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CONTENTS

TRANSMISSION, REFLECTION AND DIFFRACTION OF LOVE WAVES IN WELDED LAYERED QUARTER-SPACES WITH A PLANE SURFACE (THEORETICAL)

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Summary

In this paper we investigate the two-dismensional diffraction problem associated with the propagation of plane harmonic Love waves, normally incident (from either side) upon the plane of discontinuity in the horizontally discontinuous structure consisting of welded layered quarter-spaces with a plane surface. Formulae for complex reflection and transmission co-efficients are obtained for the plane wave approximation and their variational improvement is sought through the Schwinger-Levine variational principle by means of a technique previously used by the author (vide Kazi (1978*a*, *b*) in the treatment of a similar problem associated with a structure consisting of a half-space with a surface step.

1. Introduction

In a previous paper (see Kazi 1978*a*) the author used the method of integral representation and Schwinger-Levine variational principle to describe, by means of a scattering matrix, the diffraction of plane, harmonic, monochromatic Love Waves, incident normally (from either side) upon the plane of discontinuity in a structure consisting of a half-space with a surface step—an idealized model of a continental margin. Approximate expressions for the elements of the scattering matrix were obtained through the plane-wave approximation and their variational

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improvement was sought through the Schwinger-Levine variational principle. Complex reflection and transmission co-efficients were obtained through a transmission matrix related to the scattering matrix. Numerical computation of these results were presented in Kazi (1978b) for a model considered previously by Knopoff & Hudson (1964) and also by Alsop (1966).

In this paper we use the afore-mentioned method to derive explicit formulae for the reflection and transmission co-efficients under plane-wave as well as variational approximations for the similar Love Wave transmission problem associated with the structure consisting of welded layered quarter-spaces with a plane surface. This problem has also been considered previously by Alsop (1966). In another paper we shall present the numerical computation of our results.

2. Equations of Motion

Let us suppose that a quarter-space consisting of a material of rigidity μ_2, shear velocity β_2 and density ρ_2, overlain by a layer of depth h, density ρ_1, rigidity μ_1 ($< \mu_2$) and shear velocity β_1 ($< \beta_2$), is in welded contact with a similar quarter-space of material of rigidity μ_2', shear velocity β_2' and density ρ_2', overlain by a layer of depth h, density ρ_1', rigidity μ_1' ($< \mu_2'$) and shear velocity β_1' ($< \beta_2'$ (see fig.). We take the vertical plane of contact between the two structures to be $x = o$ in the co-ordinate system shown in the figure and regard the top plane surface $z = o$ as stress-free. All the materials are considered to be isotropic and homogeneous.

We consider two-dimensional problems of the diffraction of time-harmonic Love wave normally incident upon the plane of contact (from either side). Again, the wave-motion is entirely SH in character. The displacement fields in the regions I ($x < o$) and II ($x > o$) (see fig.) are

denoted by $e^{-i\omega t} v(x, z)$ and $e^{-i\omega t} v'(x, z)$ respectively, where

$$e^{-i\omega t} v(x, z) = e^{-i\omega t} v_1(x, z), \quad 0 \leqslant z \leqslant h, \quad x < o,$$

$$= e^{-i\omega t} v_2(x, z), \quad h \leqslant z. \qquad x < o,$$

and

$$e^{-i\omega t} v'(x, z) = e^{-i\omega t} v_1'(x, z), \quad 0 \leqslant z \leqslant h, \quad x > o.$$

$$= e^{-i\omega t} v_2'(x, z), \quad h \leqslant z, \qquad x > o,$$

<div style="text-align:right">(ω being the angular
frequency)</div>

respectively.

The conditions at the free surface $z = o$ and the plane of welded contact $x = o$ imply :

$$\frac{\partial v_1}{\partial z} = o \text{ and } \frac{\partial v_1'}{\partial z} = o \text{ at } z = o \tag{2.1a}$$

$$v = v' \text{ at } x = o, \ z \geqslant o, \tag{2.1b}$$

$$\mu(z) \frac{\partial v}{\partial x} = \mu'(z) \frac{\partial v'}{\partial x} \text{ at } x = o, \ z \geqslant o, \tag{2.1c}$$

where

$$\mu(z) = \mu_1, \ o \leqslant z < h, \ x < o,$$
$$= \mu_2, \ h < z, \ x < o, \tag{2.2}$$

and

$$\mu'(z), = \mu_1', \ o \leqslant z < h, \ x > o,$$

$$= \mu_2', \ h < z, \ x > o. \tag{2.3}$$

The complete solution for the displacement v in domain I can be expressed in terms of proper and improper eigenfunctions of the Love-wave operator for a homogeneous half-space of rigidity μ_2 and shear velocity β_2, subjacent to a homogeneous layer of depth h, rigidity μ_1

and shear velocity β_1 (formulae for proper and improper eigenfunctions are given in Kazi (1976). Like-wise, we can write the complete solution for the displacement v' in terms of the proper and improper eigenfunctions of the Love-wave operator for a homogeneous half-space of rigidity μ_2' and shear velocity β_2' overlain by a homogeneous layer of depth h, rigidity μ_1' and shear velocity β_1' . Thus

in DOMAIN I $(x < o, z \geqslant o)$:

$$
v(x, z) = -\left[\sum_{m=1}^{r} (A_m e^{-ik_m |x|} + B_m e^{ik_m |x|}) \chi_m(z) \right.
$$

$$
+ \int_0^{\omega/\beta_2} \{C(k) e^{-ik|x|} + D(k) e^{ik|x|}\} \phi(z, k)\, dk
$$

$$
\left. + \int_{0\psi}^{\infty} E(k) e^{-k|x|} \psi(z, k)\, dk \right].
$$

and in DOMAIN II $(x > o, z \geqslant o)$

$$
v'(x, z) = \left[\sum_{m=1}^{s} (A_m' e^{-ik_m' x} + B_m' e^{ik_m' x}) \chi_m'(z) \right.
$$

$$
+ \int_0^{\omega/\beta_2'} \{C(k') e^{-ik'x} + D'(k') e^{ik'x}\} \phi'(z'\, k')\, dk'
$$

$$
\left. + \int_0^{\infty} E'(k') e^{-k'x} \psi'(z, k') \right]. \tag{2.5}
$$

where (using formulae derived in Kazi (1976).

$$\chi_m(z) = \phi_1^{(m)}(z), \quad o \leqslant z \leqslant h$$

$$= \phi_2^{(m)}(z), \quad h \leqslant z, \tag{2.6}$$

$$\chi'_m(z) = \phi_1^{'(m)}(z), \quad o \leqslant z \leqslant h$$

$$= \phi_2^{'(m)}(z), \quad h \leqslant z, \tag{2.7}$$

$$\phi_1^{(m)}(z) = F_m \frac{\cos(\sigma_1^{(m)'} z)}{\cos(\sigma_1^{(m)'} h)}, \quad \phi_2^{(m)}(z) = F_m \, e \, \sigma_2^{(m)}(h-z), \tag{2.8}$$

$$F_m = \left[\frac{2 \, \sigma_2^{(m)}}{\mu_2} \cdot \frac{\left(\beta_1^{-2} - U_m^{-1} \, C_m^{-1} \right)}{\left(\beta_1^{-2} - \beta_2^{-2} \right)} \right]^{\frac{1}{2}} \tag{2.9}$$

$$\phi_1^{'(m)}(z) = F'_m \frac{\cos(\sigma_1^{(m)'} z)}{\cos(\sigma_1^{(m)'} h)}, \quad \phi_2^{(m)'}(z) = F'_m \, e^{\sigma_2^{(m)}} \cdot (h-z),$$

$$\tag{2.10}$$

$$F'_m = \left[\frac{2 \, \sigma_2^{(m)'}}{\mu'_2} \cdot \frac{U_m^{'(-1)}, \, C_m^{'(-1)}}{\beta_1^{'(-2)} - \beta_2^{'(-2)}} \right]^{\frac{1}{2}} \tag{2.11}$$

(U_m, U'_m are the group velocities and C_m, C'_m are the phase velocities in the m-th modes),

$$\sigma_1(\lambda_1) = \left[\frac{\omega^2}{\beta_1^2} - \lambda \right]^{\frac{1}{2}}, \quad \sigma_2(\lambda) = (\lambda - \frac{\omega^2}{\beta_2^2})^{\frac{1}{2}} \tag{2.12}$$

$$\sigma_1^{(m)} = \sigma_2 \ (\lambda_m), \quad \sigma_2^{(m)} = \sigma_2 \ (\lambda_m), \tag{2.13}$$

$$\lambda_m = k_m^2 \ , \quad k_m > o, \tag{2.14}$$

and similarly for $\sigma_1^{(m)\prime}$, $\sigma_2^{(m)\prime}$ and λ_m'.

The eigenvalues $\lambda = \lambda_m = k_m^2$, $m = 1, 2, \ldots \ldots r$, satisfy the Love wave dispersion equation

$$\mu_1 \ \sigma_1 \tan (\sigma_1 \ h) - \mu_2 \ \sigma_2 = o, \tag{2.15}$$

whereas $\lambda = \lambda_m' = k_m'^2$, $m = 1, 2, \ldots \ldots s$ are the roots of the dispersion equation

$$\mu_1' \ \sigma_1' \tan (\sigma_1' \ h) - \mu_2' \ \sigma_2' = o \tag{2.16}$$

$\psi \ (z, k)$, the improper eigenfunctions corresponding to the improper eigenvalues $\lambda = (ik)^2$, $k > o$ are given by

$$\psi \ (z, k) = \psi_1 \ (z, k), \quad o \leqslant z \leqslant h,$$
$$= \psi_2 \ (z, k), \quad h \leqslant z, \tag{2.17}$$

where

$$\psi_1 \ (z, k) = G_k \ \frac{\cos \ (\sigma_1^{(k)} \ z)}{\cos \ (\sigma_1^{(k)} \ h)}, \quad o \leqslant z \leqslant h, \tag{2.18}$$

and

$$\psi_2 \ (z, k) = G_k \ \frac{\sin \ \{\theta^{(k)} - s_2^{(k)} \ (z-h)\}}{\sin \ \theta^{(k)}}, \quad z \geqslant h, \tag{2.19}$$

with

$$G_k = \frac{\sqrt{2k} \ \sin \ \theta^{(k)}}{\sqrt{\pi \ \mu_2 \ s_2^{(k)}}} \tag{2.20}$$

$$s_2^{(k)} = \left(\frac{\omega^2}{\beta_2^2} - \lambda\right)^{\frac{1}{2}} \text{ real and positive } (\lambda = -k^2, k > o),$$

$$(2.21)$$

and

$$\theta^{(k)} = \tan^{-1}\left(\frac{\mu_2 \, s_2^{(k)} \, \text{Cot} \, \sigma_1^{(k)} \, h}{\mu_1 \, \sigma_1^{(k)}}\right). \qquad (2.22)$$

Owing to the factor $e^{-k\,|\,x\,|}$ in the integral containing ψ these represent non-propagated modes.

Similarly, $\psi'\,(z, k')$, the improper eigenfunctions corresponding to the improper eigenvalues $\lambda' = (i\,k'^2)$, $k' > o$ are given by

$$\psi'\,(z, k') = \psi_1'\,(z, k'), \quad o \leqslant z \leqslant h,$$

$$= \psi_2'\,(z, k'), \quad h \leqslant z, \qquad (2.23)$$

where

$$\psi_1'\,(z,\,k') = G'_{k'} \, \frac{\text{Cos}(\sigma_1'^{(k')} z)}{\text{Cos}(\sigma_1'^{(k')} h)}, \qquad (2.24)$$

and

$$\psi_2'\,(z, k') = G'_{k'} \, \frac{\text{Sin}\,\{\theta^{(k')} - s_2'^{(k')}\,(z-h)\}}{\text{Sin}'\,\theta^{(k')}} \qquad (2.25)$$

with

$$\theta'^{(k')} = \tan^{-1}\frac{\mu_2' s_2'^{(k')} \, \text{Cot}\,(\sigma_1'^{(k')} h)}{\mu_1'^{(k')} \, \sigma_1'}, \qquad (2.26)$$

$G'_{k'}$, $s_2'^{(k')}$ have expressions similar to those for G_k, $s_2^{(k)}$ but in primed notation.

$\dot\psi\,(z, k)$ and $\psi'\,(z, k')$, the improper eigenfunctions belonging to the improper eigenvalues $\lambda = k^2$, $\lambda = k'^2$,

$(o < k \leqslant \dfrac{\omega}{\beta'_2}, \; o < k' \leqslant \dfrac{\omega}{\beta'_2})$ respectively, have expressions similar to those for $\psi\,(z, k)$ and $\psi'\,(z, k')$. Owing to the form of x-dependence in the integrals containing ϕ. ϕ', these represent waves travelling in the x - direction.

The orthonormality relations amongst various proper and improper eigenfunctions are given by (*cf.* Kazi 1976) :

$$\int_0^\infty \mu\,(z)\,\chi_m\,(z)\,\chi_n\,(z)\,dz = \delta_{mn}, \quad 1 \leqslant m, n \leqslant r, \tag{27a}$$

$$\int_0^\infty \mu\,(z)\,\chi_m\,(z)\,\phi\,(z, k)\,dz = o,\; 1 \leqslant m \leqslant r,\, o < k \leqslant \frac{\omega}{\beta_2} \tag{2.27b}$$

$$\int_0^\infty \mu\,(z)\,\chi_m\,(z)\,\phi\,(z, k)\,dz = o,\; 1 \leqslant m \leqslant r,\, o < k \leqslant \frac{\omega}{\beta_2} \tag{2.27c}$$

$$\int_0^\infty \mu\,(z)\,\chi\,\phi\,(z, k)\,\psi\,(z, k)\,dz = o \tag{2.27d}$$

$$\int_0^\infty \mu(z)\psi\,(z, k)\,\psi\,(z, l)\,(dz = \delta\,(k-l), \quad o < k,\, l \leqslant \infty \tag{2.27e}$$

$$\int_0^\infty \mu\,(z)\,\phi\,(z, k)\,\phi\,(z, l)\,dz = \delta\,(k-l),\, o < k,\, l \leqslant \frac{\omega}{\beta_2} \tag{2.27f}$$

The orthonormality relations amongst

$\chi'_m\,(z)\,(m = 1, \ldots . s)$, $\phi'\,(z, k')\;(o < k' \leqslant \dfrac{\omega}{\beta_2})$, and

$\psi'\,(z, k')\,(o < k' \leqslant \infty)$ are identical to $(2.27a\text{-}f)$ but in primed quantities.

3. Integral Equation Formulation

We proceed as in Kazi (1978a). If we denote the component τ_{xy} of stress at any point of the vertical plane of contact $x=o$ by $\tau(z)$, then (2.1c) implies :

$$\tau(z)=\tau_{xy}\Big|_{x=0} = \mu(z)\frac{\partial v}{\partial x}\Big|_{x=0-} = \mu(z)\frac{\partial v'}{\partial x}\Big|_{x=0+}, \quad z>o \qquad (3.1)$$

we have both

$$\tau(z)=\mu(z)\frac{\partial v}{\partial x}\Big|_{x\to 0-} = -\mu(z)\Bigg[\sum_{m=1}^{r} ik_m(A_m-B_m)\chi_m(z)$$

$$+\int_0^{\omega/\beta_2} ik\{C(k)-D(k)\}\phi(z,k)\,dk+\int_0^{\infty} k.E(k)\psi(z,k)\,dk\Bigg] \qquad (3.2)$$

and

$$\tau(z)=\mu'(z)\frac{\partial v'}{\partial x}\Big|_{x\to 0+} = -\mu'(z)\Bigg[\sum_{m=1}^{s} ik'_m(A'_m-B'_m)\chi'_m(z)$$

$$+\int_0^{\omega/\beta_2'} ik\{C'(k')-D'(k')\}\phi'(z,k')\,dk' +\int_0^{\infty} k'\,E'(k')\psi'(z,k')\,dk'\Bigg] \qquad (3.3)$$

On multiplying equation (3.2) separately by $\chi_m(z)$ $(m=1, 2\ldots r)$, $\phi(z,k)$ $(o<k\leqslant\frac{\omega}{\beta_2})$ and $\psi(z,k)$ $(o<k\leqslant\infty)$, and integrating with respect to z from o to ∞, we obtain (using orthonormality relations (2.27) :

$$-ik_m(A_m-B_m) = \int_0^{\infty} \tau(\eta)\,\chi_m(\eta)\,d\eta, \; m=1, 2,\ldots.r, \qquad (3.4a)$$

$$-ik\{C(k)-D(k)\}=\int_0^{\infty} \tau(\eta)\,\phi(\eta,k)\,d\eta, \qquad (3.4)$$

and

$$-k \, E \, (k) = \int_0^\infty \tau \, (\eta) \quad \psi \, (\eta, \, k) \, d\eta \qquad (3.4c)$$

A similar procedure, applied to the equation (3.3), leads to the following :

$$- \, ik'_m (A'_m - B'_m) = \int_0^\infty \tau \, (\eta) \, \chi'_m \, (\eta) \, d\eta, \, m=1, \, 2\ldots\ldots s, \qquad (3.4d)$$

$$-ik' \, \{C' \, (k')-D' \, (k')\} = \int_0^\infty \tau \, (\eta) \, \psi' \, (\eta, \, k') \, d\eta \qquad (3.4e)$$

and

$$-k' \, E' \, (k') = \int_0^\infty \tau \, (\eta) \, \psi' \, (\eta, \, k') \, d\eta \qquad (3.4f)$$

Eliminating $D \, (k)$, $D' \, (k')$, $E \, (k)$, $E' \, (k')$ (assuming $C \, (k) = C' \, (k') = o$ and invoking the matching condition $(3.1c)$, we obtain

$$\sum_{m=1}^r \, (A_m+B_m) \, \chi_m \, (z) \, + \, \sum_{m=1}^s \, (A'_m + B'_m) \, \chi'_m \, (z)$$

$$= \int_0^\infty \tau \, (\eta) \, g \, (z, \, \eta) \, d\eta \qquad (3.5)$$

where

$$g \, (z, \, \eta) = G \, (z, \, \eta) + i \, g \, (z, \, \eta),$$

$$g \, (z, \, \eta) = \int_0^\infty \frac{\psi \, (z, \, k) \, \psi \, (\eta, \, k)}{k} dk + \int_0^\infty \frac{\psi' \, (z, \, k') \, \psi \, (\eta, \, k')}{k'} dk' \qquad (3.7)$$

and

$$g \, (z, \eta) = \int_0^{\omega/\beta_2} \frac{\phi \, (z, k) \, \phi \, (\eta, k)}{k} dk + \int_0^{\omega/\beta_2} \frac{\phi' \, (z, k') \, \phi' \, (\eta, k')}{k'} dk' \qquad (3.8)$$

Equations (3.4*a*, *d*) and (3.8) constitute the integral equation formulation of the problem. The scattering matrix formulation is the same as in Kazi 1978*a*. Moreover, the approximate formulae for the elements of the scattering matrix and the resulting reflection and transmission co-efficients, arising from the approximation based upon the neglect of modes corresponding to the continuous spectrum and the variational approximation based upon the Schwinger-Levine variational principle, are identical in form to the formulae derived for the continental margin problem in Kazi 1978*a*. We shall, therefore, omit details already covered in Kazi 1978*a* and restrict ourselves to the derivation of explicit expressions for the reflection and transmission co-efficients for some simple cases under both approximations.

4. Formulate for Reflection and Transmission Co-efficients.

(i) *Approximation Based Upon the Neglect of Modes Corresponding to the Continuous Spectrum*

The transmission matrix $\underline{T}$ is (cf. Kazi 1978*a*) given by the formula

$$\begin{bmatrix} -Q \\ R \end{bmatrix}^{-1} \cdot \begin{bmatrix} Q \\ R \end{bmatrix}$$

where Q and R are given by

$$\underline{Q} = \begin{bmatrix} 1 & & & & \lambda_{11}P_{11} & \cdots & \lambda_{s1}P_{s1} \\ & 1 & & 0 & \lambda_{12}P_{12} & \cdots & \lambda_{s2}P_{s2} \\ & & \ddots & & \vdots & & \\ & & & \ddots & \vdots & & \\ & 0 & & 1 & \lambda_{1r}P_{1r} & \cdots & \lambda_{sr}P_{rs} \end{bmatrix}$$

and

$$R = \begin{pmatrix} \dfrac{P_{11}}{\lambda_{11}} & \cdots & \dfrac{P_{1r}}{\lambda_{1r}} & -1 & & & \\[2ex] \dfrac{P_{21}}{\lambda_{21}} & \cdots & \dfrac{P_{2r}}{\lambda_{2r}} & & \ddots & & \\[2ex] \cdot & & & & & \ddots & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \dfrac{P_{s1}}{\lambda_{s1}} & \cdots & \dfrac{P_{sr}}{\lambda_{sr}} & & & & -1 \end{pmatrix}$$

The particular forms of $\underline{T}$ in the special cases $r = 1, s \geqslant 1$ and $r \geqslant 1$, $s = 1$ are given by

$$\underline{T} = \frac{1}{N} \begin{pmatrix} -2+N & -2\lambda_{11}\,P_{11} & \cdots\cdots\cdots\cdots & -2\lambda_{s1}P_{s1} \\[2ex] \dfrac{-2\,P_{11}}{\lambda_{11}} & -2P_{11}^2 +N & & \dfrac{-2P_{11}\,\lambda_{s1}\,P_{s1}}{\lambda_{11}} \\[2ex] \dfrac{-2\,P_{21}}{\lambda_{21}} & \dfrac{-2P_{21}\,P_{11}\,\lambda_{11}}{\lambda_{21}} & \cdot & \dfrac{-2P_{21}\,\lambda_{s1}\,P_{s1}}{\lambda_{21}} \\[1ex] \cdot & \cdot & & \cdot \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\[1ex] \dfrac{-2\,P_{s1}}{\lambda_{s1}} & \dfrac{-2P_{s1}\,P_{11}\,\lambda_{11}}{\lambda_{s1}} & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & -2\,P_{s1}^2 +N \end{pmatrix}$$

where

$$N = 1 + P_{11}^2 + P_{11}^2 + \cdots\cdots + P_{s1}^2 ,$$

and

$$\underline{T} = \frac{1}{N} \begin{bmatrix} -N + 2P_{11}^2 & \dfrac{2P_{11}\, P_{12}\, \lambda_{11}}{\lambda_{12}} & & & -2P_{11}\, \lambda_{11} \\[2ex] \dfrac{2P_{11}\, \lambda_{12}\, P_{12}}{\lambda_{11}} & -N + 2P_{12}^2 & & & -2P_{12}\, \lambda_{12} \\[1ex] \vdots & \vdots & & & \vdots \\[1ex] \dfrac{2\, P_{11}\, \lambda_{1r}\, P_{11}}{\lambda_{11}} & \dfrac{2P_{12}\, P_{1r}\, \lambda_{1r}}{\lambda_{12}} & & & -2P_{1r}\, \lambda_{1r} \\[2ex] \dfrac{2\, P_{11}}{\lambda_{11}} & \dfrac{-2\, P_{12}}{\lambda_{12}} & & & -N+2 \end{bmatrix}$$

where

$$N = 1 + P_{11}^2 + P_{12}^2 + \cdots\cdots + P_{1r}^2 \ .$$

All the formulae for the reflection and transmission co-efficients, which were derived in terms of he coupling co-efficients for various special cases in Kazi 1978a remain unchanged for the present problem. We must, however-, re-evaluate the coupling co-efficients λ_{im} P_{im}

Here

$$I = \lambda_{im}\ P_{im} = \int_0^\infty \mu(z)\, \chi'_i(z)\, \chi_m(z)\, dz, \quad \begin{array}{l} i = 1, 2, \ldots\ldots s, \\ m = 1, 2, \ldots\ldots r \end{array} \quad \lambda_{im} = \sqrt{\frac{k_i}{k_m}}, \qquad (4.1)$$

where $\chi'_i(z)$ and $\chi_m(z)$ are given by (2.6) - (2.11).

we write

$$I = \mu_1 \int_0^h \phi_1'^{(i)}(z)\, \phi_1^{(m)}(z)\, dz + \mu_2 \int_h^\infty \phi_2'^{(i)}\, \phi_2^{(m)}(z)\, dz \qquad (4.2)$$

$$= I_1 + I_2,$$

where $I_1 = \dfrac{\mu_1 F_1' \; F_m}{\text{Cos }(\sigma_1^{(i)'}h)\;\text{Cos }(\sigma_1^{(m)}\,h)} \displaystyle\int_0^h \text{Cos }(\sigma_1^{(i)}\,z)\;\text{Cos }(\sigma_1^{(m)}\,z)\;dz$

$$(\text{using } (2.8),\ (2.10))$$

$$= \frac{\mu_1 F_1' F_m}{2}\left[\frac{\tan (\sigma_1^{(m)}h) + \tan (\sigma_1^{(i)'}\,h)}{\sigma_1^{(i)'} + \sigma_1^{(m)}} \right.$$

$$\left. + \frac{\tan (\sigma_1^{(i)'}\,h) - \tan (\sigma_1^{(m)}\,h)}{\sigma_1^{(i)'} - \sigma_1^{(m)}} \right]$$

$$= \frac{F_i' \; F_m}{(\sigma_1^{(i)'})^2 - (\sigma_1^{(m)})^2}\left[\frac{\mu_1}{\mu_1'}\cdot\;\; \mu_1' \, \sigma_1^{(i)'}\tan (\sigma_1^{(i)'}\,h) - \mu_1\,\sigma_1^{(m)}\tan (\sigma_1^{(m)}h) \right]$$

$$= \frac{F_m \; F_i' \left(\mu_1 \; \mu_2' \, \sigma_2^{(i)} - \mu_1' \,\mu_2\, \sigma_2^{(m)} \right)}{\mu_1\{(k_m^2 - k'^2) + w^2\,(\frac{1}{\beta_1'^2} - \frac{1}{\beta_1^2})\}},$$

(using the dispersion equations (2.15) and (2.16) together with the relations

$$(\sigma_1^{(m)})^2 = \frac{w^2}{\beta_1^2} - k_m^2$$

and

$$(\sigma_1^{(i)})^2 = \frac{w^2}{\beta_1'^2} - k_i'^2), \qquad (4.3)$$

and

$$I_2 = \mu_2 \; F_i' \; F_m \displaystyle\int_h^\infty (\, \sigma_2^{(i)'} + \sigma_2^{(m)}\,)\; e^{(k-z)}\;dz \qquad\qquad (\text{using } (4.8),\ (4.10)$$

$$= \frac{\mu_2 \, F_l' \, F_m}{\sigma_2^{(i)} + \sigma_2^{(m)}}$$

$$= \frac{\mu_2 \, F_l' \, F_m \, \{ \sigma_2^{(i)'} - \sigma_2^{(m)} \}}{(k_i'^2 - k_m^2) + {}_w^2 (\frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2})} , \tag{4.4}$$

(using the relations

$$(\sigma_2^{(i)'})^2 = k_i'^2 - \frac{w^2}{\beta_2'^2}$$

and

$$(\sigma_2^{(m)})^2 = k_m^2 - \frac{w^2}{\beta_2^2} ,$$

whence from (4.2) to (4.4.) we obtain :

$$\lambda_{im} \, P_{im} = \frac{F_l' \quad F_m}{\mu_1'} \left[\frac{\mu_1 \mu_2' \; \sigma_2^{(i)'} - \mu_1' \; \mu_2 \; \sigma_2^{(m)}}{(k_m^2 - k_i'^2) + \frac{w^2}{b_1^2}} + \right.$$

$$\left. \frac{\mu_1' \, \mu_2 \; (\sigma_2^{(i)'} - \sigma_2^{(m)})}{(k_i'^2 - k_m^2) + \frac{w^2}{b_2^2}} \right] \tag{4.5}$$

with $\dfrac{1}{b_1^2} = \dfrac{1}{\beta_1'^2} - \dfrac{1}{\beta_1^2}$ and $\dfrac{1}{b_2^2} = \dfrac{1}{\beta_2^2} - \dfrac{1}{\beta_2'^2}$; F_m and F_l'

are given by (2.9) and (2.11) respectively.

In the particular case when $r=1$, $s=1$ in equations (2.2) and (p. 3) (*i.e.* the frequency here is such that there are single (fundamental) modes in the left-hand and right-hand domains shown in Fig.) then we find (see Kazi 1978*a*) that

$B=T.A,$

where

$$T = \frac{1}{1+P_{11}^2} \begin{bmatrix} -1 + P_{11}^2 & -2\lambda_{11} P_{11} \\ -\dfrac{2 P_{11}}{\lambda_{11}} & 1 - P_{11}^2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 \\ A_1' \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_1' \end{bmatrix}$$

(Here A_1, B_1 are the coefficients of χ_1' in equation (2.2) and A_1', B_1' are the coefficients of χ_1' in equation (2.31).

$$\lambda_{11} P_{11} = \frac{\mu_1 \sigma_1^{(1)} F_1' F_1 \sin (\sigma_1^{(1)} \delta)}{(k_1^2 - k_1'^2) \cos (\sigma_1^{(1)} h_1) \cos (\sigma_1^{(1)} h_2)} \qquad (4.10)$$

$$\lambda_{11} = \left[\frac{k_1'}{k_1} \right]^{\frac{1}{2}} \qquad (4.11)$$

(The terms appearing in (4.10) and (4.11) have the same meaning as in Section 2.)

Thus if the incident wave is travelling from right to left so that

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then the transmission coefficient

$$B_1 = \frac{-2 \lambda_{11} P_{11}}{1 + P_{11}^2} \tag{4.12}$$

and the reflection coefficient

$$B_1' = \frac{1 - P_{11}^2}{1 + P_{11}^2} \tag{4.13}$$

If the incident wave is travelling from left to right with

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then the transmission coefficient

$$B_1' = \frac{-2 P_{11}}{\lambda_{11} (1 + P_{11}^2)} = \frac{k_1}{k_1'} B_1 \tag{4.14}$$

and the reflection coefficient

$$B_1 = \frac{P_{11}^2 - 1}{1 + P_{11}^2} = (-1) B_1' \tag{4.15}$$

(ii) Variational approximation

The variational formulation of the problem can be achieved in exactly the same manner as in Kazi 1978a.

In the simple case when $r = 1$, $s = 1$ in equations (2.2) & (2.3) then (cf. Kazi 1978a)

$$A = \begin{pmatrix} A_1 \\ A_1' \end{pmatrix} \qquad B = \begin{pmatrix} B_1 \\ B_1' \end{pmatrix} \qquad (4.16)$$

$$T = \frac{1}{1+P_{11}^2 - i\,I_{11}'} \begin{pmatrix} P_{11}^2 & -i\,I_{11}' - 2P_{11}\,\lambda_{11} \\ \dfrac{-2P_{11}}{\lambda_{11}} & 1-P_{11}^2 - i\,I_{11}' \end{pmatrix} \qquad (4.17)$$

P_{11} and λ_{11} are given by (4.10) and (4.11) and

$$I_{11}' = \frac{4k_1\,\sigma_2^{(1)}}{\mu_2\,\mu_2'}\,\frac{\left(\beta_1^{-2} - U_m^{-1}\,C_m^{-1}\right)}{\left(\beta_1^{-2} - \beta_2^{-2}\right)}\left[\int_0^\infty \{H_1(k')\}^2\,dk' + i \right.$$

$$\left. \int_0^{\omega/\beta_2'} \{H_1(k')\}^2\,dk' \right. \qquad (41.8)$$

where

$$H_1(k') = \frac{(\omega^2/\beta_{2'}^2 - k'^2)^{\frac{1}{4}}}{\{\mu_1'^2\,(\dfrac{\omega^2}{\beta_1'^2}-k'^2)\tan^2\,(\dfrac{\omega^2}{\beta_1'^2} - k'^2)^{\frac{1}{2}}\,h + \mu_2'^2\,(\dfrac{\omega^2}{\beta_2'^2}-k'^2)\}^{\frac{1}{2}}} \times$$

$$\frac{\left[\{\mu_1\mu_2'\,(k_1^2 - k'^2 +\dfrac{\omega^2}{\beta_2'^2})-\mu_2\mu_1'\,(k_1^2 +k'^2+\dfrac{\omega^2}{b_1^2})\}\,\sigma_1'\,(\lambda') \right.}{}$$

$$\frac{\left. \tan\,\sigma_1\,(\lambda')\,h+\mu_2\,\mu_2'\,\sigma_2^{(1)}\,\omega^2\,(\dfrac{1}{b_1^2} - \dfrac{1}{b_2^2})\right]}{\left\{(k_1^2 - k'^2)+\dfrac{\omega^2}{b_1^2}\right\}\,\left\{(k_1^2 -k'^2)+\dfrac{\omega^2}{b_2^2}\right\}}$$

$$(4.19)$$

and

$$H_2(k') = \frac{\left(\dfrac{\omega^2}{\beta_2'^2} + k'^2\right)^{\frac{1}{2}}}{\left(\mu_1'^2 \left(\dfrac{\omega^2}{\beta_1'^2} + k'^2\right) \tan^2\left\{\left(\dfrac{\omega^2}{\beta_1'^2} + k'^2\right)^{\frac{1}{2}} h\right\} + \mu_2'^2 \left(\dfrac{\omega'^2}{\beta_2'^2} + k'^2\right)\right)^{\frac{1}{2}}} \times$$

$$\frac{\left[\left\{\mu_1\,\mu_2'\,(k_1^2 + k'^2 + \dfrac{\omega^2}{b_2'^2}) - \mu_2\,\mu_1'\,(k_1^2 + k'^2 + \dfrac{\omega^2}{b_1^2})\right\}\sigma_1'\ (\lambda')\right.}{}$$

$$\frac{\left. \tan\ \sigma_1'\,(-\lambda^1) + \mu_2\,\mu_2'\ \sigma_2^{(1)}\ \omega^2\,(\dfrac{1}{b_1^2} - \dfrac{1}{b_2^2})\right]}{\left\{(k_1^2 + k'^2) + \dfrac{\omega^2}{b_1^2}\right\}\quad \left\{(k_1^2 + k'^2) + \dfrac{\omega^2}{b_2^2}\right\}}$$

$$\tag{4.20}$$

Various notations appearing in these formulae have the same meaning as in Section 2. For an incident wave travelling from left to right with

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we obtain the reflection coefficient

$$B_1 = \frac{P_{11}^2 - 1 - i\,I_{11}'}{1 + P_{11}^2 - i\,I_{11}'} \tag{4.21}$$

and the transmission coefficient

$$B_1' = \frac{-2P_{11}}{\lambda_{11}\,(1 + P_{11}^2 - i\,I_{11}')} \tag{4.22}$$

Similarly for an incident wave travelling from right to left with

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we get the transmission coefficient

$$B_1 = \frac{-2\lambda_{11} \; P_{11}}{1+P_{11}^2 - i \, I'_{11}} = \frac{k'_1}{k'_1} \, B'_1 \qquad (4.23)$$

and the reflection coefficient

$$B'_1 = \frac{1-(P_{11}^2 + i \, I'_{11})}{1+P_{11}'^2 - i \, I'_{11}} \qquad (4.24)$$

it may be remarked that the integral $\int_0^\infty \{ H_2 \, (k') \}^2 \; dk'$ in (4.18)

corresponds to the non-propagated modes arising from the negative improper eigenvalues belonging to the continuous spectrum. The integral and $\{ (H_2 \, (k') \}^2$ is of the order of $\frac{1}{k'^3}$ for large values of k' and is regular throughout the domain. The integral is therefore convergent.

The second integral $\int_0^\infty \{ H_1 \, (k') \}^2 \; dk'$ in (4.18) corresponds to the pro-

pagated modes arising from the improper eigenvalues of the continuous spectrum belonging to the interval $(o, \, \omega^2/\beta_2'^2)$ and represents a contribution from the body waves. This integral is also convergent. Both the integrals must, however, be evaluated numerically.

Acknowledgements

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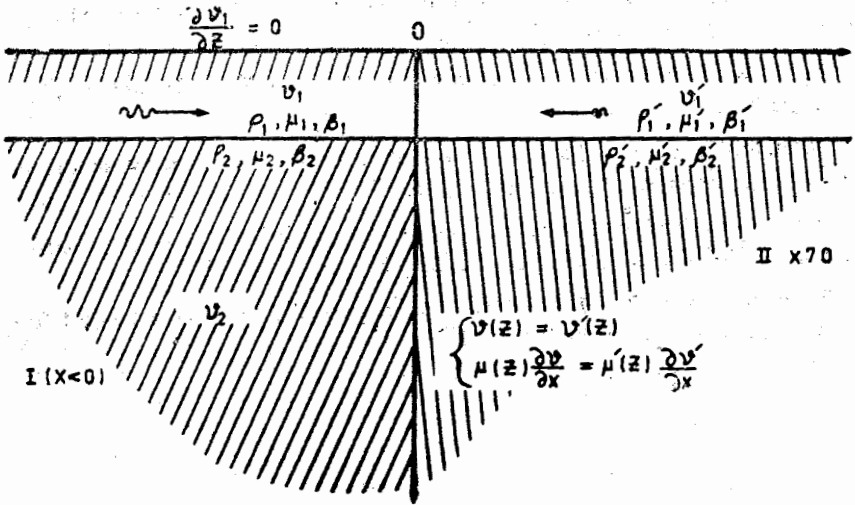
Fig. 1

ON FIXED AND COMMON FIXED POINTS OF MAPPINGS

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Introduction

In this paper some results on fixed and common fixed points of mappings in a complete metric space and in a space with two metrics are obtained. Theorem 1 of this paper generalizes many common fixed point theorems. In fact the theorem contains many classical results which are proved by the method of successive approximation. The other results are generalizations of some of the common fixed point theorems of Kiyoshi Iséki [1, 2].

Let P and Q be two mappings of a metric space (X, d) into itself, and suppose that

$$(1.1) \quad d(Px, Qy) \leqslant a_1 d(x, Px) + a_2 d(y, Qy) + a_3 d(x, Qy)$$
$$+ a_4 d(y, Px) + a_5 d(x, y) + a_6 d(x, QPy)$$
$$+ a_7 d(Px, QPx) + a_8 d(y, QPx) + a_9 d(Qy, QPx)$$

for all x, y in X,

$$(1.2) \quad a_k \in [0, 1] \text{ and } \sum_{k=1}^{9} a_k < 1,$$

$$(1.3) \quad a_3 + a_6 = a_4 + a_9.$$

The well known mappings introduced by Banach, Reich, Hardy and Rogers and others are particular cases of [1.1].

2. Common Fixed Point Theorems in a Complete Metric space

Throughout this section let (X, d) be a complete metric space. Following is the main result of this section.

Theorem 1. Let T_1 and T_2 be two mappings from X into itself. Suppose that

(2.1) the condition (1.1) holds for $P=T_1$, $Q=T_j$, $i=1, 2, j=1, 2$ with $i \neq j$, where the constants a_k are as in (1.2) and (1.3). Then T_1 and T_2 have a unique common fixed point.

Proof : Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by setting $x_1=T_1 x_0$, $x_2=T_2 x_1$, and inductively for each $n \in I^+$ (positive integers), $x_{2n+1}=T_1 x_{2n}, x_{2n+2}=T_2 x_{2n+1}$. Then taking $i=1, j=2$ in (2.1) we have $d(x_1, x_2)=d(T_1 x_0, T_2 x_1)$

$$\leqslant a_1 d(x_0, T_1 x_0)+a_2 d(x_1, T_2 x_1)+a_3 d(x_0, T_2 x_1)+a_4 d(x_1, T_1 x_0)$$

$$+a_5 d(x_0, x_1)+a_6 d(x_0, T_2 T_1 x_0)+a_7 d(T_1 x_0, T_2 T_1 x_0)$$

$$+a_8 d(x_1, T_2 T_1 x_0)+a_9 d(T_2 x_1, T_2 T_1 x_0).$$

Simplify to get

$$d(x_1, x_2) \leqslant p\, d(x_0, x_1).$$

where $p=(a_1+a_3+a_5+a_6)/(1-a_2-a_3-a_6-a_7-a_8)$.

Conditions (1.2) and (1.3) make $p<1$.

Further by taking $i=2, j=1$ in (2.1) we have

$$d(x_2, x_3) \leqslant p\, d(x_1, x_2) \leqslant p^2 d(x_0, x_1).$$

Inductively, $d(x_n, x_{n+1}) \leqslant p^n d(x_0, x_1)$.

Thus $\{x_n\}$ is a Cauchy sequence and by the completeness of X, it converges to some point u of X. We show that u is a fixed point of T_1. For each $n \in I^+$ and $i=2, j=1$ in (2.1),

$$d(u, T_1 u) \leqslant d(u, x_{2n+2})+d(T_2 x_{2n+1}, T_1 u).$$

$$\leqslant d(u, x_{2n+2})+a_1 d(x_{2n+1}, x_{2n+2})+a_2 d(u, T_1 u)+a_3 d(x_{2n+1}, T_1 u)$$

$$+a_4 d(u, x_{2n+2})+a_5 d(x_{2n+1}, u)+a_6 d(x_{2n+1}, x_{2n+3})$$

$$+a_7 d(x_{2n+2}, x_{2n+3})+a_8 d(u, x_{2n+3})+a_9 d(T_1 u, x_{2n+3}).$$

Letting $n \to \infty$, we get

$(1-a_2-a_3-a_9)\, d\,(u,\, T_1\, u) \leqslant 0,$

a contradiction to (1.2) unless $T_1\, u=u$. Similarly, we get $T_2\, u=u$.

The uniqueness of the common fixed point follows easily.

Remark : Since the Banach contraction principle was published, many special cases of the above theorem have been obtained. Here it is not possible to state these results appeared in literatures, but the author only mentions the results of C.S. Wong [7], and S. Ranganathan [5, Theorem II. 4]. Wong's result is obtained when $a_k=0$ $(k=6, 7, 8, 9)$ and (without any loss of generality) $i=1$, $j=2$. Ranganathan's result is obtained when $T_1=T_2=T$.

Corollary 1. Let $\{T_i\}$ $i \in I^+$ be a family of mappings of X into itself. Suppose that there is an $m_1 \in I^+$ for each T_i such that for all x, y in X and every pair i, j with $i \neq j$,

$$(2.2) \quad d\,(U_i\, x,\, U_j\, y) \leqslant a_1\, d\,(x,\, U_i\, x)+a_2\, d\,(y,\, U_j\, y)+a_3\, d\,(x,\, U_j\, y)$$

$$+a_4\, d\,(y,\, U_i\, x)+a_5\, d\,(x,\, y)+a_6\, d\,(x,\, U_j\, U_i\, x)$$

$$+a_7\, d\,(U_i\, x,\, U_j\, U_i\, x)+a_8\, d\,(y,\, U_j\, U_i\, x)+a_9\, d\,(U_j\, y,\, U_j\, U_i\, x)$$

holds, where $U_i=T_i^{mi}$ for each $i \in I^+$ and a's are as in (1.2) and (1.3). Then the family of mappings $\{T_i\}$ $i \in I^+$ has a unique common fixed point.

Proof : Note that the hypothesis permits that U_i and U_j can be interchanged in (2.2). For any fixed i, j we apply Theorem 1 to get a unique common fixed point of U_i and U_j. Then there exists a unique p in X such that $U_i\, p=U_j\, p=p$. We therefore, have

$$T_i\, p=T_i\,(U_i\, p)=U_i\,(T_i\, p)$$

which inmplies that $T_i\, p$ is a fixed point of U_i. Similarly $T_j\, p$ is a fixed point of U_j. By (2.2),

$$d\,(T_i\, p,\quad T_j\, p)=d\,(U_i\,(T_i\, p),\, U_j\,(T_j\, p))$$

$$\leqslant (a_3+a_4+a_5)\, d\,(T_i\, p, T_j\, p)+(a_6+a_7)\, d\,(T_i\, p,\, U_j\,(T_i\, p))$$

$$+(a_8+a_9)\, d\,(T_j\, p,\, U_j\,(T_i\, p))$$

which implies

(2.3) $\quad (1-a_3-a_4-a_5)\, d\,(T_i\,p,\,T_j\,p)$

$$\leqslant (a_6+a_7)\, d\,(T_i\,p,\,U_j\,T_i\,p)+(a_8+a_9)\, d\,(T_j\,p,\,U_j\,(T_i\,p)).$$

Again by (2.2),

$$d\,(T_i\,p,\,U_j\,T_i\,p)=d\,(U_i\,T_i\,p,\,U_j\,T_i\,p)$$

$$\leqslant (a_2+a_3+a_6+a_7+a_8)\, d\,(T_i\,p,\,U_j\,T_i\,p)$$

which means

(2.4) $\quad d\,(T_i\,p,\,U_j\,T_i\,p)=0$ i.e. $U_j\,T_i\,p=T_i\,p.$

Hence (2.3) yields $(1-a_3-a_4-a_5-a_8-a_9)\, d\,(T_i\,p,\,T_j\,p)\leqslant 0,$

a contradiction unless $T_i\,p=T_j\,p$. Therefore $T_i\,p=T_j\,p$ is a common fixed point of U_i and U_j. But the common fixed point of mappings U_i and U_j is unique. Hence $T_i\,p=T_j\,p=p$, which means that p is a common fixed point of T_i and T_j. If such a p is not unique, let there be another point q in X such that $T_i\,q=T_j\,q=q$. Then, since $p=T_i\,p=T_i^{mi}\,p=U_i\,p,$ (2.2) implies

$$d\,(p,\,q)=d\,(U_i\,p,\,U_j\,q)\leqslant (a_3+a_4+a_5+a_8+a_9)\, d\,(p,\,q).$$

Hence $p=q$. Thus T_i and T_j have a unique common fixed point in X.

To show that the 'family of mappings' under consideration has a unique common fixed point, let p, z be common fixed points of T_i, T_j and T_k, T_s respectively. To prove $p=z$, we may suppose $i\neq k$.

Then in view of (2.2), $z=U_k\,z$ and $p=U_i\,p$ imply

$$d\,(p,\,z)=d\,(U_i\,p,\,U_k\,z)\leqslant (a_3+a_4+a_5)\, d\,(p,\,z)+(a_6+a_7)\, d\,(p,\,U_k\,p)$$

$$+(a_8+a_9)\, d\,(z,\,U_k\,p).$$

By virtue of (2.4) $U_k\,p=p$. Hence $d\,(p,\,z)=0$ giving $p=z$. Thus fixed points of every pair T_i, T_j coincide uniquely.

Remark. The result of Iseki [2, Theorem 1] is obtained as a particular case of the above corollary when $a_1 = a_2, a_3 = a_4, a_k = 0$ $(k=6, 7, 8, 9)$.

Corollary 2. Let T_1 and T_2 be two mappings of X into itself. Suppose that for $i=1, 2, j=1, 2$ with $i \neq j$ and for all x, y in X

$$(2.5) \quad d(T_i T_j x, T_j T_i y) \leqslant a_1 d(x, T_i T_j x) + a_2 d(y, T_j T_i y)$$

$$+ a_3 d(x, T_j T_i y) + a_4 d(y, T_i T_j x) + a_5 d(x, y)$$

$$+ a_6 d(x, T_j T_i^2 T_j x) + a_7 d(T_i T_j x, T_j T_i^2 T_j x)$$

$$+ a_8 d(y, T_j T_i^2 T_j x) + a_9 d(T_j T_i y, T_j T_i^2 T_j x)$$

hold, where a's are as in (1.2) and (1.3). Then T_1 and T_2 have a unique common fixed point in X.

We remark that if T_1 and T_2 commute, then the relations (2.5) may be replaced by a single relation obtained by taking $i=1, j=2$.

Proof : Let $U_1 = T_1 T_2$ and $U_2 = T_2 T_1$. Then, since

$$T_2 T_1^2 T_2 = T_2 T_1 T_1 T_2 = U_2 U_1 \text{ and } T_1 T_2^2 T_1 = U_1 U_2,$$

relations (2.5) can be written in the form of

$$(2.6) \quad d(U_s x, U_t y) \leqslant a_1 d(x, U_s x) + a_2 d(y, U_t y) + a_3 d(x, U_t y)$$

$$+ a_4 d(y, U_s x) + a_5 d(x, y) + a_6 d(x, U_t U_s x)$$

$$+ a_7 d(U_s x, U_t U_s x) + a_8 d(y, U_t U_s x) + a_9 d(U_t y, U_t U_s x)$$

where $s=1, 2$ and $t=1, 2$ with $s \neq t$. Hence in view of Theorem 1, there is a unique common fixed point p (in X) of U_1 and U_2. Then

$$T_1 p = T_1 (U_2 p) = T_1 (T_2 T_1 p) = T_1 T_2 (T_1 p) = U_1 (T_1 p).$$

Therefore $T_1 p$ is a fixed point of U_1. Similarly $T_2 p$ is a fixed point of U_2. Putting $s=1, t=2, x=T_1 p$ and $y=T_2 p$ in (2.6) we get

$$d(T_1 p, T_2 p) = d(U_1 [T_1 p], U_2 [T_2 p])$$

$(2.7) \quad \leqslant (a_3+a_4+a_5) \, d \, (T_1 \, p, \, T_2 \, p) + (a_6+a_7) \, d \, (T_1 \, p, \, U_2 \, [T_1 \, p])$

$\qquad + (a_8+a_9) \, d \, (T_2 p, \, U_2 \, [T_1 \, p]).$

Since $d \, (T_1 p, \, U_2 \, [T_1 \, p]) = d \, (U_1 \, [T_1 \, p], \, U_2 \, [T_1 \, p])$, applying (2.6) again we have $d(T_1 \, p, \, U_2 \, (T_1 \, p)) \leqslant (a_2+a_3+a_6+a_7+a_8) \, d \, (T_1 \, p, \, U_2 \, [T_1 \, p])$ which implies that $T_1 \, p = U_2 \, (T_1 \, p)$. Therefore, from (2.7) we get $d(T_1 \, p, \, T_2 \, p) = O$. Hence $T_1 \, p = T_2$ p, and by the uniqueness of the common fixed point of U_1 and U_2, we obtain $p = T_1 p = T_2 \, p.$

The uniqueness of the common fixed point of T_2 and T_2 follows easily.

Remarks : In case $a_1 = a_2, a_3 = a_4, a_k = O$ $(k = 6, 7, 8, 9)$ and (without any loss of generality) $i = 1, j = 2$, we get the result of Iséki [2, Theorem 2] as a particular case of the above corollary.

Corollary 3. Let $T_i \, (i = 1, 2, \ldots, n)$ be a family of mappings of X into itself. Let

$$P = T_1 \, T_2 \ldots T_n \quad \text{and} \quad Q = T_n \, T_{n-1} \ldots T_1.$$

If $\{T_i\}$ satisfies the conditions: P commutes with every T_i, and relation (1.1) holds and also holds when P and Q are interchanged, where a's are as in (1.2) and (1.3), then $T_i \, (i = 1, \ldots, n)$ have a unique common fixed point.

Proof : By Theorem 1, P and Q have a unique common fixed point p in X. For any T_i, $T_i \, p = T_i \, (Pp) = P \, (T_i \, p)$. Therefore $T_i \, p$ is also a fixed point of P. Then by the condition (1.1), we have

$$d \, (T_i \, p, \, p) = d(P \, (T_i \, p), \, Q \, p)$$

$(2.8) \quad \leqslant (a_3+a_4+a_5) \, d \, (T_i \, p, \, p) \, + \, (a_6+a_7) \, d \, (T_i \, p, \, Q \, T_i \, p)$

$$+ (a_8+a_9) \, d \, (p, \, Q \, T_i \, p).$$

Furthermore,

$d \, (T_i \, p, \, QT_i p) = d(P \, (T_i \, p), \, Q \, (T_i \, p)) \leqslant (a_2+a_3+a_6+a_7+a_8) \, d(T_i \, p, \, QT_i \, p)$ which implies that $T_i \, p = QT_i \, p$. Hence (2.8) yields $d \, (T_i \, p, \, p) = O$. and consequently $T_i \, p = p \, (i = 1, 2, \ldots, n)$. The uniqueness follows easily.

Remark. Results of Istratescu [3, pp. 100 – 105] and Iséki. [1, Th. 2] are particular cases of the above corollary. The former is obtained by taking $a_1 = a_2$, $a_k = 0$ $(k = 3, 4, \ldots, 9)$ while the later is obtained by taking $a_1 = a_2$, $a_3 = a_4$, $a_k = 0$ $(k = 6, 7, 8, 9)$.

Corollary 4. Let T_i $(i = 1, 2, \ldots, n)$ be a family of mappings of X into itself. If $T_i (i = 1, 2, \ldots, n)$ satisfy

$$T_i \, T_j = T_j \, T_i \quad (i, j = 1, 2, \ldots, n),$$

and there is a system of positive integers $m_1, m_2, \ldots, m_n$ such that $(1.1) - (1.3)$ hold for

$$Q = P = T_1^{m1} \; T_2^{m2} \ldots\ldots\ldots\ldots\ldots T_n^{mn},$$

then T_i $(i = 1, 2, \ldots, n)$ have a unique common fixed point.

Proof : By a result of S. Ranganathan (5, Theorem II.4) (which is also obtained by taking $T_1 = T_2 = P$ in Theorem 1), P has a unique fixed point p in X. Therefore T_i $(Pp) = T_i^* p$ $(i = 1, 2, \ldots, n)$. Hence the the commutativity of $\{T_i\}$ implies that $P(T_i \, p) = T_i \, (Pp) = T_i \, p$. This shows that $T_i \, p$ is a fixed point of P. Since fixed point of P is unique, we obtain $T_i \, p = p$, $i = 1, 2, \ldots, n$. Uniqueness of the fixed point of the family $\{T_i\}$ follows easily.

Remark. We mention a few particular cases of the above corollary. If $a_1 = a_2$, $a_3 = a_4$ and $a_k = 0$ $(k = 6, 7, 8, 9)$ then we get Theorem 1 of Iséki [1]. In case $a_1 = a_2$ and $a_k = 0$ $(k = 3, 4, \ldots, 9)$, we obtain a result of Istrateşçu [3, pp. 100 – 105]. Theorem II.5 of Ranganathan [5] is obtained by taking $P = T_1{}^{m_1} T_2{}^{m_2}$.

3. Common fixed point Theorem in a space with two metrics.

Following is theorem is the main result of this section.

Theorem 2. Let X be a space with two metrics ϱ and d. Suppose that

(3.1) $\varrho\,(x, y) \leqslant d\,(x, y)$ for all x, y in X,

(3.2) X is complete with respect to ϱ,

(3.3) two mappings $T_1, T_2 : X \to X$ are continuous with respect to ϱ, and for $i=1, 2, j=1, 2, i \neq j$, the condition (1.1) holds with $P=T_i$ and $Q=T_j$ where a's are as in (1.2) and (1.3).

Then T_1 and T_2 have a unique common fixed point.

Proof : Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by setting $x_{2n+1}=T_1 x_{2n}, x_{2n+2}=T_2 x_{2n+1}$ for each positive integer n. Then proceeding as in Theorem 1, we find that $\{x_n\}$ is a Cauchy sequence with respect to d. Therefore, by (3.1), it is also Cauchy sequence with respect to ϱ. By the completeness of (X, ϱ), there exists a point p in X such that $x_n \to p$. We now use the continuity of T_1 to infer that p is a fixed point of T_1.

$$T_1 \, p = T_1 \, (\lim \, x_{2n}) = \lim \, T_1 \, x_{2n} = \lim \, x_{2n+1} = p.$$

Similarly, p is a fixed point of T_2. It is trivial that the common fixed point is unique.

Remark : In case $a_1=a_2, a_3=a_4, a_k=0$ ($k=6, 7. 8, 9$) and (without any loss of generality ($i=1, j=2$, we get Iseki's result which is mentioned in his paper [2, page 104].

Corollary 5. If $T_1=T_2=P$ then under the hypotheses of Theorem (2) P has a unique fixed point.

Remark : The result of M.G. Maia [4] is obtained as a particular case of Corollary 5 when P is a contraction.

The conclusion of the above corollary can be obtained under much less restricted condition. We do not require the continuity of P with respect to ϱ, just the continuity at a point will serve the purpose. Consequently we have the following.

Corollary 6. Let X have two metrics ϱ and d, and the following conditions be fulfilled.

(3.4) $\varrho \, (x, y) \leqslant d \, (x, y)$ for all x, y in X.

(3.5) $P : X \to X$ is such that (1.1) holds with $P=Q$,

(3.6) P is continuous at $p \in X$ with respect to ϱ,

(3.7) there exists a point $x_0 \in X$ such that the sequence of iterates

$\{P^m \, x_0\}$ has a subsequence $\{P^{m_l} \, x_0\}$ converging to p in (X, ϱ).

Then P has a unique fixed point.

Proof: It is easily seen that $\{x_n\}$, the sequence of iterates is a Cauchy sequence with respect to d. By (3.4), $\{x_n\}$ is also a Cauchy sequence in (X, ϱ). Since the subsequence $\{x_{n_l}\}$ of $\{x_n\}$ converges to p, $x_n \to p$ under the metric ϱ. Also since P is continuous at p, we have

$$\text{P}p = \text{P} \lim x_n = \lim \text{P} \, x_n = \lim x_{n+1} = p.$$

Uniqueness of the fixed point p follows easily.

Remark : The result of Singh (6, page 17) is obtained as a particular case of the above corollary, when P is a contraction. We obtain slightly new results when P is replaced by P^n in Corollaries 5 and 6 for some integer $n > 1$.

Corollary 7. Let X be a space with the metrics ϱ and d satisfying (3.1) and (3.2), and let $\{T_i\}$ $i \in I^+$ be a family of continuous mappings of (X, ϱ) into itself. Suppose that there is an $m_i \in I^+$ for each T_i such that for all x, y in X and every pair i, j with $i \neq j$, the condition (2.2) holds, where $U_i = T_i \, m_i$ for each $i \in I^+$ and a's are as in (1.2) and (1.3). Then the family of mappings $\{T_i\}$ $i \in I^+$ has a unique common fixed point.

Proof : Note that each U_l is continuous with respect to ϱ. Then in view of Theorem 2, for any fixed i, j $(i \neq j)$ there exists a unique point p such that $U_i \, p = U_j \, p = p$. The rest part of the proof follows on the corresponding lines of Corollary 1.

Remark : Theorem 3 of Isèki [2] is obtained as a particular case of the above corollary by taking $a_1 = a_2$, $a_3 = a_4$ and $a_k = 0$ $(k = 6, 7, 8, 9)$.

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COMPARISON OF POISSON SAMPLING WITH SOME SELECTION PROCEDURES FOR UNEQUAL PROBABILITIES

By

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SUMMARY

'In this paper the poisson sampling variance is compared with some selection procedures using a straight forward Horvitz and Thompson estimater and a modified form of the estimator on three artifical populations given by Yates and Grundy (1953). Stability of the variance of poisson sampling is also compared with other selection procedures.

1. Introduction

Hajek (1964) defines poisson sampling for unequal probabilties as a selection procedure in which each unit in the population is given a probability of inclusion in sample, and a set of N Bernoulli trials is carried out to determine whether each unit is actually included in the sample or not. The ith unit is included in the sample if the ith trial results in a success and excluded otherwise.

Brewer *et al* (1972) redefine the Horvitz and Thompson (1952) unbiased estimator of a population total as

$$y'_{PS} = \sum_{i=1}^{m} y_i/\pi_i \tag{1}$$

where m is sample size, y_i is the value of the ith unit in the sample, and π_i is the probability of inclusion in the sample of the ith population unit. They derived the variance formula

$$\mathrm{Var}_{ps}(y'_{HT}) = \sum_{i=1}^{N} (1-\pi_i)\frac{y_i^2}{\pi_i} \tag{2}$$

and an unbiased variance estimator of (2) which is

$$\text{var}_{ps} \, (y'_{HT}) = \sum_{i=1}^{m} (1-\pi_i) \, \frac{y_i^2}{\pi_i^2} \tag{3}$$

When the sample size is a random variable they have suggested using a the ratio estimator

$$y''_{PS} = \begin{cases} \dfrac{n}{m} \, y'_{PS} & \text{if } m \neq 0 \\[2mm] 0 & \text{otherwise} \end{cases} \tag{4}$$

where $n = E\,(m) = \sum_{i=1}^{m} \pi_i$. The mean square error of (4) is

$$\text{Var}_{ps} \, (y''_{HT}) = \sum_{i=1}^{N} \pi_i \, (1-\pi_i) \left(\frac{Y_i}{\pi_i} - \frac{Y^2}{n} \right)^2 + P_o \, Y^2 \tag{5}$$

where $P_o = \text{Pr} \, (m=0)$. A consistent estimator of (5) is

$$\text{var}_{ps} \, (y''_{HT}) = \sum_{i=1}^{m} (1-\pi_i) \left(\frac{y_i}{\pi_i} - \frac{y''_{HT}}{n} \right)^2 + P_o \, y'_{PS} \tag{6}$$

2. Empirical Study

The three typical populations widely used in literature for comparison purposes have been considered to compare Poisson sampling with the following selection procedures.

(i) PPS sampling with replacement (PPSWR)

(ii) Midzuno's selection procedure (Midzuno)

(iii) Rao, Hartly and Cochran (RHC)

(iv) Brewer, Durbin and Sampford (BDS)

(v) Goodman and Kish (G-K)

(vi) Narain's selection procedure (Narain)

(vii) Raj's selection procedure (Raj)

(viii) Murthy's selection procedure (Murthy)

(ix) Poisson sampling (Unbiased)

(x) Poisson sampling (Ratio estimator)

The three populations are reproduced below with common selection probabilities,

POPULATIONS

Unit	P_i	A	B	C
1	0.1	0.5	0.8	0.2
2	0.2	1.2	1.4	0.6
3	0.3	2.1	1.8	0.9
4	0.4	3.2	2.0	0.8

The variances of the estimates of population total have been calculated for all selection procedures and are as follows,

TABLE 1
Variances of population total of various selection procedures
Variances

Selection Procedures	A	B	C
PPSWR	0.500	0.500	0.125
Midzuno	2.88	0.384	0.240
RHC	0.333	0.333	0.083
BDS	0.270	0.275	0.058
G-K	0.370	0.370	0.033
Narain	0.323	0.270	0.057
Raj	0.365	0.365	0.088
Murthy	0.312	0.312	0.070
Poisson (unbiased)	8.67	8.67	1.40
Poisson (Ratio)	2.13	1.63	0.29

Table 1 shows that the efficiency of poisson sampling (unbiased) is poor regarding with other selection procedures in population A and B, but in population C the performance is little better. The efficiency of ratio is better than unbiased. In population A, the performance of Poisson (ratio) is better than that of Midzuno's, in population B the case is reverse and

in population C both are almost equally efficient. Although Poisson sampling is less efficient but selection is strictly proportional to size, applicable for sample size greater than 2, selection procedure is simple though lengthy, π_{ij} takes the simple form, simple for rotation and estimation is simple

3. Stability of Variance Estimator of Poisson Sampling :

The linear stochastic model

$$Y_i = B\,Z_i + U_i$$

where $E^*(U_i) = 0$, $E^*(U_i\,U_j) = 0$ for $i \neq j$ and

$$E(U_i^2) = \sigma^2\,Z_i^{2\gamma}, \tfrac{1}{2} \leqslant \gamma \leqslant 1 \tag{7}$$

Z_i is measure of size $\Sigma\,Z_i = Z$

is used to find the stability of variance estimator E^* denotes the conditional expectation over all possible hyothetical population and σ_i^2 is constant over these populations for any particular value of i. From (7) and (2) we obtain

$$E\,\mathrm{var}\,(y'_{HT}) = B^2\,Z^2\left\{\left[n + n^2\sum_{i=1}^{N}(Z_i/Z)^2 + \Sigma\left(\frac{1}{n\,P_i} - 1\right)\sigma_i^2\right.\right\} \tag{8}$$

where $P_i = Z_i/Z$

and from (2), (3) and (7) we obtain

$$E\,E^*\,[\mathrm{var}\,(y'_{HT}) - (\mathrm{Var}\,(y'_{HT})] = B^2\,Z^4\left(\frac{1}{n^3} + \frac{2}{n^2} - \frac{2}{n}\right)$$

$$+ \left(\frac{2B^4\,Z^2}{n} - \frac{B^4\,Z^2}{n^2}\right)\Sigma\,Z_i^2$$

$$+ B^4\,Z\,n^{-1}\,\Sigma\,Z_i^3 + (B^2 - B^4)\,\Sigma_{i \neq j}\,Z_i^2\,Z_j^2 - B^4\,\Sigma_{i=1}\,Z_i^4$$

$$- 2B^4\,Z_h^{-1}\,\Sigma_{i \neq j}\,Z_i^2\,Z_j^2$$

$$+4B^2 n^{-3} \quad \Sigma \quad Z_i^{-3}(Z-nZ_i)^3 \sigma_i^2 + \Sigma \; [K.n^{-3} Z_i^{-3}(Z-nZ_i)^3$$

$$+2n^{-1} Z^{-1} Z_i^{-1}(nZ_i-Z)^2] \sigma_i^4 + \underset{i\neq j}{\Sigma \; \Sigma} \frac{Z_i-Z_j}{nZ_i Z_j Z}(Z-nZ_i)(Z-nz_j) \; \sigma_i^2 \sigma_j^2 \quad (9)$$

where K is the kurtesis of U_i. Putting $B=O$, $K=3$ and $\sigma_i^2 = \sigma^2 Z_i^{2\gamma}$ in (8) and (9) the relative expected variance is

$$\frac{E^* \; [\text{var} \; (y'_{HT}) - \text{Var} \; (y'_{HT})]^2}{[E^* \; (y'_{HT})]^2}$$

$$= \frac{\Sigma \; [3n^{-3}+2n \; ZZ_i^{-1}(Z-nZ_i)^2] \; Z_i^{4\gamma} + \Sigma \Sigma \dfrac{Z_i-Z_j}{nZZ_i Z_j} \; (Z-nZ_i)\;(Z-nZ_j)}{[\; \Sigma \; Z_i^{-1} \; n^{-1}\;(Z-nZ_i)Z_i^{2\gamma}\;]^2} \quad (9)$$

Brewer and Hanif (1969b) considered the stability of various selection procedures using Horvitz and Thompson Estimator and also using special estimators. The same population of 4 units has been considered here to find the stability. The values of the population are $N=4$, $n=2$ and $Z_i= 1, 2, 3, 4$.

TABLE 2

Relative expected variance for various selection Procedures

Procedure	Relative expected variance with		
	$\gamma=1/2$	$\gamma=3/4$	$\gamma=1$
PPSWR	3.81	3.39	3.29
G-K (Systematic)	10.34	10.65	10.95
Narain	7.60	6.27	5.29
BDS	8.17	6.61	5.43
Raj	3.08	2.52	2.26
Murthy	3.20	2.57	2.27
Yates—Grundy at			
(Theoretical Optimum)	6.44	5.90	5.21
Poisson (unbiased)	9.07	7.51	6.23

The relative expected variance of Poisson is somewhat larger than Yates and Grundy variance estimator in all three cases but it is more stable than Goodman and Kish selection procedure.

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THE STRICT DUAL OF $C(X, E_n)$

By

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1. Introduction

Let $C(X,E)$ be the vector space of all bounded continuous functions from a topological space X into a topological vector space E (over the field K of real or complex numbers); when $E = K$, this space is denoted by $C(X)$. The notion of the strict topology on $C(X,E)$ was first introduced by R.C. Buck [1] in the case of X locally compact and E a locally convex space. He also proved that the strict dual of $C(X)$ is isometrically isomorphic to $M(X)$, the Banach space of all bounded regular Borel measures on X ([1], Theorem 2). This result was later extended to the case of X, a completely regular space by Giles [2] and, independently, by several other authors. In this paper we characterise the strict dual of $C(X, E_n)$, where E_n is an n-dimensional vector space. Our result generalizes a result of Oates [5] which was originally proved for X, a compact Hausdorff space.

2. Preliminaries

The *strict topology* β on $C(X,E)$ is the linear topology which as a base of neighbourhoods of O consisting of all sets of the form ;

$$U(\psi, W) = \{ f \in C(X,E) : \psi(x) f(x) \in W \text{ for all } x \in X\}.$$

where $\psi \in B_o(X)$, the set of all bounded functions on X which 'vanish at infinity.' and W belongs to a base of closed balanced neighbourhoods of

$\overset{\circ}{O}$ in E. (For details, see [4]). When $E=(E_n, \| \|_n)$, the β—topology on $C(X, E_n)$ can alternatively be defined as the locally convex topology given by the semi-norms

$$f \to \| \psi f \| = \sup_{x \in X} \| \psi(x) f(x) \|_n,$$

where ψ varies over $B_o(X)$.

We denote by $M^+(X)$ the set of all positive measures in $M(X)$.

We shall require the following result due to Giles ([2], Theorem 4.6).

Theorem 1. *Let X be a completely regular space. If L is a β—continuous linear functional on $C(X)$, then there exists a unique $\mu \in M(X)$ such that*

$$L(\phi) = \int_X \phi \, d\mu \tag{1}$$

for all $\phi \in C(X)$. Conversely, for any $\mu \in M(X)$, the equation (1) defines a β—continuous linear functional L on $C(X)$.

3. The β—dual of $C(X, E_n)$

In this section we establish the following theorem which extends ([5], Proposition 1).

Theorem 2. *Let X be a completely regular space. If L is a β-continuous linear functional on $C(X, E_n)$, then there exists a $\mu \in M^+(X)$ and an E_n-valued function h on X, each of whose components is μ-integrable, such that $\| h(x) \|_n = 1$ for all $x \in X$ and*

$$L(f) = \int_X (f, h) \, d\mu \qquad (f \in C(X, E_n)), \tag{2}$$

where $(.)$ denotes the usual inner product in E_n. Conversely, for any $\mu \in M^+(X)$ and E_n-valued function h on X with the above properties, the equation (2) defines a β—continuous linear functional L on $C(X, E_n)$.

Proof. Let L be a β—continuous linear functional on $C(X, E_n)$. If f is any function in $C(X, E_n)$, then we can express it as $f = \sum_{j=1}^{n} f_j \otimes e_j$, where $e_1 \ldots e_n$ is an orthonormal basis for E_n, $f_1 \ldots, f_n \in C(X)$ are the components of f, and $f_j \otimes c_j (x) = f_j(x) e_j$ for all $x \in X$. Therefore

$$L(f) = \sum_{j=1}^{n} L(f_j \otimes e_j) = \sum_{j=1}^{n} L_j(f_j), \tag{3}$$

where $L_j(\phi) = L(\phi \otimes e_j)$ for all $\phi \in C(X)$. Since L is β-continuous on $C(X, E_n)$, there exists a $\psi \in B_o(X)$ such that $|L(f)| < 1$ whenever $f \in C(X, E_n)$ and $\|\psi f\| < 1$. Now, if $\phi \in C(X)$ and $\|\psi \phi\| < 1$, then $\|\psi(\phi \otimes e_j)\| = \|\psi \phi\| \cdot \|e_j\|_n < 1$ $(1 \leqslant j \leqslant n)$, and so $|L_j(\phi)| = |L(\phi \otimes e_j)| < 1$. Hence each L_j is a β—continuous linear functional on $C(X)$. By Theorem 1, there exist μ_j $(1 \leqslant j \leqslant n)$ in $M(X)$ such that

$$L_j(\phi) = \int_X \phi \, d\mu_j \tag{4}$$

for all $\phi \in C(X)$.

Now, by the Jordan-decomposition theorem, we can write $\mu_j = \mu_j^{(1)} - \mu_j^{(2)} + i\mu_j^{(3)} - i\mu_j^{(4)}$, where each $\mu_j^{(k)}$ $(k = 1, \ldots, 4) \in M^+(X)$. Let $\eta = \sum_{j=1}^{n} \left(\mu_j^{(1)} + \mu_j^{(2)} + \mu_j^{(3)} + \mu_j^{(4)} \right)$. Then $\eta \in M^+(X)$. and, for each $j = 1, \ldots, n, k = 1, \ldots, 4$, $\mu_j^{(k)}$ is absolutely continuous with respect to η. It follows from the Radon-Nikodoym theorem ([3], p. 128) that there exist k-valued η-integrable functions $\phi_{j,k}$ $(j = 1, \ldots, n, k = 1, \ldots, 4)$ on X such that

$$\int_X \phi \, d\mu_j^{(k)} = \int_X \phi \, \phi_{j,k} \, d\eta \tag{5}$$

for all $\phi \in C(X)$. Define an E_n-valued function g on X by

$$g = \sum_{j=1}^{n} (\phi_{j,1} - \phi_{j,2} + i\, \phi_{j,3} - i\, \phi_{j,4}) \otimes e_j.$$

It follows from (3), (4), and (5) that

$$L(f) = \int_X \sum_{j=1}^{n} f_j\, (\phi_{j,1} - \phi_{j,2} + i\, \phi_{j,2} - i\, \phi_{j,4})\, d\eta = \int_X (f, g)\, d\eta.$$

This holds for all $f \in C(X, E_n)$. Let $A = \{x \in X : g(x) \neq o\}$, and define
$h(x) = \dfrac{g(x)}{\|g(x)\|_n}$ if $x \in A$, and $h(x) = e_1$ otherwise. Then

$$L(f) = \int_A (f, g)\, d\eta = \int_A (f(x), h(x))\, \|g(x)\|_n\, d\eta.$$

For any Borel subset B of X, define $\mu(B) = \int_A \|g(x)\|_n\, d\eta$.

Then $\mu \in M^+(X)$ and, since $\mu(X - A) = o$, if follows that

$$L(f) = \int_X (f, h)\, d\mu.$$

It is clear that the function h is μ-integrable and that $\|h(x)\|_n = 1$ for all $x \in X$.

Conversely, let μ and h be as given in the theorem. By Theorem 1, the equation

$$T(\phi) = \int_X \phi\, d\mu \qquad\qquad (\phi \in C(X))$$

defines a β—continuous linear functional T on $C(X)$. Hence there exists a $\psi \in B_o(X)$ such that $|T(\phi)| < 1$ for all $\phi \in C(X)$ with $\|\psi \phi\| < 1$. Now, tor each $x \in X$, $|(f(x), h(x))| \leqslant \|f(x)\|_n \cdot \|h(x)\|_n = \|f(x)\|_n$, and so, if $f \in C(X, E_n)$ with $\|\psi f\| < 1$, then

$$|L(f)| \leqslant \int_X \|f(x)\|_n\, d\mu = T(\|f\|_n) < 1.$$

where $\|f\|_n \in C(X)$ such that $\|f\|_n(x) = \|f(x)\|_n$ $(x \in X)$. Thus L is β—continuous, as required.

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A GENERALIZED DERIVATION OF ARITHMETIC FUNCTIONS

By

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Abstract ;—In this note we define a generalized derivative of an arithmetic function $g(n)$ by $g'(n)=g(n) \cdot f(n)$, where $f(n)$ is a completely additive arithemetic function and prove the usual properties of derivation of arithmetic fuctions leading to a generalized form of Selberg's identity.

Let $f(n)$ be a real or complex valued completely additive arithmetic function on the set Z of natural numbers so that $f(1)=0$ $f(m.n) = f(m)+f(n)$ for all $m, n \in Z$. Thus $f(1)=0$ and

$$f(p_1{}^{a_1} \dots p_k{}^{a_k}) = \sum_{i=1}^{k} a_i f(p_i)$$

for any positive integer $n = p_1{}^{a_1} \dots p_k{}^{a_k} \geqslant 2$.

Define $\Lambda_f(n) = \begin{cases} f(p), \text{ when } n=p^m, p \text{ a prime and } m \geqslant 1 \\ 0 \text{ otherwise} \end{cases}$

Theorem 1.1

$$f(n) = \sum_{d \mid n} \Lambda_f(d).$$

We have $\sum_{d \mid n} \Lambda_f(d) = \sum_{i=1}^{k} \sum_{m=1}^{a_i} f(p_i{}^m) = \sum_{i=1}^{k} \sum_{m=1}^{a_i} f(p_i) =$

$$\sum_{i=1}^{k} a_i f(p_i) = f(n)$$

Theorem 1.2

If $n > 1$, then

$$\Lambda_f(n) = \sum_{d \mid n} \mu(d) f(n/d) = - \sum_{d \mid n} \mu(d) f(d)$$

We have

$$f(n) = \sum_{d \mid n} \Lambda_f(d) \text{ by Theorem 1.1}$$

Hence inverting this by Möbius inversion formulae,

$$\Lambda_f(n) = \sum_{d \mid n} \mu(d) f(n/d)$$

$$= \sum_{d \mid n} \mu(d) (f(n) - f(d))$$

$$= f(n) \sum_{d \mid n} \mu(d) - \sum_{d \mid n} \mu(d) f(d)$$

$$= f(n) I(n) - \sum_{d-n} \mu(d) f(d),$$

where $I(n) = 1$ when $n = 1$ and $I(n) = 0$ for all $n > 1$, is the usual identity function. This proves $\Lambda_f(n) = \sum_{d \mid n} \mu(d) f(n/d) = - \sum_{d \mid n} \mu(d) f(d)$

2. Generalized Derivation.

Let $g(n)$ be any arithmetid function then $g \cdot (n) = g(n) f(n)$ is called the generalized derivative of $g(n)$.

It is evident that the usual derivative of $g(n)$ given by $g \cdot (n) = g'(n) \log n$ and the distributive function $G(n) = g(n) f(n)$, [2], in which $g(n)$ is necessarily multiplicative are both special cases of the generalized derivative $g \cdot (n)$.

Since $I(n) f(n) = 0$, we have $I \cdot (n) = 0$ for all n. Let $U(n) = 1$ be the unity arithmetic function so that $U(n) = 1$ for all n. Thus

$$U \cdot (n) = U(n) f(n) = f(n)$$

Hence the result of Theorem 1.1 can be written as

Lemma 2.1

$$\wedge_f * U = U^{\cdot}$$

where $*$ denotes Dirichlet multiplication.

Theorem 2.2

For $g(n)$, $h(n)$ arithmetic functions we have :

(a) $(g+h)^{\cdot} = g^{\cdot} + h$.

(b) $(g * h)^{\cdot} = g^{\cdot} * h + g * h^{\cdot}$

(c) $(g^{-1})^{\cdot} = -g^{\cdot} * (g * g)^{-1}$, provided $g(1) \neq 0$.

(a) follows immediately since $(g+h)^{\cdot} = (g+h)f = gf + hf = g^{\cdot} + h^{\cdot}$. To prove (b), we have

$$(g * h)^{\cdot}(n) = \sum_{d \mid n} (g(d) \, h(n/d) \, f(n)$$

$$= \sum_{d \mid n} g(d) \, h(n/d) \, (f(d) + f(n/d)$$

$$= \sum_{d \mid n} g(d) f(d) \, h(n/d) + \sum_{d \mid n} g(d) \, h(n/d) f(n/d)$$

$$= g^{\cdot}(n) * h(n) + g(n) * h^{\cdot}(n).$$

This proves (b).

To prove (c), we have

$$I^{\cdot}(n) = I(n) \, f(n) = 0 \text{ for all } n.$$

Hence, $I^{\cdot}(n) = (g * g^{-1})^{\cdot} = g^{\cdot} * g^{-1} + g * (g^{-1})^{\cdot} = 0.$

or $g * (g^{-1})^{\cdot} = -g^{\cdot} * g^{-1}$. By multiplying by g^{-1} on both sides, we have

$$(g^{-1})^{\cdot} = (-g^{\cdot} * g^{-1}) * g^{-1} = -g^{\cdot} * (g^{-1} * g^{-1})$$

$$= -g^{\cdot} * (g * g)^{-1}.$$

This proves (c). We next prove a generalized form of Selberg's identity [1] in

Theorem 2.3

For $n \geqslant 1, \wedge_f(n) f(n) + \sum_{d \mid n} \wedge_f(d) \wedge_f(n/d)$

$$= \sum_{d \mid n} \mu(d) f^2(n/d)$$

From Lemma 2.1, we have

$$\wedge_f * U = U^\cdot$$

Applying Theorem 2.2, (b) we get

$$\dot{\wedge}_f * U + \wedge_f * U^\cdot = U^{\cdot\cdot} \text{ or since } U^\cdot = \wedge_f * U,$$

$$\dot{\wedge}^f * U + \wedge_f * (\wedge_f * U) = U^{\cdot\cdot}$$

Multiplying both sides by $\mu = U^{-1}$, we get

$$\dot{\wedge}_f + \wedge_f * \wedge_f = U^{\cdot\cdot} * \mu .$$

This completes the proof.

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RINGS EMBEDDABLE IN A RIGHT FIELD

By

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Introduction

In this paper R will stand for an integral domain (not necessarily commutative) with unity element different from zero and R^* for the set of all non-zero elements of R. Furthermore, all modules will be unitary. For homological concepts and undefined terms we refer to [1] and [5].

Looking into 0. Ore's theorem [6] and the results given by E. R. Gentile in [2] and [3], the following conditions are equivalent :—

   (a) R is a right Ore domain (see next section) ;

   (b) R can be embedded into a right field of the form $R (R^*)^{-1}$;

   (c) R/A is a torsion module for every right ideal A of R ;

   (d) In every right R-module the sum of two torsion elements is a torsion element.

If R is a right Ore domain, it follows (see the proof of the theorem in the next section) that

   (e) For a torsion element x in a right R-module M, x R is a torsion submodule of M.

The aim of this paper is to show that (e) is equivalent to the requirement that R is a right Ore domain. To prove this we shall make use of the properties of the maximal right quotient ring of R in the sense of Y. Utumi [8].

2. Torsion Elements

R is called a right Ore domain, if for all a, b in R^*, a $R^* \cap bR^*$ is nonempty (Ore condition). Consider a right R-module M and write

$$T(M) = \{x \in M \mid \exists\, r \in R^* \ni x\, r = 0\}$$

and call each element of $T(M)$ a torsion element. M is called torsion (respectively torsion free) module, if $M = T(M)$ (respectively $T(M) = 0$).

Let I be the injective hull (see [5] of the right R-module R and H the ring of all R-endomorphisms of I. Then R can be regarded as a left H-module by defining, for $h \in H$, $i \in I$, hi by the image of i under h. The biendomorphism ring Q of I, ie., $Q = \text{Hom}_H(I, I)$, is called maximal right quotient ring of R. Maximal quotient rings were first defined by Y. Utumi [8]. For details we refer to [4] or [5]. What we need about Q are the following facts :

(i) R can be regarded as subring of Q ;

(ii) for $0 \neq x$, y in Q there exists $r \in R$ such that $x\, r \neq 0$ and $yr \in R$;

(iii) Q_R, i.e., Q as right R-module in injective. Regarding right Ore domains we require a result of [7], which we state as

(iv) R is a right Ore domain if and only if R can be embedded into a torsion free and injective right R-module.

Theorem. The following statements are equivalent:—

(1) R is a right Ore domain ;

(2) For a given right R-module M, whenever $x \in T(M)$ it follows that $x\, R \subseteq T(M)$.

Proof : Let us assume (1) and consider a torsion element x in a right R-module M. Then there exists an element r in R^* such that $x\, r = 0$. Now for $t \in R^*$, Ore condition yields elements r', $t' \in R^*$ satisfying $r\, r' = t\, t'$. Thus

$$(x\, t)\, t' = (x\, r)\, r' = 0, \text{ which implies } x\, t \in T(M),$$

and it follows that $x\, R$ is a torsion submodule of M.

Conversely, suppose that (2) holds and Q is the maximal right quotient ring of R. In view of (*i*), (*iii*), and (*iv*) it is enough to show that Q_R is torsion free. Suppose, on the contrary, there exists a non-zero element q in $T(Q_R)$. Then by (*ii*) and (2) there is an element $r_1 \in R$ such that $0 \neq q\, r_1 \in R \cap T(Q_R)$. Thus $(q\ r_1)\, t_1 = 0$ for some $t_1 \in R^*$. This is impossible, since R is an integral domain. Hence Q_R is torsion free.

Corollary. The following conditions are equivalent :—

(1) R is a right Ore domain;

(2) for any right R-module M, x R is a torsion submodule of M, whenever x is a torsion element of M;

(3) for any right R-module M, $T(M)$ is an Abelian subgroup of M;

(4) $T(M)$ is a submodule of M for every right R-module M.

All these characterizations of a right Ore domain are laid down in terms of torsion elements. However, several necessary and sufficient conditions are given in [7] utilising torsion free modules. Some others will be given elsewhere making use of divisible elements.

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CONVERGENCE IN REGULAR ORDERED BANACH SPACES

BY

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In this paper a relation between norm convergence and order convergence is discussed. It is known [4: IV.2.4] that the notions of norm convergence and relative uniform $*$–convergence are identical in a Banach Lattice. It is proved below that the same holds for Regular Ordered Banach Spaces.

Let X be an ordered Banach Space with positive cone X_+ .

X is called *regular* by Davies [2] if it satisfies the following two conditions :

(R_1) : $-x \leqslant y \leqslant x \rightarrow \| y \| \leqslant \| x \|$;

(R_2) : Given $y \in X$ and $\epsilon > 0$, $\exists\, x \in X_+$ such that

$$y, -y \leqslant x \text{ and } \| x \| \leqslant \| y \| + \epsilon .$$

A sequence $\{x_n\}$ in an ordered Banach space X will be called *relatively uniformly convergent* to $x_0 \in X$ if there exists an element $u \in X_+$ and a sequence $\{\lambda_n\}$ of positive real numbers decreasing to zero such that

$$x_n - x_0, x_0 - x_n \leqslant \lambda_n . u, \ n=1, 2, \ldots\ldots .$$

A sequence $\{x_n\}$ in X will be called *relatively uniformly $*$–convergent* to $x_0 \in X$ if every subsequence of $\{x_n\}$ contains a subsequence that is relatively uniformly convergent to x_0.

Proposition :

Let X be a regular Ordered Banach Space. A sequence $\{x_n\}$ in X norm converges to $x_0 \in X$ if and only if $\{x_n\}$ is relatively uniformly $*$–convergent to x_0.

Proof :

Let $\{x_n\}$ be relatively uniformly convergent to x_0. there exists $u \in X_+$ and a sequence $\{\lambda_n\}$ of real numbers which decrease to zero and

$$x_n - x_0, \ x_0 - x_n \leqslant \lambda_n . u. \quad n = 1, 2 \ldots . ;$$

i.e. $-\lambda_n \, u \leqslant x_n - x_0 \leqslant \lambda_n \, u.$

Since X satisfies (R_1) we have

$$\| x_n - x_0 \| \leqslant |\lambda_n| \cdot \| u \|.$$

Thus $\{x_n\}$ converges to x_0 in norm. Since a sequence $\{y_n\}$ in X converges in norm to v if and only if every subsequence of $\{y_n\}$ has a subsequence that converges in norm to v, we see that relative uniform $_*$–Convergence implies norm convergence.

Conversely let $\{x_n\}$ converge to zero in norm. We will show that any subsequence of $\{x_n\}$ has a subsequence which is relatively uniformly convergent to zero.

Let $\{y_n\}$ be a subsequence of $\{x_n\}$. Since X satisfies (R_2), there are $z_m \in X_+$ such that $- y_m, y_m \leqslant z_m$ and

$\| z_m \| \leqslant \| y_m \| + \dfrac{1}{m}$ for all $m = 1, 2, 3, \ldots \ldots$ Thus $z_m \to 0$ and there exists a subsequence

$\{z_{m_k}\}$ of $\{z_m\}$ with $\| k . z_{m_k} \| \leqslant \dfrac{1}{2^k}$. For given $p, \, q > 0$

$$\| \sum_{k=p+1}^{p+q} k . z_{m_k} \| \leqslant \sum_{p+1}^{p+q} \| k \, z_{m_k} \|$$

$$< \frac{1}{2^{p+1}} + \ldots \ldots + \frac{1}{2^{p+q}}$$

$$< \frac{1}{2^p} .$$

which implies that $\left\{ \sum\limits_{k=1}^{p} k \cdot z_{n_k} \right\}$ is a Cauchy sequence in X and hence converges to an element $z \in X_+$.

Thus $z_{n_k} \leqslant \dfrac{1}{k} z$ for all k ; which means that $\{z_{n_k}\}$ converges relatively uniformly to zero. This further implies that $\{x_n\}$ is relatively uniformly $_*$–convergent to zero.

In [1] and [5] compactness of certain subsets of an Ordered Banach Space is considered. Above Proposition is helpful to establish compactness of certain subsets of a regular Ordered Banach Space.

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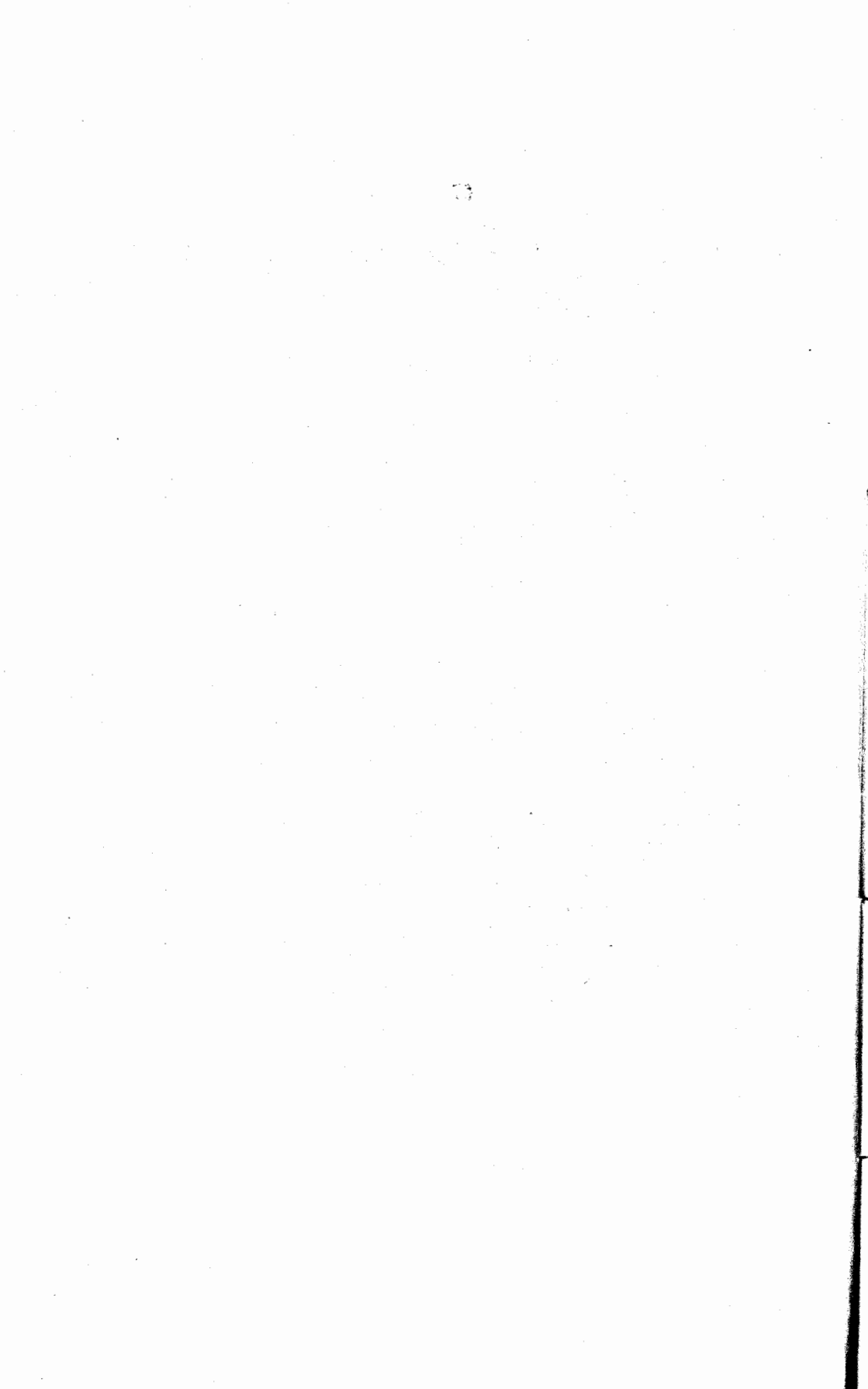
ON A FIXED POINT THEOREM OF GOEBEL KIRK AND SHIMI

BY

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Generalizations of a fixed point theorem in uniformly convex spaces of Goebel, Kirk and Shimi [2] have been obtained by Bogin [1] and Rhoades [3]. We present another extension of their result.

Theorem : Let X be a uniformly convex Banach space, K a nonempty bounded closed and convex subset of X, and F : K→K a continuous mapping satisfying for $x, y \in K$:

(1) $\|F(x) - F(y)\| \leqslant a_1 \|x-y\| + a_2 (\|x-F(x)\| + \|y-F(y)\|)$

$\qquad + a_3 (\|x-F(y)\| + \|y-F(x)\|) + a_4\|x-F^2(x)\|$

$\qquad + a_5\|F(x)-F^2(x)\| + a_6\|y-F^2(x)\| + a_7\|F(y)-F^2(x)\|$

where a_i's are nonnegative real numbers such that

(2) $\qquad a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 + a_7 = 1$ and $a_4 = a_7$ with $a_1 = 0$

implies $a_5 = 0$. Then F has a fixed point in K.

It may be mentioned that the result in [2] is obtained by taking $a_4 = a_5 = a_6 = a_7 = 0$ in (1). We shall follow the same line of argument and notations as that of [2].

Proof. Putting $y = F(x)$ in (1), we get

$\|F(x) - F^2(x)\| \leqslant a_1\|x-F(x)\| + a_2(\|x-F(x)\| + \|F(x)-F^2(x)\|)$

$\qquad + a_3\|x-F^2(x)\| + a_4\|x-F^2(x)\| + a_5\|F(x)-F^2(x)\|$

$\qquad + a_6 \| F(x)-F^2(x) \|$

which gives

$$\|F(x) - F^2(x)\| \leqslant p\|x - F(x)\|$$

where $p = \dfrac{(a_1 + a_2 + a_3 + a_4)}{(1 - a_2 - a_3 - a_4 - a_5 - a_6)}$.

By (2), $p = 1$. Hence

(3) $\quad \|F(x) - F^2(x)\| \leqslant \|x - F(x)\|$

and consequently

(4) $\quad \|F^{2+1}(x) - F^i(x)\| \leqslant \|F^i(x) - F^{i-1}(x)\|, \; i = 1, 2, 3, \ldots \ldots$

Assertion : $\quad \inf\limits_{x \, \epsilon \, K} \quad \|x - F(x)\| = 0.$

To prove the assertion, we assume $\quad \inf\limits_{x \, \epsilon \, K} \quad \|x - F(x)\| = d > 0.$

Let $\varepsilon > 0$ and choose $x \, \epsilon \, K$ such that $\|x - F(x)\| \leqslant d + \varepsilon$. Using uniform convexity and assumptions on K, we can find a real number $\alpha, 0 < \alpha < 1$, as in [2] such that

(5) $\quad \|F^{i-1}(x) - F^{i+1}(x)\| \leqslant 2\alpha\|F^{i-1}(x) - F^i(x)\|, \quad 0 < \alpha < 1.$

Writing j for $i - 1$ in (5), we obtain

(6) $\quad \|F^j(x) - F^{j+2}(x)\| \leqslant 2\alpha \|F^j(x) - F^{j+1}(x)\|, \quad 0 < \alpha < 1.$

Case 1. a_3 or $a_4 \neq 0$.

By (1),

$$\|F^i(x) - F^{i+1}(x)\| \leqslant (a_1 + a_2) \|F^i(x) - F^{i-1}(x)\|$$
$$+ (a_2 + a_5 + a_6) \|F^i(x) - F^{i+1}(x)\|$$
$$+ (a_3 + a_4) \|F^{i+1}(x) - F^{i-1}(x)\|.$$

Therefore by (5),

$$(1 - a_2 - a_5 - a_6) \|F^i(x) - F^{i+1}(x)\|$$
$$\leqslant (a_1 + a_2 + 2\alpha(a_3 + a_4)) \|F^i(x) - F^{i-1}(x)\|.$$

Hence $(1-a_2-a_5-a_6)\, d \leqslant (a_1+a_2+2\alpha(a_3+a_4))\,(d+\varepsilon)$.

Since $\varepsilon > 0$ is arbitrary, we have

$$(1-a_2-a_5-a_6)\, d \leqslant (a_1+a_2+2\alpha\,(a_3+a_4))\, d.$$

his implies

$$1-a_2-a_5-a_6 \leqslant a_1+a_2+2\alpha\,(a_3+a_4)$$

which, in view of $\alpha < 1$, contradicts (2).

Case II. $a_3=a_4=0$. Note that $a_7=a_4=0$. Let
$m=(F^i(x)+F^{i+1}(x))/2$. Then by (1),

$$\| m-F(m) \| \leqslant 2^{-1} \| F^i(x)-F(m) \| + 2^{-1} \| F^{i+1}(x)-F(m) \|$$

$$\leqslant 2^{-1}(a_1 \| F^{i-1}(x)-m \| + a_3(\| F^{i-1}(x)-F^i(x) \| + \| m-F(m) \|)$$

$$+ a_5 \| F^i(x)-F^{i+1}(x) \| + a_6 \| m-F^{i+1}(x) \|)$$

$$+ 2^{-1}(a_1 \| F^i(x)-m \| + a_2 (\| F^i(x)-F^{i+1}(x) \| + \| m-F(m) \|)$$

$$+ a_5 \| F^{i+1}(x)-F^{i+2}(x) \| + a_6 \| m-F^{i+2}(x) \|)$$

that is

$$(1-a_2) \| m-F(m) \| \leqslant 2^{-1}(a_1 (\| F^{i-1}(x)-m \| + \| F^i(x)-m \|)$$

$$+ a_6 (\| m-F^{i+1}(x) \| + \| m-F^{i+2}(x) \|)$$

$$+ a_2 (\| F^{i-1}(x)-F^i(x) \| + \| F^i(x)-F^{i+1}(x) \|)$$

$$+ a_5 (\| F^i(x)-F^{i+1}(x) \| + \| F^{i+1}(x)-F^{i+2}(x) \|)$$

$$\leqslant 4^{-1}(a_1(\| F^{i-1}(x)-F^i(x) \| + \| F^{i-1}(x)-F^{i+1}(x) \| + \| F^i(x)\ F^{i+1}(x) \|)$$

$$+ a_6(\| F^i(x)-F^{i+1}(x) \| + \| F^{i+1}(x)-F^{i+2}(x) \| + \| F^i(x)-F^{i+2}(x) \|)$$

$$+ 2^{-1}(a_2(\| F^{i-1}(x)-F^i(x) \| + \| F^i(x)-F^{i+1}(x) \|)$$

$$+ a_5(\| F^i(x)-F^{i+1}(x) \| + \| F^{i+1}(x)-F^{i+2}(x) \|)).$$

Hence applying (5) and (6) we have

$$(1-a_2)\, d \leqslant (1/4)\,(a_1+2\alpha\,a_1+a_1+a_6+a_6+2\alpha\,a_6)\,(d+\varepsilon)$$

$$+ (1/2)\,(a_2+a_2+a_5+a_5)\,(d+\varepsilon).$$

Letting $\varepsilon \to 0$ we obtain

$$(1-a_2)\, d \leqslant ((1+\alpha)a_1/2 + a_2 + a_5 + (1+\alpha)\, a_6/2)\, d$$

that is, if a_6 or $a_1 \neq 0$,

$$1-a_2 < a_1 + a_2 + a_5 + a_6$$

which contradicts (2).

    Case III. $a_3 = a_4 = a_7 = a_6 = a_1 = a_5 = 0$.

In view of Soordi's result [4] this case need not be considered. Thus the assertion is proved.

    Now for $\varepsilon \in (0, 1)$ assume

$$C\varepsilon = \{x : \| x - F(x) \| \leqslant \varepsilon\} \text{ and } D\varepsilon = \{x \in C\varepsilon : \| x \| \leqslant \bar{a} + \varepsilon\}$$

where $\bar{a} = \lim\limits_{\varepsilon \to 0} a\,(C\varepsilon)$ and $a\,(C\varepsilon) = \inf\,\{\|x\| : x \in C\varepsilon\}$.

Since F is continuous, the sets $C\varepsilon$ and (so) $D\varepsilon$ are closed and by the assertion they are nonvoid. The proof is completed by showing

$$\bigcap_{\varepsilon > 0} C\varepsilon \neq \phi.$$

    If $\bar{a} = 0$ then $0 \in \bigcap\limits_{\varepsilon > 0} C\varepsilon \neq \phi$ and we are finished. Therefore we

may assume $\bar{a} > 0$. Letting $u_1, u_2 \in C\varepsilon$ we have by (1) and the triangle inequality, for $i = 1, 2$,

$$(7) \quad \|u_i - F((u_1+u_2)/2)\| \leqslant \|u_1 - F(u_1)\| + \|F(u_1) - F((u_1+u_2)/2)\|$$

$$\leqslant \varepsilon + a_1\|u_1 - (u_1+u_2)/2\| + a_2(\|u_1 - F(u_1)\| + \|(u_1+u_2)/2$$
$$- F((u_1+u_2)/2)\|) + a_3(\|u_1 - F((u_1+u_2)/2)\|$$
$$+ \|(u_1+u_2)/2 - u_1\| + \|u_1 - F(u_1)\|) + a_4(\|u_1 - F(u_1)\|$$
$$+ \|F(u_1) - F^2(u_1)\|) + a_5\|F(u_1) - F^2(u_1)\|$$
$$+ a_6(\|(u_1+u_2)/2 - u_1\| + \|u_1 - F(u_1)\| + \|F(u_1) - F^2(u_1)\|)$$
$$+ a_7(\|F((u_1+u_2)/2) - u_1\| + \|u_1 - F(u_1)\| + \|F(u_1) - F^2(u_1)\|).$$

We note that by (3),

$$\| F(u_1) - F^2(u_1) \| \leqslant \| u_1 - F(u_1) \| \leqslant \varepsilon.$$

Also since

$$\| (u_1+u_2)/2 - F((u_1+u_2)/2) \| \leqslant \max_{i=1,2} \| u_1 - F((u_1+u_2)/2) \|,$$

we obtain

$$(1-a_2-a_3-a_7) \max_{i=1,2} \| u_1 - F((u_1+u_2)/2) \|$$

$$\leqslant (1+a_2+a_3+2a_4+a_5+2a_6+2a_7)\,\varepsilon$$

$$+(a_1+a_3+a_6)\, \| u_1-u_2 \|/2$$

so $\| u_1 - F((u_1+u_2)/2) \leqslant \alpha\,\varepsilon + \beta\, \| u_1-u_2 \|/2$

where $\alpha = (1+a_2+a_3+2a_4+a_5+2a_6+2a_7)/(1-a_2-a_3-a_7)$

and $\beta = (a_1+a_3+a_6)/(1-a_2-a_3-a_7)$.

By (2), $\beta \leqslant 1$. Therefore

$$\| u_1 - F((u_1+u_2)/2) \| \leqslant \alpha\,\varepsilon + \| u_1-u_2 \|/2.$$

Now the rest part of the proof follows as in [2].

We remark that if a_2 or a_4 or $a_5 \neq 0$ in (2) then the fixed point is unique.

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