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BANACH ALGEBRAS HOMEOMORPHIC WITH c_0

By

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Abstract.

In this note we give a necessary and sufficient condition on basis for a Banach algebra with an orthogonal unconditional basis to be isomorphic and homeomorphic with c_0.

A basis $\{x_i\}$ in a topological algebra is said to be orthogonal [3] if $x_i x_j = \delta_{ij} x_i$, in which δ_{ij} is the Kronecker's delta. Indeed, the Banach algebra c_0 of all complex sequences converging to zero, endowed with pointwise operations and sup-norm, has an unconditional (see for example [1] for definition) orthogonal basis, viz. (e_i) where $e_i = \{\delta_{ij}\}_{j \geq 1}$ for $i = 1, 2, \ldots$.

Also, the Banach algebra l_1 of all complex sequences $a = \{a_i\}$ with $\| a \|_1 = \Sigma \mid a_i \mid < \infty$ and pointwise operations has an unconditional orthogonal basis $\{e_i\}$. For l_1, we know the following:

Theorem A. ([2]). Let A be a complex normed algebra with an orthogonal unconditional basis $\{x_i\}$. Suppose there exists $e \in A'$ (topological dual of A) with $e(x_i) = 1$ for all $i \geq 1$. Then A is isomorphic with l_1 iff the basis $\{x_i\}$ is boundedly complete [1]. If A, in addition, is complete, then A is homeomorphic with l_1.

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In this note, we present an analogous result for c_o. But first recall that if A is a normed algebra with an orthogonal basis $\{x_i\}$, then for each $x \in A$, there is a unique sequence $\{\lambda_i(x)\}$ of complex numbers depending upon x such that $x = \Sigma_{i=1}^{\infty} \lambda_i(x) x_i$. Without loss of generality we may assume that $\| x_i \| = 1$ for all $i \geq 1$. Thus the convergence of the series implies that

$$\lim_{i \to \infty} | \lambda_i(x) | = \lim_{i \to \infty} \| \lambda_i(x) x_i \| = 0.$$

In other words, the sequence $\{\lambda_i(x)\} \in c_o$ for all $x \in A$, Moreover, each $\lambda_i(x) = \lambda_i$ (called a coordinate functional) is a multiplicative linear functional on A, [3]. Hence $\lambda_i \in A'$ for all $i \geq 1$, provided A is a Banach algebra. Now we prove the following:

Theorem. Let A be a complex Banach algebra with an uncondiional orthogonal basis $\{x_i\}$. Suppose there exists an $e' \in A''$ (bidual of A, see [4]) with $e'(\lambda_i) = 1$ for all $i \geq 1$, where $\{\lambda_i\}$ is the sequence of coordinate functionals associated with $\{x_i\}$. Then A is isomorphic and homeomorphic with c_o iff the basis $\{x_i\}$ is shrinking, [1].

Proof. Since $\{e_i\}$ is a shrinking orthogonal unconditional basis of c_o, in view of Theorem 1.8 [3], the "only if" part follows. For the "if" part, assume $\{x_i\}$ is shrinking. Then, by Theorem 5, § 4, Chapt. IV [1], $\{\lambda_i\}$ is an unconditional boundedly complete basis of the Banach space A'. By Proposition 4.1 [3], there exists a multiplication and an equivalent norm on A'' making it a Banach algebra with $\{\lambda_i\}$ as its orthogonal basis. Hence in view of the hypothasis, by Theorem A, A' is isomorphic and homeomorphic with the Banach algebra l_1. This isomorphism is given by $\phi' : A' \to l_1$ with $\phi'(f) = \{\lambda_i'(f)\} \in l_1$, where $f = \Sigma \lambda_i'(f) \lambda_i \in A'$, because $\{\lambda_i\}$ forms a basis of A'. On the other hand, the map $\phi : A \to c_o$ defined by $\phi(x) = \{\lambda_i(x)\}$, where $x = \Sigma \lambda_i(x) x_i$, is an injective (by the definition of basis) algebra homomorphism (easy to verify). Furthermore, each λ_i being a multiplicative linear functional on the Banach algebra A is continuous. Actually, $\| \lambda_i \| \leq 1$ for all $i \geq 1$, [5].

Therefore

$$\| \phi(x) \| = \sup_{i \geq 1} | \lambda_i(x) | \leq \| x \| ,$$

shows that ϕ is continuous. Thus ϕ is a continuous embedding of A into c_0. We show that the inverse map ϕ'^{-1}, $l_1 \to A'$ is actually the conjugate (or adjoint) map of ϕ. For this let $\{a_i\} \varepsilon l_1$. Then there is $f \varepsilon A'$ with $f = \Sigma a_i \lambda_i$ *i.e.* $\phi'^{-1}(\{a_i\}) = f$. Now if $x = \Sigma \lambda_i(x) x_i \varepsilon A$, then $\{\lambda_i(x)\} \varepsilon c_0$ and we have that $f(x) = \Sigma a_i \lambda_i(x)$.

But this equation can be written in the duality form as :

$$< x, \phi'^{-1}(\{a_i\}) > = f(x) = \Sigma a_i \lambda_i(x) = < \{\lambda_i(x)\}, \{a_i\} >$$
$$= < \phi(x), \{a_i\} > .$$

This shows that ϕ'^{-1} is, indeed, the adjoint map of ϕ. Since ϕ' is an isomorphism and homeomorphism and so is ϕ'^{-1}, by Theorem 7.8 ([4], Chapt. IV), $\phi(A) = c_0$ i.e., A is isomorphic with c_0. Since ϕ is already shown to be continuous, by the Banach Open Mapping theorem, ϕ is a homeomorphism.

REFERENCES

1. Day, M.M. : Normed linear spaces, *Springer Verlag. N.Y., Berlin,* 1958.

2. Husain, T. : Topological algebras isomorphic with l_1 or H (D), *Punjab University Journal of Mathematics Vol. XIV-XV (1981-82), 1-14 Lahore, Pakistan.*

3. Husain, T. and Watson, S. : Topological algebras with orthogonal Schauder bases ; *Pacific Journal of Mathematics* **91** (1980), 339-347.

4. Schaefer, H.H. : Topological vector spaces ; *Springer Verlag, N.Y,. Berlin* 1971.

5. Zelazko, W. : Banach algebras ; *Elsevier, Amsterdam* 1973.

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A COMMON FIXED POINT THEOREM FOR THREE MAPPINGS ON A COMPACT METRIC SPACE

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The following theorem was proved in [1].

Theorem 1. Let T and I be commuting mappings and let T and J be commuting mappings of a compact metric space (X, d) into itself satisfying the inequality.

$$d(Tx, Ty) < \max \{ d(Ix, Jy), \ d(Ix, Tx), \ d(Jy, Ty), \ d(Ix, Ty), \ d(Jy, Tx) \} \quad (1)$$

for all x, y in X for which the right hand side of the inequality is positive. If for each x in X there exists y in X such that

$$Tx = Iy = Jy$$

and if T, I and J are continuous, then T, I and J have a unique common fixed point

We now prove the following generalization of theorem 1 which shows that the condition that I and J be continuous is unnecessary.

Theorem 2. Let T and I be commuting mappings and let T and J be commuting mappings of a compact metric space (X, d) into itself satisfying inequality (1) for all x, y in X for which the right hand side of the inequality is positive.

If T is continuous, then T, I and J have a unique common fixed point z.

Proof. We note that since T is continuous, T maps compact sets into compact sets. It follows that since X is compact, $T^n X$ is compact for $n = 1, 2, \ldots\ldots$. Further, it is obvious that

$$T^{n+1} X \subseteq T^n X$$

for $n = 1, 2, 3, \ldots$ and it now follows that

$$F = \bigcap_{n=1}^{\infty} T^n X = T F$$

is a non-empty, compact subset of X.

Now let x be an arbitrary point in F. Then x is in $T^n X$ for $n = 1, 2, \ldots$, and so Ix is in $IT^n X = T^n I X$ for $n = 1, 2, \ldots$. It follows that

$$Ix \in \bigcap_{n=1}^{\infty} T^n I X \subseteq \bigcap_{n=1}^{\infty} T^n X = F.$$

Thus I maps F into F.

Similarly J maps F into F.

Since d is a continuous mapping of the compact set F^2 into the reals, there exist points z, w in **F** with Tz, Tw in $F = TF$ such that

$$d(Tz, Tw) = \sup \{ d(x, y) : x, y \in \mathbf{F} \} = \mathbf{M}.$$

Let us suppose that

$$\max \{ d(Iz, Jw), d(Iz, Tz), d(Jw, Tw) \} > 0.$$

Then inequality (1) holds for z, w and so

$$\begin{aligned} M &= d(Tz, Tw) \\ &< \max \{d(Iz, Jw), d(Iz, Tz), d(Jw, Tw), d(Iz, Tw), d(Jw, Tz) \} \\ &\leq M \end{aligned}$$

since Iz, Jw are in **F**, giving a contradiction. This implies that

$$d(Iz, Jw) = d(Iz, Tz) = d(Jw, Tw) = 0$$

or

$$Iz = Jw = Tz = Tw.$$

It now follows that

$$M = d(Tz, Tw) = 0$$

and so the set $F = TF$ consists of the single point $z = w$ where z must be a fixed point of T. Since I and J map $F = \{ z \}$ into F, z is also a fixed point of I and J.

The uniqueness of this common fixed point z is easily proved.

The corollaries follow immediately.

Corollary 1. Let T and I be commuting mappings of a compact metric space (X, d) into itself satisfying the inequality

$d(Tx, Ty) < \max \{ d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx) \}$ for all x, y in X for which the right hand side of the inequality is positive. If T is continuous, then T and I have a unique common fixed point z.

Corollary 2. Let I and J be mappings of a compact metric space (X, d) into itself satisfying the inequality

$d(x, y) < \max \{ d(Ix, Jy), d(Ix, x), d(Jy, y), d(Ix, y). d(Jy, x) \}$ for all x, y in X for which the right hand side of the inequality is positive. Then I and J have a unique common fixed point z.

Corollary 3. Let I be a mapping of a compact metric space (X, d) into itself satisfying the ineqnality

$$d(x, y) < \max \{ d(Ix, x), d(Ix, y) \}$$

for all x, y in X for which the right hand side of the inequality is positive. Then I has a unique fixed point z.

REFERENCE

1. B. Fisher, "Three mappings with a common fixed point", Math. Sem. Notes, Kobe Univ., **10** (1982), 293-302.

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Introduction.

The aim of this paper is to outline an alternative approach to Chevalley groups which is suggested by results of R. Steinberg, especially § 11 of [6], and by [1]. The approach we have in mind works with a system of axioms which involve only a root system and a commutative ring, and in a sense avoids Chevalley bases. Needless to say, this would have been impossible without knowing the contents of [2] and [6]. An advantage of this approach is that problems like those mentioned in [2, p. 64] vanish automatically. This paper is organized as follows : In § 1 we prove an analogue of [1] for a class of Lie algebras. Then, in § 2, by simply reversing a procedure given in the proof of proposition (1.1), we construct, for a given root system which has no multiple bonds, a function N, defined on pairs of independent roots (u, v) such that $N_{u,v}$ is ± 1 if and only if $u + v$ is a root, and verify the Jacobi identity for N. That such a function exists is nothing new ; see, for example [2, p. 24], [8] or [5, p. 285], which also gives the briefest solution to date of this problem. We have thought doing this worthwhile as the function N arises naturally from the root system. The construction of a Lie algebra for a given root system is then immediate. This construction may also interest those who do machine computations as definition (2.3) can be translated into an algorithm which will produce positive roots and structure constants one after the other.

In the final section we give a system of axioms for Chevalley groups over commutative rings, and making use of results of R. Steidberg together with those of the previous sections, we outline a proof of existence of these groups.

The arguments of this paper are of an elementary character and in essence involve only the Jacobi identity and some technicalities on root systems.

Our references for root systems and Chevalley groups are [2, 4, 6].

1. A Uniqueness Theorem.

Let R be an irreducible root system with no multiple bonds, R^+ a positive system of roots, S the corresponding simple system of roots and A a commutative ring. In this section we consider Lie algebras (L, [,]) over A with the following properties :

(*a*) L is generated by elements X_r ($r \in R$) such that $aX_r \neq 0$ for all non-zero $a \in A$.

(*b*) $[X_r, X_s] = N_{r,s} X_{r+s}$, if $r + s \in R$, $N_{r,s}$ being an element of A, and $[X_r, X_s] = 0$ if $r + s \neq 0$ and $r + s \notin R$.

(*c*) $[X_s, X_{-s}; X_r] = < r, s > X_{r,s}$ being a *simple* and r an arbitrary root : here $< r, s >$ is the Cartan integer corresponding to the pair of roots (r, s).

Proposition 1.1. There exist units c_r ($r \in R$) such that if we set $X_r' = c_r X_r$, $[X_r', X_s'] = N'_{r,s} X'_{r+r}$ $(r + s \neq 0)$ and $H_r = [X'_r, X'_{-r}]$ for all $r, s \in R$ then

(i) $[H'_r, X_s'] = < s, r > X_s'$ ($r, s \in R$)

(ii) $N'_{r,s} = \pm 1$, if $r + s \in R$

(iii) $N'_{r,s}$ is completely determined once an ordering on S has been fixed.

(iv) If $[X_a, X_{-a}]$ ($a \in S$) and X_r ($r \in R$) form a basis of L then every automorphism of R extends to an automorphism of L.

(v) In any case, every automorphism of R extends to an automorphism of the Lie algebra with generators Y_a ($a \in R$) and relations $[Y_a, Y_b] = N'_{a,b} Y_{a+b}$ (a, b being independent roots).

Proof. (After [1]). Fix an ordering on S. Let $\sigma \varepsilon R^+$ be a non-simple root and let α be the first simple root such that $(\sigma, \alpha) > 0$. Then $\sigma - \alpha$ is a root but $\sigma + \alpha$ is not a root.

(A) Applying the Jacobi identity to $X_\alpha, X_{-\alpha}, X_\sigma$ we find that $N_{\alpha, \sigma - \alpha} N_{\sigma, -\alpha} = 1$. Hence $N_{\alpha, \sigma - \alpha}$ is a unit ; likewise $N_{-\alpha, -\sigma + \alpha}$ is also a unit, so scaling X_σ and $X_{-\sigma}$ we can assume that $N_{\alpha, \sigma - \alpha} N_{-\alpha, -\sigma + \alpha} = -1$: this is the normalization which (i) requires, as we will soon see.

We next show that with this normalization we always have

$N_{u, v} N_{-u, -v} = -1$

u, v being positive roots such that $u + v$ is (★) a root.

Let $\sigma = u + v$, let α be the first simple root such that $(\sigma, \alpha) > 0$ and let $R_{uv\alpha}$ denote the integral closure of u, v, and α in R. If $R_{uv\alpha}$ is of type A_2 then u, v form a basis of $R_{uv\alpha}$, so u or v is α, and (*) holds by definition, and therefore also when height of σ is 2. So suppose $R_{uv\alpha}$ is of type A_3. Choose a simple system of roots, say a, b, c, corresponding to the positive system $R_{uv\alpha} \cap R^+$. We may assume that $< a, b > = < b, c > = -1$ and $< a, c > = 0$. Then σ must be the sum of these simple roots. But σ has only two decompositions as sums of two roots in $R_{uv\alpha} \cap R^+$, namely

$\sigma = a + (b + c) = (a + b) + c$, and α is a or c (so

$N_{a, b+c} N_{-a, -b-c} = -1$ or

$N_{c, a+b} N_{-c, -a-b} = -1$).

By the Jacobi identity we have

$$N_{b,c} N_{b+c, a} = N_{a,b} N_{c, a+b},$$

$$N_{-b,-c} N_{-b-c,-a} = N_{-a,-b} N_{-c,-a-b}.$$

By induction on heights we also have

$N_{a,b} N_{-a,-b} = N_{b,c} N_{-b,-c} = -1$, so multiplying the previous two equations and using the parenthetical remark above we find that $N_{u,v} N_{-u,-v} = -1$.

Let $H_r = [X_r, X_{-r}]$ $(r \in R)$, with the X_r normalized as above. By assumption, when r is simple, we have $[H_r, X_s] = \langle r, s \rangle X_s$ and $[H_{-r}, X_s] = \langle -r, s \rangle X_s$. Assume this is true for all roots of height less than N and that r is a root of height N. Let $r = \alpha + \beta$, where $\alpha \in S$ and $(r, \alpha) > 0$.

Applying the Jacobi identity to $X_\alpha, X_\beta, X_{-\alpha-\beta}$ we find that

$$N_{\alpha\beta} H_{\alpha+\beta} = N_{\beta,-\alpha-\beta} H_\alpha + N_{-\alpha-\beta,\alpha} H_\beta . \qquad (**)$$

As $[H_\alpha, H_\beta] = \langle \beta, \alpha \rangle X_\beta$ as well as $N_{\alpha,\beta} N_{\alpha+\beta,-\alpha} X_\beta$,
we have $\langle \alpha, \beta \rangle = N_{\beta,\alpha} N_{\alpha+\beta,-\beta}$.
Similarly, $\langle -\alpha, -\beta \rangle = N_{-\beta,-\alpha} N_{-\alpha-\beta,\beta}$.
By induction on heights we have
$[H_\beta, X_\alpha] = \langle \alpha, \beta \rangle X_\alpha$, so
$\langle -\beta, -\alpha \rangle = N_{-\alpha,-\beta} N_{-\alpha,-\beta,\alpha}$.
Multiplying (* *) by $N_{-\alpha,-\beta}$ and using $N_{\alpha,\beta} N_{-\alpha,-\beta} = -1$
we have :

$$-H_{\alpha+\beta} = N_{-\alpha,-\beta} N_{\beta,-\alpha\ \beta} H_\alpha + N_{-\alpha,-\beta} N_{-\alpha-\beta,\alpha} H_\beta$$
$$= \langle \alpha, \beta \rangle H_\alpha + \langle \beta, \alpha \rangle H_\beta .$$

Hence $H_{\alpha+\beta} = H_\alpha + H_\beta$, and therefore $[H_r, X_s] = \langle r, s \rangle X_s$ for all $s \in R$. This proves (i).

(B) To achieve (ii) we normalize X_σ and $X_{-\sigma}$ $(ht\, \sigma \geq 2)$ so that $N_{\alpha,\sigma-\alpha} = 1$ and $N_{-\alpha,-\sigma+\alpha} = -1$. Arguing as in (A) we find that this normalization determines all the constants $N_{u,v}$ if $u + v$ is a root and u, v are both positive or both negative. Moreover, $N_{u,v} N_{-u,-v}$ is still -1 so $[H_r, X_s] = \langle r, s \rangle X_s$ for all $r, s \varepsilon$ R. This implies that $< u, v > = N_{v,u} N_{v+u,-u}$. By considering the roots in the integral closure of u and v we find that the remaining structure constants are also completely determined.

(C) The proof of the remaining assertions is implicit in steps (A) and (B) and is left to the reader.

The following corollary has been known for quite some time : See [8, p. 51].

Corollary 1.2. [Steinberg]. *The existence problem for semi-simple Lie algebras is equivalent to the existence problem for Lie algebras whose root systems have no multiple bonds.*

Proof. Given a root system R with multiple bonds there exists a root system $\tilde{R}$ with no multiple bonds and an automorphism ρ of $\tilde{R}$ such that twisting $\tilde{R}$ according to p one obtains R : see [6, p. 175] for details.

As a semisimple Lie algebra corresponding to the root system $\tilde{R}$ is of the type considered above, we can extend the automorphism to an automorphism of this Lie algebra and consider its fixed points : this will be a Lie algebra with root system R. All of this follows from (1.1) and [7, p. 873-877]*.

Corollary 1.3 [3, p. 147] Let R be a root system with no multiple bonds, L a semi-simple Lie algebra whose root system is R, S a simple system of roots and ρ an automorphism of R which maps S into itself. If L_α ($\alpha \varepsilon$ R) are the root spaces of L then there is an automorphism σ which maps L_α into $L_{-\alpha}$ ($\alpha \varepsilon$ R) and which commutes with ρ.

Proof. We can choose a system of generators X_α ($\alpha \varepsilon$ R) such that

$[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ $(\alpha + \beta \neq 0)$ and

$[X_\alpha, X_{-\alpha}; X_\beta] = < \beta, \alpha > X_\beta$

[4, p. VI-2]. The automorphisms $\alpha \to -\alpha$ ($\alpha \varepsilon$ R) and σ commute and by (1.1) extend to commuting automorphisms of L

* See appendix.

2. A Construction

Let R, R^+ and S be as in § 1. Denote by $R_{ab\ldots}$ the integral closure of the roots a, b, ... in R. We wish to reverse the procedure given in the proof of (1.1) to construct a function N, defined on pairs of positive roots such that :

(2.1) : (a) $N_{u,v} = - N_{v,u}$;

(b) $N_{u,v} = 0$ if $u+v$ is not a root and $N_{u,v} = \pm 1$ otherwise;

(c) $N_{u,v} N_{u+v,w} + N_{v,w} N_{v+w,u} + N_{w,u} N_{w+u,v} = 0$,

for all $u, v, w \in R^+$.

We first record some properties of R which we require :

Lemma 2.2. Let u, v, w be distinct positive roots :

(i) If $< u + v, w > \; > 0$ then either $< u, w > \; = 1$ and $< v, w > \; = 0$, or $< u, w > \; = 0$ and $< v, w > \; = 1$.

(ii) If $u + v + w$ is a root then exactly two of $u + w$, $v + w$, $w + u$ are roots.

This is a consequence of the assumptions on R, namely, if a, b are distinct roots and $a + b \neq 0$ then the Cartan integer $< a, b >$ is 0, 1 or -1.

The following definition is more or less dictated by (1.1).

(2.3) Definition. Fix an ordering on S. Let u, v be positive roots such that $\sigma = u + v$ is a root. Let α be the first simple root such that $(\sigma, \alpha) > 0$. Set $N_{\alpha, \sigma - \alpha} = 1$, $N_{\sigma - \alpha, \alpha} = -1$.

If u, v are distinct from α define $N_{u,v}$ and $N_{v,u}$, by induction on height of $(u + v)$, by the identities :

(*) $N_{u - \alpha, \alpha} N_{u,v} + N_{v, u - \alpha} N_{\sigma - \alpha, \alpha} = 0$,

$N_{v,u} = - N_{u,v}$, in case $(u, \alpha) = 1$, $(v, \alpha) = 0$, and

$$(^{**})\quad N_{u,\,v-\alpha}\, N_{\sigma-\alpha,\,\alpha} + N_{v-\alpha,\,\alpha}\, N_{v,\,u} = 0,$$

$$N_{u,\,v} = -N_{v,\,u}\,,\ \text{in case } (u,\alpha) = 0,\ (v,\alpha) = 1.$$

If $u + v$ is not a root, set $N_{u,\,v} = 0$.

(2.4) Proposition. Let u, v, w be positive roots and let N be as in (2.3). Then

$$(^*)\quad N_{u,\,v}\, N_{u+v,\,w} + N_{v,\,w}\, N_{v+w,\,u} + N_{w,\,u}\, N_{w+u,\,v} = 0.$$

Proof. If $\sigma = u + v + w$ is not a root then there is nothing to prove. So let σ be a root. We may assume that $u + v$, $v + w$ are roots but $u + w$ is not a root (2.2) : call such a triple (u, v, w) an A_3 – triple. Denote the left-hand side of $(^*)$ by $J(u, v, w)$. Let α be the first simple root such that $< \sigma, \alpha > \; > 0$. If α is one of u, v or w then $(^*)$ follows from the definition of N. So assume α is distinct from u, v and w. Then, by (2.2), we have $< u + v, \alpha > = 1$ and $< w, \alpha > = 0$ or $< u + v, \alpha > = 0$ and $< w, \alpha > = 1$. Now we express, using (2.3), $J(u, v, w)$ as a linear combination of $J(u', v', w')$ with height of $(u' + v' + w')$ less than height of $(u + v + w)$ and apply induction. The details are as follows :

(A) Suppose $< u + v, \alpha > = 1$ (and $< w, \alpha > = 0$). Then $< u, \alpha > = 1$ and $< v, \alpha > = 0$ or $< v \ \alpha > = 1$ and $< u, \alpha > = 0$. In the first case $J(u, v, w)$ is, by definition of N,

$$N_{u,\,v}\, N_{u+v-\alpha,\,w} (N_{\alpha,\,u+v-\alpha})^{-1} +$$

$$N_{v,\,w}\, N_{v+w,\,u-\alpha} (N_{\alpha,\,u-\alpha})^{-1}.$$

Hence $(N_{\alpha,\,u-\alpha})\, J(u, v, w) \equiv J(u - \alpha, v, w)$ (using 2.3 $(^*)$).

In case $(u, \alpha) = 0$, $(v, \alpha) = 1$, $(w, \alpha) = 0$ we have

$$J(u, v, w) \equiv J(u, v - \alpha, w).$$

(B) Suppose $< u + v, \alpha > = 0$ and $< w, \alpha > = 1$. Then

$< u, \alpha > = < v, \alpha > = 0$ or $< u, \alpha > = 1$, $< v, \alpha > = -1$: $< v, \alpha >$ cannot be 1, else $< v + w, \alpha >$ would be 2, i.e., $v + w$ would be a simple root.

The first case follows by symmetry from (A). So suppose $< u, \alpha > = 1$, $< v, \alpha > = -1$. Then $(u-\alpha, w, v)$ and $(w-\alpha. u, v)$ are A_3 – triples. In this case

$$J(u,v,w) = N_{u,v} N_{u+v,w-\alpha} (N_{\alpha,w-\alpha})^{-1} +$$

$$N_{v,w} N_{v+w,u-\alpha} (N_{\alpha,u-\alpha})^{-1}.$$

Now $0 = J(u-\alpha, w, v) =$

$$N_{u-\alpha,w} N_{u+w-\alpha,v} + N_{w,v} N_{w+v,u-\alpha}$$

$$0 = J(w-\alpha, u,v) = N_{w-\alpha,u} N_{w-\alpha+u,v} + N_{u,v} N_{u+v,w-\alpha}.$$

Dividing the second equation by $N_{\alpha,w-\alpha}$, the first by $N_{\alpha,u-\alpha}$ setting $c = N_{u+w-\alpha,v}$ and subtracting we see that

$$0 = (N_{w-\alpha,u} (N_{\alpha,w-\alpha})^{-1} - N_{u-\alpha,w} (N_{\alpha,u-\alpha})^{-1} c + J(u,v,w)$$

i.e.,

$$0 = J(\alpha, u-\alpha, w-\alpha) c + N_{\alpha,u-\alpha} N_{\alpha,w-\alpha} J(u,v,w).$$

Since $J(\alpha, u-\alpha, w-\alpha) = 0$ we see that $J(u,v,w) = 0$.

This completes the proof of (2.4).

We now extend the function N of (2.3) to a function $\tilde{N}$, defined on all pairs of roots u, v such that $(u+v) = 0$, and having the properties (2.1) (a, b, c). This extension is again forced upon us by (1.1).

(2.5). **Definition.** Let u be a positive root and v a root such that $u+v$ is a root. If v is positive, set $\tilde{N}_{u,v} = N_{u,v}$ and define

$\tilde{N}_{-u,-v}$ by : $N_{u,v}\,\tilde{N}_{-u,-v} = -1$. If v is negative define

$\tilde{N}_{u,v}$ by the equation :

$$\tilde{N}_{u,v}\,N_{u+v,-v} + \langle v, u \rangle = 0, \text{ in case } u+v \text{ is positive,}$$

and by :

$$\tilde{N}_{u,v}\,\tilde{N}_{u+v,-u} - \langle u, v \rangle = 0, \text{ in case } u+v \text{ is negative.}$$

Set $\tilde{N}_{v,u} = -\tilde{N}_{u,v}$.

Finally, let $\tilde{N}_{a,b} = 0$ if $a+b$ is not a root.

(2.6). Corollary. Let $\tilde{N}$ be as in (2.5). If u, v, w are roots and $R_{u,v,w}$ is of rank 3 then

$$(*)\quad \tilde{N}_{u,v}\,\tilde{N}_{u+v,w} + \tilde{N}_{v,w}\,\tilde{N}_{v+w,u} + \tilde{N}_{w,u}\,\tilde{N}_{w+u,v} = 0.$$

Proof. For notational convenience, denote $\tilde{N}$ by N. It suffices to assume that $\sigma = u+v+w$ is a root. As in (2.2), we may also assume that $u+v$, $v+w$ are roots but $u+w$ is not a root. Denote the left hand side of (*), by J (u, v, w).

Now (*) is true when u, v, w are all positive or all negative, so we may assume that v is positive. As $N_{a,b} = -N_{b,a}$ for all roots a, b we may also assume that $u \in R^+$ and $w \in R^-$. So we have the following possibilities :

(A) $v + w \in R^+$: Here $J(u, v, w) \equiv N_{u,v}\,N_{\sigma,-w} + N_{v+w,-w} \cdot N_{v+w,u}$. We have $J(u, v+w, -w) = 0$. Writing this out and multiplying by $N_{u,v+w}\,N_{v,u}$ we find that the relation so obtained is equivalent to $J(u, v, w)$ being 0 .

(B) $v + w \varepsilon R^{-}$ and $u + (v + w) \varepsilon R^{+}$: Here the relation to be checked becomes $N_{u,v} N_{\sigma,-w,} + N_{-v-w,v} N_{\sigma,-v-w} = 0$. Now $J(\sigma, -v-w, v) = 0$. We multiply this by $N_{\sigma,-w} N_{\sigma,-v-w}$ to get the desired result.

(C) $v + w \varepsilon R^{-}$, $u + (v + w) \varepsilon R^{-}$: In this case the relation $J(u, v, w) = 0$ is equivalent to

$$N_{u,v} N_{\sigma,-u-v} (-1) + N_{v+w,-v} N_{u,v+w} = 0 ,$$

i. e. to $N_{u,v} N_{-\sigma, u+v} + N_{-\sigma, u} = 0$,
the left hand side of which is $J(-\sigma, u, v)$.

This completes the proof of (2.6).

(2.7). The Lie algebra $L_R(A)$. Let A be a commutative ring. Using (2.5), it is now easy to construct a Lie algebra $L_R(A)$ such that every automorphism of R extends to an automorphism of $L_R(A)$. We take $L_R(A)$ to be the free A-module with basis H_a ($a \varepsilon S$), $X_b (b \varepsilon R)$.

For u, v both positive or negative let $[X_u, X_v] = \tilde{N}_{u,v} X_{u+v}$, $\tilde{N}$ being as in (2.5). If $a \varepsilon S$, set $H_a = [X_a, X_{-a}]$, and if $\sigma \varepsilon R^{+}$ and $(\sigma, a) > 0$, set $H_\sigma = H_a + H_{\sigma - a}$, and $[X_\sigma, X_{-\sigma}] = H_\sigma$. Defining, for a simple root a and an arbitrary root b $[H_a, X_b]$ to be $< b, a > X_a$, requiring this operation to be bilinear and anti-symmetric (i. e. $[X, X] = 0$ for all $X \varepsilon L_R(A)$) the reader will find that $L_R(A)$ is now a Lie algebra over A with the stated properties.

Clearly $L_R(A) \cong L_R(Z) \otimes_Z A$. Moreover $ad\, X_a^3 = 0$ ($a \varepsilon R$) and $\frac{1}{2}\, ad\, X_a^2$ maps $L_R(Z)$ into itself. These remarks, which are trivial to to check, will play a role in the following section.

3. THE FUNCTOR G_R (A)

Let R be an irreducible root system of rank $\geqq 2$, A a commutative ring with unity and A* the group of units of A. Let G be a group with generators $x_a(u)$ ($a \varepsilon$ R, $u \varepsilon$ A) which satisfy the following relations :

(R 1) $x_a(u+v) = x_a(u)\, x_a(v)$ $\quad (u, v \varepsilon \text{A}, a \varepsilon \text{R})$

(R 2) If a, b are linearly independent roots then the commutator

$$(x_a(u), x_b(v)) = \prod_{\substack{ia + jb \,\varepsilon\, R \\ i, j > 0}} x_{ia+jb}(N_{a,b,i,j}\, u^i v^j),$$

where $N_{a,b,i,j}$ are elements of A and the product on the right hand side is taken in some ordering of the roots $i_a + j_b$ $(i, j > 0$

(R 3) If J is an integrally closed irreducible subsystem of R of rank at most 3, $\overset{+}{J}$ a positive system of roots in J and an ordering of the roots in $\overset{+}{J}$ has been fixed, then every element x of the group generated by $x_r(u)$ ($r \varepsilon \overset{+}{J}$, $u \varepsilon$ A) has a unique expression

$$x = \prod_{r \,\varepsilon\, \overset{+}{J}} x_r(u_r)$$

the product on the right hand side being taken in the chosen ordering of roots in $\overset{+}{J}$. [In case R has no multiple bonds we need only assume that rank (J) $\leqq 2$].

(R 4) If a, b are independent roots and $u \varepsilon$ A* then

$w_a(u)\, U_b\, w_a(u)^{-1} = U w_a(b)$

where $w_a(u) = x_a(u)\, x_{-a}(-u^{-1})\, x_a(u)$, w_a is the reflection along the root a, and U_r ($r \varepsilon$ R) is the group generated by $x_r(u)$ ($u \varepsilon$ A).

It is shown in [1] that every group with the above properties is a homomorphic image of a single group G_R (A), which is determined upto isomorphism by the system R and the ring A : in particular, every automorphism of R extend to an automorphism of G_R (A) (see remarks following statment of the proposition in [1]*).

* For the casc of G_2, see [9, p. 295]

To prove the existence of $G_R(A)$ we first assume that R has no multiple bonds. Let $L_R(A)$ be the Lie algebra as defined in (2.7). Recall that the Steinberg group $St_R(A)$ is the group with generators

$x_a'(u)$ $(a \in R, u \in A)$ subject to the relations

$$\begin{aligned} &(A) \quad x_a'(u+u') = x_a'(u)\, x_a'(u') \quad (u, u' \in A,\ a \in R) \\ &(B) \quad (x_a(u), x_b(v)) = x_{a+b}(N'_{ab}\, uv), \text{ if } u+v \in R \\ &\qquad\qquad\qquad\qquad = 1 \qquad\qquad\qquad\ \ , \text{ if } u+v \notin R. \end{aligned}$$

Here the N'_{ab} are as in proposition (1.1).

This group has a representation in Aut $(L_R(A))$, namely, map $x_a'(u)$ into the formal exponential

$$x_a(u) = 1 + (ad\, X_a) \otimes u + \frac{ad\,(X_a)^2}{2\,!} \otimes u^2 .$$

Here x_a is a basis element of $L_R(A)$ as given in (2.7), and the formal exponential has only two terms because R has no multiple bonds.

Straight forward calculations show that the group $G_{ad,R}(A)$ generated by $x_a(u)$ $(a \in R, u \in A)$ satisfies (R 1), (R 2) and (R 4). In fact $w_a(u)\, x_b(v)\, w_a(u)^{-1} = x_{a+b}(N'_{a,b}\, uv)$ if $a+b$ is a root. To see that (R 3) holds we need an auxiliary lemma.

Let U_r $(r \varepsilon R)$ be the group generated by $x_r(u)$ $(u \varepsilon A)$, let R^+ be a positive system of roots and let $a_1, \dots, a_N$ be all the elements of R^+ listed so that $ht(a_i) \leqq ht(a_j)$ if $i \leq j$, Let U^+ be the group generated by the subgroups U_r $(r \varepsilon R^+)$.

Lemma. [2, p. 39] Every element x of U^+ has a unique expression

$$x = \prod_{i=1,\dots,N} x_{a_i}(u_i) .$$

Proof. The commutator formula (R 2) implies that x has an expression of the above form. Let S be the simple system of roots which corresponds to R^{+} and let $L_R(A)$ be the Lie algebra as defined in (2.7) with H_a ($a \varepsilon S$), X_b ($b \varepsilon R$) as a basis. Let U^{+} and U^{-} be the subalgebras generated by x_r ($r \varepsilon R^{+}$) and $X_{r'}$ ($r' \varepsilon R^{-}$), respectively.

Now if u, v are positive roots and $ht(u) > ht(v)$ then either $u - v$ is not a root, or else it is a positive root ; and if $ht(u) = ht(v)$ then $u - v$ is not a root. Moreover, if u and v are distinct then $x_u(t)\, X_{-v} = X_{-v} + t\, N_{u,-v}\, X_{u-v}$. Therefore if $x = \prod_{i=1,\dots,N} x_{a_i}(u_i)$ then

$$x(X_{-a_1}) \equiv x_{a_1}(u_1)(X_{-a_1}) \pmod{U^{+}}$$

$$\equiv X_{-a_1} + u_1 [X_{x_1}, X_{-a_1}] \pmod{U^{+}}$$

$$\equiv u_1 H_{a_1} \pmod{(U^{+} + U^{-})}$$

As $L_R(A) = H + U^{+} + U^{-}$ we see that $u_1 H_{a_1}$ is uniquely determined by x. As rank $R \geqq 2$, there exists some root b with $< b, a > = 1$. This means that u_1 is uniquely determined by x. Therefore if $x = \prod \bar{x}_{a_i}(u_i) = \prod x_{a_i}(u_i')$ then $u_1 = u_1'$. Cancelling $x_{a_1}(u_1)$ we continue and conclude that $u_i = u_i'$ for all i.

From proposition (1.1) it is clear that if σ is an automorphism of R then it extends to an antomorphism $\tilde{\sigma}$ of $L_R(A)$ as well as of $St_R(A)$ and we have :

$$\tilde{\sigma} X_a = C_a X_{\sigma(a)},\ \tilde{\sigma}(x_a'(u)) = x'_{\sigma(a)}(c_a u)\ ,\ c_a = \pm 1 \text{ and}$$

$$c_a\, c_{-a} = 1 \text{ (because } H_a = [X_a, X_{-a}] \text{ and } \tilde{\sigma}(H_a) = H_{\sigma(a)}.$$

Moreover $\tilde{\sigma}(ad X_a)(\tilde{\sigma})^{-1} = ad(\tilde{\sigma} X_a)$ and this means that $\tilde{\sigma}$ normalizes $G_{ad,R}(A)$. Suppose $\tilde{\sigma}$ fixes a positive system of roots R^{+} in

R. It follows by using (1.1) and [6, p. 172-175] or [7, p. 875-877] that the fixed points of $\tilde{\sigma}$ in $G_{ad,\,R}(A)$ contain a group which satisfies the relations (R 1) , ... , (R 4), with R replaced by the root system obtained by twisting R according to σ^*. This proves the existence of the groups in question.

Finally, let K be the normal subgroup of $St_R(A)$ generated by

$$w'_a(t)\, x'_a(u)\, w'_a(t)^{-1}\, x'_{-a}(-t^{-2}u) \qquad \text{and}$$
$$h_a(tt')\, h_a(t')^{-1}\, h_a(t)^{-1} \qquad (a \in R,\ t,\ t' \in A^*,\ u \in A),$$

where $w'_a(t) = x'_a(t)\, x'_{-a}(-t^{-1})\, x'_a(t)$ and $h_a(t) = w'_a(t)\, w'_a(-t)$: note that $\tilde{\sigma}(K) = K$.

It is shown in [6, p. 66] that when A is a field the group $St_R(A) / K$ is isomorphic to the universal Chevalley group corresponding to the system R, and hence $(St_R(A) / K)\,\tilde{\sigma}$ is isomorphic to the universal Chevalley group corresponding to the system obtained by twisting R according to σ [cf. 6, p. 172].

Therefore the groups $(St_R(A) / K)\,\tilde{\sigma}$ – σ being any automorphism of R – are appropriate generalizations of Chevalley groups. For example, in this way, one obtains the maximal compact subgroups of some real Lie groups. In this connection, see also [2, p. 65].

Remark. For some applications it is useful to replace the relations (R 3) of § 3 by

(R 3) (a) : If J is an integrally closed irreducible subsystem of R of rank at most 2, J^+ a positive system of roots in J and an ordering of the roots in J^+ has been fixed, then every element x of the group generated by $x_r(u)$ $(r \in J^+,\ u \in A)$ has a unique expression

$$x = \prod_{r \in J^+} x_r(u_r)$$

* see appendix, [p. 22—23].

the product on the right hand side being taken in the chosen ordering of roots in J^{+}.

(b) : If a, b, c are positive roots such that $a+b$, $b+c$ and $a+c$ are not roots then every element x of the group generated by $x_r(u)$ ($r = a, b, c$, $u \in A$) has a unique expression

$x = x_a(u)\, x_b(v)\, x_c(w)$. [In case R has no multiple bonds we need only assume (R 3) (a)].

4. APPENDIX

Let L, R, S, A and X_r ($r \in R$) be as in § 1. Assume that $[X_a, X_{-a}]$ ($a \in S$) and X_r ($r \in R$) form a basis of L over A. In view of (1.1) we may, after a suitable normalization of the generators, also assume that for all roots r and s

$$(*) \quad [[X_r, X_{-r}], X_s] = \langle s, r \rangle X_s .$$

It then follows (cf. (1.1)) that if S′ is any simple system of roots in R and σ an automorphism of R, then the mapping $X_a \longrightarrow X_{\sigma(a)}$ ($a \in S' \cup -S'$) extends to an automorphism of L, and of the group $G_R(A)$ of § 3, and this extension is unique.

From now on, we assume that the generators of L have been chosen so as to satisfy (*). Furthermore, that σ is an automorphism of R which maps S into itself (so σ is of order 2 or 3). The unique extension of the mapping $X_a \longrightarrow X_{\sigma(a)}$ ($a \in S \cup -S$) will be denoted by $\tilde{\sigma}$.

4.1. $\tilde{\sigma}(X_r) = X_r$ whenever $\sigma(r) = r$, unless R is of type A_{2m}, in which case $\tilde{\sigma}(X_r) = -X_r$ whenever $\sigma(r) = r$.

Proof. First, suppose that σ is of order 2 and R is not of type A_{2m}. Let r be a positive root fixed by σ. If r is simple then $\tilde{\sigma}(X_r) = X_r$. So let $r = \alpha + \beta$ ($\alpha \in S$, $\beta \in R^{+}$). Denoting images under σ

by primes, we have $r = r' = \alpha'+\beta'$, so $R_{\alpha\beta\,\alpha'}$ is an irreducible root system, with $R_{\alpha\beta\,\alpha'} \cap R^{+}$ as a positive system of roots, and α, α' remain simple roots of this subsystem.

If $R_{\alpha\beta\,\alpha'}$, is of type A_2 then we must have $\alpha = \alpha'$, otherwise $\alpha+\alpha'$ would be root, and since α, α' are both simple, this is only possible if R is of type A_{2m}. Hence $\alpha = \alpha'$, $\beta = \beta'$ and $\tilde{\sigma}\,[X_\alpha, X_\beta] = [X_\alpha, X_\beta]$ (by induction on heights). If $R_{\alpha\beta\,\alpha'}$, is of type A_3 (so $\alpha \neq \alpha'$) then there is a root u of this subsystem such that $\underset{\alpha}{o}-\underset{u}{o}-\underset{\alpha'}{o}$ is its Dynkin diagram, and such that $r = \alpha + u + \alpha'$. As $u = u'$ we have $\tilde{\sigma}\,(X_u) = X_u$, by induction on heights. Moreover $\tilde{\sigma}\,[X_\alpha, X_u ; X_{\alpha'}] = [X_{\alpha'}, X_u ; X_\alpha] = [X_\alpha, X_u ; X_{\alpha'}]$ (by Jacobi), hence $\tilde{\sigma}\,(X_r) = X_r$.

If R is of type A_{2m} and σ of order 2, then σ does not fix any simple root. There is a unique simple root α such that $\alpha + \alpha'$ is a root and so $\tilde{\sigma}\,[X_\alpha\ X_{\alpha'}] = -\,[X_\alpha, X_{\alpha'}]$. An argument similar to the one just given shows that $\tilde{\sigma}\,(X_r) = -X_r$ whenever $\sigma(r) = r$.

There remains the case : R is of type D_4 and $\sigma^3 = 1, \sigma \neq 1$. Label the Dynkin diagram of D_4 as $\underset{a}{o}-\underset{b}{o}<^{o\,c}_{o\,d}$. The non-simple positive roots are $a + b$, $b+ c$, $b+ d$, $a+ (b + c)$, $a + (b+d)$, $c + (b + d)$, $a + (b + c + d)$, $b + (a + b + c + d)$. Fixing the order $a < b < c < d$ on S and using (1.1) (B), we may assume that $N_{a,\,b} = N_{b,\,c} = N_{b,\,d} = 1$, $N_{a,\,b+c} = N_{a\,,\,b+d} = 1$, $N_{c\,,\,b+d} = 1$, $N_{a,\,b+c+d} = N_{b,\,a+b+c+d} = 1$; moreover if u, v are roots such that $N_{u,\,v} \neq 0$ then $N_{u,\,v}\,N_{-u,-v} = -1$. The non-simple positive roots left fixed by σ are $a + b + c+d$ and $a + 2b + c + d$.

Now

$\sigma\,[X_a, [X_c, [X_b, X_d]\,]\,] = N_{d,\,a+b}\,N_{b,\,a}\,N_{c,\,a+b+d}\,X_{a+b+c+d}$

and $[X_a, [X_c, [X_b, X_d]\,]\,] = N_{a,\,c+b+d}\,N_{c,\,b+d}\,N_{b,\,d}\,X_{a+b+c+d}$

Using the above data, one can check that the right hand sides of the last two equations are equal. The verification for the root $b + (a + b + c + d)$, which is similar, completes the proof of (4.1).

The following lemma is well known : a version occurs in [2, p. 19-20], and 4.2 (i) can also be extracted from [7, p. 877, line 14]. We need it in the following form.

4.2. Let R be not of type A_{2m} and let σ be of order 2. Denote images under σ by primes :

(i) For all roots r, we have $r + r'$ is not a root.

(ii) If $r = r'$, $s \neq s'$, r and s are non-orthogonal then $R_{rss'}$ is irreducible of rank 3 and σ acts as a non-trivial permutation on $\overset{+}{R}$ $R_{rss'}$.

(iii) If $r \neq r'$, $s \neq s'$ are roots such that $r + \xi s \in R$ $(\xi = \pm 1)$ then either $r + \xi s = r' + \xi s'$, in which case $R_{rs\,r'\,s'}$ is irreducible of rank 3 and σ acts non-trivially or $R_{rs\,r'\,s'}$ $\overset{+}{R}$, or else $< r, s' > \; = \; < r', s > \; = 0$.

Proof. We may assure that r is a positive root. As σ preserves heights, it is clear that $r - r'$ is not a root. Suppose $r + r'$ is a root. As R is not of type A_{2m} , r cannot be simple, so $r = \alpha + \beta$ $(\alpha \in S,\ \beta \in \overset{+}{R})$. As $\alpha + \beta$, $\alpha' + \beta'$ and $r + r'$ are roots, we see that $R_{\alpha\,\beta\,\alpha'\,\beta'}$ is an irreducible root system of rank 4 at most hence is of type A_2, A_3, A_4 or D_4, and σ acts as a non-trivial permutation on $\overset{+}{R} \cap R_{\alpha\,\beta\,\alpha'\,\beta'}$. One checks that if τ is an involutary automorphism of a system of type A_3 or D_4 , fixing a positive system of roots, then there is no root r such that $(r + \tau r)$ is a root. Hence $R_{\alpha\,\beta\,\alpha'\,\beta'}$ must be of type A_2 or A_4, with α, α' occuring as distinct simple roots in $R_{\alpha\beta\alpha'\beta'} \cap \overset{+}{R}$. As R is not of type A_{2m} we see that $\alpha + \alpha'$ is not a root, hence the Dynkin diagram of $R_{\alpha\,\beta\,\alpha'\,\beta'} \cap \overset{+}{R}$ must be

o—o—o—o
α u v α'

, and r is then $\alpha + u$ or $v + \alpha'$. As σ must permute α, α' and u, v, respectively, we see that u is a root of lower height than r such that $(u + u')$ is a root. By induction on heights, it follows that $r + r'$ is not a root.

Let $r = r'$, $s \neq s'$ be roots such that $r + \xi s$ is a root$(\xi = \pm 1)$. Now $R_{rss'}$ is irreducible of rank 3 at most ; its rank by (i) cannot be 2 as $R_{rss'}$ R^+ admits a permutation of order 2. This proves (ii).

Finally, let r and s be non-orthogonal roots such that $r \neq r'$, $s \neq s'$. Let $r + \xi s$ be a root. As $r \pm r'$ and $s \pm s'$ are not roots, we see that $< r + \xi s, r' + \xi s' > = 2 \xi < r, s' >$. Hence either $r + \xi s = r' + \xi s'$ or else $< r, s' > = < r', s > = 0$. This proves (iii).

4.3. Remark. The proof of (i) also shows that R is of type A_{2m} and $\underset{\alpha_1}{o} - \dots - \underset{\alpha_m}{o} - \underset{\alpha_{m+1}}{o} - \dots - \underset{\alpha_{2m}}{o}$ is its Dynkin diagram then the positive roots of R such that $r = r'$ are

$$\{\alpha_m + \alpha_{m+1}\,,\ \alpha_{m-1} + \alpha_m + \alpha_{m+1} + \alpha_{m+2}\,,\ \dots,\ \alpha_1 + {} + \alpha_m + \alpha_{m+1} + \dots + \alpha_{2m}\}$$

Proposition 4.4. [7, p. 875-877] Let V denote the real span of R and fix a positive definite inner product on R relative to which elements of the Weylgroup and σ become isometries. For $v \,\varepsilon\, V$, let $\tilde{v}$ denote the orthogonal projection of V on V_σ, where $V_\sigma = \{ v \,\varepsilon\, V \mid \sigma(v) = v \}$. Then $\tilde{R} = \{\tilde{r} : r \,\varepsilon\, R\}$ is an irreducible reduced root system in V_σ and the distinct elements of $\{\tilde{\alpha} : \alpha \,\varepsilon\, S\}$ form a fundamental system of roots of $\tilde{R}$, unless R is of type A_{2m} in which case it is of type BC_m.

The reader is referred to [6, p. 172] or [7, p.875-877] for details. In the case which interests us here, namely R is not of type A_{2m}, this also follows, as we show presently, from (4.2), when $\sigma^2 = 1$, and by explicit computations as in (4.1) when $\sigma^3 = 1$. Let $\sigma^2 = 1$ $(\sigma \neq 1)$ and let $\omega_{\tilde{a}}$ denote the reflection in the hyplane orthogonal to $\tilde{a}$. In view of (4.2), to see that $\omega_{\tilde{a}}(\tilde{R}) = \tilde{R}$, we have only to verify this when R is of type A_3 or $A_2 \times A_2$ with σ interchanging the two components in the latter case : this verification is easy, using (4.2) (ii) and (iii), and will also show that $<\tilde{a}, \tilde{b}> \in Z$. Therefore $\tilde{R}$ is a root system in the sense of [4, p. V–3] and every element of $\tilde{R}$ is an integral linear combination of elements of $\tilde{S}$. Defining height with respect to $\tilde{S}$ and using the integrality condition $<\tilde{a}, \tilde{b}> \in Z$ we see that if r is a positive root and $2\tilde{r} \in \tilde{R}$ then $2\tilde{a}$ $(a \in S)$ is also in $\tilde{R}$, say $2\tilde{a} = \tilde{s}$ $(s \in R^+)$. So s must be a linear combination of the transforms of a under σ. The condition $2\tilde{a} = \tilde{s}$ implies that $R_{aa'}$ is of type A_2 and $s = a + a'$. As a, a' are both simple, this is only possible when R is of type A_{2m}.

Now $\sigma w_{\tilde{a}} \sigma^{-1} = w_{\tilde{a}}$ $(a \in R)$ so [2, p. 19, Lemma 1] or [5, p. 234, 11.1.4] implies that if $\tilde{a}$ and $\tilde{b}$ are linearly independent roots such that a is orthogonal to all transforms of b under σ then $\tilde{R}_{\tilde{a}, \tilde{b}}$ the integral closure of $\tilde{a}, \tilde{b}$ in $\tilde{R}$, is of type $A_1 \times A_1$.

Let U^+ and U^- be the subalgebras of L generated by X_r $(r \in R^+)$ and X_s $(s \in R^-)$, respectively. Let H be the subalgebra generated by H_a $(a \in S)$. Clearly $L_{\sim\sigma} = U^+_{\sim\sigma} \oplus H_{\sim\sigma} \oplus U^-_{\sim\sigma}$ For each root $\alpha \in \tilde{R}$ choose a root r such that $\alpha = \tilde{r}$ and define X_α and H_α to be the sums of the distinct transforms of X_r and H_r, respectively, under $\tilde{\sigma}$. Now using (4.2), and (4.1) in case σ is of order 3, the reader can check that $[X_\alpha, X_{-\alpha}] = H_\alpha$,

$$[H_\alpha, X_\beta] = \langle \beta, \alpha \rangle X_\beta \text{ and } [X_\alpha, X_\beta] = \begin{cases} 0 \text{ if } \alpha + \beta \notin \tilde{R} \\ N_{\alpha,\beta} X_{\alpha+\beta} \text{ if } \alpha + \beta \in \tilde{R}, \end{cases}$$

$N_{\alpha,\beta}$ being some constants.

In particular, taking A = C and using the fact that the Cartan matrix $(\langle \tilde{r}, \tilde{s} \rangle)$ is non-singular, where r, s run through a set of representatives of the orbits of S under σ, we see that $L_R(C)_{\tilde{\sigma}}$ is a semi-simple algebra whose root system is $\tilde{R}$. This proves (1.2).

Finally, consider the group $G_{ad,R}(A)$ of § 3. The automorphism σ of R extends to an automorphism $\tilde{\sigma}$ of $G_{ad,R}(A)$. For each root $\alpha \in \tilde{R}$, choose a root $\alpha \in R$ such that $\alpha = \tilde{r}$. Define $x_\alpha(a)$ to be product of the distinct transforms of $x_\alpha(a)$ under σ and let U_α be the group generated by $x_\alpha(a)$ $(a \in A)$. Using (4.2) and, in case σ is of order 3, the normalization of the structure constants of D_4 as given in (4,1), the reader can check that the group generated by $x_\alpha(a)$ $(\alpha \in \tilde{R}$, $a\alpha \in A)$ satisfies the relations (R 1), (R 2) and (R 4) of § 3. As the group generated by U_α $(\alpha \in R^+)$ is a subgroup of the group $U^+_{\tilde{\sigma}}$, the commutator formula and the lemma in § 3 imply that the generators $x_\alpha(a)$ satisfy the relations (R 3) also : see [6, § 11, p. 180, Lemma 62] for details.

REFERENCES

1. *H. Azad*, Structure Constants of Algebraic Groups, Journal of Algebra, vol. **75**, 1982.
2. *C. Chevalley*, Sur certains groupes simples, Tôhoku Math. Journal. 1955.
3. *N. Jacobson*, Lie Algebras, Interscience, New York, 1962.
4. *J. P. Serre*, Algebres de Lie semi-simples complexes, W. A. Benjamin, New York, 1966.
5. *T. A. Springer*, Linear Algebraic Groups, Birkhauser, Boston, 1980.
6. *R. Steinberg*, Lectures on Chevalley Groups, Yale University, Lecture Notes, 1968.
7. *R. Steinberg*, Variations on a theme of Chevalley, Pacific Journal of Mathematics, 1959.
8. *J. Tits*, Sur les constantes de structure et le theoreme d'existence des algebres de Lie semi-simples. Publ. Math. I. H. E. S, No 31, 1966.
9. *M. Demazure*, Schemas en Groups, Springer-Verlag, Berlin, 1970.

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IRREDUCIBLE REPRESENTATIONS OF THE COMMUTATION RELATIONS OF A SYSTEM OF OPERATORS IN A KREIN SPACE

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Abstract

In this paper the irreducible representations of the finite sets of operators $A_1, A_2, \ldots, A_n$ and $A_1^*, A_2^*, \ldots, A_n^*$ satisfying the commutation relations

$$A_r A_s - A_s A_r = 0$$

$$A_r^* A_s^* - A_s^* A_r^* = 0$$

$$A_r A_s^* - A_s^* A_r = -\delta_{rs} I$$

$$(r, s = 1, 2, 3, \ldots, n)$$

in a Krein space have been found. The sets of operators satisfy the following conditions :

(i) $A_1, A_2, \ldots, A_n$ are closed, linear operators with a common domain of definition, dense in the Krein space.

(ii) $N = \sum_{r=1}^{n} A_r^* A_r$ is self-adjoint

and so is each $A_r^* A_r$ $(r = 1, 2, \ldots, n)$.

(iii) N has a spectrum lying on the non-positive part of the real axis and so does each of $A_r^* A_r$.

(iv) N has a spectral decomposition.

It has been shown that the Krein space can be written as the direct sum of finite number of subspaces, and each of these subspaces is an irreducible reducing subspace of A_1, $A_2, \ldots\ldots, A_n$ and A_1^*, A_2^*, ..., A_n^* and has a complete orthonormal system.

Introduction.

Rellich [4] and Tillmann [5] have found the irreducible representation of finite degrees of freedom in a Hilbert space. If A is any closed operator with a dense domain in a Hilbert space, then $A^* A$ is a self-adjoint operator and has a spectral decomposition and its spectrum is positive. Further the sum of a finite number of self-adjoint operators having a common dense domain is also a self-adjoint operator. In the case of a Krein space A^*A may not be self-adjoint and further, every self-adjoint operator does not have a spectral decomposition. Furthermore, the spectrum of a self-adjoint operator lies symmetrically about the real axis. When the spectrum, however, does not lie on the real axis, there are additional difficulties, for in that case the orthonormal system is not complete and adding of some vectors in the basis becomes necessary.

Now we consider the problem of representation in a Krein space Π of finite sets of operators $A_1, A_2, \ldots\ldots, A_n$, and A_1^*, A_2^*, ..., A_n^* under the following assumptions :

(i) Let $A_1, A_2, \ldots\ldots A_n$, be closed, linear operators, with a common domain of definition, dense in the Krein space Π satisfying the relations

$$\left.\begin{aligned} & A_r A_s - A_s A_r = 0 \\ & A_r^* A_s^* - A_s^* A_r^* = 0 \\ & A_r A_s^* - A_s^* A_r = -\delta_{rs} I \\ & r, s = 1, 2, 3, \ldots\ldots, n \end{aligned}\right\} \quad (R)$$

(ii) $N = \sum_{r=1}^{n} A_r^* A_r$ is self-adjoint and so is each $A_r^* A_r$ $(r = 1, 2, \ldots\ldots\ldots, n)$

(iii) N has a spectrum lying on the non-positive part of the real axis and so does each $A_r^* A_r$.

(iv) N has a spectral decomposition.

Let the spectral decomposition of N be given by

$$N = \int \lambda \, dE_\lambda \, . \tag{1}$$

Since N has a spectrum lying on the non-positive part of the real axis, by (iii) and by (R), N cannot be the zero operator, there exists some $\mu \in sp\,(N)$ with $\mu < \sigma$. We choose elements $f_m = E\,(\Delta_m)\, f_m$ where

$$\Delta_m = (\mu - \frac{1}{m}, \; \mu + \frac{1}{m})$$

such that $\| f_m \|_J = 1 \qquad m = 1, 2, 3, \ldots\ldots\ldots$

so that $(N - \mu I)\, f_m = g_m \longrightarrow 0$ as $m \longrightarrow \infty$.

Now $A_k\,(N - \mu I)\, f_m = A_k\, g_m$ with $1 < k < n$

or $\left(\sum_{r=1}^{n} A_k\,(A_r^* A_r) - \mu A_k \right) f_m = A_k\, g_m$

or $\left(\sum_{\substack{r=1 \\ r \neq k}}^{n} A_k A_k^* \; A_r + A_k A_k^* A_k - \mu A_k \right) f_m = A_k\, g_m$

or $\left(\sum_{\substack{r=1 \\ r \neq k}}^{n} A_r^* \; A_r A_k + (A_k^* \; A_k - I)\, A_k - \mu A_k \right) f_m = A_k\, g_m$

or $\left(\sum_{r=1}^{n} A_r^* \; A_r - (\mu + 1\}\, I \right) A_k f_m = A_k\, g_m$

or $(N - (\mu + 1)\; I)\; A_k\, f_m = A_k\; g_m$. (2)

Since $| < f_m , f_m > | \leq \| f_m \|_J^2 = 1$, for each n, we may assume that $< f_m, f_m > \longrightarrow \lambda$, (extracting a subsequence of f_m's if necessary).
Now

$$\| A_k f_m \|_J^2 > | < A_k f_m , A_k f_m > | = | < f_m , A^* A f_m > |$$

$$\longrightarrow | \mu | \; | \lambda | \text{ as } m \longrightarrow \infty .$$

We have the following possibilities

(*a*) $\lambda \neq 0$

(*b*) $\lambda = 0$, then either

(b_1) $\| A_k f_m \|_J > 0$,

i.e. $A_k f_m \neq 0$ for $m = 1, 2, 3, \ldots\ldots$; or

(b_2) $\| A_k f_m \|_J \longrightarrow 0$,

i.e. $A_k f_m \longrightarrow 0$ as $m \longrightarrow \infty$.

First we consider the case (*a*) in detail.

Since $\| A_k f_m \|_J^2 \geq | \mu | \; | \lambda |$ as $m \longrightarrow \infty$.

and $\| A_k g_m \|_J^2 = < A_k g_m, JA_k g_m > = < g_m, A^{k*} JA_k g_m >$

$$\leq \| g_m \| \; \| A^{k*} JA_k g_m \| \longrightarrow 0 \text{ as } m \longrightarrow \infty ;$$

Here we assume that D is invariant under J.

Hence from (2), we have

$$\Big(N - (\mu + 1) \Big) A_k f_m = A_k g_m \longrightarrow 0 \text{ as } m \longrightarrow \infty,$$

therefore $\mu + 1$ also belong to the spectrum of N. Thus whenever $\mu \in$ sp (N) and $\mu < 0$, then $\mu + 1 \in$ sp (N). Since the spectrum of N is lying on the non-positive part of the real axis, μ must be *a* negative integer. The same argument also shows that 0 belongs to the point spectrum of N. Therefore there exists some $\varnothing \in$ D (N) and $\varnothing \neq 0$ such that N $\varnothing = 0$

$$\text{i. e. } \sum_{r=1}^{n} A_r^* A_r \varnothing = 0$$

or $\overset{*}{A}_1 A_1 \varnothing + \overset{*}{A}_2 A_2 \varnothing + \ldots\ldots + \overset{*}{A}_n A_n \varnothing = 0$ (3)

which is an equation in 2nd operators and for the solution of which we require other 2n—1 conditions. We consider the case when those conditions are as follows :

$$\overset{*}{A}_r A_r \varnothing = 0 \quad \text{and } A_r \varnothing = 0 \qquad r = 1, 2, \ldots\ldots, n \qquad (4)$$

We define

$$\varnothing_{s_1, \ldots\ldots, s_n} = (\overset{*}{A}_1)^{s_1} (\overset{*}{A}_2)^{s_2} \ldots\ldots (\overset{*}{A}_n)^{s_n} \varnothing$$

where $s_1 \ldots, s_n = 0, 1, \ldots, m, \ldots$ a system of elements in **D.** Then

$$\overset{*}{A}_r \varnothing_{s_1, \ldots\ldots, s_n} = \left(\overset{*}{A}_1\right)^{s_1} \ldots\ldots \left(\overset{*}{A}_{r-1}\right)^{s_{r-1}} \left(\overset{*}{A}_r\right)^{s_r+1} \left(\overset{*}{A}_{r+1}\right)^{s_{r+1}} \ldots\ldots \left(\overset{*}{A}_n\right) \varnothing$$

$$= \varnothing_{s_1, \ldots\ldots, s_{r-1}, s_r + 1, s_{r+1}, \ldots\ldots, s_n}$$

and

$$A_r \varnothing_{s_1, \ldots\ldots, s_n} = (\overset{*}{A}_1)^{s_1} \ldots\ldots (\overset{*}{A}_{r-1})^{s_{r-1}} A (\overset{*}{A}_r)^{s_r} (\overset{*}{A}_{r+1})^{s_{r+1}} \ldots\ldots (\overset{*}{A}_n)^{s_n} \varnothing$$

$$= (\overset{*}{A}_1)^{s_1} \ldots\ldots (\overset{*}{A}_{r-1})^{s_{r-1}} \left[(\overset{*}{A}_r)^{s_r} A - s_r (A^*)^{s_r} - 1 \right] (\overset{*}{A}_{r+1})^{s_{r+1}} \ldots\ldots (\overset{*}{A}_n)^{s_n} \varnothing$$

$$= - s_r \varnothing_{s, \ldots\ldots, s_{r-1}, s_r - 1, s_r + 1, \ldots\ldots, s_n} .$$

Let $\quad L = (\varnothing_{s_1, s_2, \ldots\ldots, s_n}, \varnothing_{u_1, u_2, \ldots\ldots, u_n})$

We further assume that $s_r > 0$, $s_r > u_r$,

and $s_{r+1} = s_{r+2} = \ldots\ldots = s_n = 0$,

$u_{r+1} = u_{r+2} = \ldots\ldots = u_n = 0$.

Then

$$L = (\overset{*}{A}_r \varnothing_{s_1, \ldots\ldots, s_{r-1}, s_r - 1, 0, \ldots\ldots, 0}, \varnothing_{u_1, \ldots\ldots, u_r, 0, \ldots\ldots, 0}$$

$$= (\varnothing_{s_1, \ldots\ldots, s_{r-1}, s_r - 1, 0, \ldots\ldots, 0}, A_r \varnothing_{u_1, \ldots\ldots u_r, 0, \ldots, 0})$$

$$= (-1) u_r (\varnothing_{s_1, \ldots\ldots, s_{r-1}, s_r - 1, 0, \ldots\ldots, 0}, \varnothing_{u_1, \ldots\ldots, u_{r-1}, u_r - 1, 0, \ldots, 0})$$

....
....

$$= (\varnothing_{s_1, \ldots, s_r - u_r, 0, \ldots\ldots 0}, \varnothing_{u_1, \ldots, u_{r-1}, 0 \ldots 0}) (-1)^{u_r} (u_r !).$$

If $s_r - u_r > 0$, in the next step we have $L = 0$.

If $s_r - u_r = 0$, we have

$$L = \left(\varnothing_{s_1, \ldots\ldots, s_{r-1}, 0, \ldots\ldots 0}, \varnothing_{u_1, \ldots\ldots, u_{r-1}, 0 \ldots\ldots 0} \right) (-1)^{u_r} (s_r !)$$

Similarly we have

$$L = \begin{cases} (-1)^{s_1 + s_2 + \ldots\ldots + s_r} (s_1 !)(s_2 !) \ldots (s_r !) & \text{if } u_i = s_i, \ i = 1, \ldots, r \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\psi_{s_1, \ldots\ldots, s_n} = \left(1 / \sqrt{(s_1 !)(s_2 !) \ldots (s_n !)} \right) \varnothing_{s_1, \ldots\ldots, s_n}$$

$$s_1, \ldots\ldots, s_n = 0, 1, 2, \ldots\ldots\ldots$$

is an orthonormal system for which we have

$$\left.\begin{aligned} A_r^* \psi_{s_1, \ldots\ldots, s_n} &= \sqrt{s_r + 1}\, \psi_{s_1, \ldots\ldots, s_{r-1}, s_r + 1, s_{r+1}, \ldots s_n} \\ \text{and } A_r \psi_{s_1, \ldots\ldots, s_n} &= -\sqrt{s_r}\, \psi_{s_1, \ldots, s_{r-1}, s_r - 1, s_r + 1, \ldots, s_n} \end{aligned}\right\} \quad (5)$$

If $\psi = \varnothing^{(c)} = \varnothing$ is not the only eigenvector of N corresponding to eigenvalue 0, then, we assume that

$$\varnothing^{(o)} = \varnothing, \varnothing^{(1)}, \varnothing^{(2)}, \ldots\ldots$$

is an orthonormal system which satisfies the condition (4) and generates the eigenspace of N corresponding to the eigenvalue 0.

Then

$$\psi^{(i)}_{s_1, \ldots\ldots, s_n} = \frac{(A_1^*)^{s_1} \ldots\ldots (A_n^*)^{s_n} \varnothing^{(i)}}{\sqrt{(s_1 !) \ldots\ldots (s_n !)}}$$

for $i = 0, 1, 2, \ldots\ldots\ldots$

$$s_1, s_2, \ldots\ldots\ldots, s_n = 0, 1, 2, 3, \ldots\ldots\ldots$$

is an orthonormal system of elements in D.

Let us denote the closed subspace generated by

$$\left\{ \psi^{(i)}_{s_1, \ldots\ldots, s_n} \right\}_{s_1, \ldots\ldots, s_n}$$

by $\Pi_{(i)}$.

Then $\Pi_{(i)}$ is orthogonal to $\Pi_{(j)}$, $i \neq j$.

For, if $u_r < s_r$ for some r such that $1 \leqslant r \leq n$, then

$$\left\langle \psi^{(i)}_{s_1, \ldots\ldots, s_n}, \psi^{(j)}_{u_1, \ldots\ldots, u_n} \right\rangle$$

$$= \frac{1}{\sqrt{s_r!}} \left\langle (A_r^*)^{s_r} \psi^{(i)}_{s_1, \ldots\ldots, s_{r-1}, 0, s_r, \ldots\ldots, s_n}, \psi^{(j)}_{u_1, \ldots\ldots, u_n} \right\rangle$$

$$= (-1)^{u_r} \sqrt{\frac{u_r!}{s_r!}} \left\langle \psi^{(i)}_{s_1, \ldots\ldots, s_{r-1}, 0, s_r, \ldots\ldots, s_n}, A^{s_r - u_r} \psi_{u_1, \ldots\ldots, u_{r-1}, 0, u_{r+1}, \ldots\ldots, u_n} \right\rangle \quad (6)$$

$= 0$ if $s_r > u_r$ for at least one r for $A_r \varnothing^{(j)} = 0$ by our assumption and when $u_r = s_r$ $r = 1, 2, \ldots\ldots, n$ then $(6) = 0$ because $\left\langle \varnothing^{(i)}, \varnothing^{(j)} \right\rangle = 0$.

Next we show that the system

$$\text{(B)} \quad \left\{ \psi^{(i)}_{s_1, s_2, \ldots\ldots, s_n} \quad \begin{array}{l} i = 0, 1, 2, \ldots\ldots \\ s_1, \ldots\ldots, s_n = 0, 1, 2, 3, \ldots\ldots \end{array} \right\}$$

is complete in Π , i.e. Π_i $i = 0, 1, 2, \ldots\ldots$ together generate the space Π .

Since our topology is derived from the Hilbert space metric and it is separable majorant (as we always consider a separable Hilbert space) there must be an orthonormal system complete with respect to this topology.

We can assume, without loss of generality, that our orthonormal system (B) is the system which can be extended to make it a complete system.

Let us assume that the system (B) is not complete. Then there exists some X such that $NX = -kX$ for some positive integer k (because we have assumed that N is decomposable and its entire spectrum consists of $0\ -1, -2, \ldots\ldots$) and is such that

$$\left\langle \psi^{(i)}_{s_1, s_2, \ldots\ldots, s_n}, X \right\rangle = 0 \qquad i = 0, 1, 2, 3, \ldots\ldots$$

$$< X, X > \neq 0 \qquad s_1, s_2, \ldots\ldots, s_n = 0, 1, 2, 3, \ldots$$

Here we assume that k is the smallest of all such k's.

Now

$$A_r NX = A_r \sum_{i=1}^{n} A_i^{*} A_i X$$

$$= -k A_r X \text{ for some } r,\ 1 < r < n.$$

i.e. $$\left(\sum_{\substack{i=1 \\ i \neq r}}^{n} A_r A_i^{*} A_i + A_r A_r^{*} A_r \right) X = -k A_r X$$

i.e. $$\left(\sum_{\substack{i=1 \\ i \neq r}}^{n} A_i^{*} A_i A_r + (A_r^{*} A_r - 1) A_r \right) X = -k A_r X$$

i.e. $$N A_r X = -(k-1) A_r X.$$

Therefore $A_r X$ is also an eigenvector of N. Also

$$\left\langle A_r X, \psi^{(i)}_{s_1, \ldots\ldots, s_n} \right\rangle = \left\langle X, A_r^{*} \psi^{(i)}_{s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_n} \right\rangle$$

$$= \sqrt{s_r + 1}$$

$$\left\langle X, \psi^{(i)}_{s_1, \ldots\ldots, s_{r-1}, s_r+1, s_{r+1}, \ldots s_n} \right\rangle$$

$$= 0$$

and the corresponding eigenvalue is $-(k-1)$, where $k-1 < k$ contradicting our assumption

Hence $\prod = \sum_i (\dot{+}) \prod^{(i)}$

Thus $\prod$ can be written as the direct sum $\prod = \sum_i (\dot{+}) \prod^{(i)}$ of spaces $\prod^{(i)}$ such that for each i, $\prod^{(i)}$ is an irreducible reducing subspace of $A_1, A_2, \ldots\ldots, A_n$, and $A_1^*, A_2^*, \ldots\ldots, A_n^*$ and having a complete orthonormal system

$$\left\{ \psi^{(i)}_{s_1, s_2, \ldots\ldots, s_n} \quad \begin{matrix} i = 0, 1, 2, \ldots\ldots\ldots \\ s_1, \ldots\ldots, s_n = 0, 1, 2, 3, \ldots\ldots \end{matrix} \right\}$$

with the property that $A_1, A_2, \ldots\ldots, A_n$ and $A_1^*, A_2^*, \ldots\ldots, A_n^*$ satisfy (5).

Little can, however, be said about the cases (b_1) and (b_2).

REFERENCES

1. M. B. Sadiq : Commutation Relations in Indefinite Metric Space. Ph. D. Thesis, London University (1975).
2. Bogner, J. : Indefinite inner product spaces. Springer Verlag, (1974).
3. Putnam, C.R. : Commutation Properties of Hilbert Space Operators and Related Topics. Springer Verlag, (1967).
4. Rellich, E. : Der Eindeutigkeitssatz fur die Losungen der quantennechanischen vertanschungorelationen. Naclor. Akad. Wiss. Gott. Math. Phys. Klasse 107—115 (1946)
5. Tillmann, H. G. : Zur Eindentigkeit der Losungen der quanten mechanischen vertauschungsrelationen-II Arch. der Math. **15**, 332-334 (1964).
6. Rierz, and Sz. Nagy, B. : Functional Analysis, Frederick Ungar Pub. Co. N.Y. (1955).

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IRREDUCIBLE REPRESENTATIONS OF THE COMMUTATION RELATIONS IN A KREIN SPACE UNDER RESTRICTIVE CONDITIONS

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Abstract

In this paper, we discuss the irreducible representations of the commutation relation

$$A A^* - A^* A = - I$$

in a Krein space uuder the following assumptions :

(i) A is a closed, linear and densely defined operator
(ii) $T = A^*A$ is self-adjoint
(iii) $T = A^*A$ is decomposable
(iv) The space is irreducible for A and A^*.

It is proved that the spectrum of A^*A is discrete and there are two irreducible representations of the above commutation relation.

Introduction

In a Krein space, if A is a closed and densely defined operator then A^*A may not be self-adjoint. Unlike a Hilbert space, where every self-adjoint operator has a spectral decomposition, a self-adjoint operator in a Krein space has a spectral decomposition under very restrictive conditions. Further, the spectrum of a self-adjoint operator in a Krein space is symmetrical with respect to the real axis [3]. Therefore if λ is in the spectrum of a self-adjoint operator, then $\bar{\lambda}$ is also in the spectrum. There are complex eigenvalues which belong to the point spectrum and the eigenvectors corresponding to such eigenvalues do not form a part of the orthonormal system of the Krein space.

Under the assumptions we have made, it has turned out that the point spectrum consists of real eigenvalues and the corresponding eigenvectors form a complete orthonormal system. The condition that the space is irreducible for the pair of operators A and A* ensures that the operator A*A does not have a continuous spectrum.

We discuss the problem of the irreducible representation of

$$AA^* - A^*A = -I \tag{C}$$

under the following conditions :

(i) A is a closed, linear, densely defined operator in a Krein Space Π and A satisfies (C), i.e.

$$AA^* - A^*A = -I \quad \text{on a dense set } D,$$

i.e. $\quad D = D(A^*A) = D(AA^*)$.

(ii) $T = A^*A$ is self-adjoint

(iii) $T = A^*A$ is decomposable,

i.e. has a spectral decomposable,

(iv) The space is irreducible for A and A*.

Let the spectral decomposition of T be given by

$$T = \int \lambda \, dE_\lambda .$$

Now by (C), T cannot be the zero operator, there exists some $\mu \in sp(T)$, such that $\mu \neq 0$.

We choose elements $f_n = E(\Delta_n) f_n$ of unit length i.e.

$\| f_n \|_J = 1$, $n = 1, 2, \ldots\ldots$ where $\Delta_n = \left(\mu - \frac{1}{n}, \mu + \frac{1}{n} \right)$, so that

$$(T - \mu I) f_n = g_n \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{1}$$

i.e. $\| (T-\mu I) f_n \|_J = \| g_n \|_J \longrightarrow 0$ as $n \longrightarrow \infty$.

Now $E(\Delta_n) g_n = E(\Delta_n)(T-\mu I) f_n$

$$= (T-\mu I) E(\Delta_n) f_n$$
$$= (T-\mu I) f_n = g_n$$

so that $g_n \in D(T)$.

From (1) we obtain

$$A(A^*A - \mu I) f_n = Ag_n$$

i.e. $\left\{ (A^*A - I) - \mu I \right\} A f_n = Ag_n$

$$\left\{ A^*A - (\mu + 1) I \right\} A f_n = Ag_n \qquad (2)$$

Also

$$\| A f_n \|^2_J \geq | \langle A f_n , A f_n \rangle | = | \langle f_n , A^*A f_n \rangle |$$
$$\longrightarrow | \mu | \, | \langle f_n , f_n \rangle | \qquad (3)$$

Since $| \langle f_n , f_n \rangle | \leq \| f_n \|^2_J = 1$, for every n, we may assume

that $\langle f_n , f_n \rangle \longrightarrow \lambda$ (extracting a subsequence of $\left\{ f_n \right\}$, if necessary)

where

(i) $\lambda \neq 0$, or (ii) $\lambda = 0$.

From (3) we have the following possibilities

(*a*) If $\lambda \neq 0$, then $\| A f_n \|^2_J \geq | \mu | \, | C |$

i.e. $A f_n \neq 0$ for $n = 1, 2, 3, \ldots\ldots$

(*b*) If $C = 0$ then either

(b_1) : $\| A f_n \|_J > 0$

i.e. $A f_n \neq 0$ for $n = 1, 2, \ldots\ldots$; or

(b_2) : $\| A f_n \|_J = 0$

i.e. $A f_n = 0$ for $n = 1, 2, \ldots\ldots$, We consider the case (*a*) first.

$$\text{Since } \| A f_n \|_J^2 \longrightarrow |\mu| \, |C|$$

$$\| Ag_n \|_J^2 = \langle Ag_n , J Ag_n \rangle = \langle g_n , A^*JAg_n \rangle$$

$$< \| g_n \| \, \| A^*J \, A g_n \|_J \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Here we assume that $D = D(A^*A) = D(AA^*)$ is invariant under J.

Hence from (2) we have

$$\left\{ A^*A - (\mu+1) I \right\} A f_n \longrightarrow 0 \text{ as } n \longrightarrow \infty ;$$

therefore $\mu + 1$ also belongs to the spectrum of T.

By assumption (iv), the space is irreducible for A and A^*, therefore we have the following two possibilities :

(1) μ is an eigenvalue of $A^* A$.

(2) μ belongs to the continuous spectrum of $A^* A$.

(1) Let μ be an eigenvalue of $A^* A$.

Let us denote the corresponding eigenvector by $\varnothing$; we have $A^* A \varnothing = \lambda \varnothing$, then

$$\begin{aligned} A^* A \, A \varnothing &= (AA^* + I) A \varnothing \\ &= A (A^*A + I) \varnothing \\ &= A (\mu + 1) \varnothing = (\mu + 1) A \varnothing , \end{aligned}$$

$$A^* AA^* \varnothing = A^* (A^* A - I) \varnothing = (\mu - 1) A^* \varnothing .$$

Therefore $\mu \pm n$ are also the eigenvalues of $A^* A$, unless there is an n such that either $A^* AA^{*n} \varnothing = 0$ or $A^* A^{n+1} \varnothing = 0$. In this case, μ must be an integer and because of the irreducibility of the space, the set

$$\left\{ A^{*n} \varnothing \right\}_{n=0}^{\infty} \left(\left\{ A^n \varnothing \right\}_{n=0}^{\infty} \right) \text{ must span } \Pi .$$

Let us write

$$A^* A \varnothing_n = (\lambda + n) \varnothing_n .$$

Then $\langle A^* A \varnothing_n , \varnothing_m \rangle = \langle \varnothing_n , A^* A \varnothing_m \rangle$.

Therefore $(\lambda + n) < \varnothing_n , \varnothing_m > = (\lambda + m) < \varnothing_n , \varnothing_m >$;
here λ is real, as a consequence of the irreducibility of the space.
We have $(n - m) < \varnothing_n , \varnothing_m > = 0$,
therefore $< \varnothing_m , \varnothing_n > = 0$ if $m \neq n$.
Here $< \varnothing_m , \varnothing_m > \neq 0$ for any m, because then we would have a degenerate space.

Let $(\varnothing_m , \varnothing_m) = \delta_m = \pm 1$.

$$A \varnothing_m = \alpha_m \varnothing_{m+1} , \quad A^* \varnothing_m = \beta_m \varnothing_{m-1}$$

$$m = 0, \pm 1, \pm 2, \ldots$$

then

$$< A \varnothing_m , \varnothing_n > = 0 \quad \text{unless } n = m + 1$$

$$< A^* \varnothing_m , \varnothing_n > = 0 \quad \text{unless } n = m - 1.$$

Now $< A \varnothing_m , \varnothing_{m+1} > = < \varnothing_m , A^* \varnothing_{m+1} >$ gives us

$$\alpha_m \delta_{m+1} = \bar{\beta}_{m+1} \delta_m .$$

If we put $\delta_{m+1} / \delta_m = \eta_m = \pm 1$, we have

$$\bar{\beta}_{m+1} = \alpha_m \eta_m .$$

If we replace $\varnothing_m$ by $\lambda \varnothing_m$ where $| \lambda | = 1$, we do not alter δ_m , α_m , and β_m are multiplied by an arbitrary constant of modulus 1. Therefore we can suppose without loss of generality that $\beta_m \geq 0$ for all m. Then α_m is also real.

Also $A^* A \varnothing_m = \alpha_m \beta_{m+1} \varnothing_m$,

therefore $\alpha_m \beta_{m+1} = \mu + m$

i.e. $\eta_m \alpha_m^2 = \mu + m$.

We have the following possibilities :

(a) There is a smallest $\mu + m$, where m is a negative integer. In this case $\varnothing_m \neq 0$ but $A^* \varnothing_m = 0$, therefore

$$0 = AA^* \varnothing_m = (A^* A - I) \varnothing_m = (\mu + m - 1) \varnothing_m$$

or $\mu = -m + 1 = k + 1$, where $k = -m$ is a positive integer. In this case all the eigenvalues of A* A are positive. We have $\eta_m = 1$ for all m, so that all δ_m are $+1$ or -1. The smallest eigenvalue of A* A is 1. Let us denote the corresponding eigenvector by $\varnothing_0$. Therefore $A^* A \varnothing_0 = \varnothing_0 \qquad < \varnothing_0 , \varnothing_0 > = \pm 1$,

then $AA^* \varnothing_0 = (A^* A - I) \varnothing_0 = 0$.

Thus 0 is the smallest eigenvalue of A* A and the corresponding eigenvector is $\varnothing_0$. By repeated application of A to $\varnothing_0$, we see that $A \varnothing_0 , A^2 \varnothing_0 , \dots , A^n \varnothing_0 , \dots$, are the eigenvectors of AA* corresponding to the eigenvalues $1, 2, \dots, n, \dots$, respectively.

If we normalize these eigenvectors, and denote by ψ_n the normalized eigenvector corresponding to the eigenvalue n, where

$$\psi_n = A^n \varnothing_0 / \sqrt{n!} ,$$

then

$$A \psi_n = A^{n+1} \varnothing_0 / \sqrt{n!} = \sqrt{n+1}\, \psi_{n+1} ,$$

$$A^* \psi_n = A^* A^n \varnothing_0 / \sqrt{n!} = (A^n A^* + n A^{n-1}) \varnothing_0 / \sqrt{n!}$$

$$= A^{n-1} A A^* \varnothing_0 / \sqrt{n!} + \sqrt{n}\, A^{n-1} \varnothing_0 / \sqrt{(n-1)!}$$

$$= \sqrt{n}\, A^{n-1} \varnothing_0 / \sqrt{(n-1)!} = \sqrt{n}\, \psi_{n-1} ,$$

(b) There is the largest $\mu + m$,
where m is a positive integer or zero.

In this case $\varnothing_m \neq 0$ but $\varnothing_{m+1} = 0$.

Then $0 = A^* \varnothing_{m+1} = A^* A \varnothing_m = (\lambda + m) \varnothing_m$ i.e. $\lambda = -m$. Therefore λ is a negative integer,

Taking $\lambda = 0$, we see that the vectors $\varnothing_1 , \varnothing_2 , \dots$, do not exist and the eigenvalues of A* A are $\dots, -2, -1, 0$.

Let us denote the normalized eigenvector of A* A corresponding to the eigenvalue 0 by $\varnothing_0$.

We have $A^* A \varnothing_0 = 0$,
then $A^* A A^* \varnothing_0 = A^* (A^* A - I) \varnothing_0 = - A^* \varnothing_0$,

....

We see that $A^* \varnothing_o$, $A^{*2} \varnothing_o$, ..., $A^{*n} \varnothing_o$, ... have the eigenvalues -1, -2, ..., $-n$ ... respectively.

If we normalize these eigenvectors, and denote by ψ_n, the normalized eigenvector of A^*A corresponding to the eigenvalue $-n$, where

$$\psi_n = A^{*n} \varnothing_o / \sqrt{n!}$$

then

$$A^* \psi_n = A^{*n+1} \varnothing_o / \sqrt{n!} = \sqrt{(n+1)!}\, \psi_{n+1}$$

$$\begin{aligned} A \psi_n &= A A^{*n} \varnothing_o / \sqrt{n!} \\ &= (A^*A - 1) A^{*n-1} \varnothing_o / \sqrt{n!} \\ &= \frac{1}{\sqrt{n}} (A^*A - 1) \psi_{n-1} \\ &= \frac{1}{\sqrt{n}} \left\{ -(n-1) - 1 \right\} \psi_{n-1} \\ &= -\sqrt{n}\, \psi_{n-1} . \end{aligned}$$

We must now take $\eta_m = -1$, δ_m alternate in sign.

(c) μ is not an integer.

We can choose μ, so that $0 < \mu < 1$.

Let us denote the corresponding eigenvector of A^*A by $\varnothing_o$, i. e.

$$A^*A \varnothing_o = \mu \varnothing_o .$$

Then $A\varnothing_o$, $A^2 \varnothing_o$, ..., $A^n \varnothing_o$, ...

are the eigenvectors of A^*A corresponding to the eigenvalues $\mu + 1$, $\mu + 2$, ..., $\mu + n$, ... respectively, and $A^* \varnothing_o$, $A^{*2} \varnothing_o$, ..., $A^{*n} \varnothing_o$, ..., are the eigenvectors of A^*A corresponding to the eigenvalues $\mu - 1$, $\mu - 2$, ..., $\mu - n$, ... respectively.

We denote by ψ_n^+ the normalized eigenvector of A*A corresponding to the eigenvalue $\lambda + n$, where

$$\psi_n^+ = A^n \varnothing_o / \sqrt{\mu(\mu+1) \dots (\mu+n-1)}$$

and by ψ_n^- the normalized eigenvector of A*A corresponding to the eigenvalue $\mu - n$, where

$$\psi_n^- = \frac{1}{i^n} \frac{A^{*^n} \varnothing_o}{\sqrt{(1-\mu)(2-\mu)\dots(n-\mu)}}$$

We have defined here $\psi_o^- = \varnothing_o = \psi_o^+$.

Now

$$A\,\psi_n^+ = \frac{A^{n+1}\varnothing_o}{\sqrt{\mu(\mu+1)\dots\dots(\mu+n-1)}}$$

$$= \frac{\sqrt{\mu+n}\;A^{n+1}\varnothing_o}{\sqrt{\mu(\mu+1)\dots\dots(\mu+n)}}$$

$$= \sqrt{\mu+n}\;\psi_{n+1}^+$$

$$= n = 0,\ 1,\ 2,\ \dots\dots$$

$$A^*\,\psi_n^+ = \frac{A^* A^n \varnothing_o}{\sqrt{\mu(\mu+1)\dots(\mu+n-1)}} = \frac{A^*A\,A^{n-1}\varnothing_o}{\sqrt{\mu(\mu+)\dots(\mu+n-1)}}$$

$$= \frac{\sqrt{\mu+n-1}\;A^{n-1}\varnothing_o}{\sqrt{\mu(\mu+1)\dots(\mu+n-2)}}$$

$$= \sqrt{\mu+n-1}\;\psi_{n-1}^+,\quad n = 1,\ 2,\ 3,\dots\dots$$

$$A\,\psi_n^- = \frac{A\,A^*\,A^{*^{n-1}}\varnothing_o}{i^n\sqrt{(1-\mu)(2-\mu)\dots(n-\mu)}}$$

$$= \frac{(A^*A-1)\,A^{*^{n-1}}\varnothing_o}{i^n\sqrt{(1-\mu)(2-\mu)\dots\dots(n-\mu)}}$$

$$= \frac{\left\{(\mu-\overline{n-1})-1\right\}A^{*^{n-1}}\varnothing_o}{i^n\sqrt{(1-\mu)(2-\mu)\dots(n-\mu)}}$$

$$= \frac{\sqrt{\mu - n}\, A^{*n-1}\, \varnothing_0}{i^{n-1}\sqrt{(1-\mu)(2-\mu)\ldots(n-1-\mu)}}$$

$$= \sqrt{\mu - n}\, \psi^-_{n-1}\, .$$

$$A^* \psi^-_n = \frac{A^{*n+1}\, \varnothing_0}{i^n \sqrt{(1-\mu)(2-\mu)\cdots(n-\mu)}}$$

$$= \frac{i\sqrt{n+1-\mu}\quad A^{*\,n+1}\, \varnothing_0}{i^{n+1}\sqrt{(1-\mu)\ldots\ldots(n+1-\mu)}}$$

$$= i\sqrt{n+1-\mu}\quad \psi^-_{n+1}$$

$$= \sqrt{\mu-(n+1)}\quad \psi^-_{n+1}\quad .$$

In this case all the ψ^+_n have the same signs and ψ^-_n have the alternate signs. Therefore

$$\eta_n = \begin{cases} 1 & \text{if } n > 0 \\ -1 & \text{otherwise.} \end{cases}$$

(2) Next we consider the case when μ belongs to the continuous spectrum. Here we show that (C) has no irreducible representation in Π. To prove this we make the following agsumption,

For any bounded intervel Δ , E (Δ) Π is in the domains of A and A*.

Then it follows that for any bounded interval Δ there exists K_Δ , such that

$$\| A f \|_J < K_\Delta \| f \|_J$$

$$\| A^* f \|_J < K_\Delta \| f \|_J$$

if $f \in E(\Delta)\,\Pi$.

Now if $f \in E(\Delta)\,\Pi$, for any $\delta > 0$, there exist $f_1, \ldots\ldots, f_n \in E(\Delta)\,\Pi$ and $\mu_1, \cdots, \mu_m\, \varepsilon\, \Delta$ such that

$$\| A^* A f_r - \lambda_r f_r \|_J < \delta \| f_r \|_J$$

and $\quad \Sigma \| f_r \|_J \leq 2 \| f \|_J .$

Now $\quad A^* A A f_r = (A A^* + 1) A f_r$

so that

$$\| A^* A A f_r - (\mu_r + 1) A f_r \|_J = \| AA^* A f_r - \mu_r A f_r \|_J$$

$$= \| A (A^* A f_r - \mu_r f_r I) \|_J$$

$$< \delta \| A \|_J \| f_r \|_J$$

$$< \delta K_\triangle \| f_r \|_J$$

So if $A f = g, g = \Sigma g_r = \Sigma A f_r$
then

$$\| A^* Ag - \Sigma \mu_r g_r \|_J < K_\triangle \delta \Sigma \| f_r \|_J$$

$$< K_\triangle \delta \| f \|_J .$$

Letting $\delta \longrightarrow 0$, we have $g \varepsilon E (\triangle) \Pi$
i.e. $\quad A f \varepsilon E (\triangle + 1) \Pi$.
Hence under these hypotheses

$$A : E (\triangle) \Pi \longrightarrow E (\triangle + 1) \Pi .$$

Similarly, $A^* : E (\triangle) \Pi \longrightarrow E (\triangle - 1) \Pi$.

It follows that $\cup_{n=-\infty}^{\infty} E (\triangle + n) \Pi$ is an invariant subspace for A and A* .

Therefore the spectrum of A* A is discrete and consists of points $\left\{ \mu \pm n \right\}_{n=0}^{\infty}$; for, if not we could find $\triangle_1$ and $\triangle_2$ contained in some interval of length L, $\triangle_1 \cap \triangle_2 = \varnothing$, $E (\triangle_1) \neq 0$, $E (\triangle_2) \neq 0$ and $\cup_{n=-\infty}^{\infty} E (\triangle_1 + n) \Pi$, $\cup_{n=-\infty}^{\infty} E (\triangle_2 + n)$ would be disjoint invariant subspaces for A and A* . This fact contradicts our assumption that the space is irreducible for the pair of operators A and A* .

Thus we have shown that the spectrum of A* A is discrete under the assumptions (i) – (iv), we have made.

REFERENCES

1. M.B. Sadiq : Commutation Relations in indefinite metric spaces. A Ph. D. Thesis, London University (1975).
2. J. Bognar : Indefinite inner product spaces· Springer-Verlag, (1974).
3. J. Bognar and A. Kramli : Operators of the form C*C in indefinite inner product spaces. Acta Sci. Math. (S-zeged) **29**, 19—29 (1968).
4. V. Jalava : On spectral decomposition of operators in J-spaces. Ann. Acad. Sci. Fenn Ser. AI, No. 4, 446 (1969).
5. C. R. Putnam : Commutation properties of Hilbert space operators and Related Topics. Springer-Verlag, Berlin, Heidelberg, New York (1967).
6. F. Riesz and B. Sz. Nagy : Functional Analysis, Frederick Ungar Pub. Co. N. Y. (1955).

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NEW EXPLICIT FORMULAS FOR THE GRUNSKY COEFFICIENTS OF UNIVALENT FUNCTIONS

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According to Grunsky's [1] and Schiffer's [2] well-known results a necessary and sufficient condition that the function

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \ (a_1 = 1) \qquad \dots (1)$$

analytic in the dise $\Delta_o = \left\{ z : | z | < 1 \right\}$ should be univalent in Δ_o is that

$$\left| \sum_{n, m=1}^{\infty} g_{nm} x_n x_m \right| \leq \sum_{n=1}^{\infty} \frac{| x_n |^2}{n} \qquad \dots (2)$$

for every sequence $\{x_n\}$ for which the right-hand side converges, where g_{nm} are the Grunsky coefficients of the function (1) determined by

$$\ln \left(\frac{z-\xi}{f(z)-f(\xi)} \right) = \sum_{n, m=0}^{\infty} g_{nm} z^n \xi^m, \qquad \dots (3)$$

Let us note that in (3) and further in our paper, where many-valued functions are observed, we consider their principal values.

From (1), (2) and (3) we obtain, respectively, that the function

$$F(z) = \frac{1}{f(1/z)}, \quad | z | > 1, \qquad \dots (4)$$

is meromorphic in the disc $\Delta_\infty = \{z : |z| > 1\}$ and is univalent in Δ_∞ if and only if (2) is fulfiled, where

$$\ln\left(\frac{z-\xi}{F(z)-F(\xi)}\right)=\sum_{n,m=1}^{\infty}\frac{g_{nm}}{z^n\,\xi^m}, \qquad z,\xi\in\Delta_\infty \qquad \ldots(5)$$

i.e. the Grunsky coefficients g_{nm} for $n,\ m \geq 1$ of the two functions (1) and (4) are the same.

We can consider, more generally, the p =symmetric functions

$$f_p(z) = \sqrt[p]{f(z^p)} = z+\sum_{n=1}^{\infty} a^{(p)}_{np+1}\, z^{pn+1}$$

$$p = 1, 2, \ldots\ldots\ldots;\ z\in\Delta_o \qquad \ldots(6)$$

and

$$F_p(z) = \frac{1}{f_p(1/z)} = z+\sum_{n=1}^{\infty}\frac{\alpha^{(p)}_{np-1}}{z^{np-1}} \qquad \ldots(7)$$

$$p = 1, 2, \ldots;\ z\in\Delta_\infty$$

for which the Grunaky coefficients g_{nm} are determined by the expansion

$$\ln\left(\frac{z^{-1}-\xi^{-1}}{F_p(z^{-1})-F_p(\xi^{-1})}\right) = \ln\left(\frac{z-\xi}{f_p(z)-f_p(\xi)}\right)\cdot\frac{f_p(z)}{2}\cdot\frac{f_p(\xi)}{2}$$

$$=\sum_{n,m=1}^{\infty} g^{(p)}_{nm}\, z^n\,\xi^m, \qquad z,\xi\in\Delta_o \qquad \ldots(8)$$

Explicit formulas for the coefficients $g^{(p)}_{nm}$ are found by Schur [3] and Hummel [4]. But their methods and results are much more complicated. In this paper we use an alternate method by which we obtain new explicit formulas for the Grunsky coefficients $g_{n\,m}^{(p)}$. Our method and results are simpler and more convenient in comparison with those of Schur [3] and Hummel [4] and they afford the possibility of making some new investigations on the expansion (8) and its coefficients. In particular, we give a new recursion formula by which the computation of the Grunsky coefficients $g_{nm}^{(p)}$ is considerably easier in comparison with the Schur [3] and Hummel [4] formulas. Our method is based on the Faa di Bruno precise formula for the *n th* derivative of composite functions, which is given in our paper [5], pp. 82-83, Theorem 1. By this formula the following result is obtained (see [5], p. 84, formulas (25-26)): If $g(z)/z \neq 0$ in $\triangle_0$, where $f(z)$ is the regular function (1), and m is an arbitrary complex number, then in $\triangle_0$ we have the expansion

$$\left(\frac{f(z)}{z}\right)^m = \sum_{n=0}^{\infty} g_n(m)\, z^n \equiv \sum_{n=0}^{\infty} c_{z^n}\left(\frac{f(z)}{z}\right)^m \cdot z^n, \qquad \dots (9)$$

in which the coefficients

$$g_n(m) \equiv C_{z^n}\left(\frac{f(z)}{z}\right)^m = \sum_{k=0}^{n} (m)_k\, C_{nk}\left(a_2, \dots, a_{n-k+2}\right) \qquad \dots (10)$$

where $(m)_k$ denotes the factorial polynomial

$$(m)_k = m(m-1)\dots(m-k+1), \quad (k \geq 1;\ (m)_0 = 1), \qquad \dots (11)$$

and $C_{nk}\left(a_2, \dots, a_{n-k+2}\right)$ denotes the polynomial

$$C_{nk}\left(a_2,\ldots,a_{n-k+2}\right)=\sum\frac{a_2^{\nu_1}\ldots\ldots(a_{n-k+2})^{\nu_{n-k+1}}}{\nu_1!\ldots\ldots(\nu_{n-k+1})!}$$

$$\left(1\le k\le n\,;\,n\ge 1\right),\ \left(C_{no}(a_2,\ldots,a_{n+2})=0,\right.$$

$$\left.n\ge 1\,;\,C_{oo}(a_2)=1\right)\quad\ldots(12)$$

where the sum is taken over all non-negative integers $\nu_1,\ldots,\nu_{n-k+1}$ satisfying

$$\nu_1+\nu_2+\ldots\ldots+\nu_{n-k+1}=k\,,$$

$$\nu_1+2\nu_2+\ldots\ldots+(n-k+1)\,\nu_{n-k+1}=4\quad\ldots(13)$$

Applications of this result are given in our papers [5] and [6].

Now we shall give here another application of this result finding the Grunsky coefficients in (8) :

Theorem If the function (1) is regular and univalent in Δ_o, then at the points $z,\xi\in\Delta_o$ the expansion (8) has the form (P = 1, 2,...)

$$\ln\frac{z^{-1}-\xi^{-1}}{F_p(z^{-1})-F_p(\xi^{-1})}=\ln\left\{\frac{z-\xi}{f_p(z)-f_p(\xi)}\cdot\frac{f_p(z)}{z}\cdot\frac{f_p(\xi)}{\xi}\right\}$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}g_{np,\,mp}(p)\,z^{np}\,\xi^{mp}$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{p-1}{}'\;g_{np+q,\,mp-q}(p)\,z^{np+q}\,\xi^{mp-q}\quad\ldots(14)$$

where the prime of the sum denotes that for $p=1$ it is replaced for 0, and the Grunsky coefficients are equal to $(n,m,p\ge 1)$

$$g_{np,\,mp}(p)=\frac{1}{p}\sum_{s=1}^{n}\frac{1}{s}g_{n-s}(s)\,g_{m+s}(-s)\quad\ldots(15)$$

respectively to ($n > 0,\ m \geqq 1,\ p \geqq 2,\ 1 \leqq q \leqq p-1$)

$$g_{np+q,\ mp-q}(p) = \sum_{s=o}^{n} \frac{1}{ps+q}\, g_{n-s}\left(s+\frac{q}{p}\right) g_{m+s}\left(-s-\frac{q}{p}\right) \quad \ldots(16)$$

where ($0 \leqq q \leqq p-1$)

$$g_{n-s}\left(s+\frac{q}{p}\right) \equiv C_{z^{n-s}}\left(\frac{f(z)}{z}\right)^{s+\frac{q}{p}}$$

$$= \sum_{k=0}^{n-s} \left(s+\frac{q}{p}\right)_k C_{n-s,\,k}\left(a_2,\ldots a_{n-s-k+2}\right) \quad \ldots(17)$$

$$g_{m+s}\left(-s-\frac{q}{p}\right) \equiv C_{z^{m+s}}\left(\frac{f(z)}{z}\right)^{-s-\frac{q}{p}}$$

$$= \sum_{k=1}^{m+s} \left(-s-\frac{q}{p}\right)_k C_{m+s,\,k}\left(a_2,\ldots a_{m+s-k+2}\right) \quad \ldots(18)$$

where $\left(\pm s \pm (q/p)\right)_k$ denotes the products (11) after substituting m by $\pm s \pm (q/p)$ and the polynomials $C_{n-s,\,k}\left(a_2,\ldots,\ a_{n-s-k+2}\right)$ and $C_{m+s,\,k}\left(a_2,\ldots,\ a_{m+s-k+2}\right)$ are given by (12 – 13) after substituting n by $n-s$ and $m+s$, respectively.

Proof. Let the function (1) be regular and univelent in Δ_o. Then for $|z| < |\xi|$, we have

$$\ln\left\{\frac{z-\xi}{f_p(z)-f_p(\xi)}\ \frac{f_p(\xi)}{\xi}\right\} = \ln\left(1-\frac{z}{\xi}\right) - \ln\left(1-\frac{f_p(z)}{f_p(\xi)}\right)$$

$$= -\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{z}{\xi}\right)^r + \sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{f_p(z)}{f_p(\xi)}\right)^r \quad \ldots(19)$$

From (6), (9) and (10) we obtain

$$\left(\frac{f_p(z)^r}{z}=\left(\frac{f(z^p)}{z^p}\right)^{\frac{r}{p}}=\sum_{n=0}^{\infty} g_n\left(\frac{r}{p}\right) z^{np} \qquad \ldots(20)$$

where

$$g_n\left(\frac{r}{p}\right)\equiv C_{z^n}\left(\frac{f(z)}{z}\right)^{\frac{r}{p}}$$

$$=\sum_{k=0}^{n}\left(\frac{r}{}\right)_k C_{nk}\left(a_{2,\ldots,}\, a_{n-k+2}\right) \qquad \ldots(21)$$

From (20—21) it follow that (below we put $r=ps+q$, $s \geq 0$, $1 \leqq q \leqq p$)

$$\ln\left(1-\frac{f_p(z)}{f_p(\xi)}\right)^{-1}=\sum_{r=1}^{\infty}\frac{1}{r\left(f_p(\xi)\right)^r}\sum_{n=o}^{\infty} g_n\left(\frac{r}{p}\right) z^{np+r}$$

$$=\sum_{q=1}^{p}\sum_{s=o}^{\infty}\frac{1}{(ps+q)\left(f_p(\xi)\right)^{ps+q}}\sum_{n=s}^{\infty} g_{n-s}\left(s+\frac{q}{}\right) s^{np+q}$$

$$=\sum_{n=o}^{\infty}\sum_{q=1}^{p} g_{np+q}(p)\left(\xi\right) z^{np+q} \qquad \ldots(22)$$

where $n=0$, $q=1$

$$g_{np+q}(p)(\xi)=\sum_{s=0}^{n}\frac{g_{n-s}\left(s+\frac{q}{p}\right)}{(ps+q)\left(f_p(\xi)\right)^{ps+q}} \qquad \ldots(23)$$

and $g_{n-s}\left((s+q/p)\right)$ are given by (17) for $1 \leqq q \leqq p$.

Thus by (22—23) and (19) we obtain the expansion

$$\ln\left\{\frac{z-\xi}{f_p(z)-f_p(\xi)}\;\frac{f_p(\xi)}{\xi}\right\}=\sum_{n=o}^{\infty}\sum_{q=1}^{\infty} G_{np+q}(p)\left(\xi\right)^{z^{np+q}}$$

$$\ldots(24)$$

where

$$G^{(p)}_{np+q}(\xi) = g^{(p)}_{np+q}(\xi) - \frac{1}{(np+q)\ \xi^{np+q}} \qquad \ldots(25)$$

If one of the two variables z or ξ is fixed in the disc Δ_o, then the left-hand side of (24) is analytic in Δ_o with respect to the other variable (in particular for $z = 4$, the left-hand side of (24) is equal to $\ln\left(f_p(\xi) / \xi f_p'(\xi) \right)$. Hence, at every fixed $\xi \in \Delta_o$ the expansion (24) remains valid for all $z \in \Delta_o$, where the coefficients $G^{(p)}_{np+q}(\xi)$ are analytic function of ξ everywhere in Δ_o and, in particular, at $\xi = 0$. Hence, the principal part of the Laurent expansion of the function (23) in the neighbourhood of the point $\xi = 0$ must contain the term $1/(np+q)\ \xi^{np+q}$ only. On this basis now, we shall obtain the Taylor expansion of the function (25) in the neighbourhood of the point $\xi = 0$. From (6), (9) and (10) again we obtain

$$\left(\frac{f_p(\xi)}{\xi}\right)^{-ps-q} = \left(\frac{f(\xi^p)}{\xi^p}\right)^{-s-\frac{q}{p}}$$

$$= \sum_{m=0}^{\infty} g_m\left(-s-\frac{q}{p}\right) \xi^{mp} \qquad \ldots(26)$$

where

$$g_m\left(-s-\frac{q}{p}\right) \equiv C_{zm}\left(\frac{f(z)}{z}\right)^{-s-\frac{q}{p}}$$

$$= \sum_{k=0}^{m} \left(-s-\frac{q}{p}\right)_k C_{mk}\left(a_2, \ldots, a_{m-k+2}\right) \qquad \ldots(27)$$

$$\left(f_p(\xi)\right)^{-ps-q} = \sum_{m=0}^{s} g_{s-m}\left(-s-\frac{q}{p}\right)\frac{1}{\xi^{mp+q}}$$

$$+ \sum_{m=1}^{\infty} g_{m+s}\left(-s-\frac{q}{p}\right) \xi^{mp-q} \qquad \ldots(28)$$

where $g_{s\mp m}\left(-s-(q/p)\right)$ are obtained by (27) after substituting m by $s \mp m$. By (28) and (23) we obtain the Laurent expansion

$$g_{np+q}^{(p)}(\xi) = \sum_{m=o}^{n} \frac{1}{\xi^{mp+q}} \sum_{s=m}^{n} \frac{1}{ps+q}$$

$$g_{n-s}\left(s+\frac{q}{p}\right) g_{s-m}\left(-s-\frac{q}{p}\right)$$

$$+\sum_{m=1}^{\infty} \xi^{mp-q} \sum_{s=o}^{n} \frac{1}{ps+q} \, g_{n-s}\left(s+\frac{q}{p}\right) \cdot$$

$$g_{m+s}\left(-s-\frac{q}{p}\right) \quad \ldots (29)$$

In the principal part of (29) the term for $m = n$ is equal to $1/(np+q)\ \xi^{np+q}$ and all terms for $m < n$ must be equal to zero. Thus we obtain the identities

$$\sum_{s=m}^{n} \frac{1}{ps+q} \, g_{n-s}\left(s+\frac{q}{p}\right) g_{s-m}\left(-s-\frac{q}{p}\right) = 0$$

$$(o \leq m \leq n-1\ ;\ n \geq 1) \qquad \ldots (30)$$

and the Laurent expansion (29) of the function (23) takes the form

$$g_{np+q}^{(p)}(\xi) = \frac{1}{(np+q)\ \xi^{np+q}} + \sum_{m=1}^{\infty} g_{np+q\,,\,mp-q}^{(p)} \xi^{mp-q}$$

$$(o < |\xi| < 1) \qquad \ldots (31)$$

where

$$g_{np+q,\,mp-q}^{(p)} = \sum_{s=o}^{n} \frac{1}{ps+q} \, g_{n-s}\left(s+\frac{q}{p}\right)$$

$$g_{m+s}\left(-s-\frac{q}{p}\right) \qquad \ldots (32)$$

Hence, by (31) and (25) we obtain the Taylor expansions

$$G^{(p)}_{np+q}(\xi) = \sum_{m=1}^{\infty} g^{(p)}_{np+q,\ mp-q}\ \xi^{mp-q} \qquad (\xi \in \Delta_o)$$

of the coefficients of the expansion (24). Thus by (33) and (24) we finally obtain the Taylor expansion ($p = 1, 2, \ldots\ldots$)

$$\ln\left\{\frac{z-\xi}{f_p(z) - f_p(\xi)}\ \frac{f_p(\xi)}{\xi}\right\}$$

$$= \sum_{n=o}^{\infty} \sum_{m=1}^{\infty} \sum_{q=1}^{p} g^{(p)}_{np+q,\ mp-q}\ z^{np+q}\ \xi^{mp-q}$$

$$(z, \xi \in \Delta_o) \qquad \ldots(34)$$

with coefficients (32). From (34), in particular, for $\xi = 0$ we obtain

$$\ln \frac{z}{f_p(z)} = \sum^{\infty} g^{(p)}_{np}\ z^{np}$$

$$(z \in \Delta_o;\ p = 1, 2, \ldots) \qquad \ldots(35)$$

where

$$g^{(p)}_{np} \equiv g^{(p)}_{np,\ o} = \sum_{s=1}^{n+1} \frac{1}{ps}\ g_{n-s}(s)\ g_s(-s)$$

and $g_{n-s}(s)$ and $g_s(-s)$ are obtained from (10) after corresponding substitution of the letters. Now, if we subtract (35) from (34), we shall obtain the expansion (14) with coefficients (15–18). This completes the proof of our Theorem.

Now we shall establish some typical characteristics of the expansion (14) and the coefficients (15–16). If we replace in (14) in the double sum n for m and m for n, and in the triple sum n for $m-1$. m for $n+1$ and q for $p-q$, we shall obtain the expansion (14) in the form

$$\ln \frac{z^{-1} - \xi^{-1}}{F_p(z^{-1}) - F_p(\xi^{-1})} \quad \ln\left\{\frac{z-\xi}{f_p(z)-f_p(\xi)}\ \frac{f_p(z)}{z}\ \frac{f_p(\xi)}{\xi}\right\}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mp,\ np}^{(p)} \; z^{mp} \; \xi^{np}$$

$$+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{q=1}^{p-1} g_{mp-q,\ np+q}^{(p)} \; z^{mp-q} \; \xi^{np+q} \qquad \dots(37)$$

If now in (14) we change the places of z and ξ and we compare the expansion obtained with (37), we obtain the equations

$$g_{np,\ mp}^{(p)} = g_{mp,\ np}^{(p)} \qquad (p, n, m \geq 1)$$

$$g_{np+q,\ mp-q}^{(p)} = g_{mp-q,\ np+q,\ np+q} \qquad \dots(39)$$

$$(n \geq 0,\ m \geq 1,\ p \geq 2,\ 1 \leq q \leq p-1)$$

which in this form express the well-know symmctry of the Grunsky coefficients. With the rcplacement of n, m and q from (15) and (16) used above, we obtain the following explicit formulas for the sy nmetric coefficients

$$g_{mp,\ np}^{(p)} = \frac{1}{p} \sum_{s=1}^{m} \frac{1}{s} \; g_{m-s}(s) \; g_{n+s}(-s) \qquad \dots(40)$$

$$g_{mp-q,\ np+q}^{(p)} = \sum_{s=1}^{m} \frac{1}{ps-q} \; g_{m-s}\left(s-\frac{q}{p}\right) g_{n+s}\left(-s+\frac{q}{p}\right)$$

$$\dots(41)$$

with (17 – 18) after changing the places of n and m.

Further: if in (36) and (15) we replacc p for Np and in (16) we replace p for Np and q for Nq (N=1, 2,...), immedlately we obtain the formulas

$$g_{Nnp}^{(N_p)} = \frac{1}{N} \; g_{np}^{(p)} \qquad (n, p, N \geq 1) \qquad \dots(42)$$

$$g^{(Np)}_{Nnp,\, Nmp} = \frac{1}{N} g^{(p)}_{np,\, mp} \qquad (n, m, p, N \geq 1) \qquad \dots (43)$$

$$g^{(Np)}_{N(np+q),\, N(mp-q)} = g^{(p)}_{np+q,\, mp-q\,/\,N}$$
$$(n \geq 0, m, N \geq 1, p \geq 2, 1 \leq q \leq p-1) \qquad \dots (44)$$

which in other notations are obtained by Aummel [4], pp, 147-148, Theorem 5 in a complicated way.

Now we shall show that the formulas (15) and (36) can be written in a simpler form. In fact, if we use the Faa di Bruno precise formula from our paper [5], pp. 82-83, Theorem 1 to the composite function $(f(z))^s \equiv t^s \circ f(z)$, $s = 1 \dots, n$, then we shall immediately conclude that the formula (17) for $g = 0$ can be replaced by the following simpler formula

$$g_{n-s}(s) \equiv C_z^n \left(f(z) \right)^s = s\,!\, C_{ns}(a_1, a_2, \dots\dots, a_{n-s+1})$$
$$(a_1 = 1, \; 1 \leq s \leq n) \qquad \dots (45)$$

where $C_{ns}(a_1, \dots, a_{n-s+1})$ denotes the polynomial

$$C_{ns}(a_1, \dots\dots, a_{n-s+1}) = \sum \frac{(a_1)^{\nu_1} \dots\dots (a_{n-s+1})^{\nu_{n-s+1}}}{\nu_1\,! \dots\dots, \nu_{n-s+1}\,!}$$
$$\dots (46)$$

and the sum is taken over all non-negative integers $\nu_1, \dots\dots, \nu_{n-s+1}$ satisfying (13) after substituting k with s. Hence, the formulas (15) and (36) can also be written in the following way $(n, m, p \geqq 1, a_1 = 1)$

$$g^{(p)}_{np,\, mp} = \frac{1}{p} \sum_{s=1}^{n} (s-1)\,!\, C_{ns}(a_1, \dots\dots, a_{n-s+1})\, g_{m+s}(-s)$$
$$\dots (47)$$

$$g^{(p)}_{np} = \frac{1}{p} \sum_{s=1}^{n+1} (s-1)\,!\, C_{ns}(a_1, \dots\dots, a_{n-s+1})\, g_s(-s)$$
$$\dots (48)$$

Similarly, the formuia (40) is written ($m, p \geq 1,\ a_1 = 1$)

$$g^{(p)}_{mp,\ np} = \frac{1}{p} \sum_{s=1}^{m} (s-1)!\, C_{ms} (a_1, \ldots\ldots, a_{m-s+1})\, g_{n+s}(-s) \qquad \ldots\ldots (49)$$

with (46) after substituting n by m.

Now we shall consider some special cases of the Theorem proved. For $p = 1$ from (14), (18) and (47) we obtain ([6], Theorem 6) the expansion for the Grunsky coefficients of the functions (1) and (4)

$f_1(z) \equiv f(z)$, $F_1(z) \equiv F(z) = 1 / f(1/z)$, $g^{(1)}_{nm} \equiv g_{nm}$, $(z \in \Delta_o)$

$$\ln \frac{z^{-1} - \xi^{-1}}{F(z^{-1}) - F(\xi^{-1})} = \ln \left\{ \frac{z - \xi}{f(z) - f(\xi)} \frac{f(z)}{z} \frac{f(\xi)}{\xi} \right\}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g_{nm}\, z^n \xi^m \qquad \ldots (50)$$

where $(a_1 = 1)$

$$g_{nm} = \sum_{s=1}^{n} \sum_{k=1}^{m+s} (-1)^k (s+k-1)!\, C_{ns} (a_1, \ldots\ldots, a_{n-s+1}) \,.$$

$$.\ C_{m+s} (a_2, \ldots\ldots, a_{m+s-k+2}) \qquad \ldots (51)$$

with (46) and (12-13) after substituting n by $m+s$ (the expansion for $\ln (z / f(z))$ is given from (35) with (48) for $p=1$). For $p=2$ from (14), (16), (47) and (51) we obtain the expansion for the Grunsky coefficients of the odd functions (6-7) ($z, \xi \in \Delta_o$).

$$\ln \frac{z^{-1} - \xi^{-1}}{F_2(z^{-1}) - F_2(\xi^{-1})} = \ln \left\{ \frac{z - \xi}{f_2(z) - f_2(\xi)} \frac{f_2(z)}{z} \frac{f_2(\xi)}{\xi} \right\}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g^{(2)}_{2n,\ 2m}\, z^{2n} \xi^{2m}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g^{(2)}_{2n-1,\ 2m-1}\, z^{2n-1} \xi^{2m-1} \qquad \ldots (52)$$

where

$$g^{(2)}_{2n,\,2m} = \frac{1}{2}\, g_{nm} \qquad \ldots (53)$$

$$g^{(2)}_{2n-1,\,2m-1} = \sum_{s=0}^{n-1} \frac{1}{2s+1}\, g_{n-1-s}\left(s+\frac{1}{2}\right) \cdot g_{m+s}\left(-s-\frac{1}{2}\right) \qquad \ldots (54)$$

with (17) after replacing n by $n-1$ and (18) (the expansion for $\ln\,(z\,/\,f_2\,(z)\,)$ is given from (35) with (48) for $p=2$).

Now we shall explain the meaning of the double and the triple sums in (14). By comparing (47) with (51) we get the following general formula

$$g^{(p)}_{np,\,mp} = \frac{1}{p}\, g_{nm} \qquad (n, m,\ p \geq 1) \qquad \ldots (55)$$

from which and from (50) it follows that ($p \geq 1$, z, $\xi \in \Delta_o$)

$$\ln\left(\frac{z^p - \xi^p}{f(z^p) - f(\xi^p)}\; \frac{f(z^p)}{z^p}\; \frac{f(\xi^p)}{\xi^p} \right) = p \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g^{(p)}_{np,\,mp}\, z^{np}\, \xi^{mp} \qquad \ldots (56)$$

From (56) and (14) we obtain ($p \geq 2$, z, $\xi \in \Delta_o$)

$$\ln\left\{ \left(\frac{z-\xi}{f_p\,(z)-f_p\,(\xi)} \right)^{p-1} \frac{\sum_{k=0}^{p-1} \left[\, f_p\,(z)\,\right]^{p-1-k} \left[\, f_p\,(\xi)\,\right]^{k}}{\sum_{k=0}^{p-1} z^{p-1-k}\xi^{k}} \right\}$$

$$= p \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{q=1}^{p-1} g^{(p)}_{np+q,\ mp-q} \, z^{np+q} \xi^{mp-q} \quad \ldots (57)$$

In particular for $p=2$, we have

$$\ln \left[\frac{z-\xi}{f_2(z)-f_2(\xi)} \quad \frac{f_2(z)+f_2(\xi)}{z+\xi} \right]$$

$$= 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g^{(2)}_{2n-1,\ 2m-1} \, z^{2n-1} \xi^{2m-1} \quad \ldots (58)$$

Now we shall examine some consequences which are obtained for $\xi = 2$. For $\xi = z$ from (50) we obtain

$$\ln \left\{ \frac{1}{f'(z)} \left(\frac{f(z)}{z} \right)^2 \right\} = \sum_{n=2}^{\infty} z^n \sum_{m=1}^{n-1} g_{n-m,\ m} \quad \ldots (59)$$

On the other hand, the Faadi Bruno precise formula from our paper [5], pp. 82-83, Theorem 1, applied to the composite functions $\ln\ (f(z)/z) \equiv \ln t \circ (f(z)/z)$ and $f'(z) \equiv \ln t \circ f'(z)$, respectively, immediately gives

$$\ln \frac{f(z)}{z} = \sum_{n=1}^{\infty} b_n z^n \quad \ldots (60)$$

where

$$b_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)!\ C_{nk} \ (a_2, \ldots\ldots, a_{n-k+2}) \quad \ldots (61)$$

and

$$\ln f'(z) = \sum_{n=1}^{\infty} C_n z^n \quad \ldots (62)$$

where

$$C_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)!\, C_{nk} (2a_2, \ldots\ldots, (n-k+2)\, a_{n-k+2}) \quad \ldots (63)$$

and keeping in mind (12-13). Hence, we have

$$\ln \left\{ \left(\frac{1}{f'(z)} \left(\frac{f(z)}{z} \right)^2 \right) \right\} = \sum_{n=2}^{\infty} (2b_n - C_n)\, z^n \quad \ldots (64)$$

whence and from (59) we obtain the relation

$$2b_n - C_n = \sum_{m=1}^{n-1} g_{n-m,\, m} \qquad (n = 2, 3, \ldots\ldots) \quad \ldots (65)$$

From (65) and (55) it follows, more generally, that

$$2b_n - C_n = p \sum_{m=1}^{n-1} g^{(p)}_{(n-m)\, p,\, mp} \qquad (n \geq 2\, ;\ p \geq 1) \quad \ldots (66)$$

For $\xi = z$ from (57) we obtain $(p \geq 2)$

$$\ln \frac{f_p(z)}{z f_p'(z)} = \frac{p}{p-1} \sum_{n=1}^{\infty} z^{np} \sum_{m=1}^{n} \sum_{q=1}^{p-1} g^{(p)}_{(n-m)\, p+q,\ mp-q} \quad \ldots (67)$$

On the other hand, from (6), (60) and (62) we have

$$\ln \frac{f_p(z)}{z f_p'(z)} = \ln \frac{f(z^p)}{z^p f'(z^p)}$$

$$= \sum_{n=1}^{\infty} (b_n - C_n)\, z^{np} \quad \ldots (68)$$

Thus from (67-68) the relation follows :

$$b_n - C_n = \frac{p}{p-1} \sum_{m=1}^{n} \sum_{q=1}^{p-1} g^{(p)}_{(n-m)\,p+q,\; mp-q}$$

$$(n \geq 1\;;\; p \geq 2) \qquad \ldots (69)$$

Let us note that the expansion (35) according to (6) and (60-61) can also be written in the form

$$\ln \frac{z}{f_p(z)} = -\frac{1}{p} \sum_{n=1}^{\infty} b_n\, z^{np} \qquad (p \geq 1\,,\; z \in \Delta_0) \qquad \ldots (70)$$

whence and (35) it follows that

$$g^{(p)}_{np} = -\frac{1}{p}\, b_n \qquad (n, p = 1, 2, \ldots\ldots) \qquad \ldots (71)$$

In comparison with (36) and (48) the formula (71) gives the simplest possible expression of the coefficients in (35).

Finally, we shall note that the polynomials (12) are computed easily by the following recursion formula from our paper [5], p. 85, Formula (27) :

$$C_{nk} = \frac{1}{k} \sum_{\mu=1}^{n-k+1} a_{\mu+1}\, C_{n-\mu,\, k-1}$$

$$(1 \leq k \leq n\;;\; n \geq 1\;;\; C_{no} = 0\;;\; C_{oo} = 1)$$

$$C_{nk} \equiv C_{nk}\,(a_2\,, \ldots\ldots, a_{n-k+2}) \qquad \ldots (72)$$

$$C_{n1} = a_{n+1}\,,\; C_{nn} = \frac{1}{n\,!}\, a_2^{n} \qquad (n \geq 1) \qquad \ldots (73)$$

In the same way the polynomials (46) are also computed, if in (72-73) we replace $a_{\mu+1}$ for a_μ .

Thus by (72) and (51) we obtain the Grunsky coefficients g_{nm} in the expansion (50) for $1 \leqq m \leqq 4$ $(1 \leqq n \leqq m)$:

$$\text{(I)}\quad -g_{11} = a_3 - a_2^2 \;;$$

$$-g_{12} = a_4 - 2a_2a_3 + a_2^3\,, \quad -g_{22} = a_5 - 2a_2a_4 - \tfrac{3}{2}\,a_3^2$$

$$+ 4a_2^2a_3 - \tfrac{3}{2}a_2^4 \;;$$

$$-g_{13} = a_5 - 2a_2a_4 - a_3^2 + 3a_2^2a_3 - a_2^4\,,$$

$$-g_{23} = a_6 - 2a_2a_5 - 3a_3\,a_4^2 + 5a_2a_3^2 + 4a_2^2a_4 - 7a_2^3a_3$$

$$+ 2a_2^5\,,$$

$$-g_{33} = a_7 - 2a_2a_6 - 3a_3a_5 - 2a_4^2 + 12a_2a_3a_4 + 10a_2^2\,a_5$$

$$+ \frac{7}{3}\,a_3^3 - 15a_2^2\,a_3^2 - 8a_2^3a_4 + 14a_2^4a_3 - \frac{10}{3}a_2^6\,,$$

$$-g_{14} = a_6 - 2a_2a_5 - 2a_3a_4 + 3a_2a_3^2 + 3a_2^2a_4 - 4a_2^3a_3 + a_2^5\,,$$

$$-g_{24} = a_7 - 2a_2a_6 - 3a_3a_5 - \tfrac{3}{2}\,a_4^2 + 10a_2a_3a_4 + 4a_2^2a_5$$

$$+ 2a_3^3 - 12a_2^2a_3^2 - 7a_2^3a_4 + 11a_2^4a_3 - \tfrac{5}{2}a_2^6\,.$$

$$-g_{34} = a_8 - 2a_2a_7 - 3a_3a_6 - 4a_4a_5 + 7a_2a_4^2 + 12a_2a_3a_5$$

$$+ 4a_2^2a_6 + 8a_3^2a_4 - 13a_2a_3^3 - 33a_2^2a_3a_4 - 8a_2^3\,a_5$$

$$+ 36a_2^3a_3^2 + 15a_2^4a_4 - 25a_2^5\,a_3 + 5a_2^7\,,$$

$$-g_{44} = a_9 - 2a_2\,a_8 - 3a_3a_7 - 4a_4a_6 - \tfrac{5}{2}\,a_5^2 + 12a_3a_3a_6$$

$$+ 16a_2\,a_4a_5 + 4a_2^2a_7 + 10a_3a_4^2 + 9a_3^2\,a_5 - 48a_2a_3^2a_4$$

$$- 36a_2^2a_3a_5 - 21a_2^2a_4^2 - 8a_2^3a_6 - \frac{19}{4}a_3^4 + 52a_2^2\,a_3^3$$

$$+ 88a_2^3 a_3 a_4 + 16a_2^4 a_5 - 90a_2^4 a_3^2 - 30a_2^5 a_4$$

$$+ 50a_2^6 a_3 - \frac{35}{4} a_2^8 .$$

By the formula (55) from Table (I) we also obtain the Grunsky coefficients $g_{np,\, mp}^{(p)}$ in the expansion (14) for $1 \leqq m \leqq 4$ $(1 \leqq n \leqq m)$ and $p \geq 2$.

By the formula (54) we obtain the Grunsky coefficients $g_{2n-1,\, 2m-1}^{(2)}$ in the expansion (52) for $1 \leqq m \leqq 4$ $(1 \leqq n \leqq m)$:

$$\text{(II)}\ -2g_{11}^{(2)} = a_2\,;\ -2g_{13}^{(2)} = a_3 - \tfrac{3}{4} a_2^2,\ -2g_{33}^{(2)} = a_4 - 2a_2 a_3 + \frac{13}{12} a_2^3\,;$$

$$-2g_{15}^{(2)} = a_4 - \tfrac{3}{2} a_2 a_3 + \tfrac{5}{8} a_2^3,$$

$$-2g_{35}^{(2)} = a_5 - 2a_3 a_4 - \tfrac{5}{4} a_3^2 + \frac{29}{8} a_2^2 a_3 - \frac{85}{4} a_2^4,$$

$$-2g_{55}^{(2)} = a_6 - 2a_2 a_5 - 3a_3 a_4 + \frac{21}{4} a_2 a_3^2 + 4a_2^2 a_4 - \frac{59}{8} a_2^3 a_3 + \frac{689}{320} a_2^5\,;$$

$$-2g_{17}^{(2)} = a_5 - \frac{3}{2} a_2 a_4 - \frac{3}{4} a_3^2 + \frac{15}{8} a_2^2 a_3 - \frac{35}{64} a_2^4,$$

$$-2g_{37}^{(2)} = a_6 - 2a_2 a_5 - \frac{5}{2} a_3 a_4 + 4a_2 a_3^2 + \frac{29}{8} a_2^2 a_4 - \frac{45}{8} a_2^3 a_3 + \frac{49}{32} a_2^5,$$

$$-2g_{57}^{(2)} = a_7 - 2a_2 a_6 - 3a_3 a_5 - \frac{7}{4} a_4^2 + \frac{45}{4} a_2 a_3 a_4 + 4a_2^2 a_5 + \frac{9}{4} a_3^3 - \frac{225}{16} a_2^2 a_3^2 - \frac{123}{16} a_2^3 a_4 + \frac{839}{64} a_2^4 a_3 - \frac{791}{256} a_2^6,$$

$$-2g^{(2)}_{77} = a_8 - 2a_2a_7 - 3a_3a_6 - 4a_4a_5 + \frac{29}{4}a_2a_4^2 + 12a_2a_3a_5$$

$$+ 4a_2^2a_6 + \frac{33}{4}a_3^2a_4 - \frac{27}{2}a_2a_3^3 - \frac{273}{8}a_2^2a_3a_4 - 8a_2^3a_5$$

$$+ \frac{603}{16}a_2^3a_3^2 + \frac{989}{64}a_2^4a_4 - \frac{421}{16}a_2^5a_3 + \frac{9517}{1792}a_2^7 .$$

Thus the Tables (I) and (II) contain the coefficients a_7, a_8 and a_9 for which the Bieberbach conjecture $|a_7| \leqq 7$, $|a_8| \leqq 8$ and $|a_9| \leqq 9$ is currently open.

REFERENCES

1. H. Grunsky, Koeffizientenbedingungen fur schlicht abbildende meromorphe Funktionen, Math. Z. **45** (1939), 29—61.
2. M. Schiffer, Faber polynomials in the theory of univalent functions, Bull. Amer. Math. Soc. **54** (1948), 503—517.
3. I. Schur, On Faber polynomials, Amer. J. Math. **67** (1945), 33—41.
4. J.A. Hummel, The Grunaky coefficients of a schlicht function, Proc. Amer. Math. Soc. **15** (1964), 142-150.
5. P. G. Todorov. New explicit formulas for the coefficients of P-symmetric functions, Proc. Amer. Math. Soc. **77** (1979), 81—86.
6. P. G. Todorov, Explicit formulas for the coefficients of Faber polynomials with respect to univalent functions of the class Σ. Proc. Amer. Math. Soc. **82** (1981), 421—438.

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ON THE KLEIN'S REGULARIZATION (USING SPLINES) FOR THE NUMERICAL SOLUTION OF CERTAIN FREDHOLM CONVOLUTION EQUATIONS OF THE FIRST KIND

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Introduction.

The general linear equation may be written as

$$h(x) f(x) + \int_a^b k(x, y) f(y)\, dy = g(x) \quad a \leq x \leq b). \quad \text{...(1.1)}$$

where the known functions $h(x)$, $K(x, y)$ and $g(x)$ are assumed to be bounded and usually to be continuous. If $h(x) \equiv 0$ the equa ion is of first kind if $h(x) \neq 0$ for $a \leq x \leq b$, the equation is of second kind if $h(x)$ vanishes somewhere but not identically, the equation is of third kind.

If the range of integration is infinite or if the kernel $k(x, y)$ is not bounded, the equation is singular. Here we will consider only non-singular lInear integral equations of the first kind.

Consider the FREDHOLM integral equation of the first kind of convolution type.

$$(K f)(x) \equiv \int_{-\infty}^{\infty} k(x - y)\, f(y)\, dy = g(x) \qquad -\infty < y < \infty \quad \text{... (1.2)}$$

where k and g are known functions in $L_2(R)$ and $f \in H^p(R)$ is to be found. There is extensive literature on equations of the second kind but literature on linear equations of the first kind is sparse. However, several methods for solving equations of the first kind numerically have been proposed [5 – 18]. No method has been very successful for arbitrary kernels when the function $g(x)$ is known with only modest accuracy. Hence we conclude that the success in solving equation (1.2) depends to a large extent on the accuracy of $g(x)$ and the shape of $k(x-y)$.

2. Description of the Technique :

We now return to the convolution equation (1.2). Klein worked in (real) x-space using natural splines. To simplify the computation we have

(A) used cardinal B-splines and

(B) worked in Fourier space.

(A) Let f be approximated by

$$f_M(x) = \Sigma_{j=o}^{M-1} \alpha_j \, B_j(H\,;x) \qquad \ldots (2.1)$$

where the $B_j(H\,;x)$ are periodic cubic cardinal B-splines with period $T = MH$ and knot spacing H. M is the number of B-splines.

The vector $\underset{-}{\alpha} = (\alpha_o, \ldots, \alpha_{M-1})^T$

Following Schoenberg [11], we have

$$B_j(H\,;x) = Q\left(\frac{x}{H} - j + 2\right) \qquad \ldots (2.2)$$

$$\text{where } Q(x) = \frac{1}{6}\,\Sigma_{K=0}^{4} (-1)^K \binom{4}{K} (x-K)_+^3 \qquad \ldots (2.3)$$

where $\quad x = \max(0, x)$.

Since $B_o(H\,;x)$ is periodic on $(0, T)$, it has the Fourier series

$$B_o(H\,;x) = \frac{1}{T}\,\Sigma_{q=-\infty}^{\infty} \hat{B}_{oq} \exp(i\,\omega_q\,x) \qquad \ldots (2.4)$$

where $\quad \omega_q = (2\pi/T)\,q$

and

$$\hat{B}_{oq} = \int_0^T B_o(H\,;x)\exp(-i\,\omega_q\,x)\;dx.$$

$$= H\left[\frac{\sin(H\,\tilde{\omega}_q/2)}{(H\,\tilde{\omega}_q/2)}\right]^4 \qquad \dots (2.5)$$

$$\tilde{\omega}_q = \begin{cases} \omega_q\,, & 0 \le q < N/2 \\ \omega_{N-q}\,, & \tfrac{1}{2}N \le q \le N-1 \end{cases}$$

($\wedge$ denotes Fourier transformation, then from the convolution theorem we have $\hat{K}(\omega)\,\hat{f}(\omega) = \hat{g}(\omega)$)

Furthermore, since $B_j(H\,;x)$ is simply a translation of $B_o(H\,;x)$ by an amount jH, we have

$$\hat{B}_{jq} = \hat{B}_{oq}\exp\left(-i\,\omega_q\,jH\right) \qquad \dots (2.6)$$

$$q = 0, \pm 1, \pm 2, \dots\dots\dots\dots$$

$$j = 0, 1, 2, \dots\dots\dots\dots\dots \quad M-1$$

The spline in equation (2.1) has the Fourier series

$$f_M(x) = \frac{1}{T}\,\Sigma_{q=-\infty}^{\infty}\,\hat{f}_{M,q}\,\exp(i\,\omega_q\,x) \qquad \dots (2.7)$$

with Fourier coefficients

$$\hat{f}_{M,q} = \sum_{j=0}^{M-1}\alpha_j\,\hat{B}_{jq} \qquad \dots (2.8)$$

(B) We consider the transition to discrete Fourier space. We shall assume (*i*) that the functions k and g are approximated on (o, T) by the trigonometric polynomials k_N and $g_N \in T_{N-1}$ defined by equations

$$\left.\begin{aligned} g_N(x) &= \frac{1}{2}\ \Sigma_{q=0}^{N-1}\ \hat{g}_{N,q} \exp(i\,\omega_q\, x) \\ k_N(x) &= \frac{1}{2}\ \Sigma_{q=0}^{N-1}\ \hat{k}_{N,q} \exp(i\,\omega_q\, x) \end{aligned}\right\} \qquad (2.9)$$

where

$$\left.\begin{aligned} \hat{g}_{N,q} &= \Sigma_{n=0}^{N-1}\ g_n \exp(-i\omega_q\, x_n) \\ \hat{K}_{N,q} &= \Sigma_{n=0}^{N-1}\ k_N \exp(-i\,\omega_q\, x_n) \end{aligned}\right\} \quad q = o, \ldots, N-1$$

with $g\ (x_n) = g_n = g_N\,(x_n)$ and $\omega_q = (\frac{2\pi}{Nh})\, q$

the order of regularization $p = 2$

The appropriate smoothing functional is then

$$C\,(f_M : \lambda) = \|\ \frac{1}{\sigma}\ [k_N(x)*f_M\ (x)]\ \|_2^2 + \lambda\ |\ f''_M\,(x)\,\|_2^2$$

where $\|\ .\ \|$ denotes the inner product norm on $L_2\,(0, T)$, (2.11)

since $K_N * f_M \in T_{N-1}$ for any square integrable periodic funtion f_M of period T, Plancherel's theorem gives

$$\|\ \frac{1}{\sigma}\,(k_N * f_M - g_N)\ \|_2^2 = \frac{T}{N^2\sigma^2}\ \Sigma_{q=o}^{N-1}\ |\ \hat{k}_{N,q}\,\hat{f}_{M,q} - \hat{g}_{N,q}\ |^2 \qquad (2.12)$$

Now. Plancherel's theorm applied to the periodic splines f''_M gives

$$\|\ f''_M\ \|^2 = \frac{1}{T}\,\Sigma_{q=-\infty}^{\infty}\ \tilde{\omega}_q^{4}\ |\ \hat{f}_{M,q}\ |^2 \qquad (2.13)$$

where

$$|\,\hat{f}_{M,q}\,|^2 \sim \tilde{\omega}_q^{(-8)} \text{ as } |\,q\,| \to \infty$$

The infinite series clearly converges.

In practice, we truncate the series to the form

$$\frac{1}{T} \sum_{q=0}^{N_1-1} \tilde{\omega}_q^4 \; | \hat{f}_{M,q} |^2 \qquad \ldots (2.14)$$

where $\quad N \le N_1 < \infty$

Now to express the functional (2.11) in matrix form we define the matrices

$$\left.\begin{array}{l}
\hat{P} \; (N \times N) : \; \hat{P}_{qr} = \sqrt{\dfrac{T}{N\sigma}} \; \delta_{qr} \\
\qquad\qquad\qquad\qquad q, r = 0, \ldots\ldots\ldots\ldots N-1 \\
\hat{K} \; (N \times N) : \hat{K}_{qr} = \hat{K}_{N,q} \; \delta_{qr} \\
\qquad\qquad\qquad\qquad q, r = 0, \ldots\ldots\ldots, N-1 \\
\hat{B} \; (M \times N) : \; \hat{B}_{jq} \text{ as in equation (2.6)} \\
\qquad\qquad\qquad\qquad j = 0, \ldots\ldots, M-1 \\
w^{(1)} (N \times M) : \; W^{(1)} = \hat{K} \, (\hat{B})^T \\
w^{(2)} (N_1 \times M) : \; W^{(2)}_{js} = \dfrac{1}{T} \, \tilde{\omega}_s^2 \, B_{js}
\end{array}\right\}$$

from which we obtain

$$C\,(f_M \, ; \lambda) = C\,(\underline{\alpha} \, ; \lambda) = \| \hat{P} \, (W^{(1)} \underline{\alpha} - \hat{\underline{g}} \|_2^2 + \lambda \| W^{(2)} \underline{\alpha} \|_2^2 \qquad \ldots (2.16)$$

where $\| \, . \, \|_2^2$ denotes the vector 2-norm in C^N and

$$\hat{\underline{g}} = (\hat{g}_{N,0}, \ldots\ldots, \hat{g}_{N,N-1})^T$$

or

$$\left.\begin{aligned} \underline{U} &= W^{(1)} H_{(\hat{p})}^{\wedge 2} \, \hat{\underline{g}} \\ W &= W^{(1)} H_{(\hat{p})}^{\wedge 2} \, W^{(1)} \\ \text{and } V &= W^{(2)\,H} \, W^{(2)} \end{aligned}\right\} \quad \ldots (2.17)$$

C $(\underline{\alpha} ; \lambda)$ has a unique minimum at

$$\underline{\alpha} = (W + \lambda V)^{-1} \underline{U} \quad \ldots (2.18)$$

3. Special properties of W & V

It is easy to show that the $r\,s$ th element of W is

$$W_{rs} = \frac{T}{N^2 \sigma^2} \sum_{q=0}^{N-1} \left| \hat{K}_{N,q} \hat{B}_{oq} \right|^2 \exp\{i\omega_q (r-s) H\},$$

$$r, s = 0, 1, 2, \ldots\ldots\ldots, M-1$$

$$= \sum_{q=0}^{N-1} a_q \exp\left(\frac{2\pi}{M} (r-s)\, iq \right) \quad \ldots (3.1)$$

$$\text{where } a_q = \frac{T}{N^2 \sigma^2} \left| \hat{K}_{N,q} \hat{B}_{oq} \right|^2 \quad \ldots (3.2)$$

It follows that W is a circulant matrix [12], since

$w_{jk} = W_{rs}$ if $j-k = (r-s) \pmod{M}$

W is also hermitian

Similarly V *is also a circulant hermition matrix, with*

$$V_{rs} = \sum_{q=0}^{N-1} b_q \exp\left(\frac{2\pi}{M} iq\,(r-s) \right) \quad \ldots (3.3)$$

where

$$b_q = \frac{1}{T} \left| \tilde{\omega}_q^2 \hat{B}_{oq} \right|^2$$

It is well known that the Model matrix ψ *of any* $M \times M$ *circulant matrix has elements*

$$\psi_{rs} = \frac{1}{\sqrt{M}} \exp\left(\frac{2\pi i}{M} rs \right) \qquad \ldots (3.5)$$

Under this normalization ψ is unitary :

$$\psi^H \psi = \psi \psi^H = I \qquad \ldots (3.6)$$

Thus if W and V have real eigenvalues μ_s and ν_s respectively where

$s = 0, \ldots\ldots, M-1$, we may write

$$\left. \begin{aligned} W &= \psi D_W \psi^H \\ V &= \psi D_V \psi^H \end{aligned} \right\} \qquad \ldots (3.7)$$

where $D_W = \text{diag}\,(\mu_s)$, $D_V = \text{diag}\,(\nu_s)$. We then have

$$(W + \lambda V)^{-1} = \psi \Lambda \psi^H$$

$$\text{where} \quad \Lambda = \text{diag}\left(\frac{1}{\mu_s + \lambda \nu_s} \right) \qquad \ldots (3.8)$$

We now show that the eigenvalues μ_s and ν_s are simply related to the coefficients a_q and b_q defined in equations (3.2) and (3.4).

Consider the eigen value equation

$$\sum_{n=0}^{M-1} W_{mn} \psi_{ns} = \mu_s \psi_{ms} \qquad \ldots (3.9)$$

From equations (3.1) the L. H. S is

$$\sum_{n=0}^{M-1} \sum_{q=0}^{N-1} a_q \exp\left[\frac{2\pi i}{M} q\,(m-n) \right] \psi_{ns}$$

$$= \frac{1}{\sqrt{M}} \Sigma_n \Sigma_q a_q \exp\left[\frac{2\pi i}{M} \{q\,(m-n) + ns\} \right]$$

$$= \frac{1}{\sqrt{M}} \Sigma_q \{ a_q \left[\exp\left[\frac{2\pi i}{M} mq \right] \right.$$

$$\left. \Sigma_n \exp\left[\frac{2\pi i}{M} (s-q)\,n \right] \right\}$$

since

$$\sum_{n=0}^{M-1} \exp\left[\frac{2\pi i}{M} j n\right] = \begin{cases} M, & j \equiv 0 \pmod{M} \\ 0, & \text{otherwise} \end{cases}$$

the L. H S. of (3.9) is

$$M \sum_{\substack{q=0 \\ q \equiv s \pmod{M}}}^{N-1} a_q \psi_{mq} = \left(M \sum_{\substack{q=0 \\ q \equiv s \pmod{M}}}^{N-1} a_q\right) \psi_{ms}$$

Hence

$$\mu_s = M \sum_{\substack{q=0 \\ q \equiv s \pmod{M}.}}^{N-1} a_q \quad \text{... (3.10)}$$

$$\nu_s = M \sum_{\substack{q=0 \\ q \equiv s \pmod{M}.}}^{N-1} b_q \quad \text{... (3.11)}$$

4. Calculation of λ and α

The *rth* element of the vector $\underline{U}$ is

$$U_r = \sum_{q=0}^{N-1} c_q \exp\left[\frac{2\pi i}{M} qr\right] \quad r = 0, \dots, M-1 \quad \text{... (4.1)}$$

where

$$c_q = \frac{T}{N^2 \sigma^2} \bar{\hat{K}}_{N,q} \hat{g}_{N,q} \hat{B}_{oq} \quad \text{... (4.2)}$$

where σ^2 is unknown, Turchin [14] suggests its estimation by the formula

$$\sigma^2 = \frac{1}{N(N-2M)} \sum_{q=M}^{N-(M+1)} |\hat{g}_{N,q}|^2 \quad \text{... (4.3)}$$

where $M \simeq N/4$

It is clear that premultiplication of a C^M vector by ψ^H is equivalent to an M-dimensional DFT. We may thus write

$$\hat{\underline{\alpha}} = \psi^H \underline{\alpha} \text{ and } \hat{\underline{U}} = \psi^H \underline{U}$$

From equations (2.18) and (3.8), therefore, we have

$$\underline{\hat{\alpha}} = \Lambda \, \underline{\hat{U}} \qquad \dots (4.4)$$

Hence $$\hat{\alpha}_s = \frac{\sqrt{M}\, \hat{U}_s}{M\,(\mu_s + \lambda\, \nu_s)} \qquad \dots (4.5)$$

whee $$\hat{U}_s = \sqrt{M} \sum_{\substack{q=0 \\ q \equiv S \ (\mathrm{mod}\ M)}}^{N-1} c_q \qquad \dots (4.6)$$

λ in (4.5) is unknown

In order to evaluate the optimal value of λ in the case of trignometric approximation we write

$$T_r \left(W(W+\lambda V) \right)^{-1} - \lambda \underline{\hat{\alpha}}^H V \underline{\hat{\alpha}} = 0 \qquad \dots (4.7)$$

which reduces to

$$\sum_{s=0}^{N-1} \frac{\mu_s}{\mu_s + \lambda v_s} - \lambda \sum_{s=0}^{N-1} \frac{|\hat{U}_s|^2 \nu_s}{(\mu_s + \lambda \nu_s)^2} = 0 \qquad \dots (4.8)$$

where the eigenvalues μ_s and v_s , $\hat{U}_s$ are calculated earlier.

(4.8) is a non-linear eq. in λ and *in some problems which we have discussed it has more than one value i.e. the equation (4.8) has more than one zero ; then we shall have to pick up the optimal value of* λ i.e. the regularization parameter.

Knowing λ, $\underline{\alpha}$ may then be calculated from the inverse DFT of equation (4.5) as

$$\underline{\alpha} = \psi \, \underline{\hat{\alpha}}$$

5. Calculation of Solution vector $\underline{f}$

(I) when M=N

i.e. when number of cardinal cubic B-splines is equal to the number of grid points.

$$f_M(J) = \sum_{J=0}^{M-1} (\alpha_{j-1} + 4\alpha_j + \alpha_{j+1}) / 6.0$$

where

$$\alpha_{-1} = \alpha_{M-1}$$

$$\alpha_o = \alpha_M \text{ and } \alpha_1 = \alpha_{M+1} .$$

(II) when $M = N/2$

$$\hat{U}_s = \sqrt{M}\,(c_s + c_{M+S})$$

$$\mu_s = M\,(a_s + a_{M+S})$$

$$v_s = M\,(b_s + b_{M+S})$$

$$o \leq s \leq M-1$$

$$s \equiv q \pmod{M}$$

$$\alpha_{-1} = \alpha_{M-1},\ \alpha_o = \alpha_M \text{ and } \alpha_1 = \alpha_{M+1}$$

$$\left.\begin{aligned} f_M(2J) &= \sum_{j=0}^{M-1} (\alpha_{j-1} + 4\alpha_j + \alpha_{j+1}) / 6.0 \\ f_M(2j+1) &= \sum_{j=0}^{M-1} (\alpha_{j-1} + 23\,\alpha_j + 23\,\alpha_{j+1} + \alpha_{j+2})/48.0 \end{aligned}\right\} \quad (A)$$

Prob'ems discussed.

P (1) : This example is given in Phillips [13] and has a noisy data function g with a mɛximum absolute error of about 0.02. We have

$$\int_{-30}^{30} k\,(x-y)\, f\,(x)\; dx = g\,(y)$$

where $k\,(x)$, $g\,(x)$ and $f(x)$ are given in Table (1). The number of gird points is 31.

TABLE 1

x_n	g_n	k_n	f_n
−30.0	0.0100	0.1184	0.0000
−28.0	0.0100	0.1311	0.0000
−26.0	0.0110	0.1464	0.0000
−24.0	0.0170	0.1651	0.0000
−22.0	0.0305	0.1883	0.0000
−20.0	0.0405	0.2179	0.0000
−18.0	0.0585	0.2563	0.0000
−16.0	0.0869	0.3077	0.0000
−14.0	0.1309	0.3788	0.0000
−12.0	0.2018	0.4816	0.0000
−10.0	0.3235	0.6380	0.0000
−8.0	0.5469	0.8914	0.0000
−6.0	0.9621	1.3333	0.0019
−4.2	1.6301	2.1483	0.0345
−2.0	2.4047	3.5108	0.0965
0.0	2.9102	4.3600	0.1321
2.0	2.8912	3.0628	0.1096
4.0	2.4586	1.6329	0.0584
6.0	1.9049	0.8806	0.0349
8.0	1.4144	0.5095	0.0173
10.0	1.0282	0.3137	0.0107
12.0	0.7411	0.2021	0.0028
14.0	0.5409	0.1341	0.0005
16.0	0.4083	0.0906	0.0000
18.0	0.3214	0.0614	0.0000
20.0	0.2623	0.0413	0.0000
22.0	0.2201	0.0269	0.0000
24.0	0.1886	0.0165	0.0000
26.0	0.1580	0.0089	0.0000
28.0	0.1270	0.0031	0.0000
30.0	0.0780	0.0013	0.0000

P 2 (A)

This problem is given in Turchin [6]. We have

$$\int_{-2}^{2} k\,(x-y)\,f\,(y)\;dy = g\,(x)$$

where f is the function of two Gaussian functions.

$$f\,(x) = 0.5 \exp\left[\frac{-\,(x+0.4)^2}{0.18}\right] + \exp\left[\frac{-\,(x-0,6)^2}{0.18}\right]$$

with essential support $-2 \leq x \leq 2$

K (x) is triangular with equation

$$K\,(x) = \begin{cases} -x + 0.5 & 0 \leq x < 0.5 \\ x + 0.5 & -0.5 \leq x < 0 \\ 0 & |x| \geq 0.5 \end{cases}$$

we have calculated the values of g (x) by the NAG Algorithm DOIABA using Romberg's method with aecuracy 10^{-7}, 41 grid values have been eonsidered.

P 2 (B)

This example is the same as P 2 (A) except that the triangular kernel is made wider.

$$K\,(x) = \begin{cases} (5/8)\,(-x + 0.8), & 0 \leq x < 0.8 \\ (5/8)\,(x + 0.8), & -0.8 \leq x < 0 \\ 0 & |x| \geq 0.8 \end{cases}$$

The wider kernel makes the problem more ill-posed. 41 grid points are again considered.

P 2 (C)

The problem is made highly ill-posed by choosing an even wider kernel

$$K\,(x) = \begin{cases} (5/12)\,(-x + 1.2), & 0 \leq x < 1.2 \\ (5/12)\,(x + 1.2), & -1.2 \leq x < 0 \\ 0 & |x| \geq 1.2 \end{cases}$$

Again 41 grid points are considered.

P 2 (BE)

The problem is the same as P 2 (B) but we have extended the support from (−2.0 to 2.0) to (−3.2 to 3.2), therefore, 64 grid pts. have been considercd

P 2 (CE)

Again the problem is the same as P 2 (C) but we have extended the support as in P 2 (BE). 64 grid points are again considered.

P (3)

This probiem has been taken from MEDGYESSY [12], with some modification. The solution is the sum of six Gaussian, and the kernel is also Gaussian.

$$\text{We have} \int_{-\infty}^{\infty} k\,(x-y)\,f\,(y)\,dy = g\,(x)$$

$$g\,(x) = \sum_{k=1}^{6} A_k \exp\left[-\frac{(x-\alpha_k)^2}{\beta_k}\right]$$

where

$A_1 = 10.0$	$\alpha_1 = 0.5$	$\beta_1 = 0.04$
$A_2 = 10.0$	$\alpha_2 = 0.7$	$\beta_2 = 0.02$
$A_3 = 5.0$	$\alpha_3 = 0.875$	$\beta_3 = 0.02$
$A_4 = 10.0$	$\alpha_4 = 1.125$	$\beta_4 = 0.04$
$A_5 = 5.0$	$\alpha_5 = 1.325$	$\beta_5 = 0.02$
$A_6 = 5.0$	$\alpha_6 = 1.525$	$\beta_6 = 0.02$

The essential support of $k\,(x)$ is $0 < x < 2$

$$k\,(x) = \frac{1}{\sqrt{\pi\lambda}} \exp\,(-x^2/\lambda), \quad \lambda = 0.015$$

The essential support of $k\,(x)$ is $(-0.26, 0.26)$

The solution is

$$f\,(x) = \sum_{k=1}^{6} \left(\frac{\beta}{\beta-\lambda}\right)^{1/2} A_k \exp\left[-\frac{(x-\alpha_k)^2}{(\beta_k-\lambda)}\right]$$

The essential support of $f\,(x)$ is $(0.26, 1.74)$.

6. Numerical Rcsults for KLEIN'S Regularization using B-splines

In this section we describe the application of the method to problems P 1 – P_3 .

In solving the problems P 2 and P 3, we have considered the data function $g(x)$ as defined eariier and also the same data functions contaminated by varying amounts of random noise.

To generate sequences of random errors of the form $\{\epsilon_n\}$ $n=0, \ldots N-1$, we have used the NAG algorithm G 05 DDA which returns pseudo-random real numbers raken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental error we have taken

$$A = 0.0$$

$$B = \frac{x}{100}\left(\max_{0 \leq n \leq N-1} |g_n|\right) \qquad \ldots 6.1)$$

where x denotes a chosen percentage, *e.g.*

$x = 0.3,\ 0.7,\ 1.7$ or 3.3 etc.

Thus the standard deviation of the random error ϵ_n added to g_n does not exceed x % of the maximum value of $g(x)$.

The actual error ϵ_n may be as high as 3B.

P (1)

The interval $[-30, 30]$ is mapped onto $[0, 60]$ which is extended to $[0, 64]$ by introducing zero values of k and g. The step length $h=2$ is given. Thus $N = 32$. The data is noisy with a maximum error 0.02 (< 0.7%)

The Algorithm was tried for the cases

$M = 32$, $M = 16$ and $M = 8$

the case $M = 32$ is shown in diagram (1) and the ease $M = 16$ is shown in diagram (8) and gives a better solution than the case $M=32$.

The case $M = 8$ gave a poor solution which is not shown.

P 2 (A)

Here, we have ehosen $M=N=64$ and $M = 32$, $N = 64$. In both cases solution resolves the two peaks.

The solution for $M = N = 64$ is shown indiagram (2) with 3.3% noise.

The solution for $N = 64$, $M = 32$ with 3.3% error is shown in diagram (9) which is slightly better than the solution for $M = N = 64$.

P 2 (B)

In the case of accurate data, a reasonable solution with clearly resolved peaks was obtained in both cases, but for the noisy case the solution for $M = 32$ is slightly better than the case $M = 64$, as shown in diagram (3) and diagram (10).

P 2 (C)

Again in the case of clean data, solution is not very good in both cases but resolve the two peaks. In case of noisy data it does not resolve the two peaks as shown in diagrams (4) and (11)

P 2 (BE)

Here, we have extended the support. In the case of clean data, solution is quite reasonable in both cases.

In the case of noisy data the solution for $M=32$ is better than $M = 64$ as shown in diagrams (5) and (12)

P 2 (CE).

Here again in the case of accurate data the solution is quite reasonable and resolves the two peaks clearly in both cases, but in the case of noisy data the solution is poor, and is shown in diagrams (6) and (13).

P 3

The essential supports of $f(x)$, $g(x)$ and $k(x)$ respectively are (0.26, 1.74), (0, 2.0) and (−0.26, 0.26). First we can consider a common interval (−0.26, 2) for all these three functions which covers all of their essential supports. This interval was translated to (0,2.26) and extended to (0, 3.2). Thus $T = 3.2$ and we took a step length $h=0.05$ so that $N = 64$.

(i) In the case $M = N$ for clean data the solution is very good; all peaks are O.K. but for noisy data the solution becomes unreasonable as shown in diagram (7)

(ii) In the case $M = N/2$ for clean data the solution is reasonable giving all the peaks. For noisy data the solution is reasonable giving 5 peaks as shown in diagram (14).

TABLE 2

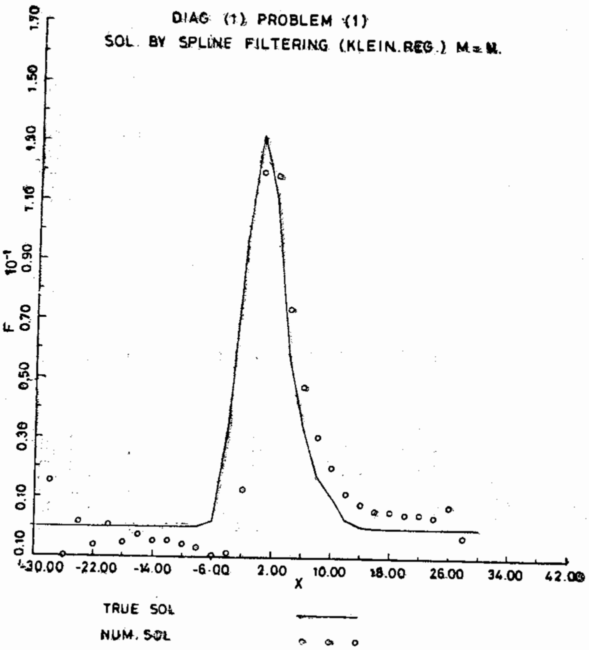
FOR M=N

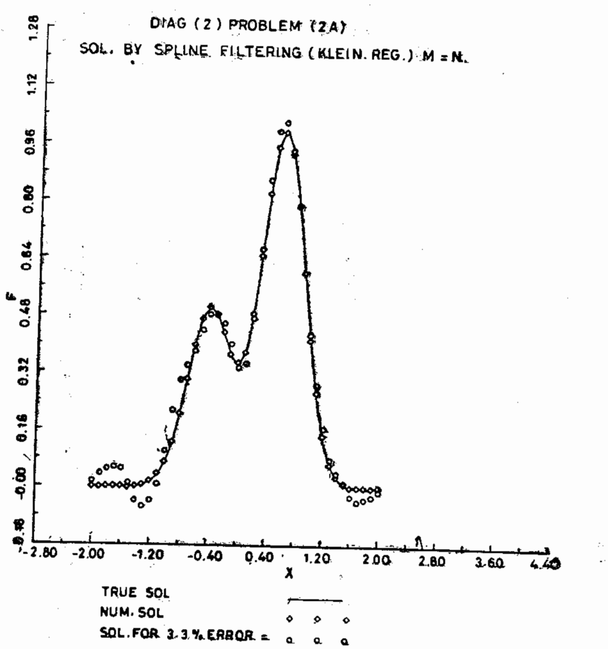
PROBLEM	N	h	Level of Noise X	σ-TURCHIN σ T	λ M = N	Roots of $f(\lambda)$	DIAGRAMS
P 1	32	2.0	0.7%	0.00853	151.51	2	DIAG (1)
P 2 (A)	64	0.1	0.0% / 3.3%	0.00000195 0.00556	0.01397 0.0033808	2 4	DIAG (2)
P 2 (B)	64	0.1	0.0% / 3.3%	0.0000461 0.0074	0.005914 0.00227	2 4	DIAG (3)
P 2 (C)	64	0.1	0.0% / 0.7%	0.000729 0.000729	0.000127 0.002283	2 2	DIAG (4)
P 2 (BE)	64	0.1	0.0% / 3.3%	0.0001785 0.01042	0.006420 0.005043	4 4	DIAG (5)
P 2 (CE)	64	0.1	0.0% / 0.7%	0.0001319 0.002780	0.0066 0.006012	4 4	DIAG (6)
P 3	64	0.5	0.0% / 1.7%	0.01838 0.41926	0.0000000102 0.000002327	2 2	DIAG (7)

TABLE 3

M=N/2

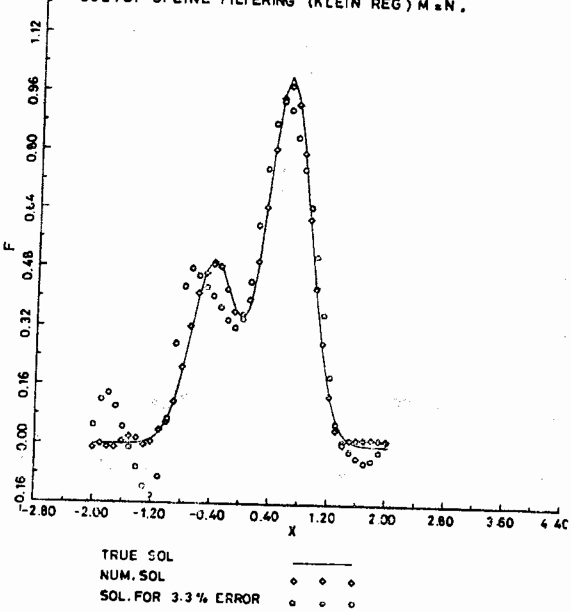
PROBLEM	N	h	Level of Noise x	σ-TURCHIN σ T	λ M=N/2	Roots of $f(\lambda)$	DIAGRAMS
P 1	32	2.0	0.0%	0.00853	954.59	2	DIAG (8)
P 2 (A)	64	0.1	0.0% 3.3%	0.0000195 0 005565	0.01657 0.008505	2 4	DIAG (9)
P 2 (B)	64	0.1	0.0% 3.3%	0.0000461 0.0074	0.01463 0.0106	2 4	DIAG (1)
P 2 (C)	64	0.1	0.0% 0.7%	0.0007291 0.002141	0.006717 0.225389	2 2	DIAG (11)
P 2 (BE)	64	0.1	0.0% 3.3%	0.0001785 0.0105753	0.014965 0.010075	4 4	DIAG (12)
P 2 (CE)	64	0.1	0.1% 0.7%	0.0001319 0.0027809	0.014244 0.024074	2 4	DIAG (13)
P 3	64	0.05	0.0% 1.7%	0.0183 0.2160	0.000000172 0.000000318	2 4	DIAG (14)


DIAG (1) PROBLEM (1)
SOL. BY SPLINE FILTERING (KLEIN.REG.) M=N.
F
10^{-1}
1.70
1.50
1.30
1.10
0.90
0.70
0.50
0.30
0.10
0.10
-30.00
-22.00
-14.00
-6.00
2.00
10.00
18.00
26.00
34.00
42.00
X
TRUE SOL
NUM. SOL


DIAG (2) PROBLEM (2A)
SOL. BY SPLINE FILTERING (KLEIN. REG.) M = N.
F
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
-0.00
-2.80
-2.00
-1.20
-0.40
0.40
1.20
2.00
2.80
3.60
4.40
X
TRUE SOL
NUM. SOL
SOL. FOR 3.3% ERROR =

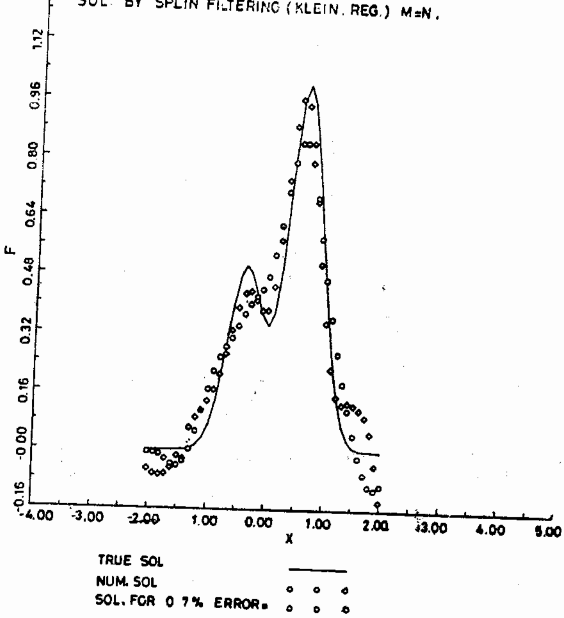
DIAG (3) PROBLEM (2B)

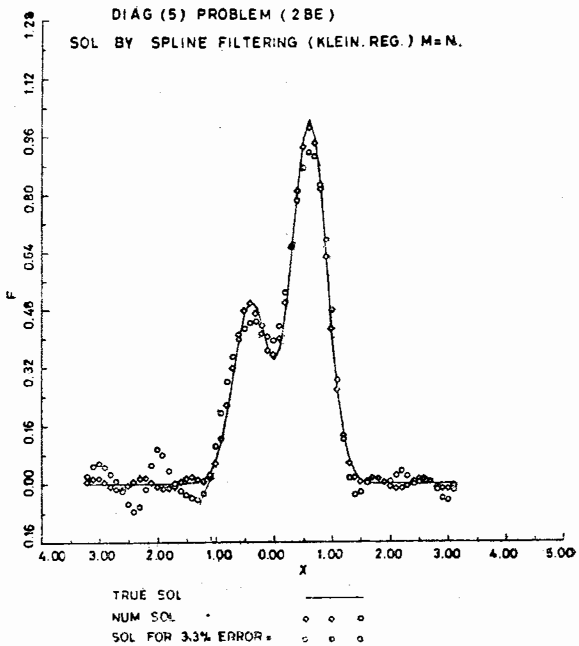
SOL. BY SPLINE FILTERING (KLEIN REG) M = N.

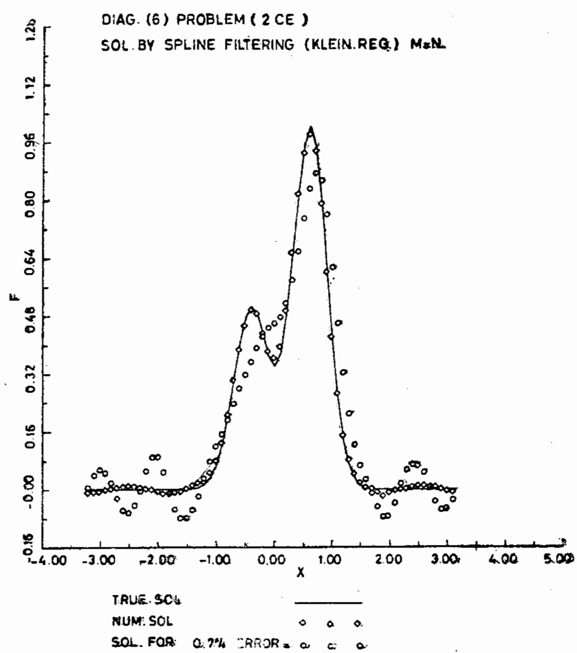


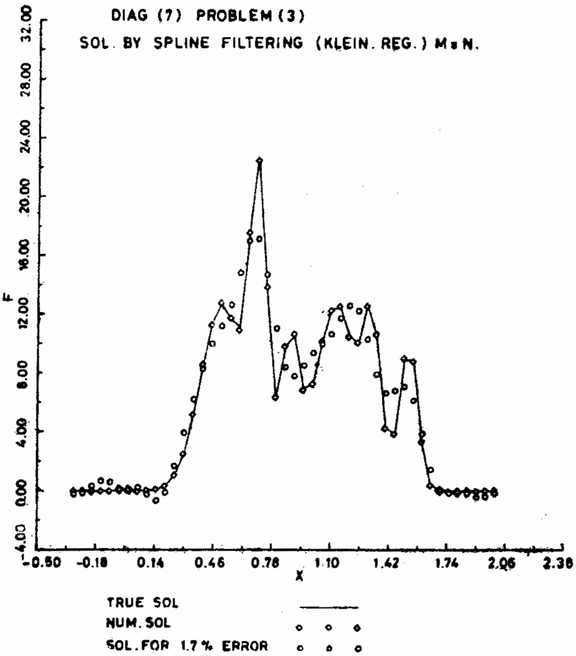
DIAG (4) PROBLEM (2C)

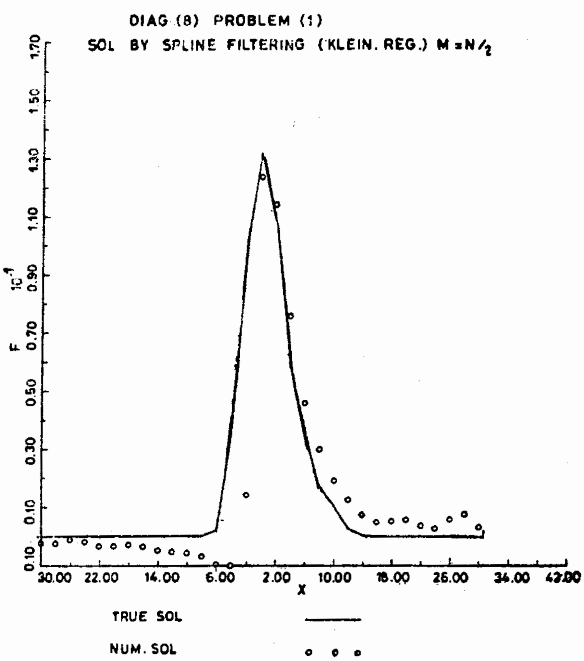
SOL. BY SPLIN FILTERING (KLEIN. REG.) M=N.




DIAG (5) PROBLEM (2BE)
SOL BY SPLINE FILTERING (KLEIN. REG.) M=N.
F
X
TRUE SOL
NUM SOL
SOL FOR 3.3% ERROR =

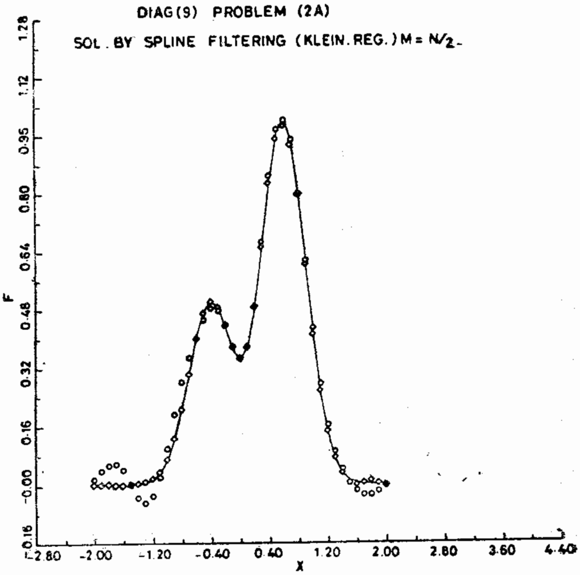

DIAG. (6) PROBLEM (2 CE)
SOL. BY SPLINE FILTERING (KLEIN. REG.) M=N.
F
X
TRUE SOL
NUM. SOL
SOL. FOR 0.7% ERROR =


DIAG (7) PROBLEM (3)
SOL. BY SPLINE FILTERING (KLEIN. REG.) M = N.
F
X
TRUE SOL
NUM. SOL
SOL. FOR 1.7 % ERROR


DIAG (8) PROBLEM (1)
SOL BY SPLINE FILTERING (KLEIN. REG.) M = N/2
F
X
TRUE SOL
NUM. SOL

DIAG (9) PROBLEM (2A)

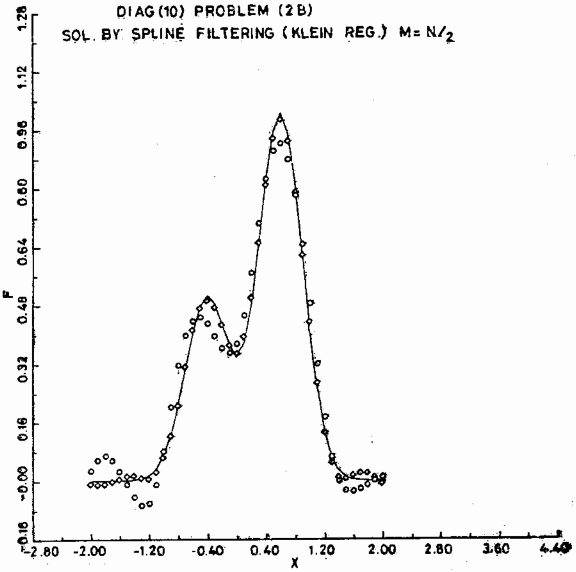
SOL. BY SPLINE FILTERING (KLEIN. REG.) M = N/2.



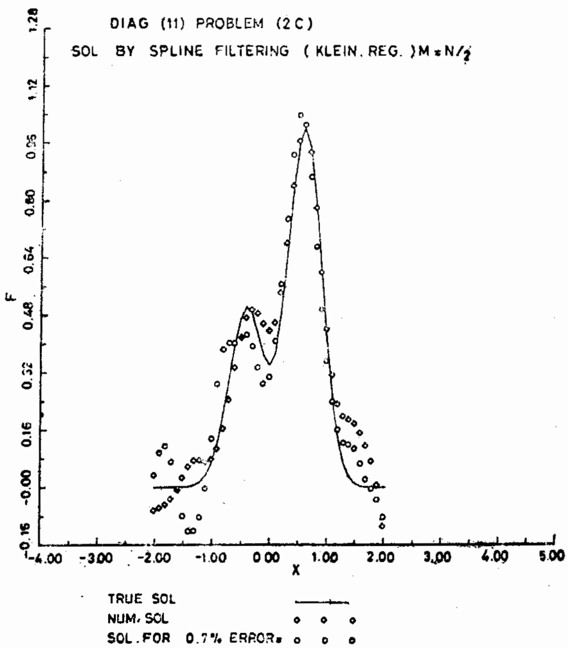
TRUE SOL ———
NUM. SOL ◇ ◇ ◇
SOL. FOR 3.3% ERROR ○ ○ ○

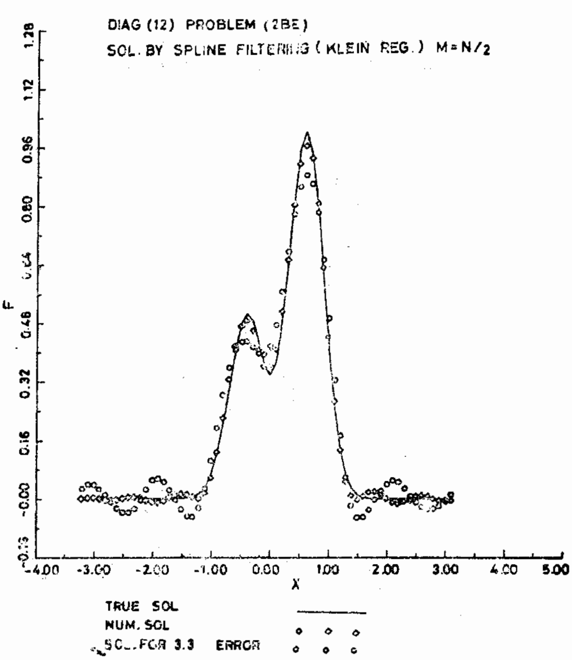
DIAG (10) PROBLEM (2 B)

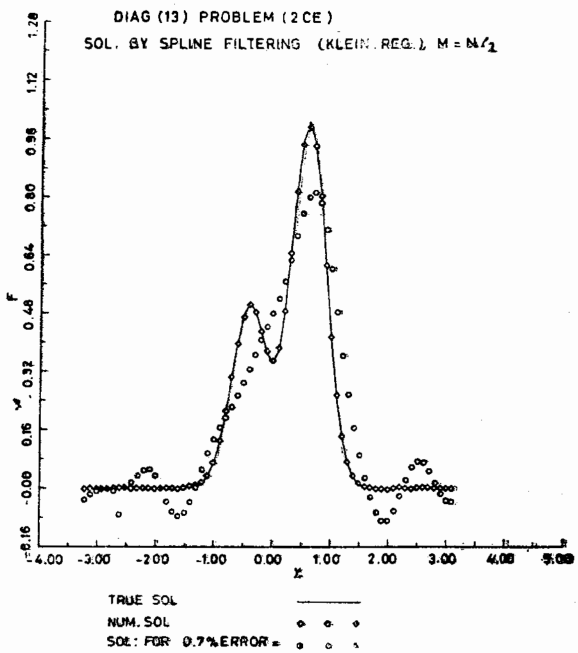
SOL. BY SPLINE FILTERING (KLEIN REG.) M = N/2

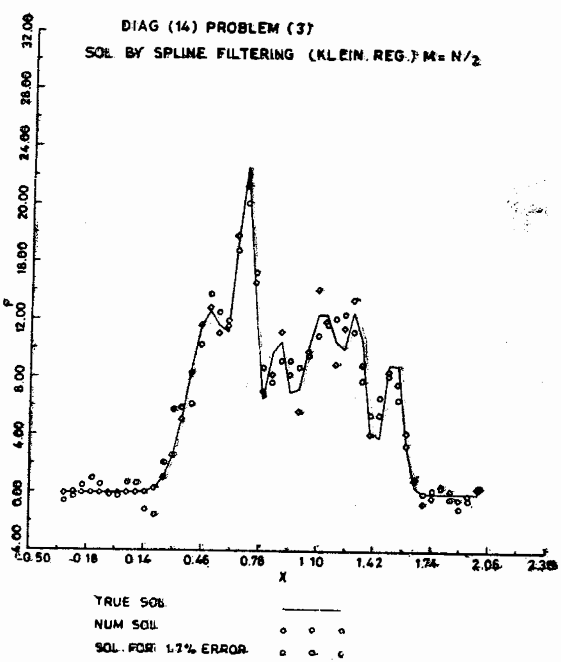


TRUE SOL ———
NUM. SOL ◇ ◇ ◇
SOL. FOR 3.3% ERROR ○ ○ ○


DIAG (11) PROBLEM (2C)
SOL BY SPLINE FILTERING (KLEIN. REG.) M = N/2
F
X
TRUE SOL
NUM. SOL
SOL. FOR 0.7% ERROR


DIAG (12) PROBLEM (2BE)
SOL. BY SPLINE FILTERING (KLEIN REG.) M = N/2
F
X
TRUE SOL
NUM. SOL
SOL. FOR 3.3 ERROR


DIAG (13) PROBLEM (2CE)
SOL. BY SPLINE FILTERING (KLEIN. REG.) M = N/2
F
X
TRUE SOL
NUM. SOL
SOL: FOR 0.7% ERROR


DIAG (14) PROBLEM (3)
SOL BY SPLINE FILTERING (KLEIN. REG.) M= N/2
F
X
TRUE SOL
NUM SOL
SOL. FOR 1.7% ERROR

REFERENCES

1. Klein, G. "On spline functions and statistical regularization of ill-posed problems", Journal of Computational and Applied Mathmatics, Vol. 5, no. 4 (1972). pp. 259-263.
2. Cox, M.G. "The Numerical Evaluation of B-Splines" J. Inst. Maths. Applcs (1972) 10, pp. 134-149.
3. Boor, C.D, "Package for calculating with B – SPLINES" SIAM. J. Numer. Anal. Vol. 14, No. 3 (1977) pp. 441-472.
4. Curry, H. B. and Schoenberg, "On splines distributions and their limits : the polya distribution", Abstract. Bul. Amer. Math. Soc. 53. (1947) pp. 11—14.
5. Cullum, J. "The effective choice of smoothing norm in regularizasion : Math. Computation, 33, 149-170 (1979).
6. Turchin, V. F., Koslov, V.P. and Malkevich, M.S. "The use of mathematical-statistics methods in the solution of incorrectly posed problems" Soviet Phys. USP. 13, pp. 681-702 (1971).
7. Tikhonov, A.N$_1$ and Arsenin, V.Y. "Solution of ill-posed problems" WILEY/WINSTON, London (1977).
8. Miller, G.F. Fredholm integral equations of the first kind pp. 175-188 of L.M. Delves and J. Walsh (Eds), "Numerical solution of integral equations" Oxford University Press, Oxford (1974).
9. Bracewell, R.N. "The Fourier transform and its applications" McGraw Hill 1978).
10. Boor, C. D. "A Practical Guide to splines" Springer-Verlag (1978) New York.
11. Schoenberg, I. J. "Cardinal Spline Interpolation SIAM, Philadelphia, 1973
12. Medgyessy, p. "D Composition of Superposition of density functions and discrete distributions" Adam Hilger, Bristol, England, 1977.

13. Phillips, D.L. "A technique for the numerical solutions of certain integral equations of the first kind" J. Ass. Comput. Mach. Vol. 9 (1962) pp. 84-96.

14. Arsenin, V. Ja., "Ill-posed Problems" Russian Math. Surveys Vol. 31 (1976) pp. 93-107

15. Hanson R.J. and Phillips, J.L. "Numerical solution of 2-dimensional Integral Equations using Linear elements." SIAM. J. Numer. Analysis 15 (1978) 113-121.

16. Elden, L. "Numerical Methods for the Regularization of Fredholm Integral equations of the first kind". Report LIH-MAT-R-1974-7, Dept. of Math. LINKÖPENG University 1974.

17. Franklin, J. N. "On Tikhonov's Method for ill-posed problems" Math. Computation Vol. 28 (1974) pp. 889-907.

18. Graves J & Prenter P.M. "Numerical Interative filters applied to first kind integral equations" Numer. Math. Vol. 30, (1978) No. 3 pp. 281-300.

The Punjab University
Journal of Mathematics
Vol. XVI. (1983), pp. 99-112.

ON THE GENERALIZED CROSS-VALIDATION METHOD TO FIRST KIND FREDHOLM INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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Summary : A numerical study of generalized cross validation technique applied to linear first kind Fredholm integral equations of convolution type :

$$(Kf)(x) = \int_{-\infty}^{\infty} k(x-y)\, f(y)\, dy = g(x), \; -\infty < y < \infty \qquad (1.1)$$

is carried out. Interpolating B-splines are used for the algebrization and smoothing regularization of linear Fredholm integral equations of the first kind.

Introduction .

The linear Fredholm integral equation of the first kind (1.1) is an example of a mathematically ill-posed problem, arising in connection with physical measurements. Slight perturbation of G might correspond to arbitrarily large perturbations of the solution F. This is due to the smoothing eharacter of the operator K.

WAHBA's [1] generalized cross validation can be used for spline approximation.

Suppose that the approximate solution $f_{n,\lambda}$ is taken to be of the form

$$f_{n,\lambda}(t) = \sum_{k=1}^{M} c_k \, B_{k,\lambda}(t) \qquad (1.2)$$

where M = No of B-spline and λ here is a regularization parameter and

$$\left[B_{k,\lambda}(t) \right]_{k=1}^{M} \quad \text{are the basis functions.}$$

For a given set of basis functions the coefficients $\{\alpha_k\}$ are determined

$$\text{to minimize } \sum_{j=1}^{n} \left[K f_{n,\lambda}(t_j) - y(t_j) \right]^2 \tag{1.3}$$

and

$$V(\lambda) = \frac{1}{n} \underline{y}^T (1 - \hat{A}(\lambda))^2 \underline{y} \Big/ \left[\frac{1}{n} \text{Trace}(I - \hat{A}(\lambda)) \right]^2 \tag{1.4}$$

which estimates the values of λ i.e. regularization parameter, which minimizes (1.3). Results related to the convergence of $\| f - f_{n,\lambda} \|$ of certain ill posed problems are obtained [15].

2. Approximation and Solution Method when M=N/2

From Iqbal [6] we know

$$f_M(x) = \sum_{j=0}^{M-1} \alpha_j B_j(H; x) \tag{2.1}$$

where $B_j(H; x)$ are periodic cubic cardinal B-splines with period $T = MH$ and knot spacing H, [7] and [8] ; M is the number of B-splines. The vector α of unknown coefficients is to be determined.

Since $B_o(H; x)$ is periodic on (0, T), it has the Fourier series

$$B_o(H; x) = \frac{1}{T} \sum_{q=-\infty}^{\infty} \hat{B}_{oq} \exp(i\omega_q x) \tag{2.2}$$

where

$$\omega_q = (2\pi/T)\, q$$

and

$$\hat{B}_{oq} = \int_o^T B_o \exp(-i\omega_q x)\, dx$$

$$= H \left[\frac{\text{Sin} (H\tilde{\omega}_q/2)}{(H\tilde{\omega}_q/2)} \right]^4$$

where

$$\tilde{\omega}_q = \begin{cases} \omega_q , & 0 \leq q < N/2 \\ \omega_{N-q} , & \frac{1}{2} N \leq q < N-1 \end{cases}$$

Also

$$\hat{B}_{jq} = \hat{B}_{oq} \exp(-i \, \omega_q \, jH) \qquad (2.3)$$

($\wedge$ denotes the Fourier Transformation) $\quad q = 0, \pm 1, \pm 2, \ldots\ldots$
$j = 0, 1, 2, \ldots\ldots M-1$

$$\hat{f}_{M, q} = \sum_{j=0}^{M-1} \alpha_j \hat{B}_{jq} \qquad (2.4)$$

considering $\sigma = 1.0$

$$\left. \begin{aligned} a_q &= \frac{T}{N^2} \mid K_q \ \hat{B}_{oq} \mid^2 , \ b_q = \frac{1}{T} \mid \tilde{\omega}_q^2 \hat{B}_{oq} \mid^2 \\ c_q &= \frac{T}{N^2} (\bar{\hat{K}} \ \hat{g}_q \hat{B}_{o, q}) \quad \text{for} \leq 0 \leq q \leq N-1 \end{aligned} \right\} \qquad (2.4)$$

$$\hat{\alpha}_s = \frac{\sqrt{M} \ U_s}{M (\mu_s + \lambda \nu_s)} , \quad \left. \begin{aligned} 0 \leq s \leq M-1 \\ s \equiv q \ (\text{mod } M) \end{aligned} \right] \qquad (2.5)$$

where

$$\hat{U}_s = \sqrt{M} \ (c_s + c_{M+s})$$

$$\mu_s = M (a_s + a_{M+s})$$

(2.5) becomes

$$\hat{\alpha}_s = \frac{\bar{\hat{K}}_s \hat{g}_s \hat{B}_{os} + \bar{\hat{K}}_{M+s} \hat{g}_{M+s} \hat{B}_{oM+s}}{\left\{ \left[\mid \hat{K}_s \hat{B}_{os} \mid^2 + \mid \hat{K}_{M+s} \hat{B}_{oM+s} \mid^2 \right] + B_2 \lambda \left[\mid \tilde{\omega}_s^2 \hat{B}_{os} \mid^2 + \mid \tilde{\omega}_{M+s}^2 \hat{B}_{o, M+s} \mid^2 \right] \right\}} \qquad \ldots(2.6)$$

$B_2 = N^2/T^2$

OR

$$\hat{\alpha}_s = A_{1,s}(\lambda)\,\hat{g}_s + A_{2,s}(\lambda)\,\hat{g}_{M+s} \qquad \ldots(2.7)$$

$$A_{1,s}(\lambda) = \overline{\hat{K}}_s \hat{B}_{os} \Big/ \Big[\,|\hat{K}_s \hat{B}_{os}|^2 + |\hat{K}_{M+s} B_{o,M+s}|^2 + B_2\lambda\,\big[\,|\tilde{\omega}^2_s \hat{B}_{o,s}|^2 + |\tilde{\omega}^2_{M+s} \hat{B}_{o,M+s}|\,\big]\Big]$$

$$A_{2,s}(\lambda) = \overline{\hat{K}}_{M+s} \hat{B}_{o,M+s} \Big/ \Big[\,|\hat{K}_s \hat{B}_{os}|^2 + |\hat{K}_{M+s} \hat{B}_{o,M+s}|^2 + B_2\lambda\,\big[\,|\hat{\omega}^2_s \hat{B}_{o,s}|^2 + |\tilde{\omega}^2_{M+s} \hat{B}_{o,M+s}|^2\big]\Big]$$

Now consider

$$\hat{g}_{\lambda,q} = \hat{K}_q \hat{f}_{M,q} \qquad 0 \le q \le N-1 \qquad \ldots(2.8)$$

Also

$$\hat{f}_{M,q} = \sum_{j=0}^{M-1} \alpha_j \hat{B}_{o,q} \exp(-i\,\tilde{\omega}_q jH) \qquad 0 \le q \le N-1$$

(2.8) takes the form

$$\hat{g}_{\lambda,q} = \left\{\sum_{j=0}^{M-1} \alpha_j \exp(-i\,\tilde{\omega}_q jH)\right\} \hat{K}_q \hat{B}_{oq} \qquad \ldots(2.9)$$

$$\text{Let } \zeta_q = \begin{cases} \hat{\alpha}_q & 0 \le q \le M-1 \\ \hat{\alpha}_{N-q} & M \le q \le N-1 \end{cases}$$

$\therefore$ (2.9) can be written as

$$\hat{g}_{\lambda, q} = \begin{cases} \hat{\alpha}_q \hat{K}_q \hat{B}_{o,q}, & 0 \le q \le M-1 \\ \hat{\alpha}_{N-q} \hat{K}_q \hat{B}_{oq}, & M \le q \le N-1 \end{cases}$$

using (2.7)

$$\hat{g}_{\lambda, q} = \begin{cases} A_{1,q}(\lambda)\hat{K}_q \hat{B}_{oq} \hat{g}_q + A_{2\,q}(\lambda) \hat{K}_q \hat{B}_{oq} \hat{g}_{M+q}, & 0 \le q \le M-1 \\ A_{1,N-q}(\lambda) \hat{K}_q \hat{B}_{oq} \hat{g}_{N-q} + A_{2,N-q}(\lambda) \hat{K}_q \hat{B}_{oq} \hat{g}_{3M-q} & M \le q \le N-1 \end{cases} \quad (A)$$

$$\therefore \quad \underline{\hat{g}}_{\lambda,q} = \hat{A}(\lambda)\, \underline{\hat{g}} \qquad \ldots(2.11)$$

$$A(\lambda) = \Gamma\, (\hat{A}(\lambda))\, \Gamma^{H} \qquad \ldots(2.12)$$

Γ is unitary matrix i.e. $\Gamma\Gamma^{H} = \Gamma^{H}\Gamma = I$

Also Trace $(A(\lambda))$ = Trace $(\Gamma^{H} A(\lambda) \Gamma)$

$$= \mathrm{Tr}\,(\hat{A}(\lambda)) \qquad \ldots(2.13)$$

$$\therefore \quad \mathrm{Tr}\,(I-A(\lambda)) = \mathrm{Tr}\,(I-A(\lambda)) \quad [14], \qquad \ldots(2.14)$$

Also by Plancheral's Theorem $\| f \|_2^2 = \text{constt}\ \| \hat{f} \|_2^2$

$$\therefore \quad \| (I-A(\lambda))\, \underline{g} \|_2^2 = \| (I-\hat{A}(\lambda))\, \underline{\hat{g}} \|_2^2 \qquad (2.15)$$

since $V(\lambda)$ in KHM's [12] is

$$V(\lambda) = \frac{\frac{1}{N} \| (I-A(\lambda))\, \underline{g} \|_2^2}{\left[\frac{1}{N} \mathrm{Tr}\,(I-A(\lambda)) \right]^2} \qquad \ldots(2.16)$$

Using (2.14) & (2.15), (2.16) can be written as

$$V(\lambda) = \frac{\frac{1}{N} \| I - \hat{A}(\lambda)\, \underline{\hat{g}} \|_2^2}{\left[\frac{1}{N} \mathrm{Tr}\,(I - \hat{A}(\lambda)) \right]^2} \qquad \ldots (2.17)$$

Using (A) and (2.11) ; (2.17) can be written as

$$V(\lambda) = \frac{1}{N} \left[\sum_{q=0}^{M-1} \left| \left(1 - a_{1q}\right) \hat{g}_q - a_{2q} \bar{\hat{g}}_{M-q} \right| \right.$$

$$\left. \times \sum_{q=0}^{M-1} \left| (1 - a_{4q}) \bar{\hat{g}}_{M-q} - a_{3q} \hat{g}_q \right| \right] \Big/ \left[1 - \frac{1}{N} \sum_{q=0}^{M-1} (a_{1_q} + a_{4_q}) \right]^2$$

(2.18)

where

$$A(\lambda) = \left[\begin{array}{c|c} \mathrm{diag}\ \mathbf{a}_1 & \mathrm{diag}\ \mathbf{a}_2 \\ \hline \mathrm{diag}\ \mathbf{a}_3 & \mathrm{diag}\ \mathbf{a}_4 \end{array} \right] \qquad (2.19)$$

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are four complex vectors $\in C^M$.

Matrix A (λ) is circulant. [4]

Computationally this is a simple function to minimize w.r.t. λ. We have used NAG algorithm Eo4ABA based on quardratic interpolation to find the optimal value of λ.

Knowing λ then

$$\hat{\alpha}_s = \frac{\sqrt{M}\ \hat{U}_s}{M(\mu_s + \lambda \nu_s)}$$

$\underline{\alpha}$ may than be calculated from the inverse DFT of equation (2.6) as

$$\underline{\alpha} = \psi\, \underline{\hat{\alpha}} \qquad (2.20)$$

3. Calculation of Solution Vector $\underline{f}$

we know that

$$f_M(x) = \sum_{j=0}^{M-1} \alpha_j \, B_j(H\,;x).$$

when M = N/2

$$\alpha_{-1} = \alpha_{M-1}, \ \alpha_o = \alpha_M \text{ and } \alpha_1 = \alpha_{M+1}$$

$$\left.\begin{aligned} f_M(2j) &= \sum_{j=0}^{M-1} (\alpha_{j-1} + 4\alpha_j + \alpha_{j+1})/6.0 \\ f_M(2j+1) &= \sum_{j=0}^{M-1} (\alpha_{j-1} + 23\,\alpha_j + 23\,\alpha_{j+1} + \alpha_{j+2})/48.0 \end{aligned}\right\} \quad \ldots(3.1)$$

Example 1

This example is given in Phillips [9] and has noisy data function g with a maximum absnlute error of about 0.02. we have

$$\int_{-30}^{30} k\,(x-y)\, f\,(y)\, dy = g\,(x).$$

where K, g and f are given in Table (1). The number of grid points is 31.

TABLE 1

x_n	g_n	k_n	f_n
−30.0	0.0100	0.1184	0.0000
−28.0	0.0100	0.1311	0.0000
−26.0	0.0110	0.1464	0.0000
−24.0	0.0170	0.1651	0.0000
−22.0	0.0305	0.1883	0.0000
−20.0	0.0405	0.2179	0.0000
−18.0	0.0585	0.2563	0.0000
−16.0	0.0869	0.3077	0.0000
−14.0	0.1309	0.3788	0.0000
−12.0	0.2018	0.4816	0.0000
−10.0	0.3235	0.6380	0.0000
−8.0	0.5469	0.8914	0.0000
−6.0	0.9621	1.3333	0.0019
−4.0	1.6301	2.1483	0.0345
−2.0	2.4047	3.5108	0.0965
0.0	2.9104	4.3600	0.1321
2.0	2.8912	3.0628	0.1096
4.0	2.4586	1.6329	0.0584
6.0	1.9049	0.8806	0.0349
8.0	1.4144	0.5095	0.0173
10.0	1.0282	0.3137	0.0107
12.0	0.7411	0.2021	0.0028
14.0	0.5409	0.1341	0.0005
16.0	0.4083	0.0906	0.0000
18.0	0.3214	0.0614	0.0000
20.0	0.2623	0.0413	0.0000
22.0	0.2201	0.0269	0.0000
24.0	0.1886	0.0165	0.0000
26.0	0.1580	0.0089	0.0000
28.0	0.1270	0.0031	0.0000
30.0	0.0780	0.0013	0.0000

Example 2. This example has been taken from MEDGYESSY [10] with some modification. The solution function is the sum of six Gaussians and the Kernel is also Gaussian :

We have

$$\int_{-\infty}^{\infty} k(x-y)\, f(y)\, dy = g(x)$$

$$g(x) = \sum_{K=1}^{6} A_K \exp\left[- \frac{(x-\alpha_K)^2}{\beta_K} \right]$$

where

$A_1 = 10.0$	$\alpha_1 = 0.5$	$\beta_1 = 0.04$
$R_2 = 10.0$	$\alpha_2 = 0.7$	$\beta_2 = 0.02$
$A_3 = 5.0$	$\alpha_3 = 0.875$	$\beta_3 = 0.02$
$A_4 = 10.0$	$\alpha_4 = 1.125$	$\beta_4 = 0.04$
$A_5 = 5.0$	$\alpha_5 = 1.325$	$\beta_5 = 0.02$
$A_6 = 5.0$	$\alpha_6 = 1.525$	$\beta_6 = 0.02$

The essential support of $g(x)$ is $0 < x < 2$

$$K(x) \frac{1}{\sqrt{\pi\lambda}} \exp(-x^2 / \lambda). \ \lambda = 0.015$$

The essential support of $K(x)$ is $(-0.26, 0.26)$.
The solution is

$$f(x) = \sum_{K=1}^{6} \left(\frac{\beta_K}{\beta_K - \lambda} \right)^{\frac{1}{2}} A_K \exp\left(- \frac{(x-\alpha_k)^2}{(\beta_k - \lambda)} \right)$$

The essential support of $f(x)$ is $(0.26, 1.74)$

4. Numerical Results for GCV Method.

In this section we describe the application of the technique to problems (1) and (2).

In solving the problems 1 and 2 we have considered the data function $g(x)$ as defined earlier and also the same data functions contaminated by varying amounts of random error. To generate sequences of random errors of form $\{\varepsilon_n\}$ $n = 0, \ldots\ldots, N-1$, we have used the NAG algorithm G 05 DDA which returns poseudo-random real numbers taken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental error we have taken

$$A = 0$$

$$B = \frac{x}{100} \left(\max_{0 \leq n \leq N-1} |g_n| \right) \qquad \ldots(4.1)$$

where x denotes a chosen percentage, e.g.

$x = 0.3$, 1.7 or 3.3 ete.

Thus the standard deviation of the random error ε_n added to g_n does not exceed $x\%$ of the maximum value of $g(x)$.

The actual error ε_n may be as high as 3B.

EXP. (1)

The interval (– 30, 30) is mapped onto (0, 60) which is extended to (0, 64) by introducing zero values of k and g; the step length $h = 2.0$ is given. Thus $N = 32$. The data is noisy with a maximum error 0.02 (0.7%). The algorithm is tried for the case $M = N/2 = 16$ and is shown in DIAG (1), which depicts a good approximation.

EXP. (2)

The essential supports of $f(x)$, $g(x)$ and $k(x)$ respectively are (0.26, 1.74), (0, 2) and (– 0.26, 0.26).

First we consider a common interval (–0.26, 2.0) for all these three functions which covers all of their essential supports, this interval we then translated to (0, 2.26) and extended to (0.0, 3.2). Thus $T = 3.2$ and step length $h = 0.05$ so that $N = 64$.

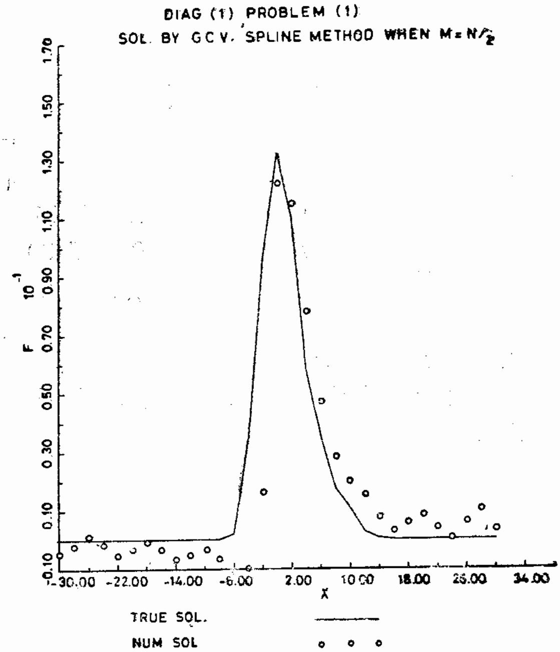
The algoirthm is tried for $M = N/2 = 32$, for clean data it resolves all the six peaks and results are quite resonable and for noisy data we could resolve only 5 peaks and the results are resonable as shown in DIAG (2).

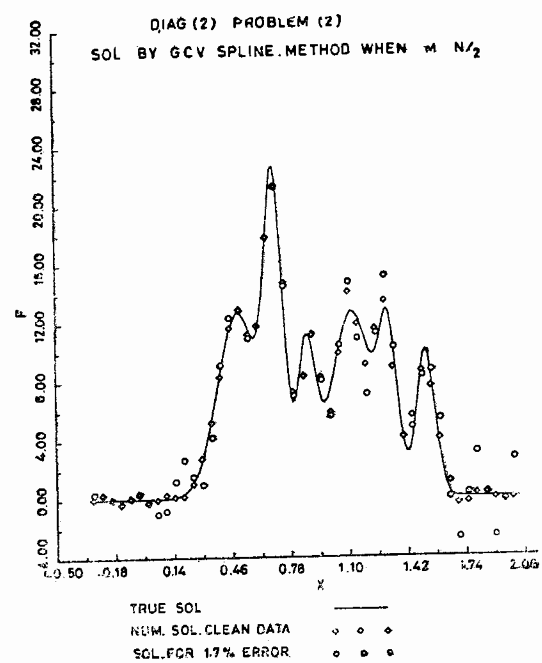
TABLE 2

M = N/2

EXAMPLE	N	h	Level of Noise x	λ	DIAGRAMS
* P (1)	32	2.0	0.7%	735347.81	DIAG (1)
P (2)	64	0.05	0·0%	0.0335980	DIAG (2)
			1.7%	0.033372	

* For P (1) $\lambda = 0.0$ gives the same solution i.e. the filter is very weak; regularized and unregularized solutions are the same.


DIAG (1) PROBLEM (1)
SOL. BY G.C.V. SPLINE METHOD WHEN M=N/2
F 10^-1
X
TRUE SOL.
NUM SOL


DIAG (2) PROBLEM (2)
SOL BY GCV SPLINE.METHOD WHEN M N/2
F
X
TRUE SOL
NUM. SOL.CLEAN DATA
SOL.FOR 1.7% ERROR

REFERENCES

1. Wahba, G. Practical approximate solutions to linear operator equations when the data are noisy.
Siam J. Numer. Anal 14, pp. 651-667 (1977).

2. Wahba, G. A survey of some smoothing problems and the method of generalized cross validation of solving them.
Proc. Conference on Application of Statistics, Dayton ohio (1976).

3. Wahba, G. "Ill-posed problems" Numerical and Statistical methods for mildly moderately and severely ill posed problems with noisy data.
Proc. Conference Ill-posed problems DELAMARE (1979).

4. Wahba, G. Constrained Regularization for ill-posed liner operator equations, with applications in Meterology and Medicine. Technical Report No. 646 (August 1981) University of Wisconsin, 1210 w Dayton Street Madison, WI 53706.

5. Wahba, G. and Wendel Berger, J.
Some new mathematical methods for variational objective analysis using splines and cross validation. Monthly weather Review 108, 1122-1143 (1980)

6. Iqbal, M. On the Klein regularization using splines for the numerical solution of certain Fredholm convolution equations of the first kind, to be published.

7. Boor, C.D. A practical guide to splines
Springer-Verlag (1978), New York.

8. Schoenberg, I.J. Cardinal Spline Interpolation
SIAM, Philadelphia, 1973.

9. Phillips, D.L. A technique for the numerical solution of certain integral equations of the first kind
J. Ass. Comput. Mach. Vol. 9. (1962) pp. 84-96.

10. Medgyessy, p.
D Composition of Superpositions of density function and discrete distributions Adam Hilger, Bristol, England, 1977.

11. Turchin, V. F., Koslov, V. P. and Malkevich, M.S.
The Use of Mathematical Statistics Methods in the solution of incorrectly posed problems *Soviet Physics* USP Vol. **13**, (1971) pp. 681-702.

12. Maleknejad, K. Ph. D.. Thesis, University of Wales, Aberystwyth (1981).

13. Al-Faour, O.M. Ph. D. Thesis, University of Wales, Aberystwyth (1981).

14. Bellman, R. Introduction to Matrix Analysis McGraw-Hill (1960).

15. H.P. Baltes, Inverse Scattering Problems in Optics Springer-Verlag (1980).

16. Hanson R.J. and Phillips J.I.
An adaptive numerical method for solving Linear Fredholm Integal equations of the first kind
Numcr Math. **24** (1975) No. 4 pp. 291-308

17. Ivanov. V.K. and Busmanova. M. V.
Stability estimates for the solution of Operator Equations of the first kind by the Method of Regularization, Soviet Math. (Iz. VUZ) 20, (1976) No. 1 pp. 39-44.

The Punjab University
Journal of Mathematics
Vol. XVI. (1983) pp. 113-128.

ANALYSIS OF THE 3×3 – MATRIX LINEAR SPECTRAL PROBLEM

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§ 1. Introduction

In this paper a generalised case

$$\frac{\partial}{\partial x}\,\mathbf{u} = \{A(\zeta) + B(x, \zeta)\}\,\mathbf{u} \qquad \ldots(1)$$

where A (ζ) and **B** (x, ζ) are 3×3 matrices, and **u** a 3-element column vector is considered. A set of spectral data which is sufficient for the reconstruction of **B** (x, ζ) is found and then using the method called the Inverse Spectral Transform introduced by Ablowitz et al., [1] the problem of reconstructing the **B** (x, ζ) from the spectral data is solved.

§ 2. The Direct Spectral Problem

Eigenvalues and eigenvectors of A (ζ) in (1) are

$$A(\zeta)\,\mathbf{v}_i(\zeta) = \lambda_i(\zeta)\,\mathbf{v}_i(\zeta) \qquad \ldots(2)$$

where $i = 1, 2, 3$. We assume that $\lambda_i(\zeta)$ and $\mathbf{v}_i(\zeta)$ are analytic throughout the complex ζ-plan (no branch points). We want to define $\underline{\varnothing}_i(x, \zeta)$, $i=1, 2, 3$ throughout the complex ζ-plane such that

$$\mathbf{u}_i = \underline{\varnothing}_i(x, \zeta)\, e^{\lambda_i(\zeta)\, x} \qquad \ldots(3i)$$

satisfies (3).

$\underline{\varnothing}_i (x, \zeta) \rightarrow \mathbf{v}_i (\zeta)$ as $x \rightarrow -\infty$ ...(3ii)

For any given ζ, $\underline{\varnothing}_i (x, \zeta)$ is bounded for $-\infty < x < \infty$

Using the conditions (3i) and (3ii) we get from (1) an integral equation

$$\underline{\varnothing}_i (x, \zeta) = \mathbf{v}_i (\zeta) + \int_{-\infty}^{x} e^{[A(\zeta) - \lambda(\zeta)](x-y)} B(y, \zeta) \, \underline{\varnothing}_i (y, \zeta) \, dy \qquad \text{...(4)}$$

Expanding $\underline{\varnothing}_i (x, \zeta)$ as a Neumann series, we see that the Neumann Series for $\underline{\varnothing}_i (x, \zeta)$ converges absolutely and uniformly and the conditions (3 i), (3 ii) and (3 iii) are satisfied provided

$$\int_{-\infty}^{\infty} \left(1 + |x|^{m_i - 1}\right) | B(x, \zeta) | \, dx = K(\zeta) < \infty \qquad \text{...(5)}$$

Further if $K(\zeta)$ is bounded in a closed region D then $\underline{\varnothing}_i (x, \zeta)$ is continuous in D and analytic in the interior of D.

We define another function

$\underline{\psi}_i (x, \zeta)$, $i = 1, 2, 3$ throughout the complex ζ-plane such that

$$\mathbf{u}_i = \psi (x, \zeta) \, e^{\lambda_i (\zeta) x} \qquad \text{...(6 i)}$$

$\underline{\psi}_i (x, \zeta) \rightarrow \mathbf{v}_i (\zeta)$ as $x \rightarrow +\infty$ ...(6 ii)

For any given ζ, $\underline{\psi}_i (x, \zeta)$ is bounded for $-\infty < x < \infty$ ...(6 iii)

using conditions (6 i), (6 ii) we get another integral equation

$$\underline{\psi}_i = \mathbf{v}_i (\zeta) - \int_{x}^{\infty} e^{[A(\zeta) - \lambda_i(\zeta)](x-y)} B(y, \zeta) \, \underline{\psi}_i (y, \zeta) \, dy \qquad \text{...(7)}$$

Now consider equation (7) and similarly expand it as a Neumann Series. We see that the Neumann series for $\underline{\psi}_i\,(x,\ \zeta)$ converges absolutely and uniformly and the conditions (6i), (6ii) and (6iii) are satisfied provided

$$\mathrm{Re}\,\lambda_i\,(\zeta)\ \leq\ \mathrm{Re}\,\lambda_j\,(\zeta)$$

For any given value of ζ we need another function. This can be defined as follows. We define the functions $\underline{\tilde{F}}_i\,(x, \zeta)$ and $\underline{\tilde{\psi}}_i\,(x, \zeta)$ of the conjugate scattering problem

$$\frac{\partial}{\partial x}\ \tilde{\mathbf{u}}_i\ =\ -\ \tilde{\mathbf{u}}_i\ \{\mathrm{A}\,(\zeta) + \mathrm{B}\,(x, \zeta)\,\} \qquad \ldots(8)$$

Left-eigenvectors of A (ζ) are $\tilde{\mathbf{v}}_i\ \ (\zeta)$

$$\tilde{\mathbf{v}}_i\ \ (\zeta)\ \mathrm{A}\ \ (\zeta) = \lambda\,(\zeta)\ \ \tilde{\mathbf{v}}_i\,(\zeta) \quad i = 1, 2, 3$$

We define a function $\underline{\tilde{\varnothing}}_i\,(x, \zeta)$, $i = 1, 2, 3$ throughout the complex $\zeta-$plane such that

$$\underline{\tilde{\mathbf{u}}}_i = \underline{\tilde{\varnothing}}_i\,(x, \zeta)\ e^{-\ \lambda_i\,(\zeta)\ x} \qquad (9\ \mathrm{i})$$

satisfies (8)

$$\underline{\tilde{\varnothing}}_i\ \ (x, \zeta) \rightarrow \underline{\tilde{\mathbf{v}}}_i\ \ (\zeta)\ \text{as} \qquad x \rightarrow -\ \infty \qquad (9\ \mathrm{ii})$$

For any given ζ, $\underline{\tilde{\varnothing}}_i\ (x, \zeta)$ is bounded for $-\ \infty <\ \ x < \infty$ (9 iii)

Using conditions (9 i), and (9 ii) we get an integral equation

$$\underline{\tilde{\varnothing}}_i\ (x, \zeta) = \underline{\tilde{\mathbf{v}}}_i\ \ (\zeta)\ -\ \int_{-\infty}^{x}\ \underline{\tilde{\varnothing}}_i\ \ (y, \zeta)\ \mathrm{B}\,(y, \zeta)\ e^{(-\mathrm{A}\,(\zeta)+\lambda_i\,(\zeta))\ \ (x-y)}dy \qquad (10)$$

Now consider (10). We see that the Neumann Series for $\underset{\sim}{\tilde{\varnothing}}_i (x, \zeta)$ converges for $\text{Re}\,\lambda_i(\zeta) \leq \text{Re}\,\lambda_j(\zeta)$, $j = 1, 2, 3$, $j \neq i$ and the conditions (9 i), (9 ii) and (9 iii) are satisfied. Similarly we can write the integral equation involving $\underset{\sim}{\tilde{\psi}}_i (x, \zeta)$ as

$$\underset{\sim}{\tilde{\psi}}_i (x, \zeta) = \tilde{\mathbf{v}}_i (\zeta) + \int_x^{\infty} \underset{\sim}{\tilde{\psi}}_i (y, \zeta)\, B(y, \zeta)\, e^{(-A(\zeta) + \lambda_i(\zeta))(x-y)}\, dy \tag{11}$$

The Neumann series for $\underset{\sim}{\tilde{\psi}}_i (x, \zeta)$ converges and satisfies the following conditions.

$$\tilde{\mathbf{u}}_i = \underset{\sim}{\tilde{\psi}}_i (x, \zeta)\, e^{-\lambda_i(\zeta) x} \tag{12 i}$$

satisfies (8).

$$\underset{\sim}{\tilde{\psi}}_i (x, \zeta) \to \tilde{\mathbf{v}}_i (\zeta) \text{ as } x \to \infty \tag{12 ii}$$

For any given ζ, $\underset{\sim}{\tilde{\psi}}_i (x, \zeta)$ is bounded for $-\infty < x < \infty$ (12 iii)

If $\mathbf{u}_i$ satisfies (1) and $\tilde{\mathbf{u}}_i$ satisfies (8) then

$$\frac{\partial}{\partial x} (\tilde{\mathbf{u}}_i \,.\, \mathbf{u}_i) = 0. \tag{13}$$

Definition 1.

$\alpha_i(\zeta) = \underset{\sim}{\tilde{F}}_i (x, \zeta) . \underset{\sim}{\psi}_i (x, \zeta) \,/\, \tilde{\mathbf{v}}_i (\zeta) . \mathbf{v}_i (\zeta)$ provided

$\text{Re}\,\lambda_i(\zeta) \leq \text{Re}\,\lambda_j(\zeta)$, $\quad i = j = 1, 2, 3 \quad j \neq i$

Definition 2.

$\tilde{\alpha}_i(\zeta) = \underset{\sim}{\tilde{\psi}}_i (x, \zeta) . \underset{\sim}{F}_i (x, \zeta) \,/\, \tilde{\mathbf{v}}_i (\zeta) . \mathbf{v}_i (\zeta)$ provided

$$\text{Re } \lambda_i (\zeta) \geq \text{Re } \lambda_j (\zeta), \qquad j = 1, 2, 3 \qquad j \neq i$$

For

$$\text{Re } \lambda_i (\zeta) \leq \text{Re } \lambda_j (\zeta), \qquad j = 1, 2, 3 \qquad j \neq i$$

$$\underline{F}_i (x, \zeta) = \frac{1}{\alpha_i (\zeta)} \underline{\psi}_i (x \ \ \zeta)$$

this has poles where $\alpha_i (\zeta) = 0$

Now we use the theory of tensors to solve the 3×3 case.

§ 3. The Direct Spectral Problem for 3 × 3 Case.

Introduce (in tensor notation) Γ_{ijk} satisfying

$$\frac{\partial}{\partial x} \Gamma_{ijk} = \left(A_i^l + B_i^l \right) \Gamma_{ljk} + \left(A_i^l + B_j^l \right) \Gamma_{ilk} + \left(A_k^l + B_k^l \right) \Gamma_{ijl} \tag{14}$$

In the case when $n = 3$ we can choose.

$$\Gamma_{ijk} = \alpha \, \varepsilon_{ijk} \tag{15}$$

where

$\varepsilon_{ijk} = 1$ if (ijk) is a + ve permutation of (123)

$= -1$ if (ijk) is a − ve permutation of (123)

$= 0$ otherwise

and α satisfies

$$\alpha_x = \text{Tr } (A (\zeta) + B (x, \zeta)) \, \alpha \tag{16}$$

The solution of this is

$$\alpha = \exp \left(\text{Tr } (A (\zeta) \, x + \int_{-\infty}^{x} B (y, \zeta) \, dy) \right)$$

We divide the whole of the comples ζ – plane into six types of regions which are labelled by the permutation (i, j, k) of $(1, 2, 3)$ where

$$\text{Re } \lambda_i (\zeta) \geq \text{Re } \lambda_j (\zeta) \geq \text{Re } \lambda_k (\zeta)$$

Region (1, 2, 3).

We need the following definitions

Def : $\underline{\phi}_1 (x, \zeta)$

$$e^{-\lambda_1(\zeta) x} \underline{\phi}_1 (x, \zeta) = \mathbf{v}_1(\zeta) + \int_{-\infty}^{x} e^{(A(\zeta) - \lambda_1(\zeta))(x-y)} \times B(y, \zeta)\, e^{-\lambda_1(\zeta) y} \underline{\phi}_1 (y, \zeta)\, dy \quad (17\text{ a})$$

Def : $\underline{\psi}_3 (x, \zeta)$

$$e^{-\lambda_3(\zeta) x} \underline{\psi}_3 (x, \zeta) = \mathbf{v}_3(\zeta) - \int_{x}^{\infty} e^{(A(\zeta) - \lambda_3(\zeta))(x-y)} \times B(y, \zeta)\, e^{-\lambda_3(\zeta) y} \underline{\psi}_3 (y, \zeta)\, dy \quad (17\text{ b})$$

Def : $\tilde{\underline{\phi}}_3 (x, \zeta)$

$$e^{\lambda_3(\zeta) x} \tilde{\underline{\phi}}_3 (x, \zeta) = \tilde{\mathbf{v}}_3(\zeta) - \int_{-\infty}^{x} e^{(-A(\zeta) + \lambda_3(\zeta))(x-y)} \times B(y, \zeta)\, e^{\lambda_3(\zeta) y} \tilde{\underline{\phi}}_3 (y, \zeta)\, dy \quad (17\text{ c})$$

Def : $\tilde{\underline{\psi}}_1 (x, \zeta)$

$$e^{\lambda_1(\zeta) x} \tilde{\underline{\psi}}_1 (x, \zeta) = \tilde{\mathbf{v}}_1(\zeta) + \int_{x}^{\infty} e^{(-A(\zeta) + \lambda_1(\zeta))(x-y)} \times B(y, \zeta)\, e^{\lambda_1(\zeta) y} \underline{\psi}_1 (y, \zeta)\, dy \quad (17\text{ d})$$

We have already seen that Neumann series solution of these integral equations converge in this region. We define some more functions.

Def : $\underline{\Gamma} (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \exp\left(\mathrm{Tr}\left(A(\zeta)\, x + \int_{-\infty}^{x} B(y, \zeta)\, dy\right)\right) \tilde{\mathbf{u}} \times \tilde{\mathbf{v}} \quad (17\text{ e})$

Def : $\tilde{\underline{\Gamma}}(\mathbf{u}, \mathbf{v}) = \exp\left(-\mathrm{Tr}\left(A(\zeta)\, x + \int_{-\infty}^{x} B(y, \zeta)\, dy\right)\right) \mathbf{u} \times \mathbf{v}$ (17 f)

Def : $\underline{\rho}_{31}(x, \zeta) = \underline{\Gamma}(\tilde{\underline{\phi}}_3(x, \zeta), \tilde{\underline{\psi}}_1(x, \zeta))$ (17 g)

Def : $\tilde{\underline{\rho}}_{13}(x, \zeta) = \tilde{\underline{\Gamma}}(\underline{\phi}_1(x, \zeta), \underline{\psi}_3(x, \zeta))$ (17 h)

Def : $\alpha_3(\zeta) = \tilde{\underline{\phi}}_3(x, \zeta) \cdot \underline{\psi}_3(x, \zeta)$ (17 k)

Def : $\tilde{\alpha}_1(\zeta) = \tilde{\underline{\psi}}_1(x, \zeta) \cdot \underline{\phi}_1(x, \zeta)$ (17 l)

Def : $F_1(x, \zeta) = e^{-\lambda_1(\zeta)\, x}\, \underline{\phi}_1(x, \zeta)$ (17 m)

Def : $F_2(x, \zeta) = \dfrac{(\mathbf{v}_1(\zeta), \mathbf{v}_2(\zeta), \mathbf{v}_3(\zeta))}{\tilde{\alpha}_1(\zeta)\, \tilde{\mathbf{v}}_3(\zeta) \cdot \mathbf{v}_3(\zeta)}\, e^{-\lambda_2(\zeta)\, x}\, \underline{\rho}_{31}(x, \zeta)$ (17 n)

Def : $F_3(x, \zeta) = \dfrac{\tilde{\mathbf{v}}_3(\zeta) \cdot \mathbf{v}_3(\zeta)}{\alpha_3(\zeta)}\, e^{-\lambda_3(\zeta)\, x}\, \underline{\psi}_3(x, \zeta)$ (17 p)

We can deduce some results from the above definitions.

$\tilde{\Gamma}(\underline{\phi}_1(x, \zeta), \underline{\rho}_{31}(x, \zeta)) = \tilde{\underline{\alpha}}_1(\zeta)\, \tilde{\underline{\phi}}_3(x, \zeta)$ (18 a)

$\tilde{\underline{\Gamma}}(\tilde{\underline{\psi}}_2(x, \zeta), \underline{\rho}_{31}(x, \zeta)) = -\alpha_3(\zeta)\, \tilde{\underline{\psi}}_1(x, \zeta)$ (18 b)

$\underline{\Gamma}(\tilde{\underline{\phi}}_3(x, \zeta), \tilde{\underline{\rho}}_{13}(x, \zeta)) = \alpha_3(\zeta)\, \underline{\phi}_1(x, \zeta)$ (18 c)

and

$$\tilde{\Gamma}\,(\underline{\tilde{\psi}}_1(x,\zeta).\ \underline{\tilde{\rho}}_{13}(x,\zeta)) = -\tilde{\alpha}_1(\zeta)\ \underline{\psi}_3(x,\zeta) \tag{18 d}$$

the crucial functions in the region (1, 2, 3) are

$$F_1(x,\zeta) = e^{-\lambda_1(\zeta)\,x}\,\underline{\phi}_1(x,\zeta) \tag{19}$$

$$F_2(x,\zeta) = \frac{(v_1(\zeta),\ v_2(\zeta),\ v_3(\zeta))}{\underline{\tilde{\alpha}}_1(\zeta)\ \tilde{v}_3(\zeta).\ v_3(\zeta)}\,e^{-\lambda_2(\zeta)\,x}\,\underline{\rho}_{31}(x,\zeta) \tag{20}$$

$$F_3(x,\zeta) = \frac{\tilde{v}_3\ ,\ v_3}{\alpha_3(\zeta)}\,e^{-\lambda_3(\zeta)\,x}\,\underline{\psi}_3(x,\zeta) \tag{21}$$

Residues in region (1, 2, 3)

(i) poles of $F_1(x,\zeta)$ do not exist.

(ii) poles of $F_2(x,\zeta)$ exist where $\tilde{\alpha}_1(\zeta) = 0$

(we assume that the zero is simple)

Now since $\tilde{\alpha}_1(\zeta) = 0$ and $\underline{\tilde{\Gamma}}\,(\underline{\phi}_1(x,\zeta),\ \underline{\rho}_{31}(x,\zeta)) = \tilde{\alpha}_1(\zeta)\ \underline{\tilde{\phi}}_3(x,\zeta)$

it follows that

$$\underline{\rho}_{31}(x,\zeta) = \underline{\alpha}\ \phi_1(x,\zeta)$$

$$\text{Residue} = \frac{(v_1(\zeta),\ v_2(\zeta),\ v_3(\zeta))}{\frac{d}{d\zeta}(\tilde{\alpha}_1(\zeta))\ \tilde{v}_3(\zeta)\ .\ v_3(\zeta)}\,e^{-\lambda_2(\zeta)\,x}\,\rho_{31}(x,\zeta)$$

$$= \frac{(v_1(\zeta),\ v_2(\zeta),\ v_3(\zeta))}{\frac{d}{d\zeta}(\tilde{\alpha}_1(\zeta))\ \tilde{v}_3(\zeta)\ .\ v_3(\zeta)}\,e^{-\lambda_2(\zeta)\,x}\,\alpha\ \underline{\phi}_1(x,\zeta)$$

$$= \frac{(\mathbf{v}_1(\zeta),\ \mathbf{v}_2(\zeta),\ \mathbf{v}_3(\zeta))}{\frac{d}{d\zeta}(\tilde{\alpha}_1(\zeta)\ \tilde{\mathbf{v}}_3(\zeta)\cdot\mathbf{v}_3(\zeta))}\, e^{-\lambda_2(\zeta)\,x}\ \alpha\, e^{\lambda_1(\zeta)\,x}\ \mathbf{F}_1(x,\zeta)$$

$$= \beta\, e^{(\lambda_1(\zeta) - \lambda_2(\zeta))\,x}\ \mathbf{F}_1(x,\zeta) \tag{22}$$

where

$$\beta = \frac{\alpha\,(\mathbf{v}_1(\zeta),\ \mathbf{v}_2(\zeta),\ \mathbf{v}_3(\zeta))}{\frac{d}{d\zeta}(\tilde{\alpha}_1(\zeta)\ \tilde{\mathbf{v}}_3(\zeta)\cdot\mathbf{v}_3(\zeta))}$$

(iii) Poles of $\mathbf{F}_3(x,\zeta)$ occur where $\alpha_3(\zeta) = 0$

(assume simple zero)

Now since $\alpha_3(\zeta) = 0$ and $\tilde{\underline{\Gamma}}(\underline{\psi}_3(x,\zeta),\ \underline{\rho}_{31}(x,\zeta)) = -\,\alpha_3(\zeta)\,\tilde{\underline{\psi}}_1(x,\zeta)$

it follows that

$$\underline{\rho}_{31}(x,\zeta) = \alpha\,\underline{\psi}_3(x,\zeta)$$

If $\alpha \neq 0$

$$\text{Residue} = \beta\, e^{(\lambda_2(\zeta) - \lambda_3(\zeta))\,x}\ \underline{\phi}_2(x,\zeta)$$

where

$$\beta = \frac{\tilde{\mathbf{v}}(\zeta)\cdot\mathbf{v}(\zeta)}{\alpha_3}\cdot\frac{1}{\alpha}\cdot\tilde{\alpha}_1(\zeta)\ \frac{\tilde{\mathbf{v}}_3(\zeta)\cdot\mathbf{v}_3(\zeta)}{(\mathbf{v}_1(\zeta),\ \mathbf{v}_2(\zeta),\ \mathbf{v}_3(\zeta))}$$

(where ($'$) denotes $\frac{d}{d\zeta}$)

If $\alpha = 0$, $\underline{\rho}_{31}(x,\zeta) \equiv 0$

and

$$\underline{\rho}_{31}\ (x, \zeta) = \underline{\Gamma}\ (\ \tilde{\underline{\phi}}_3\ (x, \zeta),\ \tilde{\underline{\psi}}_1\ (x, \zeta)\ )$$

it follows

(a) $\tilde{\underline{\psi}}_1\ (x, \zeta) = \tilde{\underline{\phi}}_3\ (x, \zeta)$

(b) $\tilde{\underline{\Gamma}}\ (\ \underline{\phi}_1\ (x, \zeta),\ \underline{\rho}'_{31}\ (x, \zeta)\ ) = \tilde{\alpha}'_1\ (\zeta)\ \tilde{\underline{\phi}}_3\ (x, \zeta)$

and

(c) $\tilde{\underline{\Gamma}}\ (\ \underline{\psi}_3\ (x, \zeta),\ \underline{\rho}'_{31}\ (x, \zeta)\ ) = -\ \alpha'_3\ (\zeta)\ \tilde{\underline{\psi}}_1\ (x, \zeta)$

From (a), (b) and (c) we get

$$\alpha'_3\ (\zeta)\ \alpha_1\ \underline{\phi}_1\ (x, \zeta) + \tilde{\alpha}'_1\ (\zeta)\ \underline{\psi}_3\ (x, \zeta) = C_2\ \underline{\rho}_{31}\ (x, \zeta)$$

also

$$\underline{\phi}_3\ (x, \zeta) = \frac{(\ \mathbf{v}_1\ (\zeta),\ \mathbf{v}_2\ (\zeta),\ \mathbf{v}_3\ (\zeta)\ )}{\tilde{\mathbf{v}}_3\ (\zeta)\ .\ \mathbf{v}_3\ (\zeta)\ \tilde{\alpha}'_1\ (\zeta)}\ \underline{\rho}'_{31}\ (x, \zeta)$$

$$\text{Residue} = \frac{\tilde{\mathbf{v}}_3\ (\zeta)\ .\ \mathbf{v}_3\ (\zeta)}{\alpha'_3\ (\zeta)}\ e^{-\ \lambda_3\ (\zeta)\ x}\ \frac{1}{\tilde{\alpha}'_1\ (\zeta)}\ (\ C_2\ \underline{\rho}'_{31}\ (x, \zeta) - C_1\ a'_3\ (\zeta)) \times$$

$$\mathbf{F}_1\ (x, \zeta) = \beta\ e^{(\lambda_2\ (\zeta) - \lambda_3\ (\zeta)\ )\ x}\ \mathbf{F}_2\ (x, \zeta) + \gamma\ e^{(\lambda_1\ (\zeta) - \lambda_3\ (\zeta)\ x}\ \mathbf{F}_3\ (x, \zeta)) \tag{23}$$

where

$$\beta = \frac{(\ \tilde{\mathbf{v}}_3\ (\zeta)\ .\ \mathbf{v}_3\ (\zeta)\ )^2}{\alpha'_3\ (\zeta)} \cdot \frac{C_2}{(\ \mathbf{v}_1\ (\zeta)\ ,\ \mathbf{v}_2\ (\zeta)\ ,\ \mathbf{v}_3\ (\zeta))}$$

and

$$\gamma = - \tilde{\mathbf{v}}_3 (\zeta) \cdot \mathbf{v}_3 (\zeta) \frac{C_1}{\alpha'_1 (\zeta)}$$

The corresponding functions and residues in other regions can be found by suitable permutations of the suffices.

On the boundary between regions where $\operatorname{Re} \lambda_1 (\zeta) \geq \operatorname{Re} \lambda_2 (\zeta) \geq \operatorname{Re} \lambda_3 (\zeta)$ and $\operatorname{Re} \lambda_1 (\zeta) \geq \operatorname{Re} \lambda_3 (\zeta) \geq \operatorname{Re} \lambda_2 (\zeta)$.

$\Delta\, \mathbf{F}_1 (x, \zeta) = \mathbf{O}$

$$\Delta\, \mathbf{F}_2 (x, \zeta) = e^{- \lambda_2 (\zeta)\, x} \left[\frac{(\mathbf{v}_1 (\zeta), \mathbf{v}_2 (\zeta), \mathbf{v}_3 (\zeta))}{\tilde{\alpha}_1 (\zeta)\, \tilde{\mathbf{v}}_3 (\zeta) \cdot \mathbf{v}_3 (\zeta)} \underline{\rho}_{31} (x, \zeta) - \frac{\tilde{\mathbf{v}}_2 (\zeta) \cdot \mathbf{v}_2 (\zeta)}{\alpha_2 (\zeta)} \underline{\psi}_2 (x, \zeta) \right]$$

now

$$\tilde{\underline{\Gamma}} \left(e^{\lambda_2 (\zeta)\, x} \Delta\, \mathbf{F}_2 (x, \zeta), \underline{\rho}_{21} (x, \zeta) \right) = \tilde{\underline{\Gamma}} \left[e^{\lambda_2 (\zeta)\, x} \Delta\, \mathbf{F}_2 (x, \zeta), \Gamma \left(\tilde{\underline{\phi}}_2 (x, \zeta), \tilde{\underline{\psi}}_1 (x, \zeta) \right) \right] = \mathbf{O}$$

$$\Delta\, \mathbf{F}_2 (x, \zeta) = \mathrm{P} (\zeta)\, e^{- \lambda_2 (x, \zeta)} \underline{\rho}_{21} (x, \zeta)$$

$$= \mathrm{R}_{23} (\zeta)\, e^{(\lambda_3 (\zeta) - \lambda_2 (\zeta))\, x}\, \mathbf{F}_3^{(2)} (x, \zeta) \qquad (24)$$

where $\mathbf{F}_3^{(2)}$ is evaluated on the side of the border in the region where

$\operatorname{Re} \lambda_1 (\zeta) \geq \operatorname{Re} \lambda_3 (\zeta) \geq \operatorname{Re} \lambda_2 (\zeta)$

and

$$R_{23}(\zeta) = - P(\zeta)\, \frac{\tilde{a}_1(\zeta)\, \tilde{\mathbf{v}}_3(\zeta) \cdot \mathbf{v}_3(\zeta)}{(\mathbf{v}_1(\zeta), \mathbf{v}_2(\zeta), \mathbf{v}_3(\zeta))}$$

also

$$e^{\lambda_3(\zeta) x}\, \mathbf{F}_3(x, \zeta) = \left[\frac{\tilde{\mathbf{v}}_3(\zeta) \cdot \mathbf{v}_3(\zeta)}{\alpha_3(\zeta)}\, \underline{\psi}_3(x, \zeta) + \frac{(\mathbf{v}_1(\zeta), \mathbf{v}_2(\zeta), \mathbf{v}_3(\zeta))}{\tilde{\alpha}_1(\zeta)\, \tilde{\mathbf{v}}_2(\zeta) \cdot \mathbf{v}_3(\zeta)}\, \underline{\rho}_{21}(x, \zeta) \right]$$

$$= R_{32}(\zeta)\, e^{(\lambda_2(\zeta) - \lambda_3(\zeta))x}\, \overset{(1)}{\mathbf{F}_2}(x, \zeta) \qquad (25)$$

on the boundary between regions where

Re $\lambda_1(\zeta) \geq$ Re $\lambda_2(\zeta) \geq$ Re $\lambda_3(\zeta)$ and Re $\lambda_2(\zeta) \geq$ Re $\lambda_1(\zeta) \geq$ Re $\lambda_3(\zeta)$

$$\triangle\, \underline{\varnothing}_1(x, \zeta) = e^{-\lambda_1(\zeta) x} \left[\underline{\phi}_1(x, \zeta) + \frac{(\mathbf{v}_1(\zeta), \mathbf{v}_2(\zeta), \mathbf{v}_3(\zeta\lambda))}{\tilde{\alpha}_2(\zeta)\, \tilde{\mathbf{v}}_3(\zeta) \cdot \mathbf{v}_3(\zeta)}\, \underline{\rho}_{32}(x, \zeta) \right]$$

$$= R_{12}(\zeta)\, e^{(\lambda_2(\zeta) - \lambda_1(\zeta))x}\, \overset{(3)}{\mathbf{F}_2}(x, \zeta) \qquad (26)$$

and

$$\triangle\, \mathbf{F}_2(x, \zeta) = e^{-\lambda_2(\zeta) x} \left[\frac{(\mathbf{v}_1(\zeta), \mathbf{v}_2(\zeta), \mathbf{v}_3(\zeta)}{\tilde{\alpha}_1(\zeta)\, \tilde{\mathbf{v}}_3(\zeta) \cdot \mathbf{v}_3(\zeta)}\, \underline{\rho}_{31}(x, \zeta) - \underline{\phi}_2(x, \zeta) \right]$$

$$= R_{21}(\zeta)\, e^{(\lambda_1(\zeta) - \lambda_2(\zeta))x}\, \overset{(1)}{\mathbf{F}_1}(x, \zeta) \qquad (27)$$

also

$$\triangle \mathbf{F}_3 (x, \zeta) = \underline{0}$$

on the boundary between regions where

$\mathrm{Re}\,\lambda_1 (\zeta) \geq \mathrm{Re}\,\lambda_2 (\zeta) \geq \mathrm{Re}\,\lambda_3 (\zeta)$ and $\mathrm{Re}\,\lambda_3 (\zeta) \geq \mathrm{Re}\,\lambda_2 (\zeta) \geq \mathrm{Re}\,\lambda_1 (\zeta)$

$$\triangle \mathbf{F}_1 (x, \zeta) = e^{-\lambda_1 (\zeta) x} \left[\underline{\phi}_1 (x, \zeta) - \frac{\tilde{\mathbf{v}}_1 (\zeta) \cdot \mathbf{v}_1 (\zeta)}{a_1 (\zeta)} \underline{\psi}_1 (x, \zeta) \right]$$

$$= R_{12} (\zeta)\, e^{(\lambda_2 (\zeta) - \lambda_1 (\xi))\, x} \underline{\phi}_2^{(4)} (x, \zeta) + R_{13} (\zeta)\, e^{(\lambda_3 (\zeta) - \lambda_1 (\zeta))\, x} \underline{\phi}_3^{(4)} (x, \zeta) \tag{28}$$

$$\triangle \mathbf{F}_2 (x, \zeta) = e^{-\lambda_2 (\zeta) x} \left[\mathbf{v}_1 (\zeta) , \mathbf{v}_2 (\zeta) , \mathbf{v}_3 (\zeta) \right]$$

$$\left[\frac{1}{\tilde{\alpha}_1 (\zeta)\, \tilde{\mathbf{v}}_3 (\zeta) \cdot \mathbf{v}_3 (\zeta)} \underline{\rho}_{31} (x, \zeta) + \frac{1}{\tilde{\alpha}_3 (\zeta)\, \tilde{\mathbf{v}}_1 (\zeta) \cdot \mathbf{v}_1 (\zeta)} \underline{\rho}_{13} (x, \zeta) \right]$$

$$= R_{21} (\zeta)\, e^{(\lambda_1 (\zeta) - \lambda_2 (\zeta)\,)\, x} \mathbf{F}_1^{(1)} (x, \zeta)$$

$$+ R_{23} (\zeta)\, e^{(\lambda_3 (\zeta) - \lambda_2 (\zeta)\, x} \mathbf{F}_3^{(4)} (x, \zeta) \tag{29}$$

and

$$\triangle \mathbf{F}_3 (x. \zeta) = e^{-\lambda_3 (\zeta) x} \left[\frac{\tilde{\mathbf{v}}_3 (\zeta) \cdot \mathbf{v}_3 (\zeta)}{\alpha_3 (\zeta)} \underline{\psi}_3 (x, \zeta) - \underline{\phi}_3 (x, \zeta) \right]$$

$$= R_{31} (\zeta)\, e^{(\lambda_1 (\zeta) - \lambda_3 (\zeta)\,)\, x} \mathbf{F}_1^{(1)} (x, \zeta)$$

$$+ R_{32} (\zeta)\, e^{(\lambda_2 (\zeta) - \lambda_3 (\zeta)\,)\, x} \mathbf{F}_2^{(1)} (x, \zeta) \tag{30}$$

The quantities R_{ij} (ζ) along all the boundaries constitute the continum part of the spectral data. The spectral data is

$$s = \left[\zeta_i^{(T)} , \gamma_{ij}^{(T)} , R_{ij} \ (\zeta) \ i , j = 1, 2, 3, \ i \neq j, \ \ T = 1, 2, \ldots\ldots , m_i \right]$$

4. The Inverse Spectral Problem.

The inverse spectral problem is that of the reconstruction of the matrix $B(x, \zeta)$ from the spectral data S. We notice that the quantities

$$F_i \ (x, \zeta) = \exp \{ - \lambda_i \ (\zeta) \ ) \ x \} \ F_i \ (x \ \zeta) \qquad (32)$$

(from definition)

have the following properties

$$F_i \ (x, \zeta) - v_i \ (\zeta) = 0 \qquad \text{as} \qquad \zeta \to \infty \qquad (33)$$

From (4) and (5)

F_i (x, ζ) has simple pales at $\zeta = \zeta_i^{(T)}$

$T = 1, 2, \ldots\ldots , m_i$ with residues

$$\text{Res } F_i \left[x, \zeta_i^{(T)} \right] = \sum_{\substack{j=1 \\ j \neq i}}^{3} \gamma_{ij}^{(T)} \left[\exp \lambda_j \left(\zeta_i^{(T)} \right) - \lambda_i \left(\zeta_i^{(T)} \right) \right] \times F_j \left(x, \zeta_i^{(T)} \right) \qquad (34)$$

on the boundary

$$\triangle \ F_i \ (x, \zeta) = \sum_j R_{ij} \ (\zeta) \exp \{ (\lambda_j \ (\zeta) - \lambda_i \ (\zeta) \ ) \ x \} \ F_j \ (x, \zeta) \qquad (35)$$

These properties are sufficient to define F_i (x, ζ)

$$F_i \ (x, \zeta) = v_i \ (\zeta) - \sum_{k=1}^{m_i} \sum_{\substack{j=1 \\ i \neq i}}^{3} \frac{\gamma_{ij}^{(T)} \exp \{ (\lambda_j \ (\zeta)_i^{(T)} - \lambda_i \ (\zeta_i^{(T)}) \ ) \} \ x}{\zeta_i^{(T)} - \zeta} \times F_j \ (x, \zeta_i^{(T)})$$

$$+ \frac{1}{2 \Pi i} \int \sum_j \frac{R_{ij} \ (\zeta_o) \exp \{ (\lambda_j \ (\zeta_o) - \lambda_i \ (\zeta_o) \ ) \ x \}}{\zeta_o - \zeta} \ F_j^{\pm} \ (x, \zeta_o) \ d\zeta_o$$

where the integral is along all the boundaries the direction of integration being so that the + ve side is on the left. By choosing appropriate values for ζ and replacing i by j the left hand side can be

$$\mathbf{F}_j \; (x, \zeta_i^{(\mathrm{T})}), \; i, j = 1, 2, 3. \qquad i \neq j, \qquad \mathrm{T} = 1, 2, \ldots\ldots, m_i$$

or by allowing ζ to approach the boundaries from the appropriate sides $\mathbf{F}^{\pm}(x, \zeta_o)$. Thus we have a set of linear matrix Fredhölm equations in the unknowns $\mathbf{F}_j(x, \zeta_i^{(\mathrm{T})})$ and $\mathbf{F}_j^{\pm}(x, \zeta_o)$. The question of the existence and uniqueness of the solution to these equations has yet to be investigated but in many cases of practical interest there appears to be no difficulty. Equations (32) and (36) give $\mathbf{F}_i\,(x, \zeta)$, $i = 1, 2, 3$ throughout the complex ζ – plane and hence $\beta\,(x, \zeta)$ can be found from (1).

REFERENCES

1. Ablowitz M.J., Kaup D.J., Newell A.C., and Segur H., Studies in Appl Maths **53** No. 4. 249 (1974).

ERRATUM

ON THE WARING FORMULA FOR THE POWER SUMJ, PUJM, VOL. XIV-XV (1981-82), pp. 165-174,

By

P.G. TODOROV

(1) Page 166, line (4) from above :

The correct relation is

$$S_m (n) = m \quad (- 1)^{\nu_1 + \cdots + \nu_n} \frac{(y_1 + \cdots + \nu_{n-1})}{y_1 ! \ldots \nu_n !} a_1^{y_1} \ldots a_n^{y_n} \tag{5}$$

(2) Page 166, line 13 above :

The relation

$k ! = \nu_2$

must be read as

$k : = \nu_2$

(3) Page 168, line 8 above :

$\gamma_1 = m - 2k + \gamma = 0$

must be read as

$\nu_1 = m - 2k + \nu = 0$

(4) Page 169, line 5 above :

Corollary 2, x_1 , x_2 x_3

must be read as

Corollary 2. If x_1 , x_2 , x_3

(5) Page 169, line 14 below :

[2]

must be read as

(2)

(6) Page 170, line 3 above :

$$B_{m,\, m-k} = \frac{a_1^{\nu_1} \ldots a_{k+1}^{\nu_{k+1}}}{\nu_1 ! \ldots \nu_{k+1} !} .$$

must be read as

$$B_{m,\, m-k} = \sum \frac{a_1^{\nu_1} \ldots a_{k+1}^{\nu_{k+1}}}{\nu_1 ! \ldots \nu_{k+1} !}$$

(7) Page 170, line 6 above :

$\ldots + (k + 1)\, v_{+1} = m$

must be read as

$\ldots + (k + 1)\, v_{k+1} = m$

(8) Page 170, line 7 above :

If $m \geq n$

must be read as

If $m > n$

(9) Page 170, line 13 above :

$\ldots + v_{k+l}$

must be read as

$\ldots + v_{k+1}$

(10) Page 170, line 4 below :

$= m = k$

must be read as

$= m - k$

(11) Page 171, line 3 above :

$m \geq n$

must be read as

$m > n$

(12) Page 172, line 1 above :

$v_3 \in [\, 0, k / 2 \,]$

must be read as

$v_3 \in [\, 0, k / 2 \,],\ v_2 \in [\, 0, k \,],$

(13) Page 172, line 11 below :

$\mathbf{B}_{m1\ m-3}$

must be read as

$\mathbf{B}_{m,\ m-3}$

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