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ON ISOMORPHISM OF SOLUTIONS FOR CERTAIN SEMIBIPLANES USING HUSSAIN'S TECHNIQUE

By

SHOAIB UDDIN

Mathematics Department
Punjab University, Lahore—20, Pakistan

1. Introduction.

A *design* π is a pair (**P**, **B**), where, **P** is a finite set of v objects, called *points*, and **B** is a collection of subsets of **P**, called *blocks*, each consisting of k points, where k is a constant, $(0 < k < v)$. Normally, the total number of blocks is denoted by b. A design is said to be *symmetric* if $v = b$.

Two designs with the same parameters are *isomorphic* if there is a bijection between their points under which blocks correspond to blocks.

Order the points and also the blocks of a design π. The $b \times v$ *incidence matrix* of π has (i, j) th entry 0 or 1 according as the i th block is not or is on the j th point. Zeros are often ignored when writing an incidence matrix.

A design π is said to be *resolvable* if its set of blocks can be partitioned into subsets, called parallel classes, such that each parallel class partitions the set of points of π. In this case, two blocks are said to be *parallel* if they are in the same parallel class and *nonparallel* otherwise. If π is resolvable so that any two nonparallel blocks meet in a constant number, say μ, of points, π is said to be *affine resolvable*.

The *dual* π^* of π is obtained by interchanging the roles of points and blocks and reversing the incidence relation. π is said to be *self-dual* if π^* and π are isomorphic. Two points of π are called *parallel* if π^* is resolvable and these points are parallel as blocks of π^*.

Definition. A design π is an $H_m(\mu)$ if π is a symmetric affine resolvable design with μm^2 points and μm points on each block such that its dual design is also affine resolable. An $H_m(2)$ is referred to as a *semibiplane*.

Biplanes are designs in which :

(a) two distinct points are in exactly 2 distinct blocks ;

(b) two distinct blocks contain exactly 2 common points.

Semibiplanes have the following properties which can be compared with those of biplanes :

(a′) two nonparallel points are in exactly 2 distinct blocks ;

(b′) two nonparallel blocks contain exactly 2 common points.

Hussain [5] uses a method of building up a design step by step to enumerate thc nonisomorphic solutions of biplanes with smaller values of v and k. In this paper we modify the method of [5] ; using this modification it is possible to verify whether or not two given solutions of a semibiplane are isomorphic. The uniquenes of an $H_3(2)$ is proved as an application of the method.

2. The Chains

Take any block of a semibiplane $H_m(2)$; this will be referred to as the *initial block* (*or i-block*). Label the points on the block 1, 2,..., 2m; these will be referred to as the *initial points* or (*i-points*). Blocks not parllel to the initial block will be referred to as the *j-blocks*. points which are not on the initial block and are not parallel to the point 1 will be referred to as *j-points*. Points parellel to 1 and blocks parallel to the initial block will be referred to as *z-points* and *z-blocks*, respectively.

Note that any $H_m(2)$ will have $2m^2-3m+1$ *j-points*.

Any *j-block* has 2 points in common with the initial block and given 2 distinct *i-points* there is exactly one *j-block* containing both of them. Thus the *j-block* containing the *i*-points p and q may be represented by the unordered pair $\{p, q\}$. A standered order for the *j-blocks* is : $\{1, 2\}$ $\{1, 3\}$ ... $\{1, 2m\}$ $\{2, 3\}$ $\{2, 4\}$ ... $\{2, 2m\}$ $\{3, 4\}$ ... $\{3, 2m\}$ ... $\{2m-1, 2m\}$.

Observe that there is a placing for all the *i-points.* Thus the *i-point* 1 is on the initial block and the *j-blocks* {1, 2} , {1, 3} , ... , {1, 2*m*}, *i-point* 2 is on the initial block and the *j-clocks* {1, 2}, {2, 3},..., {2, 2*m*}, and so on.

Given any *j-point* it occurs in $2m-1$ *j-blocks*, which are represented by $2m-1$ unordered pairs $\{r_1, s_1\}$, ... , $\{r_{2m-1}, s_{2m-1}\}$, say. Now any given *j-point* and each of the *i-points* not parrllel to it must lie togather on exaetly two j-blooks. Hence $2m-1$ of the $2m$ *i*-points must occur exactly twice in these pairs. The $2m-1$ pairs can be formed into a chain of $2m-1$ elements consisting of one or more cycles in the following way.

Arrange the pairs such that the last element of a pair is the same as the first element of the next pair. As soon as the last element of a pair is the same as the first element of the startıng pair a cycle is closed. If all the pairs have not already been used in the cycle, start again with a pair not included in the first cycle, and complete a new cycle-

Note that these cycles must contain three or more elements, since in the process of formation of cycles out of a given number of pairs as outlined above, only one element cannot be left out ; nor can one pair be left out, for this would mean that the points of this pair have not occured more than once, which cannot be true.

Notation

The chain ossociated with the *j*-points common to *j*-blocks $\{1, p\}$, and $\{1, q\}$, $1 < p, q \leq 2m$, is denoted by $c(p, q)$. Thus $c(p, q)$ must have p, 1 and q as consecutive elements.

Example. Consider an $H_3(2)$. Take one of its blocks as the initial block and lable the points and blocks according to the above procedure. Suppose that one of the *j*-points occur in the following *j*-blocks :

{1, 2}, {1, 3}, {2, 4}, (3, 5}, {4, 5}.

Then the chain associated to this *j*-point is (12453), and is denoted by $c(2, 3)$.

Similar to the placing for all *i*-points, there is also a placing for the $2m-1$ *j*-blocks {1, 2}, {1, 3}, ... , {1, 2*m*}. According to this placing the block $\{1, 2\}$ contains the points 1, 2, $c(2, 3)$, $c(2, 4)$, ..., $c(2, 2m)$, the block $\{1, 3\}$ contains the points 1, 3, $c(2, 3)$, $c(3, 4)$, ..., $c(3, 2m)$, and so on.

Remark. If the incidence matrices of different solutions of an $H_m(2)$ are written so that the first $2m^2 - m + 1$ rows correspond to blocks in the following order :

Initial block, $\{1, 2\}, \ldots, \{1, 2m\}, \{2, 3\}, \quad , \{2, 2m\}, \ldots, \{2m-1, 2m\}$, and the first $2m^2 - m + 1$ columns correspond to points in the following order ;

$1, 2, \ldots, 2m, c(2, 3), \ldots, c(2, 2m), c(3, 4), \ldots, c(3, 2m), \ldots, c(2m-1, 2m)$, then the first $2m$ rows and the first $2m$ columns will be the same for every solution. See, for example, the incidence matrix of an $H_3(2)$ (Figure 1).

The above order for points and blocks shall be called the *standard order*, and the corresponding matrix the *ordered incidence matrix* of the $H_m(2)$.

Lemma. Any set of $2m^2 - 3m + 1$ chains associated with the j-points of a semibiplane $H_3(2)$ satisfies the following properties.

(a) Three consecutive numbers of any chain do not coincide with three consecutive numbers of any other chain in the set.

(b) The set can be partitioned into $m-1$ subsets, called families, such that two chains belong to the same family if and only if exactly one pair of consective numbers of one chain coincides with a pair of consecutive numbers of the other chain.

(c) The set can also be partitioned into $2m-1$ subsets, called parallel classes, of $m-1$ chains each, taking exactly one chain from each family such that,

   (i) no two consecutive numbers of any chain coincide with two consective numbers of any other chain in the same parallel class.

   (ii) exactly two pairs of consecutive numbers of any chain coincide with two pairs of consecutive numbers of a chain from a different family and a different parallel class.

Proof.

(a) The violation of (a) implies existance of two j-points, say, j_1 and j_2 which have three consecutive numbers abc in common. Then the blocks $\{a, b\}$ and $\{b, c\}$ have the points j_1, j_2, b in common. This is not possible, since any two nonparallel blocks of a semibiplane meet in exactly 2 points.

(b) Each of the $m-1$ z-blocks is on exactly $2m-1$ j-points. The points on the same z-block form a family.

(c) The $m-1$ chains which correspond to $m-1$ j-points parallel to each of the $2m-1$ initial points, excluding the point 1, form a parallel class.

Definition. A set of $2m^2 - 3m + 1$ chains is consistent if it satisfies the properties (a), (b) and (c). Chains in a consistant set are said to be mutually consistant.

3. Applications.

Notice that a consistant set of chains remains consistent after any permutation of the points of the initial block and no new solution is obtained by any such permutation. Also two semibiplanes are isomorphic if the set of chains associated with the one can be obtained from the set of chains associated with the other by a bijection between the points of the initial block.

(a) *Uniqueness of the semibiplane* H_3 *(2).*

The existence of an H_3 (2) was established by Bose et al [1]. Other examples have been given since by different authors (see, for example, [2], [3] or [4]; the semibiplane with 6 points on a line of [6] is also an H_3 (2)). We verify here that these are the same; that is, upto isomorphism there is only one H_3 (2).

Since we are only interested in non-isomorphic solutions, therefore, it is possible to choose one of the chains arbitrarily and then find a consistent set of chains containing this chain. Take $c(2, 5) = (12345)$ as the arbitrary selected chain. Then $c(2, 5)$ is parallel to the i-point 6 and the third point in the parallel class must be $c(3, 4) = (13524)$.

Now the j-point represented by the chain $c(2, 3)$ cannot be parallel to the point 6. Hence there are the following four choices for $c(2, 3)$:

1 – (12653), 2 – (12643), 3 – (12463), 4 – (12563).

The chain (12653) is not consistent with $c(3, 4)$, since the two chains have three consecutive elements 135 in common. If $c(2, 3)$ is taken to be (12643) then $c(2, 3)$ is parallel to the point 5 and the third point in the parallel class must be $c(4, 6) = (14236)$. Now, since the j-point $c(2, 4)$ can be parallel to neither the initial point 5 nor 6, therefore, there is only one choice (12564) for $c(2, 4)$, which is consistent with $c(2, 5)$, $c(3, 4)$, $c(2, 3)$ and $c(4, 6)$. However, there is no possible choice left for j-point $c(2, 6)$. Hence $c(2, 3)=(12643)$ is not possible. The coice $c(2, 3)$ can be eliminated similarlv.

Finally, $c(2, 3) = (12563)$ was taken and all possible choices were examined for the remaining j-points and it was found that the following is the unique set of 10 consistent chains for an $H_3(2)$.

$c(2, 3) = (12563)$	$c(3, 5) = (14365)$
$c(2, 4) = (12634)$	$c(3, 6) = (13246)$
$c(2, 5) = (12345)$	$c(4, 5) = (14625)$
$c(2, 6) = (12456)$	$c(4, 6) = (14536)$
$c(3, 4) = (13524)$	$c(5, 6) = (15326)$

The set can be partitioned into the following two families :

$$F_1 = \{ c(4, 6),\ c(4, 5),\ c(2, 3),\ c(3, 6),\ c(2, 5) \},$$

$$F_2 = \{ c(3, 5),\ c(2, 6),\ c(5, 6),\ c(2, 4),\ c(3, 4) \};$$

where the nth element of F_1 and the nth element of F_2, $n=1, 2, \ldots, 5$, are in the same parallel class.

The possible choice for the two z-points in the semibiplane which would be consistent with the above set of chains is (23645) and (24356).

Figure 1. The ordered incidence matrix of the unique H_3 (2).

1	1	1	1	1	1												
1	1					1	1	1	1								
1		1				1				1	1	1					
1			1				1			1			1	1			
1				1				1			1		1		1		
1					1				1			1		1	1		
	1	1						1				1			1	1	
	1		1						1	1		1					1
	1			1		1				1			1			1	
	1				1		1						1		1		1
		1	1				1	1			1						1
		1		1						1				1	1		1
		1			1	1	1							1		1	
			1	1				1	1					1		1	
			1		1						1	1	1			1	
					1	1	1		1		1						1
							1		1			1	1	1			1
								1		1	1	1			1	1	

(b) **Self-dual solutions of H_4 (2) and H_5 (2) designs.**

(i) Rajkundlia has given an H_4 (2) design in [7]. The set of chains associated with this design is :

(1368754)	(1583476)	(1735648)
(1652478)	(1548627)	(1285764)
(1536728)	(1265873)	(1683257)
(1382647)	(1423768)	(1278436)
(1248375)	(1435287)	(1327458)
(1485326)	(1254638)	(1342865)
(1234567)	(1357246)	(1473625)

The chains are arranged so that rows correspond to parallel classes and columns correspond to the families of chains.

(ii). The semibsplane with 10 points on a line of [6] is in fact an $H_5(2)$ design. The set of chains associated with the design is:

(153709846)	(130687594)	(193856740)	(179634508)
(127496085)	(176842509)	(162954078)	(146579820)
(128305976)	(185732690)	(152063987)	(135680729)
(170) (268) (349)	(129804) (367)	(138) (246097)	(169) (230) (478)
(189) (257) (340)	(120793) (458)	(147) (235908)	(150) (249) (378)
(148) (239) (560)	(159) (264380)	(124530) (689)	(136) (258) (490)
(137) (240) (569)	(160) (253479)	(123649) (570)	(145) (267) (390)
(132546780)	(156307428)	(143720586)	(126048357)
(142635879)	(165498327)	(134829675)	(125937468)

Again rows correspond to parallel classes and columns correspond to families of chains.

The set of chains corresponding to the duals of the designs (i) and (ii) were also obtained and it was found that in both cases the set of chains corresponding to the dual designs were obtainable from the set of chains associated with the original designs by suitable permutations. Hence the two designs are self-dual.

These results are contained in the author's doctoral thesis submitted to University College of Wales, Aberystwyth. The author wishes to express his gratitude to Dr. V.C. Mavron for all his assistance and encouragement in carrying out this research.

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ON GROUP - VALUED SUBMEASURES

By

ABDUL RAHIM KHAN

Department of Mathematics
Bahauddin Zakariya University, Multan
Pakistan

1. Introduction.

The notion of $[0, \infty]$ – valued submeasure [1] is of interest since submeasures can be used to generate a ring topology, thereby creating a natural setting for the consideration of continuity in relation to topological group-valued set functions. In recent years developments have also taken place in certain areas of group-valued measures. These facts motivate us to look into group-valued submeasures. In this paper we introduce such submeasures and in particular we have proved an exhaustion principle which is turn enables us to establish a Lebesgue decomposition theorem for these submeasures. Our methods are those used by Drewnowski [2] and main theorems of this note generalize theorems 4.7 and 6.7 of [2].

2. Notation and Terminology.

Let (G, τ) be a commutative Hausdorff topological group (written additively) and R a ring of subsets of a set X. Let μ be a G-valued function on R. We say that μ is

(i) *Finitely additive* if $\mu(E \cup F) = \mu(E) + \mu(F)$ for all E, F in R with $E \cap F = \phi$ and if, in addition, $\mu(\phi) = 0$, we say μ is a *measure*.

(ii) σ*- additive* if for any sequence $\{E_n\}$ of disjoint sets in R such that $\bigcup_{n=1}^{\infty} E_n \in R$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$, where the convergence is relative to the group topology on G.

(iii) *Order continnous* if for each decreasing sequence $\{E_n\}$ in **R** such that $\lim_{n \to \infty} E_n = \phi$, then $\lim_{n \to \infty} \mu(E_n) = 0$.

(iv) *exhaustive* if for each disjoint sequence $\{E_n\}$ of sets in **R** $\lim_{n \to \infty} \mu(E_n) = 0$ " .

Let A be the σ–algebra of subsets of **R**. Then the Lebesgue measure $\mu : A \to [0, \infty]$, is neither order continuous nor exhaustive. It is straight forward to show that a G-valued measure is σ-additive if and only if it is order continuous and if μ is σ-additive on **R** , then it is exhaustive.

A real-valued function q on G is said to be a *quasi-norm* on G if

(i) $q(x) \geqq 0$ $(x \in G)$,
(ii) $q(0) = 0$
(iii) $q(x) = q(-x)$
(iv) $q(x+y) \leq q(x) + q(y)$ $(x, y \in G)$ " . "

If, in addition, $q(x) = 0$ implies $x = 0$, then q is said to be a *norm* on G.

If (G, q) is a quasi-normed group and μ is a G-valued function, then *semi-Variation*, $\bar{\mu}$ (F) of μ on $F \subseteq X$ is defined by

$$\bar{\mu}(F) = \sup \{q \circ \mu(E) : E \in \mathbf{R}, E \subseteq F\}$$

Let G be a commutative lattice group [5], abbreviated to l-group. A quasi-norm (norm) on G is said to be an *l-quasi-norm* (*l-norm*) if $q(x) \leq q(y)$ for all x, y in G with $|x| \leq |y|$. A G-valued function μ on **R** is said to be a *submeasure* if $\mu(\phi) = 0$, $\mu(E \cup F) \leq \mu(E) + \mu(F)$ for all E, F in **R** with $E \cap F = \phi$ and μ is monotone i.e. $\mu(E) \leq \mu(F)$ for E, F in **R** with $E \subseteq F$. Note that if μ is a G-valued submeasure on **R**, then $\mu(E) \geq 0$ for all E in **R**. Further if (G, q) is an l-quasi-normed group, then $q \circ \mu$ is an $\mathbf{R}_+$-valued submeasure in the sense of [1] and if μ is exhaustive, then we can show by an indirect argument that $\bar{\mu}$ is also exhaustive.

A subset V of G is said to be *solid* if $a \varepsilon V$ and $|x| \leq |a|$ implies that $x \varepsilon V$. In particular, a solid set V is symmetric (i.e. $V = -V$). A group topology τ is said to be locally solid if it has a base of τ-neighbourhoods of 0 consisting of solid sets. A family of l-quasi-norms determines a locally solid group topology on G ; on the other hand, if τ is a locally solid group topology on G, then τ may be determined by the family of all τ-continuous l-quasi-norms on G (see [4], 22C).

3. Preparatory Lemmas

Lemma 1.

Let μ be a G- valued σ-additive measure on **R**. Then $\bar{\mu}$ is an order continuous submeasure.

Proof :

The only non trivial part is to show that $\bar{\mu}$ is order continuous. Let $\{E_n\}$ be a decreasing sequence in **R** such that $\lim_{n \to \infty} E_n = \phi$.

Suppose the result is not true. Then there exist a positive number δ and a subsequence $\{n_p\}$ such that, $\bar{\mu}(E_{n_p}) > \delta$ for $p = 1, 2, 3, \ldots$; Since $\bar{\mu}$ is monotone it follows that $\bar{\mu}(E_n) > \delta$ for all n. In particular $\bar{\mu}(E_1) > \delta$ $(n_1 = 1)$ and so there exists a set $A_1 \varepsilon$ **R** such that $A_1 \subseteq E_1$, and $q(\mu(A_1)) > \delta$. Since μ is σ-additive it is easy to show that μ is continuous from above and so there exists a positive integer n_2 such that $q(\mu(A_1 \cap E_{n_2})) < \delta/2$,

Let $F_1 = (E_{n_1} \setminus E_{n_2}) \cap A_1$. Then

$$q(\mu(F_1) \geq | q(\mu(E_n \cap A_1)) - q(\mu(E_n \cap A_1)) | > \delta / 2$$

As $\bar{\mu}(E_{n_2}) > \delta$ we can use a similar argument to show that there exist sets A_2, E_{n_3} in **R** with $A_2 \subseteq E_{n_2}$ such that if $F_2 = (E_{n_2} \setminus E_{n_3}) \cap A_2$, then $q(\mu(F_2)) > \delta/2$. So a sequence $\{F_n\}$ of disjoint sets in **R** can be found such that $q(\mu(F_n)) > \delta/2$ ($n = 1, 2, 3, \ldots$). Now since μ is σ-additive so

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \cdots$$

Since the right hand side converges, $\lim_{n \to \infty} q(\mu(F_n)) = 0$. This contradiction proves the result.

A G-valued measure μ is said to be bounded if $\{\mu(E) : E \varepsilon R\}$ is a bounded subset of G. The semi-variation of group-valued measures in general may be unbounded [6] but for exhaustive measures we have the following.

Lemma 2.

Let (G, q) be a quasi-normed group and μ be an exhaustive G-valued measure on **R**. Then $\bar{\mu}(E)$ is finite for all subsets E of **X.**

Proof :

Since μ is exhaustive of $\bar{\mu}$ is exhaustive. Now if $\{E_n\}$ is disjoint sequence of subsets of X, then the increasing sequence $\{E_n \triangle E_m\}$ of X is $\bar{\mu}$-cauchy (i.e. $\lim_{n \to \infty} \bar{\mu}(E_n \triangle E_m) = 0$). Otherwise for some $\varepsilon > 0$, there exists an increasing sequence $\{n_k\}$ of integers such that $\bar{\mu}(E_{n_{k+1}} \triangle E_{n_k}) > \varepsilon$ ($k = 1, 2, 3, \ldots$). This contradicts the exhaustive property of $\bar{\mu}$. Now suppose that $\bar{\mu}(E) = +\infty$ for some $E \subsetneq X$. Then there exists an increasing sequence $\{G_n\}$ such that $\bar{\mu}(G_n) > n$ for $n = 1, 2, 3, 4, \ldots\ldots$ By the above fact the sequence $\{G_n\}$ is $\bar{\mu}$-cauchy and so $\bar{\mu}$-bounded. This yields a contradiction and so $\bar{\mu}(E) < +\infty$ for all subsets E of **X.**

Lemma 3.

Let (G, q) be an l-quasi-normed group and let $\{\mu_i\}$ ($i \varepsilon I$) be a family of G-valued uniformly order continuous sub-measure s on a σ-ring **R.** Then the family $\{\mu_i\}$ is uniformly exhaustive.

Proof :

Let $\{E_n\}$ be a sequence of disjoint sets in **R**. Define $F_n = \bigcup_{k \geq n} E_k$. Then $\{F_n\}$ is a decreasing sequence in **R** and $\lim_{n \to \infty} F_n = \phi$. Clearly $q(\mu_i(E_n)) \leq q(\mu_i(F_n))$ for all $i \varepsilon I$ and $n = 1, 2, 3, \ldots\ldots$, and so by the order continuity of the family $\{\mu_i\}$ it follows that the family $\{\mu_i\}$ is uniformly exhaustive.

4. The Exhaustion Principle and Some Applications.

We start this section with an 'exhaustion principle' in the following.

Theorem 1.

Let (G, q) be an l-quasi-normed group and μ an exhaustive G-valued submeasure on **R**. If $\mathbf{M} \subseteq \mathbf{R}$, then there exists a sequence $\{M_n\}$ of sets in **M** such $\lim_{n \to \infty} q(\mu(M \setminus \bigcup_{k=1}^{n} M_k)) = 0$ uniformly with respect to $M \varepsilon \mathbf{M}$.

Proof :

Let $\mathbf{M}_1 = \{E \varepsilon \mathbf{R} : E \subseteq M \text{ for some } M \varepsilon \mathbf{M}\}$. Clearly $\mathbf{M} \leq \mathbf{M}_1$. Let ε_1 be a positive number such that $\varepsilon_1 < \sup_{E \varepsilon \mathbf{M}_1} q(\mu(E))$, and let E_1^1 be a set in $\mathbf{M}_1$ such that $q(\mu(E_1^1)) > \varepsilon_1$. We now choose successive disjoint sets E_k^1 $(k = 1, 2, \ldots)$ in $\mathbf{M}_1$ such that $q(\mu(E_k^1)) > \varepsilon_1$; since μ is exhaustive we can extract a finite disjoint sequence $E_1^1, \ldots\ldots, E_{n_1}^1$ in $\mathbf{M}_1$ such that $q(\mu(E)) < \varepsilon_1$ for all $E \varepsilon \mathbf{M}_2$, where $m_2 = \{E \varepsilon \mathbf{M}_1 : E \cap E_k^1 = \phi \ (k = 1, \ldots\ldots, n_1)\}$. Let ε_2 be a positive number such that $\varepsilon_2 < \min(E_1 / 2, \sup_{E \varepsilon \mathbf{M}_2} (q(\mu(E))))$.

We similarly find a finite sequence of disjoint sets $E_1^2, \ldots\ldots, E_{n_2}^2$ in $\mathbf{M}_2$ such that $q(\mu(E_k^2)) > \varepsilon_2$ $(k = 1, \ldots\ldots, n_2)$ and $q(\mu(E)) < \varepsilon_2$

for all E ε $\mathbf{M}_3$, where $\mathbf{M}_3 = \{E \, \varepsilon \, \mathbf{M}_2 : E \cap E^2_k = \phi, (k=1, \ldots\ldots, n_2)\}$.
Continuing with this process we obtain a disjoint sequence $E^1_1, \ldots\ldots, E^1_{n_1}; E^2_1, \ldots\ldots, E^2_{n_2}, \ldots\ldots$, in $\mathbf{M}_1$ and a sequence $\{\varepsilon_k\}$ of positive numbers such that $\varepsilon_k \to 0$, as $k \to \infty$. To obtain the sequence $\{M_n\}$ we choose sets in m which contain in turn the sets $E^1_1, \ldots, E^1_{n_1}; E^2_1, \ldots, E^2_{n_2}, \ldots$. The sequence $\{M_n\}$ has the required properties, as follows ;

Let M be any element in **M**. Then, for any integer p,

$$M \setminus \bigcup_{n=1}^{\sum_{k=1}^{p} n_k} M_n \subseteq M \setminus \bigcup_{k=1}^{p} \bigcup_{i=1}^{n_k} E^k_{n_i}$$

, and so, since μ is a sub-measure and q is an l-quasi-norm, it follows that

$$q \circ \mu \left(M \setminus \bigcup_{n=1}^{\sum_{k=1}^{p} n_k} M_n \right) \leq q \circ \mu \left(M \setminus \bigcup_{k=1}^{p} \bigcup_{i=1}^{n_k} E^k_{n_i} \right) < \varepsilon_p \to 0, \text{ as } p \to \infty .$$

This proves the theorem.

Corollary 1.

With q, μ and **R** as in Theorem 1, there exists a sequence $\{E_n\}$ in **R** such that $\lim_{n \to \infty} q(\mu(E \setminus \bigcup_{k=1}^{n} E_k)) = 0$ uniformly with r espect to E ε **R**.

Corollary 2.

With q and μ as above and **R** a σ-ring, there exists a set E_o in **R** such that $q(\mu(E \setminus E_o)) = 0$ for all E ε **R**.

Corollary 3.

Let (G, q) be an l-normed group and μ an order continuous G-valued submeasure on a σ-ring **R**. Then there exists a set E_o ε **R** sush that $\mu(E \setminus E_o) = 0$ for E ε **R**.

Proof :

By Lemma 3, μ is exhaustive and so by Corollary 2, there exists a set E_o in **R** such that $q(\mu(E \setminus E_o)) = 0$ for all E ε **R**. Now (G, q) is a normed group and so $\mu(E \setminus E_o) = 0$ for each E ε **R**, as required.

Corollary 4.

Let (G, q) be a normed group and μ be a G-valued σ-additive measure on a σ-ring **R**. Then there exists a set $E_0 \varepsilon$ **R** such that $\mu (E \setminus E_0) = 0$ for each $E \varepsilon$ **R**.

Proof :

Since μ is σ-additive, it is exhaustive, and so by Lemmas 1 and 2, $\bar{\mu}$ is an order continuous R_+ -valued submeasure on **R**. By corollary 3 there exists a set $E_0 \varepsilon$ **R** such that $\bar{\mu} (E \setminus E_0) = 0$ for all $E \varepsilon$ **R**. It follows that $q(\mu (E \setminus E_0)) = 0$ for all $E \varepsilon$ **R** and so, since q is a norm on G, $\mu (E \setminus E_0) = 0$ for all $E \varepsilon$ **R**, as required.

Definition 1.

Let (G, q) be an l-quasi-normed group and μ a G-valued submeasure on **R**. A set $E \varepsilon$ **R** *is μ-null* if and only if $\mu (E) = 0$.

We note that for G-valued sudmeasures the above notion of a μ-null set agrees with that defined by Traynor ([7], *p*. 136) for group-valued measures.

Definition 2.

With G and μ as in definition 1, μ is said to be *σ-subadditive* if and only if, for each sequence $\{ E_n \}$ of disjoint sets in **R** such that

$$\bigcup_{n=1}^{\infty} E_n \varepsilon \mathbf{R}, \quad q\left(\mu\left(\bigcup_{n=1}^{\infty} E_n\right)\right) \leq \sum_{n=1}^{\infty} q(\mu(E_n)).$$

Clearly for G-valued submeasure μ, $H_\mu = \{ E \varepsilon \mathbf{R} : \mu (E) = 0 \}$ is an ideal in **R** and that H_μ is a σ-ideal if μ is σ-sub-additive.

Theorem 2.

Let (G, q) and (Z, p) be l-quasi-normed groups and μ be an exhaustive G-valued submeasure on a σ-ring **R**. Then, for any Z-valued submeasure ν on **R**, there is a sequence $\{ E_n \}$ of ν-null sets such that $q(\mu (E)) = 0$ for all E in **R** such that $E \cap \bigcup_{n=1}^{\infty} E_n = \phi$.

Proof :

Let $H_\nu = \{ E \in R : \nu(E) = 0 \}$. By Theorem 1 there exists a sequence $\{E_n\}$ in H_ν such that $\lim_{n \to \infty} q(\mu(E \setminus \bigcup_{k=1}^{n} E_k)) = 0$ uniformly with respect to E in **R**. Thus, if $E_0 = \bigcup_{k=1}^{\infty} E_k$, we have that $q(\mu(E \setminus E_0)) = 0$ for all $E \in R$, and so $q(\mu(E)) = 0$ for all E in **R** with $E \cap E_0 = \phi$.

The following proves that exhaustive submeasures are bounded.

Theorem 3

Let (G, q) be an l-quasi-normed group and μ an exhaustive G-valued submeasure on **R**. Then $\sup_{E \in R} q(\mu(E)) < +\infty$.

Proof :

By Theorem 1 there exists a sequence $\{E_n\}$ in **R** such that $\lim_{n \to \infty} q(\mu(E \setminus \bigcup_{k=1}^{n} E_k)) = 0$ uniformly for $E \in R$. Thus there exists a positive integer N such that $q(\mu(E \setminus \bigcup_{k=1}^{n} E_k)) < 1$ for all $E \in R$ and $n \geq N$. Now, for any $E \in R$,

$$E = (E \setminus \bigcup_{k=1}^{N} E_k) \cup \bigcup_{k=1}^{N} E_k,$$

and so

$$q(\mu(E)) \leq q(\mu(E \setminus \bigcup_{k=1}^{N} E_k)) + \sum_{k=1}^{N} q(\mu(E_k))$$

$$\sum_{k=1}^{N} q(\mu(E_k)).$$

It follows that $q \circ \mu$ is bounded on **R** as required.

As a consequence of Theorem 3 we have the following result due to Drewnowski ([2], Cor. 4.11).

Corollary 5.

Let (G, q) be a quasi-normed group and μ an exhaustive **G**-valued measure on **R**. Then $\sup_{E \varepsilon \mathbf{R}} q(\mu(E)) < \infty$.

Proof :

By lemma 2, $\bar{\mu}$ is a finite-valued function and so $\bar{\mu}$ is an exhaustive $\mathbf{R}_+$-valued submeasure on **R**. Thus by Theorem 3 $\sup_{E \varepsilon \mathbf{R}} q(\mu(E)) < +\infty$.

Definition 3 :

Let **H** be an ideal in **R**. A submeasure μ on **R** is said to be *nearly supported on* **H** if and only if, for every $\varepsilon > 0$ there exists a set E in **H** such $q(\mu(A)) < \varepsilon$ whenever $X \setminus E \supseteq A \varepsilon \mathbf{R}$.

Definition 4 :

Let **H** and μ be as above. We say that μ *vanishes on* **H**, if and only if $\mu(E) = 0$ for all $E \varepsilon \mathbf{H}$.

We now use theorem 1 to derive a Lebesgue type decomposition theorem for *l*-group-valued submeasures ; our results generalize a theorem of Drewnowski ([2], Theorem 6.7) and Traynor ([7], Theorem 2.2).

Theorem 4 :

Let G be an *l*-quasi-normed group and μ be an exhaustive G-valued submeasure on **R**. If **H** is a σ-subring of **R**, then there exists a set H in **H** such that

(i) the submeasure μ_1 defined by $\mu_1(E) = \mu(E \cap H)$ is nearly supported on **H** ,

(ii) the submeasure μ_2 defined by $\mu_2(E) = \mu(E \setminus H)$, $(E \varepsilon \mathbf{R})$ vanishes on **H**.

Proof :

By Theorem 1 there exists a sequence $\{ H_n : n = 1, 2, 3, \ldots \}$ in **H** such that $\lim_{n \to \infty} q(\mu(A \setminus \bigcup_{k=1}^{n} H_k)) = 0$ uniformly for A in **H**.

Since **H** is a σ-ideal in **R**, $H = \bigcup_{k=1}^{\infty} H_k \in$ **S** and so $q(\mu(A \setminus H) = 0$ for all A in **H**. It is easy to see that the submeasure μ_2 is exhaustive and for any A in **S**, $q(\mu_2(A)) = q(\mu(A \setminus H)) = 0$, which implies that μ_2 vanishes on **S**. If $E \in$ **R** and $E \cap H = \phi$, then $\mu(E \cap H) = 0$ and so the submeasure μ_1 is nearly supported on **H**.

Following [7], we shall say that μ is ν-continuous if and only if μ vanishes on ν-null sets and μ is ν-singular if and only if there exists a ν-null set E such that $\mu(A \setminus E) = 0$ for all $A \in$ **R**. We say μ is equivalent to ν, written as $\mu \sim \nu$ if and only if μ is ν-continuous and ν is μ-continuous.

Theorem 5.

Let G and H be l-quasi-normed groups. Suppose that μ is an exhaustive G-valued submeasure on **R** and ν is a σ-subadditive H-valued submeasure on **R**. Then there exists a ν-null set E_o such that

(i) the submeasure μ_1 defined by $\mu_1(E) = \mu(E \cap E_o)$, ($E \in$ **R**) is ν-singular ; and

(ii) the submeasure μ_2 defined by $\mu_2(E) = \mu(E \setminus E_o)$, ($E \in$ **R**) is ν-continuous.

Moreover the submeasure $\mu_1 + \mu_2$ is equivalent to μ.

Proof :

Let **H** be the collection of all ν-null sets in **R**. Since ν is σ-subadditive, **H** is a σ-ideal in **R**. By Theorem 4 there exists a set E_o in **H** such that $\mu_1 : E \to \mu(E \cap E_o)$ is nearly supported on **H** which implies that μ_1 is ν-singular and $u_2 : E \to \mu(E \setminus E_o)$ ($E \in$ **R**) vanishes on **H** which gives that μ_2 is ν-continuous. Suppose that E is in **R**.

Then $(\mu_1 + \mu_2)(E) = \mu(E \cap E_o) + \mu(E \setminus E_o)$ and so, since μ is monotone and q is an l–quasi-norm, we have $(q(\mu_1 + \mu_2)(E)) \leq 2q(\mu(E))$, which implies that $\mu_1 + \mu_2$ is μ-continuous. Conversely we have

$$q(\mu(E)) \leq q(\mu(E \cap E_o) + \mu(E \setminus E_o)) \leq q((\mu_1 + \mu_2)(E));$$

and so μ is $(\mu_1 + \mu_2)$ -continuous. Thus $\mu_1 + \mu_2 \sim \mu$.

Remarks :

(i) The proof of theorem 1 indicates that the theorem is valid for submeasures with values in any locally solid l-group (G, τ).

(ii) The decomposition of submeasure μ in theorem 5 is unique up to equivalent submeasures (see [6], *p.* 97).

Finally for group-valued submeasures we have following 'uniform boundedness theorem' which generalizes a result due to Drewnowski ([3], Theorem 1).

Theorem 6

Let (G, q) be an l-quasi-normed group and let M be a family of G-valued submeasures on a σ-ring **R** such that

$$\sup_{\mu \varepsilon M} q(\mu(E)) < +\infty$$

for each E in **R**. Then $\sup_{\substack{\mu \varepsilon M \\ E \varepsilon \mathbf{R}}} q(\mu(E)) < +\infty$

Proof :

Let H be the group of all G-valued mappings on M. Clearly H is a commutative partially ordered group, the ordering being $f \leq g$ if and only if $f(\mu) \leq g(\mu)$ for all $\mu \varepsilon M$. We define the functional ϕ on H by

$$\phi(f) = \sup_{\mu \varepsilon M} q(f(\mu))$$

and note that ϕ is an $\mathbf{R}_+^*$ –valued quasi-norm on H such that $\phi(f) \leq \phi(g)$ if $0 \leq f \leq g$. We define a mapping $\nu : \mathbf{R} \rightarrow H$ by

$$\nu(E)(\mu) = \mu(E).$$

Clearly ν is an H-valued submeasure on **R**.

Suppose that the theorem is not true. Then with the above notation, $\sup_{E \varepsilon \mathbf{R}} \phi(\nu(E)) = +\infty$. Thus for each positive integer n, there exists a set E_n in **R** such that $\phi(\nu(E_n)) > n$. Let $E = \bigcup_{n=1}^{\infty} E_n$.

Since **R** is a σ-ring, $E \varepsilon \mathbf{R}$ and $\phi(\nu(E)) = +\infty$; this implies that $\sup_{\mu} q(\mu(E)) = +\infty$, which contradicts the hypothesis. Thus

$$\sup_{\substack{\mu \varepsilon M \\ E \varepsilon \mathbf{R}}} q(\mu(E)) \text{ is finite, as required.}$$

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PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS OF L_i (s) TYPE

By

SYED MEHBOOB ALI SHAH

Department of Statistics.
University of Sind, Jamshoro, Pakistan

1. Summary and Introduction.

Partially balanced incomplete block designs with two associate classes and having the latin square type of association scheme are presented, The general treatment brings out relations between the class studied and certain other closely related designs. Including balanced incomplete block designs, regular and semiregular group divisible designs, and various square lattice designs.

We shall use s to reprsent the size of the square array of $v = s^2$ treatments in the association scheme. The association scheme is specified by this square array and $i-2$ mutually orthogonal latin squares $(i \geq 2)$ which are superposed on to the square array to identify first and second associates of any treatment. Treatment pairs which appear in the same row or column of the square array or which correspond to the same letter of a superposed latin square are first associates, otherwise a treatment pair is related as second associates.

For latin square type designs

$$v = s\,, \quad n_1 = i\,(s-1),\; n_2 = (s-1)\,(s-i+1), \tag{1.1}$$

and the parameters of the second kind are given by

$$P_1 = (p^1_{ij}) = \begin{bmatrix} i^2 - 3\,i + s & (i-1)\,(s-i+1) \\ (i-1)\,(s-i+1) & (s-i)\,(s-i+1) \end{bmatrix},$$

$$P_2 = (p^2_{ij}) = \begin{bmatrix} i\,(i-1) & i\,(s-i) \\ i\,(s-i) & (s-i)^2 + i - 2 \end{bmatrix} \tag{1.2}$$

and are determined by the association scheme.

The symbol $L_i(s)$ is used to designate a latin square type design of $v = s^2$ treatments and utilizing $i - 2$ mutually orthogonal latin squares in the specification of its association scheme. Thus a latin square type $L_2(s)$ design requires no latin squares in the definition of the association scheme but defines first and second associates of all treatments by use of the square array of s^2 treatment numbers. A latin square type $L_3(s)$ design is always possible for any finite integral value of $s \geq 2$ because it is always possible to construct a latin square of side (or order) s. However, for $i \geq 4$ the availability of mutually orthogonal latin squares depends upon the parameter s. The $L_4(s)$ designs have definable association schemes for all s except $s = 2$ and 6 [4]. It is well known that $s = p^n$, p a prime and n a positive integer, a full set of $s-1$ mutually orthogonal latin squares of side s exist. Hence when $s = p^n$ we can always define the association scheme for an $L_i(s)$ type design for $2 \leq i \leq s + 1$. However, when s is a composite member, the number of available mutually orthogonal latin squares varies with the value of s, presently being 2 for $s = 10$ and 5 for $s = 12$, for example.

2. Latin Square Type Designs with $k \leq s$.

Consider a latin square type association scheme defined by a square array of size s containing $v = s^2$ treatments and i constraints i.e. utilizing $i - 2$ mutually orthogonal latin squares, $2 \leq i \leq s + 1$. Treatment pairs which are second associates will never appear together in any block, so that blocks will be formed only from treatments which are first associates. Fix attention on any row (column) of s treatments in the association scheme and form the C^s_k distinct blocks that are possible by taking all combinations of s treatments k at a time ($k \leq s$). This provides $2s\,C^s_k$ blocks, each of size k, when we use the above procedure for every row and every column of the association Scheme. Now superpose one of the $i - 2$ mutually orthogonal latin squares on to the association scheme and fix attention on the s treatments corresponding to a latin letter. As before, use this set of s treatments to construct all the C^s_k distinct blocks of size k, do this for every latin letter of the square, and then repeat the procedure for each of the remaining latin squares, thus obtaining $(i-2)\, s\, C^s_k$ additional blocks. Clearly, we will have $b = i\, s\, C^s_k$ blocks of size k involving $v = s^2$ treatments. In each subset of blocks, a treatment will occur C^{s-1}_{k-1} times if it appears at all, and it occurs in subsets corresponding to a row, a column, and to one of the letters of each of the $i-2$ latin squares, so that $r = i\, C^{s-1}_{k-1}$. A treatment-pair related as first associate appears in blocks corresponding to either a row or a column or to a letter of one of the $i-2$ latin squares and in this subset of blocks occurs C^{s-2}_{k-2} times, hence $\lambda_1 = C^{s-2}_{k-2}$. Thus we have constructed a latin square type design specified by the following set of integral parameters

$$v = s^2 \quad , \quad b = i\, s\, C^s_k \;, \; n_1 = i\,(s-1) \quad , \; \lambda_1 = C^{s-2}_{k-2},$$
$$\ldots \text{(A)}$$
$$r = i\, C^{s-1}_{k-1} \;, \; k = k \qquad n_2 = (s-1)\,(s-i+1), \lambda_2 = 0 \quad ,$$

where $2 \leq i \leq s+1$ and $2 \leq k \leq s$. One may check that this three parameter family of designs satisfied the necessary conditions

$$v\, r = bk \qquad , \; n_1\, \lambda_1 + n_2\, \lambda_2 = r\,(k-1) \qquad (2.1)$$

which apply to any partially balanced incomplete block design with two associate classes. The other arithmetic conditions are automatically satisfied due to the fact that the design was constructed with all requirements of the $L_i(s)$ association scheme satisfied special case $i = s+1$ and $k = s$. When s is a prime or prime power, i can attain the maximum value and the latin square type $L_{s+1}(s)$ design may be constructed as described. When $k = s$ also, the parameter values of (A) reduce to

$$v = s^2 \,, \; b = s\,(s+1) \,, \; r = s+1, \; k = s, \; \lambda = 1 \qquad (2.2)$$

which is the well known family of balanced square lattice designs. This is so because $n_2 = 0$ and then there is only one kind of association between treatments, namely that in which every pair of treatments occurs together in exactly one block. Special case $i = s + 1$, $k < s$. When $k < s$ and $i = s + 1$ the parameters of (A) reduce to

$$v = s^2,\ b = s(s+1)\, C_k^s\ ,\ r = (s+1)\, C_{k-1}^{s-1}\ ,\ k < s,$$

$$\lambda = \lambda_1 = C_{k-2}^{s-2}\ , \qquad (2.3)$$

a family of balanced incomplete block designs, because $n_2 = 0$ and we have only one kind of association relation between treatments.

Special case $i = s$. When $i = s$, $p^2_{12} = 0$ which identifies the designs as group divisible. Construct the design as described but afterwards rename the associate classes. This leads to the parameter values.

$$v = s^2\ ,\quad b = s^2\ C_k^s\ ,\quad n = n_2 + 1 = s\ ,\quad \lambda_1 = 0,$$

$$(2.4)$$

$$r = s\ C_{k-1}^{s-1}\ ,\ k = k\ ,\qquad m = \frac{n_1}{n} + 1 = s\ ,\quad \lambda_2 = C_{k-2}^{s-2}\ ,$$

$$P_1 = \begin{bmatrix} s-2 & 0 \\ 0 & s(s-1) \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & s-1 \\ s-1 & s(s-2) \end{bmatrix} \qquad (2.5)$$

Now $r\,k - v\,\lambda_2 = s\,(s-k)\,/\,(k-1)\ C_{k-2}^{s-2}\ ,$

When $s = k$, $rk = v\,\lambda_2 = 0$ and $r - \lambda_1 > 0$, so that, by definition, the designs are a family of semi-regular group divisible designs. The cases $s = 6$ and 10 are not constructable (by this method at least) but the cases $s = 2, 3, 4, 5, 7, 8$, and 9, having $r \leq 10$ are known [2]. Many designs of this subfamily having $r > 10$ may be constructed by the method of this section.

Again we note that for $i = s$ it is necessary to have $i - s = s - 2$ mutually orthogonal latin squares available in order to define the association scheme and construct the design. This requirement is surely met when $s = p^n$, a prime or prime power.

Special case $k = s$, when $k = s$, the maximum possible value, we obtain from the family of designs (A) the subfamily with parameters.

$$\begin{aligned} v &= s^2, \quad b = is, \quad n_1 = i(s-1), \quad \lambda_1 = 1, \\ r &= i, \quad k = s, \quad n_2 = (s-1)(s-i+1), \quad \lambda_2 = 0, \end{aligned} \tag{2.6}$$

where $2 \leq i \leq s+1$. This is recognized as the well-known family of square lattice designs with i replications.

Special case $i = 2$. When $i = 2$ the family of designs (A) simplifies to those specified by

$$\begin{aligned} v &= s^2, \quad b = 2s\, C^{s}_{k}, \quad n_1 = 2(s-1), \quad \lambda_1 = C^{s-2}_{k-2}, \\ r &= 2\, C^{s-1}_{k-1}, \quad k = k, \quad n_2 = (s-1)^2, \quad \lambda_2 = 0, \; k \leq s, \end{aligned} \tag{2.7}$$

When $k = s$ we have from the preceding paragraph, the subfamily of simple square lattice designs with $r = 2$.

Special case $i = 3$. The designs of the subfamily of (A) with $i = 3$, are given by

$$\begin{aligned} v &= s^2, \quad b = 3s\, C^{s}_{k}, \quad n_1 = 3(s-1), \quad \lambda_1 = C^{s-2}_{k-2} \\ r &= 3\, C^{s-1}_{k-1}, \quad k = k, \quad n_2 = (s-1)(s-2), \quad \lambda_2 = 0 \end{aligned} \tag{2.8}$$

where $2 \leq k \leq s$ and we place the restriction $s \geq 4$ in order to avoid group divisible designs. Again the designs having $k = s$ are recognized to be the square lattices in $r = 3$ replication cases $i \geq 4$. Designs of family (A) having $i \geq 4$ are generally large $(r > 10)$ and except for the cases $k = s$, $i = s$ and $i = s + 1$ are thought to be new.

3. Latin Square Type Designs with $k = 2s$.

Consider the general association scheme of a latin square type design with $v = s$ treatments and involving $i - 2$ multually orthogonal latin squares, $2 \leq i \leq s + 1$, in the definition of first and second associates. We desire to construct blocks of size $k = 2s$ and do this in two stages. First, we form blocks by combining all possible pairs of rows (columns) of treatments in the square array of the association scheme. This yields $2\ C_2^s$ blocks, each of size $k = 2s$. Next superpose one of the mutually orthogonal latin squares are square array and form blocks from the treatments corresponding to pairs of latin letters, obtaining C_2^s block, each of size $k = 2s$. Repeat this procedure of block formation for each of the $i - 2$ mutually orthogonal latin squares. This yields a total of $b = i\ C_2^s$ blocks, each of size $k = 2s$, from the $v = s^2$ treatments. An arbitrarily chosen treatment is seen to occurs/ in $s - 1$ blocks of each of the i subroutines of block formation, so that $r = i\,(s - 1)$. A pair of first associates appears together in $s - 1$ blocks in the subroutine of block formation in which the treatment pair occurs in the same row or column of the association seheme or correspond to the same letter of a latin square, further, each pair of first associates occurs in one block in each of the other $i - 1$ subroutines of block formation. Thus $\lambda_1 = s + i - 2$. A treatment pair related as second associates occurs in exactly one block in each of the i subroutines of block formation, so that $\lambda_2 = i$. We have thus constructed a latin square type design with parameters.

$$v = s^2 \quad , \quad b = i\ C_2^s \quad , \quad n_1 = i\,(s - 1) \quad , \quad \lambda_1 = s + i - 2$$

$$\ldots \text{(B)}$$

$$r = i\,(s - 1) \quad , \quad k = 2\,s \quad , \quad n_2 = (s - 1)\,(s - i + 1) \quad , \quad \lambda_2 = i$$

where $2 \leq i \leq s + 1$. Clearly, the necessary conditions specified by (2.1) are satisfied.

Special case $i = s + 1$. Again the designs may be constructed when i attains the maximum value, $s + 1$, provided $s = p^n$, a prime or prime power. The subfamily with $i = s + 1$ has parameters

$$v = s^2 , b=(s+1)(s)(s-1)/2, r= s^2-1, k=2s, \lambda=\lambda_1=2s-1 \quad (3.1)$$

because $n_2 = 0$ so that there is only one kind of association relation among treatments. For these designs to be balanced incomplete block designs it is necessary that $r \geq k$, which in this instance requires that $s \geq 3$. However, the case $s = 2$ is clearly constructible and is in fact the randomized complete block design with $v = k = 4$ and $b = r = \lambda = 3$. The balanced incomplete block designs corresponding to $s=3$ and 4 are known and those with $s \geq 5$ have $r \geq 24$. Special case $i = s$. When $i = s$ the designs $L_s(s)$ are surely constructible if $s = p^n$, a prime or prime power. Setting $i = s$ we see from (1.2) that $p^2_{12} = 0$ which, upon renaming associate classes, leads to

$$v = s^2 , b=s^2(s-1)/2 , n=n_2+1=s , \lambda_1 = s,$$
$$\ldots \quad (3.2)$$
$$r = s(s-1) , k=2s , m=n_1/n+1=s, \lambda_2=2(s-1)$$

and the parameters of the second kind are automatically determined by setting $i = s$ in (1.2). The quantity $rk - v\lambda_2 = 0$ implies that if the designs have two associate classes, they are of the semi-regular type of group divisible designs, this requires $s \geq 3$. The case $s = 3$ is known [2] but those arising from $s \geq 4$ have $r \geq 12$ and may be new. In the case $s = 2$, $n_1 = 2$, $n_2 = 1$ and $\lambda_1 = \lambda_2 = 2$ so that the two associate classes collapse to one and the design exists as the randomized complete blocks design.

Special case $i = 2$. When $i = 2$ we obtain the subfamily of (B) of type $L_2(s)$ with parameters.

$$v = s^2 , b = s(s-1) , n_1 = 2(s-1) , \lambda_1 = s$$
$$\ldots \quad (3.3)$$
$$r = 2(s-1), k = 2s , n_2 = (s-1)^2 , \lambda_2 = 2, s \geq 3$$

The design arising from $s = 3$ is given in [2], the cases $s = 4$ and 5 were constructed by [5], and those cases with $s \geq 6$

($r \geq 10$, $k \geq 12$) have not been previously reported in the literature. Special case $i = 3$. When construction utilizes a simple latin square, we have the subfamily of type $L_3(s)$ with parameters.

$$v = s^2 \quad , b = 3s(s-1)/2, \; n_1 = 3(s-1) \quad , \lambda_1 = s+1,$$
$$r = 3(s-1), \; k = 2s \quad , n_2 = (s-1)(s-2), \; \lambda_2 = 3 , \quad \ldots (3.4)$$

Where $s \geq 4$ and all designs are possible for finite s since it is always possible to construct one latin square of side s.

Special case $i = 4$. The subfamily $L_4(s)$ of designs of (B) with $i = 4$ and $s \geq 5$ utilize 2 mutually orthogonal latin squares. These designs are always constructible for $s = 5$ and $s \geq 7$ as [4] have shown that it is always possible to construct a pair of mutually orthogonal latin squares when $s \geq 3$ except for the special case $s = 6$ when Euler's conjecture holds true. The designs of the subfamily $L_4(s)$ are large, the smallest having $r = 16$.

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ON THE ELLIPTIC DIFFERENTIAL EQUATION

$$\delta_2^{\dagger} \alpha = -1 - \varepsilon e^{2\alpha} \; ; \varepsilon = 0, \pm 1$$

By
M—K. GABR

Tanta University
Egypt

Abstract

The constant solutions of the elliptic differential equation $\delta_2 \alpha = -1 - \varepsilon e^{2\alpha}$; $\varepsilon = 0, \pm 1$ are Niveaulines of different types. We are concerned in this article with one of them, namely. the concentric circles $|z| = r < 1$. In association with this type, the positive solutions, $f(\alpha)$, of the equation

$$-ff'' + f'^2 - 3Lf' - 2f + 3L'f + 2L^2 = 0, \text{ where}$$

$$L = -1 - \varepsilon e^{2\alpha}; \; \varepsilon = 0, \pm 1$$

which lead to a family of convex funtions $w(z)$ are represented and those having closed curves are drawn.

1. Introduction

Consider the hyperbolic metric in the z-plane and the ε-metric $\dfrac{|dw|}{1 + \varepsilon w\bar{w}}$; $\varepsilon^{*} = 0, \pm 1$ in the w-plane. Let us also suppose that $w(z)$, with nonvanishing derivative, is regular in $z\bar{z} < 1$. Hence, we have the differential invariant $\alpha = \log \dfrac{|w'|(1 - z\bar{z})}{1 + \varepsilon w\bar{w}}$ as a solution of

(†) $\delta_2(\,) := (1 - z\bar{z})^2 (\,)_{z\bar{z}}$ is the second Beltrami operator.

(*) According to $\varepsilon = 0, +1, -1$, we have the Euclidean the elliptic and the hyperbolic metric respectively.

the elliptic partial differential equation $\delta_2 \alpha = L(\alpha)$, where δ_2 is the second Beltrami operator and $L(\alpha) = -1 - \varepsilon e^{2\alpha}$ [1]. The equation $\delta_2 \alpha = L(\alpha)$ is a nonhomogeneous potential equation and its constant solutions represent different types of Niveaulines ([7], p. 363, 377).

We are concerned here with an important model of these Niveaulines, namely, the concentric circles $|z| = r < 1$ with a charge at centre. The lines of forces are the family of straight lines originating from the charge. This represents the field of a line charge situated perpendicular to the plane of the page at the origin and surrounded by an insulated co-axial cylinder with radius $r = 1$.

The nonlinear differential equation

$$-ff'' + f'^2 - 3Lf' - 2f + 3L'f + 2L^2 = 0 \qquad \ldots\ldots (1)$$

plays a particular role in classifying three types of Niveaulines, as shown in section 1. In accordance with each type we derive a class of positive solutions of Eq. (1).

Our aim in this article is to represent those positive solutions, $f(\alpha)$, of Eq. (1) which are related to the considered type of Niveaulines, and lead in the same time to a family of convex functions $w(z)$. Furthermore, some of the obtained solutions, $f(\alpha)$, having closed curves, are depicted graphically.

2. The main approach

Under the conditions [1] :

1. $f(\alpha)$ is a positive solution for Eq. (1).
2. $f'' - 2L' + 2 \neq 0$.
3. $f(\alpha) > \delta_1 \alpha$ in the neighbourhood of $|z| = 1$,

it has been proved that $f(\alpha) \geqq \delta_1 \alpha$ is valid in $|z| \leqq 1$. The solutions of Eq. (1) are given in [6].

Let us consider those positive solutions of Eq. (1) such that the inequality $f(\alpha) \geqq \delta_1 \alpha$ reduces to an equality i.e. we consider the majorant functions of the inequality. For $\alpha = \alpha(\theta)$, where θ is a parameter to be selected such that $\delta_1 \theta = 1$, the condition $\delta_1{}^* \alpha \leqq f(\alpha)$ implies

$$\theta + c = \int \frac{d\alpha}{\sqrt{f(\alpha)}}, \qquad \text{...... (2)}$$

where c is an arbitrary constant.

Hence, calculating $\delta_2 \theta := (1 - z\bar{z})^2 \theta_{z\bar{z}}$

$$= (1 - z\bar{z})^2 \left\{ \frac{1}{\sqrt{f}} \cdot \alpha_{z\bar{z}} - \frac{1}{f} \cdot \frac{d\sqrt{f}}{d\alpha} \alpha_z \alpha_{\bar{z}} \right\}$$

$$= \frac{L(\alpha)}{\sqrt{f(\alpha)}} - \frac{d\sqrt{f(\alpha)}}{d\alpha},$$

which can be formulated as a function of θ. Let us denote this function by $\psi(\theta)$. Thus, differentiating w.r.t. θ the expression

$$\psi(\theta) \quad := \frac{L(\alpha)}{\sqrt{f(\alpha)}} - \frac{d\sqrt{f(\alpha)}}{d\alpha}, \qquad \text{...... (3)}$$

and noting that $f(\alpha)$ satisfies Eq. (1), we obtain :

$$\frac{d\psi}{d\theta} = \frac{d\psi}{d\alpha} \cdot \frac{d\alpha}{d\theta}$$

$$= \sqrt{f} \left\{ \frac{1}{\sqrt{f}} \cdot \frac{dL}{d\theta} - \frac{L}{f} \cdot \frac{d\sqrt{f}}{d\theta} - \frac{d^2\sqrt{f}}{d\theta^2} \right\}$$

$$= 1 - \psi^2 \qquad \text{...... (4)}$$

From (4) it follows immediately that $\psi = \pm 1$, $\tanh(\theta + c)$ and take over $(\theta + c)$, $\theta + c \neq 0$ according to $|\psi| = 1$, $|\psi| < 1$ and $|\psi| > 1$, which can be deduced by using Eq. (3).

Thus, by making use of Eq. (3) all the positive solutions of Eq. (1) are classified in three classes. Each class yields consequently one of the following types of Niveaulines [5] :

(*) $\delta_1(\ ) := (1 - z\bar{z})^2 (\ )_z (\ )_{\bar{z}}$ is the first Beltrami operator.

(i) Circular lunes lying in the unit disk of the z - plane with vertices at $z = \pm 1$.

(ii) Circles lying inside $|z| = 1$ and touching it at $z = +1$.

(iii) Concentric circles with centre at $z = 0$ and radius $r < 1$.

The first type has been studied in [2] and the second type is considered only for $\varepsilon = 0$ in [3]. In this article we are concerned with the third type considering $\varepsilon = 0, \pm 1$.

To characterize this type note that $w = \log z$ maps the concentric circles $|z| = r < 1$ conformally onto the straight lines which are parallel to the imaginary axis and lying in the left half-plane. Hence, let $\alpha = \alpha(z, \bar{z}) = \alpha(r^2)$. For the majorant function $\delta_1 \alpha \leqq f(\alpha)$, we have

$$\delta_1 \alpha = (1 - z\bar{z})^2 \alpha_z \alpha_{\bar{z}} = r^2 (1-r^2) \alpha'^{\,2} \qquad \left(r^2 = z\bar{z}, ' = \frac{d^2}{dr^2} \right)$$

$$= f(\alpha)$$

which implies

$$\frac{d\alpha}{\sqrt{f(\alpha)}} = \frac{dr}{1 - r^2} \,.$$

From Eq. (2) we obtain

$$\theta + c = -2 \tanh^{-1} r$$

On calculating $\delta_2 \theta$, we have

$$\delta_2 \theta = \frac{1 + r^2}{2r} = \coth(\theta + c), \ \theta + c \neq 0.$$

Thus, we conclude that the third type of Niveaulines is characterized by $|\psi| > 1$.

Upon using Eq. (3), we determine ψ according to each $f(\alpha) > 0$. Doing this we obtain the set of positive solutions of Eq. (1) related to this type. Another set of $f(\alpha)$ is induced by selecting those leading to a family of convex functions $w(z)$. The intersection of these two sets is then to be determined.

3. A family of convex functions

The definition of the differential invariant α is $\log \dfrac{|w'|(1-z\bar{z})}{1+\varepsilon w\bar{w}}$;

$\varepsilon = 0, \pm 1$, $w(z)$ is regular in $|z| < 1$ with nonvanishing derivative. Corresponding to each positive solution of Eq. (1), the curvature k_ε of the boundary of $w(|z|<1)$ is defined and bounded from below as follows [1] :

$$k_\varepsilon = \frac{1+\varepsilon w\bar{w}}{|w'|} \operatorname{Re}\left\{ 1 + z\left(\frac{w''}{w'} - 2\varepsilon \frac{\bar{w}w'}{1+\varepsilon w\bar{w}} \right) \right\}$$

$$= \lim \frac{1 - \delta_1 \alpha}{|w'|(1-z\bar{z})}$$

$$\geqq k^*,$$

where

$$k^* = \lim \frac{1 - f(\alpha)}{e^\alpha} \qquad \text{as} \qquad \alpha \to -\infty, \; f \to 1$$

$$= -2 \left.\frac{dg}{dx}\right|_{\substack{x=0\\ g=1}} \qquad \text{with} \quad x = e^\alpha \qquad g = +\sqrt{f}$$

It is clear that all $w(z)$ for which $k^* \geqq 0$ represent a family of convex functions.

Furthermore, estimations of $|w(z)|$, for each $f(\alpha) > 0$, are useful to obtain examples of convex functions $w(z)$. Thus, note that the inequality $f(\alpha) \geqq \delta_1 \alpha$ can be written in the form

$$\log \frac{1-r}{1+r} \leqq \int_{\alpha(0)}^{\alpha} \frac{d\alpha}{\sqrt{f(\alpha)}} \leqq \log \frac{1+r}{1-r}$$

and integrating we have an estimation for α, corresponding to each positive solution $f(\alpha)$ for Eq. (1).

Substituting $\frac{e^{\alpha}}{1-r^2} = \frac{|w'|}{1+\varepsilon w\bar{w}}$ from the definition of α, and integrating again to obtain an estimation for $\left|\frac{w(z)-w(0)}{1+\varepsilon\, w(z)\,\bar{w}(0)}\right|$, where the integration is carried out along the shortest distance between 0, $r = |z|$ and its image in the w-plane. Without loss of generality, let $w(0) = 0$ and then we get an estimation for $|w(z)|$, say

$$R_1(r) \leqq |w(z)| \leqq R_2(r), \qquad \ldots\ldots (5)$$

where $R_1(r)$ and $R_2(r)$ depend on the selected $f(\alpha)$.

4. Results

For each positive solution $f(\alpha)$ of Eq. (1) [6], we calculate k^*. Then, we derive that class of $f(\alpha)$ leading to positive k^* (i.e. a family of convex functions is induced). From this class, by using the approach discussed in section 1 we determine all $f(\alpha)$ which, moreover, are in corresponence with the third type of Niveaulines. These calculations yield the following solutions which are represented parametrically $(x = e^{\alpha},\ g = +\sqrt{f})$:

I. $\varepsilon = 0$

(1) $x = a.\ \sinh(a(t+b)).\ e^{-t},\quad g = \cosh(a(t+b)) - x.\ e^{t},\ a > 0, \neq 1;\ t, b \in R$

(2) $g^2 = Cx + 1$
$C \in R^-$

II. $\varepsilon = +1$

(3) $x = \frac{\sinh(a(t+b))}{a\cosh t},\qquad g = \cosh(a(t+b)) - x.\ \sinh t,\ a>0, \neq 1, b \in R^-,\ t \in R$

(4) $g^2 = 1 + Cx - x^2$
$C \in R^-$

III. $\varepsilon = -1$

(5) $x = \frac{t+b}{\sinh t},\qquad g = 1 - x.\ \cosh t$
$b \in R,\ t \in R^+$

$$(6)\ x = \frac{\sin t}{t+b}, \qquad g = \cosh t - x$$
$$b \in R,\ t \in R^{+}$$

$$(7)\ g^2 = 1 + Cx + x^2$$
$$C \in R^{-}$$

Now, we are in a position to formulate the following theorem :

Theorem.

In correspondence with each positive solution of the nonlinear differential equation

$$-ff'' + f'^2 - 3Lf' - 2f + 3L'f + 2L^2 = 0,$$

where $L = -1 - \varepsilon e^{2\alpha}$; $\varepsilon = 0, \pm 1$, we have

i. A family of convex functions $w(z)$, defined in $|z| < 1$, if

$$-2 \frac{d\sqrt{f(\alpha)}}{de^{\alpha}} \geqq 0.$$

ii. The Niveaulines α =const., which are solutions of $\delta_2 \alpha = L(\alpha)$ with $\delta_1 \alpha = f(\alpha)$ are the concentric circles $|z| = r < 1$ if

$$\left| \frac{L(\alpha)}{f(\alpha)} - \frac{d\sqrt{f(\alpha)}}{d\alpha} \right| > 1.$$

These conditions are satisfied, simultaneously, by the above seven types of $f(\alpha)$.

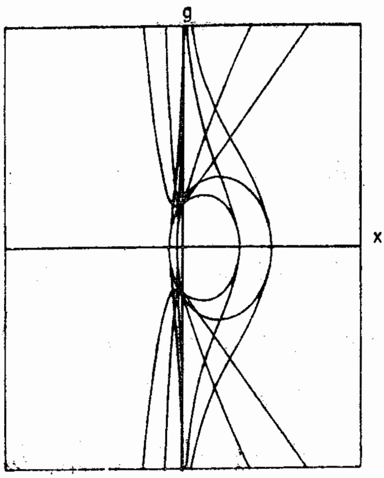
Related to some $f(\alpha)$ of above types, e.g. (1) and (2), we obtain examples of convex functions $w(z)$ as follows :

By evaluating $R_1(r)$ and $R_2(r)$ in inequality (5), we obtain interval estimations for $|w(z)|$. The majorant functions, where the inequalities reduce to equalities, represent the desired functions $w(z)$; they are respectively :

$$w(z) = A_1 \left(\frac{1}{l - mz} - \frac{1}{l} \right) \text{ and } w(z) = A_2 \left\{ \left(\frac{l + nz}{l + z/n} \right)^{1/a} - 1 \right\},$$

where A_1, A_2, l, m and n are constants ; by the second function Riemann surface is considered [1].

Lastly, the solutions (3) $\cup$ (4) and (3), (4) which are represented by closed curves are depicted graphically. This is of particular interest since Eq. 1 is nonlinear.



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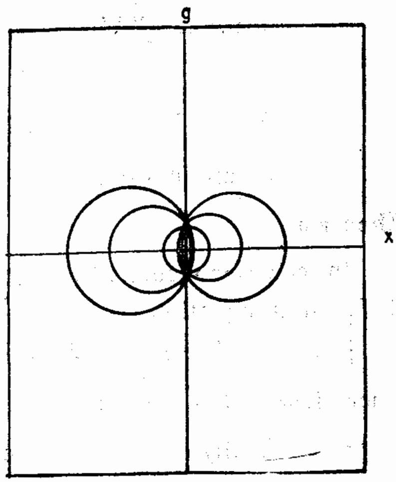

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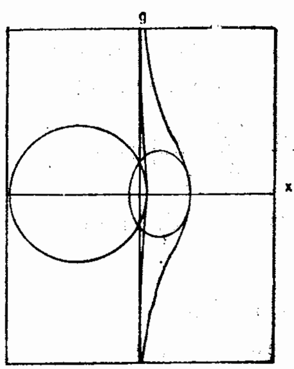

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SOME FAMILIES OF ANALYTIC FUNCTIONS CONSIDERING HYPERBOLIC METRIC

By

M.—K. GABR

Department of Mathematics,
Faculty of Science,
Tanta University, Egypt.

Abstract

Considering hyperbolic metric, some families of analytic functions are defined and an interval for a differential invariant is estimated. For one family, Bloch constant is evaluated and for another, Julia theorem together with an obtained theorem are used to get a covering theorem.

Introduction

In the unit disk of the z-plane $D_z = \{ z / z \bar{z} < 1 \}$, suppose that F denotes the family of all holomorphic functions $w(z)$ which have normalized power series development $w(z) = z + \Sigma_2^{\infty} C_\nu z^\nu$. In the z- and w-plane let us consider the hyperbolic metric which is defined through :

$$d S_z^2 = \frac{d z\, d \bar{z}}{(1 - z \bar{z})^2} \quad \text{and} \quad d S_w^2 = \frac{d w\, d \bar{w}}{(1 - w \bar{w})^2}.$$

Hence, we are concerned with hyperbolic geometry, where any point is represented by dual number [16]. The hyperbolic geometry is characterized by the fact that the sum of the angles in a triangle is less than π and is obtained from the euclidean geometry by replacing the fifth

Hilbert's axiom by the statement "through any point not lying on a line there are at least two parallel lines to it" [9]. The length of an arc of the curve $z(t)$ in the considered geometry is given by

$$\int_{t_0}^{t} \frac{|z'(t)|}{1 - z(t)\,\bar{z}(t)}\, dt.$$

For any unimodular bounded function $w(z) \in F$ one can show that the following function, $\phi(\zeta)$, belongs also to F [3] :

$$\phi(\zeta) = \frac{(1 - w\bar{w})}{(1 - z\bar{z})\, w'}\, e^{-i\theta} \left[\frac{w\left(\dfrac{z + e^{i\theta}\zeta}{1 + \bar{z}e^{-i\theta}\zeta}\right) - w(z)}{1 - \bar{w}(z)\, w\left(\dfrac{z + e^{i\theta}\zeta}{1 + \bar{z}e^{-i\theta}\zeta}\right)} \right],$$

where $\theta \in R$ and $z, \zeta \in D_z$.
This follows by applying the hyperbolic motions.

Let us now consider the following differential invariants :

$$e^{\alpha} \; := dS_w/dS_z = \frac{|w'|\,(1 - z\bar{z})}{1 - w\bar{w}},$$

$$\gamma \; := \delta_1 \alpha = (1 - z\bar{z})^2 \alpha_z \alpha_{\bar{z}} :$$

$$= \beta\bar{\beta} \quad \text{(say)},$$

$$\text{where} \quad \beta = (1 - z\bar{z})\,\alpha_z .$$

For $u = \delta_1 \alpha - f(\alpha)$, where $f(\alpha) \in C_2$ is to be obtained. Calculation of $\delta_2 u := (1 - z\bar{z})^2 u_{z\bar{z}}$ yields :

$$\delta_2 u = \frac{1}{\gamma}\Big[\delta_1 u + (1 + f' - e^{2\alpha})\,\{\delta_1(\alpha, u) + \delta_1(u, \alpha)\}^{*)} \Big] + D_1 u + D_0 ,$$

where

$$D_1 = -f'' + 4e^{2\alpha} - 2$$

(*) $\delta_1(\alpha, u) = \delta_1(u, \alpha) = (1 - z\bar{z})^2 \alpha_z u_{\bar{z}}$ is the mixed Beltrami operator.

and

$$D_0 = -ff'' + (f'+1)(f'+2) - 3f'e^{2\alpha} + 4(f-1)e^{2\alpha} - 2f + 2e^{4\alpha}.$$

We are now in a position to get an inequality, which is obtained from Peschl's principle [10]. It represents one of the fundamental tools in this article. The next section is devoted to this aim.

1. Principle and inequality of Peschl.

Using the previous notations and definitions suppose that :

1. $w(z) \in F$
2. $f(\alpha, \lambda) \in C_2$ and $\frac{\partial f}{\partial \lambda} > 0$ for $\lambda \in [\lambda_0, \lambda_1]$, $\alpha \in (-\infty, \sup_{D_z} \alpha)$
3. $f(\alpha, \lambda)$ satisfies, simultaneously, $D_0 = 0$ aud $D_1 \neq 0$.
4. $u(z, \bar{z}; \lambda_0) < 0$ on $z\bar{z} = 1$ and $u(z, \bar{z}; \lambda_1) \leqq 0$ in $z\bar{z} \leqq 1$.

Then one obtains in $z\bar{z} \leqq 1$ the inequality $u(z, \bar{z}; \lambda_0) \leqq 0$ which leads to $\delta_1 \alpha \leqq f(\alpha)$ and this implies

$$|\beta| \leqq \sqrt{f(\alpha)} \qquad \ldots\ldots (1)$$

This result is known as Peschl's principle. Its proof is given in [2], [3], [10] by showing that $\delta_1 \alpha > 0$ leads to a contradiction. To get the desired inequality recall the definition of β and note that $z\alpha_z = \alpha_{\log z}$; then one has :

$$|\beta| = |\alpha_z \cdot (1 - z\bar{z})| = \frac{1-r^2}{2r} | r\alpha_r - i\alpha_\phi |, \text{ (for } z = re^{i\phi})$$

The last equality together with inequality (1) shows that

$$|\beta| \geqq \frac{1-r^2}{2r} | \alpha_r |. \qquad \ldots\ldots (2)$$

It follows from (1) and (2) :

$\frac{1-r^2}{2r} \mid \alpha_r \mid \leqq \mid \beta \mid \leqq \sqrt{f(\alpha)}$ which impiies

$$\frac{-2\,dr}{1-r^2} \leqq \int \frac{d\alpha}{\sqrt{f(\alpha)}} \leqq \frac{+2\,dr}{1-r^2}\,.$$

Given a solution $f(\alpha) > 0$ for the nonlinear differential equation $D_0 = 0$ in the parametric representation $\alpha = \alpha(t)$ and $f = f(t)$ then integration shows :

$$\log \frac{1-r}{1+r} \leq \int \frac{d\alpha(t)}{\sqrt{f(t)}} dt \leq \log \frac{1+r}{1-r} \qquad \text{...... (3)}$$

which is the aimed inequality.

Corresponding to any positive solution of $D_0=0$ [12], and applying inequality (3) one obtains estimations for the differential invariant α and $\mid w'(z) \mid / (1-w\bar{w}) := e^{\alpha} / (1-r^2)$ to get, by integration, an estimate interval for $\mid w(z) \mid$.

Using this approach number of works, with interesting results and families of functions have been developed [2], [3], [4], [5], [11], [14].

2. The family $\tilde{F}$ and Bloch Constant

Let $\tilde{F}$ denote the subclass of the family F whose functions are characterized by :

1. The hyperbolic curvature, K_h , of the boundary of the image of D_z , under $w(z) \in F$, is lower bounded.

2. $\mid w' \mid (1-z\bar{z}) / (1-w\bar{w})$ is upper bounded.

Now, we select one of the positive solutions of $D_0 = 0$ which yields an estimation for $\mid w(z) \mid$, $w(z) \in \tilde{F}$, when applying the above method. The selected solution is given in parametric form as follows [12] :

$$e^{\alpha} := x(t) = \frac{\sinh(a(t+b))}{a \sinh(t)} = \frac{e^{ab}}{a} \cdot e^{a-1} \{1 + \ldots\} \qquad t > 0$$

$$\sqrt{f} := g(t) = -\cosh(a(t+b)) + x\cosh(t), \quad b \in R^{-}$$

$$= \frac{e^{ab}}{2a} \cdot e^{at}(1-a)\{1 + \cdots\}$$

$$a \in (0, 1) \cup (1, \infty)$$

Since inequality (3) requires that $x \geqq 0$, $g \geqq 0$ [3], then t must be on $[t_0, \infty]$, where $a.\tanh(t_0) = \tanh(a(t_0 + b))$. On the other hand by applying inequality (3) one obtains the following interval to the same parameter t : $\left[\frac{2}{a} \cdot \tanh^{-1}(AR) - b, \frac{2}{a} \cdot \tanh^{-1}(A/R) - b\right]$,

where $R = \frac{1-r}{1+r}$, $A = \tanh(a(t_1 + b)/2)$ and t_1 is determined through the condition $w'(0)=1$; namely $a.\sinh(t_1)=\sinh(a(t_1+b))$. To avoid any contradiction between the obtained intervals of t, we should have :

$$\frac{2}{a} \cdot \tanh^{-1}(AR) - b \leqq \infty \Rightarrow r \leqq \rho_1 := e^{-a}(t_1 + b)$$

and

$$\frac{2}{a} \cdot \tanh^{-1}(A/R) - b \geqq t_0 \Rightarrow r \leqq \rho_2 :$$

$$= \frac{\sinh(a(t_1 - t_0)/2)}{\sinh(a(t_1 + t_0 + 2b)/2)}.$$

Since $\frac{dx}{dt} < 0$, calculations show that

$$\frac{\sinh\{2\tanh^{-1}(A/R)\}}{a\sinh\left\{\frac{2}{a}\tanh^{-1}(A/R)-b\right\}} \underset{\leqq}{\overset{r \leqq \rho_1}{}} e^{\alpha} \underset{\leqq}{\overset{r \leqq \rho_2}{}} \frac{\sinh\{2\tanh^{-1}(AR)\}}{a\sinh\left\{\frac{2}{a}\tanh^{-1}(AR)-b\right\}}.$$

Hence, an estimation of $\frac{|w'|}{1-w\bar{w}} := \frac{e^{\alpha}}{1-r^2}$ follows immediately and integration from 0 to r yields *) :

$$\log \sqrt{\frac{\tanh \frac{1}{2} \left\{ \frac{2}{a} \tan^{-1} (A/R) - b \right\}}{\tanh (t_1 / 2)}} \overset{r \leqq \rho_1}{\leqq} | w(z) | \overset{r \leqq \rho_2}{\leqq}$$

$$\log \sqrt{\frac{\tanh (t_1 / 2)}{\tanh \frac{1}{2} \left\{ \frac{2}{a} \tanh^{-1} (AR) - b \right\}}} \quad \cdots (4)$$

To determine Bloch constant of all functions satisfying formula (4) recall the following definition :

If B_f is the radius of the largest one-sheeted disk lying on the Riemann surface, onto which the functions $w = f(z)$ map the disk D_z bijectively, then Bloch constant B, for all $f(z)$, is defined by

$$\inf_{f(z)} B_f \; ^{**)}.$$

Hence, Bloch constant of the functions satisfying the inequality (4) is obtained by taking the limit of the L.H.S. as $r \to \rho_1$; which gives $\log \sqrt{\coth (t_1 / 2)}$.

Moreover, the hyperbolic curvature K_h for the boundary of the image of D_z under any estimated function, by (4), is given by [3] :

$$K_h = \frac{1 - w\bar{w}}{| w' |} \operatorname{Re} \left\{ 1 + z \left(\frac{w''}{w'} + \frac{2 \bar{w} w'}{1 - w\bar{w}} \right) \right\}$$

(*) Note that the shortest arc from 0 to any other point $z = re^{i\phi}$, $\phi \in R$, is along a radius. Hence the geodesics are circles orthogonal to the unit circle ; they can be considered straight lines in the hyperbolic geometry of the disk.

(**) Bloch constant is studied and estimated for some families of functions in [6], [13], [15].

$$= \lim_{\substack{\alpha \to -\infty \\ z\bar{z} \to 1}} (1-\gamma)\, e^{-\alpha} \; \dagger$$

Since $\gamma \leqq f(\alpha)$, it follows that K_h has, for the considered functions, the lower bound k, where

$$k = \lim_{\substack{\alpha \to -\infty \\ z\bar{z} \to 1}} (1-f)\, e^{-\alpha} = -2 \left. \frac{d\sqrt{f}}{d e^{\alpha}} \right|_{\substack{\alpha = -\infty \\ r = 1}}$$

$$= 2 \cosh (b).$$

Lastly, $\dfrac{|w'|\,(1 - z\bar{z})}{1 - w\bar{w}} := e^{\alpha} \leqq e^{\alpha^*}$, $\alpha^* = \sup_{D_z} \alpha$. It can be shown that $\alpha^* \in (\log(\operatorname{sech} \tau), b)$, where $\tau = \tanh(\tau) - b$.

We are now in a position to formulate the following distortion theorem to the family $\tilde{F}$ corresponding to the selected solution of $D_0 = 0$:

Theorem (1).

Each function $w(z) \in \tilde{F}$, satisfies the following :

1. $+\log \sqrt{\dfrac{\tanh(\{2 \tanh^{-1}(A/R) / a - b\} / 2)}{\tanh(t_1/2)}} \underset{r \leqq \rho_1}{\leqq} |w(z)| \underset{r \leqq \rho_2}{\leqq} -\log \sqrt{\dfrac{\tanh(\{2 \tanh^{-1}(AR) / a - b\}/2)}{2 \tanh(t_1/2)}}$,

where t_1, A, R, ρ_1, ρ_2 are defined as above.

2. $|w'|\,(1 - z\bar{z}) / (1 - w\bar{w})$ has an upper bound, lies in the interval ($\operatorname{sech} \tau$, e^b), where $\tau = \tanh \tau - b$.
3. The hyperbolic curvature for the boundary of the image of D_z, under $w(z)$, $\geqq 2 \cosh(b)$.
4. Bloch constant equals to $\log \coth(t_1/2)$.

(†) Here, the second branch e^{α}, $-\sqrt{f}$, where $f \nrightarrow 1$ as $e^{\alpha} \to 0$, is considered.

Corollary.

It can be proved that the majorant function of the first condition of theorem 1 ; i.e. for which the equality sign holds, does not belong to the family $\tilde{F}$.

3. Applying Julia Theorem to get a Covering Theorem.

Let $\tilde{\tilde{F}} \subset F$ be the family of functions, for which theorem (1) holds in the whole unit disk D_z ; this means that $\tilde{\tilde{F}}$ is an extension of $\tilde{F}$ on D_z .

For any sequence z_ν on the real axis such that $w(z_\nu) \to 1$, the angular and boundary derivatives are defined, respectively, as follows :

$$\delta = \lim_{\nu\to\infty} \frac{1-|w_\nu|}{1-|z_\nu|} \quad \text{and} \quad \lambda = \lim_{\nu\to\infty} \frac{1-e^{\alpha_\nu}}{1-|z|^2}.$$

By applying Julia theorem [7], [8] one obtains

$$\frac{\delta}{4\sqrt{2\lambda}} \leqq \frac{|1-w|^2}{1-|w|^2} \leqq \frac{\delta}{2\sqrt{2\lambda}} \qquad \dots (5)$$

For each $w(z) \in \tilde{\tilde{F}}$; the above inequality means that the image of the hyperbolic disk $\frac{|1-z|^2}{1-|z|^2} \leq \frac{1}{2\sqrt{2\lambda}}$, under $w(z)$, lies in

$$\left(\frac{|1-w|^2}{1-|w|^2} \leqq \frac{\delta}{2\sqrt{2\lambda}}\right) \cap \left(\frac{|1-w|^2}{1-|w|^2} \geqq \frac{\delta}{4\sqrt{2\lambda}}\right).$$

To make use of this result suppose that

$$H_1(r) \leqq |w| \leqq H_2(r) \quad \text{and} \quad h_1(r) \leq e^{\alpha} \leqq h_2(r).$$

Since $\frac{\delta^2}{\lambda} \to \frac{(1-|w|^2)^2}{1-e^{\alpha}}$, one obtains as $r \to 1$:

$$\frac{(1-H_2^2)^2}{1-h_1} \leqq \frac{\delta^2}{\lambda} \leqq \frac{(1-H_1^2)^2}{1-h_2}$$

Using (5) this inequality reduces to

$$\frac{1-H_2^2}{4\sqrt{(1-h_1)}} \leqq \frac{|1-w|^2}{1-|w|^2} \leqq \frac{1-H_1^2}{2\sqrt{2(1-h_2)}}\ .$$

Following this approach together with theorem (1) one gets :

$$\frac{1}{4\sqrt{2}}\left\{1-\left(\log\sqrt{\frac{\tanh(1-b/2)}{\tanh(t_1/2)}}\right)^2\right\} \leqq \frac{|1-w|^2}{1-|w|^2}$$

$$\leqq \frac{1}{2\sqrt{2}}\left\{1-\left(\log\sqrt{\frac{\tanh(t_1/2)}{\tanh\frac{2-ab}{2a}}}\right)^2\right\}.$$

Hence the following covering theorem follows :

Theorem (2).

Each $w(z)\ \varepsilon\ \tilde{\tilde{F}}$ maps, at least the hyperbolic disk $\frac{|1-z|^2}{1-|z|^2} \leqq \frac{1}{2\sqrt{2\lambda}}$ in $|w| < 1$, bijectively, and its image covers a domain such that

$$\frac{1}{4\sqrt{2}}\left\{1-\left(\log\sqrt{\frac{\tanh(-b/2)}{\tanh(t_1/2)}}\right)^2\right\}$$

$$\leqq \frac{|1-w|^2}{1-|w|^2} \leqq \frac{1}{2\sqrt{2}}\left\{1-\left(\log\sqrt{\frac{\tanh(t_1/2)}{\tanh\frac{2-ab}{2a}}}\right)^2\right\}$$

for each $b\ \varepsilon\ R^-$, $a\ \varepsilon\ (0,1) \cup [1,\infty]$ and t_1 is obtained from $w'(0)=1$.

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AUTOMORPHISM GROUPS OF CERTAIN METABELIAN P-GROUPS OF MAXIMAL CLASS

By
G. Q. ABBASI
Islamia University
Bahawalpur

The order of the automorphism groups of metabelian of maxima l class have been discussed by Miech [1970, 1978] and [1977]. However they are not too explicit. In this paper we introduce a simple method to calculate the order of the automorphism groups for certain metabelian p-groups of maximal class and order p^n, $n \geq 4$ and prove that :

"for every divisor d of $p-1$ there exists a metabelian p-group G of maximal class and order p^n , $n \geq 4$ such that the order of its automorphism group Aut (G) restricted on the centre Z (G) of G is exactly d."

Before any details are given background material is,

Let G be a group. The *lower central series*

$$G \geq \gamma_1 (G) \geq \gamma_2 (G) \geq \ldots\ldots\ldots\ldots \geq \gamma_{i-1} (G) \geq \gamma_1 (G) \geq \ldots\ldots\ldots\ldots$$

of G is defined inductively by $\gamma_2 (G) = [G , G]$ and γ_i (G) $= [\gamma_{i-1} (G), \gamma_{i-1} (G)]$ for $i \varepsilon \{3, 4, \ldots \}$. If γ_2 (G) is abelian, then G is said to be *metabelian*. If there exists an nteger c such that $\gamma_c (G) = E$, then G is said to be *nilpotent* and if c is the least positive such integer then G is said to have *nilpotency class* $c - 1$.

It is well known that a group G of order p^n, $n \geq 4$ have nilpotency class no larger than $n-1$. Groups of order p^n having nilpotency class exactly $n - 1$ are called groups of *maximal nilpotency*. From now on groups of maximal nilpotency class will be called simple groups of *maximal class*.

1. References are given by the author and the date of publication appears in square brackets.

If G is a group of maximal class having order p^n, then the subgroup $\gamma_1(G)$ can be defined as the largest subgroup of G such that $[\gamma_1(G), \gamma_2(G)] \leq \gamma_4(G)$., and it exists if $n \geq 4$. The general theory of these groups can be found in Huppert [1967].

A metabelian p-group G of maximal class and order p^n, $n \geq 4$ can be described in terms of a set of parameters

$$(\alpha(p-2), \alpha(p-1), \ldots\ldots, \alpha(1), \beta, \gamma)$$

(see for example Miech [1978], theorem 1) with the following presentation.

(1) $G = < u_0, u_1, \ldots\ldots, u_{n-1} >$ where $u_0 \in G$ but not in $\gamma_1(G)$, $u_1 \in \gamma_1(G)$ but not in $\gamma_2(G)$ and $u_i = [u_{i-1}, u_0]$ for $i = 2, 3, \ldots\ldots, n-1$.

(2) $\left[u_1, u_2 \right] = u_{n-p+2}^{\alpha(p-2)} \ldots\ldots u_{n-1}^{\alpha(1)}$ where $0 \leq \alpha(p-2)$, $\ldots\ldots, \alpha(1) \leq p-1$.

(3) $u_0^p = u_{n-1}^{\beta}$ where $0 \leq \beta \leq p-1$.

(4) $(u_0 u_1)^p = u_{n-1}^{\gamma}$ where $0 \leq \gamma \leq p-1$.

(5) $u_i^{\binom{p}{1}} u_{i+1}^{\binom{p}{2}} \ldots\ldots u_{i+p-1}^{\binom{p}{p}} = e$ for $i = 2, 3, \ldots\ldots, n-1$.

Note that $\gamma_1(G)$ is abelian if all $\alpha(p-2), \alpha(p-1), \ldots\ldots, \alpha(1)$ are zero ; and $\gamma_1(G)$ is non abelian if at least one of them is non zero. To achieve our result it suffices to consider $\alpha(p-2) = \alpha(p-1) = \ldots = \alpha(2) = 0$, $\alpha(1) = \alpha$ either zero or 1. Thus, from now on, a metabelian p-group G of maximal class and order p^n where $n \geq 4$, is described by the set of parameters (α, β, γ) and the notation $G(\alpha, \beta, \gamma)$ is used ; that is ;

$$G(\alpha, \beta, \gamma) = < u_0, u_1, \ldots\ldots, u_{n-1} > \text{ such that}$$

(1) $\left[u_1, u_2 \right] = u_{n-1}^{\alpha}$ where α is either 0 or 1.

(2) $u_0^{p} = n_{n-1}^{\beta}$, $(u_0\ u_1)^{p} = u_{n-1}^{\gamma}$ where $0 \leq \beta,\ \gamma \leq p-1$.

(3) $u_i^{\binom{p}{1}}\ u_{i+1}^{\binom{p}{2}}\ \cdots\cdots\ u_{i+p-1}^{\binom{p}{p}} = e$ for $i = 2, 3, \ldots\ldots, n-1$.

It is known that an isomorphism between two groups, say, G (α, β, γ) and G $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is of the form

$$u_0 \rightarrow u_0^{\lambda}\ u_1^{a_1}\ \cdots\cdots\ u_{n-1}^{a_{n-1}}$$

$$u_1 \rightarrow u_1^{\mu}\ u_2^{b_2}\ \cdots\cdots\ u_{n-1}^{b_{n-1}}$$

where $0 < \lambda,\ \mu < p$ and $0 \leq a_i, b_j \leq p-1$ for $i \in \{1, \ldots\ldots, n-1\}$ and $j \in \{2, 3, \ldots\ldots, n-1\}$.

Lemma 1. Every map θ : G $(\alpha, \beta, \gamma) \rightarrow$ G (α, β, γ) defined as

$$u_0\ \theta = u_0\ u_2^{a_2}\ \cdots\cdots\ n_n^{a_{n-1}}$$

$$u_1\ \theta = u_1\ u_2^{b_2}\ \cdots\cdots\ u_{n-1}^{b_{n-1}}$$

is an automorphism.

Proof. Let $u_0\ \theta = v_0$ and $u_1\ \theta = v_1$. Define, inductively, $v_i = [\,v_{i-1}, v_0\,]$ for $i = 2, 3, \ldots\ldots, n-1$. It is easy to check that $v_0, v_1, \ldots\ldots, v_{n-1}$ satisfy all defining relations of G (α, β, γ) ; and Ker θ = E. Hence θ is an automorphism of G (α, β, γ).

Thus for $\lambda = 1 = \mu$ and $a_1 = 0$, p^{2n-4} autcmorphisms are fixed for each G (α, β, γ).

To calculate more automorphisms of these groups we consider only the maps of the type :

$$u_0 \rightarrow u_0^{\lambda}\ u_1 \quad \text{and} \quad u_1 \rightarrow u_1^{\mu}$$

where $0 < \lambda,\ \mu < p$ and $0 \leq a_1 \leq p-1$.
It is easy to prove that :

Lemma 2. The map $\psi : G(\alpha, \beta, \gamma) \rightarrow G(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ defined as

$$u_0\,\psi = u_0^{\lambda}\, u_1^{a_1} \quad u_1\,\psi = u_1^{\mu}$$

is an isomorphism if and only if

$$\text{(i)} \quad \lambda^{n-2}\,\mu\,\beta = \bar{\beta}\lambda + a_1(\bar{\gamma} - \bar{\beta}) - 2a_1^2\,\lambda\,\bar{\alpha} \quad \text{for } p = 3$$

$$\bar{\beta}\lambda + a_1(\bar{\gamma} - \bar{\beta}) \qquad \text{otherwise}$$

$$\text{(ii)} \quad \lambda^{n-2}\,\mu\,\gamma = \bar{\beta}\lambda + (a_1 + \mu)(\bar{\gamma} - \bar{\beta}) - 2(a + \mu)^2\,\lambda\,\bar{\alpha} \text{ for } p = 3$$

$$\bar{\beta}\lambda + (a_1 + \mu)(\bar{\gamma} - \bar{\beta}) \qquad \text{otherwise}$$

$$\text{(iii)} \quad \lambda^{n-2}\,\mu\,\alpha = \lambda\,\mu^2\,\bar{\alpha}.$$

Case 1. $\alpha = 0$; that is $\gamma_1(G)$ is abelian.

In this case total number of maps of the type ψ is $(p-1)^2\,p$. For $p \geq 3$

$$G(0, \beta, \gamma)\,\psi = G(0, \lambda^{n-3}\,\mu\,\beta + a_1\,\lambda^{n-3}(\beta - \gamma),$$
$$\lambda^{n-3}\,\mu\,\beta + a_1\,\lambda^{n-3}(\beta - \gamma) + \lambda^{n-2}(\gamma - \beta))$$

Since $0 \leq a_1 \leq p - 1$,

$$G(0, \beta, \gamma) = G(0, \lambda^{n-3}\,\mu\,\beta, \lambda^{n-3}\,\mu\,\beta + \lambda^{n-2}(\gamma - \beta))$$
$$\cong G(0, \lambda^{n-3}(\mu + 1)\,\beta - \gamma\,\lambda^{n-3},$$

$\lambda^{n-3}(\mu + 1)\,\beta - \gamma\,\lambda^{n-3} + \lambda^{n-2}(\gamma - \beta)) \cong \ldots\ldots\ldots\ldots \cong$
$G(0, \lambda^{n-3}(\mu - 1)\,\beta + \gamma\,\lambda^{n-3}, \lambda^{n-3}(\mu - 1)\,\beta + \gamma\,\lambda^{n-3} + \lambda^{n-2}(\gamma - \beta))$.
If $\beta = 0 = \gamma$, then $G(0, 0, 0)\,\psi = G(0, 0, 0)$; that is all $(p-1)^2\,p$ isomorphisms are automorphisms for $G(0, 0, 0)$ and therefore the order of automorphism group of this group is $(p - 1)^2\,p^{2n-3}$.

If $\beta = \gamma$ & $\beta \neq 0$, then $G(0, \beta, \beta)\,\psi = G(0, \lambda^{n-3}\,\mu\,\beta, \lambda^{n-3}\,\mu\,\beta)$, so $G(0, 1, 1) \cong G(0, 2, 2) \cong \ldots\ldots \cong G(0, p - 1, p - 1)$:

that is $p - 1$ groups are isomorphic to each other ; and, therefore the order of automorphism group of each of these group is $(p - 1)\,p^{2n-3}$. Lastly for $\beta \neq \gamma$ there are $p^2 - p$ groups described by $G(0, \beta, \gamma)$ which splits up into $(n - 2, p-1)$ different up to isomorphism groups under the action of ψ ; and therefore the order of automorphism group of each of these group is $(n-2, p-1)(p-1)p^{2n-4}$. Thus

Theorem 3. For $p \geq 3$, there are $(n-2, p-1)+2$ metabelian p-groups of maximal class and order p^n, $n \geq 4$ with $\gamma_1(G)$ abelian such that the order of automorphism group of each of $(n-2, p-1)$ is $(n-2, p-1)(p-1)p^{2n-4}$ and the order of automorphism group of the other two is $(p-1)^2 p^{2n-3}$ and $(p-1)p^{2n-3}$, respectively.

For $p=2$, the total number of maps of the type ψ is 2 : and $\lambda = 1 = \mu$ and a_1 is either zero or 1. Now

$$G(0, \beta, \gamma)\psi = G(0, \gamma, \beta).$$

If $\beta = 0 = \gamma$, then $G(0, 0, 0)\psi = G(0, 0, 0)$ and therefore the order of automorphism group of $G(0, 0, 0)$ is 2^{2n-3}.

If $\beta = 1 = \gamma$, then $G(0, 1, 1)\psi = G(0, 1, 1)$ and therefore the order of automorphism group of $G(0, 1, 1)$ is 2^{2n-3}.

If $\beta \neq \gamma$, then $G(0, 0, 1)$ is isomorphic to $G(0, 1, 0)$ under ψ and therefore the order of automorphism group of each of these group is 2^{2n-4}. Thus

Thborem 4. For $p=2$ there are three different up to isomorphism metabelian 2-groups of maximal class and order 2^n, $n \geq 4$ with $\gamma_1(G)$ abelian such that the order of automorphism group of two of them is 2^{2n-3} and the order of automorphism group of the other one is 2^{2n-4}.

Case II. $\alpha = 1$; that is $\gamma_1(G)$ is non abelian. In this case $\mu = \lambda^{n-3}$; and therefore total number of maps of the type ψ reduces to $(p-1)p$. For $p > 3$ and $0 \leqslant a_1 \leqslant p-1$ we have

$$G(1, \beta, \gamma) \cong G(1, \lambda^{n-3}\mu\beta, \lambda^{n-3}\mu\beta + \lambda^{n-2}(\lambda - \beta))$$

$$\cong G(1, \lambda^{n-3}(\mu+1)\beta - \gamma\lambda^{n-3}, \lambda^{n-3}(\mu+1)\beta - \gamma\lambda^{n-3} + \lambda^{n-2}(\gamma - \beta))$$

$$\cong \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \cong$$

$$G(1, \lambda^{n-3}\mu\beta + \lambda^{n-3}(\gamma-\beta), \lambda^{n-3}\mu\beta + \lambda^{n-3}(\gamma-\beta) + \lambda^{n-2}(\gamma-\beta)).$$

For $\beta = 0 = \gamma$, $G(1, 0, 0)\psi = G(1, 0, 0)$, therefore, the order of the automorphism group of this group is $(p-1)p^{2n-3}$.

For $\beta = \gamma$ and $\beta \neq 0$, $G(1, \beta, \beta)\psi = G(1, \beta\lambda^{2n-6}, \beta\lambda^{2n-6})$. Since $1 \leqslant \beta \leq p-1$, these $p-1$ groups split up into $(2n-6, p-1)$ distinct up to isomorphism groups and, therefore, order of the automorphism group of each of these group is $(2n-6, p-1)\, p^{2n-3}$.

For $\beta \neq \gamma$ there are $(p^2 - p)$ groups described by $G(1, \beta, \gamma)$ which under the action of ψ split up into $(n-2, p-1)$ different up to isomorphism groups; and, therefore, the order of automorphism group each of these group is $(n-2, p-1)\, p^{2n-4}$. Thus

Theorem 5. For $p > 3$ there are $1 + (2n-6, p-1) + (n-2, p-1)$ metabelian p-groups of maximal class with $\gamma_1(G)$ uon abelian such that the order of automorphism group of each of the $(2n-6, p-1)$ group is $(2n-6, p-1)\, p^{2n-3}$, the order of automorphism group of each of $(n-2, p-1)$ group is $(n-2, p-1)\, p^{2n-4}$; and the order of automorphism group of remaining one is $(p-1)\, p^{2n-3}$.

For $p = 3$ the number of maps of the type ψ is 6; and $0 \leq a_1 \leq 2$ implies

$$G(1, \beta, \gamma) \cong G(1, \mu^2\beta, \mu^2\beta + \lambda\mu(\gamma - \beta) + 2\mu\lambda + \mu^2) \cong$$

$$G(1, \mu^2\beta + \mu(\beta - \gamma) + 2 + \mu, \mu^2\beta + \mu(\beta - \gamma) + \lambda\mu(\gamma - \beta) + 2(2 + \mu)(\lambda - \mu)) \cong$$

$$G(1, \mu^2\beta + 2\mu(\beta - \gamma) + 2(1 + \mu), \mu^2\beta + 2\mu(\beta - \gamma) + \lambda\mu(\gamma - \beta) + 2(1 + \mu)(\lambda - \mu)).$$

Now $0 < \beta, \gamma < 2$, we have

$$G(1, 0, 0) \cong G(1, 0, 1) \cong G(1, 1, 0);$$

that is the order of the automorphism group of each these groups is 2.3^{2n-4}. Similarly $G(1, 1, 1) \cong G(1, 2, 1) \cong G(1, 1, 2)$ and $G(1, 2, 2) \cong G(1, 0, 2) \cong G(1, 2, 0)$; and, therefore the order of automorphism group of each of these group is 2.3^{2n-4}. Thus

Theorem 6. For $p = 3$ there are three different up to isomorphism metabelian p-group of maximal class and order p^n, $n \geq 4$ with $\gamma_1(G)$ non abelian such that the order of automorphism group of each of these group is 2.3^{2n-4}.

Now we prove our required result :

Theorem 7. For every divisor d of $p-1$, $p>3$ there exists a metabelian p-group G of maximal class and order p^n, $n \geq 4$ with $\gamma_1(G)$ non abelian and $[\gamma_1(G), \gamma_2(G)] \leq \gamma_4(G)$ such that the order of automorphism group Aut (G) of restricted on centre Z (G) of G is exactly d.

Proof. Take $n = p + d + 1$. The group $G(1, \beta, \gamma)$ where $\beta \neq \gamma$, by theorem 5 has automorphism group of order $(p-1+d, p-1) = dp^{2n-3}$. The automorphism of order d in $G(1, \beta, \gamma)$ is defined as follows :

$$u_0 \rightarrow u_0^{\lambda} u_1^{a_1} \ldots\ldots u_{n-1}^{a_{n-1}}$$

$$u_1 \rightarrow u_1^{\mu} u_2^{b_2} \ldots\ldots u_{n-1}^{b_{n-1}}$$

where $\lambda^d \equiv 1 \mod (p)$, $\mu = \lambda^{n-3} = \lambda^{-1}$ as $d / n-2$ and $0 \leq a_i, b_j \leq p-1$. The action of this automorphism on Z (G) is

$$u_{n-1} \rightarrow u_{n-1}^{\mu\lambda^{n-2}} = u_{n-1}^{\lambda^{-1}}$$

which has order exactly d. Hence the order of Aut (G) restricted on Z (G) is exactly d.

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EXTRA-SPECIAL q-GROUPS AND CENTRAL DECOMPOSITIONS

By
G.Q. ABBASI
Mathematics Department,
Islamia University,
Bahawalpur, Pakistan.

Abstract.

In this paper we prove that :
"an extra-special group of order p^{2r+1} with $r \geq 1$ has

$$1/r!\, p^{r(r-1)} (p^{2r-2} + \ldots\ldots + p^2 + 1)$$
$$(p^{2r-4} + \ldots\ldots + p^2 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (p^2 + 1)$$

central decompositions into non-abelian subgroups each of order p^3 ".

The concept of "central decompositions" of groups is a special case of the notion of "generalized direct decompositions" of groups which was introduced by B.H. Neumann and H. Neumann [6]. A quick glance through literature shows that the structure theorems involving the concept of central decompositions are about the groups of prime power order ; of structure of extra-special p-groups and P. Hall theorem [7] Hall-Senior-James theorem [3] & [5] ; structure of groups of prime power order with cyclic Frattini subgroups [1].

We start with the definition of a Central Decomposition of a finite group G.

A finite set $\{ H_1, \ldots\ldots, H_n \}$ of subgroups of a group G is said to be a *central decomposition* (*c*, *d*) of G if

(1) $G = \langle H_1, \ldots\ldots\ldots\ldots, H_n \rangle$,

(2) $[H_i, H_j] = E$ for $i \neq j$ in $\{1, 2, \ldots\ldots, n\}$,

(3) $Z(G) \leq H_i$ for all $i = 1, 2, \ldots\ldots, n$;

and we write $G = H_1 \gamma \ldots\ldots\ldots \gamma H_n$. Each H_i is called a *central factor* of G.

If such a set of subgroups of G exists, then G is said to be *centrally decomposable* ; otherwise G is *centrally indecomposable.*

Note that if $G = H_1 \gamma \ldots\ldots \gamma H_n$ is a c.d. of G, then $Z(G) = Z(H_i)$ for all $i = 1, 2, \ldots\ldots, n$; and it induces a direct decomposition of the factor group $G/Z(G)$; that is,

$$G/Z(G) = H_1/Z(H_1) \times \ldots\ldots \times H_n/Z(H_n).$$

A c.d. $G = H_1 \gamma \ldots\ldots \gamma H_n$ of G is said to be an *unrefinable* c.d. (which we write as u.c.d.) of G if each H_i is centrally indecomposable ; otherwise it is *refinable.*

A nonabelian p-group G is said to be *extra-special* if $G' = Z(G) = \bar{\phi}(G)$ has order p, where $\bar{\phi}(G)$ is the Frattini subgroup of G and G' is the derived subgroup of G.

Note that if G is extra-special, then $G/Z(G)$ is elementary abelian.

It is well known (see for example [2]) that a non-abelian group of order p^3 is extra-special ; and is isomorphic to one of the following groups :

$$\left.\begin{array}{l} M = \langle x, y, z \,;\, x^p = y^p = z^p = [x, z] = [y, z] = e, z = [x, y] \rangle \\ N = \langle a, b \,;\, a^{p^2} = b^p = e, b^{-1}ab = a^{1+p} \rangle \end{array}\right\} p \text{ odd}$$

$$\left.\begin{array}{l} D_8 = \langle u, v \,;\, u^{p^2} = e = v^p, v^{-1}uv = u^{-1} \rangle \\ Q_2 = \langle c, d \,;\, c^{p^2} = e, c^p = d^p, d^{-1}cd = c^{-1} \rangle \end{array}\right\} p = 2.$$

Lemma 1. Let G be extra-special p-group. Then, either G has order p^3 ; or $G = A \gamma B$ where A is non-abelian subgroup of order p^3 and $B = G(A)$ is extra-special.

Proof. Let $x \in G \setminus Z(G)$. Then there exists $y \in G$ such that $e \neq [x, y] = z$; and $z \in Z(G)$. Since $G' = Z(G) = \bar{\phi}(G)$ has order p and $G/Z(G)$ is elementary abelian, x^p and y^p lie in $Z(G)$; and we have a non-abelian group $A = \langle x, y, z \rangle$ of order p^3. To prove that $G = A \gamma B$, we only need to prove that $G = AB$. Clearly $AB \leq G$. For the converse let $[g, x] = z^m$ & $[g, y] = z^n$, where $0 \leq m, n \leq p-1$, for $g \varepsilon G$, consider

$$\left[gx^{-n} y^m , x \right] = \left[g, x \right] \left[y^m , x \right] = z^m z^{-m} = e ,$$

$$\left[gx^{-n} y^m , y \right] = \left[g, y \right] \left[x^{-n} , y \right] = z^n z^{-n} = c ;$$

that is, $c = gx^{-n} y^m \varepsilon B$ and so $g = c (x^{-n} y^m)^{-1} = ca$ for $a \in A$. Thus, each $g \varepsilon G$ is of the form $g = ca$, so $G \leq AB$; and we have $G = AB$.

If B is abelian, then $B = z(B) = z(G)$; and we have $G = A$ is a nonabelian subgroup of order p^3. If B is non-abelian, then $G' = B' = Z(B) = Z(G)$ has order p ; and $B/Z(B) \leq G/Z(G)$ is elementary abelian, therefore B is extra-special.

Lemma 2. A nonabelian group of order p^3 is centrally indecomposable.

Proof. Suppose contrary and let $G = A \gamma B$. Then one of A and B has order p^2 and so $Z(A) = Z(B) = Z(G)$ has order p^2, a contradiction.

Theorem 3. An extra-special p-group G has a u.c.d. into non-abelian subgroups each of order p^3 ; and, therefore, has order p^{2r+1} with $r \geq 1$.

Proof. Follows by induction on lemma 1 ; and by lemma 2.

Now we prove the following result which plays a significant role in determining the total number of u.c.d.s of extra-special group of order p^{2r+1} with $r \geq 1$,

Lemma 4. Let G be an extra-special group of order p^{2r+1} with $r \geq 1$; and let $g \varepsilon G \setminus Z(G)$. Then $H = G(g)$.

Proof. Note that the total number of conjugates of g in G is $[G : H]$, because if $g^{h_1} = g^{h_2}$ for $h_1, h_2 \varepsilon G$, then $g^{h_1 h_2^{-1}} = g$ and, therefore, $h_1 h_2^{-1} \varepsilon H$. This implies $Hh_1 = Hh_2$. Further each conjugate of g in G lies in gG', because $g^h = g[g, h]$ for $h \varepsilon G$;

so g has at most $|G'|$ conjugate in G ; that is, we have $1 \leq [G : H] \leq |G'| = p$. This implies $[G : H] = p$ so H has order p^{2r}.

Lemma 5. An extra-special group G of order p^5 has exactly $\frac{1}{2} p^2 (p^2 + 1)$ u.c.ds.

Proof. We note in the proof of lemma 1 that each non commuting pair of elements in G generates a nonabelian subgroup of order p^3, Since $Z(G) = G'$ has order p, the number of elements in $G \setminus Z(G)$ is $p^5 - p$. If x is any such element, then the number of elements which do not commute with x is

$p^5 - |G(x)| = p^5 - p^4$, by previous lemma 4.

To determine the total number of u.c.ds in G, we calculate the total number of subgroups of order p^3 in G. An arbitrary nonabelian subgroup p^3 has $(p^3 - p)(p^3 - p^2)$ non commuting pairs of elements, therefore, thc number of nonabelian subgroups of order p^3 in G is

$$\frac{(p^5 - p)(p^5 - p^4)}{(p^3 - p)(p^3 - p^2)} = p^2 (p^2 + 1).$$

If $G = U \gamma V$ is a u.c.d. of G, then each central factor of G is a nonabelian subgroup of order p^3. Moreover for each U in $G = U \gamma V$, there is only one V which is equal to $G(U)$. As we do not distinguesh between $G = U \gamma V$ and the $G = V \gamma U$, therefore, total number of such u.c.ds in G is $\frac{1}{2} p^2 (p^2 + 1) = \frac{1}{2!} p^2 (p^2 + 1)$.

Theorem 6. Let G be extra-special group of order p^{2r+1} with $r \geq 1$. Then G has exactly

$$\frac{1}{r!} p^{r(r-1)} (p^{2r-2} + \ldots\ldots + p^2 + 1)(p^{2r-4} + \ldots\ldots + p^2 + 1)$$

$\ldots\ldots (p + 1)$ u.c.ds.

Proof. For $r = 2$ the result follows from the above lemma. For $r = 3$ we proceed as follows :

Call a c.d of type A if it is u.c.d ; (that is, it has three central factors each of order p^3) and a c.d of type B if G has two central factors one of order p^5 and the other of order p^3. Let a and b be, respectively, the numbers of such decompositions of G. Then

$$a = \frac{b c_1}{c_2}; \text{ where}$$

c_1 = number of refinements of a c.d. of type B (which is the number of u.c.ds of G of order p^5);

c_2 = number of c.d of type B such that a given u.c.d of type A is a refinement (which is equal to 3).

Now b is equal to the number of nonabelian subgroups of order p^3 in G having order p^7 ; that is,

$$b = \frac{(p^7 - p)(p^7 - p^6)}{(p^3 - p)(p^3 - p^2)} = p^4 (p^4 + p^2 + 1).$$

Thus we have

$$a = \frac{1}{3\,!} p^6 (p^4 + p^2 + 1)(p^2 + 1) .$$

Assume that the result is true for $r = n \geq 2$; that is, an extra-special group of order p^{2n+1} has exactly

$$\frac{1}{n\,!} p^{n(n-1)} (p^{2n-4} + \ldots\ldots + p^2 + 1) \ldots\ldots\ldots\ldots (p^2 + 1) \quad \text{u.c.ds.}$$

We prove that the result is true for $r = n + 1$,

Consider an extra-special group G of order p^{2n+3} . Following the same notation, we call a c.d of G of the type A if it is u.c.d (if it has $n + 1$ central factors each of order p^3) ; and a c.d of the type B if it has only two factors one of order p^{2n+1} and the other of order p^3 . Let "a" and "b" denote the numbers of decompositions of type A and type B, respectively. Then $a = bc_1 / c_2$, where

c_1 = number of refinements of a c.d. of type B (which is the number of u.c.ds of group of order p^{2n+1}),

c_2 = number of c.ds of type B such that a u.c.d of type A is a refinement (which is equal to $n + 1$).

Again "b" is equal to the number of nonabelian subgroups of order p^3 in G of order p^{2n+1} ; that is,

$$b = \frac{(p^{2n+1} - p)(p^{2n+1} - p^{2n})}{(p^3 - p)(p^3 - p^2)} \;;$$

and we have

$$a = 1 / (n + 1)\,!\,.\,p^{n(n+1)} (p^{2n} + \ldots\ldots + p^2 + 1) \ldots\ldots (p^2 + 1) .$$

This completes the proof.

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ESTIMATION FOR THE ERRORS-IN-VARIABLES MODEL

By
G. R. PASHA

Department of Statistics
Bahauddin Zakariya University
Multan, Pakistan.

Abstract

Parmeters involved in a linear model are estimated when both independent and dependent variables in a replicatd case are subject to error. It it also assumed that errors are normally distributed with mean zero and we discuss here two cases with common and different variances.

1. Introduction.

Variables in a regression model may be masked by measurement errors which arise from different factors or hidden sources, Nevertheless, it may be possible to make inferences about the parameters relating to the regression of variables. This problem has been examined extensively by many authors, e. g., Kendall and Stuart (1979), Cochran (1968), Mandansky (1959), Moran (1971), Sprent (1966), and Villegas (1961, 1964). These authors draw attention to the variety of ad hoc methods of estimation available, including "grouping" methods and the use of instrumental variables, cumulants, and components of variance.

The first detailed application of general methods of estimation in the two variable linear errors-in-variables problem is that of Lindley (1947). He resolved many of the earlier uncertainties and anomalies in demonstrating the breakdown of the maximum likelihood method, reflecting in the unidentifiability of the parameters and the inconsistency of the estimators for the linear structural, and linear functional, models respectively. But this was specifically for the unreplicated case. Not much is written on the replicated case.

We develop the maximum likelihood estimates for the replicated case. As Barnett (1970) points out, "In principle there is no reason why the maximum likelihood method should not be used in the replicated case, apart from the computational problems of unravelling the sometimes awakward ML equations, but the author is not aware of any published results on this."

2. Maximum Likelihood Estimators (Equal Variances Case).

Consider the linear case of two variables

$$X_i = \eta_i + \delta_i \qquad \dots (1)$$

$$Y_{ij} = \xi_i + \varepsilon_{ij} \qquad \dots (2)$$

$$\xi_i = \beta_0 + \beta_1 \eta_i \qquad \dots (3)$$

$$Y_{ij} = \beta_0 + \beta_1 X_i + (\varepsilon_{ij} - \beta_1 \delta_i),\ i=1, 2, \dots, k\ ;\ j=1, 2, \dots, m. \qquad \dots (4)$$

where,

$$\varepsilon_{ij} \sim \text{IN}\ (0, \sigma^2_\varepsilon)\ ;\ \delta_i \sim \text{IN}\ (0, \sigma^2_\delta)\ ;\ \text{Cov}\ (\varepsilon_{ij}, \delta_i) = 0. \qquad \dots (5)$$

X_i and Y_{ij} are observed values ; η_i and ξ_i are true values and δ_i and ε_{ij} are errors in observations which are mutually independent.

The likelihood function for the sample observations is

$$L = \prod_{i=1}^{k} \prod_{j=1}^{m} \frac{1}{\sigma_\varepsilon \sqrt{2\pi}} \exp \left[-\frac{1}{2\sigma^2_\varepsilon} (Y_{ij} - \beta_0 - \beta_1 \eta_i)^2 \right]$$

$$\prod_{i=1}^{k} \frac{1}{\sigma_\delta \sqrt{2\pi}} \exp \left[\frac{1}{2\sigma^2_\delta} (X_i - \eta_i)^2 \right] \qquad \dots (6)$$

so the log likelihood is

$$l = \text{const.} - \tfrac{1}{2} km \log (\sigma^2_\varepsilon) - \tfrac{1}{2} \frac{\sum_{i=1}^{k} \sum_{j=1}^{m} (Y_{ij} - \beta_0 - \beta_1 \eta_i)^2}{\sigma^2_\varepsilon}$$

$$- \tfrac{1}{2} \sum_{i=1}^{k} \frac{(X_i - \eta_i)^2}{\sigma^2_\delta} \qquad \dots (7)$$

We take the derivatives of Eq. (7) w.r. to $\hat{\beta}_0$, $\hat{\beta}_1$ $\hat{\sigma}^2_\delta$ and $\hat{\sigma}^2_\varepsilon$ and equate to zero. These yield the following equations :

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{\eta} = \bar{Y} - \hat{\beta}_1 \bar{x} \quad \ldots (8)$$

$$\hat{\beta}_1 = \frac{\Sigma\Sigma\,(Y_{ij} - \bar{Y})(\hat{\eta}_i - \bar{\eta})}{\Sigma\,(\hat{\eta}_i - \bar{\eta})^2} = \frac{m\,\Sigma\,(\bar{Y}_i - \bar{Y})(\hat{\eta}_i - \bar{\eta})}{\Sigma\,(\hat{\eta}_i - \bar{\eta})^2} \quad \ldots (9)$$

$$\hat{\sigma}^2_\delta = \frac{1}{k}\Sigma\,(X_i - \hat{\eta}_i)^2 = \frac{1}{k}\,\Sigma\,[\,(X_i - \bar{X}) - (\hat{\eta}_i - \bar{\eta})\,]^2 \quad \ldots (10)$$

$$\hat{\sigma}^2_\varepsilon = \frac{1}{km}\,\Sigma\Sigma\,(Y_{ij} - \hat{\beta}_0 - \hat{\beta}_1\hat{\eta}_i)^2 = \frac{1}{km}\,\Sigma\Sigma\,[Y_{ij} - \bar{Y} - \hat{\beta}_1(\hat{\eta}_i - \bar{\eta})]^2$$

$$\ldots (11)$$

and

$$\left[\frac{1}{\hat{\sigma}^2_\delta} + \frac{\hat{\beta}_1^2}{\hat{\sigma}^2_\varepsilon}\right](\hat{\eta}_i - \bar{\eta}) = \frac{\hat{\beta}_1}{\hat{\sigma}^2_\varepsilon}(\bar{Y}_i - \bar{Y}) + \frac{(X_i - \bar{X})}{\hat{\sigma}^2_\delta}$$

$$\ldots (12)$$

Substituting (8) in Eq. (12) we get

$$(\bar{Y}_i - \hat{\beta}_0 - \hat{\beta}_1\hat{\eta}_i) = \frac{\hat{\sigma}^2_\varepsilon}{\hat{\beta}_1\hat{\sigma}^2_\delta}(X_i - \hat{\eta}_i). \quad \ldots (13)$$

Substituting Eq. (13) in Eq. (11) and using (10)

$$\hat{\sigma}^2_\varepsilon = \frac{1}{km}\,\Sigma\Sigma\,(Y_{ij} - \bar{Y}_i + \bar{Y}_i - \hat{\beta}_0 - \hat{\beta}_1\hat{\eta}_i)^2.$$

$$\hat{\sigma}^2_\varepsilon = \frac{1}{km}\,\Sigma\Sigma\,[\,(Y_{ij} - \bar{Y}_i) + \frac{\hat{\sigma}^2_\varepsilon}{\hat{\sigma}^2_\delta\,\hat{\beta}_1}(X_i - \hat{\eta}_i)\,]^2$$

which yields

$$\hat{\sigma}^2_\varepsilon = T\,yy/km\left[1 - \frac{1}{\lambda\hat{\beta}^2_1}\right], \quad \ldots (14)$$

$$\ldots (15)$$

where $\quad Tyy = \Sigma\Sigma\,(Y_{ij} - \bar{Y}_i)^2$

and $\lambda = \hat{\sigma}_\delta^2 / \hat{\sigma}_\varepsilon^2$ ... (16)

From Eq. (12) substituting the value of $(\hat{\eta}_i - \bar{\eta})$ in Eq. (10)

$$k\hat{\sigma}_\delta^2 \left[\frac{1}{\hat{\sigma}_\delta^2} + \frac{\hat{\beta}_1^2}{\hat{\sigma}_\varepsilon^2} \right]^2 = \frac{\hat{\beta}_1^2}{\hat{\sigma}_\varepsilon^4} [\hat{\beta}_1^2 S_{XX} + S_{\bar{Y}\bar{Y}} - 2\hat{\beta}_1 S_{X\bar{Y}}]$$

$$\frac{k\hat{\sigma}_\varepsilon^2}{\lambda} (1 + \hat{\beta}_1^2 \lambda)^2 = \hat{\beta}_1^2 [\hat{\beta}_1^2 S_{XX} + S_{\bar{Y}\bar{Y}} - 2\hat{\beta}_1 S_{X\bar{Y}}]$$

or $$\hat{\sigma}_\varepsilon^2 = \frac{\hat{\beta}_1^2 \lambda}{k(1 + \hat{\beta}_1^2 \lambda)^2} [\hat{\beta}_1^2 S_{XX} + S_{\bar{Y}\bar{Y}} - 2\hat{\beta}_1 S_{X\bar{Y}}] \quad \text{... (17)}$$

where $\Sigma (X_i - \bar{X})^2 = S_{XX}$; $\Sigma (\bar{Y}_i - \bar{Y})^2 = S_{\bar{Y}\bar{Y}}$

and $$\Sigma (X_i - \bar{X}) (\bar{Y}_i - \bar{Y}) = S_{X\bar{Y}} \quad \text{... (18)}$$

From Eq. (12) substituting the value of $(\hat{\eta}_i - \bar{\eta})$ in Eq. (9)

$$\hat{\beta}_1 = \frac{\Sigma [\bar{Y}_i - \bar{Y}] \left[\frac{\hat{\beta}_1}{\hat{\sigma}_\varepsilon^2} (\bar{Y}_i - \bar{Y}) + \frac{X_i - \bar{X}}{\hat{\sigma}_\delta^2} \right] \left[\frac{1}{\hat{\sigma}_\delta^2} + \frac{\hat{\beta}_1^2}{\hat{\sigma}_\varepsilon^2} \right]}{\Sigma \left[\frac{\hat{\beta}_1}{\hat{\sigma}_\varepsilon^2} (\bar{Y}_i - \bar{Y}) + \frac{X_i - \bar{X}}{\hat{\sigma}_\delta^2} \right]^2} .$$

Substituting Eq. (18) and Eq. (16) we obtain

$$\lambda = \frac{S_{X\bar{Y}} - \hat{\beta}_1 S_{XX}}{\hat{\beta}_1 (\hat{\beta}_1 S_{X\bar{Y}} - S_{\bar{Y}\bar{Y}})} . \quad \text{... (19)}$$

Substituting Eq. (14) in Eq. (17)

$$\frac{1}{m} T_{YY} = \frac{(\lambda \hat{\beta}_1^2 - 1)(\hat{\beta}_1^2 S_{XX} + S_{\bar{Y}\bar{Y}} - 2\hat{\beta}_1 S_{X\bar{Y}})}{(1 + \hat{\beta}_1^2 \lambda)^2} . \quad \text{... (20)}$$

Substituting Eq. (19) in Eq. (20) we get

$$\frac{1}{m}T_{YY} = \frac{(S_{\bar{Y}\bar{Y}} - \hat{\beta}_1{}^2 S_{XX})(\hat{\beta}_1 S_{X\bar{Y}} - S_{\bar{Y}\bar{Y}})}{(\hat{\beta}_1{}^2 S_{XX} + S_{\bar{Y}\bar{Y}} - 2\hat{\beta}_1 S_{XY})}.$$

Eq. (20) may be written as

$$(mS_{XX}\, S_{X\bar{Y}})\, \hat{\beta}_1^3 + (S_{XX}\, T_{YY} - mS_{XX}\, S_{\bar{Y}\bar{Y}})\, \hat{\beta}_1^2$$

$$- (2S_{X\bar{Y}}\, T_{YY} + mS_{X\bar{Y}}\, S_{\bar{Y}\bar{Y}})\, \hat{\beta}_1 + (mS^2_{\bar{Y}\bar{Y}} + S_{\bar{Y}\bar{Y}}\, T_{YY}) = 0 \qquad \dots (21)$$

Once the solution to (21) is obtained, we compute λ from (19) and hence $\hat{\sigma}^2_{\varepsilon}$ from (17) and $\hat{\sigma}^2_{\delta}$ from (16)

3. Maximum Likelihood Estimators (Different Variances Case)

Equations (1 — 4) are the same, save that we now assume

$$\varepsilon_{ij} \sim N(0, \sigma^2_{\varepsilon i})\ ;\ \delta_i \sim N(0, \sigma^2_{\delta})\ ;\ \mathrm{Cov}\,(\varepsilon_{ij}, \delta_i) = 0. \qquad \dots (22)$$

Taking this log likelihood function

$$l = \text{const} - \frac{1}{2}\, m \log (\sigma^2_{\varepsilon i}) - \frac{1}{2} k \log (\sigma^2_{\delta})$$

$$- \frac{1}{2}\sum_i \left[\sum_j \frac{(Y_{ij} - \beta_0 - \beta_1 \eta_i)^2}{\sigma^2_{\varepsilon i}} \right] - \frac{1}{2}\sum_i \frac{(X_i - \eta_i)}{\sigma^2_{\delta}} \qquad \dots (23)$$

Take the derivative of Eq. (23) w.r.t. $\beta_0, \beta_1, \eta_i, \sigma^2_{\delta}$, and $\sigma^2_{\varepsilon i}$ and equate to zero. This yields the following equations

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{\eta} = \bar{Y} - \hat{\beta}_1 \bar{X} \qquad \dots (24)$$

$$\hat{\beta}_1 = \frac{\Sigma\Sigma\,(Y_{ij} - \bar{Y})(\hat{\eta}_i - \bar{\eta})}{\Sigma\,(\hat{\eta}_i - \bar{\eta})^2} = \frac{\sum_i (\bar{Y}_i - \bar{Y})(\hat{\eta}_i - \bar{\eta})}{\sum_i (\hat{\eta}_i - \bar{\eta})^2} \qquad \dots (25)$$

$$\hat{\sigma}_{\delta}^{2} = \frac{1}{k} \sum_{i} (X_i - \hat{\eta}_i)^2 = \frac{1}{k} \Sigma [(X_i - \bar{X}) - (\hat{\eta}_i - \bar{\eta})]^2 \quad \ldots (26)$$

$$\hat{\sigma}_{\varepsilon i}^{2} = \frac{1}{m} \sum_{i} (Y_{ij} - \hat{\beta}_0 - \hat{\beta}_1 \hat{\eta}_i)^2$$

$$= \frac{1}{m} \sum_{j} (Y_{ij} - \bar{Y}_i + \bar{Y}_i - \hat{\beta}_0 - \hat{\beta}_1 \hat{\eta}_i)^2 \quad \ldots (27)$$

and
$$\left[\frac{1}{\hat{\sigma}_{\delta}^{2}} + \frac{\hat{\beta}_1^{2}}{\hat{\sigma}_{\varepsilon i}^{2}} \right] (\hat{\eta}_i - \bar{\eta}) = \frac{\hat{\beta}_1}{\hat{\sigma}_{\varepsilon i}^{2}} (\bar{Y}_i - \bar{Y}) + \frac{(X_i - \bar{X})}{\hat{\sigma}_{\delta}^{2}} ; \quad \ldots (28)$$

After substituting Eq. (24) into Eq. (28) we get

$$(\bar{Y}_i - \hat{\beta}_0 - \hat{\beta}_1 \hat{\eta}_i) = \frac{\hat{\sigma}_{\varepsilon i}^{2}}{\hat{\beta}_1 \hat{\sigma}_{\delta}^{2}} (X_i - \hat{\eta}_i) . \quad \ldots (29)$$

Substituting Eq. (28) into Eq. (25) we get

$$\hat{\beta}_1 = \frac{\sum_{i} [\hat{\beta}_1 \hat{\sigma}_{\delta}^{2} (\bar{Y}_i - \bar{Y})^2 + \hat{\sigma}_{\varepsilon i}^{2} (X_i - \bar{X}) (\bar{Y}_i - \bar{Y})] [\hat{\sigma}_{\varepsilon i}^{2} + \hat{\beta}_1^{2} \sigma_{\delta}^{2}]}{\sum_{i} [\hat{\beta}_1^{2} \sigma_{\delta}^{4} (\bar{Y}_i - \bar{Y})^2 + \hat{\sigma}_{\varepsilon i}^{4} (X_i - \bar{X})^2 + 2\hat{\beta}_1 \hat{\sigma}_{\varepsilon i}^{2} \hat{\sigma}_{\delta}^{2} (\bar{Y}_i - \bar{Y})(X_i - \bar{X})]} \quad \ldots (30)$$

Substituting Eq. (28) into Eq. (26) we obtain

$$\hat{\sigma}_{\delta}^{2} = \frac{1}{k} \sum_{i} \frac{\hat{\beta}_1^{2} \sigma_{\delta}^{4}}{(\hat{\sigma}_{\varepsilon i}^{2} + \hat{\beta}_1^{2} \hat{\sigma}_{\delta}^{2})^2} [\hat{\beta}_1 (X_i - \bar{X}) - (\bar{Y}_i - \bar{Y})]^2 . \quad (31)$$

Substituting Eqs. (28) (29) and (24) into Eq. (27)

$$\hat{\sigma}_{\varepsilon i}^{2} = \frac{T_{ii}}{m} + \frac{[\hat{\beta}_1 (X_i - \bar{X}) - (\bar{Y}_i - \bar{Y})]^2}{\left[1 + \frac{\hat{\beta}_1^{2} \hat{\sigma}_{\delta}^{2}}{\hat{\sigma}_{\varepsilon i}^{2}} \right]^2} \quad \ldots (32)$$

where $T_{ii} = \sum_{j} (Y_{ij} - \bar{Y}_i)^2$

Further simplification of these results does not appear to be possible and we must find the ML estimators by iterative solution from Eqs. (30-32) starting with the results for the equal variances case and substituting into Eq. (32), then (30) and (31) in that order. The interations continue until the solution converges.

4. Variance-Covariance Matrix of M.L. Estimates (Common Variance Case).

The variances-covariances of M.L.E. of β_0, β_1, σ^2_ε and σ^2_δ can be obtained as :

$$\left.\begin{aligned} \text{Var}(\hat{\beta}_0) &= \hat{\sigma}^2_\varepsilon / mk \\ \text{Var}(\hat{\beta}_1) &= \hat{\sigma}^2_\varepsilon / mk\,(\sigma^2_x - \sigma^2_\delta) \\ \text{Var}(\hat{\sigma}^2_\varepsilon) &= 2\hat{\sigma}^4_\varepsilon / mk \\ \text{Var}(\check{\sigma}^2_\delta) &= 2\hat{\sigma}^4_\delta / k \end{aligned}\right\} \quad \dots (34)$$

The estimates are consistent as $k \rightarrow \infty$.

5. Variance-Covariance Matrix of M.L. Estimates (Unequal Va r ance Case).

The variances-covariances of the paramaters can easily be found as :

$$\left.\begin{aligned} \text{Var}(\hat{\beta}_0) &= (D - B) / (AD - B^2) \\ \text{Var}(\hat{\beta}_1) &= (A - B) / (AD - B^2) \\ \text{Var}(\hat{\sigma}^2_{\varepsilon i}) &= 2\hat{\sigma}^4_{\varepsilon i} / m \\ \text{Var}(\hat{\sigma}^2_\delta) &= 2\hat{\sigma}^2_\delta \; / k \end{aligned}\right\} \quad \dots (35)$$

where $A = \sum_i (m / \hat{\sigma}^2_{\varepsilon i})$; $B = m \sum_i [(X_i - \bar{X}) / \hat{\sigma}^2_{\varepsilon i}]$

and $D = m \sum_i [(\hat{\sigma}^2_x - \hat{\sigma}^2_\delta) / \hat{\sigma}^2_{\varepsilon i}]$.

Note : $\hat{\sigma}^2_{\varepsilon i}$ is consistent if $m \to \infty$ and $\hat{\sigma}^2_\delta$ is consistent if $k \to \infty$.

6. Method of Least Squares.

We now consider how the approach from LS regression analysis breaks down when applied to the estimation of β_0 and β_1 in Eq. (4) even if the errors δ_i and ε_{ij} are assumed to be mutually independent with constant variances, and also to be independent of the true values η_i and ξ_i. The application of least squares to Eq. (4) to get estimates of β_0 and β_1 is not valid, since the factor $(\varepsilon_{ij} - \beta_1 \delta_i)$ in Eq. (4) is not independent of X_i. The covariance of X_i and $(\varepsilon_{ij} - \beta_1 \delta_i)$ is

$$\text{Cov}\,[X_i , (\varepsilon_{ij} - \beta_1 \delta_i)] = - \beta_1 \text{Var}(\delta_i) \qquad \ldots (36)$$

using Eq. (5)

Since the covariance does not vanish, there is a dependence between error term and explanatory variables in Eq. (4).

Due to this dependence the application of LS to Eq. (4) would yield biased estimates of the β_0 and β_1. Furthermore, the bias will not disappear as the sample size becomes infinitely large ; so the LS estimates are inconsistent. The bias in the replicated case is the same as in the unreplicated case, as we now show.

6.1. Inconsistency of L.S. Estimators.

The least square estimators of β_1 on the basis of j observations in k samples is

$$\beta^*_1 = \frac{\sum\sum_{i\,j} (X_i - \bar{X})(Y_{ij} - \bar{Y})}{\sum_i (X_i - \bar{X})^2} = \frac{m \sum_i (X_i - \bar{X})(\bar{Y}_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2} \qquad \ldots (37)$$

Taking limits in probability as follows :

$$\underset{k\to\infty}{p \lim} \left[\frac{1}{k} \Sigma (X_i - \bar{X})(\bar{Y}_i - \bar{Y}) \right]$$

$$= p \lim \frac{1}{k} \Sigma (\eta_i - \bar{\eta}) (\beta_0 + \beta_1 \eta_i + \bar{\varepsilon}_i)$$

$$= p \lim \left[0 + \frac{1}{k} \beta_1 \Sigma (\eta_i - \bar{\eta})^2 + \Sigma \bar{\varepsilon}_i (\eta_i - \bar{\eta}) \right]$$

$$= \beta_1 \lim [\Sigma (\eta_i - \bar{\eta})^2 / k] = \beta_1 \sigma^2_\eta \qquad \dots (38)$$

$$\underset{k\to\infty}{p \lim} \frac{1}{k} \Sigma (X_i - \bar{X})^2 = p \lim \frac{1}{k} \Sigma [(\eta_i - \bar{\eta}) + (\delta_i - \bar{\delta})]^2 = \sigma^2_\delta + \sigma^2_\eta \qquad \dots (39)$$

Substituting Eqs. (38-39) into (37) we obtain

$$\underset{k\to\infty}{p \lim} \hat{\beta}^*_1 = \beta_1 \sigma^2_\eta (\sigma^2_\eta + \sigma^2_\delta)$$

or

$$\underset{k\to\infty}{p \lim} \beta_1^* = \beta_1 (1 - \sigma^2_\delta / \sigma^2_x). \qquad \dots (40)$$

Thus $\underset{k\to\infty}{p \lim} \beta_1^* \neq \beta_1$, but is in fact an under estimate of β_1 .

The asymptotic mean square errors for the ML and OLS estimators are as follows

$$\text{MSE (ML)} = \hat{\sigma}^2_\varepsilon / mk (\hat{\sigma}^2_x - \sigma^2_\delta) \qquad \dots (41)$$

and

$$\text{MSE (OLS)} = (\hat{\beta}_1 \hat{\sigma}^2_\delta / \hat{\sigma}^2_x)^2 + \hat{\sigma}^2_\varepsilon / mk \hat{\sigma}^2_x \qquad \dots (42)$$

Approximately, MSE (MLE) may be greater than MSE (OLSE) if

$$\hat{\sigma}^2_\varepsilon / mk (\hat{\sigma}^2_x - \hat{\sigma}^2_\delta) > \hat{\beta}^2_1 \sigma^4_\delta / \sigma^4_x + \hat{\sigma}^2_\varepsilon / mk \hat{\sigma}^2_x$$

or

$$\hat{\sigma}^2_\varepsilon / mk (\hat{\sigma}^2_x - \hat{\sigma}^2_\delta) > \hat{\beta}^2_1 \hat{\sigma}^2_\delta / \hat{\sigma}^2_x \qquad \dots (43)$$

$$\text{if } |\hat{\beta}_1| < \sigma_{\varepsilon} / \sigma^2_{\delta} \left(1 - \frac{\sigma^2_{\delta}}{\sigma^2_x} \right)^{\frac{1}{2}}$$

i.e., errors-in-variables are relatively small or if *mk* is small.

We undertake an empirical study to compare the performance of the ML, OLS and WLS estimators for both common and different varia ces.

7. Empirical Results and Conclusion.

For this empirical investigation, the model of Jacquez, et al. (1968) is considered in which α and β are assumed to be equal to one. The values of K are chosen to be 4, and 10 and the corresponding chosen values for X_i are (1, 4, 7, 10), and (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) respectively. The values of *m* considered are 4 and 10. We generated 200 samples for each set of (*m*, *k*). For each (*m*, *k*) pair, three σ patterns are chosen ;

(1) $\sigma_{\varepsilon_i} = 1$; $\sigma_{\delta} = 1$, and $\sigma_{\delta} = 2$

(2) $\sigma_{\varepsilon_i} = (\frac{1}{2} X_i + 1) / 3$; $\sigma_{\delta} = 1$ and $\sigma_{\delta} = 2$

(3) $\sigma_{\varepsilon_i} = (X_i + 8) / 9$; $\sigma_{\delta} = 1$ and $\sigma_{\delta} = 2$

Computations were carried out on the IBM 3033 at the Pennsylvania State University, USA.

The study presented in Tables (1) and (2) shows that : (*i*) The ML estimator of β has very little bias in comparison to WLS and OLS but has high MSE becausc of high variability in the errors in the X variables when small samples are used. (ii) When σ^2_{δ} is small, then the OLS estimates have a small bias as expected. However, when σ^2_{δ} is large, the MSE for OLS is still smaller than that for ML unless the number of replications is increased. (iii) When σ^2_{δ} is large and large size samples with a large number of replicates are used, ML gives better estimates.

Table I. A compartive study of ML, WLS and OLS for $\hat{\beta}$ in the "errors-in-variables" case when error variances are different.

	$[\sigma_{\varepsilon i} = 1 ; \sigma_\delta = 1]$			$[\sigma_{\varepsilon i} = (0.5\, x_i + 1) / 3 ; \sigma_\delta = 1]$			$[\sigma_{\varepsilon i} = (X_i + 8) / 9 ; \sigma_\delta = 1]$		
	$\bar{(\hat{\beta})}$	Var $(\hat{\beta})$	MSE $(\hat{\beta})$	$\bar{(\hat{\beta})}$	Var $(\hat{\beta})$	MSE $(\hat{\beta})$	$\bar{(\hat{\beta})}$	Var $(\hat{\beta})$	MSE $(\hat{\beta})$
				$(k = 4 ; (m = 10)$					
ML	1.00602	0.18831	0.18835	1.05137	0.22038	0.22302	1.08487	0.25744	0.26464
WLS	0.97730	0.02333	0.02385	0.97772	0.03132	0.03808	0.97599	0.02789	0.02847
OLS	0.90780	0.02157	0.03005	0.93773	0.02295	0.02683	0.91801	0.02353	0.02361
				$[k = 10 ; m = 10]$					
ML	0.95708	0.01181	0.01365	1.01262	0.01840	0.01856	1.05892	0.02898	0.03245
WLS	0.92724	0.00886	0.01415	0.91815	0.01326	0.01664	0.92558	0.01053	0.01607
OLS	0.92541	0.00883	0.01439	0.92488	0.00986	0.01550	0.92484	0.01644	0.01609
	$[\sigma_{\varepsilon i} = 1 ; \sigma_\delta = 2]$			$[\sigma_{\varepsilon i} = (0.5\, x_i + 1) / 3 ; \sigma_\delta = 2]$			$[\sigma_{\varepsilon i} = (X_i + 8) / 9 ; \sigma_\delta = 2]$		
				$(k = 4 ; m = 10)$					
ML	0.96101	0.13651	0.13803	1.00333	0.13021	0.13022	1.07127	0.17011	0.17519
WLS	0.93121	0.09330	0.09806	0.93110	0.08992	0.09441	0.98121	0.10022	0.10054
OLS	0.93878	0.08518	0.08902	0.93839	0.08552	0.08924	0.96710	0.09307	0.09409
				$[k = 10 ; m = 10]$					
ML	0.98170	0.01911	0.01947	0.99354	0.02515	0.02519	1.00231	0.03121	0.03122
WLS	0.89780	0.01860	0.02921	0.90077	0.02098	0.03058	0.88218	0.02975	0.04344
OLS	0.76253	0.01872	0.32538	0.76240	0.02001	0.07665	0.74381	0.03133	0.03788

Table 2. A comparative study of ML, WLS, and OLS for $\hat{\beta}$ in the "errors-in-variables" case when error variances are equal.

	$(\sigma_\varepsilon = 1 ; \sigma_\delta = 1)$			$[\sigma_\varepsilon = (0.5x_i + 1) / 3 ; \sigma_\delta = 1]$			$[\sigma_\varepsilon = (X_i + 8) / 9 ; \sigma_\delta = 8]$		
	$(\hat{\beta})$	Var $(\hat{\beta})$	MSE $(\hat{\beta})$	$(\hat{\beta})$	Var $(\hat{\beta})$	MSE $(\hat{\beta})$	$(\hat{\beta})$	Var $(\hat{\beta})$	MSE $(\hat{\beta})$
				$[k = 4 ; m = 10]$					
ML	1.00602	0.18830	0.18835	1.05137	0.22038	0.22302	1.08486	0.25747	0.26467
WLS	0.96883	0.02289	0.02386	0.96991	0.02898	0.02988	0.96616	0.02668	0.02784
OLS	0.93881	0.02293	0.02666	0.94007	0.02403	0.02763	0.92181	0.02402	0.08010
				$[k = 10 ; m = 10]$					
ML	0.95708	0.01181	0.01365	1.01259	0.01837	0.01859	1.05883	0.02897	0.03244
WLS	0.92618	0.00893	0.01438	0.90955	0.01442	0.02252	0.92713	0.01069	0.01602
OLS	0.92333	0.00880	0.01468	0.91810	0.00992	0.01664	0.92681	0.01055	0.01588
	$(\sigma_\varepsilon = 1 ; \sigma_\delta = 2)$			$(\sigma_\varepsilon = (0.5 x_i + 1) / 3 ; \sigma_\delta = 2)$			$(\sigma_\varepsilon = (X_i + 8) / 9 ; \sigma_\delta = 2)$		
				$[k = 4 ; m = 10]$					
ML	0.96101	0.13644	0.13803	1.00337	0.13022	0.13022	1.07129	0.17060	0.18019
WLS	0.93271	0.09401	0.09852	0.93110	0.08991	0.09481	0.98108	0.10020	0.10056
OLS	0.94122	0.08478	0.08824	0.93842	0.08553	0.08913	0.96561	0.09386	0.09503
				$[k = 10 ; m = 10]$					
ML	0.98180	0.01909	0.01941	0.99354	0.02515	0.02519	1.00230	0.03117	0.03118
WLS	0.88991	0.01877	0.03087	0.90033	0.02002	0.02996	0.88322	0.02890	0.04259
OLS	0.75986	0.01993	0.07753	0.76366	0.02107	0.07724	0.74296	0.03186	0.09791

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ESTIMATING THE PARAMETERS OF BURR POPULATIONS FROM SOME ORDERED STATISTICS

By

MUNIR AHMAD

Department of Mathematical Sciences
University of Petroleum and Minerals
Dhahran, Saudi Arabia

Abstract.

We consider the problem of estimating the parameters of the Burr distribution using logarithms or some order statistics. We eliminate one of the parameters by finding the ratio of the logarithm of some ordered statistics and estimate the second parameter. Asymptotic variances and covariances of the estimators are obtained and an efficiency of the estimator of the parameter relative to its maximum likelihood estimate when one of the parameters is known, is obtained. A table is constructed for computational purposes. An example is given to illustrate the application of the estimating equation.

1. Introduction.

In recent years considerable interest has been shown in the range, median, and other order statistics for solution of various problems, e.g. in flood and drought prediction, engineering (particularly electrical engineering), etc. Although these statistics are inefficient, they are very simple to use. In this paper a simple method of estimating Burr parameters is given together with an example illustrating the results.

2. Estimation of Parameters.

Burr (1942) introduced a family of distributions given by

$$F(x) = 1 - (1 + x^{\alpha})^{-\beta}, \quad x \geq 0, \ \alpha, \beta > 0 \qquad \ldots (1)$$

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Burr (1968), Burr and Cislak (1968), Hatke (1949), Khaliq (1973) and Austin (1973) discuss (1). The probability function is unimodal if $\alpha > 1$ and L-shaped if $\alpha \leq 1$. Since it is used in many fields, problems of estimations of the Burr parameters are invariably encountered, and in particular quick and simple methods should be available for engineering problems.

Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics for a sample of size n of a random variable X having distribution function (1). The q-quantile of X is defined by

$$F(x_q ; \theta) = q \qquad \dots (2)$$

where x_q is the q-th quantile and the equation (2) implies that 100 q percent of all possible values of the random variable X lie below x_q and θ is a set of unknown parameters.

2.1. Estimation based on two ordered statistics.

Suppose that the r-th observation from a sample of size n is chosen so that its proportion $\frac{r}{n}$ is equal to or just greater than q. With $X_{(r)}$ replacing x_q, (2) may then serve as an estimating equation for the parameter θ, and the order q may be chosen to minimize the variance of the corresponding estimator. If $\theta = (\theta_1, \theta_2, \dots, \theta_k)$, then k values of q and the corresponding sample approximations of x_q give a set of k simultaneous equations of the form

$$F(x_{(r)}, \theta) = q$$

for determining the k estimators of the parameter components, θ_i, $i = 1, 2, \dots, k$.

Let

$$F_\beta (x_q ; \alpha, \beta) = q$$

where $F_\beta (x ; \alpha, \beta)$ is given at (1).

Solving for x_q, we have

$$x_q = [(1-q)^{-1/\beta} - 1]^{1/\alpha}.$$

Since there are two parameters α and β, two estimating equations are required in terms of sample order statistics

$$x_{(r)} \doteq [\,(1 - q_1)^{-1/\tilde{\beta}} - 1]^{1/\tilde{\alpha}} \qquad \dots (3)$$

$$x_{(s)} \doteq [\,(1 - q_2)^{-1/\tilde{\beta}} - 1]^{1/\tilde{\alpha}} \qquad \dots (4)$$

where $\frac{r}{n}$ and $\frac{s}{n}$ are, respectively equal to or just greater than q_1 and q_2 and $r < s$. We eliminate α to obtain an estimating equation for β.

$$\frac{\ln X_{(r)}}{\ln X_{(s)}} = \frac{\ln [\,(1 - q_1)^{-1/\tilde{\beta}} - 1\,]}{\ln [\,(1 - q_2)^{-1/\tilde{\beta}} - 1\,]} \qquad \dots (5)$$

The estimating equation for α can be either (3) or (4) or a combination of (3) and (4). Note that $X_{(r)}$ and $X_{(s)}$ are random variables for a given order q. Let $Y_r = \ln X_{(r)}$. Because $\ln X_{(r)}$ is a monotone function of $X_{(r)}$, then $\ln X_{(r)} = (\ln X)_{(r)} = Y_r$ holds.

2.2. Variance of $\hat{\beta}$ from two ordered statistics.

When X has a distribution function (1), the distribution function of Y is given by

$$F(y) = 1 - (1 + e^{\alpha y})^{-\beta}, \qquad -\infty < y < \infty . \qquad \dots (6)$$

In determination of var $(\hat{\beta})$, we will need the following asymptotic joint distribution of sample quantiles.

If we consider y_i $(i = 1, 2, \dots, n)$ to be a random sample from the continuous distribution with distribution function given in (6), the asymptotic joint distribution of the sample quantiles [see David (1981)]

$$[\sqrt{n}\,(Y_r - \mu_{q_1}),\ \sqrt{n}\,(Y_s - \mu_{q_2})\,], \qquad Y_r \leq Y_s \quad \dots (7)$$

is bivariate normal with zero mean and variance-covariance matrix

$$C_{jj'} = \frac{q_j (1 - q_{j'})}{p(\mu_{q_j})\, p(\mu_{q_{j'}})}, \qquad j \leq j',\ j, j' = 1, 2. \quad \text{... (8)}$$

where $p(\mu_q)$ is defined in David (1981).

Consider the estimating equation for β,

$$\frac{y_r}{y_s} = \frac{\ln [(1 - q_1)^{-1/\tilde{\beta}} - 1]}{\ln [(1 - q_2)^{-1/\tilde{\beta}} - 1]} = \frac{\ln t_1}{\ln t_2} \quad \text{... (9)}$$

where $t_i = (1 - q_i)^{-1/\tilde{\beta}} - 1, \quad i = 1, 2$ ... (10)

Taking logarithmic differentiation of (9), we have

$$\frac{1}{y_r}\, \partial y_r - \frac{1}{y_s}\, \partial y_s = \partial\, (\ln \ln t_1 - \ln \ln t_2) = \frac{\partial t_1}{t_1 \ln t_2} - \frac{\partial t_2}{t_2 \ln t_2} \quad \text{... (11)}$$

Taking differentiation of (10), we obtain

$$\partial t_i = (1 - q_i)^{-1/\tilde{\beta}} \ln (1 - q_i) \left(\frac{1}{\tilde{\beta}^2}\right) \partial \tilde{\beta}$$

$$= - \frac{1}{\tilde{\beta}} (1 - q_i)^{-1/\beta} \ln [(1 - q_i)^{-1/\tilde{\beta}}]\, \partial \tilde{\beta} \quad (12)$$

$$= - \frac{1}{\tilde{\beta}} (t_i + 1) \ln (t_i + 1)\, \partial \tilde{\beta}.$$

Squaring and taking expectation of both sides of (12), we obtain the variances and covariance of t_i and t_j :

$$\text{Var}\,(t_i) \doteq \frac{1}{\tilde{\beta}^2} (t_i + 1)^2 [\ln (t_i + 1)]^2\, \text{Var}\,(\tilde{\beta})$$

$$\text{Cov}(t_i\,,\,t_j) \doteq \mathbf{E}(\partial t_i\,\partial t_j)$$

$$= \frac{1}{\beta^2}(t_i+1)(t_j+1)\ln(t_i+1)\ln(t_j+1)\,\text{Var}(\tilde{\beta}).$$

From equation (8), we have

$$\text{Var}(y_r) \doteq \frac{q_1(1-q_1)}{n f^2(y_{q_1})} \qquad \ldots (13)$$

$$\text{Var}(y_s) \doteq \frac{q_2(1-q_2)}{n f^2(y_{q_2})} \qquad \ldots (14)$$

$$\text{Cov}(y_r\,,\,y_s) \doteq \frac{q_1(1-q_2)}{n f(y_{q_1}) f(y_{q_2})} \qquad \ldots (15)$$

Squaring and taking expectation of both sides of (11), we obtain

$$\frac{1}{y^2_{q_1}}\text{Var}(y_r) + \frac{1}{y^2_{q_2}}\text{Var}(y_s) - \frac{2}{y_{q_1} y_{q_2}}\text{Cov}(y_r\,,\,y_s)$$

$$= \frac{\text{Var}(t_1)}{(t_1 \ln t_1)^2} + \frac{\text{Var}(t_2)}{(t_2 \ln t_2)^2} - 2\,\frac{\text{Cov}(t_1\,,\,t_2)}{t_1 t_2 (\ln t_1)(\ln t_2)} \qquad \ldots (16)$$

$$\doteq \frac{A}{\beta^2}\text{Var}(\tilde{\beta}),$$

where

$$A = \left[\frac{[(t_1+1)\ln(t_1+1)]^2}{(t_1 \ln t_1)^2} + \frac{[(t_2+1)\ln(t_2+1)]^2}{(t_2 \ln t_2)^2} - \frac{2(t_1+1)(t_2+1)\ln(t_1+1)\ln(t_2+1)}{t_1 t_2 \ln t_1 \ln t_2}\right]$$

Also we know from (6) that

$$y_{q_1} = \alpha^{-1} \ln t_1 \,. \qquad \ldots (17)$$

$$y_{q_2} = \alpha^{-1} \ln t_2 \,. \qquad \ldots (18)$$

Thus

$$f(y_{q_i}) = \exp(\alpha y_{q_i}) = \frac{\alpha\beta\, t_i}{[(t_i+1)]^{\beta+1}}\,, \qquad i = 1, 2.$$

Using the values at (13), (14), (15), (17) and (18) and substituting in (16), we obtain, after simplifications,

$$\text{Var}(\tilde{\beta}) = \frac{\beta^2}{nA}\left[\frac{\alpha^2}{(\ln t_1)^2}\,\frac{q_1(1-q_1)(t_1+1)^{2\beta+2}}{\alpha^2\beta^2 t_1^2}\right.$$

$$+\frac{\alpha^2}{(\ln t_2)^2}\,\frac{q_2(1-q_2)(t_2+1)^{2\beta+2}}{\alpha^2\beta^2 t_2^2}$$

$$\left.-\frac{2\alpha^2}{\ln t_1 \ln t_2}\,\frac{q_1(1-q_2)(t_1+1)^{\beta+1}(t_2+1)^{\beta+1}}{\alpha^2\beta^2 t_1 t_2}\right]$$

$$=\frac{1}{nA}\left\{q_1(1-q_1)\left[\frac{(t_1+1)^{\beta+1}}{t_1\ln t_1}\right]^2\right.$$

$$+q_2(1-q_2)\left[\frac{(t_2+1)^{\beta+1}}{t_2\ln t_2}\right]^2$$

$$\left.-2q_1(1-q_2)\frac{(t_1+1)^{\beta+1}(t_2+1)^{\beta+1}}{t_1 t_2 \ln t_1 \ln t_2}\right\}. \quad \ldots (19)$$

We assumed $E(\tilde{\beta}) \simeq \beta$ and other similar expectations.

A numerical evaluation of $\text{Var}(\tilde{\beta})$ in (19) has been conducted on IBM 370/- for various values of q_1, q_2 and β and has shown that the efficiency of the estimator given by $\hat{\beta} / n\,[\,\text{Var}(\tilde{\beta})\,]$ where $\hat{\beta}$ is the maximum likelihood estimator of β ranges between 50.4% and 111.88% and is 101.96% for $q_1 = 0.60$, $q_2 = 0.95$ when $\beta = 2$ and varies between 62.16% and 99.4% for $\beta < 5$.

The equation

$$t_0 = \frac{\ln(t_1)}{\ln(t_2)}$$

is used to estimate β.

3. Estimation based on three ordered statistics.

An estimate based on 3 suitably selected ordered statistics may have small variance than an estimate based on two ordered statistics.

One may use ratio of sample median to a quasi-range, or ratio of some ordered statistics to a quasi-range. These statistics are selected so as to minimize the variance of the estimators of the parameters. Let y_p, y_r and y_s be any three ordered statistics. The ratio $y_r / (y_s - y_p)$ where $y_p \leq y_r \leq y_s$ eliminates α while estimating β.

Suppose we have

$$T = \frac{y_r}{y_s - y_p} = \frac{\ln t_2}{\ln t_3 - \ln t_1} \quad \dots (20)$$

where $t_i = (1 - q_i)^{-1/\beta} - 1, \quad i = 1, 2, 3.$

Some Special Cases.

If $p = i$ and $s = n + 1 - i$, $2 \leq i \leq \left[\frac{1}{2} n\right]$, then W$i$)(

$= Y_{(n+1-i)} - Y_{(i)}$ is the quasi-range and is an estimator of the standard deviation in normal samples and Cadwell (1953) has shown that $W_{(1)}$ is more efficient than any quasi-range for $n \geq 17$ but that $W_{(3)}$ is more efficient for $n \geq 32$ and so on. Quasi-ranges are useful in censored samples and have some robustness against outliers. In complete samples, their efficiency falls off but has computational and other simplicities. The ratio T in (20) is a statistic which leads to the elimination of the parameter α when estimating β. If r is chosen so that y_r becomes the median, then the numerator of (20) is an estimate of the median and the denominator is an estimate of standard deviation. The exact and limiting distribution of the ratio T has been studied by Birnbaum and Vineze (1973) for some values of p, r and s. Since the exact distribution of T was quite involved, Monte Carlo estimates of these probabilities are obtained by Tague (1969). A general class of related statistics is considered by Siddiqui (1960) and Birnbaum (1970, 1972) for many non-normal populations.

If we consider sample median as an estimate of mean and quasi-range as an estimate of standard deviation, we may use

$$T_1 = \frac{Y_{(\text{median})}}{Y_{(n+1-i)} - Y_{(i)}} \quad \dots (21)$$

and choose i such that Var $(\tilde{\beta})$ is minimum. Alternatively, one can use

$$T_2 = \frac{\frac{1}{2}\left[Y_{(n+1-i)} + Y_{(i)} \right]}{Y_{(n+i-1)} - Y_{(i)}} \quad \ldots (22)$$

One can choose some other combinations af available sample observations such that the variance of $\tilde{\beta}$ is minimum. Equation (21) becomes a special case of (20), when $q_1 = \frac{i}{n+1}$, $q_2 = \frac{1}{2}$ and $q_3 = 1 - q_1$.

We have

$$\mu_{q_2} = \alpha^{-1} \ln (2^{1/\beta} - 1)$$

$$\mu_{q_1} = \alpha^{-1} \ln [(1 - q_1)^{-1/\beta} - 1] ,$$

and

$$\mu_{q_3} = \alpha^{-1} \ln [q_1^{-1/\beta} - 1].$$

Variances and covariance of $\tilde{\alpha}$ and $\tilde{\beta}$ come from (8).

3.1. Variance of estimation of β based on three ordered statistics.

Suppose $\tilde{\beta}_1$ is the solution of the equation (20). We obtain variances as follows :

The logarithmic differentiation of the estimating equation

$$\frac{y_r}{y_s - y_p} = \frac{\ln t_2}{\ln t_3 - \ln t_1}$$

is

$$\frac{1}{y_r} \partial y_r - \frac{\partial y_s - \partial y_p}{y_s - y_p} = \frac{\partial t_2}{t_2 \ln t_2} - \frac{\left(\frac{\partial t_3}{t_3} - \frac{\partial t_1}{t_1} \right)}{(\ln t_3 - \ln t_1)}$$

where $t_i = (1 - q_i)^{-1/\beta} - 1$. Squaring both sides and taking the expectations, we have

$$\frac{\mathrm{Var}\,(y_r)}{y^2_{q_2}} + \frac{1}{(y_{q_3} - y_{q_1})^2}\{\,\mathrm{var}\,(y_p) + \mathrm{var}\,(y_s) - 2\,\mathrm{cor}\,(y_p\,, y_s)\,\}$$

$$- 2\,\frac{1}{y_{q_2}(y_{q_3} - y_{q_1})}\,[\mathrm{cov}\,(y_p\,, y_r) - \mathrm{cov}\,(y_r\,, y_s)\,]$$

$$= \Bigg[\frac{\mathrm{Var}\,(t_2)}{(t_2\,\ln\,t_2)^2} + \left(\frac{1}{\ln t_3 - \ln t_1}\right)^2 \Bigg[\frac{\mathrm{var}\,(t_3)}{t^2_3} + \frac{\mathrm{var}\,(t_1)}{t^2_1}$$

$$- 2\,\frac{\mathrm{cov}\,(t_1\,, t_3)}{t_1\,t_3} + 2\,\frac{1}{t_2 \ln t_2\,(\ln t_3 - \ln t_1)}$$

$$\left[\frac{\mathrm{cov}\,(t_1\,, t_2)}{t_1} - \frac{\mathrm{cov}\,(t_2\,, t_3)}{t_3}\right]\Bigg]$$

where

$$\mathrm{Var}\,(t_i) = \frac{1}{\beta^2}\,(t_i + 1)^2\,[\,\ln\,(t_i + 1)\,]^2\,\mathrm{var}\,(\tilde{\beta})$$

$$\mathrm{Cov}\,(t_i\,, t_j) = \frac{1}{\beta^2}\,(t_i + 1)\,(t_j + 1)\,\ln\,(t_i + 1)\,\ln\,(t_j + 1)\,\mathrm{var}\,(\hat{\beta}).$$

Thus

$$n\,\mathrm{Var}\,(\hat{\beta}) = \frac{\beta^2}{B}\left\{\frac{\mathrm{var}\,(y_r)}{y^2_{q_2}} + \frac{\mathrm{var}\,(y_s) + \mathrm{var}\,(y_p) - 2\,\mathrm{cov}\,(y_p, y_s)}{(y_{q_3} - y_{q_1})^2}\right.$$

$$\left. - 2\,\frac{\mathrm{cov}\,(y_p\,, y_r) - \mathrm{cov}\,(y_r\,, y_s)}{y_{q_2}(y_{q_3} - y_{q_1})}\right\}$$

where

$$B = \left[\frac{(t_2 + 1)\ln\,(t_2 + 1)}{t_2\,\ln\,t_2}\right]^2 + \left(\frac{1}{\ln t_3 - \ln t_1}\right)^2$$

$$\left[\left\{\frac{(t_3 + 1)\ln\,(t_3 + 1)}{t_3}\right\}^2 + \left\{\frac{(t_1 + 1)\ln\,(t_1 + 1)}{t_1}\right\}^2\right.$$

$$\left. - 2\,\frac{(t_1 + 1)\,(t_3 + 1)\,\ln\,(t_1 + 1)\,\ln\,(t_3 + 1)}{t_1\,t_3}\right]$$

$$+ \frac{2}{t_2 \ln t_2 [\ln t_3 - \ln t_1]} \left[\frac{(t_1+1)(t_2+1)\ln(t_1+1)\ln(t_2+1)}{t_1} - \frac{(t_2+1)(t_3+1)\ln(t_2+1)\ln(t_3+1)}{t_3} \right]$$

A table has been constructed to estimate β from the equation

$$t_0 = \frac{\ln t_2}{\ln t_3 - \ln t_1}$$

when t_0 is given for various values of q_1, q_2 and q_3. A numerical evaluation of Var $(\tilde{\beta})$ has been made for various values of q_1, q_2, q_3 and β and have shown that the efficiency of the estimator given by $\hat{\beta} / n$ [Var $(\tilde{\beta})$] where $\tilde{\beta}$ is the maximum likelihood estimator β, ranges between 66.2% and 93.1% for $\beta < 5$.

In this section, we specially deal with T and use $q_1 = \frac{\tilde{1}}{4}$ and $q_2 = \frac{3}{4}$ for the estimates of β. An example is given to illustrate this procedure.

For solution of $\tilde{\beta}$ from (21), Table 1 has been constructed for computation of $\tilde{\beta}$ for various values of q_1 and $\tilde{\beta}$. The values of T_1 are given for selected values q_1 and $\tilde{\beta}$. t is a decreasing function and a linear interpolation can give a good approximation for any intermediary values. For $\tilde{\beta} \rightarrow 0$, $T_1 \rightarrow \ln 2 / \ln (q_1{}^{-1} - 1)$ and $\tilde{\beta} \rightarrow \infty$, $T_1 \rightarrow -\infty$. If $\tilde{\beta} \rightarrow 1$, $T_1 \rightarrow 0$. As such $\tilde{\beta} \leq 1$ for $T_1 \geq 0$ and $\tilde{\beta} > 1$ for $T_1 < 0$.

TABLE 1. VALUES OF T_1 FUNCTION

q_1	$\tilde{\beta}$ = 0.01	0.1	0.2	0.5	0.9	2	3	5	10
.01	.1508	.1435	.1321	.0838	.0155	−.1177	−.1928	−.2878	−.4138
.05	.2354	.2283	.2119	.1337	.0242	−.1800	−.2930	−.4354	−.6245
.10	.3155	.3094	.2891	.1817	.0325	−.2390	−.3877	−.5753	−.8241
.25	.6309	.6275	.5959	.3712	.0653	−.4723	−.7638	−1.1308	−1.6179

Table 1 can be used for any $r \leq n / 4$. If $n = 100$ and $r = 4$, then $q_1 = 4 / 101 = 0.04$. If $n = 70$ and $r = 12$ then $q_1 = 12 / 71 = 0.17$ for which a double interpolation gives a value of $\tilde{\beta}$.

For a given value of $\tilde{\beta}$, $\tilde{\alpha}$ can best be estimated from the equation using the median function $y(m)$,

$$\tilde{\alpha} = \ln(2^{1/\tilde{\beta}} - 1) / y_{(m)} .$$

The asymptotic variances and covariance of $\tilde{\beta}$ and $\tilde{\alpha}$ (writing here-after y_r for $y_{(r)}$ and x_r for $x_{(r)}$), up to order n^{-1}, have the forms (see Kendall and Stuart [9, p. 253]).

$$\text{Var}(\tilde{\beta}) = a^{-2} V(t)$$

$$\text{Var}(\tilde{\alpha}) = \alpha^2 [b_m^2 \text{ var}(\tilde{\beta}) + (y_m x_m)^{-2} \text{ var}(x_m)$$
$$+ 2b_m (y_m x_m)^{-1} \text{ cov}(\tilde{\beta}, x_m)]$$

$$\text{Cov}(\tilde{\alpha}, \tilde{\beta}) = -\alpha [b_m \text{ var}(\tilde{\beta}) + (y_m x_m)^{-1} \text{ cov}(\tilde{\beta}, x_m)]$$

where

$$V(t) = y_m^{-2} \operatorname{var}(y_m) + (y_s - y_r)^{-2} \operatorname{var}(y_s - y_r) - 2y_m^{-1} (y_s - y_r)^{-1}$$

$$\operatorname{var}(y_m, y_s) + 2y_m^{-1} (y_s - y_r)^{-1} \operatorname{cov}(y_m, y_r)$$

$$\operatorname{Cov}(\tilde{\beta}, x_m) = (2a)^{-1} x_m y_m [V(t) - (y_s - y_r)^{-2}$$

$$\operatorname{var}(y_s - y_r) + y_m^{-2} \operatorname{var}(y_m)],$$

$$a = b_r - b_s - b_m,\ s = n - r + 1,\ a_r = R^{-1/\tilde{\beta}} - 1,\ a_s$$

$$= (1 - R)^{-1/\tilde{\beta}} - 1,$$

$$a_m = 2^{1/\tilde{\beta}} - 1,\ b_r = [\tilde{\beta}\, a_r \ln(a_r / a_s)]^{-1} (a_r + 1) \ln(a_r + 1),$$

$$b_s = [\tilde{\beta}\, a_s \ln(a_r / a_s)]^{-1} (a_s + 1) \ln(a_s + 1), \text{ and}$$

$$b_m = (\tilde{\beta}\, a_m \ln a_m)^{-1} (a_m + 1) \ln(a_m + 1).$$

Var ($x_{(u)}$) and cov ($x_{(u)}$, $x_{(v)}$) may be approximated by the formulae (Kendall and Stuart [9, p. 253]). The asymptotic variance of the maximum likelihood estimate, $\hat{\beta}$ for a given value of α is β^2 / n. The efficiency of $\tilde{\beta}$ for a given value of α compared to $\hat{\beta}$ is $(\beta)^2 / n\, V(\tilde{\beta})$.

3. Example.

To illustrate the method, a random sample of size 100 is drawn from the Burr distribution with $\beta = 1$, $\alpha = 1$. Using $q = 1/4$ the values of $x_{(n/4)} = x_1$, $x_{(m)}$ and $x\,(3n/4+1) = x_2$ are 0.186, 0.860 and 2.321 respectively. Then $y_1 = -1.68201$, $y_{(m)} = -0.14041$, $y_2 = 0.84200$. For $q_1 = 0.25$, $t = -0.05563$ and the value of $\tilde{\beta}$

from table 1 is $\tilde{\beta} = 1.091$, and from (5), $\tilde{\alpha} = 0.884$. For known $\alpha = 1$, the maximum likelihood estimate of $\tilde{\beta} = 1.1023$.

Estimates of the asymptotic variance and covariance, as given by equations at (6), are obtained by replacing the parameters by its corresponding samples values. In the example, we find $a_1 = 2.56322$, $a_2 = 0.30172$, $a_m = 0.88765$, $b_1 = 0.15704$, $b_2 = 0.48737$, $b_m = -10.39095$, $a = 10.06062$ var $(\hat{\alpha}) = 0.477$; cov $(\hat{\alpha}, \hat{\beta}) = -0.409$ and n var $(\tilde{\beta}) = 1.62263$ for all values of α. The asymptotic variance of the maximum likelihood estimate of β is 0.01215 for $\alpha = 1$. The efficiency of $\tilde{\beta}$ comes to about 74.86%.

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EXPLICIT 4 CYCLIC 4-STEP METHOD OF ORDER 5 TO SOLVE THE INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

By

R. ANSORGE

Inst. F. Angew. Mathematik
Der Universitat Hamburg,
Bundesstr, 55, 2000 Hamburg 13, West Germany.

and

NAZIR A. MIR,

Department of Mathematics
Bahauddin Zakariya University
Multan (Pakistan).

Abstract

In 1956 Dahlquist [5] showed that stable linear k-step methods cannot exceed a certain consistency order. In 1971 Donelson and Hansen [6] introduced cyclic methods which violate this restriction. Ansorge and Taubert [1] showed that strongly-stable cyclic methods can be applied to smooth as well as non-smooth problems, where the question of order is not important.

In this paper, the question of higher order of the cyclic strongly-stable methods for smooth problems is studied. First of all consistency conditions for m cyclic k-step methods in general for arbitrary order $q \geq 1$ are algebraically formulated and then 4 cyclic 4-step strongly stable methods of order 5 are constructed. Some numerical test cases are given. The results of these tests show unfavourable rounds off-error propagation. This unfavourable behaviour is then studied. As a result it seems that explicit 4 cyclic 4-step methods of order 5 inspite of strong stability are of no use for practical applications, even if smooth problems are under consideration.

Introduction :

Let us consider the initial value problem

$$y' = f(t, y)$$
$$y(0) = y_0 \,, \; 0 \leq t \leq T \qquad (1.1)$$

For the numerical approximate solution to the initial value problem (1.1) m cyclic k-step methods are to be considered. These methods are generalization of the linear k-step methods of the type

$$\sum_{\nu=0}^{k} a_\nu \, y_{n+\nu} + h \sum_{\nu=0}^{k} b_\nu \, f(t_{n+\nu}\,,\, y_{n+\nu}) = 0 \qquad (1.2)$$

$$(n = 0, 1, 2, \dots)$$

$a_\nu \,, b_\nu \in \mathrm{IR}$ $a_k \neq 0$ and $|a_0| + |b_0| > 0,$

Dahlquist [5] has shown that a stable, explicit k-step method ($b_k = 0$) of the form (1.2) has the maximal order $q = k$, a stable, implicit k-step method ($b_k \neq 0$) has the maximal order $q = k + 2$, when k is odd and $q = k + 1$, when k is even.

To avoid this restriction of "maximal order" m cyclic k-step methods have been introduced (compare the example given by Donelson and Hansen [6]).

$$Z_{m(n+1)} = A\, Z_{mn} + mh\, \tilde{\varphi}\,(t_{mn}\,, h\,, Z_{m(n+1)}\,, Z_{mn}) \qquad (1.3)$$

$$(n = 0, 1, 2, \dots)$$

with

$$Z_p = \begin{bmatrix} y_{p+k-1} \\ y_{p+k-2} \\ \dots \\ y_p \end{bmatrix} \in R^k \qquad (p = 0, 1, 2, \dots)$$

and $\tilde{\varphi}\,(t_{mn}\,, h\,, Z_{m(n+1)}\,, Z_{mn}) = \tilde{\beta}_0\,(t_{mn}\,, h)\, Z_{m(n+1)}$

$$+ \tilde{B}_1\,(t_{mn}\,, h)\, Z_{mn}\,,$$

Where $\tilde{B}_0$, $\tilde{B}_1$ are formal matrices, the elements of which are for every fixed t and every fixed h non-linear mappings from IR into IR. The linear k-step method (1.2) is a special case of (1.3), namely for $m = 1$, when in addition to equation (1.2) also the $k-1$ trivial equations

$$
\begin{aligned}
y_{n+k-1} &= y_{n+k-1} \\
y_{n+k-2} &= y_{n+k-2} \\
&\vdots \\
y_{n+1} &= y_{n+1}
\end{aligned}
$$

are considered (Urabe [7]). We then obtain

$$
\begin{bmatrix} y_{n+k} \\ y_{n+k-1} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+1} \end{bmatrix}
=
\begin{bmatrix}
\frac{-a_{k-1}}{a_k} & \frac{-a_{k-2}}{a_k} & \cdots & \frac{-a_1}{a_k} & \frac{-a_0}{a_k} \\
1 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\begin{bmatrix} y_{n+k-1} \\ y_{n+k-2} \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}
$$

$$
+ h \begin{bmatrix}
\frac{-b_k}{a_k} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix} f(t_{n+k}, y_{n+k}) \\ f(t_{n+k-1}, y_{n+k-1}) \\ \cdot \\ \cdot \\ f(t_{n+1}, y_{n+1}) \end{bmatrix}
$$

$$
+ h \begin{bmatrix}
\frac{-b_{k-1}}{a_k} & \frac{-b_{k-2}}{a_k} & \cdots & \frac{-b_0}{a_k} \\
0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix} f(t_{n+k-1}, y_{n+k-1}) \\ f(t_{n+k-2}, y_{n+k-2}) \\ \cdot \\ \cdot \\ f(t_n, y_n) \end{bmatrix}
$$

i.e.

$$Z_{n+1} = A\, Z_n + h\, (\tilde{B}_0 (t_n , h)\, Z_{n+1} + \tilde{B}_1 (t_n , h)\, Z_n),$$

where

$$A = \begin{bmatrix} \frac{-a_{k-1}}{a_k} & \frac{-a_{k-2}}{a_k} & \cdot\;\cdot\;\cdot & \frac{-a_1}{a_k} & \frac{-a_0}{a_k} \\ 1 & 0 & \cdot\;\cdot\;\cdot & 0 & 0 \\ 0 & 1 & \cdot\;\cdot\;\cdot & 0 & 0 \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ 0 & 0 & \cdot\;\cdot\;\cdot & 1 & 0 \end{bmatrix}$$

$$\tilde{B}_0 (t_n , h)\, Z_{n+1} = B_0 \begin{bmatrix} f(t_{n+k} , y_{n+k}) \\ f(t_{n+k-1} , y_{n+k-1}) \\ \cdot \\ \cdot \\ f(t_{n+1} , y_{n+1}) \end{bmatrix}$$

$$\tilde{B}_1 (t_n , h)\, z_n = B_1 \begin{bmatrix} f(t_{n+k-1} , y_{n+k-1}) \\ f(t_{n+k-2} , y_{n+k-2}) \\ \cdot \\ \cdot \\ f(t_n , y_n) \end{bmatrix}$$

with

$$B_0 = \begin{bmatrix} \frac{-b_k}{a_k} & 0 & \cdot\;\cdot\;\cdot & 0 \\ 0 & 0 & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0 & 0 & \cdot\;\cdot\;\cdot & 0 \end{bmatrix} \quad (k \times k \text{ matrices})$$

$$B_1 = \begin{bmatrix} \frac{-b_{k-1}}{a_k} & \frac{-b_{k-2}}{a_k} & \cdots & \frac{-b_0}{a_1} \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The generalization (1.3) can analogously be written as :

$$Z_{m(n+1)} = A\, Z_{mn} + mh\, B_0 \begin{bmatrix} f(t_{m(n+1)+k-1}, y_{m(n+1)+k-1}) \\ f(t_{m(n+1)+k-2}, y_{m(n+1)+k-2}) \\ \cdot \\ \cdot \\ f(t_{m(n+1)}, y_{m(n+1)}) \end{bmatrix} \quad (1.4)$$

$$+ mh\, B_1 \begin{bmatrix} f(t_{mn+k-1}, y_{mn+k-1}) \\ f(t_{mn+k-2}, y_{mn+k-2}) \\ \cdot \\ \cdot \\ f(t_{mn}, y_{mn}) \end{bmatrix},$$

where A, B_0, B_1 are $k \times k$ matrices which are now fully occupied (P. Albrecht [3]). The m cyclic k-step method can also be defined as follows :

Definition 1.1.

If m linear k-step methods are applied cyclically in a fixed order to calculate successively the approximate vectors Z_{in} $(i = 1\,(1)\,m)$ where the initial vector Z_0 is assumed to be already calculated by a certain other method then the method is said to be m cyclic k-step method.

Definition 1.2.

The method (1.4) is called explicit, when $B_0 = \theta$ ($k \times k$, – zero matrix), otherwise implicit.

Dahlquist's stability condition [5] for linear k-step methods (1.2), namely that the roots of the polynomial

$$\rho(\mu) = \sum_{\nu=0}^{k} a_\nu \mu^\nu$$

must not lie outside the unit circle and that the roots on the unit circle must be simple, is equivalent to the condition

$$\exists\, k_0 < \infty\ \forall\, n \in \mathrm{IN} : \| A^n \| < k_0 \tag{1.5}$$

(Ansorge [2]) for any norm on IR^k. We also use (1.5) as definition of stability in case $m \geq 1$.

Definition 1.3.

The linear m cyclic k-step method is said to be convergent, if

$$\lim_{\substack{n \to \infty \\ n \to 0 \\ nh \to t}} Z_n = \begin{bmatrix} y(t) \\ y(t) \\ \cdot \\ \cdot \\ y(t) \end{bmatrix} = y(t)\, e \tag{1.6}$$

for permissible initial vector fields, where $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathrm{IR}^k$

Definition 1.4.

The method (1.4) is said to be strong-stable, when the matrix A in addition to simple eigenvalue $\mu_1 = 1$ possesses only eigenvalues which lie in the inside of the unit circle.

Convergence theorems for cyclic methods have been proved in the case of globally lipshitz-continuous right - hand - sides of equation (1.1). Ansorge and Taubert [1] showed that in the case of non-smooth problems cyclic methods can also be convergent, if they are strongly stable and the consistency conditions are weaker than the classical ones in the case of real cyclic methods ($m > 1$). In this case the question of high order is not important (Taylor-Expansion is not possible).

In this paper, however, we shall construct explicit 4 cyclic 4-step methods of order 5 (violating Dahlquist's maximal order condition) which are strongly stable (inorder to ensure that non-smooth problems can also be included).

2. Algebraic consistency conditions for higher order of m cyclic k-step methods.

Let there exist a unique solution of the initial value problem (1.1) and $y(t)$ the exact solution of (1.1). Let $f(t, y(t))$ be sufficiently smooth. We prove the following theorem.

Theorem 2.1.

The m cyclic linear k-step method (1.4), namely

$$Z_{m(n+1)} = A\, Z_{mn} + mh\, B_0 \begin{bmatrix} f(t_{m(n+1)+k-1}, y_{m(n+1)+k-1}) \\ f(t_{m(n+1)+k-2}, y_{m(n+1)+k-2}) \\ \cdot \\ \cdot \\ \cdot \\ f(t_{m(n+1)}, y_{m(n+1)}) \end{bmatrix} \quad (2.1)$$

$$+ mh\, B_1 \begin{bmatrix} f(t_{mn+k-1}, y_{mn+k-1}) \\ f(t_{mn+k-2}, y_{mn+k-2}) \\ \cdot \\ \cdot \\ \cdot \\ f(t_{mn}, y_{mn}) \end{bmatrix}$$

With $k \times k$ matrices A, B_0, B_1 is consistent with the initial value problem (1.1) if the following conditions hold :

$$Ae = e, \qquad e = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \in R^k \tag{2.2}$$

and $(A-I)\, c = m\,(I - \varphi)\, e$,
where $\varphi = B_0 + B_1$,

$$c = \begin{bmatrix} k-1 \\ k-2 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \text{ I is } k \times k \text{ identity matrix.}$$

The method (2.1) is of order q if simultaneously the following equations hold :

$$(A-I)\, c^p = pm\,(I-\varphi)\, c^{p-1} + \sum_{v=0}^{p-2} \binom{p}{v} m^{p-v}\, (I-(p-v)\, B_0)\, c^v$$

$$(p=0, 1, 2, \dots, q) \tag{2.3}$$

where the sum on the right-hand-side for $p \leq 1$ is considered to be θ (empty sum). We define also

$$c^p = \begin{bmatrix} (k-1)^p \\ (k-2)^p \\ \cdot \\ \cdot \\ \cdot \\ 1^p \\ 0 \end{bmatrix} \text{ with } c^0 = e \qquad (2.4)$$

Proof :

The method (2.1) can be written as :

$$Z_{m(n+1)} - A\,Z_{mn} - mh\,B_o\,(f(t_{m(n+1)+k-1},\; y_{m(n+1)+k-1}),\; \dots,\; f(t_{m(n+1)},\, y_{m(n+1)}))^T$$

$$-mh\,B_1\,(f(t_{mn+k-1}, y_{mn+k-1}),\; \dots,\; f(t_{mn},\, y_{mn}))^T = 0 \qquad (2.5)$$

The method (2.5) is said to be consistent with initial value problem (1.1), if for all $t + (m+k-1)\,h \in [0, T]$ the inequality

$$\| z(h, t+mh) - Az(h, t) - mh\,B_0\,(y'(t+(m+k-1)h),$$

$$y'(t+(m+k-2)h)\; \dots,\; y'(t+mh))^T - mh\,B_1\,(y'(t+(k-1)h),$$

$$y'(t+(k-2)h),\; \dots,\; y'(t))^T \| \leq \varepsilon(h, y_0) \qquad (2.6)$$

hold, where ε is independent of all $t \in [0, T]$ and $\varepsilon(h, y_0) = 0(h)$, i.e.

$$\lim_{h \to 0} \frac{\varepsilon(h, y_0)}{h} = 0.$$

The method of order q if for (2.5) in particular holds

$$\varepsilon(h, y_0) \leq (\text{constant})\, h^{q+1} \qquad (2.7)$$

The vector $z(h, t)$ originates from z_p if the value of y_i in the definition of z_p are replaced by the unknown values of the solution $y(t)$ of (2.1) :

$$z(h, t) = \begin{bmatrix} y(t + (k-1)h) \\ y(t + (k-2)h) \\ \cdot \\ \cdot \\ \cdot \\ y(t) \end{bmatrix} \tag{2.8}$$

Applying Taylor's expansion to $z(h, t + mh)$ at the point $(0, t)$ and using (2.8), we obtain

$$z(h, t + mh) = y(t)\, e + h\,(\mathrm{I}c + me)\, y'(t)$$
$$+ \frac{h^2}{2!} \sum_{\nu=0}^{2} \left(\binom{2}{\nu} m^\nu \, c^{2-\nu} \, y''(t) \right)$$
$$+ \ldots\ldots\ldots$$
$$+ \frac{h^q}{q!} \sum_{\nu=0}^{q} \left(\binom{q}{\nu} m^\nu \, c^{q-\nu} \, y^{(q)}(t) \right) + \ldots$$

Applying Taylor's expansion also to the values of y' in (2.6) and then comparing the coefficients of individual powers of h on both sides, we get, first of all for h^0 :

$$\mathrm{A}e = e. \tag{2.10}$$

A possesses therefore the eigenvalue $\mu_1 = 1$ with corresponding eigen-vector e, compairing for h^1 we get

$$(\mathrm{A} - \mathrm{I})\, c = m\,(\mathrm{I} - \phi)\, e, \tag{2.11}$$

Dahlquist's consistency conditions for linear k-step methods are

$$\rho(1) = 0 \tag{2.12}$$

and $\sigma(1) + \rho'(1) = 0$

with

$$\rho(\mu) = \sum_{\nu=0}^{k} a^\nu, \quad \sigma(\mu) = \sum_{\nu=0}^{k} b_\nu \mu^\nu .$$

(2.10) corresponds to the first condition in (2.12) and (2.11) corresponds to the 2nd, condition in (2.12). Equation (2.10) and (2.11) quaranty the order one. Campa-ring the coefficients of h^p and using Binomial theorem, we have

$$(A-1)\, c^p = pm\,(I-\phi)\, c^{p-1} + \sum_{\nu=0}^{p-2} \binom{p}{\nu} m^{p-\nu} (I-(p-\nu)\, B_0)\, c^\nu \tag{2.13}$$

$$(p = 0, 1, 2, \dots, q)$$

3. Construction of strongly stable methods.

In this section we shall construct explicit 4 cyclic 4-step methods of order 5. Therefore, we put $B_0 = \theta$ in equation (2.14) and solve the following problem :

We determine the 4×4 matrices (i.e. $m = k = 4$) A, ϕ ($= B_1$), so that they satisfy the first 6 linear equations in (2.13) (namely for $p = 0, 1, \dots, 5$). Each of these equations represents 4 single equations. Therefore, we have to solve 24 equations for 32 unknowns i.e. 4 systems having each 6 equations with 8 unknowns. At the same time it is also achieved that $\mu_1 = 1$ is an eigenvalue of A with the corresponding eigenvèrctor e. Besides that we are also at liberty to choose the surplus unknowns in such a way that the absolute values of the other three eigenvalues of A remain less than one (making the method strongly stable).

Adopting the above mentioned procedure, we have computed the elements of A and $\varnothing$ for two methods, where it was demanded that except for $\mu_1 = 1$ the other three eigenvalues of A are zero and are compiled in the following tables :

Method 3.1 **Table 3.1**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\doteq \begin{bmatrix} -795 & 36 & -260 & 1020 \\ -1010 & 36 & 954 & 21 \\ -898.5057916 & 36 & 1149.8240196 & -286.3182280 \\ -906.7070466 & 36 & 1261.5310662 & -389.8240196 \end{bmatrix}$$

$$
\varnothing = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{bmatrix}
= \begin{bmatrix} 90 & 180 & 285 & 90 \\ 90.75 & 319.5 & 92.25 & 0 \\ 74.6013344 & 306.60049 & 16 & -27.1340005 \\ 72.5295037 & 317.3827665 & -6 & -36.2647518 \end{bmatrix}
$$

Just for control the eigenvalues of A (which must have the values 1, 0, 0, 0) were computed again with a programme of computer centre library of the university Hamburg. This yields the eigenvalues.

Table 3.2

	Real Part	Imaginary Part
μ_1 :	1.000000434 D + 00	0.0
μ_2 :	–6.9471401384 D – 03	0.0
μ_3 :	3.4733526754 D – 03	6.3924651436 D – 03
μ_4 :	3.4733526754 D – 03	6.3924651436 D – 03

which inspite of round off error lie in the inside of the unit circle.

Table 3.3

Method 3.2

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & o_{42} & a_{43} & a_{44} \end{bmatrix}
= \begin{bmatrix} -435 & 360 & -908 & 984 \\ -194 & 360 & -342 & 177 \\ -538.5057916 & 360 & 501.8240196 & -322.3182280 \\ -546.7070466 & 360 & 613.5310662 & -425.8240196 \end{bmatrix}
$$

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 18 & 204 & 90 \\ 27.75 & -4.5 & 11.25 & 18 \\ 47.6013344 & 144.2060049 & -65 & -27.1340005 \\ 45.5295057 & 155.3827665 & -87 & -36.2647518 \end{bmatrix}$$

Again the eigenvalues of A (which must have the values 1, 0, 0, 0) were calculated with a computer programme.

Table 3.4

	Real Part	Imaginary Part
μ_1 :	9.9999735527 D−01	0.0
μ_2 :	1.3647709203 D−02	0.0
μ_3 :	−6.8225322390 D−03	1.1885533095 D−02
μ_4 :	−6.8225322390 D−03	1.1885533095 D−02

Here the eigenvalues lie again in the inside of a unit circle.

4. Numerical example and reasoning for unfavourable round off error propagation.

The methods (3.1) and (3.2) were applied to the test-equation

$$y' = qy, \; y_0 = 1 \qquad (4.1)$$

$$qt \; \varepsilon \; \mathrm{IR}$$

with the solution $y = e^{qt}$.

The results show unfavourable round off error reproduction. In-order to give the reasoning for this unfavourable behaviour, consider a 4 cyclic, explicit, linear k-step method

$$z_{4(n+1)} = Az_{4n} + 4h\,\phi\, f(z_{4n}) \qquad (4.2)$$

where $f(z_n) \begin{bmatrix} f(t_{n+k-1}, y_{n+k-1}) \\ f(t_{n+k-2}, y_{n+k-2}) \\ \cdot \\ \cdot \\ \cdot \\ f(t_n, y_n) \end{bmatrix}$

Let $\hat{z}_{4(n+1)} = A\,\hat{z}_{4n} + 4h\,\phi\, f(\hat{z}_{4n}) + \varepsilon_{4n}$ (4.3)

where ε_{4n} is the local round of error by nth step for the method (4.2), and the symbol "∧" over the vectors in (4.3) means that really calculated vectors are considered.

Subtracting equation (4.2) from (4.3), we obtain :

$$\hat{z}_{4(n+1)} - z_{4(n+1)} = A(\hat{z}_{4n} - z_{4n}) + 4h\,\varnothing\,(f(\hat{z}_{4n}) - f(z_{4n})) + \varepsilon_{4n}.$$

Applying the mean value theorem, and using the test-equation

we obtain with $H = 4hq$ (4.4)

$$\hat{z}_{4(n+1)} - z_{4(n+1)} = (A + H\varnothing)(\hat{z}_{4n} - z_{4n}) + \varepsilon_{4n} \quad (4.5)$$

(4.5) gives recursively

$$\hat{z}_{4n} - z_{4n} = (A + H\varnothing)^n(\hat{z}_o - z_o) + (A + H\varnothing)^{n-1}\varepsilon_o + (A + H\varnothing)^{n-2}\varepsilon_4 + \ldots + (A + H\varnothing)\,\varepsilon_{4(n-2)} + \varepsilon_{4(n-2)} \quad (4.6)$$

Let $= \max\limits_{0 \le v \le n-1} \| \varepsilon_{4(n-1-v)} \|$. Equation (4.6) then reduces to

$$\| \hat{z}_{4n} - z_{4n} \| \le \| (A + H\varnothing) \|^n \, \| \hat{z}_o - z_o \| + \varepsilon \frac{\| (A+H) \|^n - 1}{\| (A+H) \| - 1}$$

If there are no round off error at the start i.e. $\hat{z}_0 = z_0$ then it follows that

$$\| \hat{z}_{4n} - z_{4n} \| \leq \varepsilon \frac{\| (A+H) \|^{n} - 1}{\| (A+H) \| - 1} \tag{4.7}$$

It is now obvious that for the round off error reproduction in the case of non-zero H not the eigenvalues of A are responsible but the spectral radius of $(A + H \varnothing)$. We have considered a series of values of H between-0.1 and 0.1, and the corresponding spectral radii of $A + H \varnothing$ for the methods (3.1) and (3.2) are calculated :

Method 3.1

H	Spectral radius of A + H ∅
$-0.1 = -100.10^{-3}$	167.435
-90.10^{-3}	157.42
-80.10^{-3}	146.99
-70.10^{-3}	136.061
-60.10^{-3}	124.517
-50.10^{-3}	112.191
-40.10^{-3}	98.825
-30.10^{-3}	83.983
-20.10^{-3}	66.81
-10.10^{-3}	48.325
0	1
10.10^{-3}	47.576
20.10^{-3}	56.867
30.10^{-3}	81.655
40.10^{-3}	94.09
50.10^{-3}	105.014
60.10^{-3}	114.857
70.10^{-3}	123.877
80.10^{-3}	132.244
90.10^{-3}	140.831
$0.1 = 100.10^{-3}$	147.453

(Compare with **Fig. 4.1**)

Method 3.2

H	Spectral radius of A + H ∅
$-0.1 = -100.10^{-3}$	189.379
-90.10^{-3}	181.379
-80.10^{-3}	172.883
-70.10^{-3}	163.785
-60.10^{-3}	153.94
-50.10^{-3}	143.139
-40.10^{-3}	131.048
-30.10^{-3}	117.095
-20.10^{-3}	100.128
-10.10^{-3}	77.016
1	1
10.10^{-3}	72.84
20.10^{-3}	93.745
30.10^{-3}	108.939
40.10^{-3}	121.351
50.10^{-3}	132.054
60.10^{-3}	141.577
70.10^{-3}	150.222
80.10^{-3}	158.187
90.10^{-3}	165.601
$0.1 = 100.10^{-3}$	172.562

(Compare with Fig. 4.2)

From tables and the corresponding figures 4.1, 4.2 it is obvious that except for $h = 0$ the spectral radii of $A + H \varnothing$ are extraordinary large. That is why the round off error goes on increasing. As a result it seems that explicit 4 cyclic 4-step method of order 5 inspite of strong stability are of no use for practical applications, also if only smooth problems are under consideration.

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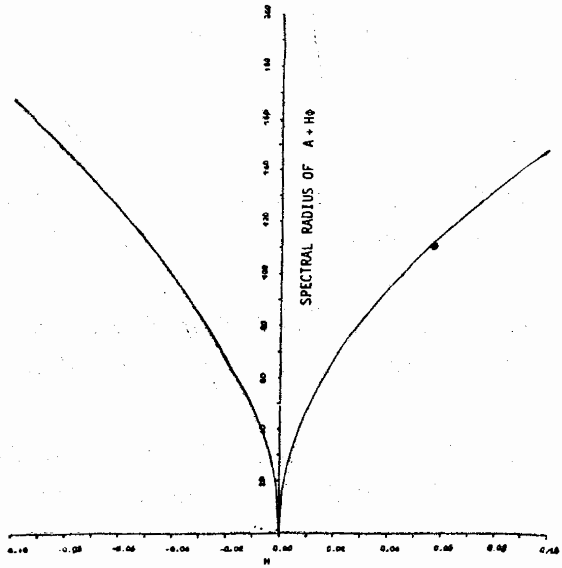

Fig. 4.1

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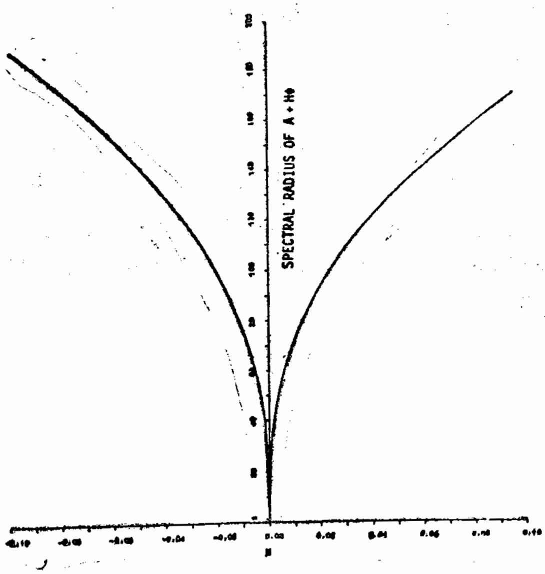

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MAXIMAM LIKELIHOOD METHOD FOR SOLVING FREDHOLM CONVOLUTION TYPE OF EQUATIONS OF THE FIRST KIND USING TRIGONOMETRIC APPROXIMATIONS

By
M. IQBAL

Department of Mathematics,
Punjab University, New Campus,
Lahore-20, Pakistan.

1. Introduction.

Consider the Fredholm Integral equation of the first kind of convolution type :

$$(Kf)(x) \equiv \int_{-\infty}^{\infty} k(x-y)\, f(y)\, dy = g(x) \quad -\infty < y < \infty \quad (1.1)$$

where k and g are known functions in L_2 (R), and $f \varepsilon H^p$ (R) is to be found. If $\wedge$ denotes Fourier Transformation, then from the convolution theorem we have

$$\hat{k}(\omega)\, \hat{f}(\omega) = \hat{g}(\omega) \quad \ldots (1.2)$$

whence

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\check{g}(\omega)}{\check{k}(\omega)} \exp(i\, \omega\, y)\, d\omega \quad \ldots (1.3)$$

The ill-posedness of (1.1) is reflected by the fact that any small perturbation ε in g, whose transform $\hat{\varepsilon}(\omega)$ does not decay faster than $\hat{k}(\omega)$ as $|\omega| \rightarrow \infty$, will result in a perturbation In $\hat{g}(\omega) / \hat{k}(\omega)$

which will grow without bound. When g is inexact, therefore, we may seek a stable or filtered approximation to f given by

$$f_{\lambda}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega\, ; \lambda) \frac{\hat{g}(\omega)}{\hat{k}(\omega)} \exp(i\, \omega\, y)\, d\omega \qquad \ldots (1.4)$$

where $z(\omega, \lambda)$ is a filtered function dependent on a parameter λ. Filters may be constructed in several ways, either directly for the convolution Kernel [1], or as a special case of general Fredholm integral equations [2], provided in the latter case it is realized that in (1.1) the operator K is not compact and the fourier transform (FT) here plays the role of singular function expansion in the context of compact operators.

It this paper we construct a maximum likelihood (ML) method which determines the regularization parameter λ optimally. Our construction of the method is a simple extension of the ideas of Anderssen and Bloomfield [4, 5], who consider the problem of numerically differentiating noisy data.

2. Description of the Method.

We assume that the support of each function, f, g and k is essentially finite and contained within the interval [0, T]. Let T_{N-1} denote the space of trigonometric polynomials of degree at most $N-1$ and period T. We shall seek a filtered solution of (1,1) within the space T_{N-1}

Let $g_n = g_N(x_n) + \varepsilon_n$

We assume that $g_N(x)$ and ε_n are stationary stochastic processes with zero mean, where

$$g_N(x) = \int_0^T \exp(i\, \omega\, x)\, d\zeta_{g_N}(\omega) = \frac{1}{N} \sum_q \hat{g}_{N,q} \exp(i\, \omega_q\, x). \qquad \ldots (2.2)$$

and

$$\varepsilon_n = \int_0^T \exp(i\, \omega\, x)\, d\zeta_{\varepsilon}(\omega) \qquad \ldots (2.3)$$

the relevant features of the functions ζ_{g_N} and ζ_{ε} is that the variance of an integral

$$\int_o^T \theta(\omega)\, d\zeta_{g_N}(\omega) \text{ is } \int_o^T |\theta(\omega)|^2 P_u(\omega)\, d\omega \qquad \dots (2.3)$$

Suppose now that we have a filter $\left\{ l_k \right\}_{k=-\infty}^{\infty}$ such that (the detrended) $f_N(x_k)$ is estimated by

$$\Sigma\, l_{k-m}\, g_m \qquad (A)$$

Since
$$f_N(x) = \int_o^T \exp(i\,\omega\, x)\, d\zeta_{g_N}(\omega) / \hat{k}_N(\omega) \qquad \dots (2.4)$$
where
$$\hat{k}_N(\omega) = \Sigma\, k_n \exp(-\, i\, \omega\, x_n)$$

The error of estimate is

$$f_N(x_n) - \Sigma\, l_{n-m}\, g_m = \int_o^T \exp(i\,\omega\, x_n) \times \left[\frac{1}{\hat{k}_N(\omega)} - \hat{l}_N(\omega) \right] d\zeta_{g_N}(\omega) - \int_o^T \exp(i\,\omega\, x_n)\, \hat{l}_N(\omega)\, d\zeta_{\varepsilon}(\omega) \qquad \dots (2.5)$$

where

$$\hat{l}(\omega) = \sum_{r=-\infty}^{\infty} l_r \exp(-\, i\, \omega\, x_r) \qquad \dots (2.6)$$

then the variance of (2.5) is

$$\int_o^T \left| \frac{1}{\hat{K}_N(\omega)} - \hat{l}_N(\omega) \right|^2 P_{g_N}(\omega)\, d\omega + \int_o^T \left| \hat{l}_N(\omega) \right|^2 P_{\varepsilon}(\omega)\, d\omega \qquad \dots (2.7)$$

which is minimized when

$$\hat{l}_N(\omega)\,\hat{k}_N(\omega) = \frac{P_{g_N}(\omega)}{P_{g_N}(\omega) + P_\varepsilon(\omega)} = \beta(\omega) \qquad \text{... (2.8)}$$

We now find the relationship between the filter $\hat{l}(\omega)$ given by (2.6) and the filter $z(\omega)$ [7].

We require that filtered solution has Fourier Transform

$$\hat{f}_{N,\,q\,;\,\lambda} = \hat{l}_N(\omega_q)\,\hat{g}_N(\omega_q) = z_{q,\,\lambda}\,\hat{g}_N(\omega_q)\,/\,\hat{k}_N(\omega_q) \qquad \text{(2.9)}$$

where

$$\hat{g}_N(\omega) = \sum_n g_n \exp(-i\,\omega\,x_n).$$

We can compare this with the F.T. of (A) to obtain

$$Z(\omega)\,\hat{V}(\omega)\,/\,\hat{K}(\omega) = \hat{l}(\omega)\,\hat{V}(\omega)$$

or

$$Z(\tilde{\omega}) = \hat{l}(\omega)\,\hat{k}(\omega) \qquad \text{... (2.10)}$$

Thus in our method the ratio $\beta(\omega)$ in (2.8) in A and B's work [4, 5] is equivalent to our filter $Z(\omega)$.

3. Optimization by Maximum Likelihood.

To optimize the filter with respect to λ, we now modify A and B's work accordingly [4, 5]. This involves choosing an error distribution of the form $P_\varepsilon(\omega) = b\,\phi(\omega)$ where b is an unknown constant and $\phi(\omega)$ is a known function. Also, for second order filter (i.e. $p = 2$) we choose as the distribution for g_N

$$P_{g_N}(\omega) = \frac{b\phi(\omega)\,|\,\hat{K}(\omega)\,|^2}{\lambda\tilde{\omega}^4} \qquad \text{... (3.1)}$$

so that $z(\tilde{\omega}) = |\hat{K}(\omega)|^2 / (|\hat{K}(\omega)|^2 + B\lambda\tilde{\omega}^4)$

$$= \beta(\tilde{\omega}) \qquad \text{... (3.2)}$$

Thus the distribution for g_n is given by $P_{g_n} = P_{g_N} + P_\varepsilon$.

$$P_{g_n}(\omega\,;\,b,\lambda) = b\,\phi(\omega) + \frac{b\,\phi(\omega)\,|\hat{K}(\omega)|^2}{B\lambda\tilde{\omega}^4}$$

$$= b\,\phi(\omega)\left[1 + \frac{|\hat{K}(\omega)|^2}{B\lambda\tilde{\omega}^4}\right], \left(\because B = \frac{N^2}{T^2}\right)$$

$$\text{... (3.3)}$$

(Let $\phi(\omega) = 1$), $P_\varepsilon = b\,\phi(\omega) = b$

$$\therefore\ P_{g_n}(\omega\,;\,b,\lambda) = b\left[1 + \frac{|\hat{K}(\omega)|^2}{B\lambda\tilde{\omega}^4}\right] \qquad \{B = \text{const}\}$$

$$= \frac{1}{1 - z_q} \qquad \text{... (3.4)}$$

$$z_q = \frac{|\hat{K}_q|^2}{|\hat{K}_q|^2 + B\lambda\tilde{\omega}_q^4} \qquad \text{... (3.5)}$$

and

$$I(\omega_q) = \left|\sum_{k=0}^{N-1} g_k \exp(-i\,\omega\,x_k)\right|^2 = |\hat{g}_q|^2 \qquad (3.6)$$

Anderssen and Bloom-field show how to eliminate the constant b from the problem.

First they approximate the likelihood function of the parameters λ, b by using a formula due to whittle [16]. This says that the logarithm of the likelihood function of P_{g_n} is approximately.

$$L \equiv \text{Constt.} - \tfrac{1}{2} \sum_{q=0}^{N-1} \left[\log P_{g_n}(\omega_q) + I(\omega_q) / P_{g_n}(\omega_q) \right] \quad \ldots (3.7)$$

(3.7) can be maximized w.r.t. λ which is equivalent to MINIMIZING (For minimizing we have used NAG Routine EO 4 ABA based upon quadratic interpolation technique.)

$$V(\lambda) = (N/2) \log \left[\sum_{q=0}^{N-1} |\hat{g}_q|^2 (1 - z_q) \right] \sum_{q=1}^{N-1} \log (1 - z_q) \quad \ldots (3.8)$$

Knowing λ from (3.8)

$$\hat{f}_{\lambda, q} = \sum_{q=0}^{N-1} z_q \, | \hat{g}_q / \hat{k}_q | \quad \ldots (3.9)$$

then by inverse F.T. of (3.9) we can find the solution function f.

Problems Discussed.

P (IA). This problem is given in [8] we have

$$\int_{-2}^{2} k(x-y) f(y) \, dy = g(x)$$

where f is the function of two gaussian functions

$$f(x) = 0.5 \exp \left[\frac{-(x+0.4)^2}{0.18} \right] + \exp \left[\frac{-(x-0.4)^2}{0.18} \right]$$

with essential support $-1.3 < x < 1.5$

K (x) is triangular with equation

$$K(x) = \begin{cases} -x + 0.5 & 0 \le x < 0.5 \\ x + 0.5 & -0.5 \le x < 0 \\ 0 & |x| \ge 0.5 \end{cases}$$

We have calculated the values of $g(x)$ by the NAG Algorithm DOIABA Using Rombergs' Method with accuracy 10^{-7}. 41 grid valuess have been considered.

P (1 B).

This example is the same as P (1 A) except that the triangular Kernel is made wider

$$K(x) = \begin{cases} (5/8)(-x+0.8), & 0 \le x < 0.8 \\ (5/8)(x+0.8), & -0.8 \le x < 0 \\ 0 & |x| \ge 0.8 \end{cases}$$

The wider Kernel makes the problem more ill-posed, 64 grid points are considered.

P (1 C).

The problem is made highly ill-posed by choosing an even wider Kernel 64 grid points are considered.

$$K(x) = \begin{cases} (5/12)(-x+1.2), & 0 \le x < 1.2 \\ (5/12)(x+1.2), & -1.2 \le x < 0 \\ 0 & |x| \ge 1.2 \end{cases}$$

P (2). This problem has been taken from [9]. The solution function is the sum of six Gaussians and the kernel is also Gaussian. We have

$$\int_{-\infty}^{\infty} k(x-y)\, f(y)\, dy = g(x)$$

$$g(x) = \sum_{k=1}^{6} A_k \exp\left[\frac{-(x-\alpha_k)^2}{\beta_k}\right]$$

where

$A_1 = 10.0$	$\alpha_1 = 0.5$	$\beta_1 = 0.04$
$A_2 = 10.0$	$\alpha_2 = 0.7$	$\beta_2 = 0.02$

$A_3 = 5.0$	$\alpha_3 = 0.875$	$\beta_3 = 0.02$
$A_4 = 10.0$	$\alpha_4 = 1.125$	$\beta_4 = 0.04$
$A_5 = 5.0$	$\alpha_5 = 1.325$	$\beta_5 - 0.02$
$A_6 = 5.0$	$\alpha_6 = 1.525$	$\beta_6 = 0.02$

The essential support of $g(x)$ is $0 < x < 2$

$$k(x) = \frac{1}{\sqrt{\lambda \pi}} \exp(-x^2 / \lambda), \qquad \lambda = 0.015$$

The essential support of $k(x)$ is $(-0.26.0.26)$

The solution is

$$(x) = \sum_{k=1}^{6} \left(\frac{\beta_k}{\beta_k - \lambda} \right)^{\frac{1}{2}} A_k \exp \left[- \frac{(x-\alpha_k)^2}{(\beta_k - \lambda)} \right]$$

The essential support of $f(x)$ is $(0.26, 1.74)$.

4. Numerical Results.

Random noise in the Problems is used.

P (1 A).

The interval $[-2, 2]$ which is the essential support of $f(x)$ was translated to $[0, 4]$ and then extended to $[0, 6.4]$ thus $T = 6.4$, $h = 0.1$ and $N = 64$.

The numerical solutions with clean data and noisy data have been compared with the true solution in the essential support of $f(x)$.

P (1 A) (i) with clean data the algorithm gave a value of regularization parameter $\lambda = (5.8)\ 10^{-13}$. The solution shown in DIAG (1) is very good.

(ii) Data with 3.3% noise. Here $\lambda = (8.1)\ 10^{-7}$, the solution resolves two peaks, is good as shown in DIAG (1),

P (1 B).

(i) The algorithm with clean data gave $\lambda = (2.8)\ 10^{-12}$. The solution, resolves two peaks and is quite good as shown in DIAG (2).

(ii) Data with 1.7% error. Here $\lambda = (6.8)\ 10^{-9}$; again the solution resolves two peaks and is reasonable and solution improves when we increose $P = 2$ to $P = 3$ as shown in DIAG (2).

P (1 C).

(i) Algorithm with clean data gave $\lambda = (1.3)\ 10^{-10}$.
The solution resolves two peaks but there are wild oscillations as shown in Diag (3).

(ii) Data wite 0.7% noise. Here $\lambda = (3.1)\ 10^{-9}$; again the solution has oscillations and is not good but when P is increased i.e. for $P = 4$ the solution improved a lot as shown in DIAG (3).

P (2). (i) Algorithm with clean data gave $\lambda = (2.1)\ 10^{-14}$. The solution is quite good resolving all the six peaks as shown in DIAG (4).

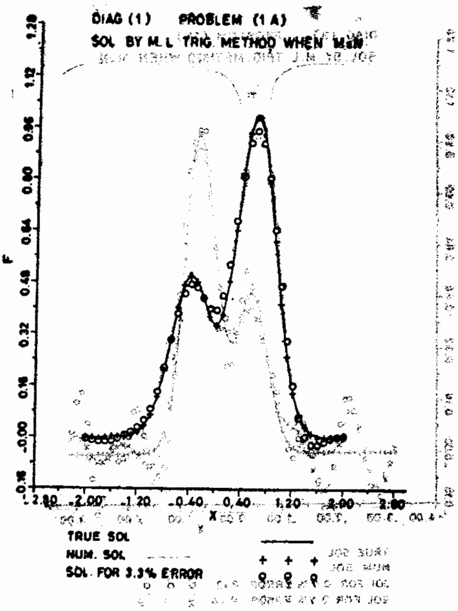
(ii) Data with 1.7% noise gave $\lambda = (8.1)\ 10^{-11}$. The solution gives only 5 peaks as shown in DIAG (4).

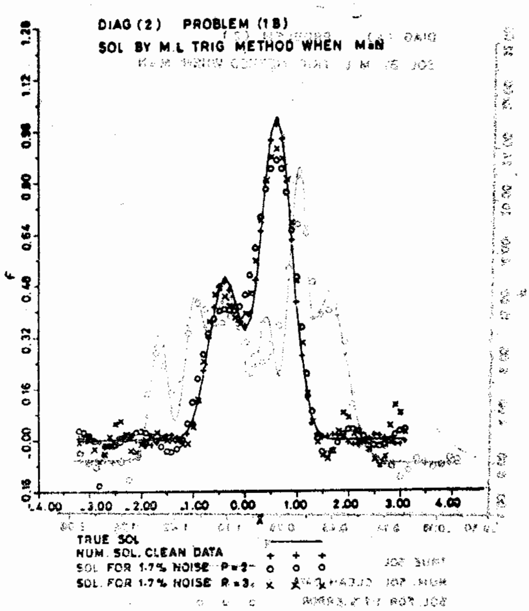
Acknowledgement.

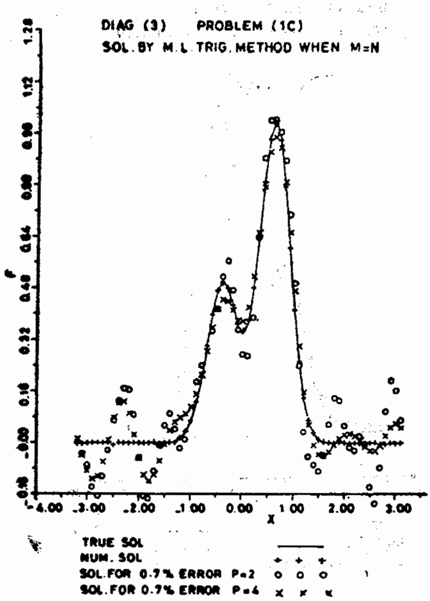
The auther is very much indebted to his supervisor Dr. A.R. Davies, Dept. of applied Maths. U.C.W Aberystwyth, Wales, U.K. who has suggested the problems and helped a lot in developing the theory of the method.

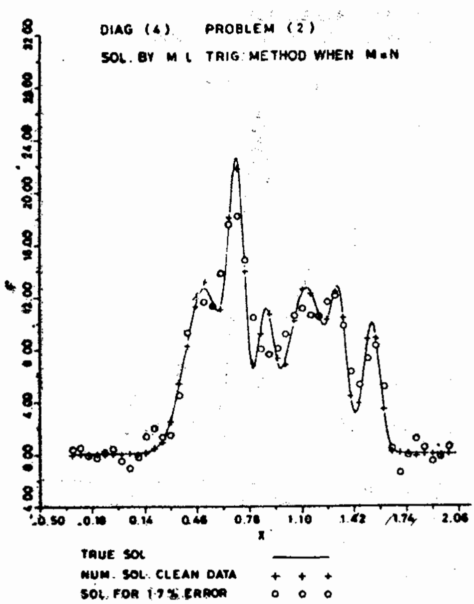
TABLE

PROBLEM	N	h	LEVEL OF NOISE= ×	λ	DIAGS
P (1 A)	64	0.1	0.0% / 3.3%	(5.8) 10^{-13} (8.1) 10^{-7}	DIAG (1)
P (1 B) P=2 P=3	64	0.1	0.0% / 1.7%	(2.8) 10^{-12} (6.8) 10^{-9} (2.70) 10^{-10}	DIAG (2)
P (1 C) P=2 P=4	64	0.1	0.0% / 0.7%	(1.3) 10^{-10} (3.1) 10^{-9} 2.70×10^{-10}	DIAG (3)
P (2)	64	0.05	0.0% / 1.7%	(2.1) 10^{-14} (8.1) 10^{-11}	DIAG (4)


DIAG (1) PROBLEM (1 A)
SOL BY M.L TRIG METHOD WHEN M=N
F
X
TRUE SOL
NUM. SOL
SOL. FOR 3.3% ERROR


DIAG (2) PROBLEM (1 B)
SOL BY M.L TRIG METHOD WHEN M=N
F
X
TRUE SOL
NUM. SOL. CLEAN DATA
SOL. FOR 1.7% NOISE P=2
SOL. FOR 1.7% NOISE R=3


DIAG (3) PROBLEM (1C)
SOL. BY M.L. TRIG. METHOD WHEN M=N
F
X
TRUE SOL
NUM. SOL
SOL. FOR 0.7% ERROR P=2
SOL. FOR 0.7% ERROR P=4


DIAG (4) PROBLEM (2)
SOL. BY M.L. TRIG. METHOD WHEN M=N
F
X
TRUE SOL
NUM. SOL. CLEAN DATA
SOL. FOR 1.7% ERROR

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ESTIMATING SOLUTIONS OF FIRST KIND INTEGRAL EQUATIONS OF CONVOLUTION TYPE USING GENERALIZED CROSS VALIDATION METHOD WITH TRIGONOMETRIC POLYNOMIAL APPROXIMATION

By

M. IQBAL

Department os Mathematics
Punjab University, New Campus,
Lahore—20 (Pakistan).

Abstract.

A method is presented in this paper for estimating solutions of Fredholm Integral Equation of the first kind, given noisy data.

Regularization is effected by GCV technique using trigonometric polynomial approximation.

We propose a technique by which an approximately optimal amount of smoothing may be computed, based only on the data and the assumed known noise variances. Numerical examples are given.

1. Introduction.

In many branches of science, problems arise, in which it is desired to solve *ILLPOSED* problems in the form of integral equations of the first kind. Introducing the mathematical problem more specifically, assume that we are given N discrete measurements g_i taken ai various points x_i. These are noisy versions of an integral of the product of a known kernel $k(x, y)$ and an unknown function $f^o(y)$; that is, we have

$$g(x_i) = \int_D k(x, y) f^o(y) \, dy + \varepsilon_i \qquad \dots (1)$$

$$i = 0, 2, \dots\dots, N-1$$

where ε_i is the random noise associated with the ith measurement and D is the domain appropriate to the physical situation. The problem is to estimate the unknown function f^o by some function f which depends upon the observed data. This problem is known to bc ill-posed, since radically different f^o s' could have given rise to very similar data, there is always the danger that small random variations in the data will cause unacceptably different estimate $f(x)$ of $f^o(x)$.

This danger follows heuristically from the fact that high frequency components in the true solution f^o will be highly attenuated in their contribution to the data g, depending on the smoothness of the kernels.

Subsequent to the theoretical work on the properties of ill-posed, inverse problems by Tikhonov [1] a method of obtaining practical solutions of such problems was presented by Tikhonov [2] as well as by Phillips [8] and Twomey [4].

In place of a straight forward inversion of the noisy data of [1] producing a wildly oscillatory solution, these authors presented a method of smoothing or "Regularizing" the solutions.

The development of these and other methods are reviewed in several articles, Turchin [3], Morozov [5], Miller [6] and others from [9] to [16].

2. Approximation and Solution Method.

Consider the Fredholm integral equation of the first kind of convolution type

$$(k f)(x) \equiv \int_{-\infty}^{\infty} k(x-y)\, f(y)\, dy = g(x), \; -\infty < y < \infty \quad (2.1)$$

where k and g are known functions in L_2 (R) and $f \varepsilon H^{p}$ (R) which is to be determined.

We assume that the support of each function f, g and k is finite and contained within the intrival (0, T). Let T_{N-1} denote the space of trigonometric polynomials of degree $N-1$ and period T.

We shall find a filtered solution of (2.1) within the space T_{N-1} for the following reasons :

(i) The discretization error in the convolution may be made precisely zero at the grid pts.

(ii) Fast Fourier Transform (FFT) routines are easily employed in the solution procedure.

(iii) The choice of T_{N-1} as the approximating function space is itself a regularizing feature.

The ill-posedness of (2.1) is reflected by the fact that any small perturbation ε in data function g, whose transform $\hat{\varepsilon}(\omega)$ does not decay faster than $\hat{k}(\omega)$ as $|\omega| \to \infty$ will result in a perturbation in $\frac{\hat{g}(\omega)}{\hat{k}(\omega)}$ which will grow without bound, when g is inexact. Usually the ill-posedness of (2.1) is measured by the width of the instrument (transform) function K. The more wider is function K, the more ill-posed is the problem, therefore, we seek a filtered approximation to f which is given by

$$f_{\lambda}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\omega;\lambda) \frac{\hat{g}(\omega)}{\hat{k}(\omega)} \exp(i\omega y)\, d\omega \qquad \dots (2.2)$$

where $z(\omega;\lambda)$ is a filter function dependent on regularization parameter λ. Now the filtered solution $f_{N,\lambda}(x) \varepsilon T_{N-1}$, which minimizes

$$\sum_{n=0}^{N-1} \left[(K_N * f)(x_n) - g_n \right]^2 + \lambda \| f^{(p)}(x) \|_2^2$$

where P the order of regularization, is given by

$$f_{N,\lambda}(x) = \frac{1}{N} \sum_{q=0}^{N-1} \hat{f}_{N,\lambda,q} \exp\left(\frac{2\pi i}{T} qx \right) \qquad \dots (2.3)$$

where

$$\hat{f}_{N,\lambda,q} \equiv z_q \frac{\hat{g}_{N,q}}{\hat{k}_{N,q}} \quad \text{with}$$

$$z_q = \frac{|\hat{k}_{N,q}|^2}{|\hat{k}_{N,q}|^2 + \lambda B_2 \tilde{\omega}_q^{2p}} \quad \text{where } (B_2 = N^2 / T^2) \qquad \ldots (2.4)$$

writing

$$\underline{f}_{N,\lambda} = \left(f_{N,\lambda}(x_0), \ldots, f_{N,\lambda}(x_{N-1}) \right)^T$$

we can define

$$\underline{g}_{N,\lambda} = (g_{N,\lambda,0}, \ldots, g_{N,\lambda,N-1})^T \quad \text{where}$$

$$g_{N,\lambda,n} \equiv g_{N,\lambda}(x_n) \equiv (K_N * f_{N,\lambda})(x_n) = (K \underline{f}_{N,\lambda})_n$$

or

$$\underline{g}_{N,\lambda} = K \underline{f}_{N,\lambda} = \psi \hat{K} \psi^H \underline{f}_{N,\lambda} = A(\lambda)\, \underline{g} \qquad \ldots (2.5)$$

where

$$A(\lambda) = \psi \hat{z} \psi^H, \ \hat{z} = \text{diag}(z_q) \qquad \ldots (2.6)$$

where ψ is $N \times N$ criculant matrix with elements

$$\psi_{rs} = \frac{1}{N} \exp\left(\frac{2\pi i}{N} rs \right), \ r, s = 0, 1, 2, \ldots, N-1$$

ψ is unitary $\psi\psi^H = \psi^H \psi = I$.

The idea of GCV is very simple. Suppose we ignore the jth data point g_j and define the filtered solution

$f_{N,\lambda}^{(j)}(x) \in T_{N-1}$ as the minimizer of

$$\sum_{n=0}^{N-1} \left[(k_N * f)(x_n) - g_n \right]^2 + \lambda \, \| f^{(p)}(x) \|_2^2 \qquad \text{... (2.7)}$$

we then obtain a vector $g^{[j]}_{N,\lambda} \,\varepsilon\, R^N$ defined by

$$g^{(j)}_{N,\lambda} = k \underline{f}^{(j)}_{N,\lambda}$$

Clearly the jth element $\left[g^{(j)}_{N,\lambda} \right]_j$ should "Predict" the missing value g_j. We may thus construct the weighted mean square prediction error over all j;

$$V(\lambda; p) = \frac{1}{N} \sum_{j=0}^{N-1} \omega_j(\lambda) \left[(g^{(j)}_{N,\lambda})_j - g_j \right]^2 \qquad \text{... (2.8)}$$

Thus the optimum λ minimizes $V(\lambda; p)$ and does not depend in any way on a knowledge of σ^2.

WAHBA [10, 11] has shown in a more general context that the choice of weights

$$\omega_j(\lambda) = \left[\frac{1 - a_{jj}(\lambda)}{\frac{1}{N} \operatorname{Tr}(I - A(\lambda))} \right]^2 \qquad \text{(A)}$$

$$j = 0, 1, 2, \ldots, N-1$$

where the matrix $A(\lambda)$ given in equations (2.5) and (2.6), enables the expression in equation (2.8) to be written in the much simpler form

$$V(\lambda; p) = \frac{\frac{1}{N} \| I - A(\lambda) \underline{g} \|^2}{\left(\frac{1}{N} \operatorname{Tr}(I - A)(\lambda) \right)^2} \qquad \text{... (2.9)}$$

which under transformation to discrete Fourier space is

$$V(\lambda; p) = \frac{\frac{1}{N} \| (I - \hat{z}) \hat{\underline{g}} \|_2^2}{\left(\frac{1}{N} \operatorname{Tr}(I - \hat{z}) \right)^2}$$

$$= \frac{\frac{1}{N} \sum_{q=0}^{N-1} (1-z_q)^2 \mid \hat{g}_{N,q} \mid^2}{\left(1- \sum_{q=0}^{N-1} z_q \right)^2} \quad \ldots (2.10)$$

Computationally this is a very simple function to minimize w.r.t. λ (p is usually taken as 2) but as we vary p some problems gave better results with noise as is shown in section 3).

For minimization of (2.10) we have used NAG Routine E 04 ABA based on quadratic interpolation technique. For completeness we observe that since the matrix A (λ) is circulant, the weights calculated from equation (A) are equal to unity.

After finding optimum value of λ from (2.10) then, the pth order filter

$$z(\omega_q ; \lambda) = \frac{\mid k_q \mid^2}{\mid k_q \mid^2 + \lambda B_2 \tilde{\omega}_q^{2p}}$$

can be found, when

$$\tilde{\omega}_q = \begin{cases} \omega_q , & 0 \le q \le \frac{1}{2} N \\ \omega_{N-q}, & \frac{1}{2} N \le q < N-1 \end{cases}$$

(usually P = 2 but in some problems we have increased p and they yielded better results). Ultimately the filtered solution is given by

$$f_{N;\lambda}(x) = \sum_{q=0}^{N-1} z_{q;\lambda} \frac{\hat{g}_{N,q}}{\hat{k}_{N,q}} \exp(i\, \omega_q\, x)$$

Problems Discussed.

P (1) This example is given in Phillips [8] and has an inherent noisy data function g with a maximum absolute error of about 0.02 (0.7%). We have

$$\int_{-30}^{30} k(x-y) f(y)\, dy = g(x)$$

where $k(x)$, $g(x)$ and $f(x)$ are given in table (1).
The number of grid points is 31.

TABLE (1)

x_n	g_n	k_n	f_n
−30,0	0.0100	0.1184	0.0000
−28.0	0.0100	0.1311	0.0000
−26.0	0.0110	0.1464	0,0000
−24.0	0.0170	0.1651	0.0000
−22.0	0.0305	0.1883	0.0000
−20.0	0.0405	0.2179	0.0000
−18.0	0.0585	0.2563	0.0000
−16.0	0.0869	0.3077	0.0000
−14.0	0.1309	0.3788	0.0000
−12.0	0.2018	0.4816	0.0000
−10.0	0.3235	0.6380	0.0000
−8.0	0.5469	0.8914	0.0000
−6.0	0.9621	1.3333	0.0019
−4.0	1.6301	2.1483	0.0345
−2.0	2.4047	3.5108	0.0965
0.0	2.9104	4.3600	0.1321
2.0	2.8912	3.0628	0.1096
4.0	2.4586	1.6329	0.0584
6.0	1.9049	0.8806	0.0349
8.0	1.4144	0.5095	0.0173
10.0	1.0282	0.3137	0.0107
12.0	0.7411	0.2021	0.0028
14.0	0.5409	0.1341	0.0005
16.0	0.4083	0.0906	0.0000
18.0	0.3214	0.0614	0.0000
20.0	0.2623	0.0413	0.0000
22.0	0.2201	0.0269	0.0000
24.0	0.1886	0.0165	0.0000
26.0	0.1580	0.0089	0.0000
28.0	0.1270	0.0031	0.0000
30.0	0.0780	0i0013	0.0000

P (2A) This problem is given in Truchin [13] we have

$$\int_{-2}^{2} k(x-y)\, f(y)\, dy = g(x)$$

where f is the function of two Gaussian functions

$$f(x) = 0.5 \exp\left(\frac{-(x+0.4)^2}{0.18}\right) + \exp\left(\frac{-(x-0.6)^2}{0.18}\right)$$

with essential support $-1.3 < x < 1.5$

$k(x)$ is triangular with equation

$$k(x) = \begin{cases} -x + 0.5, & 0 \le x < 0.5 \\ x + 0.5, & -0.5 \le x < 0 \\ 0, & |x| \ge 0.5 \end{cases}$$

We have calculated the values of $g(x)$ by the NAG Routine DOI AGA using Clenshaw-curtis quadrature method with accuracy 10^{-7}.

41 grid points have been considered.

P (2 BE)

This problem is the same as P (2 A) except that the triangular kernel is made wider which makes the problem more ill-posed.

$$k(x) = \begin{cases} (5/8)(-x + 0.8), & 0 \le x < 0.8 \\ (5/8)(x + 0.8), & -0.8 \le x < 0 \\ 0 & |x| \ge 0.8 \end{cases}$$

We have extended the support from (−2.0 to 2.0), to (−3.2, 3.2), therefore 64 grid points have been considered.

P (2 CE)

The problem is made highly ill-posed by choosing an even wider kernel

$$k(x) = \begin{cases} (5/12)(-x + 1.2), & 0 \le x < 1.2 \\ (5/12)(x + 1.2), & -1.2 \le x < 0 \\ 0 & |x| \ge 1.2 \end{cases}$$

again we have extended the support from $(-2.0, 2.0)$ to $(-3.2, 3.2)$, therefore, 64 grid points are considered.

P (3).

This problem has been taken from Medgyessy [7] with some modification. The solution function is the sum of six Gaussians and the kernel is also Gaussian. We have

$$\int_{-\infty}^{\infty} k(x-y)\, f(y)\, dy = g(x)$$

$$g(x) = \sum_{k=1}^{6} A_k \exp\left[-\frac{(x-\alpha_k)^2}{\beta_k}\right]$$

where

$A_1 = 10{,}0$	$\alpha_1 = 0.5$	$\beta_1 = 0.04$
$A_2 = 10.0$	$\alpha_2 = 0.7$	$\beta_2 = 0.02$
$A_3 = 5.0$	$\alpha_3 = 0.875$	$\beta_3 = 0.02$
$A_4 = 10.0$	$\alpha_4 = 1.125$	$\beta_4 = 0.04$
$A_5 = 5.0$	$\alpha_5 = 1.325$	$\beta_5 = 0.02$
$A_6 = 5.0$	$\alpha_6 = 1.525$	$\beta_6 = 0.02$

The essential support of $g(x)$ is $0 < x < 2$

$$k(x) = \frac{1}{\sqrt{\pi\lambda}} \exp(-x^2/\lambda), \qquad \lambda = 0.015$$

The essential support of $k(x)$ is $(-0.26, 0.26)$. The solution is

$$f(x) = \sum_{k=1}^{6} \left(\frac{\beta_k}{\beta_k - \lambda}\right)^{\frac{1}{2}} A_k \exp\left(-\frac{(x-\alpha_k)^2}{(\beta_k - \lambda)}\right)$$

The essential support of $f(x)$ is $(0.26, 1.74)$.

3. Numerical Results for GCV Regularization using Trigonometric Polynomial Approximation.

In this section we describe the numerical results with the application of generalized cross validation (GCV) method to problems P (1) to

P (3).

In solving the problems P (2 A), P (2 BE), P (2 CE) and P (3), we have considered the data function $g(x)$ as defined earlier and also the same data functions contaminated by varying amounts of random noise.

To generate the sequence of random errors of the form $\{\varepsilon_n\}$, $n = 0, 1, 2, \ldots, N-1$. We have used the NAG Algorithm G05 DDA which returns pseudo random real numbers taken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental errors we have taken $A = 0.0$

$$B = \frac{x}{100} (\max |g_n|), \qquad 0 \le n \le N-1 \qquad \ldots (3.1)$$

where x denotes a chosen percentage *e.g.*

$x = 0.7, 1.7$ or 3.3 etc.

Thus the standard deviation of the random error ε_n added to g_n does not exceed $x\%$ of the maximum value of $g(x)$.

The actual error ε_n may be as high as 3B.

P (1)

The interval $(-30, 30)$ is mapped onto $(0, 60)$ which is extended to (0.64) by introducing zero values of k and g. The step length $h=2.0$ is given. Thus $N = 32$. $T =$ period $= 64$.

The algorithm is tried and is shown in Diag (1).

P (2 A)

(a) Here $N = 64$, $h := 0.1$, $T =$ period $= 6.4$
We tried the algorithm for $p = 2$ and it gives a good solution as is shown in DIAG (2).

(b) The solution for $N = 64$, $h = 0.1$; $T = 6.4$
with 3.3% noise is also good as shown in Diag. (2).

P (2 BE)

(a) In the case of accurate data (without adding any noise) a reasonable solution with clearly resolved peaks was obtained as shown in DIAG (3).

(b) In case of noisy data, when $p = 2$ it resolves two peaks clearly and solution is good as shown in DIAG (3).

(c) For $p = 3$ a slightly better solution was obtained which is shown in DIAG (3).

P (2 CE)

(a) Again in the case of clean data a very good solution is obtained resolving two peaks very clearly as shown in DIAG (4).

(b) In case of noisy data when $p = 2$ it does not yield a good solution, however when $p = 6$ a reasonable solution is obtained as shown in DIAG (4).

P (3)

The essential supports of $f(x)$, $g(x)$ and $k(x)$ are (0, 26, 1.74), (0, 2) and (—0.26, 0.26) respectively. First we can consider a common interval (—0.26, 2.0) for all these three functions which covers all of their essential supports. This interval was translated to (0,2.26) and then extended to (0, 3.2). Thus T = period = 3.2 and we took a step length $h = 0.05$ so that $N = 64$.

(a) For clean data the solution is very good, giving all peaks O.K. as shown in DIAG (5).

(b) In case of 1.7% noise the solution is not very good but reasonable when $p = 2$ as shown in DIAG (5). The increase in p (the order of regularization) does not help in this case.

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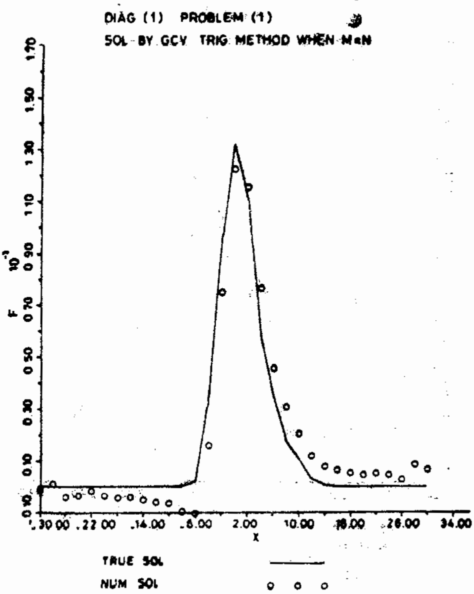
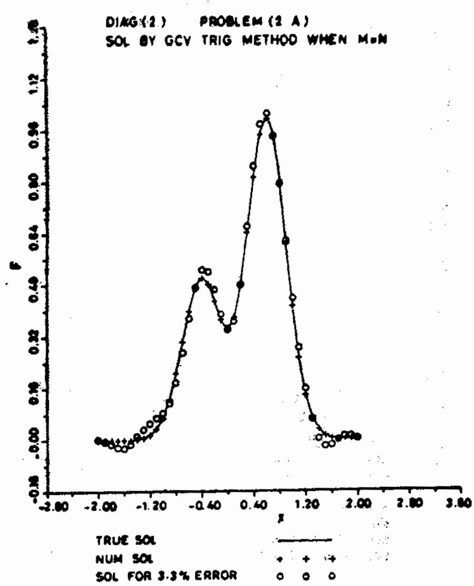
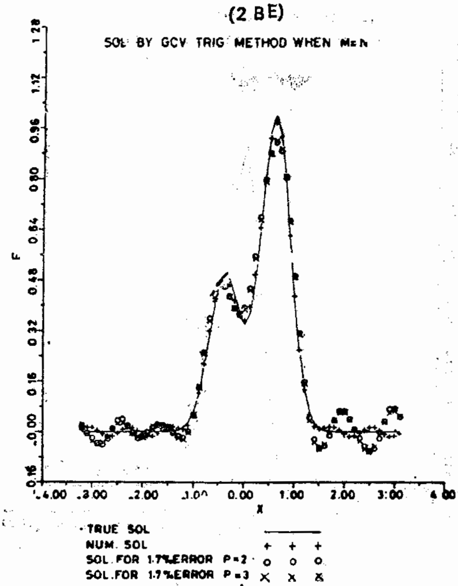
TABLE (2)

PROBLEM	N	h	Level of Noise	λ	v (λ)	No. of Minima	DIAGS
P (1)	32	2.0	0.7% inherent in data	4.8962	0.002461	one	(1)
P (2 A)	64	0.1	0.0%	(2.0) 10^{-12}	(1.0) 10^{-9}	one	(2)
			3.3%	(8.4) 10^{-8}	0.00255	one	
P (2 BE)	64	0.1	0.0%	(6.8) 10^{-13}	(1.6) 10^{-9}	one	(3)
$p=2$			0.7%	(1.0) 10^{-9}	(5.3) 10^{-4}	one	
$p=3$				(8.0) 10^{-10}	4.082×10^{-4}	one	
P (2 CE)	64	0.1	0.0%	(4.3) 10^{-14}	(4.4) 10^{-10}	one	(4)
$p=2$			0.7%	(1.7) 10^{-7}	(4.48) 10^{-4}	one	
$p=6$				2.71×10^{-14}	4.42×10^{-4}	one	
P (3)	64	0.05	0.0%	(6.0) 10^{-19}	(7.4) 10^{-9}	one	(5)
			1.7%	(1.0) 10^{-19}	4.14044	one	

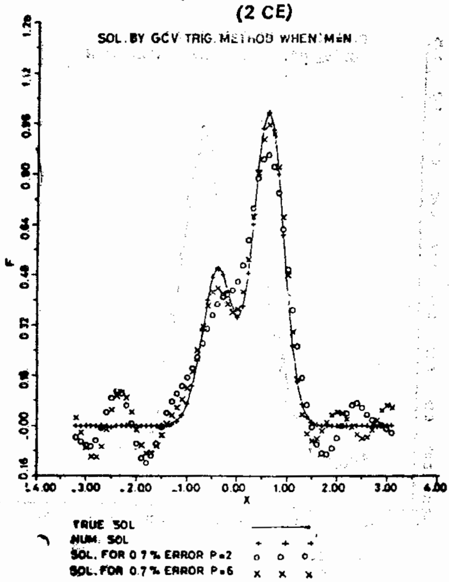

DIAG (1) PROBLEM (1)
SOL BY GCV TRIG METHOD WHEN M=N
F 10^{-1}
X
TRUE SOL
NUM SOL

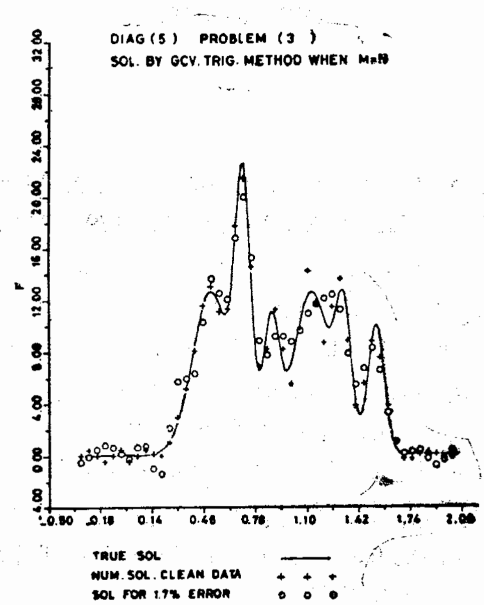

DIAG (2) PROBLEM (2 A)
SOL BY GCV TRIG METHOD WHEN M=N
F
X
TRUE SOL
NUM SOL
SOL FOR 3.3% ERROR

DIAG (3) PROBLEM
(2 BE)



DIAG (4) PROBLEM
(2 CE)




DIAG (5) PROBLEM (3)
SOL. BY GCV. TRIG. METHOD WHEN M=19
F
TRUE SOL
NUM. SOL. CLEAN DATA
SOL FOR 1.7% ERROR

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ERRATUM

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