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CONTENTS

ON TWO DIMENSIONAL FAITHFUL REPRESENTATIONS OF THE GROUP OF TREFOIL KNOT

By

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(Dedicated to Professor B.H. Neumann on his Seventieth Birthday.)

In this paper we determine faithful representations of the group

$$T = < a, b \; : \; ba^2 ba^{-1} b^{-1} a^{-1} = 1 >$$

of Trefoil Knot (c.f. [2]) in GL (2, C) and show that T has no faithful representation in SL (2, C).

All notations and terms are standard and can be found in [1]; Z, R and C respectively denote the sets of integers, reals and of complex numbers.

We need the following lemmas :

Lemma 1. [2].

The matrices

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

generate the modular group SL (2, Z) which is the generalised free product of the groups $A = < a : a^2 = -I >$, $B = < b : b^3 = -I >$ amalgamating $H = \{ \pm I \}$.

Lemma 2.

Let $a, b \in SL(2, C)$ be such that $a^m = -I = b^n$ and $< a, b >$ be the generalised free product of $< a >$ and $< b >$ amalgamating

$H = \{\pm I\}$. For any $0 \neq \alpha \in C$, which is not a root of unity, take $\beta = \alpha^{m/n}$. If

$$A_\alpha = <\alpha a>, \quad B_\beta = <\beta b>$$

then $<A_\alpha, B_\beta>$ is the generalised free product of A_α, B_β amalgamating

$$H_{\alpha\beta} = < -\alpha^m I = -\beta^n I >$$

Proof

Since $G = <a, b>$ is the generalised free product of $A = <a : a^m = -I>$, $B = <b : b^n = -I>$ amalgamating $H = \{\pm I\}$, each element of G is uniquely of the form

$$w = a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_k} b^{\beta_k} \tag{2}$$

where $o \leqslant \alpha_i < m$, $o \leqslant \beta_j < n$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$. The only non-trivial relations in G are $a^m = -I$, $b^n = -I$ and their consequence namely $a^m b^{-n} = I$. So for $k \geqslant 1$, $w \neq I$. Also

$$(\alpha a)^m \cdot (\beta b)^{-n} = \alpha^m \beta^{-n} a^m b^{-n} = I,$$

which is consequence of

$$(\alpha a)^m = -\alpha^m I = -\beta^n I = (\beta b)^n,$$

is a non-trivial relation in $<A_\alpha, B_\alpha>$. Let

$$w' = (\alpha a)^{\alpha_1} \cdot (\beta b)^{\beta_1} \dots (\alpha a)^{\alpha_k} (\beta b)^{\beta_k}$$
$$= \alpha^p \beta^q a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_k} b^{\beta_k} = I,$$

$k \geqslant 1$, be a non-trivial relation in $<A_\alpha, B_\beta>$, $o \leqslant \alpha_i < m$ $o \leqslant \beta_j < n$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$. Then

$$w = a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_k} b^{\beta_k} = \alpha^{-p} \beta^{-q} I$$

is an element of $G \subseteq SL(2, C)$. Hence

$\det w = \alpha^{-2p} \cdot \beta^{-2q} = 1$

so that $(\alpha\,\beta)^{pq} = \pm 1$. Therefore $\alpha\,\beta$ is a root of unity. But then $(\alpha\,\beta)^n = \alpha^{m+n}$ s a root of un'ty, a contradiction. Hence $w' = I$ is not a non-trivial relation in $< A_\alpha , B_\beta >$. Thus $< A_\alpha , B_\beta >$ is the generalised free product of A_α , B_β amalgamating $H_{\alpha\beta} = A_\alpha \cap B_\beta$ $= < -\alpha^m I = -\beta^n I >$

Next we prove the main theorem of the paper.

Theorem 1

The group T of trefoil knot has a faithful matrix representation in GL (2, C).

Proof

The group of trefoil knot has a presentation

$$T = < a , b = ba^2 \quad ba^{-1} \quad b^{-1} \quad a^{-1} = 1 >$$

Now $ba^2\, ba^{-1}\, b^{-1}\, a^{-1} = 1$ implies $ba^2b = aba = u$ (say)

and $u^2 = aba^2\, ba = aua$ so that $u^3 = (au)^2$

Put $au = v$. Then

$$T = < u , v : u^3 = v^2 >$$

which is the free preouct of $U = < u >$ and $V = < v >$ amalgamating $W = < u^3 = v^2 >$. Consider now the matrices a and b given in (1). Then $< a, b >$ is generalised free product of $A = < a : a^2 = -I >$ and $B = < b : b^3 = -I >$ amalgamating $H = \{\pm I\}$. For any $o \neq \alpha \in C$ which is not an nth root of unity for any n, take $\beta = \alpha^{2/3}$ Then $A_\alpha = < \alpha\, a >$, $B_\beta = < \beta\, b >$ are such that

$$\alpha^2\, a^2 = -\,\alpha^2 I = -\beta^3 I = \beta^3\, b^3.$$

so, by lemma 2, $< A_\alpha , B_\beta >$ is the generalised free product of A_α , B_β amalgamating $H_{\alpha\beta} = A_\alpha \cap B_\beta = < -\alpha^2 I = -\beta^3 I >$

$= < \alpha^2 a^2 = \beta^3 b^3 >$ and is isomorphic to T. Hence $< A_\alpha , B_\beta >$ is a faithful representation of T.

It is clear that the faithful representation of T obtained above is in GL (2, C). One can ask whether T has a faithful representation in SL (2, C). We answer this question in the negative by proving the following theorem.

Theorem 2

The group T of trefoil knot has no faithful representation in SL (2, C).

Proof :

T is a two generater knot group. We take T in the form

$$T = < u, v = v^2 = u^3 >.$$

By a theorem of B.H. Neumann [5], T being the generalised free product of torsion free groups is torsion free. Suppose that $\phi : T \to SL(2, C)$ is a faithful representation of T such that

$$\phi(v) = a, \quad \phi(u) = b.$$

So that $a^2 = b^3$ in $< a, b >$ and a, b are matrices in SL (2, C) having infinite order. $< a, b >$ is irreducible because reducible subgroups of SL (2, C) are abelian or at most metabelian and $< a, b >$ contains free subgroups of rank 2 because T does. By corollary II.1.4 [3] there is a matrix c such that

$$cac^{-1} = a' = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \quad c\,b\,c^{-1} = b' = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

with $a'^2 = b'^3$ and $< a', b' >$ isomorphic to $< a, b >$. But then

$$a'^2 = \begin{pmatrix} \lambda^2 & 0 \\ \xi(\lambda+\lambda^{-1}) & \lambda^{-2} \end{pmatrix}, \quad b'^3 = \begin{pmatrix} \mu^3 & \eta(\mu^2+1+\mu^{-2}) \\ 0 & \mu^{-3} \end{pmatrix}$$

so that $a'^2 = b'^3$ implies

$$\xi(\lambda + \lambda^{-1}) = 0 \quad \text{and} \quad \eta(\mu^2 + 1 + \mu^{-2}) = 0.$$

If $\lambda + \lambda^{-1} = 0$ then $a'^2 = -I$ so that a' has finite order. But then a has finite order, a contradiction, because $< a , b >$ is torsion free. If $\xi = 0$, then $< a' , b' >$ and hence $< a , b >$ is reducible, again a contradiction. Hence $< a , b >$ is not isomorphic to $T = < u , v >$. Thus T has no faithful representation in SL (2, C).

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COMPACTNESS OF POSITIVE MAPS MAJORIZED BY COMPACT MAPS

By

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Conditions were given in (1) under which K (x, y), the space of compact maps between two ordered Banach spaces X and Y, is an order ideal in L (x, y) the space of linear maps between X and Y. It was shown there that in such cases a positive map in L (x, y) majorized by a compact map, is itself compact. In this paper a similar problem is discussed. Some cases are considered where the above is still true but additional conditions are needed on the majorizing map and the space involved.

Let X be an ordered Banach space with closed positive cone X_+. A subset A of X is called *order convex* if $a \leqslant c \leqslant b$ with $a, b \in A$, implies that $c \in A$. The smallest order convex set containing a subset B is called the *order convex cover* of B and is denoted by [B]. In fact $[B] = (B + X_+) \cap (B - X_+)$. X is α-*normal* if $x \leqslant z \leqslant y$ implies that $\| z \| \leqslant \alpha \max \{ \| x \|, \| y \| \}$. Equivalently X is α-normal if $[U] \subseteq \alpha \,.\, U$ where U is the unit ball in X. X_+ is α-*generating* iff for each $x \in X$ there exist $u, v \in X_+$ with $x = u - v$ and $\| u \| + \| v \| \leqslant \alpha . \| x \|$. X is *directed* upwards if for $x, y \in X$ there is $z \geqslant x, y$.

X is said te be a *base normed* space if there is a convex subset B of X such that for $x \in X_+$, $x \neq 0$ there is a unique positive number $f(x)$ with $x/f(x) \in B$, and the Minkowski functional of Co $(-B \cup B)$ defines the norm on X. B is called the *base* of X_+. An *approximate order unit*

X is an upward directed set $\{e_\lambda : \lambda \in \Lambda\}$ in X such that for each $x \in X$, their exist $\delta \in \Lambda$ and $\alpha > O$ with $-\alpha e_\delta \leqslant x \leqslant \alpha e_\delta$. If the Minkowski functional of $\{x : \text{there exists } \lambda \in \Lambda \text{ with } -e_\lambda \leqslant x \leqslant e_\lambda\}$ is a norm, then X with this norm is called *approximate order unit* (*a.o.u.*) *normed.*

A linear map $T : X \rightarrow Y$ is called *positive* if $Tx \in Y_+$ whenever $x \in X_+$.

LEMMA

Let X and Y be ordered Banach spaces and T be a linear map on X into Y which maps X_+ onto Y_+. Then T (A) is order convex in Y whenever A is order convex in X.

Proof

Let A be an order convex subset of X and $u, w \in T(A)$. There exist $x, y \in A$ with $Tx = u$ and $Ty = w$.

Let $u \leqslant v \leqslant w$ where $v \in Y$. Then $o \leqslant v - u \leqslant w-u$ and by hypothesis there are $a, b \in X_+$ such that $Ta = v-u$ and $Tb = w-u$; i.e. $o \leqslant Ta \leqslant Tb$. Thus $u \leqslant Ta + u \leqslant Tb + u$ i.e. $u \leqslant T(a+x) \leqslant w$. We obtain $Tx \leqslant T(a + x) \leqslant Ty$ and $v = T(a + x)$. This implies that $x \leqslant a + x \leqslant y$. But A is order convex and therefore $a + x \in A$ i. e. $v \in T(A)$. Thus T (A) is order convex.

Theorem

Let X and Y be ordered Banach spaces, $T \in L(X, Y)$, $S \in K(X, Y)$ and $o \leqslant T \leqslant S$. If S maps order convex sets in X onto order convex sets in Y then $T \in K(X, Y)$ provided one of the following conditions is satisfied :

(*a*) X is approximate order unit normed ;

(*b*) X is base normed ;

(*c*) X is a Banach lattice ;

(*d*) X is 1-normal and X_+ is 1-generating.

(*e*) X is 1-normal and the open unit ball in X is directed upwards.

Proof

(*a*) Since X is *a.o.u.* normed, X is 1-normal and V the open unit ball in X is directed upwards [4 : Lemma 2]. Therefore $[V] \subseteq 1. V \subseteq [V]$ which implies that V is order convex.

Let $y \in T[V]$, then $y = Tx$ for some $x \in V$. As V is directed there is $u \in V_+$ with $x, -x \leqslant u$ and hence $T(-u) \leqslant Tx \leqslant Tu$. Thus $S(-u) \leqslant Tx \leqslant Su$. But S (V) is order convex by hypothesis and therefore $y = Tx \in S(V)$; i.e. T (V) is a subset of S (V).

Now the compactness of S implies that T is a compact map.

(*b*) Let B denote the base of X_+ and U be the closed unit ball in X. Then $U = Co(-B \cup B)$.

Since X_+ is 2-normal [4-Lemma 1], $[U] \subseteq 2 . U$ and therefore $M = S[U] \subseteq 2 . S(U)$. This shows that M is relatively compact. We also note that M is order convex.

Let $b \in B$, then $Tb \leqslant Sb$ and $T(-b) \geqslant S(-b)$ i.e. $S(-b) \leqslant T(\pm b) \leqslant Sb$. Since $\pm B \subseteq U$, we see that $S(\pm b) \in M$ and therefore order convexity of M implies that $T(\pm b) \in M$.

Next let $y \in T(U)$; $y = Tx$, $x \in U$ and $x = \lambda b - \lambda' b'$ for $b, b' \in B$ and $o \leqslant \lambda, \lambda' \leqslant 1$. Then $-b' \leqslant x \leqslant b$ and $T(-b') \leqslant Tx \leqslant Tb$. But Tb and $T(-b')$ belong to M which implies that $Tx \in M$ i.e. $T(U) \subseteq M$. Thus T(U) is relatively compact and therefore $T \in K(X, Y)$.

(*c*) Let U be the closed unit ball in X as in (*b*). Since X is a Banach lattice, X_+ is 2-normal and $[U] \subseteq 2 . U$ [3 : pg. 153].

Let $M = S[U]$. Then $M \subseteq 2 . S(U)$ and therefore M is relatively compact. It is also order convex.

If $z \in U_+$ then $o \leqslant Tz \leqslant Sz$ so that $Tz \in M$. Similarly $T(-z) \in M$.

Now let $y \in T(U)$ and $y = Tx$ for some $x \in U$. Then $x = x^+ - x^-$ and $\|x^+\|, \|x^-\| \leqslant 1$; i.e. $^+$, $xx^- \in U_+$. As $-x^- \leqslant x \leqslant x^+$ we have $T(-x^-) \leqslant Tx \leqslant Tx^+$. But $T(-x^-)$, $T(x^+) \in M$ which implies that $y = Tx \in M$. Thus $T(U) \subseteq M$ and $T \in K(X, Y)$.

(*d*) Proof as in part (*b*).

(*e*) Proof as in part (*a*).

Corollary

Let $T \in L(X, Y)$ and $o \leqslant T \leqslant W$ where W is a w-compact map on X into Y. If X satisfies one of the conditions in theorem and W maps order convex subsets onto order convex subsets then T is also w-compact map.

Proof

The Proof is similar to that of the theorem.

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ON A STONE-WEIERSTRASS THEOREM FOR VECTOR-VALUED FUNCTIONS

By

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Let X be a topological space, E a topological vector space with a base W for closed balanced neighbourhoods of o and C (X, E) the vector space of all bounded continuous E-valued functions on X. Let C_o (X, E) be the subspace of C (X, E) consisting of those functions which 'vanish at infinity' ; that is, if $f \in C_o$ (X, E), then, for any $w \in$ W, the set $\{x \in X : f(x)$ does not belong to $w\}$ is compact in X. When E is the real or complex field, these spaces are denoted by C (X) and C_o (X). If X is a compact Hausdroff space, then clearly C_o (X, E) = C (X, E). We shall denote by C (X) $\otimes$ E the vector space spanned by the set of all functions of the form $\phi \otimes a$, where $\phi \in$ C (X) $a \in$ E, and $(\phi \otimes a)(x) = \phi(x)\, a$ $(x \in X)$. The *uniform topology* σ on C (X, E) is the linear topology which has a base of neighbourhoods of o consisting of all sets of the form :

$$N(o, w) = \{ f \in C(X, B) : f(x) \in w \text{ for all } x \in X \}$$

where w varies over W.

In this paper we establish a Stone-Weierstress type theorem for C_o (X, E) which extends the results of Buck [1] and Shuchat [4].

We begin with the following definition.

Definition ([3], p. 9) Let U be a collection of subsets of a topological space X. For any $x \in$ X, we define ord_x U, the order of U at x, as the number of members of U which contain x ; ord_x U = sup $\{\text{ord}_x \text{U}\}$, $x \in$ X. The *covering dimension* of X is defined as the least positive integer n such that every finite open covering of X has a open refinement of order $\leqslant n+1$. If no such finite n exists, then we say that X has an infinite covering dimension.

Theorem 1

Let X be a locally compact Hausdorff space of finite covering dimension and E a Hausdorff topological vector space. Let A be a $C(X)$-submodule of $C_0(X, E)$, and let $f \in C_0(X, E)$. Then the following are equivalent :

(i) f belongs to the σ-closure of A ;

(ii) for each $x \in X$, $f(x)$ belongs to the closure of $A(x) = \{g(x) : g \in A\}$.

Proof

(i) implies *(ii)*. Suppose that f belongs to the σ-closure of A, and let x be any point in X. Let $\{f_\alpha\}$ be a net in A such that $f_\alpha \xrightarrow{\sigma} f$. Then, in particular, $f_\alpha(x) \to f(x)$ in E. Since $\{f_\alpha(x)\} \subseteq A(x)$, it follows that

$f(x) \in \overline{A(x)}$.

(ii) implies *(i)*. Suppose that, for each $x \in X$, $f(x) \in \overline{A(x)}$. Suppose X has a covering demension of order n, and let $w \in W$. We show that there exists a function g in A such that $g - f \in N(o, W)$. Choose a V such that $V + V \ldots. + V$ $(n+2)$-terms) $\subseteq W$. Since $f \in C_0(X, E)$, there exists a compact set K in X such that $f(x) \in V$ if $x \in X—K$. It follows from *(ii)* that, for each $x \in X$, we can choose a function g_x in A such that $g_x(x) - f(x) \in V$. Now $g_x - f$ is continuous, and so there exists an open neighbourhood $U(x)$ of x in X such that

$$g_x(y) - f(y) \in V \text{ for all } y \in U(x).$$

Since K is compact, the open covering $\{U(x) : x \in K\}$ of K has a finite subcovering, $\{U(x_i) : i = 1, \ldots, m\}$ say. The collection $U = \{X—K, U(x_i) : i = 1, \ldots, m\}$ form a finite open covering of X, and so, by hypothesis, there exists an open refinement V of order $\leqslant n+1$. Choose a finite number of members $U_1, U_2, \ldots, U_r$ (say) of V which cover K. Moreover, for each $1 \leqslant j \leqslant r$, there exists a i_j, $1 \leqslant i_j \leqslant m$, such that $U_j \leqslant U(x_{i_j})$. Let $\{\phi_j : j = 1, \ldots, r\}$ be a collection of functions in C (X) such that $o \leqslant \phi_j \leqslant 1$, $\phi_j = o$ outside of U_j, $\sum_{j=1}^{r} \phi_j(x) = 1$ for $x \in K$, and $\sum_{j=1}^{r} \phi_j(x) \leqslant 1$ for $x \in X$ ([2], p. 69, Lemma 2).

Let g be an E-valued function on X defined by

$$g(x) = \sum_{j=1}^{r} \phi_j(x)\, g_{x_{i_j}}(x),$$

where $g_{x_{i_j}}$'s are the functions in A choosen earlier. Then $g \in A$ since A is a C (X)-submodule. Let y be any point in X. If $y \in K$, then

$$g(y) - f(y) = \sum_{j=1}^{r} \phi_j(y) \left(g_{x_{i_j}}(y) - f(y)\right) \in \sum_{j=1}^{r} \phi_j(y)\, V \subseteq w.$$

If $y \in X$—K, then

$$g(y) - f(y) = \sum_{j=x} \phi_j(y) \left(g_{x_{i_j}}(y) - f(y)\right) + \left(\sum_{j=1}^{r} \phi_j(y) - 1\right) f(y)$$

$$\subseteq V + V + \ldots + V \text{ (at most } (n+1)\text{-times)} + V \subseteq w.$$

Thus $g - f \in N(o, w)$, and so it follows that f belongs to the σ-closure of A.

Corollary 2

Let X and E be given as in the theorem, and let A be a C (X)-submodule of $C_o(X, E)$ such that, for each $x \in X$, $A(x)$ is dense in E. Then A is σ-dense in $C_o(X, E)$.

Proof

Let $f \in C_0(X, E)$. It follows from the hypothesis that, for each $x \in X$, $f(x) \in \overline{A(x)}$. Hence, by the theorem, f belongs to the σ-closure of A, and so A is σ-dense in $C_0(X, E)$.

Corollary 3

Let X and E be as given in the theorem. Then $C_0(X) \otimes E$ is σ-dense in $C_0(X, E)$.

Proof

Since X is locally compact, it is easy to see that, for each $x \in X$, $(C_0(X) \otimes E(x) = E$. Hence, by Corollary 2, $C_0(X) \otimes E$ is σ-dense is $C_0(X, E)$.

Remark

If E is assumed to be locally convex (with a base W for closed balanced 'convex' neighbourhoods of o) then the above results hold without restricting X to have a finite covering dimension.

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ERROR ESTIMATES FOR THE FINITE ELEMENT APPROXIMATION OF VARIATIONAL INEQUALITIES

BY

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Abstract : For the piecewise linear and conforming elements, we prove that the error estimate for the finite element approximations of mildly nonlinear variational inequalities is of order h in the energy norm.

1. Introduction

Variational concept play a fundamental role in the theory of partial differential equations. Variational formulations can serve not only to unify diverse fields, but also to suggest new theories. Variational methods are usually used for approximation. Recently variational theory has been enriched by the development of the theory of variational inequalities. Stampacchia [15] has shown the equivalence of the weak and the variational formulations of linear elliptic boundary value problems in the constrained case. Since then it has been shown that the theory of variational inequalities has had a significant impact in the theory of partial differential equations, mechanics contact problems, optimal control systems, convex programming, and many other branches of mathematical and engineering sciences, see for example :

Lions [7], Fichera [5], Noor [10, 13],

and many other research workers.

In this paper, we derive the *error estimates for the finite element approximations of mildly nonlinear elliptic boundary value problems having auxiliary constraint conditions.* A much used approach with any elliptic problem is to reformulate it in a weak or variational form and to approximate these. When a constraint is present, such approach leads to a variational inequality, which is the weak formulation, see Noor [12]. An approximate formulation of the variational inequality is then defined, and the error estimates involving the difference between the solution of the exact and the approximate formulation in the W_2^1—norm is obtained which is in fact of order h. This result is an extension of that obtained by Falk [4] and Mosco and Stang [9] for the constrained linear problems.

The general and basic theory of mildly nonlinear variational inequalities has been studied by Noor [10], where one finds the inequalities bounding the error in the approximation and the convergence theorems regarding the internal approximation of these inequalities. Also for related results on variational inequalities, see Janovsky and Whiteman [6] and Noor [11].

2. Preliminaries

We are concerned with the numerical solutions of nonlinear problems of the type :

$$\left.\begin{aligned} -\Delta u(x) &= f(x, u), & x \in \Omega \\ u(x) &= 0, & x \in \partial\Omega \end{aligned}\right\} \tag{1}$$

Where Ω is a simply connected open domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ and its closure $\bar{\Omega} \equiv \Omega \cup \partial\Omega$, $f(u) = f(x, u(x))$, is a nonlinear function of x and u. It is assumed that the boundary $\partial\Omega$ and $f(u)$ are smooth enough to ensure the existence and uniqueness of the solution u of (1). We study this problem in the usual sobler space $W_2^1(\Omega) \equiv H^1$, the space of functions which together with their generalized derivatives of order one are in $L_2(\Omega)$, The subspace of functions from H^1, which in a generalized sense satisfy the homogeneous boundary conditions on $\partial\,\Omega$ is $\overset{0}{W}{}_2^1\,(\Omega) \equiv H_0^1$.

It has been shown by Tonti [17] that in its direct variational formulation, (1) is equivalent to finding $u \in H_0^1$, such that

$$I\,[u] \leqslant I\,[v], \quad \text{for all } v \in H_0^1,$$

where

$$I\,[v] = \int_\Omega \left\{ \left(\frac{\partial\, v}{\partial\, x}\right)^2 - 2\int_0^v f(\eta)\, d\eta \right\} d\,\Omega \equiv a\,(v, v)$$
$$-2F\,(v), \tag{2}$$

is the energy functional associated with (1).

We here consider the case, when the solution u of (1) is required to satisfy the condition $u \geqslant \psi$ where ψ is a given function on Ω. In this situation, our problem is to find

$$u \in K \overset{def}{\equiv} \{v\ ;\ v \in H_0^1,\ v \geqslant \psi \ on\ \Omega\},$$

is a closed convex subset of H_0^1, see Mosco [8], such that u minimizes $I\,[v]$ on K. It has been shown by Noor [10, 12] that the minimum of $I\,[v]$ on K can be characterized by a class of variational inequalities.

$$a\,(u,\, v-u) \geqslant \langle F'\,(u),\, v-u \rangle, \quad \text{for all } v \in K, \tag{3}$$

where $F'\,(u)$ is thec Frechet differential of $F(u)$ and is in fact, see [14],

$$\langle F'\,(u),\, v \rangle = \int_\Omega f\,(u)\; v\; d\,\Omega, \tag{4}$$

and the pairing $< -\Delta u, v >$ after integration by parts gives the bilinear forms

$$a(u, v) = \int_{\Omega} \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} \right) d\Omega$$

Concerning the regularity of $u \in K$, we assume the following hypothesis ;

(A) : { For $f\, L_2(\Omega)$, $\psi \in H_0^1 \cap H^2$, $u \in K$ satisfying (4) also lies in H^2 }.

3. Main Result

We assume that Ω is a pologonal domain of $\mathbb{R}^2$. Let $\{T_h\}_{h>0}$ be a regular family, see Ciarlet [3], of triangulation of Ω and define :

$$S_h = \{v_h : v_h \in C^\circ(\Omega),\ v_h \big|_{\partial\Omega} = 0,\quad v_h \big|_{T} \in P_1 \text{ for all } T \in T_h\}$$

where P_1 is a set of all polynomials on $\mathbb{R}^2$ of degree $\leqslant 1$. Clearly S_h is a finite dimensional subspace of H_0^1. The set K_h is defined as :

$$K_h = \{v_h \in S_h : v_h \geqslant \psi \quad \text{at every vertex of triangulations } T_h\}.$$

It is obvious that K_h is a closed convex subset of S_h. In this paper, we consider the case $K_h = K \cap S_h$, for other choices, see Falk [4], Noor [11], and Janovsky and Whiteman [6].

The approximate problem is defined by ;

$$a(u_h, v_h - u_h) \geqslant < F'(u_h), v_h - u_h >, \text{ for all } v_h \in K_h. \tag{5}$$

where

$$< F'(u_h), v_h > = \int_{\Omega} f(u_h)\, v_h\, d\Omega. \tag{6}$$

We now state and prove the main result of this paper, which shows that error estimate $u - u_h$ is of order h.

Theorem 1

Let $u \in K$ and $u_h \in K_h$ be respectively solutions of (3) and (5). If F' is antimonotone and the hypothesis (A) holds, then

$$\| n - u_h \| = O(h).$$

Proof

Since $u_h \in K_h \subset K$, it follows that

$$a(u, u - u_h) \leqslant < F'(u), u - u_h >$$

and

$$a(u_h, u_h - v_h) \leqslant < F'(u_h), u_h - v_h >, \text{ for all } v_h \in K_h.$$

Adding these inequalities and rearranging terms, we get

$$a(u - u_h, u - u_h) \leqslant a(u_h, v_h - u) + < F'(u) - F'(u_h), u - u_h > + < F'(u_h), u - v_h >.$$

$\leqslant a(u_h, v_h - u) + < F'(u_h) u - v_h >$ by the antimonotonicity of F'. Thus we have :

$$a(u - u_h, u - u_h) \leqslant a(u - u_h, u - v_h) + a(u, v_h - u) + < F'(u_h), u - v_h >$$

In case of problems (1), the above inequality can be written as follows :

$$a(u - u_h, u - u_h) \leqslant a(u - u_h, u - v_h) + a(u, v_h - u) + \int_\Omega f(u_h)(u - v_h)\, d\Omega$$

Since by hypotheses (A), $u \in H^2$, it is possible to integrate by parts so that :

$$a(u - u_h, u - u_h) \leqslant a(u - u_h, u - v_h) + \int_\Omega \{-\Delta u - f(u_h)\}(v_h - u)\, d\Omega \tag{7}$$

from which it follows that

$$\| u - u_h \|^2_{H^1_0} \leqslant \| v_h - u \|^2_{H^1_0} + \left\{ \| \Delta u \|_{L_2(\Omega)} + \| f(u_h) \|_{L_2(\Omega)} \right\} \times$$

$$\times \| v_k - u \|_{L_2(\Omega)} \leqslant \| v_h - u \|^2_{H^2_0} + C \| v_h - u \|_{L_2(\Omega)},$$

see [13]. (8)

Let I_h be the operator of S_h – interpolation. Then, since $\Omega \subset \mathcal{R}^2$, we have $H^2 \subseteq C^0(\Omega)$ and $u \in H^2, u \geqslant \psi$ on Ω imply that $I_h u \in K_h$. Taking $v_h = I_h u$ in (8), we have

$$\| u-u_h \|^2_{H^1_0} \leqslant \| I_h u-u \|^2_{H^1_0} + C \| I_h u-u \|_{L_2(\Omega)} \tag{9}$$

Since $u \in H^2$, it follows from the strandred approximation theory results, see Ciarlet [3] and Strang and Fix [16] that

$$\| I_h u-u \|_{r, \Omega} \leqslant C_1 h^{2-r} \| u \|_{2, \Omega}, \quad (r=0, 1) \tag{10}$$

where C's are constant independent of h and u. Thus form (9) and (10), we obtain

$$\| u - u_h \| = O(h),$$

the required estimate.

Remark 1

We also note that form K = H, we have the following error bound

$$\| u-u_h \| \leqslant C_2 \| u - v_h \|$$

a well known result for mildly nonlinear elliptic boundary value problems without obstacle, see Noor and Whiteman [14].

Remark 2

For the linear variational inequalities involving the abstacle problems, using the quardatic elements, Brezzi and Sacchi [2] have proved the $O(h^{3/2-\epsilon})$ convergence. We here conjecture that for the mildly nonlinear variational inequalities, the error estimate for $u-u_h$ would be $O(h^{3/2})$ as in the linear case.

The problems of deriving the L_2 and L_∞ error estimates for the mildly nonlinear problems having constraint conditions are still open.

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GENERALIZATION OF THE HORVITZ AND THOMPSON ESTIMATOR

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Summary

In this paper a general theory of sampling with unequal probability is presented which allows population units to appear more than once in sample. The only condition which is imposed on the selection procedure is that the total number of appearances in sample is fixed. Selection with replacement (multinomial sampling) and without replacement are special cases of this. Two possible variance estimators are presented which may be used in both single stage and multi-stage sample designs. The application of this general theory is illustrated by a numerical example.

1. Introduction

Hansen and Hurwitz (1943) developed a theory for multinomial sampling 'sampling with replacement'. The variance of their unbiased estimator $y'_{\mu\mu} = \frac{1}{n} \Sigma \frac{y_i}{p_i}$ of population total Y is

$$(1) \quad V\left(y'_{HH}\right) = \frac{1}{n}\left(\sum_{I=1}^{N} \frac{Y_I^2}{P_1} - Y^2\right).$$

An unbiased variance estimator of (1) is

$$(2)\quad v\left(y'_{HH}\right) = \frac{1}{n(n-1)} \sum_{i=1}^{N} \left(\frac{y_i}{p_i} - y'_{HH}\right)^2,$$

where p_i is the probability of selection of the ith unit to be in the sample and Y_i is the estimand variable.

A general theory of sampling with unequal probabilities without replacement was also given by Horvitz and Thompson (1952). Their unbiased estimator of population total Y is

$$(3)\quad y'_{HT} = \sum_{i=1}^{n} \frac{y_i}{\pi_i},$$

where π_i is the a priori probability of inclusion in sample of the ith unit in that sample. The variance of y'_{HT} is

$$(4)\quad V\left(y'_{HT}\right) = \sum_{I=1}^{N} \frac{Y_I^2}{\pi_I} + \sum_{\substack{I,J=1 \\ J \neq I}}^{N} \sum \pi_{IJ} \frac{Y_I}{\pi_I} \frac{Y_J}{\pi_J} - Y^2$$

For fixed n, the following variance formula was given by Yates and Grundy (1953).

$$(5)\quad V_{YG}\left(y'_{HT}\right) = \tfrac{1}{2} \sum_{\substack{I,J=1 \\ J \neq I}}^{N} \sum \left(\pi_I \pi_J - \pi_{IJ}\right)\left(\frac{Y_I}{\pi_I} - \frac{Y_J}{\pi_J}\right)^2$$

with the unbiased variance estimator (given also by Sen, 1953)

$$(6)\quad v_{SYG}\left(y'_{HT}\right) = \tfrac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \sum \frac{\pi_i \pi_j - \pi_{ij}}{\pi_{ij}} \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2.$$

Some selection procedures cannot be categorized as either 'without replacement' or as 'with replacement' in the usual sense (that is multinomial). The most important of these are intermediate cases where, for example, one or more of the population units may appear at most once. In this paper a general theory of sampling with unequal probabilities is presented which allows population units to appear more than once in sample. The only condition which will be placed on the selection procedure is that the total number of appearance in sample must be fixed.

The type of sample design for which this general theory may be of particular interest include

(*i*) crdinary systematic selection where one or more of the population units is large enough to be certain of selection at least once.

(*ii*) Deming's (1960) procedure which selects systematic samples with different random starts

(*iii*) contsrained methods of selection, such dumbbeld selection, where one or more units are subject to multiple selection.

This theory is also applicable in principal to a very wide range of sample design such as simple stratified sampling with the unbiased estimator. In particular it is possible in a multistage design, to evaluate the probability of selection of each possible final stage sample and then to treat the sampling procedure as though it were single stage. In practice, however, stratified and multistage samples will probably continue to be treated best as special cases. An example of this explicit use of multistage properties is the derivation of the multistage variance estimator which will be considered in Section 4.

2. A Generalized Horvitz-Thompson (GHT) Estimator.

Let S_I be the number of times the I th population unit appears in sample and S_{IJ} the number of times the ordered pair (I, J) appears in the set of $n\,(n-1)$ ordered pair of sample units. Then

$$S_{IJ} = \begin{cases} S_I \; S_J & J \neq I \\ S_I \; (S_I - 1) & \text{otherwise.} \end{cases}$$

The expected values of S_I and S_{IJ} will be written as μ_I and μ_{IJ} respectively. Generalized Horvitz-Thompson (GHT) estimator may be defined as :

$$y'_{GHT} = \sum_{I=1}^{N} \frac{S_I \; Y_I}{\mu_I},$$

which is clearly unbiased. However, many optimal properties possessed by the Horvitz-Thompson etimator are not carried over to the GHT. The Hansen-Hurwitz estimator, for example, though convenient and widely used, is well known to be inadmissible, and this will generally be true of any estimator for which the S_I can take values other than 0 and 1.

The variance of the GHT estimator is

$$V\left(y'_{GHT}\right) = E\left[\sum_{I=1}^{N} \frac{S_I \; Y_I}{\mu_I} - Y\right]^2$$

$$= \sum_{I=1}^{N} \left(\mu_{II} + \mu_I\right) \frac{Y_I^2}{\mu_I^2} + \sum\sum_{I,J=1}^{N} \mu_{IJ} \frac{Y_I}{\mu_I} \frac{Y_J}{\mu_J} - Y^2$$

(9) $$= \sum_{I=1}^{N} \frac{Y_I^2}{\mu_I} + \sum\sum_{I,J=1}^{N} \mu_{IJ} \frac{Y_I}{\mu_I} \frac{Y_J}{\mu_J} - Y^2.$$

The expression (9) may be written

(10) $$V\left(y'_{GHT}\right) = \tfrac{1}{2} \sum\sum_{I,J=1}^{N} \left(\mu_I \; \mu_J - \mu_{IJ}\right) \left(\frac{Y_I}{\mu_I} - \frac{Y_J}{\mu_J}\right)^2.$$

This is similar in form to (5) but more general in its meaning. When selection is strictly without replacement $\mu_I = \pi_I$, $\mu_{IJ} = \pi_{IJ}$ for $J \neq I$ and $\mu_{II} = 0$, then (9) is identical with (4).

Writing for convenience $P_I = \mu_I / n$, $P_{IJ} = \mu_{IJ} / n\ (n-1)$, (9) and (10) may be written

$$(11)\quad V\left(y'_{GHT} \right) = \frac{1}{n}\left(\sum_{I=1}^{N} \frac{Y_I^2}{P_I} - Y^2 \right) + \frac{n-1}{n} \sum_{I,J=1}^{N}\sum \left(\frac{P_{IJ}}{P_I P_J} Y_I\, Y_J - Y^2 \right)$$

$$(12)\quad V\left(y'_{GHT} \right) = \tfrac{1}{2} \sum_{I,J=1}^{N}\sum \left(P_I\, P_J - \frac{n-1}{n} P_{IJ} \right) \left(\frac{Y_I}{P_I} - \frac{Y_J}{P_J} \right)^2$$

For sampling 'with replacement' (multinomial sampling) $\mu_{IJ} = n(n-1) P_I\ P_J$ and (11) reduces to expression (2).

Expression (11) may be written as

$$(13)\quad V\left(y'_{GHT} \right) = V\left(y'_{HH} \right) - \frac{n-1}{n} D^2\,(y'),$$

where

$$(14)\quad D^2\,(y') = - \sum_{I,J=1}^{N}\sum \frac{P_{IJ}}{P_I P_J} Y_I\ Y_J + Y^2.$$

Expression (14) is nearly independent of sample size ; since the P_I are not, and the P_{IJ} need not be, functions of n. To simplify the discussions we will assume that a sampling procedure is being used for which the P_{IJ} remain constant as n increases and hence that $D^2(y')$ is not a function of n.

The generalization of the Sen-Yate -Grundy variance estimator is

$$(15) \quad v\left(y'_{GHT} \right) = \frac{1}{2n^2} \sum_{i,j=1}^{n}\sum \left(\frac{n}{n-1} \frac{p_i p_j}{p_{ij}} - 1 \right) \left(\frac{y_i}{p_i} - \frac{y_j}{p_j} \right)^2.$$

If $P_{IJ} > 0$ for $J \neq I$, this estimator is unbiased for (12). If, however, $P_{IJ} = 0$ for some $\{ I, J \}$, $J \neq I$, then the bias is non-zero and

$$(16) \quad E\, v\left(y'_{GHT} \right) - V\left(y'_{GHT} \right) = \frac{1}{2} \sum_{I,J=1}^{N}\sum \frac{n-1}{n} P_I P_J \left(\frac{Y_I}{P_I} - \frac{Y_J}{P_J} \right)^2$$

3. An Alternative Estimator of Variance.

Since the π_{ij} are involved in (15), that e pression is not usually easy to calculate. A simpler but biased estimator is

$$(17) \quad v_b(y') = \frac{1}{n(n-1)} \sum_{i=1}^{n} \left(\frac{y_i}{p_i} - y' \right)^2.$$

The expectation of (17) is

$$(18) \quad E\, v_b(y') = \frac{1}{n} \left(\sum_{I=1}^{N} \frac{Y_I^2}{P_I} - Y^2 \right) - \frac{1}{n} \cdot \left(\sum_{I,J=1}^{N}\sum \frac{P_{IJ}}{P_I P_J} Y_I Y_J - Y^2 \right)$$

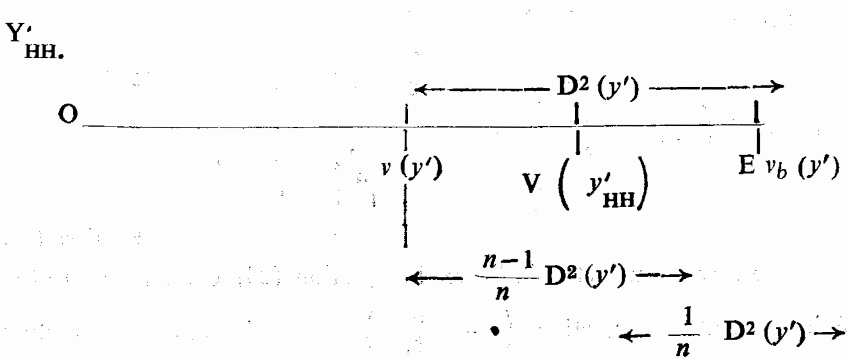
Assuming that the P_{IJ} are independent of n, the magnitude of the resulting bias is also independent of n. Hence

$$(9)\quad E\, v_b(y') - V\left(y'_{GHT}\right) = D^2(y') = \frac{n}{n-1}\left[V\left(y'_{HH}\right) - V\left(y'_{GHT}\right)\right].$$

The bias of the simple variance estimator (17) is seen from (19) to be directly proportional to the difference in variance between the estimator actually employed and the corresponding multinomial sampling estimator. Paradoxically the lower the variance of the estimator employed, the higher the expectation of its variance estimator. Further, whenever this estimator is more efficient than the corresponding multinomial sampling estimator, it will always tend to appear less efficient and vice versa. This result was obtained for the special case of sampling without replacement by Raj (1954).

A practical application of the above result is that the efficiencies of Y′ under various sampling procedures may be compared using this biased estimator, the actual efficiencies bearing an inverse relation to the apparent efficiencies.

The following diagram may be used to illustrate the relationship between the actual variance of Y′, the expected value of its biased estimator, and the variance of the corressponding Hansen-Hurwitz estimator



Since $D^2(y')$ remains constant over n, it functions in the same way as the finite population correction does in simple random sampling without replacement.

The correction factor may be obtained by using the following super population model :

$$(20)\quad \begin{cases} Y_I = \beta \mu_I + \epsilon_I, \\ E^* \epsilon_I = 0, \quad E^* \epsilon_I \epsilon_J = \begin{cases} \sigma_I^2 & J = I \\ 0 & \text{otherwise} \end{cases} \\ \sigma_I^2 = \sigma^2 Z_I^{2\gamma}, \qquad \tfrac{1}{2} \leqslant \gamma \leqslant 1. \end{cases}$$

where β, γ and σ^2 are constant and E^* denotes the expectation overall possible hypothetical populations. Then

$$(21)\quad E^* V\left(y'_{GHT} \right) = \sigma^2 \left(\frac{Z}{N} \right)^{2\gamma} \sum_{I=1}^{N} (1-\mu_I)\, \mu_I^{2\gamma-1}$$

$$(22)\quad E^* D^2 (y') = \sigma^2 \left(\frac{Z}{n} \right)^{2\gamma} \sum_{I=1}^{N} \left(\mu_I^{2\gamma} - \mu_{II}\, \mu_I^{2\gamma-2} \right),$$

and

$$(23)\quad E^* E\, v_b (y') = \sigma^2 \left(\frac{Z}{n} \right)^{2\gamma} \sum_{I=1}^{N} \mu_I^{2\gamma-1} .$$

From (21), (22) and (23) we obtain

$$(24)\quad E^* V\left(y'_{GHT} \right) = \left\{ 1 - \frac{\sum_{I=1}^{N} \left(\mu_I^{2\gamma} - \mu_{II}\, \mu_I^{2\gamma-2} \right)}{\sum_{I=1}^{N} \mu_I^{2\gamma-1}} \right\} E^* E\, v_b (y')$$

The correction factor, in braces in equation (24). corresponds to the finite population correction $\left(1 - \frac{n}{N}\right)$ in simple random sampling without replacement ; when $\gamma = \frac{1}{2}$, and $\mu_{II} = 0$ for all I, it actually takes that value.

4. Application of the GHT Estimator to Multistage Sampling

Selection with unequal probabilities is very frequently used in multistage sampling. A multistage variance estimator suitable for use in sampling without replacement was used by Durbin (1967) and given in explicit form by Brewer and Hanif (1970). The idea underlying this multistage estimator is very general in application, and may be expressed as follows :

> 'An unbiased estimator of variance in multistage designs may be written as the sum of three terms, of which the second must be prefixed by a minus sign. The first term is equal to the estimator of variance calculated on the assumption that the first stage units have been measured without error. The second term is an unbaised estimator of the contribution made to this first term by variances from lower stages of sampling. The third term is an unbaised estimator of the veriance from these lower stages.'

When this principle is applied to the Generalized Horvitz and Thompson estimator, the resulting variance estimator may be written as follows :

$$v(y') = \frac{1}{2n^2} \sum_{i,j=1}^{n}\sum \left(\frac{n}{n-1}\frac{p_i p_j}{p_{ij}} - 1\right)\left(\frac{y_i'}{p_i} - \frac{y_j'}{p_j}\right)^2$$

$$(25) \qquad - \frac{1}{2n^2} \sum_{i,j=1}^{n}\sum \left(\frac{n}{n-1}\frac{p_i p_j}{p_{ij}} - 1\right)\left(\frac{s_i^2}{p_i^2} + \frac{s_j^2}{p_j^2}\right)$$

$$+ \frac{1}{2n^2} \sum_{i,j=1}^{n}\sum \frac{1}{n-1}\left(\frac{s_i^2}{p_i^2} + \frac{s_j^2}{p_j^2}\right).$$

where y_i' is the contribution to y' from the ith first stage sample unit in the population, and s_i^2 is an unbiased estimator of S_I^2 the variance of y_i' due to sampling at the second and lower stages.

When sampling is without replacement, expression (25) is the same as that given by Brewer and Hanif (1970). For 'sampling with replacement' it reduces to the familiar formula :

$$(26) \quad v\left(y'_{HH} \right) \frac{1}{2n^2(n-1)} \sum_{i,j=1}^{n}\sum \left(\frac{y_i'}{p_i} - \frac{y_j'}{p_j} \right)^2.$$

5. Numerical Example

Consider the case N = 5, n = 3, μ_I = .3, .4, .5, .6, 1.2, Y_I = 5, 6 8, 10, 11, 19. Using the Randomized Systematic Procedure the probabilities of selection of each possible sample of three units are given by π_{IJK} where

$$\pi_{125} = \frac{10}{120}, \quad \pi_{135} = \frac{10}{120}, \quad \pi_{145} = \frac{14}{120},$$

$$\pi_{155} = \frac{2}{120}, \quad \pi_{235} = \frac{14}{120}, \quad \pi_{245} = \frac{18}{120},$$

$$\pi_{255} = \frac{6}{120}, \quad \pi_{345} = \frac{30}{120}, \quad \pi_{355} = \frac{6}{120},$$

$$\pi_{455} = \frac{10}{120}.$$

The μ_{IJ} are therefore

$$\mu_{12} = \frac{10}{120}, \quad \mu_{13} = \frac{10}{120}, \quad \mu_{14} = \frac{24}{120},$$

$$\mu_{15} = \frac{38}{120}, \quad \mu_{23} = \frac{14}{120}, \quad \mu_{24} = \frac{18}{120},$$

$\mu_{25} = \frac{54}{120}$, $\mu_{34} = \frac{30}{120}$, $\mu_{35} = \frac{66}{120}$,

$\mu_{45} = \frac{82}{120}$, $\mu_{55} = \frac{48}{120}$.

The variance of the GHT estimator using (12) is 0.5563. This may be compared with the variance of the Hansen-Hurwitz estimator fo multinomial sampling with the same values of P_I, which is 0.8333.

A sample was selected by cumulating the π_{IJK} above and choosing a random number in the interval [0,1). This sample contained the 1st, 4th and 5th population units. The unbiased estimator of the population total (Y = 48) was Y′ = 49.167. The generalized Sen-Yates-Grundy variance estimator using (15) was 0.1120. The biased variance estimate using (17) was 1.2732. The correction factor (24) to be applied to this biased variance estimator was calculated using three values of γ. For $\gamma = \frac{1}{2}$ it was 0.4667 ; for $\gamma = 3/4$ it was 0.4230 ; and for $\gamma = 1$ it was 0.3667. The three corresponding estimates of variance were 0.5941, 0.5385, and 0.4668. These happened to be, by chance, remarkably close to the true variance. A second sample was therefore selected consisting of the 3rd unit once and the 5th unit twice. The unbiased variance estimate for this sample was 0.0505, and the biased estimate 0.25. The correction factors, being independent of the particular sample selected, remained as before, yielding estimates of variance 0.1167, 0.1058, and .0917. These are underestimates of the true variance but still closer than the unbiased estimate.

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BESSEL POTENTIALS WITH WEIGHTS

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1. Introduction

In recent years the Bessel potentials first introduced by Aronszaju and Smith [1] have sttracted much interest. Calderon [2] has investigated Bessel potentials in L^p-spaces and Stein [6] has discussed its characterisation.

In this paper we introduce Bessel potentials with weight functions and prove some its of properties. These are related to Fourier operators with weight functions of the type $(b^2 + t^2)^{\lambda}$; with $b = o$, these operators reduce to those studied by Okikiolu [3]. A few results involving relative boundedness of Fourier operators and Bessel potentials are also obtained.

2. Bessel potentials

Given reals numbers $\alpha > o, b > o, n \geqslant 1$, let

$$G^{\alpha}(x, b) = \pi^{-\frac{1}{2}n} \frac{2^{-n\alpha}}{\Gamma(\frac{1}{2}n\alpha)} \int_{o}^{\infty} u^{\frac{1}{2}n(1-\alpha)-1} e^{-|x|^2 u - b^2/4u} du. \qquad (2.1)$$

It posseses the following properties

(i) $G^{\alpha}(.,b)$ is everywhere positive, decreasing and integrable function on n-dimensional Euclidean space E_n.

(ii) $G^{\alpha}(b^{-1}x, b) = b^{(1-n\alpha)}\, G(x, 1)$; (2.2)

(iii) $\{G^{\alpha}(.,b)\}^{\wedge}(x) = (2\pi)^{-\frac{1}{2}n}\,(b^2 + |x|^2)^{-\frac{1}{2}n\alpha}$, (2.3)

where $\wedge$ denotes the Fourier transform;

(iv) $\| G^{\alpha}(.,b) \|_1 = b^{-n\alpha}$; (2.4)

(v) $G^{\alpha+\beta} = G^{\alpha} * G^{\beta}$, (2.5)

where $*$ denotes the convolution.

Now for a function f measurable on E_n, we define the weighted Bessel potential by

$$J^{\alpha,\lambda}_{\nu,b} = (b^2 + |x|^2)^{\frac{1}{2}(\lambda-\alpha-\nu)\,n} \int_{E_n} (b^2 + |y|^2)^{\frac{1}{2}\nu n}\, G^{\alpha}(x-y,b) f(y)\, dy \qquad (2.6)$$

and denote $J^{\alpha,o}_{\nu,b}$ by $J^{\alpha}_{\nu,b}$.

Theorem 1

Let p, q, α, ν and λ be real numbers such that

$$p > 1, \quad \frac{1}{\ } = \frac{1}{p} - \lambda,$$

(i) $0 \leqslant \lambda < \alpha < \frac{1}{p}, \frac{1}{p} - 1 < \nu < \frac{1}{p} - \alpha$,

or

(ii) $0 \leqslant \lambda < \alpha + \nu < \frac{1}{p} < 1, \ \frac{1}{p} - 1 < \nu, \lambda < \alpha$.

Then

$$J^{\alpha,\lambda}_{\nu,p} : L^{p} \to L^{q},$$

and there is a finite constant $k = k\ (n, \lambda, \alpha, \nu, p)$ independent of b such that $f \in L^p\ [= L^p\ (E_n)\ ]$ such that

$$\| J^{\alpha, n}_{\nu, p}\ (f)\ \|_q \leqslant k\ \| f \|_p\ . \tag{2.7}$$

Proof

Using the fact that, for $0 < \alpha < 1$,

$$|\ G^{\alpha}\ (x,b)\ |\ \leqslant\ |\ x\ |^{n\,(\alpha-1)}, \qquad \text{(see 4.2)}$$

we have

$$|\ J^{\alpha,\lambda}_{\nu,b}\ \ (f)\ (x)\ |\ \leqslant k\ (b^2 + |\ x\ |^2)^{\frac{1}{2}\ (\lambda-\alpha-\nu)\,n}$$

$$\int_{E_n} (b^2 + |\ y\ |^2)^{\frac{1}{2} n \nu}\ \ |\ x - y\ |^{\,n\,(\alpha-1)}\ |\ f\,(y)\ |\ dy$$

$$= \ I^{\alpha,\lambda}_{\nu,b}\ \ (f)\ (x), \text{ say.}$$

If $\lambda < \alpha + \nu, \nu < o$, then $I^{\alpha,\,\lambda}_{\nu,\,b}\ (f) \leqslant I^{\alpha,\,\nu}_{\nu,\,o}$ and the result follows from Theorem 4.4.15 of [4].

If $\nu \geqslant o, o \leqslant \lambda < \alpha$, then

$$I^{\alpha,\,\lambda}_{\nu,\,b}\,(f)\ (x) = k\ \ (b^2 + |\ x\ |^2)^{\frac{1}{2}\ (\lambda-\alpha)\,n} \int_{E_n} |\ t - x\ |^{(\alpha-1)\,n}$$

$$\left[\frac{b^2 + |\ t\ |^2}{b^2 + |\ x\ |^2}\right]^{\frac{1}{2}\nu n}\ \ |\ f\ (t)\ |\ dt \leqslant k\ |\ x\ |^{(\lambda-\alpha)\,n} \int_{E_n}$$

$$|\ t - x\ |^{(\alpha-1)n} \left[1 + \left|\ \frac{t}{x}\ \right|^2\right]^{\frac{1}{2}\nu n}\ \ |\ f(t)\ |\ dt.$$

Since the kernel in the last integral is radial and the function

$$\psi\ (u,\ v) = u^{(\lambda-\alpha)n}\ \ |\ u - v\ |^{(\alpha-1)n} \left[1 + \frac{v^2}{u^2}\right]^{\frac{1}{2}\nu n}$$

is homogeneous of degree $(\lambda - 1)\, n$ and

$$k_1 = \int_o^\infty |\, r \,|^{\frac{n}{(1-\lambda)p'} - 1} \; |\, r - 1 \,|^{\frac{(\alpha-1)\, n}{1-\lambda}} \; |\, 1 + r^2 \,|^{\frac{1}{2} \frac{n\lambda}{1-\lambda}} \, dr$$

is finite if $\alpha < \frac{1}{p} - v$, the application of Theorem 3.3 of [5] yields that if $\alpha < \frac{1}{p} - \nu,\ \nu \geqslant o,\ \frac{1}{} = \frac{1}{p} - \lambda$ and $f \in L^p$, then

$$\| J^{\alpha,\lambda}_{\nu,b}(f) \|_q < k \, \| f \|_p \qquad (2.8)$$

Similarly if $\nu < o;\ o \leqslant \lambda < \alpha$ we have

$$I^{\alpha,\lambda}_{\nu,b}(f)(x) = k(b^2 + |\, x \,|^2)^{\frac{1}{2}(\lambda-\alpha)\, n} \int_{E_n} |\, t - x \,|^{(\alpha-1)n}$$

$$\left[\frac{b^2 + |\, x \,|^2}{b^2 + |\, t \,|^2} \right]^{\frac{1}{2}\nu n} |\, f(t) \,| \ dt \leqslant k \,|\, x \,|^{(\lambda-\alpha)n} \int_{E_n} |\, t - x \,|^{(\alpha-1)n}$$

$$\left[1 + \left| \frac{x}{t} \right|^2 \right]^{-\frac{1}{2}\nu n} |\, f(t) \,| \quad dt.$$

Since

$$k_2 = \int_o^\infty |\, r \,|^{\frac{n}{(1-\lambda)p'} - 1} \left[1 + \frac{1}{p} \right]^{\frac{1}{2} \frac{\nu n}{1-\lambda}} |\, r - 1 \,|^{\frac{(\alpha-1)}{1-\lambda}} \ dr$$

is finite if $\nu = \frac{1}{p} - 1,\ \alpha < \frac{1}{p}$, using the same theorem it follows that when $f \in L^p$, $\frac{1}{q} = \frac{1}{p} - \lambda$

$$\| J^{\alpha,\lambda}_{\nu,b} \|_q \leqslant k \, \| f \|_q \, . \qquad (2.9)$$

We now obtain the result by combining (2.8) and (2.9).

Theorem 2

Let α, β, ν, p be real numbers such that $\alpha > o$, $\beta > o$, $\alpha + \beta < 1$, $p > 1$, $\frac{1}{p} - 1 < \nu < \frac{1}{p} - \alpha - \beta$. Then for $f \in L^p$

$$J^{\beta}_{\alpha+\nu,\, b} \; J^{\alpha}_{\nu,\, b} = J^{\alpha+\beta}_{\nu,\, b} \qquad (2.10)$$

Proof

Since $G^{\alpha}(.,b) \in L'$, the result follows by changing the order of integration and applying (2.5)

3. Fourier Operators

We define fourier operators by the formula

$$F^{\sigma}_{\nu,\, b}(f)(x) = (2x)^{-\frac{1}{2}n} \; (b^2 + |x|^2)^{\frac{1}{2}(\nu+\sigma)n} \int_{E_n} (b^2 + |t|^2)^{\frac{1}{2}n\nu} \; e^{ix.t} \, f(t) \; dt. \qquad (3.1)$$

where $x.t = x_1 t_1 + x_2 t_2 + \ldots + x_n t_n$.

Since $F^{\sigma}_{\nu,\, b}$ with $b = o$ have been studied by Okikiolu [3] we shall restrict ourself to the case $b \neq o$ which assumed in the rest of paper.

Theorem 3 :

Let p, q, ν and σ be real number such that $1 < p < q < \infty$, $\frac{1}{p} - 1 < \nu < \min(0, -\sigma)$, $\frac{1}{q} = 1 - \frac{1}{p} - \sigma$. (3.2)

Then $F^{\sigma}_{\nu,\, b}$ can be extended to a bounded operator from L^p into L^q and there is finite constant $k = k(p, \nu, \sigma, n)$ independent of b such that, for $f \in L^p$,

$$\| F^{\sigma}_{\nu,\, b}(f) \|_q = k \, \| f \|_p \qquad (3.3)$$

Moreover, there is a constant k independent of b such that for $g \in L^{q'}, \frac{1}{q} + \frac{1}{q'} = 1,$

$$\| F^{-\sigma}_{\nu+\sigma,\, b}(g) \|_{p'} \leqslant k \, \|g\|_{q'} \tag{3.4}$$

Proof

Okikiolu [3, Theorem 6.5.11] proved that with $b = o$ and conditions (3·2)

$$\| F_{\nu,\, o}(f)\|_q \leqslant k \, \|f\|_p. \tag{3.5}$$

If $\psi(t) = (b^2 + |t|^2)^{\frac{1}{2}n}$ and $\phi(t) = |t|^n$, then

$$F^{\sigma}_{\nu,\, b}(f)(x) = \psi^{\nu+\sigma}(x) \int \phi^{\nu}(t)\, e^{ix.t}\, \phi^{-\nu}\, \psi^{\nu}(t) f(t)\, dt.$$

Since $\nu \leqslant 0$, $\nu + \sigma \leqslant 0$, (3.5) yields

$$\| F^{\sigma}_{\nu,\, b}(f) \|_q \leqslant \| \phi^{-\nu}\, \psi^{\nu} f \|_p$$

$$\leqslant \| \phi^{-\nu}\, \phi^{\nu} f \|_p$$

$$= \| f \|_p.$$

Alternatively, we can use analogue of Stein's proof [5] of Pitt's theorem by interpolation. To prove (3.4) replace $f(t)$ by

$$| F^{-\sigma}_{\nu+\sigma,\, b}(g) |^{p'-1} \left\{ \operatorname{sgn} F^{-\sigma}_{\nu+\sigma,\, b}(g) \right\} \chi_{(-N,N)},$$

rearrange the terms and let $N \to \infty$.

4. Relative Boundedness

We shall now prove some identities and using them deduce certain estimates which show that $F^{\sigma}_{\nu,\, b}$ with different indices and also operators $J^{\sigma}_{\nu,\, b}$ exhibit relative boundedness.

Theorem 4

Let p, ν_0, ν_1, σ_0, σ_1 be real number such that

$$p > 1,\ \nu_o > \frac{1}{p} - 1,\ -1 < \nu_1 + \nu_o + \sigma_o < 0.$$

If $f \in L^p$ is a step function, then we have

$$F^{\sigma_1}_{\nu_1, b} \quad F^{\sigma_o}_{\nu_o, b} \quad (f)(x) = (b^2 + |x|^2)^{\frac{1}{2}(\sigma_1 - \sigma_o)\, n}$$

$$J^{-\nu_1 - \nu_o - \sigma_o}_{\sigma_o, b}(f)(-x). \tag{4.1}$$

Moreover if $F^{\sigma_o}_{\nu_o, b}$ is bounded from L^p into $L^q, \frac{1}{q} = 1 - \frac{1}{p} - \sigma_o$

and $F^{\sigma_1}_{\nu_1, b}$ is bounded from L^q into $L^r, \frac{1}{r} = 1 - \frac{1}{q} - \sigma_1$, then the result holds for all $f \in L^p$.

Proof

For $\lambda > o$, $o < \alpha < 1$, use

$$\pi^{\frac{1}{2}n} \quad u^{-\frac{1}{2}n} \quad e^{-\frac{|x|^2}{4u}} = \int_{E_n} e^{-u|t|^2} \quad e^{ix.t}\ dt.$$

to obtain, with b_1

$= |x|^2 b_1$,

$$\lambda_1 = |x|^2 \lambda, \int_{E_n} e^{-\lambda|t|^2} (b^2 + |t|^2)^{-\frac{1}{2}n\alpha} \quad e^{ix.t} \quad dt$$

$$= |x|^{n(\alpha - 1)} \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n\alpha)} \int_0^\infty u^{\frac{1}{2}n\alpha - 1} \quad e^{-b_1^2\, u} (\lambda_1 + u)^{-\frac{1}{2}n}$$

$$e^{-\frac{1}{4(\lambda_1 + u)}}\ du$$

$$\to 0 \quad \text{as} \quad \lambda \to 0. \tag{4,2}$$

Let f be step function. Using Fubinis theorem it follows that for $\lambda > 0$

$$F^{\sigma_1}_{\nu_1}\left\{ F^{\sigma_o}_{\nu_o}(f)\, e^{-\lambda |x|^2} \right\}(x) = (2\pi)^{-n}(b^2+|x|^2)^{\frac{1}{2}(\sigma_1+\nu_1)\,n} \int_{E_n} (b^2+|t|^2)^{\frac{1}{2}(\nu_1+\nu_o+\sigma_o)\,n} x$$

$$x\, e^{-\lambda |t|^2}\, e^{ix.t} \int_{E_n} f(y)\,(b^2+|y|^2)^{\frac{1}{2}n\nu_o}\, e^{i+y} dy\, dt$$

$$= (2\pi)^{-n}(b^2+|x|^2)^{\frac{1}{2}(\sigma_1+\nu_1)\,n} \int_{E_n} f(x+y)\,(b^2+|x-y|^2)^{\frac{1}{2}n\nu_o} x$$

$$x \int e^{iy.t}\, e^{-\lambda |t|^2}\, (b^2+|t|^2)^{\frac{1}{2}n(\nu_1+\nu_o+\sigma_o)}\, dt\, dy.$$

Since f is a step function, the integrand on the right is in L, using (4.2) and Lebesegue Convergence Theorem we prove the result.

Theorem 5

Let p, σ, α and ν be real numbers such that $p > 1, o < \alpha < 1$, $\frac{1}{p} - 1 < \nu < \frac{1}{p} - \alpha$. Let $f \in L^p$ be a step function. Then we have

$$F^{\sigma}_{\nu+\alpha}\{ J^{\alpha}_{\nu,b}(f)\} = F^{\sigma}_{\nu,b}(f). \tag{4.3}$$

If $F^{\sigma}_{\nu+\alpha,b}$ can be extended to a bounded operator in L^p then $F^{\sigma}_{\nu,b}$ can be also extended to a bounded operator in L^p and the result holds for all $f \in L^p$.

Proof.

The result follows by using Fubini – Tonelli's theorem and (2.3) Similarly we prove

Theorem 6

Let α, ν, p, q and σ be real numbers such that

$$p > 1, o < \alpha < 1, \nu > \frac{1}{p} - 1, \frac{1}{q} = 1 - \frac{1}{p} - \sigma.$$

If $f \in L^p$ be a step function, then

$$J^{\alpha}_{-\nu-\sigma, b} \; F^{\sigma}_{\nu,b} \; (f) = F^{\sigma}_{\nu-\alpha, b}(f). \tag{4.4}$$

If $F^{\sigma}_{\nu,b}$ can be extended to a bounded operator from L^p into L^q and if $\frac{1}{p} - 1 < \nu - \alpha < \frac{1}{p} - \alpha$, then the result holds for all $f \in L^p$.

Theorem 7

Let α, ν, σ, p and q be real numbers such that

$$p > 1, o < \alpha < 1, \frac{1}{p} - 1 < \nu < \frac{1}{p} - \alpha, \frac{1}{q} = 1 - \frac{1}{p} - \sigma > 0.$$

Assume that $F^{\sigma}_{\nu, \alpha, b}$ can be extended to a bounded operator from L^p into L^q. Then there is a finite constant $k = k\ (n, p, \nu)$ independent of b such that, for $f \in L^p$.

$$\|F^{\sigma}_{\nu,b} \; (f)\|_q \leqslant k\|J^{\alpha}_{\nu, b} \; (f)\|_p .$$

Proof

Using Theorem 5 we obtain

$$\|F^{\sigma}_{\nu, b}(f)\|_q = \|F^{\sigma}_{\nu+a,b} \; J^{\alpha}_{\nu,b} \; (f)\|_q$$
$$\leqslant k\|F^{\nu} \; (f)\|_p .$$

•

Theorem 8

Let α, v, σ, σ_1, p, q and r be real numbers such that

$$p > 1,\ o < \alpha < 1,\ v > \frac{1}{p} - 1,\ \frac{1}{q} = 1 - \frac{1}{p} - \sigma > o,$$

$$\frac{1}{r} = 1 - \frac{1}{q} - \sigma_1 > o.$$

Assume that $F^{\sigma_1}_{-v-\sigma-a,b}$ can be extended to a bounded operator from L^q into L^r.

Then there is a finite constant $k = k\ (v,\ \alpha,\ p,\ \sigma,\ \sigma_1,\ n)$ independent of b such that, for $f \in L^p$,

$$\| (b^2 + |x|^2)^{\frac{1}{2}(\sigma_1 - \sigma)\, n}\, J^{\sigma}_{v,\,b}\ (f)\ (x) \|_r = k \| F^{\sigma}_{v,b}\ (f) \|_q .$$

Proof

Using Theorem 4 we have

$$\| (b^2 + |x|^2)^{\frac{1}{2}(\sigma_1 - \sigma)\, n}\, J^{\sigma}_{v,b}\ (f)\ (x) \|_r$$

$$= \| F^{\sigma_1}_{-v-\sigma-\alpha,\, b}\ F^{\sigma}_{v,\, b}\ (f) \|_r \leqslant \| F^{\sigma}_{v,\, b}\ (f) \|_q .$$

Theorem 9

Let σ_1, σ, α, v, p, q and r be real numbers such that

$$p > 1,\ o < \sigma_1 - \sigma < \alpha < 1,\ \frac{1}{p} - 1 < v - \alpha < \frac{1}{p} - \alpha,$$

$$\frac{1}{q} = 1 - \frac{1}{p} - \alpha,\ \frac{1}{r} = 1 - \frac{1}{p} - \sigma_1 .$$

Let f be any step function in L^p. Then there is a finite constant $k = k\ (v,\ \alpha,\ \sigma_1,\ \sigma,\ p,\ n)$ independent of b such that

$$\| F^{\sigma_1}_{v-\alpha,\, b} (f) \|_r \leqslant k \| F^{\sigma}_{v,\, b} (f) \|_q$$

If $F^{\sigma}_{v,\,b}$ can the extended to a bounded operator from L^p into L^q, then the result holds for all $f \in L^p$.

Proof

The given conditions implies that

$$\frac{1}{p} - 1 < v < \frac{1}{p}, \quad \frac{1}{q} - 1 < -(v+\sigma) < \frac{1}{q} - \alpha,$$

$$o < \sigma_1 - \sigma < \frac{1}{q} < 1, \quad \sigma_1 - \sigma < \alpha < 1, \frac{1}{r} = \frac{1}{q} - (\sigma_1 - \sigma)$$

Use Theorems 1, 6 to obtain

$$\| F^{\sigma_1}_{v-\alpha}(f) \|_r = \| (b^2 + | x |^2)^{\frac{1}{2}(\sigma_1 - \sigma)\, n} \; J^{\alpha}_{-v-\sigma,\, b} \; F^{\sigma}_{v,\, b}(f) \|_r$$

$$\leqslant k \, \| F^{\sigma}_{v,\, b}(f) \|_q .$$

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ESTIMATION OF Pr $\{Y < X\}$ FOR THE POWER FUNCTION DISTRIBUTION

BY

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Abstract : The Blackwell-Rao and Lehmann-Scheffe theorems are used to derive the minimum variance unbiased estimator of Pr $(Y < X)$ when the independent random variables X and Y follow the power function distribution.

1. Introduction

An extensive amount of work has been done on the problem of estimating $P = \Pr\{Y < X\}$ in both distribution-free and parametric frameworks (see, e.g., Birnbaum [1956], Church and Harris [1970], Downton [1973], Enis and Geisser [1971], Tong [1974], and others). The problem originated in the context of reliability of a component of strength X subjected to a stress Y. The component fails if at any time the applied stress is greater than its strength and there is no failure when Y is less than X. Thus the problem here is to find an estimate of the probability that Y is less than X where X and Y are both random variables having some known or unknown probability distribution.

In this paper, we estimate P for the power function distribution by applying the Blackwell-Rao and Lehmann-Scheffe theorems. The distribution has been studied before by Rider [1964], Likes [1967] and Malik [1967].

2. Minimum Variance Unbiased Estimation of P

Let the random variables X and Y follow the power function distribution with probability density functions (p. d. f.'s)

$$f_1(x\,;\alpha,\theta) = \theta\alpha^{-\theta}\, x^{\theta-1}, \quad 0<x\leqslant\alpha, \quad \theta>0, \quad \alpha>0, \tag{2.1}$$

$$f_2(y\,;\beta,\phi)=\phi\beta^{-\phi}\;y^{\phi-1},\quad 0<y\leqslant\beta,\quad \phi>0,\quad \beta>0, \tag{2.2}$$

It can be shown that

$$P=\begin{cases}1-(\beta/\alpha)^{\theta}\;\phi/(\theta+\phi), & \text{if } \beta<\alpha\\ (\alpha/\beta)^{\phi}\;\theta/(\theta+\phi), & \text{if } \alpha<\beta\\ \theta/(\theta+\phi), & \text{if } \alpha=\beta\end{cases} \tag{2.3}$$

Suppose that $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ are two independent random samples of sizes n and m from the p. d. f.'s (2.1) and (2.2) respectively. Further, let $X_{(1)}\leqslant\dots\leqslant X_{(n)}$ and $Y_{(1)}\leqslant\dots\leqslant Y_{(m)}$ be the corresponding order statistics of the samples. If we make the transformation $W=-\ln X$ in (2.1), then the p. d. f. of W is

$$g\,(w;\,\alpha,\theta)=\theta\exp\{-\theta\,(w-\ln(1/\alpha))\}\;,\quad w\geqslant\ln(1/\alpha).$$

Thus following Epstein and Sobel [1954] it can be shown that for the p. d. f. (2.1)

(i) if α is known, $U_1=\sum_{i=1}^{n}\ln(\alpha/X_i)$ is a complete, sufficient estimator of θ with the p. d. f.

$$g_1(u_1\,;\alpha,\theta)=[\theta^{n}/\Gamma(n)]\;u_1^{n-1}\exp.\{-\theta u_1\},\;u_1>0 \tag{2.4}$$

(ii) if θ is known, $X_{(n)}$ is a complete, sufficient estimator for α with the p.d.f.

$$g_2(x_{(n)}\,;\,\alpha,\theta)=n\theta\alpha^{-n\alpha}\;x^{n\theta-1},\quad 0<x_{(n)}\leqslant\alpha \tag{2.5}$$

(iii) if both θ and α are unknown, $(X_{(n)},Z_1)$ where $Z_1=\sum_{i=1}^{n}\ln(X_{(n)}/X_{(i)})$, is a complete, sufficient estimator for (α,θ) and Z_1 is stochastically independent of $X_{(n)}$ with the p.d.f. of $X_{(n)}$ being (2.5) while that of Z_1

$$g_3(z_1\,;\theta)=[\theta^{n-1}/\,\Gamma(n-1)]\,z_1^{n-2}\exp\{-\theta\,z_1\},\quad z_1>0 \tag{2.6}$$

Similarly for the p.d.f. (2 2) in an obvious notification

$$U_2 = \sum_{i=1}^{m} \ln(\beta/Y_i), \quad Y_{(m)}, \quad (Y_{(m)}, Z_2),$$

where $$Z_2 = \sum_{i=1}^{m} \ln(Y_{(m)} / Y_{(i)}),$$

are complete, sufficient estimators for ϕ, β, (β, ϕ) for cases (*i*), (*ii*), (*iii*), respectively, and with the p. d. f.'s analogous to (2.4) (2.5), (2.6).

Case i) : α, β Known, θ, ϕ Unknown

The sam les $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ can be summarized by the complete sufficient statistics U_1 and U_2 respectively. The conditional p. d. f.'s of $t_1 = \ln X_1$ given $U_1 = u_1$ and of $t_2 = \ln Y_1$ given $U_2 = u_2$ are

$$h_1(t_1 \mid u_1) = (n-1)(t_1 + u_1 - \ln\alpha)^{n-2} / u_1^{n-1},$$

$$\ln\alpha - u_1 < t_1 < \ln\alpha \qquad (2.8)$$

$$h_2(t_2 \mid u_2) = (m-1)(t_2 + u_2 - \ln\beta)^{m-2} / u_2^{m-1},$$

$$\ln\beta - u_2 < t_2 < \ln\beta \qquad (2.9)$$

An unbiased estimate of (2.3) is

$$p(t_1, t_2) = \begin{cases} 1, & \text{if } t_1 > t_2 \\ 0, & \text{Otherwise.} \end{cases}$$

Using Blackwell-Rao and Lehmann-Scheffe theorems, the MVU estimator of P is :

$$\tilde{P} = \iint p(t_1, t_2)\, h_1(t_1 \mid u_1)\, h_2(t_2 \mid u_2)\, dt_1\, dt_2$$

$$= \int_{\ln\theta - u_1}^{\ln\theta} f_1(t_1 \mid u_1) \left\{ \int_{\ln\phi - u_2}^{\min(\ln\phi,\, t_1)} f_2(t_2 \mid u_2)\, dt_2 \right\} dt_1 \qquad (2.9^*)$$

For $t_1 < \ln \phi - u_2$, the integral in (2.9*) is zero, and $\tilde{P}=0$. On the other hand, for $t_1 > \ln \theta - u_1 > \ln \phi$, the integral in (2.9*) is unity, which means $\tilde{p} = 1$. For remaining cases we have

$$\tilde{P} = (1+\xi/u_2)^{m-1} \; [1- \sum_{j=0}^{n-1} \{(1-n)_j/(m)_j\} \; \{(u_2+\xi)/u_1\}^j], \qquad (2.10a)$$

$$\text{if} \;\; \ln \alpha - u_1 < \ln \beta - u_1 < \ln \alpha < \ln \beta$$

$$= (1+\xi/u_2)^{m-1} \sum_{j=0}^{m-1} \{(1-m)_j/(n)_j\} \; \{u_1/(u_2+\xi)\}^j, \qquad (2.10b)$$

$$\text{if} \;\; \ln \beta - u_2 < \ln \alpha - u_1 < \ln \alpha < \ln \beta$$

$$= 1-(1-\xi/u_1)^{n-1} \sum_{j=0}^{n-1} \{(1-n)_j/(m)_j\} \; \{u_2/(u_1 - \xi)\}^j, \qquad (2.10c)$$

$$\text{if} \;\; \ln \alpha - u_1 < \ln \beta - u_2 < \ln \beta < \ln \alpha$$

$$= 1-(1-\xi/u_1)^{n-1} [1- \sum_{j=0}^{m-1} \{(1-m)_j/(n)_j\} \; \{(u_1-\xi)/u_2\}^j], \qquad (2.10d)$$

$$\text{if} \;\; \ln \beta - u_2 < \ln \alpha - u_1 < \ln \beta < \ln \alpha$$

where $(a)_j \equiv a\,(a+1) \ldots (a+j-1)$, $(a)_0 \equiv 1$, $\xi = \ln(\alpha/\beta)$.

When $\alpha=\beta$, $\xi=0$ and (2.10 a) – (2.10 d) reduce to

$$\tilde{P} = 1- \sum_{j=0}^{n-1} \{(1-n)_j \,/\, (m)_j\} \; (u_2/u_1)^j, \text{ if } u_2 < u_1$$

$$= \sum_{j=0}^{m-1} \{(1-m)_j \,/\, (n)_j\} \; (u_1/u_2)^j, \text{ if } u_1 < u_2$$

which agree with the results obtained by Tong (1974, 75).

Case (ii) : θ, ϕ Known, α, β Unknown

The samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ can be summarized by the complete sufficient statistics $X_{(n)}$ and $Y_{(m)}$ respectively. The conditional p.d.f.'s of t_1 given $X_{(n)}=x_{(n)}$ and of t_2 given $Y_{(m)} = y_{(m)}$ are

$$h_1(t_1 \mid x_{(n)}) = \begin{cases} (1-1/n)\,\theta\, x_{(n)}^{-\theta} \exp(\theta t_1), & \text{if } -\infty < t_1 < \ln x_{(n)} \\ 1/n, & \text{if } t_1 = \ln x_{(n)} \end{cases} \quad (2.11a)$$

$$h_2(t_2 \mid y_{(m)}) = \begin{cases} (1-1/m)\,\phi\, y_{(m)}^{-\phi} \exp(\phi t_2), & \text{if } -\infty < t_2 < \ln y_{(m)} \\ 1/m, & \text{if } t_2 = \ln y_{(m)} \end{cases} \quad (2.11b)$$

The MVU estimator of P in this case is :

$$\tilde{P} = \iint p(t_1, t_2)\, h_1(t_1 \mid x_{(n)})\, h_2(t_2 \mid y_{(m)})\, dt_1\, dt_2$$

$$= \int_{(-\infty,\, \ln x_{(n)}]} h_1(t_1 \mid x_{(n)}) \left\{ \int_{(-\infty,\, \min(\ln y_{(m)},\, t_1)]} h_2(t_2 \mid y_{(m)})\, dt_2 \right\} dt_1$$

$$\tilde{P} = (1-1/m)\,(x_{(n)}/y_{(m)})^{\phi}\, \{(1/n)+(1-1/n)(\theta/(\theta+\phi))\}, \quad (2.11c)$$
$$\text{if } \ln x_{(n)} < \ln y_{(m)}$$

$$= 1 - (1-1/n)\,(y_{(m)}/x_{(n)})^{\theta}\, \{1-(1-1/m)\,\theta/(\theta+\phi)\}, \quad (2.11d)$$
$$\text{if } \ln y_{(m)} < \ln x_{(n)}$$

When $\alpha = \beta$, the estimators (2.11 c) and (2.11 d) remain unchanged.

Case (iii) : α, β, θ, ϕ, Unknown

The samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ can be summarized by the complete sufficient statistics $(X_{(n)}, Z_1)$ and $(Y_{(m)}, Z_2)$ respectively. The conditional p.d.f.'s of t_1 given $(X_{(n)}, Z_1)$ and of t_2 given $(Y_{(m)}, Z_2)$ are

$$h_1(t_1 \mid x_{(n)}, z_1) = \begin{cases} (1-1/n)(n-2)\,(t_1+z_1-\ln x_{(n)})^{n-3}/z_1^{\,n-2}, \\ \qquad \text{if } \ln x_{(n)} - z_1 < t_1 < \ln x_{(n)} \\ 1/n, \qquad \text{if } t_1 = \ln x_{(n)} \end{cases} \quad (2.12)$$

$$h_2(t_2 \mid y_{(m)}, z_2) = \begin{cases} (1-1/m)(m-2)(t_2+z_2-\ln y_{(m)})^{m-3}/z_2^{m-2}, & \\ \qquad \text{if } \ln y_{(m)}-z_2<t_2<\ln y_{(m)} & \\ 1/m, \qquad \text{if } t_2=\ln y_{(m)} & \end{cases} \quad (2.13)$$

The MVU estimator of P in this case is

$$\tilde{P} = \iint p(t_1, t_2)\, h_1(t_1 \mid x_{(n)}, z_1)\, h_2(t_2 \mid y_{(m)}, z_2)\, dt_1\, dt_2$$

$$= \int_{(\ln x_{(n)}-z_1,\ \ln x_{(n)}]} h_1(t_1 \mid x_{(n)}, z_1) \left\{ \int_{(\ln y_{(m)}-z_2,\ \min(\ln y_{(m)}, t_1)]} h_2(t_2 \mid y_{(m)}, z_2)\, dt_2 \right\} dt_1 \quad (2.13^*)$$

For $t_1 < \ln y_{(m)} - z_2$, the integral in (2.13*) is zero, and $\tilde{P}=0$. On the other hand, for $t_1 > \ln x_{(n)} - z_1 > \ln y_{(m)}$, the integral in (2 13*) is unity, which means $\tilde{P}=1$. For remaining case we have :

$$\tilde{P} = (1-1/m)(1+\eta/z_2)^{m-2}\left[1-(1-1/n)\sum_{j=0}^{n-2}\{(2-n)_j/(m-1)_j\}\{(z_2+\eta)/z_1\}^j\right], \quad (2.14a)$$

if $\ln x_{(n)} - z_1 < \ln y_{(m)} - z_2 < \ln x_{(n)} < \ln y_{(m)}$

$$= (1-1/m)(1+\eta/z_2)^{m-2}\left[(1/n)+(1-1/n)\sum_{j=0}^{m-2}\{(2-m)_j/(n-1)_j\}\{z_1/(z_2+\eta)\}^j\right], \quad (2.14\text{ b})$$

if $\ln y_{(m)} - z_2 < \ln x_{(n)}) - z_1 < \ln x_{(n)} < \ln y_{(m)}$

$$= 1-(1-1/n)(1-\eta/z_1)^{n-2}\left[(1/m)+(1-1/m)\sum_{j=0}^{n-2}\{(2-n)_j/(m-1)_j\}\{z_2/(z_1-\eta)\}^j\right], \quad (2.14\text{ c})$$

if $\ln x_{(n)} - z_1 < \ln y_{(m)} - z_2 < \ln y_{(m)} < \ln x_{(n)}$

$$= 1-(1-1/n)(1-\eta/z_1)^{n-2}\left[1-(1-1/m)\sum_{j=0}^{m-2}\{(2-m)_j/(n-1)_j\}\{(z_1-\eta)/z_2\}^j\right], \quad (2.14\text{ d})$$

if $\ln y_{(m)} - z_2 < \ln x_{(n)} - z_1 < \ln y_{(m)} < \ln x_{(n)}$

where $\eta = \ln(x_{(n)}/y_{(m)})$.

Again when $\alpha=\beta$, the results (2.14a) – (2.14 d) remain unchanged.

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HARMONIC CONSTANTS IN THE NORTHERN ARABIAN SEA

BY

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Abstract. Knowledge of harmonic constants is important to predict water elevation at a particular point inside a sea or on a coastal point. Harmonic constants for a coastal point can be calculated through the Fourier Analysis of a waterlevel fluctuation data. These values are known only at a very few points on the coasts of Pakistan, Iran and Oman, whereas, no information regarding Harmonic constants is available in off-shore areas. To fill the gaps, mathematical model of the northern Arabian Sea is developed to reproduce the partial tides. Major partial tides M_2, S_2, K_1 and O_1 are used in the study. Harmonic constants for all four partial tides are presented for important points on the coastal and in the off-shore areas.

Introduction :

The sea area north of 20° N of the Arabian Sea is used in the mathematical model,/Fig. 1/. Hydrodynamical—Numerical method is used to reproduce velocity components and waterlevels as function of time in the area. This information is used to evaluate values of harmonic constants as function of frequency for different partial tides.

Numerical Model :

Hydrodynamical-Numerical method by (Hansen 1957) is used to solve the system of equations :

—the equation of continuity and

—the Navier-Stoke's equations.

The equations can be used to reproduce the tidal processes in the areas for which the bottom topography and coastal geometery are known. Vertically integrated form of this system including the whole fluid mass is :

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \epsilon_{ij} v_j + \frac{r}{h+\zeta} (v_j . v_j)^{\frac{1}{2}} \quad v_i - A_h \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

$$+ g \frac{\partial \zeta}{\partial x_i} = 0$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x_j} ((h+\zeta) v_j) = 0$$

$$(i, j = 1, 2)$$

Where V_i are the components of the vertically integrated horizontal velocity, ζ the water elevation, h the water depth, ϵ_{ij} Coriolis tensor, r a friction coefficient (0.003). A_h an eddy coefficient (2.7×10^7 cm^2/sec) and g the gravity acceleration.

In addition to these equations following initial and boundary conditions have been considered.

As an initial condition the waterlevel and the velocity components are taken equal to zero.

No-slip condition is satisfied on the solid boundary.

Waterlevels are prescribed as a function of time on the open boundary:

$$\zeta (t) = A \cos (\sigma t + \kappa)$$

Where A is the amplitude and κ the phase with respect to frequency σ.

The velocity gradients in the normal direction are taken to be equal to zero at the open boundary. The solution of this initial boundary value problem is independent of initial conditions after a sufficient long computations.

This initial boundary value problem is solved with the help of explicit finite difference technique discussed in / 1 /. The sea area is covered with a computational grid with 450 computational points /Fig. 1/. Grid size is 0.5°. This grid size results in a time step of 120 sec. A value of 2.7×10^7 cm^2/sec is used for a coefficient horizontal eddy viscosity. Tidal propagation is taken as the only driving force in the model and is prescribed on the open boundaries. One of the open boundaries is the inlet of the Persian Gulf and the other is situated between DIU HEAD and MASIRAH.

Results

44 computational points shown in Fig. 2, are selected for discussion of the results. 20 out of these lie in the off-shore area and 24 on the coastal line. Out of the computational point lying on the coastal line, 5 are the existing tidal gauges, PORBANDR, KARACHI, ORMARA, PANSI and MUSKAT. These are used to check the accuracy of the computed results. 19 points are located on the coasts of PAKISTAN, IRAN and OMAN.

From 20 computational points lying in the off-shore area : 7 points from A to G are situated in the Bay of OMAN in direction of the entrance of the Persian Gulf, 7 points from H to N are lying parallel to Pakistan—India coast, 6 points from 0 to T are taken parallel to Pakistan-Iran coast.

Results are presented in three tables. The results given in Table I depict the reproduction ability of the mathematical model. The values of harmonic constants known at the guages : PORBANDAR, KARACHI, ORMARA, PASNI and MUSKAT are used to check the accuracy of the computed results. The results reproduced for the guages PORBANDAR, KARACHI and PASNI are in very good agreement, whereas there is a little difference at the gauges ORMARA and MUSKAT. At the gauge MUSKAT this difference is due to its geometrical situation, as it is under the influence of two open boundaries, defined in the mathematical model. One can doubt about the observed value at guage

ORMARA, as the numerical results are in good agreement at neighbouring guages.

Table II contains the values of harmonic constants for the computational points on the coast. These are important coastal locations where no observation regarding the collection of data of waterlevel fluctuation has been made. These values can be used for prediction of waterlevels on these places and to check the accuracy of the results of short sets of observation at a temporary tidal gauges.

Table III contains the values for the computational points in open sea areas. The values at the points (0—T) are under direct influence of the values prescribed on the open boundary between MASIRAH and DIU HEAD. The values at the points (A—G) in the bay of OMAN are under the influence of the values on both the open boundaries. The degree of accuracy of these values can be examined only when more observational values may be available for comparison. The values of harmonic constants at the points (H—N) can be having a very high degree of accuracy, as the value at representative coastal guage KARACHI for these values has been reproduced very accurately.

The values of waterlevels at time of meridian passage at Greenwich ($t=o$) and at $t=\frac{T}{4}$ respectively ξ_1 and ξ_2 are also given in tables. These values are used to compute the amplitudes and phases (in degree based on meridian passage at Greenwich).

$$A = \sqrt{\xi_1^2 + \xi_2^2}, \qquad \kappa = \tan^{-1} \frac{\xi_1}{\xi_2}$$

The values of the harmonic constants can be used for the places on the coasts and in open sea, where no measurements are available but on one's own risk.

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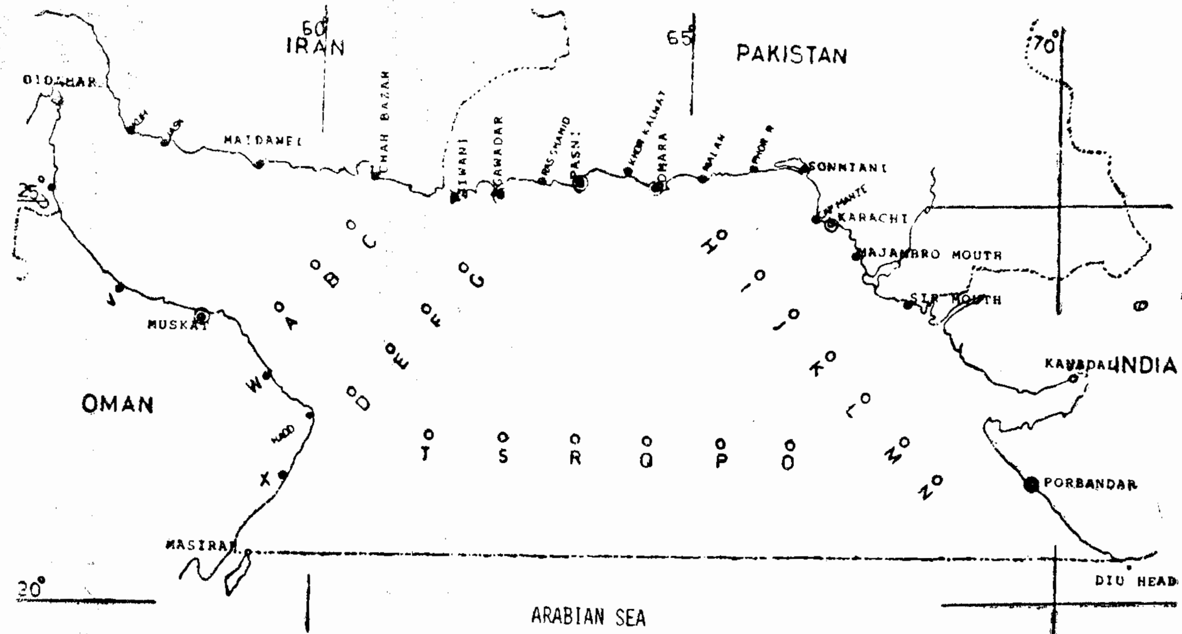

Fig.1. Special points in the Northern Arabian Sea. Coastal Points (●). Existing Gauges (◎) and Points in the open Sea area (o).

Tidal Constiuent		M_2		S_2		K_1		O_1	
Site		C	O	C	O	C	O	C	O
Porbandar	A	67.92	65.0	27.57	24.0	45.17	46.0	24.11	24.0
	κ	161.43	157.0	191.51	220.0	345.89	336.0	342.10	342.0
Karachi	A	79.91	79.8	31.47	29.6	39.22	41.1	23.68	20.0
	κ	166.34	163.7	194.67	193.9	347.51	342.2	343.93	343.2
Ormara	A	69.54	70.0	27.27	24.0	37.72	43.0	22.94	18.0
	κ	166.29	156.1	194.24	176.0	346.47	340.0	343.68	343.3
Pasni	A	68.98	69.0	27.07	26.0	37.38	31.0	22.91	24.0
	κ	166.37	165.0	194.48	192.0	346.42	346.0	343.79	346.3
Muskat	A	69.75	63.3	27.27	23.7	36.00	38.8	22.48	20.2
	κ	171.74	159.8	199.33	189.8	347.76	341.4	345.71	342.4

Tab. I. Computed (C) and Observed (O) Amplitude (A) and Phase (κ) of the major Tidal Constituents.

Tab. II.

NO.	SITE	POSITION		A	SEMI-DIURNAL TIDE		DIURNAL TIDE	
		LAT.	LONG.	κ	M_2	S_2	K_1	O_1
1.	Sir Mouth	23.66	68.12	A	77.15	30.28	39.63	23.53
				κ	165.37	193.83	347.99	343.59
				ζ_1	-74.65	-29.40	38.96	22.57
				ζ_2	19.49	-7.15	-8.29	-6.63
2.	Majambro Mouth	24.10	67.32	A	77.50	30.48	39.82	23.57
				κ	166.34	194.03	347.37	343.66
				ζ_1	-75.31	-29.57	38.33	22.62
				ζ_2	18.31	-7.39	-8.59	-6.65
3.	Cap Manze	24.80	66.90	A	37.32	28.07	38.60	23.34
				κ	163.04	190.68	344.89	342.05
				ζ_1	-70.13	-28.37	37.26	22.20
				ζ_2	21.39	-5.35	-10.08	-7.19
4.	Sonmiani	25.27	66.34	A	74.2	28.3	34.0	21.0
				κ	175.3	205.6	351.7	258.6
				ζ_1	-74.0	-29.6	33.7	-4.2
				ζ_2	6.1	-12.2	-4.9	-20.6
5.	Phor River	25.5	65.8	A	73.51	28.87	38.36	23.26
				κ	166.54	194.34	346.84	343.76
				ζ_1	-71.49	-27.97	37.35	22.33
				ζ_2	17.11	-7.15	-8.74	-6.51
6.	Malan	25.4	65.3	A	70.74	27.74	38.00	23.08
				κ	166.09	193.99	346.54	343.59
				ζ_1	-68.67	-26.92	36.96	22.14
				ζ_2	17.01	-6.71	-8.85	-6.52
7.	Khor Kalmat	25.3	64.2	A	69.10	27.11	37.54	22.94
				κ	166.32	194.27	346.40	343.70
				ζ_1	-67.14	-26.27	36.49	22.02
				ζ_2	16.35	-6.68	-8.83	-6.44

CONT.

Tab. II.

		° '	° '					
8.	Ras Shahid	25.3	61.0	A	68.96	27.08	37.23	22.88
				κ	166.69	194.70	346.42	343.88
				ζ_1	-67.11	-26.20	36.19	21.98
				ζ_2	15.85	-6.87	-8.74	-6.35
9.	Gawadar	25.11	62.33	A	69.12	27.16	37.11	22.86
				κ	166.95	194.99	346.47	344.00
				ζ_1	-67.33	-26.23	36.08	21.98
				ζ_2	15.61	-7.03	-8.68	-6.30
10.	Jiwani	25.20	61.70	A	69.41	27.28	36.81	22.92
				κ	167.22	195.31	346.55	344.13
				ζ_1	-67.69	-26.31	36.00	22.00
				ζ_2	15.36	-7.10	-8.61	-6.25
11.	Chah Bazar	25.30	60.70	A	70.68	27.75	36.81	22.92
				κ	168.79	197.06	347.08	344.81
				ζ_1	-69.33	-26.53	35.88	22.09
				ζ_2	13.74	-8.14	-8.23	-6.00
12.	Maidanal	25.40	59.20	A	73.24	28.60	36.84	23.04
				κ	176.06	199.61	348.25	345.87
				ζ_1	-72.35	-26.94	36.06	22.34
				ζ_2	11.39	-9.60	-7.50	-5.62
13.	Jask	25.7	57.9	A	81.03	31.43	37.89	23.86
				κ	176.96	206.39	351.90	348.74
				ζ_1	-80.91	-28.15	37.51	23.40
				ζ_2	4.30	-13.97	-5.34	-4.66
14.	Kuh	25.8	57.4	A	81.50	31.22	38.16	224.25
				κ	179.29	209.42	354.02	350.18
				ζ_1	-81.50	-27.20	37.95	23.90
				ζ_2	1.01	-15.34	-3.98	-4.14
15.	Hadd	23.55	59.80	A	65.79	25.84	35.56	22.19
				κ	169.06	197.08	346.50	344.81
				ζ_1	-64.60	-24.70	34.58	22.95
				ζ_2	12.49	-7.59	-8.30	-5.82

CONTD.

16.	U	25.0	56 50	A	75.21	28.15	32.98	21.38
				K	182.88	212.25	352.71	351.11
				ζ_1	-75.12	-23.81	32.71	21.12
				ζ_2	-3.78	-15.02	-4.18	-3.30
17	V	24.75	57 75	A	73.15	25.25	35.56	22.44
				K	175.20	203.76	349.76	347 72
				ζ_1	-72.89	-25.86	34.99	21.93
				ζ_2	6.13	-11.39	-6.32	-4.77
18.	W	23.25	59.75	A	68.06	26.67	35.86	22.37
				K	170.22	198.30	347.26	345.36
				ζ_1	-67.07	-25.32	34.98	21.64
				ζ_2	11.56	-8.38	-7.91	-5.65
19.	X	22.00	59 5	A	61.75	24.38	34.31	21.64
				K	168.66	196.62	345.95	344.79
				ζ_1	-60.54	-23.36	33.28	20.88
				ζ_2	12.14	-6.97	-8.33	-5.68

Tab. II

Major Tidal Constituents on
Computational Points on the Coast of Northern Arabian Sea.

Tab. III.

NO.	SITE	POSITION LAT. N ° '	POSITION LONG. E ° '	A / κ	SEMI-DIURNAL TIDE M_2	SEMI-DIURNAL TIDE S_2	DIURNAL TIDE K_1	DIURNAL TIDE O_1
1.	A	24.0	59.5	A	69.04	27.02	36.00	22.46
				κ	170.44	198.59	347.41	345.46
				ζ_1	-68.08	-25.61	35.14	29.74
				ζ_2	11.47	-8.62	-7.85	-5.64
2.	B	24.5	60.0	A	69.43	27.20	36.28	22.60
				κ	169.70	197.90	347.17	345.13
				ζ_1	-68.31	-25.89	35.38	21.84
				ζ_2	12.42	-8.36	-8.06	-5.80
3.	C	25.0	60.5	A	69.82	27.39	36.63	22.77
				κ	168.77	196.99	346.96	344.77
				ζ_1	-68.48	-26.20	35.68	21.97
				ζ_2	13.69	-8.00	-8.26	-5.98
4.	D	23.0	60.5	A	65.54	25.74	35.69	22.21
				κ	168.49	196.53	346.09	344.47
				ζ_1	-64.22	-21.68	34.62	21.40
				ζ_2	+13.08	-7.32	-8.58	-5.95
5.	E	23.5	61.0	A	66.62	26.15	36.10	22.39
				κ	168.18	196.25	346.07	344.31
				ζ_1	-65.20	-25.11	35.04	21.56
				ζ_2	13.64	-7.32	-8.69	-6.06
6.	F	24.0	61.5	A	67.39	26.45	36.47	22.55
				κ	167.76	195.83	346.10	344.16
				ζ_1	-65.86	-25.45	35.40	21.70
				ζ_2	14.29	-7.22	-8.76	-6.16
7.	G	24.5	62.0	A	68.01	26.70	36.79	22.69
				κ	167.32	195.38	346.18	344.06
				ζ_1	-66.35	-25.74	35.72	21.81
				ζ_2	14.92	-7.08	-8.78	-6.24

CONT:

Tab. III.

8.	H	24.5	65.5	A	70.05	27.45	38.04	23.07
				κ	165.61	193.52	346.18	343.33
				ζ_1	-67.85	-26.70	36.94	22.10
				ζ_2	17.41	-6.42	-9.08	-6.62
9.	I	24.0	66.0	A	69.90	27.39	38.26	23.11
				κ	164.48	192.37	345.58	342.78
				ζ_1	-67.35	-26.76	37.05	22.07
				ζ_2	18.70	-5.87	-9.53	-6.84
10.	J	23.5	66.5	A	69.11	27.00	38.52	23.10
				κ	163.37	191.31	345.71	342.42
				ζ_1	-66.22	-26.48	37.33	22.02
				ζ_2	19.77	-5.30	-9.50	-6.98
11.	K	23.0	67.0	A	69.16	26.95	39.19	23.21
				κ	162.93	191.04	345.66	342.22
				ζ_1	-66.11	-26.45	37.97	22.10
				ζ_2	20.31	-5.16	-9.71	-7.09
12.	L	22.5	67.5	A	67.15	26.07	39.72	23.12
				κ	161.09	189.40	345.64	341.70
				ζ_1	-63.53	-25.72	38.48	21.95
				ζ_2	21.76	-4.26	-9.85	-7.26
13.	M	22.0	68.0	A	66.02	25.66	40.85	23.28
				κ	159.33	187.95	344.53	340.88
				ζ_1	-61.77	-25.42	39.37	22.00
				ζ_2	23.31	-3.55	-10.90	-7.63
14.	N	21.5	68.5	A	62.65	24.46	42.20	23.31
				κ	158.40	187.77	344.57	340.81
				ζ_1	-58.25	-24.23	40.68	22.01
				ζ_2	23.07	-3.31	-11.23	-7.60
15.	O	22.5	66.5	A	65.00	25.21	35.84	22.20
				κ	163.73	191.98	345.04	343.67
				ζ_1	-62.39	-24.66	34.63	21.31
				ζ_2	18.20	-5.23	-9.25	-6.24

CONT

16.	P	22.5	65.5	A	64.76	25.17	36.41	22.37
				κ	164.55	192.59	344.77	343.28
				ζ_1	-62.42	-24.56	35.13	21.43
				ζ_2	17.25	-5.49	-9.57	-6.44
17.	Q	22.5	64.5	A	64.42	25.12	37.02	22.52
				κ	165.19	193.15	344.53	342.95
				ζ_1	-62.18	-24.46	35.68	21.54
				ζ_2	16.47	-5.71	-9.87	-6.60
18.	R	22.5	63.5	A	64.19	25.11	37.71	22.69
				κ	165.85	193.80	344.55	342.69
				ζ_1	-62.25	-24.39	36.34	21.66
				ζ_2	15.69	-5.99	-10.05	-6.75
19.	S	22.5	62.5	A	64.03	25.10	38.47	22.88
				κ	166.47	194.43	344.70	342.48
				ζ_1	-62.25	-24.31	37.10	21.82
				ζ_2	14.98	-6.26	-10.15	-6.89
20.	T	22.5	61.5	A	64.01	25.14	39.32	23.08
				κ	167.22	195.24	344.93	342.26
				ζ_1	-62.43	-24.26	37.97	21.98
				ζ_2	14.16	-6.61	10.22	-7.63

Tab.III. Major Tidal Constituents on the Computational Points in the open Sea Area of the Northern Arabian Sea.

JULY 5 - 11th, 1982

CONFERENCE ON ORDERED SETS AND ITS APPLICATIONS

LYON (France)

A conference on ordered sets, and its applications will be held in Lyon. The purpose of the conference is to present the most significant and the most recent results in these fields :

Ordered Sets and Set Theory (infinite combinatorics, partition calculus, confinality, chain conditions, topology on ordered sets and lattices, . .)

Ordered Structures (connections with model theory, ordered groups, Boolean algebras, lattices, . . .),

Algebra and Ordered Sets (algebraic methods, chain conditions in algebra, clones, . . .),

Combinatorics of Ordered sets (dimension, Dilworth number, jump number, Sperner properties, fixed point property, retracts, enumeration, . . .),

Ordered Sets and Computer Science (recursivity and algorithms, computational complexity, scheduling, sorting, linear and discrete programming, fixed point methods and semantic of programmation, . . .),
Applications of Ordered Sets to Social and Economic Sciences (Social choice, . . .)

The programme will include a few selected lectures intended to survey some broad areas. As well there will be specialized lectures, contributed lectures, and problem sessions.

Speakers : At the present time, the list of speakers includes :

C. Benzaken	(Grenoble)	E. Harzheim	(Dusseldorf)
E. Corominas	(Lyon)	E.C. Milner	(Calgary)
P. Erdos	(Budapest)	D. Monk	(Boulder)
C. Flament	(Aix-Marseille)	I. Rival	(Calgary)
F. Fraisse	(Marseille)	I.G. Rosenberg	(Montreal)
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