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CONTENTS

ELLIPTIC CURVES OF PRIME POWER CONDUCTOR

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In [3] we were able to find a large class of primes which are impossible conductor. We factorized the equation :

$$y^2 = x^3 - 1728 \epsilon p^d, \tag{1}$$

$\epsilon = \pm 1$, in the cubic field Q (θ), $\theta = p^3$, to obtain those solution of this equation called "Good Solutions" for which the equations :

$$x = (a_1^2 + 4a_2)^2 - 24\ (a_1\ a_3 + 2a_4). \tag{2}$$

$$y = -\ (a_1^2 + 4a_2)^3 + 36(a_1^2 + 4a_2)\ (a_1\ a_3 + 2a_4)$$

$$= 216\ (a_3^2 + 4a_6) \tag{3}$$

are also soluble in a_1, a_2, a_3, a_4, a_6, as rational integers. These integers a_1, a_2, a_3, a_4, a_6, are the co-efficients in the Weierstrass minimal model

$$y^2 + a_1\ x\,y + a_3\ y = x^3 + a_2\ x^2 + a_4\ x + a_6,\ a_i \in Z$$

of the elliptic curve E of conductor $N = p^i$ and discriminant $\triangle = \epsilon\ p^d$, where $1 \leqslant d \leqslant i$. Ogg [1] has shown that if $p \neq 2, 3$ then $p^3 \dagger N$ so that $i = 1$ or 2. By a suitable change of co-ordinates, we can assume that $a_1 = 1, 0$, $a_3 = 0, 1$ and $a_2 = 0, \pm 1$ (Tate [2]).

In this paper we solve the equation (1) completely for good solutions when $p = 7$, 17 and 31 and list the curves if they exist. We also prove that there are no elliptic curves of conductor p if $p = 47$, 71. We factorize the equation (1) in the quadratic fields $\sqrt{\pm 3}$, $\sqrt{\pm 3p}$. We deal

These results were a part of the Ph.D. thesis submitted by the author to the University of Manchester in 1966. The author wishes to thank Dr. B.J. Birch for his help and guidance.

separately with the four cases : d even, $\epsilon=\pm 1$: d odd, $\epsilon=\pm 1$. We find that the good solutions for $p=7$, 17 arise as special cases. In fact, these solutions for $p=7$, 17 were first found by Francis Coghlan, another research student of B.J. Birch.

We prove the following :

Theorem :

There are no curves with conductor p if $p \neq 2, 3, 17$, $p \not\equiv \pm 3 \pmod 8$, $p \neq c^2 + 64$, the class number of the fields $R(\sqrt{3p})$ and $R(\sqrt{-3p})$ are not divisible by 3, and $(m, 3) = 1$, where $1+m\sqrt{3p}$ is the unit of the field $R(\sqrt{3p})$.

In particular there are no curves with conductor p if $p=47$ and 71.

The class numbers and units of the field $R(\sqrt{a})$ are listed in the book by Borevich-Shafarevich [4].

We first suppose that $d = 2k$ $(k > 0)$ and $\epsilon = -1$, so that equation (1) can be written as

$$y^2 = x^3 + 1728.\ p^{2k} \quad (k > 0) \tag{4}$$

We note that x is odd, otherwise the equation is impossible modulo power of 2. We prove the following.

Lemma 1

If p is a rational prime such that $p \neq 2, 3, 17$, $p \not\equiv \pm 1 \pmod{12}$ and $p \not\equiv c^2 + 64$ for integer c, then equation (4) has no good solution.

Proof

Factorizing the equation in the field $R(\sqrt{3})$ we have

$$x^3 = (y - 24\ p^k\sqrt{3})\ (y + 24\ p^k\sqrt{3})$$

Since $p \not\equiv \pm 1 \pmod{12}$ by hypothesis, p does not factorize in the field $R(\sqrt{3})$. The common ideal divisor of $(y-24p^k\sqrt{3})$ and $(y+24p^k\sqrt{3})$ also divides $144p^k$ and so since x is odd and p does not factorize in $R(\sqrt{3})$, we have the following two cases to distinguish

Case 1° : $\pm y \pm 24p^k \sqrt{3} = (a+b\sqrt{3})^3$

Case 2° : $\pm y \pm 24p^k \sqrt{3} = (2+\sqrt{3})(a+\sqrt{3})^3$

where a, b are rational integers.

Case 1° implies

$$\pm y = a(a^2 + 9b^2) \tag{5}$$

$$\text{and} \pm 8p^k = b(a^2 + b^2) \tag{6}$$

We first suppose that $(p, y) = 1$, so that $(p, a) = 1$ and either $p^k \| b$ or $p^k \| (a^2+b^2)$. If $p^k \| b$ then $pb = \pm 2^{\lambda} p^k$ where $0 \leqslant \lambda \leqslant 3$ and accordingly $a^2 = 2^{3-\lambda} - 4^{\lambda} p^{2k}$ which gives $p=2$. Therefore $p^k \| (a^2+b^2)$ and $(p, b) = 1$. If $b = \pm 1, \pm 2, \pm 4$, then $a^2 = 8p^k - 1$, $4(p^k - 1)$, $2(p^k - 8)$ respectively,all of which are easily seen to be impossible. So $b = \pm 8$, $a^2 = p^k - 64$. If k is even, then $k = 2K$ and so

$$p^{2K} - 64 = a^2 \tag{6.1}$$

which can be written as

$$(p^K - a)(p^K + a) = 64.$$

The highest common factor of $p^K - a$ and $p^K + a$ is 2 and hence

$$p^K - a = \pm 2, \; p^K + a = \pm 32$$

which gives $a = 15$, $p = 17$ and $K = 1$ and we have $x = 33$, $\pm y = 12015$ as the solution of the equation $y^2 = x^3 + 1728.17^4$.

If k is odd, then we may write $k = 2K + 1$ and we have

$$(a + 8i)(a - 8i) = p^{2K+1}$$

where $i^2 = -1$. This gives

$$(a \pm 8i) = (A + Bi)^{2K+1}$$

where $p = A^2 + B^2$. We write

$$A_n + B_n i = (A + Bi)^n$$

for all $n \geqslant 0$, so that $A = A_1$, and $B = B_1$. We have $a \pm 8i = A_{2K} + iB_{2K}$, which gives

$\pm a = AA_{2K} + BB_{2K}$

and $\pm 8 = AB_{2K} + A_{2K} B$.

Obviously $2B \mid B_{2K}$ and so $B \mid 8$. If $B=\pm 1$, then A is even and so a is even which implies that $2 \mid p$. Therefore B is even and A, A_{2K} are odd, so $\pm 8 = B\left[\frac{AB_{2K}}{B} + A_{2k}\right]$ implies that $B=\pm 8$. Consequently $p = A^2 + 64$.

So we have proved that if p does not divde x, y then there exist good solutions of the equation (4) only if either $p=17$ or $p=A^2+64$.

Now we assume that $p \mid y$, then from the equations (5) and (6) we find that $p \mid a$ and $p \mid b$, so that $p^3 \mid y$ and we can write the equation (4) as

$$\left(\frac{y}{p^3}\right)^2 = \left(\frac{x}{p^2}\right)^3 + 1728.\ p^{2k-6}$$

Obviously $2k-6>0$, because if $k=3$, then the equation will not have any good solution. If $y|p^3$ is divisible by p, then by the above method we can find that $p^6 \mid y$ and hence by repeating the procedure we will finally arrive at an equation of the form $Y^2=X^3+1728\ p^{2e}$ $(e>0)$ where X, Y are both prime to p ; this will have solutions only if $p=17, p=A^2+64$ and $p\equiv\pm 1 \pmod{12}$. We have $x=33.17^{2(k-2)/3}$, $\pm y= 12015.17^{k-2}$ as the solution of the equation $y^2=x^3+1728.17^{2k}$.

Case 2° gives

$$\pm\ y=2\ (a^3+9ab^2)+9\ (a^2b+b^3)$$

and $\pm\ 24p^k=(a+9ab^2)+6\ (a^2\ b-b^3)$

which shows that $3 \mid a$, $9 \mid y$ and so $27 \mid y$, so $3 \mid b$ and $27 \mid 24p$ Since $p\neq 3$, the above equation has no solution.

This completes the proof of the lemma.

We remark that if $p\equiv\pm 1 \pmod{12}$ but $p\neq c^2+64$ then solutions have $p \mid x$.

Let $d=2k$, $\epsilon=1$, then equation (1) becomes

$$y^2 = x^3 - 1728p^{2k} \tag{7}$$

If x, y are both even then $16 \mid x$ and $8 \mid y$ and the equation is impossible modulo 8, therefore x, y are both odd. We prove

Lemma 2

The diophantine equation (7) has no good solutions if $p \neq 2, 3, 17$ and $p \not\equiv 1 \pmod 3$.

Proof

We write equation (7) as

$$[y + 24p^k (2\omega + 1)]\,[y - 24p^k (2\omega + 1)] = x^3$$

where $\omega^3=1$, $\omega \neq 1$. Since $p \not\equiv 1 \pmod 3$, p does not split in the field Q (ω), so we get

$$\pm y \pm 24p^k \ (2\omega + 1) = (a + b)^3,\ \omega(a + b\omega)^3,\ \omega^2(a + b\omega)^3$$

so that

$$\pm 48p^k = 3a\,b\,(a-b),\ a^3+b^3-3a^2b,\ -a^3-b^3+3ab^2$$

In the last two cases $2 \mid (a, b)$ which implies that $4 \mid (a, b)$ and so $64 \mid 48p^k$. Therefore

$$\pm 16p^k = ab\,(a - b)$$

This gives $p^k=17^{1+3t}$, and $a, b, a-b$ are $\pm 17^t$, $\pm 16.17^t$, $\pm 17^{t+1}$ in some order with appropriate signs so that we get a solution $x=273.17^{2t}, \pm y=4455.17^{3t}$ of the equation $y^2=x^3-1728.17^{2+6t}$.

This completes the proof of lemma 2. We note that if $p\equiv 1 \pmod 3$ and $p \nmid (x, y)$, the equation (7) has no good solution, so all the good solutions of the equation will have x, y divisible by p.

Now suppose that $d=2k+1$ and $\epsilon=-1$, so that equation (1) becomes

$$y^2 = x^3 + 1728p^{2k+1} \tag{8}$$

where $k \geqslant 0$. We now prove

Lemma 3

The equation (8) has no good solution if $p \neq 2, 3, 7$, $p \not\equiv 3 \pmod 8$, $(h, 3)=1$ and $(m, 3)=1$, where h is the class number and $(\mathrm{I}+m\sqrt{3p})/2$ is the unit of the field $R(\sqrt{3p})$.

Proof

We write

$$(y+24p^k\sqrt{3p})(y-24p^k\sqrt{3p})=x^3.$$

Since $p \not\equiv 3 \pmod 8$, 2 is either a prime or is ramified in the field $R(\sqrt{3p})$. Since $(h, 3)=1$, we have

$$\text{either } \pm y \pm 24p^k\sqrt{3p}=\left(\frac{a+b\sqrt{3p}}{2}\right)^3$$

$$\text{or } \pm y \pm 24p^k\sqrt{3p}=\left(\frac{\mathrm{I}+m\sqrt{3p}}{2}\right)\left(\frac{a+b\sqrt{3p}}{2}\right)^3$$

$$\text{If } \pm y \pm 24p^k\sqrt{3p}=\left(\frac{\mathrm{I}+m\sqrt{3p}}{2}\right)\left(\frac{a+b\sqrt{3p}}{2}\right)^3$$

$$\text{then } \pm 16\,y = \mathrm{I}\,(a^3+9p\,ab^2)+3mp\,(3a^2b+3p\,b^3)$$

$$\pm 16.24p^k = m\,(a^3+9p\,ab^2)+\mathrm{I}\,(3a^2\,b+3p\,b^3)$$

Since $(m, 3)=1$, $3 \mid a$, $3 \mid y$, so $27 \mid y$ and hence $3 \mid b$ and $27 \mid 24p^k$ which is absurd.

$$\text{If } \pm y \pm 24p\sqrt{3p}=\left(\frac{a+b\sqrt{3p}}{2}\right)^3, \text{ then}$$

$$\pm 8y = a\,(a^2+9pb^2).$$

$$\text{and } \pm 64p^k = b\,(a^2+pb^2).$$

If $(p, b)=1$, then p^k exactly divides a^2+pb^2, so $k=1$ and $p \mid a$ and after considering the various possible values for b, we get $b=\pm 1$, $a=\pm 21$, $k=1$, $p=7$ and so $x=105$, $\pm y = 1323$ as a good solution of the equation $y^2=x^3+1728.343$. ($b=\pm 2$, $a=\pm 14$ leads to a bad solution).

If $(p, a) = 1$, then p^k exactly divides b. Writing $b = \pm 2^\lambda p^k$ $(0 \leqslant \lambda \leqslant 6)$, we get $a^2 = 2^{6-\lambda} - 2^{2\lambda} p^{2k+1}$, which is impossible unless $p=2$.

Finally if $p \mid (a, b)$ then $p^3 \mid y$ and the equation (8) can be reduced to an equation $Y^2 = X^3 + 1728. \quad p^{2e+1}$
where p^3 does not divides Y. We get

$$x = 105.7^{2(k-1)/3}, \quad y = 1223.7^{k-1}$$

as the solution of the equation $y^2 = x^3 + 1728.7^{2k+1}$.

Finally, we suppose tht that $d = 2k+1$, $\epsilon = 1$, so that

$$y^2 = x^3 - 1728\, p^{2k+1} \quad (k \geqslant 0) \tag{9}$$

Lemma 4

If p is a rational prime such that $p \neq 2, 3, 7, 17$, $p \not\equiv 5 \pmod 8$, $p \neq c^2 + 64$, and $(h, 3) = 1$ where h is the class number of the field $R(\sqrt{-3p})$, then equation (9) has no good solution.

Proof

We write as before

$$x^3 = (y + 24p^k \sqrt{-3p})\ (y - 24p^k \sqrt{-3p}).$$

Since $p \not\equiv 5 \pmod 8$, 2 is not split in $R(\sqrt{-3p})$; also $(h, 3) = 1$, so we get

$$8\,(\pm y \pm 24p^k \sqrt{-3p}) = (a + b\sqrt{-3p})^3$$

where a, b are rational integers. Equating the co-efficients of the corresponding terms on both the sides, we obtain

$$\pm 8\, y = a\,(a^2 - 9pb^2) \tag{9.1}$$

and

$$\pm 64p^k = b\,(a^2 - pb^2) \tag{9.2}$$

If $(p, a) = 1$, then by (9.2) p^k exactly divides b. First suppose that $p^k \mid b$ and we get $b = \pm 2\ p^k$ and

$a^2 = 4^\lambda.\ p^{2k+1} \mp 2^{6-\lambda}$ where $0 \leqslant \lambda \leqslant 6$. If $\lambda = 2$, $4 \mid b$, $8 \mid a$ and equation (3.4) implies that $128 \mid 8y$ which is not a good solution ;

the same applies if $\lambda=1$. If $\lambda=3$, $b=\pm 8p^k$, $a^2=8\,(8p^{2k+1}\mp 1)$, which is impossible. If $\lambda=5$, $b=\pm 32p^k$, $a^2\equiv$ (mod 8) and if $\lambda=6$, then $a^2=4096p^{2k+1}+1$, a is odd and equation (9.1) is impossible modulo 2. So we are left with $\lambda=0$ or $\lambda=4$. If $\lambda=0$, we get

$$a^2 - p^{2k+1} = \pm 64.$$

If there is $-$ve sign on the right hand side then this equation is same as (6.1) which has solutions if $p=c^2+64$; and if $a^2-p^{2k+1}=64$, then we write

$$(a-8)\,(a+8) = p^{2k+1}$$

and $a-8=\pm p^f$, $a+8=\pm p^g$ for $f, g\geqslant 0$ and $f+g=2k+1$. Consequently $16=p^f-p^g$, which implies that $f=0$, $g=1$, $p=17$ and so $a=\pm 9$, $b=\pm 1$, $k=0$ and the equation $y^2=x^3-1782.17$ has a good solution $x=33$, $y=\pm 81$. If $\lambda=4$, then putting $a=2a'$, we get

$$\grave{a}^2 - 64\,p^{2k+1} = \pm\ 1.$$

Considering the equation modulo 4, we find that only $-$ve sign holds and so we write

$$(a'-1)\,(a'+1) = 64p^{2k+1}$$

Since a' is prime to $2p$, we may write

$$(a'-1) = \pm\, 2^f,\ \ a'+1 = \pm\, 2^g.\, p^{2k+1}$$

where $f+g=6$; therefore $\pm\, 2 = 2^g\, p^{2k+1} - 2^f$. If $f=1$, then $3\mid p$, so $g=1$, $p=17$. We find $x=4353$, $\pm\, y = 287199$, another solution of the equation $y^2=x^3-1728.17$.

Now suppose that $(p, b)=1$, so that from the equation (9.2)we get

$$b = \pm\, 2^f \text{ and } a^2 = p(2^{2f} \pm 2^{6-f}.\ \ p^{k-1})$$

where $0\leqslant f\leqslant 6$; we have $k=1$, $a^2=p\,(2^{2f}\pm 2^{6-f})$. It is easy to find that only possible value for f is 4 and so $a=\pm 16$, $b=16$, $p=7$. We obtain $x=1785$, $y=\pm\, 75411$ as the solution of the equation $y^2=x^3-1728.343$.

There remains the case $p \mid a$, $p \mid b$, then by (9.1) $p^3 \mid y$. By dividing (x, y) by suitable powers of p we can assume that p^3 does not divide y. Thus we have the following three sets of solutions of the of the equation (9) for $p=7$ and 17

$$x=1785.7^{2(k-1)/3}, \quad y=\pm 75411.7^{k-1}$$

$$x=4343.17^{2k/3}, \quad y=287199.17^{k}$$

$$x=33.17^{2k/3}, \quad y=81.17^{k}.$$

This completes the proof of lemma 4.

Therem follows immediately from lemmas 1, 2, 3, 4 and

Lemma 5

If p does not divide (x, y), then $N=p$ and if p divides (x, y) and the corresponding elliptic curve is in its n minimal form, then $N=p^2$.

When $p=7$, theorems 1 and 2 in [3] show that the equation (1) has good solutions only if d is odd and $d\equiv 0 \pmod 3$; also we have just proved that for d odd, the equation (1) has exactly two families of solutions, namely

$$x=15.7^{d/3}, \quad y=\pm 27.7^{(d+1)/2}$$

$$x=225.7^{d/3}, \quad y=\pm 1539.7^{(d+1)/2}$$

Solving the equations (2) and (3) for a_i $i=1, 2, 3, 4, 6$, we get the the following two families of curves

$$y^2+x\,y=x^3-x^2+\left(\frac{3-5.7^{d/3}}{16}\right)x+\frac{1}{64}(-1+5.7^{d/3}\pm 2.7^{(d+1)/2})$$

$$y^2+xy=x^3-x^2+\left(\frac{3-85.7^{d/3}}{16}\right)x+\frac{1}{64}(-1+85.7^{d/3}\pm 114.7^{(d+1)/2}).$$

These curves are in their minimal modle with a_i integers and $|\triangle|$ as small as possible when $d=3$ or 9. So in their minimal form we have following four curves with discriminants $-7^3, -7^9, 7^3, 7^9$ respectively.

$$y^2+xy=x^3-x^2-2x-1$$

$$y^2+xy=x^3-x^2-107x+552$$

$$y^2+xy=x^3-x^2-37x-78$$

and $$y^2+xy=x^3-x^2-1822x+30393.$$

When written in the form

$$\bar{x}^2=\bar{x}^3-35x-98$$

$$\bar{y}^2=\bar{x}^3-1715x+2.7^5$$

$$\bar{y}^2=\bar{x}^3-595x-5586$$

$$\bar{y}^2=\bar{x}^3-85.7^3x+114.7^5$$

with $\bar{y}=y+\frac{1}{2}x$, $\bar{x}=x+\frac{1}{4}$, all have multiplicative reduction mod 7 and so have conduc or 49. It is known that all the curves have poins of order 2 and are isogeneous.

So we have proved that there are no curves with conductor 7 and there is one isogeny class of four elliptic curves with conductor 49.

It follows from Lemmas (1), (2), (3) and (4) that the equation (1) has exactly four families of solutions for $p=17$, namely

$$x=273.17^{(d-2)/3}, \qquad y=4455.17^{(d-2)/2}$$

$$x=33.17^{(d-4)/3}, \qquad y=12015.17^{(d-4)/2}$$

$$x=4353.17^{(d-1)/3}, \qquad y=287199.17^{(d-1)/2}$$

$$x=33.17^{(d-1)/3}, \qquad y=81.17^{(d-1)/2}.$$

These solutions give the following eight curves in their minimal forms, with discriminant $\pm 17^d$.

$$y^2+xy+y=x^3-x^2=\frac{5+91.17^{(d-2)/3}}{16}x-\frac{1}{64}(17-91.17^{(d-2)/3}+330.17^{(d-2)/2}) \qquad d=2,\ 8$$

$$y^2+xy+y=x^3-x^2-\frac{5+11.17^{(d-4)/3}}{16}x-\frac{1}{64}(17-11.17^{(d-4)/3}+890.17^{(d-4)/2}) \qquad d=4,\ 10$$

$$y^2+xy+y=x^3-x^2-\frac{5+1451.17^{(d-1)/3}}{16}x-\frac{1}{64}(17-1451.17^{d-1)/3}$$
$$+2.10637.17^{(d-1)/2})\quad d=1,7$$

$$y^2+xy+y-x^3-x^2-\frac{5+11.17^{(d-1)/3}}{16}x-\frac{1}{64}(17-11.17^{(d-1)/3}$$
$$-6.17^{(d-1)/2})\quad d=1,17$$

When $d=2, 4, 1$, the good solutions of the equation (1) are not divisible by 17 and hence the curves have conductor 17; and when $d=8, 10, 7$, the corresponding curves have conductor 17^2.

So we have exactly four curves with conductor 17 and four curves with conductor 289.

We deduce from section 2.2 in [3] that when $p=31$, there is no good solution of the equation (1) for $(d, 3)=1$; also when $d\equiv 0 \pmod 3$, section (2.1) implies that (1) will be soluble only if d is odd and x, y both odd. So suppose that d is odd. When $\epsilon=1$, section 5 shows that there are no good solutions of the equation (1) for $p=31$ and d odd. Therefor $\epsilon=-1$. We go back to section 4 and follow the argument for $p=31$. A fundamental unit of the field

R $(\sqrt{93})$ is $\frac{29+3\sqrt{93}}{2}$, so we have from case 2°.

$$\pm 128.31^k = (a + 279a\, b^2) + 29\,(a^2\, b + 31b^3).$$

If a, b are both odd, then the equation is not true modulo 16, so a, b are both even. In fact $4 \mid a, b)$. Putting $a=4a'$, $b=4b'$ we get

$$\pm 2.31 = (a^2 + 279a'\, b'^2) + 29\,(a'^2\, b' + 31b'^3)$$

which implies that a', b' are of the same parity and so right hand side is divisible by 8 which is not possible. Hence that equation (1) has no good solution for $p=31$.

We claim that there are no elliptic curves with conductor 31 and 961.

Summariz'ng the results, we claim that there are no curves with conductor p or p^2, when $p=5, 13, 23, 29, 31, 41, 59, 97$ and there are no curves with conductor p when $p=7, 47, 71$.

When $p=7, 17$ the following are the only curves with conductors 17, 19, 49, 289.

Curve	Discriminant	Conductor
$y^2+xy+y=x^3-x^2-x$	17	
$y^2+xy+y=x^3-x^2-6x-4$	289	
$y^2+xy+y=x^3-x^2-91x-310$	17	17
$y^2+xy+y=x^3-x^2-x-14$	-83521	
$y^2+xy=x^3-x^2-2x-1$	-343	
$y^2+xy=x^3-x^2-107x+552$	-79	
$y^2+xy=x^3-x^2-37x-78$	343	49
$y^2+xy=x^3-x^2-1822x+30393$	79	
$y^2+xy+y=x^3-x^2-2644x-24922$	178	
$y^2+xy+y=x^3-x^2-199x-17.4016$	-17^{10}	
$y^2+xy+y=x^3-x^2-26209x-17.95680$	17^7	289
$y^2+xy+y=x^3-x^2-199x+17.120$	17^7	

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THEORETICAL STUDIES ON THE PROPAGATOIN OF SEISMIC SURFACE WAVES IN HORIZONTALLY VARYING STRUCTURES

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Abstract :

In recent years various problems associated with transmission, reflection and diffraction of seismic surface waves propagating in horizontally varying structures have been investigated by a number of authors. Such studies are important to understand the physical processes associated with energy transfer at continental margins, other horizontal transitions and with multipathing. Mathematical treatment of these problems is very difficult, even under highly simplifying assumptions and a variety of analytical, approximate and numerical methods have been used to achieve some measure of success. We present here a brief review of these methods and the seismological problems to which each method has been applied.

1. Introduction

One of the principal aims of theoretical seismology is to investigate the effects of velocity changes, and especially of discontinuities or boundaries on the propagation of elastic waves through the earth, and to identify its subsurface structure in terms of elastic parameters by comparing the theoretical results with seismological observations (cf. Jeffreys, 1970 Ch. III). As observations become increasingly precise, owing to the great improvements in techniques of data acquisition and analysis, so one may hope to learn more by these means.

Seismic surface waves play an important role in this context. A useful property of surface waves is that the energy is spread out in two

*This work was carried out at the Department of Applied Mathematics and Theoretical Physics, Cambridge University, U.K.

dimensions, rather than in three as in the case of body waves. Consequently, surface waves are relatively more prominent on seismograms, particularly on those of distant earthquakes (cf. Jeffreys, 1970, 2.03). Moreover, their restriction to the neighbourhood of the surface of the earth, a region which is not easily investigated in body wave studies, is of great advantage in determining the velocity structure in the crust.

The method employed in the investigation of the crustal and upper mantle structure of a given region by means of surface waves, is based upon the comparison of observed dispersion curves with those computed for various theoretical models (cf. Ewing, Jardetsky and Press, 1957, 4-5). Knopoff (1961) has, however, pointed out the considerable difficulty in securing uniqueness in applications of surface-wave data to investigate the detailed structure of the earth. Nevertheless, this inverse method, if used with auxiliary geophysical information, continues to be useful. Among recent studies are those of Goncz et. al. (1975), Panza and Calcagnile (1975), Knopoff et. al. (1974) and Knopoff et. al. (1973) for isotropic layering and Crampin (1975) for anisotropic layering.

The construction of dispersion curves from seismograms has been an area of active research for a long time (c.f. Ewing et. al. 1957, 4-5, for early work). Significant advances have been made in this field in recent years with the use of seismic arrays, improved instrumentation and sophisticated techniques of data analysis (such methods as cross-correlation (Landisman et al., 1969), auto-correlation (Dziewonski and Landisman 1970, Dziewonski et.al., 1971), bandpass filtration (Alexander 1963) ; Archambeau et. al., 1966 ; Kanamori and Abe 1968 ; Dziewonski et. al., 1969), the methods of sums and differences (Bloch and Hales 1968) and cross-multiplication (Bloch and Hales 1968), time-variable filtration (Pilant and Knopoff 1964), high-resolution technique (Capon 1969) and maximum likelihood techniques (Capon et. al., 1969). The observed dispersion curves together with their structural interpretations have yielded a large amount of information about the earth's crust and the regional departures from the average global picture. The most important result is the pronounced difference found between oceanic and continental structures (see

Oliver, 1962). Horizontal variations in structure also occur within oceans and continents ; e.g. across mid-ocean ridges, island arcs, mountain ranges and tectonic faults (see for example : Molnar et. al., 1975 ; Choudhary 1975 ; Smith and Bott 1975 ; Kono 1974 ; Cann 1974).

One of the most interesting phenomena associated with the propagation of surface waves is that of multipathing, such as observed by Capon (1970, 71) and Bungum and Capon (1974) at LASA. Capon has shown that multiple arrivals of surface waves can be observed, owing to refractions and reflections at continental margins. Evernden (1953, 54) seems to have been the first to observe this in connection with the propagation of Rayleigh waves ; it has also been considered by Pilant and Knopoff (1964) and Knopoff et. al. (1966). The propagation of surface wave microseisms across major structures (the Rocky Mountains) has been studied by Haubrich and McCamy (1969) and Hjortenberg (1968). Propagation of microseisms across continental margins has been reviewed by Darbyshire (1962) and also studied by Darbyshire and Okeke (1969).

In order to construct a theoretical analysis of surface wave reflection and diffraction, it is necessary to formulate the problem mathematically and to obtain exact or approximate solutions for reflected and transmitted waves at for instance, continent/ocean margins with large impedance contrasts. However, the mathematical structure of the problems is complicated by the number of parameters involved in the description of the discontinutity. These are the elastic properties of the continental and the oceanic regions, the thickness ratio of the two regions, the distance over which the thickness transition is made and the nature of the water layer over-laying the oceanic region. The complicated geometry of the structure of the transition zone is another source of difficulty. Hence only idealized models of the continental margin have been used so far. Even then the mathematical treatment of these problems is difficult.

Nevertheless, theoretical studies of transmission, reflection and diffraction of surface waves at continental margins and other horizontal transitions are important to gain further insight into the physical processes associated with energy transfer at boundaries and with multipathing.

2. Theoretical Techniques for Surface Wave Transmission Problems

The task of finding exact solutions for various problems concerning the passage of seismic surface waves through laterally varying structures is formidable, even under highly simplifying assumptions. A number of authors have used a variety of analytical, approximate and numerical methods. We present here a brief review of these methods and the seismological problems to which each method has been applied.

(i) Wiener-Hopf Technique

Sato (1961) set up the two-dimensional boundary value problem associated with the propagation of monochromatic Love waves in a structure consisting of a homogeneous half-space, overlain by a homogeneous surface-layer which undergoes an abrupt change in thickness (i.e. a surface-step, see fig. 1 (*a*)) and solved it exactly by the Wiener-Hopf

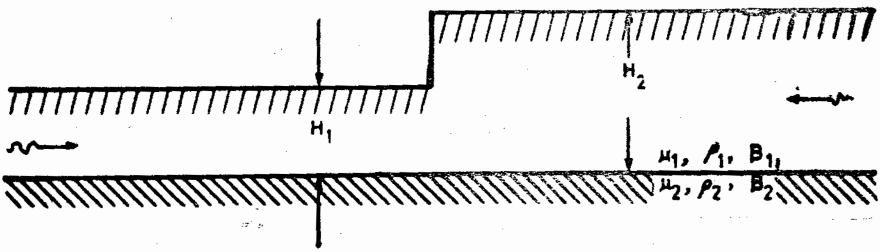


FIG. 1 (*a*). The Geometry of the problem of the propagation of Love waves past a step.

technique. The factorisation involved in the Wiener-Hopf procedure leads to an infinite set of simultaneous linear algebraic equations to be solved ; the numerical evaluation of the transmitted and the reflected Love waves is thus very difficult. Consequently, Sato derives approximate but straightforward algebraic expressions for transmission and reflection co-efficients, when the change in thickness of the surface-layer is small compared to the wave-length. Using a similar technique, Kazi (1975) has solved the problem of diffraction of Love Waves, normally incident on a perfectly rigid or a perfectly weak half-plane which lies in the surface layer (of uniform thickness) of a two-layered half-space and which is parallel to the interface (see fig. 1 (*b*)). The following conclusions are reached :

1. Love waves incident on the screens are diffracted by the screens into Love-type modes propagating in the lower half-strip (adjacent to the half-space), and channel waves in the upper half-strip which has free surface above and free or rigid surface below. Most of the channel modes die out rapidly with distance from the edge of the screens.

2. The problem of Love waves past a weak screen is connected to Sato's problem of Love waves incident on a vertical step discontinuity in the surface layer, from the thicker to the thinner side. Whereas Sato's solution involved a set of infinite, simultaneous, linear algebraic equations to be solved, our approximate solution involves a finite set.

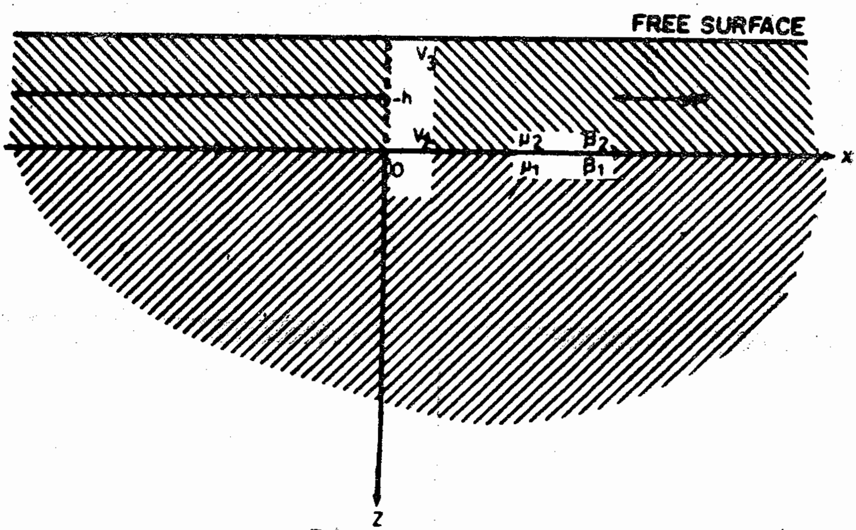


FIG. 1 (b)

FIG. 1 (*b*) The geometry of the problem. Love waves are normally incident on the screen from right to left.

(ii) Green's function techniques

(a) Kirchhoff's method

The starting point in Kirchhoff's method is to express the displacement at any point as an integral over a closed surface surrounding that point. This is accomplished by means of elastodynamic representation theorems. The solution of any diffraction problem may then be obtained by making part of the closed surface coincide with the diffracting surface, and evaluating the integral with suitable boundary conditions. To do this one needs to know the displacement and traction over the whole surface. Usually, only the displacement or the traction (but not both) are known ; and to solve the problem certain simplifying assumptions must be made.

Ghosh (1963) used a Green's function technique, essentially equivalent to Kirchhoff's method, to find the displacement of Love waves due to a point source placed at the interface between a substratum, consisting of a homogeneous half-space and a homogeneous surface layer with a gradually sloping top (see fig. 2). Ghosh showed that Love waves of small periods are strongly attenuated in the continental margin due to the slcpe of its boundary.

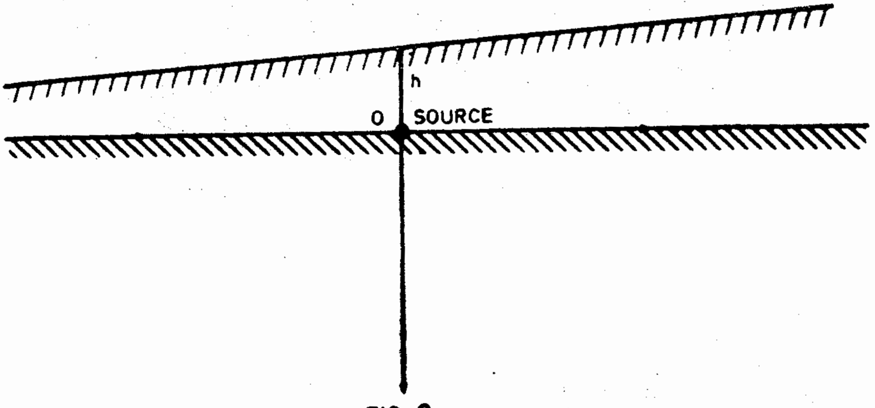


FIG. 2

Knopoff and Hudson (1964) used Kirchhoff's method to investigate the same problem as that of Sato (1961) (earlier, Hudson and Knopoff (1964 *a*, *b*) used this method for finding approximate values of complex transmission and reflection co-efficients for surface waves, normally incident upon the corner of a homogeneous elastic wedge formed by two stress-free planes). Transmission co-efficients were derived for monochromatic Love waves incident normally upon the discontinuity from either direction. Taking the surface integral to be over the vertical plane containing the step, Knopoff and Hudson used Kirchhoff's approximation by substituting displacements and tractions of the incident wave for the unknown displacements and tractions within the material, while on the step the traction is zero and displacements were estimated in different ways for the two different directions of transmission. The problem was thus reduced to quadratures. The transmission co-efficients found by Knopoff and Hudson differ greatly according to the direction of travel. When the wave travels to the thicker layer, the co-efficient is large for low frequencies and small for high frequencies (with a transition frequency). For waves travelling in the opposite direction, the co-efficients vary very little over the whole range of frequencies.

By a similar method, Mal and Knopoff (1965) found the transmission and reflection co-efficients for plane harmonic Rayleigh waves, normally incident (from either side) upon a step change in the elevation of the surface of a homogeneous half-space (see fig. 3).

Theoretical and experimental studies of transmission and reflection of Rayleigh waves round corners (de Bremaecker 1958 ; Knopoff and Gangi 1960 ; Lapwood 1961 ; Kane and Spence 1963 ***a***; Hudson and Knopoff 1964 ***b***; Pilant, Knopoff and Schwab 1964 ; Mal and Knopoff 1966 ; Lewis and Dally 1970 ; Viswanathan, Kuo and Lapwood 1971) have revealed considerable discrepancies between the observed and the predicted transmission co-efficients. These discrepancies are partly due to the fact that wave motions resulting from the diffraction by the corner are, to a large extent, ignored in the theoretical studies. The step problem is related to the wedge problem in the sense that two corners are present and much energy is considerably scattered into body waves. The Kirchhoff method is one in which diffraction at the corners is inevitably neglected.

Mal and Knopoff, in their attack on the step problem, present a modified Kirchhoff's method, making several types of approximation each applicable in a different range of the ratio of height of the step to wavelength. When the height of the step is less than one-half of the wavelength, a fairly good estimate of the transmitted waves is obtained. The transmitted wave does not suffer any appreciable phase shift, but there is

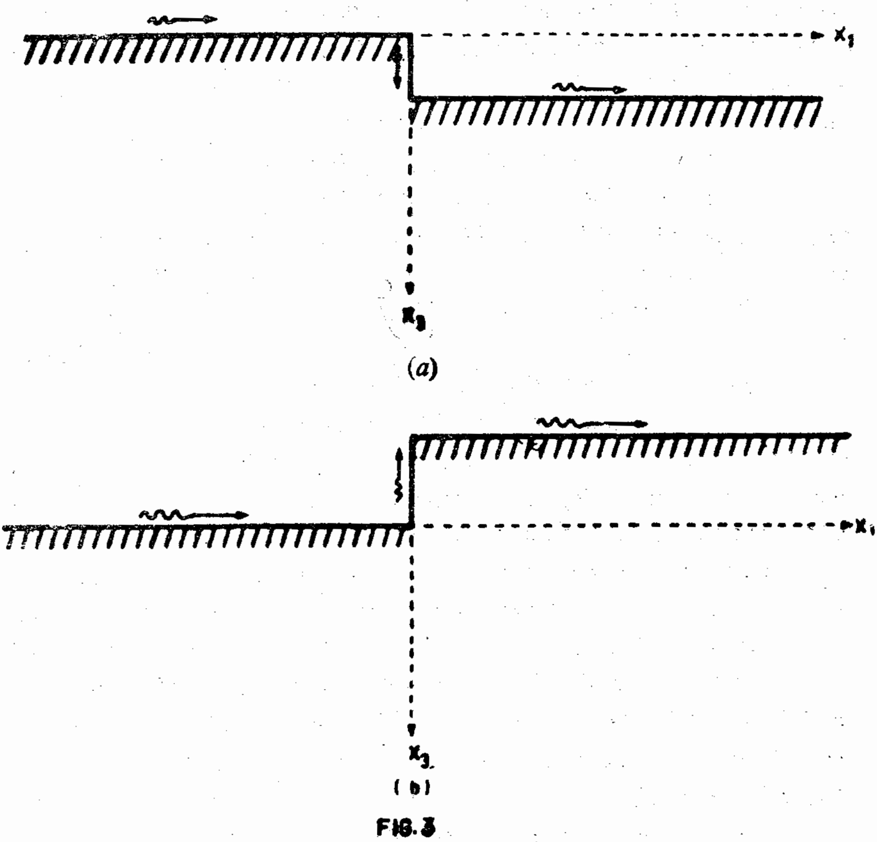


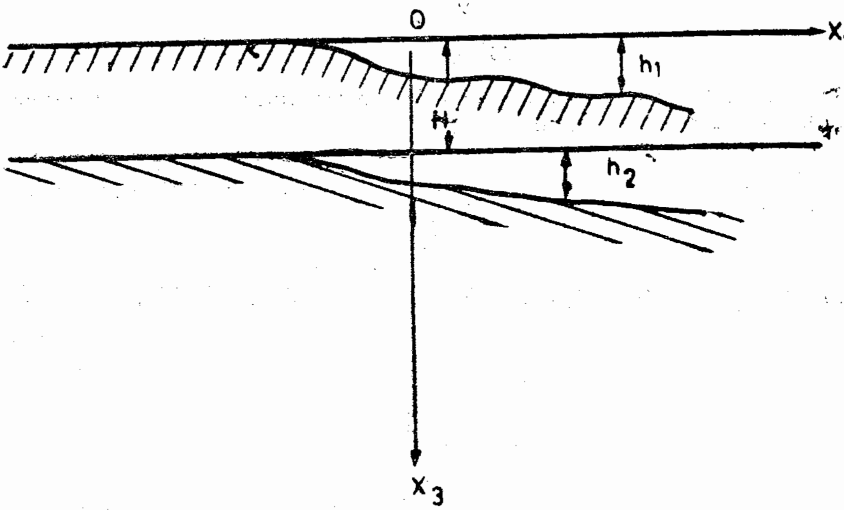
FIG. 3

FIG. 3. Geometry of the Rayleigh wave propagation problem investigated by Mal and Knopoff (1965) : (*a*) when the waves are incident from the elevated side, (*b*) when the waves are incident from the depressed side.

a rapid attenuation in the amplitude with increasing frequency. When the height of the step is greater than one-half of the wavelength, the phase as well as the amplitude is significantly perturbed and the theoretical results are not in good agreement with experimental results.

(*b*) Perturbation Methods

Herrera (1964*a*) developed a perturbation scheme for applications of Kirchhoff's method to any problem formulated in a region whose geometry and physical properties deviate only slightly from those of a region for which the corresponding Green's function and the solution of an auxiliary problem are known. In this method, Kirchhoff's surface integral is replaced by a volume integral over the region in which the structure deviates from the basic model. Within this region, displacements and stresses are replaced by their unperturbed values. It is similar to the Born's method of Atomic Physics. Herrera illustrated the method by investigating the effect of changes of the thickness of the crustal layer in mountainous regions on the transmission of Love waves (see fig. 4). Using the same technique,



FIG·4

FIG. 4. Geometry of Love wave propagation in crustal layer of variable thickness (Herrera, 1964*a*).

Mal and Herrera (1965) gave a simple mathematical expression for transmitted Love waves across a constriction in the crust in a single-layered half-space (see fig. 5). Herrera and Mal (1965) reformulated the method to treat the problem of scattering by small inhomogeneities for media which deviate slightly from homogeneity in a large region and then modified it to treat media with large deviations from homogeneity in a thin region. The scattering of Love waves by a dike (see fig. 6) was evaluated by way of illustration.

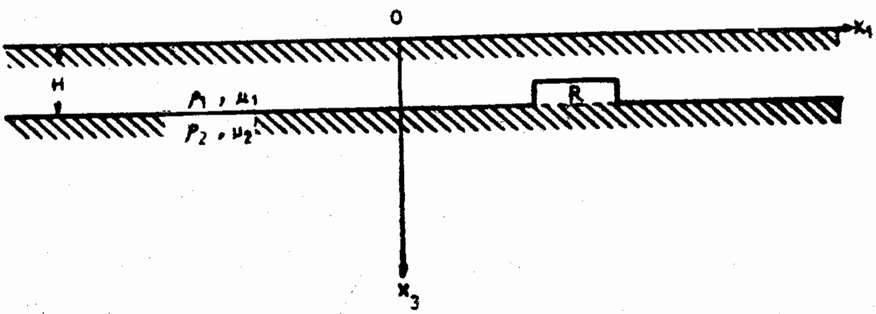


FIG. 5

FIG. 5. Geometry of the Love wave transmission problem across a constriction in the crust (Mal and Herrera, 1965).

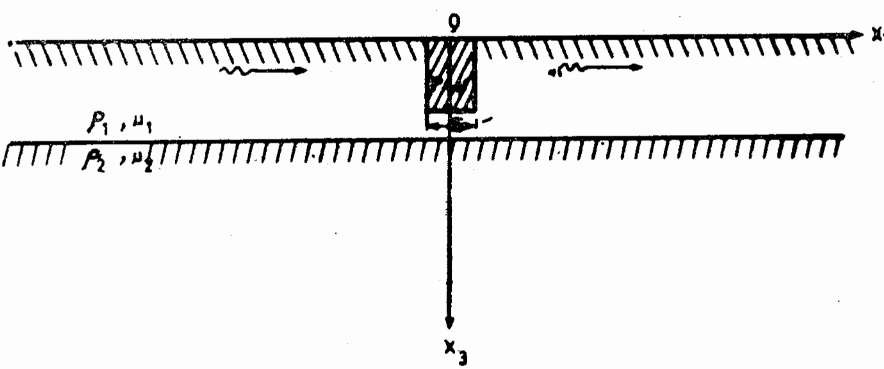


FIG 6

FIG. 6. Geometry of the problem of scattering of Love waves by a dike (Herrera and Mal, 1965).

Gilbert and Knopoff (1960) proposed a similar peturbation scheme to investigate the scattering of two-dimensional seismic waves from topographic irregularities of small magnitude and slope. In this way the problem was reduced to Lamb's problem for distributed surface sources by replacing the irregularity by an equivalent stress distribution. Scattering of Rayleigh waves incident upon a topographic irregularity was analysed by way of illustration. Hudson (1967) extended the application of the method to three-dimensional obstacles of an arbitrary nature and used it to interpret a surface wave arrival at the Eskdalemuir seismological array.

Using the formalism of propagator matrices, Kennett (1972) extended the perturbation scheme to calculate the scattering effects of slight lateral inhomogeneities of small lateral extent in a multi-layered elastic medium. Scattering effects produced by near-surface inclusions in simple half-space models are calculated as an example.

Kennett (1973) extended this method further to treat problems of the propagation of seismic waves through a multi-layered medium containing a layer within which there is a step horizontal discontinuity in elastic parameters (see fig. 7). The surface of discontinuity may be inclined to

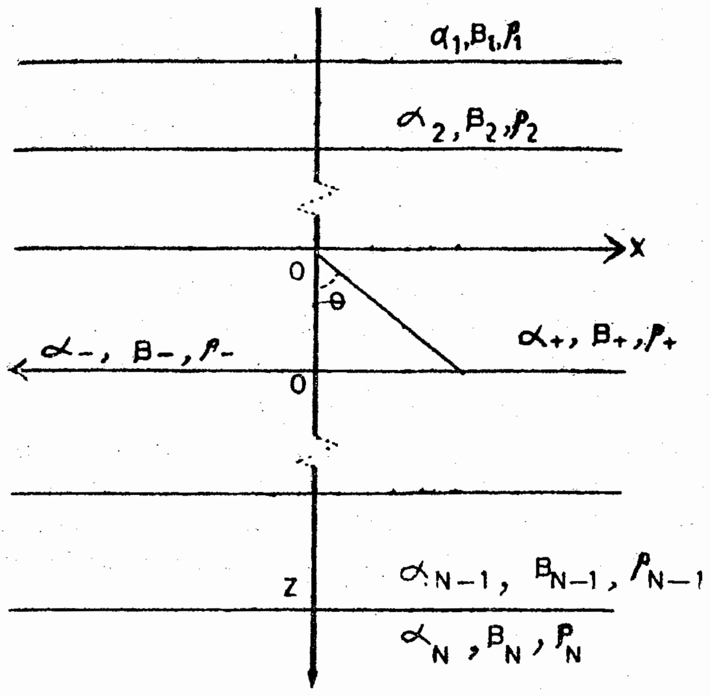


FIG. 7

FIG. 7. The geometry of the problem of propagation of seismic waves through a multi-layered medium with one layer containing a horizontal discontinuity (Kennett, 1973).

the vertical. Kennett pointed out that the perturbation method leads to a system of singular integral equations which may be solved iteratively.

(*c*) The method of local imbedding

Using a modified version of "the principle of localization" (Bellman and Kalaba 1959), Knopoff and Mal (1967) computed the effects of an inclined upper surface and an inclined Moho (see fig. 8) on the phase velocity of Love waves propagating in a single homogeneous layer of varying thickness overlying a homogeneous half-space.

According to the principle of localization, it is assumed that the sloping boundary is made up of an infinite number of small steps and that the wave undergoes instantaneous transmission and reflection at each step ; the reflection and transmission occur as if the layers to the left and

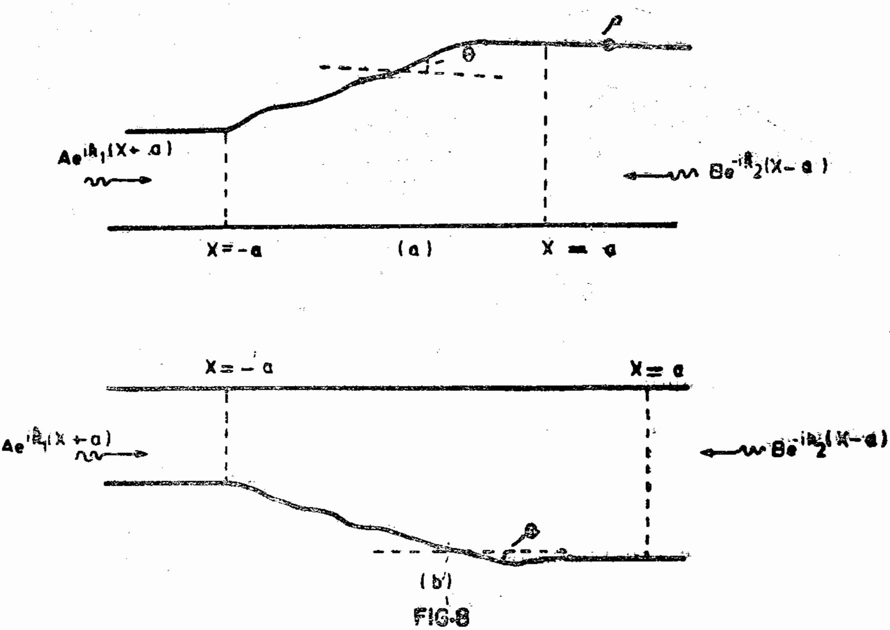


FIG. 8. The geometries of the crust in the transition regions considered by Knopoff and Mal (1967) :

(*a*) irregular surface.

(*b*) irregular interface.

to the right of the step were actually of uniform thickness, differing by an infinitesimally small amount. Hence the method depends upon the solution of an auxiliary problem, namely the determination of the transmission and reflection co-efficients for waves propagating past a step. Since the exact solution to the step problem does not exist, Knopoff and Mal use the Green function procedure of Knopoff and Hudson (1964) to obtain the reflection and transmission co-efficients of the auxiliary problem. For forward transmission, variation in elevation of the upper surface was found to produce no calculable effect. For both directions of propagation past an inclined Moho and for backward transmission for an inclined upper surface a period range was determined in which the effect is maximum and has a phase shift. Knopoff and Mal concluded that the phase velocity of surface waves in the transition zone of a thickening layer is severely influenced by back reflections from the structure beyond the seismic observatory. Hudson and Knopoff (1967) reached a similar conclusion in their investigation of surface wave scattering by topographic irregularities : forward scattering is much less important process for removing energy from a surface wave train than back-scattering. Knopoff and Mal point out that this feature is more important in producing phase shifts than is the shift in the eigenvalue as found by Takahashi (1964), who solved the eigenvalue problem (using JWKB approximation) for Love waves with a hyperbolic interface (see fig. 9) between the upper layer and the mantle.

(iii) Complex-variable methods

Wolf (1967, 70) has developed a method for determining the scattered field which results when a Love wave is incident on a layer having a small irregularity (see fig. 10*a*) in the free surface and overlying a rigid half-space. The scattered field is represented as an integral in the complex plane over the modes of the unperturbed structure and then the choice of the kernel of the integral is made in such a way that the boundary condition is approximately satisfied on the irregular part of the boundary. Wolf (1967) shows that the scattered field may be described by a super-position

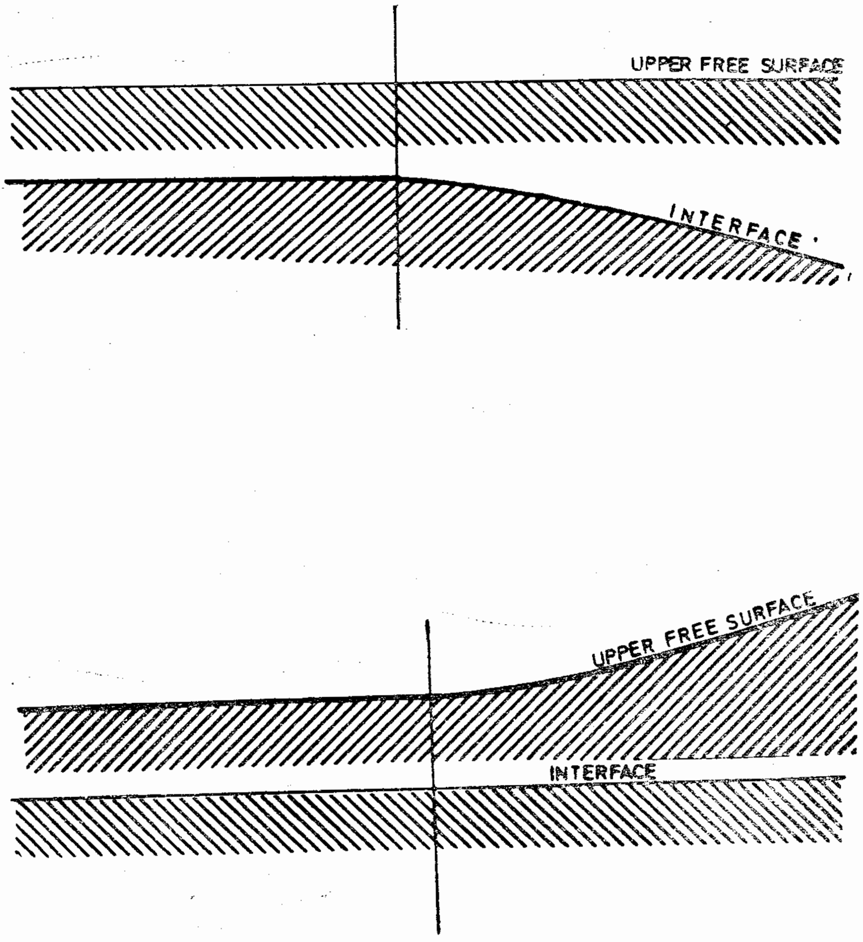
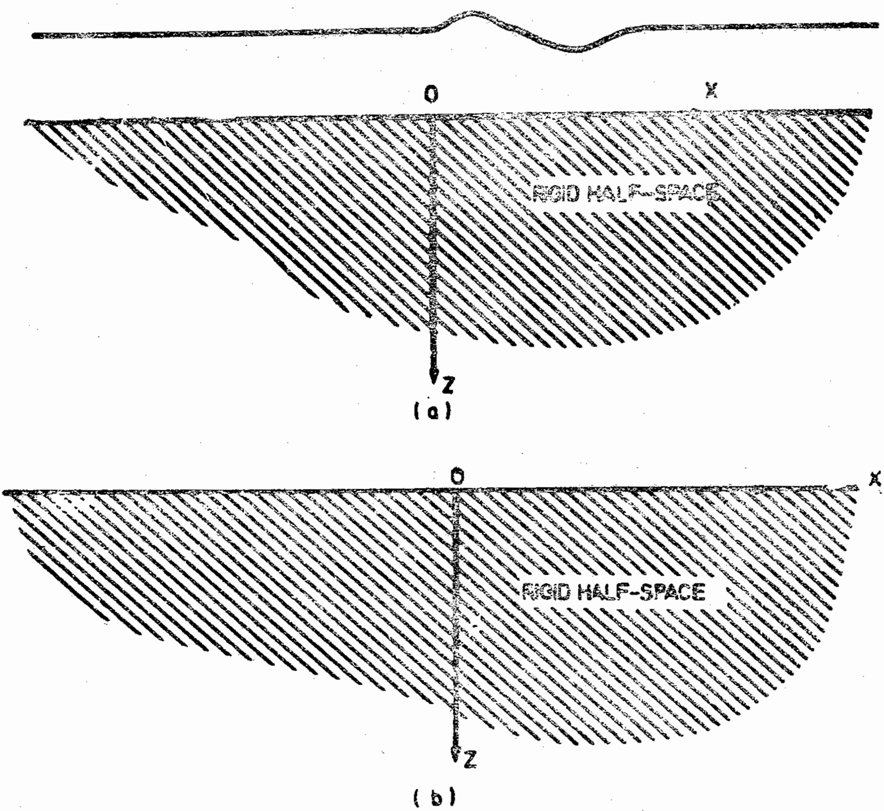


FIG 9

FIG. 9. Two of the models in the problem considered by Takahashi (1964). Here one-half of the hyperbola representing the interface or the upper free-surface is replaced by a straight line.

of Love waves and non-propagating disturbances (since the lower material is rigid, the continuous spectrum of eigenvalues is absent). Scattering by a triangular trough (see fig. 10*b*) is obtained as an illustrative example.

Slavin and Wolf (1970) have extended Wolf's method to treat the case in which the irregularity cannot be considered small. The method uses a least square procedure to approximately satisfy the traction-free boundary



FIG·10

FIG. 10. Geometry of the Love wave scattering problems considered by Wolf (1967, 70) : (*a*) a layer having a small irregularity in the free surface and overlying a rigid half-space.

(*b*) a layer having a triangular trough in the free surface and overlying a rigid half-space.

condition at the free surface irregularity. Numerical results were obtained for surface irregularities shown in figures 11.

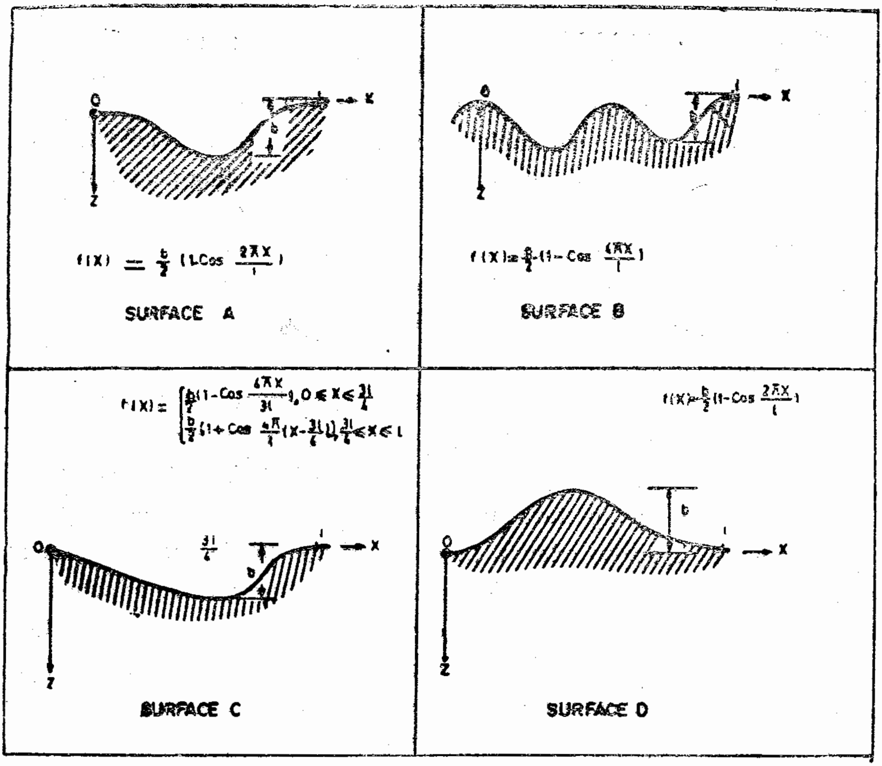


FIG. (11) Surface irregularities in the Love wave scattering problems investigated by Slavin and Wolf (1970).

(iv) Numerical methods

The propagation of Love waves across non-horizontally layered structures has been treated directly by methods of numerical analysis. Boore (1970) considered a model consisting of a surface layer with a sloping base (see fig. 8), overlying a half-space. Using a finite difference scheme, together with Fourier analysis of a pulse, Boore calculated the transmission co-efficients and velocities of reflected and transmitted phases over a wide range of wave numbers.

Lysmer and Drake (1971) developed a finite element technique for the analysis of Love wave propagation across an irregular structure separating two horizontal structures and applied it to the following models : a surface layer with a step elevation at its base, overlying a half-space (see fig. 13) ; a surface layer, with a sinusoidal free surface, overlying a half-space (see fig. 11) ; a continental boundary, and a structure including a slab of lithosphere dipping downward at an angle of 45°.

For a detailed account of the finite difference and the finite element methods in seismic wave propagation, reference may be made to the articles by Boore, and Lysmer and Drake in Bolt et. al. (1972).

(*v*) *Statistical methods*

In order to obtain estimates of the amplitudes of surface waves scattered by complex but small variation in structure, Knopoff and Hudson (1964*b*) applied Chernov's (1960) statistical methods to results obtained by a perturbation method similar to that of Herrera. By assuming the space-correlation functions of the elastic parameters to be Gaussian and isotropic, Knopoff and Hudson were able to calculate the mean-square amplitude of the scattered waves, averaged over a statistical ensemble of inhomogeneities. Hudson (1970) applied this approach to investigate the attenuation of surface waves by inhomogeneities whose size is small compared with the wavelengths of the incident waves.

Some statistical properties of Rayleigh waves due to scattering by complex topography, assumed to have certain statistical properties, have been given by Hudson and Knopoff (1967), using a similar scheme based on the perturbation method of Gilbert and Knopoff (1960).

(*vi*) *Asymptotic Methods*

(*a*) **Boundary layer methods**

The boundary layer method (also called parabolic equation method) is applicable to boundary diffraction problems (acoustic, electromagnetic, elastic), in which we may consider that a wave is propagated as if within a thin layer near the boundary. Certain boundary conditions must be

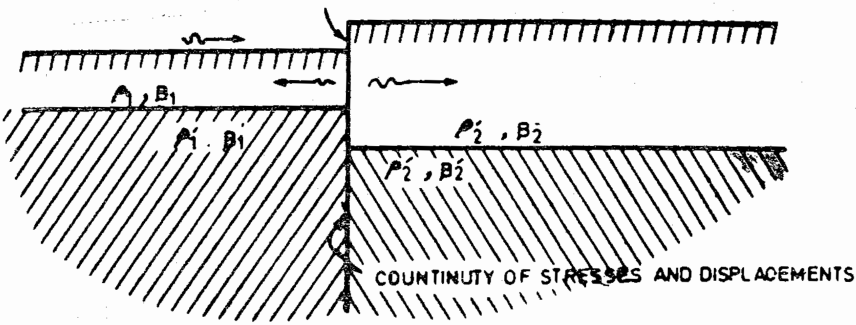


FIG. 12

FIG. (12) The Geometry of the problem of Love wave propogation in laterally varying structures considered by Alsop (1966).

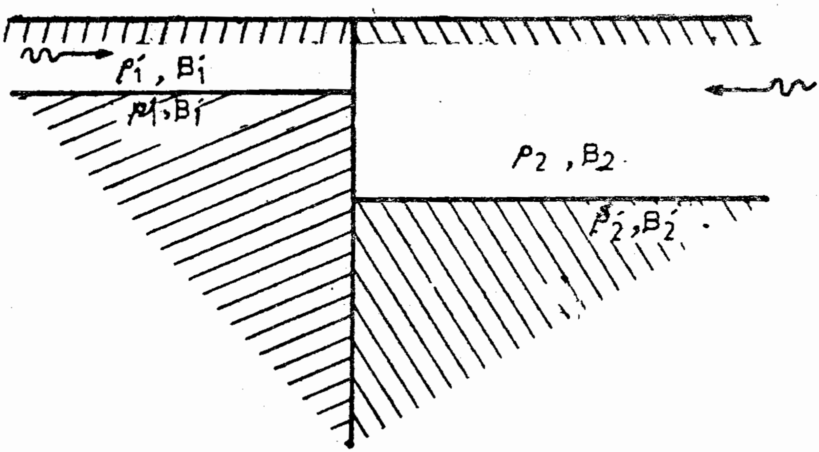


FIG . 13

FIG. (13) The configuration of the structure representing the M-discontinuity step, Alsop (1966).

specified on this boundary. Surface waves present an example — waves are propagated near the surface and decay rapidly with depth. In the application of the method to surface wave propagation, we may allow the elastic constants to vary as functions of position, but the method can be carried out only for high frequencies.

Babich and Molotkov (1966) considered the propagation of high-frequency, time-harmonic Love waves in an elastic half-space, which is inhomogeneous in the direction of two co-ordinates (one normal to the surface). The upper surface of the half-space is supposed to be stress-free. Under the assumption that the velocity of propagation of transverse waves increases with increasing depth close to the boundary of the half-space, these waves are concentrated in the surface wave-guide. The boundary layer method is used to establish formulae showing the dependence of Love wave displacements on the co-ordinates and the frequency. The depth of penetration and the phase velocities of these waves are also obtained.

Mukhina and Molotkov (1967) extended the method to high-frequency Rayleigh waves in an elastic half-space. The Rayleigh waves are concentrated in a near-surface wave guide, if it is assumed that the velocity of the transverse waves increases near the boundary of the half-space (the velocity of longitudinal wave may an arbitrary depth dependence). Mukhina and Molotkov also discuss the differences between the propagation properties of Rayleigh and Love waves.

Molotkov and Zhuze (1969) introduced into the problem a discontinuity in the properties of the medium at a vertical boundary and studied the effect on the propagation of *high frequency* Love waves. It is shown that the incidence of a Love wave in mth-mode on the vertical boundary generates a reflected wave, consisting mainly of a Love wave of mode m, and refracted Love waves of all modes, in general.

(*b*) Ray-theoretic methods

Keller and Karal (1964) developed a geometrical theory for the propagation of elastic surface waves on boundaries or interfaces of elastic solids. The theory applies to periodic waves of high frequency and short wavelength and is an extension of geometric optics. Keller and Karal introduced the concept of complex rays that travel from the source to the surface, than along the surface, and finally from the surface to points in the solid, and derived geometrical formulae for the determination of phase and amplitude associated with each point on each ray. The total displacement at a point is the sum of the displacements on all the rays through the point, each of which is constructed from the corresponding phase and amplitude. The theory is applicable to curved surfaces and to inhomogeneous media.

Gregory (1970) has developed a formal asymptotic theory for the propagation of high-frequency, time-harmonic Rayleigh waves over the general smooth free surface Σ of a homogeneous elastic solid. It is shown that on Σ these Rayleigh waves can be described by a system of surface rays, which are shown to be geodesics of the surface Σ. An explicit first order dispersion formula is derived and the amplitude of the waves on Σ is shown to vary in such a way that the energy propagated along a strip of surface rays is constant. The general theory is applied to the propagation of Rayleigh waves over the curved surface of a circular cylinder, down a circular cylinderical bore hole and over a sphere.

(*c*) The method of Whitham's averaged Lagrangian

Whitham (1965*a*, *b* ; 1967*a*, *b*) developed an approximate and general theory for the study of propagation of almost harmonic wave trains in terms of an average Lagrangian, and applied it to water wave problems. The theory, is, however, applicable to elastic surface wave problems also.

Gjevik (1973) applied Whitham's averaged Lagrangian method to the study of the effects of non-horizontally-layered crustal structures on surface wave propagation. He formulated a variational principle for Love waves based on the averaged Lagrangian and derived equations governing the slow variations in frequency, wave number and amplitude of the waves for problems where variations in structure are small within a wavelength.

Woodhouse (1974) developed a ray theory for surface wave propagation in a layered elastic structure, in which there are gradual lateral variation the thickness of the layers and in the elastic parameters characterising each layer. The theory is based upon a perturbation procedure given by Bretherton (1968), which leads to the same set of equations for linear waves in a slowly varying wave-guide as that obtained through the averaged Lagrangian. Propagator matrix formalism is used by Woodhouse to derive equations governing the slow variations in amplitude,frequency and wave number in a nearly uniform and approximately sinusoidal wavetrain. An equation governing the slow variation in phase is also deduced. Woodhouse shows that the solution of these equations by the method of characteristics gives ray-tracing equations and an amplitude equation similar to those given by the standard ray theory for body waves. His results reduce to those of Gjevik (1973) and DeNoyer (1961) (who was the first to determine the approximate effect of a sinusoidally varying layer on the propagation velocity of Love waves) in the restricted cases they consider.

(*vii*) **Variational mathods** :

Variational methods have also been used to find approximate solutions of surface wave propagation in laterally varying structures. Alsop (1966) has given a method for calculating reflection and transmission co-efficients for a monochromatic Love wave, normally incident on the vertical discontinuity between two multi-layered quarter-spaces in welded contact (a step may exist between the two quarter-spaces), see (fig. 12). The method is based upon the fact that the motion, as a function of depth, in the medium on either side of the plane of discontinuity can be completely expressed as a sum, with proper co-efficients, over the eigenfunctions that would be appropriate to the medium if it were a half-space instead of a quarter-space. However, in view of the difficulties involved in working with a continuum of eigenvalues, Alsop neglects body wave contributions and uses only the discrete modes, as a first approximation. But then the partial sums on the two sides of the plane of discontinuity cannot be matched exactly and it becomes necessary to use a variational method to get the best possible fit. The variational procedure involves the minimization of the meansquare error between the two partial sums as integrated over the boundary between the two quarter-spaces-a procedure used by Kane and Spence (1963*b*) in their theoretical studies of the propagation of surface wave across wedges. Transmission and reflection co-efficients are calculated for the surfacestep model used by Knopoff and Hudson (1964) and the M-discontinuity step (see fig. 13).

The results obtained by Alsop agree qualitatively with those of Knopoff and Hudson in general, but there are considerable differences in detail particularly at intermediate and long periods. At zero frequency and for both directions of propagation, the transmission co-efficients obtained by Alsop are unity, whereas the transmission co-efficients obtained by Knopoff and Hudson are not. Knopoff et. al. (1970) looked into the reasons for this discrepancy and noted that the amplitudes of Love modes as a function of depth depend strongly upon the layer thickness, even at very long periods, and that Love waves crossing the region of changing thickness may have a transmission co-efficient different from unity at zero frequency. This result does not apply to Rayleigh waves; for Rayleigh waves in a medium with single surface layer, the behaviour at the longest periods is as though the layer was absent.

McGarr and Alsop (1967) used a similar method to determine the reflection and transmission co-efficients for two-dimensional monochromatic Rayleigh waves, normally incident on the plane of discontinuity of the following laterally discontinuous structures—a homogeneous half-space with a step discontinuity of the free surface, and two quarter-spaces with different densities and elastic constants, welded together to form a half-space. Two-dimensional model experiments of these cases were conducted and it was claimed that the experimental data show good agreement with the computed results, as long as the structural changes at the discontinuities are not extreme.

Malichewsky (1974) has considered the interaction of seismic surface waves with curved discontinuities using Alsop's method with a curvilinear co-ordinate system. Malichewsky and Neunhöfer (personal communication) have also used Alsop's method to investigate the propagation of surface waves, obliquely incident on a vertical discontinuity, by using a representation on either side of the plane of discontinuity which is supposed to allow mutual conversion of Rayleigh and Love waves.

Gregersen and Alsop (1974) have given a method, based upon Alsop's (1966) work, of treating similar problems numerically when the incidence of Love waves on the vertical interface is different from normal. They conclude that the amplitude transmission curves show almost no dependence on the angle of incidence except when it is large. In this method the boundary conditions on the surface and on the horizontal interfaces are disregarded and the boundary condition at the vertical interface only is taken into account.

The variational method due to Schwinger and Levine (see Moiseiwitsch 1966) which has been used with success to treat wave problems in theoretical physics has only recently been applied (by Lapwood et. al. 1973, 1975*a*, *b*) in studies on the passage of elastic waves through an anomalous region, Miles (1967), in his paper on the diffraction of gravity waves in water at a discontinuity in depth, describes a method, based upon an integral equation formulation which leads to a description of the diffraction of gravity waves by means of a scattering matrix and the use of the Schwinger-Levine variational principle to estimate the elements of the scattering matrix. This method in conjunction with the spectral representation of the Love wave operator (Kazi, 1976) has been used by Kazi (1977) to investigate the two-dimensional problems of diffraction of plane harmonic Love waves, incident normally (from either side) upon the plane of discontinuity in the laterally discontinuous structure consisting of a half-space with a surface step-an idealized model of a continental margin. Diffraction of Love waves is described by means of a scattering matrix and approximate expressions for its elements are sought through the variational method. Complex reflection and transmission co-efficient can then be obtained through a transmission matrix related to the scattering matrix. This method has the additional advantage that body wave contributions (neglected in earlier methods) which are of considerable importance are also taken into account.

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SOME RESULTS ON $\overset{*}{Z}$ CLASSIFICATION STATISTIC

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1. Introduction

Given p-variate normal populations $\pi_1 : N(\mu_1, \Sigma_{11})$ and $\pi_2 : N(\mu_2, \Sigma_{11})$ with unknown means μ_1, μ_2 but specified covariance matrix Σ_{11}, independent random samples $(x_{11}, x_{12}, \ldots, x_{1N1})$ and $(x_{21}, x_{22} \ldots, x_{2N2})$ from π_1 and π_2, and a vector observation x arising from π_1 or π_2, one of the criteria for identification of x with its relevant population is the statistic

$$Z = \frac{N_1}{N_1+1}(x-\bar{x}_1)'\Sigma_{11}^{-1}(x-\bar{x}_1) - \frac{N_2}{N_2+1}(x-\bar{x}_2)'\Sigma_{11}^{-1}(x-\bar{x}_2)$$

where $\bar{x}_1$, $\bar{x}_2$ are the best linear unbiased estimators of μ_1, μ_2. Sometimes in addition to above knowledge of discriminators it is possible to secure information on a q dimensional covariate y that has the same mean in both populations. The concomitant variable when available may not be ignored but utilised somehow for improving the precision of classification. The problem with augmented information is then as follows: The observation $\begin{pmatrix} x \\ y \end{pmatrix}$ is known to have come from one of two populations $\pi_i : N\left[\begin{pmatrix} \mu_i \\ \omega \end{pmatrix}, \Sigma\right]$, $i = 1, 2$ with specified covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix};$$

the samples drawn independently from π_1 and π_2 are

$$\begin{pmatrix} x_{i1} \\ y_{i1} \end{pmatrix}, \begin{pmatrix} x_{i2} \\ y_{i2} \end{pmatrix}, \ldots\ldots, \begin{pmatrix} x_{iNi} \\ y_{iNi} \end{pmatrix} \; i=1, 2;$$

and it is required to classify $\begin{pmatrix} x \\ y \end{pmatrix}$ into its original population. For it, Memon [3] proposes a modified statistic

$$\overset{*}{Z} = \frac{N_1}{N_1+1} (X-\bar{x}_1)' Q (X-\bar{x}_1) - \frac{N_2}{N_2+1} (X-\bar{x}_2)' Q (X-\bar{x}_2)$$

Q being $(\Sigma_{11}-\Sigma_{12}\ \Sigma_{22}^{-1}\ \Sigma_{21})^{-1}$; which he obtains by replacing x by $X = x-\Sigma_{12}\ \Sigma_{22}^{-1}\ y$ in Z statistic. According to it the observation is classified into π_1 if $\overset{*}{Z}$ has a value not exceeding zero, otherwise into π_2. This paper investigates $\overset{*}{Z}$ making studies as in Memon [4], Memon and Okamoto [5] for the case of Z statistic. We discover following properties of the statistic under consideration.

Theorem 1 : The statistic $\overset{*}{Z}$ has an alternate form

(*i*) $b\ \{[X-\bar{x}_1+a\ (\bar{x}_2-\bar{x}_1)]' Q\ [X-\bar{x}_1+a\ (\bar{x}_2-\bar{x}_1)]-a\ (a+1)\ T\}$ if $N_1 \neq N_2$

where $T=(\bar{x}_1-\bar{x}_2)'\ Q\ (\bar{x}_1-\bar{x}_2)$
$a=N_2\ (N_1+1)\ /\ (N_1-N_2)$,
$b=(N_1-N_2)\ /\ (N_1+1)\ (N_2+1)$

(*ii*) $-2N/(N+1)\ \overset{*}{W}$ if $N_1 = N_2 = N$

where $\overset{*}{W}=[X-(\bar{x}_1+\bar{x}_2)\ /\ 2]'\ Q\ (\bar{x}_1-\bar{x}_2)$ is a criterion due to Cochran and Bliss [2] for the same classification problem.

Theorem 2. The asymptotic distribution of the statistic $\overset{*}{Z}$ is normal with variance $4\ D^2$ and mean $-D^2$ or D^2 accordingly as $\begin{pmatrix} x \\ y \end{pmatrix} \varepsilon\ \pi_1$ or π_2; D^2 being the Mahalanobis distance between the populations π_1 and π_2.

Theorem 3. The statistic $\overset{*}{Z}$ remains invariant under any linear transformation $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A \begin{pmatrix} x \\ y \end{pmatrix} + C$ where A is a $(p+q)$ by $(p+q)$ nonsingular matrix and C is a $(p+q)$ by 1 vector.

The above results are not very difficult to obtain, and so we omit their proofs.

Theorem 4: If the sample means $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2$ are fixed and $\begin{pmatrix} x \\ y \end{pmatrix} \varepsilon\ \pi_1$, the statistic $\overset{*}{Z}$ has the characteristic function

$$(i)\ \exp\left[-\varphi\, a\, b\, (a+1)\, T + \frac{\phi\, b}{1 - 2\phi\, b} \sum_1^p \omega_i^2 - \frac{p}{2} \log\, (1 - 2\,\varphi\, b)\right.$$

if $N_1 \neq N_2$

where $\varphi = i\, t$

$$W = M\left[\mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\,\omega - X_1 + a\ (X_2 - X_1)\right];$$ the vector comprising the components ω_i ;

and M is a nonsingular matrix such that

$M\, Q\, M' = I.$

$$(ii)\ \exp\left[\varphi\, g\left\{\mu_1 - \Sigma_{12}\,\Sigma_{22}^{-1}\,\omega - \tfrac{1}{2}(\bar{X}_1 - \bar{X}_2)\right\}' Q\,(\bar{X}_1 - \bar{X}_2) + \tfrac{1}{2}\varphi^2 g^2\, T\right]$$

if $N_1 = N_2 = N$

where $g = -\,2N\,/\,(N+1)$.

Proof (*i*) Using an alternate form of $\overset{*}{Z}$ from Theorem 1 and Lemma (2.3) in Memon and Okamoto [5], the result follows.

(*ii*) Under the given restriction, the statistic $\overset{*}{W}$ has a normal distribution with mean $\left[\mu_1 - \Sigma_{12}\,\Sigma_{22}^{-1}\,\omega - \tfrac{1}{2}\,(\bar{X}_1 - \bar{X}_2)\right]'\ Q\ (\bar{X}_1 - \bar{X}_2)$ and variance T.

Since by Theorem 1 the statistic under study is linearly related to $\overset{*}{W}$, so its distribution is also normal with mean and variance as g and g^2 times that of W.

Theorem 5 : If $D > 0$ and $\begin{pmatrix} x \\ y \end{pmatrix} \varepsilon\ \pi_1$, then an asymptotic expansion of the distribution function of $(\overset{*}{Z} + D^2)/2D$ is given by

$$(i)\ \left(1 + \frac{c_1}{N_1} + \frac{a_2}{N_2} + \frac{a_{11}}{N_1^2} + \frac{a_{22}}{N_2^2} + \frac{a_{12}}{N\,N} + \ldots\ldots\right)\phi\,(u)$$

if $N_1 \neq N_2$

where

$a_1 = \frac{1}{2} D^{-2} (d^4 + Dd^3 + pd^2),$

$a_2 = \frac{1}{2} D^{-2} [d^4 + Dd^3 - (D^2 - p) d^2 - D^3 d],$

$a_{11} = \frac{1}{8} D^{-4} [d^8 + 2D d^7 + (D^2 + 2p + 4) d^6 + 2(p+2) Dd^5$
$- (4D^2 - p^2 - 2p) d^4 - 4D^3 d^3 - 4p D^2 d^2],$

$a_{22} = \frac{1}{8} D^{-4} [d^8 + 2Dd^7 - (D^2 - 2p - 4) d^6 + 2(2D^2 - p - 2) Dd^5$
$- \{D^4 + 2(p+4) D^2 - p^2 - 2p\} d^4 + 2(D^2 - p - 4) D^3 d^3$
$+ (D^4 + 4D^2 - 4p) D^2 d^2 + 4D^5 d],$

$a_{12} = \frac{1}{4} D^{-4} [d^8 + 2Dd^7 + 2(p+2) d^6 - 2(D^2 - p - 2) Dd^5$
$- \{D^4 + (p+4) D^2 - p^2 - 2p\} d^4 - (p+4) D^3 d^3 - 2p D^2 d^2],$

$\phi(u)$ is the cumulative normal distribution function of N (0, 1) and

$$d = \frac{d}{du}.$$

(*ii*) $\left(1 + \frac{b_1}{2N} + \frac{b_2}{4N^2} + \ldots\ldots\ldots\right) \phi(u)$ if $N_1 = N_2 = N$

where

$b_1 = D^{-2} [2d^4 + 2Dd^3 - (D^2 - 2p) d^2 - D^3 d],$

$b_2 = \frac{1}{2} D^{-4} [4d^8 + 8D d^7 + 8(p+2) d^6 - 8(D^2 - p - 2) Dd^5$
$- \{3D^4 + 4(p+5) D^2 - 4p^2 - 8p\} d^4 + 2(D^2 - 2p - 10) D^3 d^3$
$+ (D^4 + 4D^2 - 12p) D^2 d^2 + 4D^5 d],$

$\phi(u)$ and d are same as above.

Proof: As $\overset{*}{Z}$ is invariant under any nonsingular transformation in Theorem 3, we may take $\mu_1 = 0$, $\mu_2 = \mu_0$, $\omega = 0$, $\Sigma = I$ where μ_0 is a $1 \times p$ vector (D, o, o, ..., o). With these substitutions, the conditional characteristic function of $\overset{*}{Z}$ is then left as a function of $\bar{x}_1$, $\bar{x}_2$ only. The resulting expression and the one obtained by Memon[4] become identical except of course that the Mahalanobis distances are different. So the same arguments as in [4] are applicable to this situation as well. These considerations immediately lead to above theorem.

The case (*ii*) is also dealt with similarly.

Corollary: The probability of misclassifying an observation $\begin{pmatrix} x \\ y \end{pmatrix} \varepsilon \pi_1$

into population π_2 by the use of the classification statistic $\overset{*}{Z}$ is given by

$$(i)\left[1-\left(\frac{a_1}{N_1}+\frac{a_2}{N_2}+\frac{a_{11}}{N_1^2}+\frac{a_{22}}{N_2^2}+\frac{a_{12}}{N_1N_2}+\ldots\ldots\right)\phi(u)\right]_{u=D/2}$$

if $N_1 \neq N_2$

$$(ii)\left[1-\left(\frac{b_1}{2N}+\frac{b_2}{4N^2}+\ldots\ldots\ldots\right)\phi(u)\right]_{u=D/2}$$

if $N_1 = N_2 = N$

where $a_1, a_2, .., b_1, b_2$ are as in above theorem.

Proof : The probability of correctly classifying $\begin{pmatrix} x \\ y \end{pmatrix}$ into π_1 is

$\text{Prob}\,(\overset{*}{Z} \leqslant 0) = \text{Prob}\left(\frac{\overset{*}{Z}+D^2}{2\,D} \leqslant \frac{D}{2}\right)$ that is, the value of the distribution function of $(Z+D^2)/2D$ at $D/2$. So the results follow.

Since the interchange of N_1 and N_2 in $\overset{*}{Z}$ leads to $-\overset{*}{Z}$, the probability of the second kind can also be immediately obtained from the above corollary.

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EXTENSIONS OF CERTAIN SYMMETRIC OPERATORS*

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1. Introduction

One of the most important problems in the theory of operators is that of extension of a symmetric operator to a self adjoint one, so that the spectral theorem can be applied. Von Neumann (see for example[1] and [5]) gave the necessary and sufficient conditions for a densely defined symmetric operator to have a self adjoint extension in the same Hilbert Space. In case of Hermitian Operator when the domain of the operator is not dense in the Hilbert Space, it is not possible to apply Von Neumann theory as such. Coddington [3] overcomes this difficulty by applying the theory of linear relations defined on a Hilbert Space (Arens [2]). We use Coddington's approach to construct a proof of the extension theorem (see [3]), on the same lines as Von Neumann's proof [5]. In section (3) we use Coddington's extension theorem to prove some results stated by Michael [5] concerning finite dimensional extensions.

2. Coddington's Extension Theory

(2.1) Some Preliminaries

Let H be a Hilbert space with inner-product $<,>$ and $H^2 = H \times H$ be the product Hilbert space having usual linear structure and innerproduct defined in it. We call a linear subspace A of H^2 a linear relation on H. The domain D (A), range R (A), and null space N (A), of a linear relation A are defined as follows :

$$D(A) = \{x : x \in H, (x, y) \in A\},$$
$$R(A) = \{y : y \in H, (x, y) \in A\},$$
$$N(A) = \{z : z \in H, (z, o) \in A\}.$$

* The results presented here form a part of author's M.Sc. dissertation, written under supervision of Dr. I.M. Michael, presented to the University of Dundee, U.K.

If A is single valued, then A becomes graph of a linear operator. We define orthogonal complement A of a linear relation A by

$$A^{\perp} = \{x : \langle x, y \rangle = 0 \quad \forall\, y \in A\}.$$

For a linear relation A on H, the adjoint of A, A^* is defined to be the subspace

$$A^* = \{ (h, k) \in H^2 : \langle g, h \rangle = \langle f, k \rangle \,\forall\, (f, g) \in A\}.$$

A is said to be symmetric if $A \subseteq A^*$, and self adjoint if $A = A^*$.

Define

$$A_{\infty} = \{(o, g) \in H^2 : (o, g) \in A\}$$

and $A_s = A \ominus A_{\infty}$, so that

$$A = A_s \oplus A_{\infty}$$

A_s is called the single valued part of A because we have the following :

Theorem

A_s is the graph of an operator, and $D(A_s)$ is dense in $(A^*(o)^{\perp})$.
From above theorem it can be shown (see [7])
$(A_s)^* = (A^*)_s$ if and only if $A(o)=A^*(o)$.

(2.2) *Extension Theorem*

For a symmetric subspace A of H^2, we define subspaces

$$M^{\pm} = \{ (h, k) \in A^* : k = \pm\, i h \}$$

$M^{\pm}$ are called Deficiency subspaces of A^*, and dim $(M^{\pm})$ are called Deficiency indices.

We have (see [7] for details)

(1) $M^{\pm}$ are graphs of single valued operators.

(2) $M^{\pm}$ are closed.

(3) If $M = A^* \ominus A$ then $M = M^+ \oplus M^-$.

Theorem

Let S be a self adjoint subspace in H^2 with $S = S_s \oplus S_\infty$. Then $(S^*)_\infty = S_\infty$, and S_s restricted to R^2, where $R = (S(o))^{\perp}$, is a self-adjoint densely defined operator in R.

For proof see [7].

The following theorem is due to Coddington [3], we present here a proof based upon the von Neumannextension theorem.

Theorem : Extension Criterion

Let S be a symmetric subspace of H^2, $M = S^* \ominus S$. Then $S_1 \subset S$, where S_1 is a self adjoint subspace of H^2, if and only if, there exists an isometry U of M^+ onto M^- such that $S_1 = S \oplus (I - U) M^+$.

In other words, S has a self adjoint extension in H^2, if and only if $\dim(M^+) = \dim(M^-)$.

Proof

(1) Consider some subspace T of H^2, such that $S \subseteq T \subseteq S^*$. We assert that T is symmetric if and only if $T \ominus S$ is symmetric.

(*a*) If T is symmetric then $T \ominus S$ being restriction of T is also symmetric.

(*b*) Now suppose that $T \ominus S$ is symmetric and, by assumption S is symmetric. The elements of $T \ominus S$ can be written in the form $(f - h,\ g - k)$, $(f,\ g) \in T$, $(h,\ k) \in S$ and $< (f - h,\ g - k,)\ (h,\ k) > = 0$.

We have

$$<g' - k', f - h> = <f' - h', g - k>$$

for all $(f' - h', g' - k') \in T \ominus S$.

Or $<g', f> - <k', f> - <g', h> + <k', h>$

$$= <f', g> - <h', g> - <f', k> + <h', k>. \qquad (i)$$

Also by symmetry of S

$<k', h> = <h', k>$ holds for all $(h', k') \in S$.

Now $(f, g) \in S^*$, so far any $(h', k') \in S$

$$<k', f> = <h', g>.$$

Hence (i) reduces to

$$<g', f> = <f', g> \text{ for all } (f', g') \in \top.$$

This implies that $\top$ is symmetric.

(2) $M = S^* \ominus S$, So

$$\top \ominus S \subseteq M^+ \oplus M^- = M$$

Hence (f, g), $(h, k) \in \top \ominus S$ are of the form

$$(f, g) = (f^+, i f^+) + (f^-, -i f^-)$$

$$(h, k) = (h^+, i h^+) + (h^-, -i h^-)$$

Then $\top \ominus S$ is symmetric if and only if

$$<f^+, h^+> = <f^-, h^-> \text{ for } f \in D(\top \ominus S).$$

If $\top \ominus S$ is symmetric, then

$$(f, g) \in \top \ominus S, (h, k) \in \top \ominus S \text{ imply } <k, f> = <h, g>$$

Or $<i h^+ - i h^-, f^+ + f^-> = <h^+ + h^-, i f^+ - i f^->$

$$= -i<h^+ + h^-, f^+ - f^->$$

$$= -<i h^+ + i h^-, f^+ - f^->$$

So that, cancelling terms, we are left with

$$<h^+, f^+> = <h^-, f^->.$$

For converse retrace the steps.

So in particular, for $(f, g) = (h, k)$

$$\|f^+\|^2 = \|f^-\|^2 \text{ for all } f \in D(\top \ominus S)$$

Thus symmetric relations (subspaces) of $M^+ \oplus M^-$ correspond to isometries from some subspace of M^+ into M^-.

The converse of this statement is that for any isometry from a subspace of M^+ into M^-, there corresponds some symmetric subspace of $M^+ \oplus M^-$.

For, f^+, f^- defined as above, if

$$\|f^+\|^2 = \|f^-\|^2, \text{ when } (f^+, i f^+) \in M^+, (f^-, -i f^-) \in M^-,$$

we have

$$\|f^{+} + h^{+}\|^{2} = \|f^{-} + h^{-}\|^{2}$$

and $$\|f^{+} - h^{+}\|^{2} = \|f^{-} - h^{-}\|^{2}, \text{ so that}$$

$$\mathcal{R} < f^{+}, h^{+} > = \frac{\|f^{+} + h^{+}\|^{2} - \|f^{+} - h^{+}\|^{2}}{4}$$

$$= \frac{\|f^{-} + f^{-}\|^{2} - \|f^{-} - h^{-}\|^{2}}{4}$$

$$= \mathcal{R} < f^{-}, h^{-} >,$$

where $\mathcal{R} < f^{\pm}, h^{\pm} >$ is the real part of $< f^{\pm}, h^{\pm} >$.

Similarly $\operatorname{Im} < f^{+}, h^{+} > = \operatorname{Im} < f^{-}, h^{-} >$. These results together imply that $< f^{+}, h^{+} > = < f^{-}, h^{-} >$.

So that $\top \ominus S$ is symmetric by the necessary and sufficient condition shown before.

(3) Finally, we have to prove that a self adjoint subspace corresponds to an isometry with domain whole of M^{+} and range M^{-}.

Suppose that $\top$ is self adjoint and that the V is the corresponding isometry introduced above whose domain a proper subset of M^{+}. This means that there exists a non zero element of M^{+}, say $(f^{+}, i f^{+})$, which is orthogonal to D(V).

Now any $(x, y) \in \top$ is of the form

$$(x, y) = (h, k) + (h^{+}, i h^{+}) + (h^{-}, -i h^{-})$$

$$(h, k) \in S, (h^{+}, i h^{+}) \in M^{+}, (h^{-}, -i h^{-}) \in M^{-}.$$

Nothing that $(f^{+}, i f^{+})$ is orthogonal to the domain of V, and that $(h^{+}, i h^{+})$ belongs to the subset of M^{+}, which constitutes the domain of V, we have

$$<(x, y), (f^{+}, i f^{+})> = 0$$

So that $<x, f^{+}> = i < y, f^{+}>$

Or $<x, i f^{+}> = <y, f^{+}>$ for all $(x, y) \in \top$.

This implies that $(f^{+}, i f^{+}) \in \top^{*}$.

Since $(f^{+}, i f^{+})$ is orthogonal to D(V), it can not be in $\top$, which shows that $\top$ is not self adjoint, a contradiction.

Similarly if the range of the isometry V is a proper subset of M^-, we have a non zero element $(f^-, -i f^-)$ which is orthogonal to the range of V. As before

$$<(x, y), (f^-, - i f^-)> = 0 \text{ for all } (x, y) \in \top,$$

from which it follows that $(f^-, - i f^-) \in \top^*$, which again shows that $\top$ cannot be self adjoint.

To complete the proof, suppose that V is an isometry from M^+ onto M^- and let $\top$ be the restriction of $\top^*$, which in this case coincides with S_1.

Since $S \subseteq S_1 \subseteq S_1^* \subseteq S^*$, (x, y) is of the form

$$(x^+, i x^+) + (x^-, - i x^-).$$

With $(x^+, i x^+) \in M^+$,

$$(x^+, i x^+) \pm V (x^+, i x^+) \in S_1,$$

and so $<(x^+, i x^+) + V<(x^+, i x^+), (x^+, i x^+) + (x^-, - i x^-) > = 0$

Or

$$< (x^+, i x^+), (x^+, i x^+) > + < V (x^+, i x^+), (x^+, i x^+) >$$
$$+<(x^+, i x^+), (x^-, - i x^-)> + < V (x^+, i x^+), (x^-, - i x^-)> = 0$$

Since $V (x^+, i x^+) \in M^-$,

$$< (x^+, i x^+), (x^+, i x^+) > + < V (x^+, i x^+), (x^-, - i x^-)> = 0$$

Similarly

$$<(x^+, i x^+), (x^+, i x^+)> + < V (x^+, i x), (x^-, - i x^-)> = 0$$

and so $< (x^+, i x^+), (x^+, i x^+) > = 0$.

A similar argument, using V^{-1}, shows that $<(x^-, i x^-), x^- - i x^{-1})> = 0$, and so $(x, y) = 0$.

This completes the proof of the theorem.

3. Finite Dimensional Extensions

In the extension theorem in section (2), we considered extensions of the symmetric subspaces in the same Hilbert space H. If, however, the deficiency indices of S are not equal, then there always exists a Hilbert space H_1, such that $H \subsetneq H_1$, and S has a self adjoint extension in

H_1. Such an extension is called finite dimensional if dim $(H_1 \ominus H) < \infty$. The results presented in this section were given by Coddington [4], but we give here an independent and much simpler proofs of these results.

Let S be the graph of a Hermitian operator in H, i.e.

$$< Sf, g > = < f, Sg > \quad \text{for all f and g in} \quad D(S),$$

such that D(S) is not necessarily dense in H but such that $R(S) \subseteq \overline{D(S)}$. Suppose that an extension of S, A say, exists. Then A is the graph of an operator self adjoint in $\overline{D(A_s)}$. All the self adjoint extensions of S in H can be described in this way, since we have the following [6]

Lemma

Let K be a closed subspace of H, and let A be the graph of an operator such that $\overline{D(A)} = K$ and A is self adjoint in K. In particular $R(A) \subseteq K$. Let

$$B = A \oplus \{(o, g) \in H^2 : g \in K^{\perp}\}$$

Then B is a self adjoint subspace in H^2.

Proof

First note that since K is closed, we can write $H = K \oplus K^{\perp}$, so that any element $x \in H$ can be written as $x = x_1 + x_2$, where $x_1 \in K, x_2 \in K^{\perp}$ and $< x_1, x_2 > = 0$.

Now let $(x, y) \in B$, (x, y) is of the form $(h, Ah + g)$, where $h \in D(A)$ and $g \in K^{\perp}$. For any element $(h', Ah' + g) \in B$ we have

$$<Ah' + g', h> = <Ah', h> + <g', h>$$
$$= <h', Ah> = <h', Ah + g>$$

Since $<Ah', h> = <h', Ah>$, $(h, Ah) \in A$, which is self adjoint.

Hence $(x, y) \in B^*$ so that $B \subseteq B^*$. Conversely, let $(x, y) \in B^*$ so that

$$<Ah + g, x> = <h, y> \quad \text{for all} \quad (h, Ah + g) \in B$$

$$\text{where } (h, Ah) \in A, g \in K^{\perp}. \qquad (1)$$

Also $B^* \subseteq A^*$ implies that

$$\langle Ah, x\rangle = \langle h, y\rangle \quad \text{for all } h \in D(A). \tag{2}$$

So that using (2) we get from (1)

$$\langle g, x\rangle = 0 \quad \text{for all } g \in K^{\perp}. \tag{3}$$

$x \in D(A)$ if $x \neq 0$, so that $(x, Ax) \in A$.

Suppose that $g' = y - Ax$.

To show that $g' \in K^{\perp}$:

For $h \in D(A)$

$$\langle g', h\rangle = \langle y - Ax, h\rangle = \langle y, h\rangle - \langle Ax, h\rangle$$

(1) implies $\langle y, h\rangle = \langle x, Ah+g\rangle$ so that

$$\langle g', h\rangle = \langle x, Ah+g\rangle - \langle Ax, h\rangle = \langle x, g\rangle = 0 \text{ by (3).}$$

Hence $g' \in K^{\perp}$. So any $(x, y) \in B^*$ is of the form $(x, y) = (x, Ax+g')$, where $g' \in K^{\perp}$, so that $(x, y) \in B$. Hence $B^* \subseteq B$.

This proves the lemma.

To find connection between possible extensions of S in $\overline{D(S)}$ and the extensions in the Hilbert space H, we denote the adjoint of S in $\overline{D(S)}$ by S'. Let D_+, D_- be the spaces defined by

$$D_{\pm} = \{g, S'g) \in S' : S'g = \pm i g\}.$$

Then according to the extension theorem, S a has a self adjoint extension in $\overline{D(S)}$ if and only if $\dim D_+ = \dim D_-$.

$$\text{Let } X^{\pm} = \{(h, \pm i h) : h \in H \ominus D(S)\}.$$

Then we have the following results

$$M^+ = D_+ \oplus X^+$$

$$M^- = D_- \oplus X^-$$

We show first of these, whereas the second follows on exactly similar lines.

(*i*) D^+, X^+ are orthogonal :

For any $(g, i g) \in D^+$, $g \in D(S')$ and any $(h, i h) \in X^+$, $h \in H \ominus D(S)$, we have

$$<(h, i\,h), \; (g, i\,g)> \; = \; <h, g> + <h, g> = 0$$

since $<h, g> = 0.$

(*ii*) For any $(h, i\,h) \in X^+ \; h \in H \ominus D(S)$, and $(x, y) \in S$

$<y, h> = <x, i\,h> = 0$ for all $(x, y) \in S$.

This is because $R(S) \subseteq \overline{D(S)}$ and $h \perp \overline{D(S)}$; so that $X^+ \subseteq M^+$. The fact that $D_+ \subseteq M^+$ needs no proof.

(*iii*) Let $(f, if) \in M^+$, where $f \in H$, and so

$$f = g + h, \; g \in D(S), \; h \in H \ominus D(S).$$

So that

$$(f, i\,f) \in S^* \text{ implies } < y, f> = <x, i\,f> \text{ for } (x, y) \in S.$$

Or $<y, g + h> = <x, i\,(g+h)>$

But $<y, h> = <x, h> = 0$

since $y, x \in \overline{D(S)}$ and $h \in H \ominus D(S)$,

hence $<y, g> = <x, ig>$ for all $(x, y) \in S$, and so $(g, i\,g) \in S'$.

This shows that every element $(f, i\,f)$ of M^+ can be written as sum of an element of D^+ and an element of X^+.

From (*i*), (*ii*), and (*iii*) follows that

$$M^+ = X^+ \oplus D^+.$$

From the above discussion we deduce

Theorem

Let S be a densely defined closed symmetric operator in H with finite but unequal deficiency indices, and let H_1 be a Hilbert space such that $H \subset H_1$ and such that S has a self adjoint extension in H_1.

Then this extension is not finite.

Proof

We have $\dim X^+ = \dim X^- = \dim (H \ominus D(S'))$. Thus if $\dim D_+ = \dim D_-$, both being finite, we see that $\dim M^+ = \dim M^-$ is not possible unless $\dim (H \ominus D(S))$ is infinite.

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ESTIMATION OF THE SCALE-PARAMETER FROM THE TYPE II CENSORED RAYLEIGH DISTRIBUTION

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1. Introduction

In the literature, samples are said to be truncated whenever all the record in the truncation portion is omitted and censored, when the count is known and not their individual values. It occasionally happens that some observations are not available. For example, we may not be able to record observations above a certain level T on account of the limitations of our recording instrument. The only information we gather is that the unrecorded observations are greater than the recorded observations and we know their count. We thus have a distribution which has been truncated on the right and the number of sample observations in the truncated portion is known. Sometimes, we may terminate the experiment after observing some fixed percentage of observations simply because of cost. Gupta (1952) calls the former case as Type I censored samples and the latter as the Type II censored samples. In the latter case, the point of truncation is a random variable while in the former case the point of truncation is not specified to be random. The samples can be singly or doubly censord. Hirai (1971) considered the estimation of the scale parameter from the complete one parameter Rayleigh distribution whose p.d.f. is given as :

$$p(x) = \frac{2x}{\lambda^2} e^{-x^2/\lambda^2} \quad 0 \leqslant x < \infty; \;\; \lambda > 0. \tag{1}$$

The following methods of estimation were discussed to estimate the unknown scale parameter λ of the complete Rayleigh distribution.

(*i*) Maximum Likelihood Estimate (M.L.E.)

(*ii*) Best Linear Unbiased Estimate (B.L.U.E.)

(*iii*) Approximate linear Estimates.

The sampling distributions of these estimates were also studied for a sample size $n \leqslant 8$.

In this paper we discuss the estimation of the scale parameter from the Type II singly censored Rayleigh distribution.

2. Estimation of the scale parameter from the censored Rayleigh distribution.

Rayleigh distribution is extremely important in communication engineering. For example, the envelope of a narrow-band Gaussian random process and the amplitude of atmospheric radio noise caused by the radiation due to lighting discharges in storms have p.d.f's given by (1). For small enough values of x, the reliability of a Rayleigh component decreases with time more slowly than the reliability of a commonly used component (a component whose failure rate is constant). For large x, the reliability of a Rayleigh component decreases with time more rapidly than in the case of an exponential component.

Consider an experiment in which n "identical" Rayleigh components are to be simultaneously put into operation at time $x=o$ and the failure times are to be recorded. The sample data will be ordered. If initial observations are lost (for example, due to the fact that initial observations are often used solely for making checks and adjustment on all devices used to perform the experiment to assure that they are functioning properly), the sample is said to be censored from the left. If final observations are lost (for example, due to the experiment's being suspended before all components have failed), the sample is said to be censored from the right.

We thus estimate the scale parameter from Type II censored samples from the Rayleigh distribution for the following two cases.

(*i*) Right tail censored.

(*ii*) Left tail censored.

Case (*i*) *Right tail censored* : We suppose $x_{(1)}^{(n)} < x_{(2)}^{(n)} < \ldots < x_{(r_2)}^{(n)}$ are the observed observations where $x_{(r_2)}^{(n)}$ is the largest observation out of sample of size n where r_2 is fixed in advance from the Rayleigh distribution given in (1). The likelihood function is given as

$$L \propto \prod_{i=1}^{r_2} \frac{2\, x_{(i)}^{(n)}}{\lambda^2} \exp\left[-\sum_{i=1}^{r_2} \frac{\left(x_{(i)}^{(n)}\right)^2}{\lambda^2}\right]\left[\int_{x_{(r_2)}^{(n)}}^{\infty} \frac{2\,x}{\lambda^2} e^{-x^2/\lambda^2}\, d\,x\right]^{n-r_2} \tag{2}$$

Differentiating log L *w.r.* to λ and equating to zero we obtain the maximum-likelihood estimating equation

$$\frac{\partial \log L}{\partial \lambda} = \frac{2\, r_2}{\lambda} + \frac{2}{\lambda^3} \sum_{i=1}^{r_2} \left(x_{(i)}^{(n)}\right)^2 + \frac{2\,(n-r_2)}{\lambda^3}\left(x_{(r_2)}^{(n)}\right)^2 = 0 \tag{3}$$

Hence we get

$$\hat{\lambda}' = \sqrt{\frac{\sum_{i=1}^{r_2}\left(x_{(i)}^{(n)}\right)^2 + (n-r_2)\, x_{(r_2)}^{(n)}}{r_2}} \tag{4}$$

Let

$$E\left(x_{(r_2)}^{2(n)}\right) = \lambda^2\, w_{r_2\, r_2}^{(n)};\quad E\left(x_{(i)}^{2(n)}\right) = \lambda^2\, w_{i\,i}^{(n)} \tag{5}$$

Thus we get

$$-\,E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = \frac{-2\, r_2}{\lambda^2} + 6 \sum_{i=1}^{r_2} \frac{w_{i\,i}^{(n)}}{\lambda^2} + \frac{6\,(n-r_2)}{\lambda^2}\, w_{r_2 r_2}^{(n)} \tag{6}$$

The asymptotic variance of $\hat{\lambda}'$ is given by

$$\operatorname{var}(\hat{\lambda}') = \frac{\lambda^2}{-2\, r_2 + 6 \sum_{i=1}^{r_2} w_{i\,i}^{(n)} + 6\,(n-r_2)\, w_{r_2\, r_2}^{(n)}} \tag{7}$$

The following recurrence relation is true for the Rayleigh distribution i.e.

$$\sum_{i=1}^{r_2} w_{i\,i}^{(n+1)} + (n-r_2+1)\, w_{r_2\,r_2}^{(n+1)} = \sum_{i=1}^{r_2} w_{i\,i}^{(n)} + (n-r_2)\, w_{r_2\,r_2}^{(n)} \tag{8}$$

The table (2.1) shows the variance of $\hat{\lambda}'$ for the right tail censored for different values of r_2 for a sample of size $n \leqslant 8$ from the Rayleigh distribution. The variance of $\hat{\lambda}$ from the complete distribution is given as

$$\text{Var}\ (\hat{\lambda}) = \frac{\lambda^2}{4\,n}. \tag{9}$$

The var $(\hat{\lambda}')$ given in (7) is greater than var $(\hat{\lambda})$ given in (9). The recurrence relation given in (8) can be utilized for other desired censored values for example when $r_2 = 4, n=8$ we have the same result as $r_2 =4$, and $n=5$ given in the table (2.1).

Table 2.1 : Showing the variance of $\hat{\lambda}'$ for different values of r_2. Each value should be multiplied by λ^2.

r_2	1	2	3	4	5	6	7
n=2	0.2500						
n=3	0.2500	0.1250					
n=4	0.2500	0.1250	0.0833				
n=5	0.2500	0.1250	0.0833	0.0625			
n=6	0.2500	0.1250	0.0833	0.0625	0.0500		
n=7	0.2500	0.1250	0.0833	0.0625	0.0500	0.04167	
n=8	0.2500	0.1250	0.0833	0.0625	0.0500	0.04167	0.0357

Case (*ii*) *Left tail censored :* We suppose that out of random sample of size n, r_1 are missing observations in the left tail. It is assumed that the missing observations are the smallest one and the observed observations are arranged according to ascending order of magnitude i.e.

$$x_{(r_1+1)}^{(n)} < x_{(r_1+2)}^{(n)} < \cdots < x_{(n)}^{(n)} \text{ such that } x_{(r_1+1)}^{(n)}$$

is the known smallest observation.

The likelihood function of this is given as

$$L \propto \left[\int_{0}^{x^{(n)}_{(r1+1)}} \frac{2x}{\lambda^2} e^{-x^2/\lambda^2} dx \right]^{r_1} \prod_{i=r_1+1}^{n} \frac{2x^{(n)}_{(i)}}{\lambda^2} e^{-\sum_{i=r_1+1}^{n} \left(x^{(n)}_{(i)}\right)^2/\lambda^2}. \tag{10}$$

Differentiating log L. *w.r.* to λ and equating to zero we obtain the maximum likelihood estimating equation.

$$\frac{\partial \log L}{\partial \lambda} = \frac{-2r_1 e^{-\left(x^{(n)}_{(r1+1)}\right)^2/\lambda^2} \cdot \left(x^{(n)}_{(r1+1)}\right)^2}{\lambda^3 \left(1 - e^{-x^{(n)}_{(r1+1)}/\lambda^2}\right)} - \frac{2(n-r_1)}{\lambda} +$$

$$2 \sum_{i=r1+1}^{n} \left(x^{(n)}_{(i)}\right)^2/\lambda^3 = 0 \tag{11}$$

The solution of (II) yields.

$$\overset{*}{\lambda} = \left[\sum_{i=r_1+1}^{(n)} \left(x^{(n)}_{(i)}\right)^3 - \frac{2r_1 \left(x^{(n)}_{(r1+1)}\right)^2 e^{-\left(x^{(n)}_{(r1+1)}\right)^2/\lambda^2}}{1 - e^{-\left(x^{(n)}_{(r1+1)}\right)^2/\lambda^2}} \right]^{\frac{1}{2}} \frac{1}{\sqrt{n-r_1}}. \tag{12}$$

Taking the second derivative of log L and putting

$$x^{(n)}_{(r1+1)}/\lambda = Z_j \; ; \; j = 1, 2, \ldots, n.$$

such that

$$E\left[\left(x^{(n)}_{(r1+1)}\right)^2\right] = \lambda^2 E(Z^2) \text{ and } E\left[\left(x^{(n)}_{(r1+1)}\right)^4\right] = \lambda^4 E(Z^4).$$

Thus we get,

$$-E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = \frac{4r_1}{\lambda^2} E\left(\frac{z^4 e^{-z2}}{1 - e^{-z2}}\right) - \frac{6r_1}{\lambda^2} E\left(\frac{z^2 e^{-z2}}{1 - e^{-z2}}\right)$$

$$+ \frac{4r_1}{\lambda^2} E\left(\frac{z^4 e^{-z2}}{(1 - e^{-z2})^2}\right) - \frac{2(n-r_1)}{\lambda^2} + 6 \sum_{i=r1+1}^{n} E\left(z_i^2/\lambda^2\right) \tag{13}$$

Gupta (1960) has constructed the moments of the k th order statistics of the Gamma distribution.

$$g(x) = \frac{e^{-x}\, x^{r-1}}{\Gamma(r)} \qquad 0 \leqslant x < \infty. \tag{14}$$

where r is the parameter assumed positive. The exponential distribution is a special case of (14) when $r=1$. If we denote the k th moment about the origin of the rth order statistics of Rayleigh distribution as $\mu'_k(r\, n,)$ then we have

$$\mu'_k\,(r, n) = \frac{n\,!\;\Gamma(1+k/2)}{(n-r)!\,(r-1)!} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \frac{1}{(i+n-r+1)^{1+k/2}}. \tag{15}$$

If we put $k=2k$ in (15) then we obtain the kth moment about the origin of the rth order statistics from the exponential distribution which we denote as $\upsilon'_k\ (r, n)$.

Hence we have

$$\upsilon'_k\ (r, n) = \frac{n\,!\;\Gamma(1+k)}{(n-r)!\,(r-1)!} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \frac{1}{(i+n-r+)^{1+k}} \tag{16}$$

Thus we have

$$-E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = \frac{4\, r_1}{\lambda^2\,(r_1+1)}\ \upsilon'_2\,(r_1+1,\ n) - \frac{6\, r_1(n-r_1-1)}{(r_1+1)\,\lambda^2}\ \upsilon'_1\,(r_1+1,\ n)$$

$$+ \frac{4r_1\ (n-r_1-1)\,(n-r_1-2)}{(r_1+1)\ (r_1+2)\lambda^2}\ \upsilon'_2\,(r_1+1,\ n) - \frac{2\,(n-r_1)}{\lambda^2}$$

$$+\ 6 \sum_{i=r_1+1}^{n} \frac{\upsilon'_i\,(r_1+1,\ n)}{\lambda^2}. \tag{17}$$

Variance of $\overset{*}{\lambda}$ for large sample is given by

$$-\left[E\,\left(\frac{\partial^2 \log L}{\partial\, \lambda^2}\right)\right]^{-1}. \tag{18}$$

Table 2.2 has been constructed to give the variance of $\overset{*}{\lambda}$ for a sample size $n \leqslant 8$ when the missing observations are in the left tail. These variances are also greater than the variances of $\hat{\lambda}$ from the complete distribution for given values of n. We can also compare the Var ($\hat{\lambda}'$) and Var ($\overset{*}{\lambda}$) from the Table 2.1 and 2.2 and conclude that as n becomes large the efficiency of $\hat{\lambda}'$ increases with more missing values as compared with $\overset{*}{\lambda}$.

Table 2.2 Showing the variance of $\overset{*}{\lambda}$ for $n \leqslant 8$ for various values of r_1. Each value may be multiplied by λ^2.

r_2	1	2	3	4	5	6	7
n=2	0.1428						
n=3	0.0861	0.1111					
n=4	0.0635	0.0784	0.0952				
n=5	0.0505	0.0518	0.0557	0.0855			
n=6	0.0420	0.0425	0.0439	0.0482	0.0787		
n=7	0.0359	0.0362	0.0368	0.0384	0.0429	0.0738	
n=8	0.0314	0.0316	0.0319	0.0326	0.0343	0.0389	0.0699

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CONTENTS

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