frac{\bar{V}(y,t)}{(y^2-x^2)^{\frac{1}{2}}}\,dy = N(t, x)$$

and assuming that

$$(23)\quad \underset{X \to \infty}{Lt}\; N(t, x) = 0 \text{ (zero)}$$

we obtain

$$(24)\quad G(t) = m(t) - \frac{4}{\pi^2}\int_1^{\infty}\frac{d}{dx}\int_x^{\infty}\frac{z\,m(z)}{(z^2-x^2)^{\frac{1}{2}}}\,dz\quad N(t,x)\,dx$$

Since the vector $m(z)$ is bounded in $(1, \infty)$, the integral in (24) can be calculated by parts. Therefore

$$(25)\quad \int_1^{\infty}\frac{d}{dx}\int_x^{\infty}\frac{z\,m(z)}{(z^2-x^2)^{\frac{1}{2}}}\,dz\quad N(t,x)\,dx$$

$$= \int_x^{\infty}\frac{z\,m(z)}{(z^2-x^2)^{\frac{1}{2}}}\,dz\,N(t,x)\Big|_1^{\infty} - \int_1^{\infty}\int_x^{\infty}\frac{z\,m(z)}{(z^2-x^2)^{\frac{1}{2}}}dz\,N'_x(t,x)\,dx$$

$$= -\;N(t,1)\int_1^{\infty}\frac{z\,m(z)}{(z^2-1)^{\frac{1}{2}}}\,dz - \int_1^{\infty}\int_x^{\infty}\frac{z\,m(z)}{(z^2-x^2)^{\frac{1}{2}}}\,dz\,N'_x(t,x)\,dx$$

$$= -\;N(t,1)\int_1^{\infty}\frac{z\,m(z)}{(z^2-1)^{\frac{1}{2}}}\,dz - \int_1^{\infty}\left\{\int_1^{z}\frac{z\,m(z)}{(z^2-x^2)^{\frac{1}{2}}}N'_x(t,x)\,dx\right\}dz$$

Substituting (25) in (24)

$$(26)\quad G(t) = m(t) + \frac{4}{\pi^2}\left\{\; N(t,1)\int_1^{\infty}\frac{z\,m(z)}{(z^2-1)^{\frac{1}{2}}}\,dz\right.$$

$$\left. + \int_1^{\infty} z\,m(z)\,dz\int_1^{z}\frac{N'_x(t,x)}{(z^2-x^2)^{\frac{1}{2}}}\,dx\right\}$$

$$= m(t) + \frac{4}{\pi^2}\left\{\int_1^{\infty} z\,m(z)\left\{\frac{N(t,1)}{(z^2-1)^{\frac{1}{2}}} + \int_1^{z}\frac{N'_x(t,x)}{(z^2-x^2)^{\frac{1}{2}}}dx\right\}\right\}dz$$

Putting

$$(27)\quad z\;\left\{\frac{N(t,1)}{(z^2-1)^{\frac{1}{2}}} + \int_1^{z}\frac{N'_x(t,x)}{(z^2-x^2)^{\frac{1}{2}}}dx\right\} = K(t,z)$$

we finally obtain

$$(28)\quad G(t) = m(t) + \frac{4}{\pi^2}\int_1^{\infty} K(t, z)\, m(z)\, dz$$

This is a set of Fredholm's integral equations of the second kind which is known to be equivalent to a single integral equation of that kind. So, we can determine the coordinates of the vector $m(t)$ and hence that of $l(x)$. Therefore $\chi(\alpha)$ is known.

Step II.

Consider the set of n-dual integral equations given by :

$$(29a)\quad \int_0^{\infty} D(\alpha)\,\phi(\alpha)\,\sin \alpha t\, d\alpha = f(t) \qquad 0 < t < 1$$

$$(29b)\quad \int_0^{\infty} E(\alpha)\,\phi(\alpha)\,\sin \alpha t\, d\alpha = 0 \qquad t > 1$$

Putting $\alpha E(\alpha)\,\phi(\alpha) = \psi(\alpha)$, reduce the set of equations (29) to the form

$$(30a)\quad \int_0^{\infty} \frac{1}{\alpha}\, J(\alpha)\,\psi(\alpha)\,\sin \alpha t\, d\alpha = f(t) \quad 0 < t < 1$$

$$(30b)\quad \int_0^{\infty} \frac{1}{\alpha}\,\psi(\alpha)\,\sin \alpha t\, d\alpha = 0 \qquad t > 1$$

where $J(\alpha) = D(\alpha)\, E^{-1}(\alpha)$

Consider the solution in the form

$$(31)\quad \psi(\alpha) = \int_0^{1} u(x)\, J_0(\alpha x)\, dx$$

where $u(x)$ is a vector to be determined later. Differentiate equation

(30*b*) and substitute (31) in it. Making use of lemma 2, equation (12) and the following integral representation

$$(32)\quad J_o(\alpha x) = \frac{2}{\pi}\int_0^{\zeta} \frac{\cos z\alpha}{(x^2-z^2)^{\frac{1}{2}}}\, dz$$

we obtain a set of Fredholm's integral equations of the second kind. The procedure is exactly similar as in step 1. The form of the integral equations is given by

$$(33)\quad \nu(t) + \frac{4}{\pi^2}\int_0^1 K(t,\omega)\,\nu(\omega)\,d\omega = f'(t) = P(t)$$

where

$$(34)\quad \nu(t) = \int_0^t \frac{u(x)}{(t^2-x^2)^{\frac{1}{2}}}\, dx \quad .$$

The problem is therefore formally solved by superposing the two results. The simple vector addition may be used for this purpose and hence $\phi(\alpha)$ is determined.

Consider the following sets of *n* dual integral equations with trigonometric kernels

$$(35a)\quad \int_0^{\infty} D(\alpha)\ \phi(\alpha)\ \cos\alpha t\ d\alpha = f(t) \qquad 0 < t < 1$$

$$(35b)\quad \int_0^{\infty} E(\alpha)\ \phi(\alpha)\ \cos\alpha t\ d\alpha = g(t) \qquad t > 1$$

and

$$(36a)\quad \int_0^{\infty} D(\alpha)\ \phi(\alpha)\ {\sin \atop \cos}\ \alpha t\ d\alpha = f(t) \qquad 0 < t < 1$$

$$(36b)\quad \int_0^{\infty} E(\alpha)\ \phi(\alpha)\ {\cos \atop \sin}\ \alpha t\ d\alpha = g(t) \qquad t > 1$$

The equations can be solved in the similar way by reducing the equations into proper forms by integrating or differentiating the equations and by proper substitution for the matrix involved.

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VARIATIONAL INEQUALITIES AND APPROXIMATION

BY

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Abstract :—The existence and uniqueness of the solution of a class of nonlinear variational inequalities is considered, and methods of approximation of the solution are given using the linearization of variational inequalities.

Some elementary results concerning bilinear forms are given in the appendix.

Let H be a real Hilbert Space with its dual H′, whose inner product and norm are denoted by ((.)) and $\|.\|$ respectively. The pairing between $f \in H'$ and $u \in H$ is denoted by (f, u). Let F′ be the Frechet differential of a nonlinear functional F on a closed convex set M in H.

Consider also a coercive continuous bilinear from a (u, v) on H, *i.e.* there exist constants $\alpha > 0$, $\beta > 0$ such that

$$a(v, v) \geqslant \alpha \|v\|^2 \qquad \text{for all } v \in H, \tag{1}$$

$$|a(u, v)| \leqslant \beta \|u\| \, \|v\| \qquad \text{for all } u, v \in H. \tag{2}$$

Furthermore let F be a given element of H′. We now consider a functional I [v] defind by

$$I[v] = a(v, v) - 2F(v) \qquad \text{for all } v \in H.$$

Many mathematical problems either arise or can be formulated in this form. Here one seeks to minimize the functional I [v] over a whole space H or on a convex set M in H. It is well-known [I] that if F is a linear functional, then the element u which minimize I [v] on M is given by

$$a(u, v-u) \geqslant (F, v-u) \qquad \text{for all } v \in M. \tag{3}$$

For a nonlinear Frechet differentiable functional F, it was shown [3] that the minimum of the functional $I[v]$ on M is given by $u \in M$ such that

$$a(u, v-u) \geqslant (F'(u), v-u) \qquad \text{for all } v \in M. \qquad (4)$$

Such type of inequalities are known as variational inequalities [I]. Lions-Stampacchia [I] have studied the existence of a unique solution of (3). The motivation for this report is to show that under certain conditions there does exist a unique solution of a more general variational inequality of which (4) is a special case.

Let us consider the following problem.

Problems 1

Find $u \in M$ such that

$$a(u, v-u) \geqslant (Au, v-u) \qquad \text{for all } v \in M, \qquad (5)$$

where A is a nonlinear operator such that $Au \in H'$.

For $M=H$, the inequality (5) is equivalent to finding $u \in H$ such that

$$a(u, v) = (Au, v) \qquad \text{for all } v \in H,$$

and thus our results include the Lax-Milgram lemma as a special case.

Definition :

The operator $T : M \rightarrow H'$ is called *antimonotone*, if

$$(Tu - Tv, u-v) \leqslant 0 \qquad \text{for all } u, v \in M,$$

and is said to be *hemicontinuous* [4], if for all $u, v \in M$, the mapping $t \in [0,1]$ implies that $(T(u+t(v-u)), u-v)$ is continuous, Furthermore, T is *Lipschitz continuous*, if there exists a constant $0 < \gamma \leqslant 1$ such that

$$\|Tu - Tv\| \leqslant \gamma \|u-v\| \qquad \text{for all } u, v \in M.$$

Theorem 1

Let $a(u, v)$ be a coercive continuous bilinear form and M a closed convex subset in H. If A is a Lipschitz continuous antimonotone operator with $\gamma < \alpha$, then there exists a unique $u \in M$ such that (5) holds.

The following lemmas are needed for the proof.

Lemma 1.

If A is an antimonotone hemicontinuous operator, then $u \in M$ is a solution of (5) if and only if u satisfies

$$a(u, v-u) \geqslant (Av, v-u) \qquad \text{for all } v \in M \qquad (6)$$

Proof

If for a given u in M, (5) holds, then (6) follows by the antimonotonicity of A.

Conversely, suppose (6) holds, then for all $t \in [0, 1]$ and $w \in M$, $v_t \equiv u + t\,(w-u) \in M$, since M is a convex set. Setting $v = v_t$ in (6) we have

$$a(u, w-u) \geqslant (Av_t, w-u) \qquad \text{for all } w \in M.$$

Now let $t \to 0$. Since A is hemicontinuous, $Av_t \to Au$.
It follows that

$$a(u, w-u) \geqslant (Au, w-u) \qquad \text{for all } w \in M.$$

The map $v \to a(u, v)$ is linear continuous on H, so by Reisz-Frechet theorem, there exists an element $\eta = Tu \in H'$ such that

$$a(u, v) = (Tu, v) \qquad \text{for all } v \in H \qquad (7)$$

Let $\wedge$ be a canonical isomorphism from H′ onto H defind by

$$(f, v) = ((\wedge f, v)) \qquad \text{for all } v \in H, f \in H' \qquad (8)$$

Then $\|\wedge\|_{H'} = \|\wedge^{-1}\|_{H} = 1$. We note first that by (1), (2) and (7), it follows that

$$(i) \quad \|T\| \leqslant \beta$$
$$(ii) \quad \alpha \leqslant \beta$$

The next lemma is a generalization of a lemma of Lions-Stampacchia [1].

Lemma 2

Let ζ be a number such that $0 < \zeta < \frac{2(\alpha-\gamma)}{\beta^2-\gamma^2}$ and $\zeta < \frac{1}{\gamma}$. Then there exists a θ with $0<\theta<1$ such that

$$\| \phi (u_1)-\phi (u_2) \| \leqslant \theta \|u_1-u_2\| \qquad \text{for all } u_1, u_2 \in \mathrm{H},$$

where for $u \in \mathrm{H}$, $\phi (u) \in \mathrm{H}'$ is defined by

$$(\phi(u), v)=((u, v))-\zeta\, a\, (u, v)+\zeta\, (\mathrm{A}u, v) \qquad \text{for all } v \in \mathrm{H}. \tag{9}$$

Proof :

For all $u_1, u_2 \in \mathrm{H}$.

$$(\phi (u_1)-\phi (u_2), v)=((u_1-u_2, v))-\zeta\, a\, (u_1-u_2, v) + \zeta(\mathrm{A}u_1-\mathrm{A}u_2, v) \quad \text{for all } v \in \mathrm{H}$$

$$=((u_1-u_2, v))-\zeta(\mathrm{T}(u_1-u_2), v)+\zeta(\mathrm{A}u_1-\mathrm{A}u_2, v), \quad \text{by (7)}$$

$$=((u_1-u_2, v))-\zeta((\wedge \mathrm{T}(u_1-u_2), v))+ \zeta ((\wedge \mathrm{A}u_1-\wedge \mathrm{A}u_2 ,v)) \quad \text{by (8)}$$

$$=((u_1-u_2-\zeta\wedge \mathrm{T}(u_1-u_2), v))+\zeta((\wedge \mathrm{A}u_1-\wedge \mathrm{A}u_2, v))$$

Thus

$$|(\phi(u_1)-\phi(u_2), v)| \leqslant \|u_1-u_2-\zeta\wedge \mathrm{T}(u_1-u_2)\|\, \|v\|+\zeta\, \|\mathrm{A}u_1-\mathrm{A}u_2\|\, \|v\| \quad \text{for all } v \in \mathrm{H}.$$

Now using (7) and (8) we have

$$\|u_1-u_2-\zeta\wedge \mathrm{T}(u_1-u_2)\|^2 \leqslant \|u_1-u_2\|^2+\zeta^2\|\mathrm{T}\|^2\|u_1-u_2\|^2-2\zeta\, a(u_1-u_2,u_1-u_2)$$
$$\leqslant (1+\zeta^2 \beta^2-2\zeta\alpha)\|u_1-u_2\|^2, \text{ by coercivity of } a\,(u, v).$$

Then

$$|(\phi (u_1)-\phi (u_2), v| \leqslant \sqrt{1+\zeta^2 \beta^2-2\zeta\alpha})\, \|u_1-u_2\|\, \|v\|+\zeta\, \|\mathrm{A}u_1-\mathrm{A}u_2\|\, \|v\| \quad \text{for all } v \in \mathrm{H}.$$

$$\leqslant \theta \|u_1-u_2\|\, \|v\|, \text{ by the Lipschitz continuity of A,}$$

and $\theta = \sqrt{1+\zeta^2 \beta^2-2\zeta\alpha}+\gamma\zeta<1$ for $0<\zeta<2\dfrac{\alpha-\gamma}{\beta^2-\gamma^2}$ and $\zeta <\dfrac{1}{\gamma}$,

because $\alpha<\gamma$. Hence for all $u_1, u_2 \in \mathrm{H}$

$$\|\phi (u_1)-\phi (u_2)\| = \sup_{v\in \mathrm{H}} \frac{|(\phi(u_1)-\phi(u_2), v)|}{\|v\|} \leqslant \theta \|u_1-u_2\|.$$

The following results are proved by Mosco [2].

Lemma 3

Let M be a convex subset of H, Then, given $z \in H$ we have

$$x = P_M z,$$

if and only if

$$x \in M : \quad ((x-z, y-x)) \geqslant 0 \text{ for all } y \in M$$

where P_M is projection of H in M.

Lemma 4

P_M is non-expansive, *i.e.*

$$\|P_M z_1 - P_M z_2\| \leqslant \|z_1 - z_2\| \qquad \text{for all } z_1, z_2 \in H.$$

Using the technique of Lions-Stampacchia [1], we now prove theorem 1.

Proof of theorem 1

(*a*) *Uniqueness*

Let u_i, $i=1, 2$ be solutions in M of

$$a(u_i, v-u_i) \geqslant (Au_i, v-u_i) \qquad \text{for all } v \in M.$$

Setting $v \equiv u_{3-i}$ $i=1, 2$ in the above inequality, by addition we have

$$a(u_1-u_2, u_1-u_2) \leqslant (Au_1-Au_2, u_1-u_2).$$

Since $a(u, v)$ is a coercive bilinear form, there exists a constant $\alpha > 0$ such that

$$\alpha\|u_1-u_2\|^2 \leqslant (Au_1-Au_2, u_1-u_2) \leqslant 0,$$

by the antimonotonicity of A. From which the uniqueness of the solution $u \in M$ follows.

(*b*) *Existence*

For a fixed ζ as in Lemma 2, and $u \in H$, define $\phi(u) \in H'$ by (9). By lemma 3, there exists a unique $w \in M$ such that

$$((w, v-w)) \geqslant (\phi(u), v-w) \qquad \text{for all } v \in M,$$

and w is given by

$$w = P_M \wedge \phi(u) = Tu,$$

which defines a map H into M.

Now for all $u_1, u_2 \in H$,

$$\|Tu_1 - Tu_2\| = \|P_M \wedge \phi\,(u_1) - P_M \wedge \phi\,(u_2)\|$$
$$\leqslant \|\wedge \phi\,(u_1) - \wedge \phi\,(u_2)\|, \text{ by lemma 4,}$$
$$\leqslant \|\phi(u_1) - \phi\,(u_2)\|,$$
$$\leqslant \theta\,\|u_1 - u_2\|, \text{ by lemma 2.}$$

Since $\theta < 1$. Tu is a contraction and has a fixed point $u = Tu$, which belongs to M, a closed convex set and satisfies

$$((u, v-u)) \geqslant (\phi(u), v-u) = ((u, v-u)) - \zeta[a(u, v-u) - (Au, v-u)]$$

Thus for $\zeta > 0$,

$$a(u, v-u) \geqslant (Au, v-u) \qquad \text{for all } v \in M$$

showing that u is a unique solution of problem 1.

Remarks

1 : It is obvious that for $Au = F'(u)$, the existence of a unique solution of a variational inequality (4) follows under the assumptions of theorem 1.

2 : If A is independent of u, that is $Au = A'$ (say), then the Lipschitz constant γ is zero, and lemma 2 reduces to a lemma of Lions-Stamacchia [1] and ζ is a number such that $0 < \zeta < \frac{2\alpha}{\beta^2}$. Consequently theorem I is exactly the same as one proved by Lions-Stampacchia for the linear case. It is obvious that our result not only generalizes their result, but also includes it as a special case.

Method of Approximation

Suppose that the bilinear form is non-negative, *i.e.*

$$a\,(v, v) \geqslant 0 \qquad \text{for all } v \in H. \qquad (10)$$

Assume that there exists at least one solution $u \in M$ of

$$a\,(u, v-u) \geqslant (Au, v-u) \qquad \text{for all } v \in M \qquad (11)$$

and X is the set of all solutions of (11), Let, finally, $b\,(u, v)$ be a coercive bilinear form on H, that is there exists a constant $\alpha > 0$ such that

$$b\,(v, v) \geqslant \alpha\,\|v\|^2 \qquad \text{for all } v \in H \qquad (12)$$

First of all we prove some elementary but important lemmas.

Lemma 5

If $a(u, v)$ is a non-negative bilinear form and $u \in M$ then the inequality (5) is equivalent to the inequality

$$a(v, v-u) \geqslant (A(u), v-u) \qquad \text{for all } v \in M. \tag{13}$$

Proof :

Let (5) hold, then

$$a(v, v-u) \geqslant (A(u), v-u) + a(v-u, v-u) \geqslant (A(u), v-u) \quad \text{by (10)}$$

Thus (13) holds :

Conversely let (13) hold, then for all $t \in [0, 1]$ and $w \in M$, $v_t \equiv u + t(w-u) \in M$. Setting $v = v_t$ in (13) it follows that

$$a(u, w-u) + t\, a(w-u, w-u) \geqslant (A(u), w-u), \qquad \text{for all } w \in M.$$

Letting $t \to 0$, (5) follows.

As a consequence of lemma 1 and lemma 5 we have the following result.

Lemma 6

If $a(u, v)$ is non-negative bilinear form and A is hemicontinuous antimonotone operator, then the inequality (5) is equivalent to

$$a(v, v-u) \geqslant (A(v), v-u) \qquad \text{for all } v \in M$$

Theorem 2

If $b(u, v)$ is a coercive continuous bilinear form and B is a Lipschitz continuous antimonotone operator with $\gamma < \alpha$ then there exists a unique solution $u_0 \in X$ such that

$$b(u_0, v-u_0) \geqslant (Bu_0, v-u_0) \qquad \text{for all } v \in X \tag{14}$$

Proof :

Obviously X is closed. In order to prove theorem (2), it is enough to show that X is convex. Since $a(u, v)$ is non-negative, so (11) is equivalent to

$$a(v, v-u) \geqslant (Av, v-u), \quad \text{by lemma 6.}$$

Now for all $t \in [0, 1]$ $u_1, u_2 \in X$,

$$a(v, v-u_2-t(u_1-u_2)) = a(v, v-u_2) - t\, a(v, u_1-u_2)$$
$$= a(v, v-u_2) - t\, a(v, u_1-v+v-u_2)$$
$$= a(v, v-u_2) + t\, a(v, v-u_1) - t\, a(v, v-u_2)$$
$$= (1-t)\, a(v, v-u_2) + t\, a(v, v-u_1)$$
$$\geqslant (1-t)(Av, v-u_2) + t(Av, v-u_1), \text{ by lemma 6.}$$

Thus for all $t \in [0, 1]$, $u_1, u_2 \in X$, $t\, u_1 + (1-t)u_2 \in X$, which implies that X is a convex set. Hence by theorem (1), there does exist a unique solution $u_0 \in X$ satisfying (14).

Theorem 3

Assume that (10) and (12) hold. If $a(u, v) + \epsilon b(u, v)$ is a continuous bilinear from and A, B are both antimonotone Lipschitz continuous with $\gamma < \alpha$, then there exists a unique solution $u_\varepsilon \in M$ such that

$$a(u_\varepsilon, v-u_\varepsilon) + \varepsilon b(u_\varepsilon, v-u_\varepsilon) \geqslant (Au_\varepsilon + \varepsilon Bu_\varepsilon, v-u_\varepsilon) \quad \text{for all } v \in M \qquad (15)$$

Proof :

Since for $\varepsilon > 0$ and by (10), (12), the continuous bilinear form $a(u, v) + \varepsilon b(u, v)$ is coercive on H, then by theorem 1, there exists a unique $u_\varepsilon \in M$ satisfying (15).

Using lemma 1 and the methods of Sibony [4] and Lions-Stampacchia [1], we prove that the elements of X can be approximated.

Theorem 4

Suppose A, B : M→H′ are both hemicontinuous operators and the assumptions of theorems (2) and (3) hold. If u_0 is the element of X defined by (14) satisfying

$$a(u_0, v-u_0) \geqslant (Au_0, v-u_0) \qquad \text{for all } v \in X. \qquad (16)$$

and u'_ε is the element of M defined by (15), then

$u_\varepsilon \to u_0$ strongly in H as $\varepsilon \to 0$.

Proof :

This is proved in three steps.

(*i*) u_ε is bounded in H.

Setting $v=u_0$ in (15) and $v=u_\varepsilon$ in (16), we get

$$a(u_\varepsilon, u_0-u_\varepsilon)+\varepsilon\, b(u_\varepsilon, u_0-u_\varepsilon)\geqslant(Au_\varepsilon+\varepsilon Bu_\varepsilon, u_0-u_\varepsilon)$$

and

$$a(u_0, u-v_0)\geqslant(Au_0, u_\varepsilon-u_0)$$

By addition of these inequalities, it follows from (10) and the anti-monotonicity of A that

$$b(u_\varepsilon, u_0-u_\varepsilon)\geqslant(Bu_\varepsilon, u_0-u_\varepsilon). \tag{17}$$

Since $b(u_\varepsilon, u_\varepsilon)$ is a coercive bilinear form, there exists a constant $\alpha>0$ such that

$$\alpha\,\|u_\varepsilon\|^2\leqslant b(u_\varepsilon, u_0)+(Bu_\varepsilon, u_\varepsilon-u_0).$$

It follows that $\|u_\varepsilon\|\leqslant$ constant, independent of ε. Hence there exists a subsequence u_ε which converges to ξ, say.

(*ii*) ξ belongs to X.

Since A and B are antimonotone operators, by (15) and the application of lemma 1, we get

$$a(u_\varepsilon, v-u_\varepsilon)+\varepsilon\, b(u_\varepsilon, v-u_\varepsilon)\geqslant(Av+\varepsilon\, Bv, v-u_\varepsilon) \qquad \text{for all } v\in M.$$

Now let $\varepsilon\to0$, $u_\varepsilon\to\xi$ and $\liminf a(u_\varepsilon, u_\varepsilon)\geqslant a(\xi,\xi)$, [1]. We have

$$a(\xi, v-\xi)\geqslant(Av, v-\xi) \qquad \text{for all } v\in X,$$

which is by lemma 1 equivalent to

$$a(\xi, v-\xi)\geqslant(A\xi, v-\xi) \qquad \text{for all } v\in X.$$

Thus $\xi\in X$.

(*iii*) Finally $\|u_\varepsilon\|\to\|\xi\|$ when $\varepsilon\to0$.

Setting $v=u\in X$ in (15) and $v=u_\varepsilon\in X$ in (11).

We obtain.

$$a(u_\varepsilon, u-u_\varepsilon)+\varepsilon\, b(u_\varepsilon,\ u-u_\varepsilon)\geqslant(Au_\varepsilon+\varepsilon\, Bu_\varepsilon, u-u_\varepsilon),$$

which is, by lemma 1, equivalent to

$$a\,(u_\varepsilon, u-u_\varepsilon)+\varepsilon\, b(u_\varepsilon,\ u-u_\varepsilon)\geqslant(Au+\varepsilon\, Bu-u_\varepsilon).$$

Also,

$$a\,(u, u_\varepsilon-u)\geqslant(Au,\ u_\varepsilon-u)$$

By addition one has

$$a\,(u_\varepsilon-u, u-u_\varepsilon)+\varepsilon\, b(u_\varepsilon,\ u-u_\varepsilon)\geqslant\varepsilon\,(Bu, u-u_\varepsilon)$$

Using (10), and for $\varepsilon>0$, we get

$$b\,(u_\varepsilon,\ u-u_\varepsilon)\geqslant(Bu, u-u_\varepsilon) \qquad \text{for all } u\in X.$$

Letting $\varepsilon\to0$, $u_\varepsilon\to\xi$, we have

$$b\,(\xi, u-\xi)\geqslant(Bu, u-\xi)$$

$$\geqslant(B\xi,\ u-\xi) \qquad \text{by lemma 1.}$$

Thus $\xi\in X$ is a solution of (14) and since the solution is unique, it follows that $\xi=u_0$.

Also from (17), by the coercivity of $b(u_\varepsilon, u_\varepsilon)$, it follows that there exists a constant $\alpha\geqslant0$ such that

$$\alpha\|u_\varepsilon-u_0\|^2\leqslant b\,(u_\varepsilon-u_0, u_\varepsilon-u_0)$$

$$\leqslant(Bu_\varepsilon,\ u_\varepsilon-u_0)-b(u_0, u_\varepsilon-u_0)$$

$$\leqslant(Bu_0,\ u_\varepsilon-u_0)-b(u_0, u_\varepsilon-u_0), \quad \text{by lemma 1,}$$

which$\to0$, as, $\varepsilon\to0$. Hence it follows that $u_\varepsilon\to u_0$ strongly in H.

Theorem 5

If $a(u, v)$, $b(u, v)$ are coercive continuous bilinear forms, M is a closed convex set in H, and A, B are hemicontinuous antimonotone Lipschitz continuous operators with $\alpha > \gamma$, then problem 1 has a unique solution if and only if there exists a constant L, independent of ϵ, such that the solution of (15) satisfies

$$\|u_\varepsilon\| \leqslant L \tag{18}$$

Proof :

If there exists a solution, then from theorem 4, it follows that (18) holds. Conversely suppose that (18) holds, then there exists a subsequence u_η of u_ε which converges to w weakly in H. Since M is a closed convex set, $w \in M$. Further writing (15) in the form

$$a(u, u-v)+\varepsilon\, b(u_\varepsilon, u_\varepsilon - v) \leqslant (Av+\varepsilon\, Bv, u_\varepsilon - v) \qquad \text{for all } v \in M$$

and taking $\varepsilon = \eta = 0$, we find that

$$a(w, w) \leqslant a(w, v)+(Av, w-v) \qquad \text{for all } v \in M,$$

which is by lemma 1, equivalent to

$$a(w, w-v) \leqslant (Aw, w-v) \qquad \text{for all } v \in M.$$

Thus w is the solution satisfying (11).

EXISTENCE OF SOLUTIONS

In this section, the existence of the solution satisfying (10) for the cases, when M is bounded or an unbounded convex subset of H is considered.

Theorem 6

If M is a bounded closed convex subset, and A is a hemicontinuous Lipschitz antimonotone operation, then there exists a unique solution of problem (1).

Proof :

Let $u_\varepsilon \in M$ be the element defind by (15). Since M is bounded, then $\|u_\varepsilon\|$ is bounded, and theorem (6) follows from theorem (5).

Consider now the case when the set M is bounded. Let $M_R = \{k; k \in M, \| k \| \leqslant R\}$ with R large enough so that $M_R \neq \phi$. Assume that A is hemicontinuous antimonotone operator, then by theorem (6), there exists a non-empty set,

$$X_R \equiv \text{set of all solution of } w \in M_R \text{ with} \tag{19}$$

$$a(w, v-w) \geqslant (Aw, v-w) \qquad \text{for all } v \in M_R$$

Theorem 7

Suppose $a(u, v)$ is a continuous bilinear form and A is a hemicontinuous antimonotone operator. If $u \in X_R$ with $\|u\| < R$, then u satisfies (11).

Proof :

In fact, let w be any solution in M. Then for $0 < \varepsilon < 1$, $u + \varepsilon(w-u) \in M$ and $\|u + \varepsilon(w-u)\| \leqslant \|u\| + \varepsilon \|w-u\| < R$ for sufficiently small ε. Thus for $0 < \varepsilon < \varepsilon_1$, $v = u + \varepsilon\ (w - u) \in M_R$. Consequently such a v is allowed in (19) with $w = u$ and it follows that

$$a(u, w-u) \geqslant (Au, w-u) \qquad \text{for all } w \in M.$$

This proves theorem 7.

APPENDIX

Let $a(u, v)$ be a coercive continuous bilinear form on H. The Cauchy-Schwarz inequality holds for $a(u, v)$ and is given by

$$| a(u, v) |^2 \leqslant a(u, u)\, a(v, v) \qquad \text{for all } u, v \in H,$$

Theorem 8

A bounded bilinear form is continuous with respect to the norm convergence.

Proof :

Let $u_n \to u$ and $v_n \to v$, these sequences are bounded. We let γ be their bound, and then $\| u_n \| \leqslant \gamma$.

Now

$$| a(u_n, v_n) - a(u, v) | = | a(u_n, v_n) - a(u_n, v) + a(u_n, v) - a(u, v) |$$
$$\leqslant | a(u_n, v_n - v | + | a(u_n - u, v) |$$
$$\leqslant C \gamma \|v_n - v\| + C_1 \|u_n - u\| \ \|v\|,$$

by the Cauchy-Schwarz inequality. But $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$ as $n \to \infty$, and therefore

$$| a(u_n, v_n) - a(u, v) | \to 0, \textit{ i.e.},$$
$$a(u_n, v_n) \to a(u, v).$$

From now on we use the norm $\|u\|^2 = a(u, u)$ on H. We now prove the following results which are more general than and include the classical results as special cases.

Theorem 9

Let v be in H and M be a closed convex subset of H. If $a(u, v)$ is a continuous and symmetric bilinear form on H, then $u \in M$ satisfies

$$a(u - v, w - u) \geqslant 0, \qquad \text{for all } w \in M, \qquad (20)$$

if and only if

$$\|u - v\| \leqslant \|w - v\| \qquad \text{for all } w \in M \qquad (21)$$

Proof :

If $u \in M$ satisfies (21), then we have to show that (20) holds.

Suppose to the contrary that there is a vector $v_1 \in M$ such that $a(u - v, u - v_1) = \epsilon > 0$. Now for all $t \in [0, 1]$ and $v_1 \in M$, $v_t = u + t(v_1 - u) \in M$, we have

$$\| v_t - v \|^2 = \|u + t(v_1 - u) - v\|^2$$
$$= a(u - v + t(v_1 - u), u - v + t(v_1 - u))$$
$$= a(u - v, u - v) + t^2 a(v_1 - u, v_1 - u) + 2t\, a(u - v, v_1 - u)$$
$$= \|u - v\|^2 + t^2 \|v_1 - u\|^2 + 2t\, a(u - v, v_1 - u)$$
$$< \| u - v \|^2,$$

for small positive t, which contradict (21). Hence no such $v_1 \varepsilon M$ can exist.

Conversely let $u \varepsilon M$ such that (20) holds, then for any $w \neq u$, $w \varepsilon M$, we have

$$\| v-w \|^2 = \|v-u + u-w\|^2$$
$$= a(v-u + u-w, v-u + u-w)$$
$$= \|v-u\|^2 + 2a(v-u, u-w) + \|u-w\|^2$$
$$> \| v-u \|^2.$$

Thus $u \varepsilon M$ satisfies (21).

We note that for $a(u, v) = (u, v)$, theorem 9 reduces to the following well known minimum norm problem [6].

Corollary :

Let v be in H and M be a closed convex subset of H. Then $u \varepsilon M$ satisfies

$$(u-v, w-u) \geqslant 0, \qquad \text{for all } w \in M,$$

if and only if

$$\|u-v\| \leqslant \|w-v\|, \qquad \text{for all } w \varepsilon M.$$

Our next result is a variant from of the projection theorem, which has many applications in numerical analysis, see *e.g.* Strang and Fix [5], but has not been proved.

Theorem 10

Let $a(u, v)$ be a continuous symmetric bilinear form on H and S be a closed proper subspace of H. If $u \varepsilon H$, then there exists a unique $u_0 \varepsilon S$ such that

$$a(u-u_0, v) = 0, \qquad \text{for all } v \varepsilon S.$$

Here u_0 can be considered as a projection of u on S.

Proof :

Let $S^{\perp} = \{u \in H,\ a(u, v)=0,$ for all $v \in S\}$. For each $v \in S$, the set $\{u \in H,\ a(u, v)=0,$ for all $v \in S\}$ is closed, because it is the inverse image of the set $\{0\}$ under the continuous functioual $a(v.,)$ and $S^{\perp}$ is the intersection of all these sets over $v \in S$. Hence $S^{\perp}$ is closed. Let u be any point of H. Since S is closed, there exists a point nearest to u, say $u_0 \in S$. We shall show that $u-u_0 \in S^{\perp}$ or in otherwords $a(u-u_0, v)=0$, for all $v \in S$.

Let $v \neq 0$ be any arbitrary point of S. The $u_0+\alpha v \in S$ for all $\alpha \in R$. Thus by theorem 1 [6, p. 50], it follows that

$$\begin{aligned} \|u-u_0\|^2 &\leqslant \| u-u_0-\alpha v \|^2 \\ &= a(u-u_0-\alpha v,\ u-u_0-\alpha v) \\ &= a(u-u_0,\ u-u_0)-2\alpha\, a(u-u_0,\ v)+\alpha^2 a(v, v) \\ &= \|u-u_0\|^2-2\alpha\, a(u-u_0,\ v)+\alpha^2 \|v\|^2 \end{aligned}$$

that is

$$-2\alpha\, a(u-u_0,\ v)+\alpha^2 \| v \|^2 \geqslant 0.$$

Letting $\alpha \to 0$, it follows that

$$a(u-u_0,\ v) = 0, \qquad \text{for all } v \in S$$

Remark :

For the special case $a(u, v) = (u, v)$, we get the well known projection theorem [6].

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ELLIPTIC CURVES WITH POINTS OF ORDER 3

by

SHAMIM AKHTAR

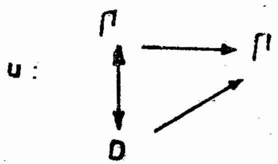
1. In a list of elliptic curves with small conductors obtained experimentally by Swinnerton-Dyer it is found that many curves have rational points of finite order. It is desirable to determine the groups of rational points of these curves, and the rational points of finite order make it easier. There are standard methods for using points of order 2 (see Cassels' report § 24 [1]), here we will use points of order 3.

It is well known that an elliptic curve Γ with rational point on it can be written in the homogeneous form

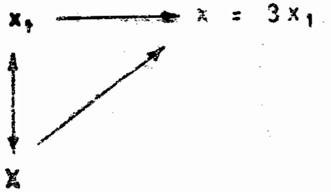
$$y^2 z = x^3 - A x z^2 - B z^3$$

If we write θ for the point, (0, 1, 0), then the points of Γ form an abelian group with θ as unit and the rational points form a subgroup A which is finitely generated.

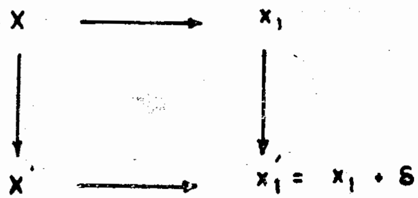
Following Cassels [2] we say that there is a 3-covering of Γ if there is Curve D defined over the rational field Q and a commutative triangle



with associated generic points.



where the map $X \rightarrow x$ is over the rationals and $x_1 \longleftrightarrow X$ is over the complex numbers. Another curve D′ with generic X′ gives the same 3-coverings if and only if there is a birational mapping $X \to X'$ over the rationals and a point δ on Γ with $3\,\delta = 0$ such that the diagram is commutative. There is a natural structure of abelian group on the

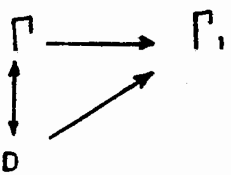


3-coverings [see Cassels [2]; under this law the 3-coverings form an abelian group G. We are interested in its subgroup G′, the group of coverings for which D has a rational point. Weil has shown that G′ is finite and is isomorphic to A/3 A.

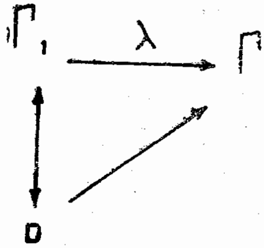
In general, it is not practicable to compute the 3-coverings of a curve Γ. However, sometimes Γ will be 3-isogeneous to a curve Γ_1 so that there are maps λ, λ' defined over Q

$$\Gamma \xrightarrow{\ \lambda\ } \Gamma_1 \xrightarrow{\ \lambda'\ } \Gamma$$

which are group homomorphisms with kernel of order 3, and with $\lambda' \lambda\, x = 3\, x$ for x on Γ. In this case, we compute the group G_λ of



λ-coverings of Γ_1 and the group $G_{\lambda'}$ of λ'-coverings



of Γ; and we have $G \simeq G_\lambda \times G_{\lambda'}$.

In section 2 we work out when such maps λ, λ' exist. We find that an elliptic curve with points of order 3 has a λ-covering of the form $x^3+y^3+2\eta z^3=bxyz$. We note that when dealing with equations of the type $u^3+v^3+w^3=kuvw$, we have a choice of using either of two systems of isogenies

$$\Gamma_1 \xrightarrow{\lambda'} \Gamma_2 \xrightarrow{\lambda} \Gamma_1$$

$$\Gamma_1 \xrightarrow{\mu'} \Gamma_3 \xrightarrow{\mu} \Gamma_1$$

with $\lambda\,\lambda'$ and $\mu\,\mu'$ multiplications by 3.

In section 3 we are concerned with the equations of the type $x^3+y^3+z^3=dxyz$.

We prove

Theorem 1

If d is a rational integer such $(d, 3)=1$, $d \neq -1, 5$, $d-3$ has no factors of form $6n+1$, and d^2+3d+9 is prime, then the equation $x^3+y^3+z^3=dxyz$ has no non-trivial rational integral solutions.

The particular case $d=1$ has been treated by Sansone and Cassels [3], by more or less the same method as ours, using the isogenies

$$\Gamma_1 \xrightarrow{\lambda'} \Gamma_2 \xrightarrow{\lambda} \Gamma_1$$

described above. Their proof is completely elementary, using the classical method of descent. We find it more convenient to use the isogenies

$$\Gamma_1 \xrightarrow{\mu'} \Gamma_3 \xrightarrow{\mu} \Gamma_1$$

We give a completely elementary proof using the classical descent method. By good luck, this works; however one would be safer to use coverings and so forth, as described in Cassels' report.

2. Let Γ be an elliptic curve Q with a rational point such that there is a curve Γ_1 and maps λ, λ' defined over Q.

$$\Gamma \xrightarrow{\lambda} \Gamma_1 \xrightarrow{\lambda'} \Gamma$$

which are group homomorphism with kernels of order 3, and with

$$\lambda' \lambda x = 3x \quad \text{for} \quad x \in \Gamma.$$

We may suppose that Γ is given by an equation of form

$$yz^2 = x^3 + a_1 x^2 z + a_2 x z^2 + a_3 z^3 ;$$

the zero of the group Γ is θ the point at ∞, and the points of order 3 are the inflections. The kernel of λ is defined over Q ; it consists of θ and a pair $(\xi, \pm \eta)$ of inflections ; then ξ must be rational and by a change of co-ordinates we may suppose that $\xi = 0$.

Suppose that at the inflection $y' = m$; then we have $\eta^2 = a_3$, $2\eta \; m = a_2$, $m^2 = a_1$; so $a_2^2 = 4a_1 \; a_3$ and we may write Γ as

$$z\, y^2 - a\, (x + \eta\, z)^2\, z = x^3$$

Given Γ of this form, it is easy to find a curve Γ_1 and a map $\lambda' : \Gamma_1 \to \Gamma$. Factoring in $Q(\sqrt{a})$, we take

$$y \pm \sqrt{a}\, (x + \eta\, z) = (u \pm \sqrt{a}\; v)^3$$

$$z = t^3$$

$$\text{so that} \quad x = (u^2 - a v^2) t$$

$$(u^2 - a\, v^2)\, t + \eta\, t^3 = 3\, u^2\, v + a\, v^3$$

Write $v - \dfrac{t}{3} = w$; then (t, u, w) lies on the curve

$$\Gamma_1 : -3\, u^2 \;\; w = \left(\frac{4\, a}{27} - \eta\right) t^3 + a w (w + t)^2$$

essentially of the same form as Γ.

If in particular Γ has two rational inflections, then a is a square, say $a=b^2$. If we set

$$y+b\,x+b\,z=u^3$$
$$-y+b\,x+b\,z=v^3$$
$$z=-w^3$$
$$\text{then } x=u\,v\,w;$$

the point (u, v, w) lies on a curve

$$\Gamma_1 : u^3+v^3+2\eta\, b\, w^3 = 2\,buvw,$$

and we have a map $\lambda' : \Gamma_1 \rightarrow \Gamma$.

It is more convenient to take a different form for Γ. The equation for Γ_1 may be written as

$$(3u+3v+2bw)\ (3u\rho+3v\,\rho^2+2bw)\ (3u\rho^2+3v\rho+2bw) = (8b^3-54\eta\, b)\, w^3$$

where $\rho=\dfrac{-1+\sqrt{-3}}{2}$. Now let

$$8b^3\,(3u+3v+2bw)=(8b^3-54\,\eta\, b)\,(A+B+C)^3$$
$$3u\rho+3v\,\rho^2+2bw=(A\rho+B\rho^2+C)^3$$
$$3u\rho^2+3v\rho+2bw=(A\rho^2+B\rho+C)^3$$

then $$2bw=A^3+B^3+C^3-3\,ABC.$$

and the point (A, B, C) lies on

$$\Gamma_2 : 4b^3\, A\, B\, C=\eta\,(A+B+C)^3.$$

The curve Γ_2 is actually equivalent to Γ; we have just given a map

$$\lambda : \Gamma_2 \rightarrow \Gamma_1 \text{ ; we get } \lambda' : \Gamma \rightarrow \Gamma_2 \text{ by}$$
$$A=u^3,\ B=v^3,\ C=2\eta\, w^3$$

We may check that the composite map $\lambda'\lambda : \Gamma_1 \rightarrow \Gamma_1$ is multiplication by 3.

Even more particularly, when we take Γ_1 in the form $u^3+v^3+2\eta\, w^3=2uvw$, 2η may be a cube. It is then sensible to absorb the 2η into w^3, so that Γ_1 become $\Gamma_1 : u^3+v^3+w^3=kuvw$, where k is not necessarily integral. As before we have

$$\lambda : \Gamma_2 \rightarrow \Gamma_1 \text{ by writing } \Gamma_1 \text{ as}$$

$$(3u+3v+kw)\,(3u\rho+3v\rho^2+kw)\,(3u\rho^2+3v\rho+kw) = (k^2-27)\,w^3$$

and setting $k^3\,(3u+3v+kw)=(k^3-27)\,(A+B+C)^3$

$$3u\rho + 3v\rho^2 + kw = (A\rho+B\rho^2+C)^3$$

$$3u\rho^2 + 3v\rho + kw = (A\rho^2+B\rho+C)^3$$

so that $$kw = A^3+B^3+C^3-3ABC$$

and (A, B, C) lies on

$$\Gamma_2 : \quad (A + B + C)^3 = k^3\,ABC.$$

We get $\lambda' : \Gamma_1 \to \Gamma_2$ by $A = u^3$, $B = v^3$ $C = w^3$. We may also write Γ_1 as

$$(3u + 3v + kw)\,(3u + kv + 3w)\,(ku + 3v + 3w)$$
$$= (k^2 + 3k + 9)\,(u + v + w)^3.$$

Write $$3u+3v+kw=(k^2+3k+9)Z^3$$

$$3u+kv+3w=Y^3$$

$$ku+3v+3w=X^3$$

so $$u+v+w=XYZ$$

and the point (X, Y, Z) lies on the curve

$$\Gamma_3 : \quad X^3+Y^3+(k^2+3k+9)\,Z^3=(k+6)XYZ$$

and we have constructed an isogeny $\mu : \Gamma_3 \to \Gamma_1$. As before, we can get a map $\mu' : \Gamma \to \Gamma_3$ with $\mu\,\mu'$ multiplication by 3, by writing Γ_3 as

$$(3X+3Y+(k+6)\,Z)\,(3\rho X+(k+6)\,Z + 3\rho Y^2)\,((k + 6)Z + 3\rho Y + 3X\rho^2)$$
$$=(k-3)^3\,Z^3$$

and setting $3X+3Y+(k+6)\,Z=(U+V+W)^3$

$$3\rho X+3\rho^2\,Y+(k+6)\,Z=(U\rho+V\rho^2+W)^3$$

$$3\rho^2\,X+3\rho Y+(k+6)\,Z=(U\rho_2+V\rho+W)^3$$

so that $\quad (k-3)\,Z=U^3+V^3+W^3-3\,UVW$

and $\quad (k+6)\,Z=U^3+V^3+W^3+6\,UVW$

therefore $\quad Z=UVW$ and consequently $U^3+V^3+W^3=kUVW$

so (U, V, W) lies on Γ_1.

Accordingly, when we deal with curves of the type

$$u^3 + v^3 + w^3 = kuvw$$

we have the choice of using either of the isogenies mentioned above.

3 ; In this section we give a proof of theorem 1.

We suppose that $x^3 + y^3 + z^3 = dxyz$ has an integral solution (x, y, z) such that $|xyz| \neq 0$. Suppose that $|xyz|$ is minimal, so that $(x, y, z) = 1$. We have

$$(3x + 3y + dz)(3x + dy + 3z)(dx + 3y + 3z)$$
$$= (d^2 + 3d + 9)\ (x + y + z)^3$$

we may suppose

$$3x + 3y + dz = cw^3$$
$$3x + dy + 3z = bv^3$$
$$dx + 3y + 3z = au^3$$

where $$abc = (d^2 + 3d + 9)\, e^3 \qquad (1)$$

and a, b, c have no cube factors. Then

$$x + y + z = euvw$$

and $$au^3 + bv^3 + cw^3 = (d + 6)\, euvw \qquad (2)$$

If p is a common factor of (a, b, c), we may remove it from $a, b. c, e$ without disturbing (1) and (2), and so we may suppose that $(a, b, c) = 1$. If now $p \mid e$, we may suppose that $p \nmid a$, and then that $p \parallel b, p^2 \parallel c$; but this makes (2) insoluble modulo powers of p.

Hence we may suppose $e = 1$, and we have a solution of

$$au^3 + bv^3 + cw^3 = (d + 6)uvw$$

with $abc = d^2 + 3d + 9$. By hypothesis $d^2 + 3d + 9$ is prime, so we may suppose $a = b = 1$,

$$u^3 + v^3 + (d^2 + 3d + 9)\ w^3 = (d + 6)uvw$$

This is $(3u)^3 + (3v)^3 + ((d + 6)\ w)^3 - 27\ (d + 6)\ uvw = (d - 3)^3\ w^3$;
write $$\gamma = 3u + 3v + (d + 6)w$$
$$\sigma = 3u\rho + 3v\rho^2 + (d + 6)w$$

then $\gamma\sigma\bar{\sigma}=((d-3)\,w)^3=\left(\frac{(d-3)\ (\gamma+\sigma+\bar{\sigma})}{3\,(d+6)}\right)^3.$

Proceeding much as before, but now working in $Q(\sqrt{-3})$, we may suppose

$\gamma=\lambda a^3$, $\sigma=\mu\,\beta^3$, $\bar{\sigma}=\bar{\mu}\,\bar{\beta}^3$, where $\lambda\in Q$, $\lambda\,\mu$, γ are cube free that except we may have $9\mid\lambda$, and $\lambda\,\mu\,\bar{\mu}$ is a cube, say

$$\lambda\,\mu\,\bar{\mu}=f^3,\ f\in Q \qquad (3)$$

Then $[3\,(d+6)\,f\,a\,\beta\,\bar{\beta}]^3=[(d-3)\ (\gamma+\sigma+\bar{\sigma})]^3$, so

$$3\,(d+6)\,f\,a\,\beta\,\bar{\beta}=(d-3)\,(\lambda a^3+\mu\beta^3+\bar{\mu}\,\bar{\beta}^3) \qquad (4)$$

A common factor of λ, μ, $\bar{\mu}$ is either rational, or a rational multiple of $\sqrt{-3}$; we can throw away common rational factors, so that $(\lambda,\mu,\bar{\mu})=(1)$ or $(\sqrt{-3})$. Suppose that a prime $\Pi\neq\sqrt{-3}$ divides $\lambda\,\mu\,\bar{\mu}$. The power of $\bar{\Pi}$ in $\bar{\mu}$ is equal to the power of Π in μ, $(\lambda,\mu,\bar{\mu})$ is not divisible by Π, and none of λ, μ, $\bar{\mu}$ is divisible by Π^3; the only possibilities are

$$\Pi\,\chi\,\bar{\mu},\ \Pi\,\|\,\mu,\ \Pi^2\,\|\,\lambda,\ \Pi\,\|\,f$$

and $$\Pi\,\chi\,\bar{\mu},\ \Pi^2\,\|\,\mu,\ \Pi\,\|\,\lambda,\ \Pi\,\|\,f$$

these are congruentially impossible unless $\Pi\mid(d-3)$. This possibility is excluded by the hypothesis that $(d-3)$ has no factor of form $6n+1$.

Hence, either $(\lambda)=(\mu)=(\bar{\mu})=1,\ f=1$

or $\lambda=9,\ (\mu)=(\bar{\mu})=(\sqrt{-3}),\ f=3$;

we have either $3\,(d+6)\ a\ \beta\ \bar{\beta}=(d-3)\ (a^3+\rho^j\ \beta^3+\bar{\rho}^j\ \bar{\beta}^3)$ (5)

or $9\,(d+6)\ a\ \beta\ \bar{\beta}=(d-3)\ (9a^3+\rho^j\sqrt{-3}\ \beta^3-\bar{\rho}^j\sqrt{-3}\ \bar{\beta}^3)$ (6)

with $j=0,\ \pm1$. We may reduce case (6) to case (5) by multiplying through by 3, and writing a for $3a$ and β for $\sqrt{-3}\,\beta$; so we have to consider

$$3(d+6)\,a\,\beta\,\bar{\beta} = (d-3)\,(a^3+\rho^j\beta^3+\bar{\rho}^j\bar{\beta}^3) \qquad (7)$$

If $3 \not| \, a$ but $\sqrt{-3} \mid (\beta, \bar{\beta})$ or if $3 \mid a$ but $\sqrt{-3} \not| \, (\beta, \bar{\beta})$ then the left hand side of (7) is divisible by 3 but the right hand side is not. If $\sqrt{-3} \not| \, a\,\beta\,\bar{\beta}$, then $a^3, \beta^3, \bar{\beta}^3 \equiv \pm 1 \pmod 9$ and the left hand side of (7) is exactly divisible by 3 ; this is only possible if $j \equiv 0 \pmod 3$. If $\sqrt{-3} \mid (a, \beta, \bar{\beta})$ then $3 \mid a$; we may suppose $3 \not| \, (\beta, \bar{\beta})$, as such a factor may be thrown away; we thus have

$$\rho^j\ \beta^3 + \bar{\rho}^j\ \bar{\beta}^3 \equiv 0 \pmod{27}$$

with $\beta, \bar{\beta}$ both exactly divisible by $\sqrt{-3}$; again, we must have $j \equiv 0 \pmod 3$. So that only possibility is

$$(d-3)\ (a^3+\beta^3+\bar{\beta}^3) = 3\,(d+6)\,a\,\beta\,\bar{\beta}$$

i.e. $$(d-3)\,(a^3+\beta^3+\bar{\beta}^3-3\,a\,\beta\,\bar{\beta}) = 27\,a\,\beta\,\bar{\beta}$$

Hence at least one of $a+\beta+\bar{\beta}$, $a+\rho\,\beta+\rho^2\,\bar{\beta}$, $a+\rho^2\,\beta+\rho\,\bar{\beta}$ is divisible by 3 ; hence all of them are divisible by $\sqrt{-3}$ and consequently all of them are divisible by 3. So we can find rational integers A, B, C given by

$$a = A + B + C$$

$$\beta = A + B\rho + C\rho^2$$

$$\bar{\beta} = A + B\rho^2 + C\rho$$

and then $$A^3+B^3+C^3 = d\,ABC.$$

If $ABC=0$, then $A+B+C=0$, so $a=0, \gamma=0, \omega=0, x+y+z=0, xyz=0$. This has been excluded so $|ABC| \geqslant 1$, and by the minimality of $|xyz|$, $|ABC| \geqslant |xyz|$.

But $a\beta\bar{\beta} = (d-3)$ A B C, so remembering that we removed common factors from $a, \beta, \bar{\beta}, \gamma\, \sigma\, \bar{\sigma}$ is a multiple of $[(d-3)\,ABC]^3$; so ω is a multiple of A B C; so $3x + 3y + dz$ is a multiple of $(d^2 + 3d + 9)\,(A\,B\,C)^3$.

Hence

$$|(x+y+z)^3\,(d^2+3d+9)| \geqslant |3x+3y+dz| \geqslant |(d^2+3d+9)\,(xyz)^3|$$

so $|x+y+z| \geqslant |xyz| \geqslant 1$. It follows that $|x\,y\,z| \leqslant 6$, and all such cases may easily be enumerated; we get cases with

$(x\,y, z, d) = (1, 1, -1, -1)$; $(1, 1, 1, 3)$; $(1, 1, 2, 5)$;
$(1, -1, 2, -4)$; $(-1, -1, 2, 3)$; $(1, -1, 3, -9)$;
$(-1, -2, 3, 3)$; $(1, 2, 3, 6)$.

The cases with $d=-1, d=5$ are not really exceptions, since the curves

$$x^3 + y^3 + z^3 + xyz = 0$$
$$x^3 + y^3 + z^3 - 5xyz = 0$$

have precisely 6 points, as is proved (for instance) by Mordell [4]. The case $d=-4$ is not exceptional at all, since $(-4-3)$ has a factor 7 of form $6n+1$.

This completes the proof of the theorem 1.

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