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THE LOVE WAVE SCATTERING MATRIX FOR THREE-LAYERED STRUCTURES CONSISTING OF WELDED LAYERED QUARTER-SPACES WITH A PLANE SURFACE

By

M.H. KAZI AND A. NIAZY

ABSTRACT

In this paper we use spectral representation of the Love wave operator for a three-layer model comprising two homogeneous, infinite strips over-lying a uniform half-space, along with a method based on an integral equation formulation and Schwinger-Levine variational principle to describe, by means of a scattering matrix, the diffraction of plane, harmonic, monochromatic Love waves, incident normally (from either side) upon the vertical plane of discontinuity in the three-layered structure consisting of welded layered quarter-spaces with a plane surface. Approximate expressions for the elements of the scattering matrix are obtained through the plane-wave approximation and their variational improvement is sought through the Schwinger-Levine variational principle in such a way as to incorporate the contributions caused by body-wave conversion. Complex reflection and transmission coefficients can be obtained through a transmission matrix related to the scattering matrix. We obtain the form of the transmission matrix (under both approximations) in some simple cases.

1. Introduction

In our previous work [Kazi (1978a,b), Niazy and Kazi (1980, 1982)] we used a method based on an integral equation formulation and Schwinger-Levine variational principle to describe, by means of a scattering matrix, the diffraction of plane, harmonic, monochromatic Love waves, incident normally upon the vertical planes of discontinuity in laterally discontinuous structures such as a half-space with a surface step and welded layered quarter-spaces (involving single top layers) with a plane surface. The method presupposes the existence of a complete set of proper and improper eigenfunctions, in terms of which the displacement fields on either side of the vertical plane of discontinuity may be expressed. Such a set of functions for the two-dimensional Love wave operator, associated with the propagation of monochromatic SH waves in a half-space overlain by a single layer, has been given in Kazi (1976). In order to be able to extend the method to laterally varying structures involving two layers over a half space, we need explicit spectral representation of the Love wave operator associated with monochromatic SH waves for a three-layer model comprising two homogeneous, infinite strips overlying a uniform half-space. Such a representation has been found in Kazi and Abu-Safiya (1982). In this paper we use this spectral representation to extend the method of integral representation and Schwinger-Levine variational principle to investigate the two-dimensional diffraction problem of plane harmonic Love waves, incident normally (from either side) upon the plane of discontinuity in the three-layered structure consisting of welded layered quarter spaces with a plane surface. The wave field is described by means of a scattering matrix, and approximate expressions for its elements are obtained through the plane-wave approximation and their variational improvement is sought through the variational principle of Schwinger and Levine. Complex reflection and transmission coefficients are obtainable through a transmission matrix related to the scattering matrix. The form of the transmission matrix in some simple special cases under the variational approximation indicates that the variational procedure incorporates the effects of propagated and non-propagated modes arising out of the continuous spectrum, which corresponds to body waves, and is, therefore, of considerable importance. Numerical computation of the reflection and transmission coefficients for backward as well as forward transmission in the welded quarter-spaces problem and other related problems will be given in another paper.

Equations of Motion

Let us suppose that a quarter-space consisting of a material of rigidity μ_3, shear velocity β_3, and density ρ_3, overlain by a layer of uniform depth H_2, density ρ_2, rigidity μ_2 ($<\mu_3$) and shear velocity $\beta_2(<\beta_3)$ and another layer of uniform depth $H_1(<H_2)$, density ρ_1, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$, is in welded contact with a similar quarter-space of material of rigidity μ'_3, shear velocity β'_3 and density ρ'_3, overlain by a layer of uniform depth H_2, density ρ'_2, rigidity $\mu'_2(<\mu'_3)$ and shear velocity $\beta'_2(<\beta'_3)$ and another layer of uniform depth H_1, density ρ'_1, rigidity $\mu'_1(<\mu'_2)$ and shear velocity β'_1 $(<\beta'_2)$ (see Figure 1). We take the vertical plane of welded contact between the two structures to be $x=0$, the plane of welded contact between the upper two layers to be the xy-plane in the co-ordinate system shown in the figure and regard the top plane surface $z=-H_1$ to be stress free. All materials are considered to be isotropic and homogeneous.

We consider two dimensional problems of the diffraction of time-harmonic Love waves normally incident upon the vertical plane of contact (from either side). Again, the wave motion is entirely SH in character. The y-components of the seismic displacement fields in the regions I$(x<0)$ and II$(x>0)$ (see Figure 1) are denoted by $e^{-i\omega t}v(x, z)$ and $e^{-i\omega t}v'(x, z)$, respectively, where

$$\begin{aligned} e^{-i\omega t}v(x, z) &= e^{-i\omega t}v_1(x, z), \ -H_1 \leqslant z \leqslant 0, \ x<0, \\ &= e^{-i\omega t}v_2(x, z), \ 0<z \leqslant H_2, \ x<0, \\ &= e^{-i\omega t}v_3(x, z), \ H_2<z, \ x<0, \end{aligned}$$

and

$$\begin{aligned} e^{-i\omega t}v'(x, z) &= e^{-i\omega t}v'_1(x, z), \ -H_1 \leqslant z \leqslant 0, \ x>0 \\ &= e^{-i\omega t}v'_2(x, z), \ 0<z \leqslant H_2, \ x>0 \\ &= e^{-i\omega t}v'_3(x, z), \ H_2<z, \ x>0, \end{aligned}$$

(ω being the angular frequency) are the solutions of the Love wave differential equation

$$\rho(z) \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left[\mu(z) \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial z}\left[\mu(z) \frac{\partial v}{\partial z} \right]$$

in the two regions on either side of the vertical plane $x=0$.

The conditions at the free surface $z=-H_1$ and the plane of welded contact $x=0$ imply

$$\frac{\partial v_1}{\partial z}=0 \text{ and } \frac{\partial v'_1}{\partial z}=0 \text{ at } z=-H_1, \tag{1a}$$

$$v=v' \text{ at } x=0,\ z\geq -H_1, \tag{1b}$$

$$\mu(z)\frac{\partial v}{\partial x}=\mu'(z)\frac{\partial v'}{\partial x} \text{ at } x=0,\ z\geq -H_1, \tag{1c}$$

where

$$\begin{aligned}\mu(z)&=\mu_1,\ -H_1\leqslant z<0,\ x<0,\\ &=\mu_2,\ 0<z<H_2,\ x<0,\\ &=\mu_3,\ H_2<z,\end{aligned} \tag{2}$$

and

$$\begin{aligned}\mu'(z)&=\mu'_1,\ -H_1\leqslant z<0,\ x>0,\\ &=\mu'_2,\ 0<z<H,\ x>0,\\ &=\mu'_3,\ H_2<z,\ x>0.\end{aligned} \tag{3}$$

The complete solution for the displacement $v(x, z)$ in domain I (see Figure 1) can be expressed in terms of proper and improper eigenfunctions of the Love wave operator for a homogeneous half-space of rigidity μ_3 and shear velocity β_3, overlaid by two infinite strips consisting of a layer of depth H_2, rigidity $\mu_2(<\mu_3)$, and shear velocity β_2 $(<\beta_3)$, and another layer of depth H_1, rigidity μ_1 $(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$. Kazi and Abu-Safiya (1982) have found explicit formulas for these proper and improper eigen function and have shown that the spectrum of the corresponding two-dimensional Love wave operator is the disjoint union of the discrete spectrum, which corresponds to the ordinary Love modes, and a continuous spectrum (corresponding to body waves) which is the interval $(-\infty, \omega^2/\beta^2{}_3)$ on the real axis of the complex λ-plane, where $\lambda=k^2$, k being the wave number and ω the angular frequency. Likewise, we can write the complete solution for the displacemeut $v'(x, z)$ in domain II in terms of proper and improper eigenfunctions of the Love wave operator for a homogenous half-space of rigidity μ'_3 and shear velocity β'_3, overlaid by two infinite strips consisting of a layer of depth H_2, rigidity μ'_2 $(<\mu'_3)$, and shear velocity β'_2 $(<\beta'_3)$ and another layer of depth H_1, rigidity $\mu'_1(<\mu'_2)$ and shear velocity

$\beta'_1(<\beta'_2)$. Using the formulas derived in Kazi and Abu-Safiya (1982),

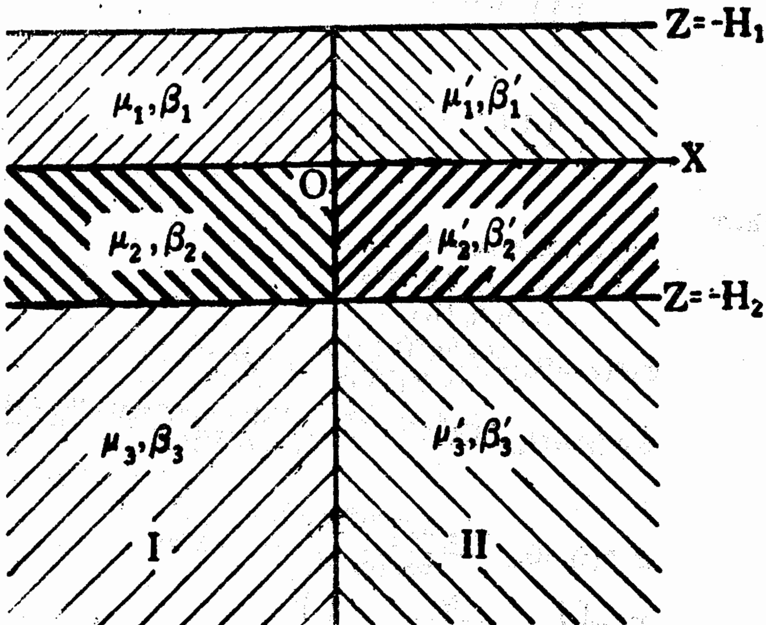

Figure 1 : The Geometry of the Problem

we have in Domain I ($x<0$, $z\geq -H_1$)

$$v(x, z)=-\left(\sum_{m=1}^{r} \left(A_m e^{-ik_m |x|} + B_m e^{ik_m |x|}\right) x_m(z)\right.$$

$$+\int_0^{\omega/\beta_3} \left\{ C(k) e^{-ik|x|} + D(k) e^{ik|x|} \right\} \phi(z, k)dk$$

$$\left. +\int_0^{\infty} E(k) e^{-k|x|} \psi(z, k)dk \right) \tag{4}$$

and in Domain II ($x>0$, $z\geq -H_1$)

$$v'(x, z)=\left(\sum_{m=1}^{s}\left(A'_m e^{-ik'_m x} + B'_m e^{ik'_m x}\right) x'_m(z)\right.$$

$$+\int_0^{w/\beta'_3}\left\{C'(k')e^{-ik'x}+D'(k')e^{ik'x}\right\}\phi'(z,k')dk'$$

$$+\int_0^{\infty}E'(k')e^{-k'x}\psi'(z,k')dk' \qquad (5)$$

where

$$\begin{aligned} \chi_m(z) &= \phi_1{}^m(z), \ -H_1 \leqslant z \leqslant 0 \\ &= \phi_2{}^m(z), \ 0 \leqslant z \leqslant H_2 \\ &= \phi_3{}^m(z), \ H_2 \leqslant z \end{aligned} \qquad (6)$$

$$\phi_1{}^m(z)=F_m\frac{\cos\{\sigma_1{}^m(z+H_1)\}}{\cos(\sigma_1{}^mH_1)},\ -H_1\leqslant z\leqslant 0, \qquad (7a)$$

$$\phi_2{}^m(z)=G_m\ \frac{\mu_2\sigma_2{}^m\cos\{\sigma_2{}^m(z-H_2)\}-\mu_3\sigma_3{}^m\sin\{\sigma_2{}^m(z-H_2)\}}{\cos(\sigma_2{}^mH_2)},$$

$$0\leq z\leqslant H_2, \qquad (7b)$$

$$\phi_3{}^m(z)=G_m\ \frac{\mu_2\sigma_2{}^me^{-\sigma_3{}^m(z-H_2)}}{\cos(\sigma_2{}^mH_2)},\ z\geq H_2 \qquad (7c)$$

$$F_m=\left[\left\{\frac{M}{\frac{\partial}{\partial\lambda}(-\Delta)_{\lambda=\lambda m}}\right\}\right]^{\frac{1}{2}} \qquad (8)$$

$$G_m=\frac{1}{M}F_m, \qquad (9)$$

$$M=\mu_2\sigma_2+\mu_3\sigma_3\tan(\sigma_2H_2), \qquad (10)$$

$$\begin{aligned}\Delta=\ &\mu_1\sigma_1\mu_2\sigma_2\tan(\sigma_1H_1)+\mu_1\sigma_1\mu_3\sigma_3\tan(\sigma_2H_2)\tan(\sigma_1H_1)\\ &-\mu_3\sigma_3\mu_2\sigma_2+(\mu_2\sigma_2)^2\tan(\sigma_2H_2)\end{aligned} \qquad (11)$$

$$\sigma_1(\lambda)=\left(\frac{\omega^2}{\beta_1{}^2}-\lambda\right)^{\frac{1}{2}},\ \sigma_2(\lambda)=\left(\frac{\omega^2}{\beta_2{}^2}-\lambda\right)^{\frac{1}{2}},\ \sigma_3(\lambda)=\left(\lambda-\frac{\omega^2}{\beta_3{}^2}\right)^{\frac{1}{2}} \qquad (12)$$

$$\sigma_i(\lambda_m)=\sigma_i{}^m,\ i=1,2,3 \qquad (13)$$

and $\lambda_n=k_n{}^2$, $k_n>0$ are the roots of $\Delta=0$, (14)

which is the dispersion equation for Love wave propagation in two layers over a half-space (see Ewing et al., 1957, p. 229), and where

$$\psi(z,\lambda)=\psi_1(z,\lambda)=G^k\mu_2\sigma_2{}^k\ \frac{\cos\{\sigma_1{}^k(z+H_1)\}}{\cos(\sigma_1{}^kH_1)\cos(\sigma_2{}^kH_2)},$$

$$-H_1\leqslant z\leqslant 0 \qquad (15a)$$

$$=\psi_2(z,\lambda)=\frac{G^k}{\cos(\sigma_2^k H_2)}\{\mu_2\sigma_2^k\cos(\sigma_2^k z)-\mu_1\sigma_1^k\sin(\sigma_2^k z)\times$$

$$\tan(\sigma_1^k H_1)\},\ 0\leqslant z\leqslant H_2, \qquad (15b)$$

$$=\psi_3(z,\lambda)=-\frac{\sin\{\theta^k+s_3^k(z-H_2)\}}{\sqrt{\pi\mu_3 s_3^k}},\ H_2\leq z, \qquad (15c)$$

where

$$G^k=\frac{\sqrt{2k}\mu_3 s_3^k\cos\theta}{\rho\sqrt{\pi\sigma_3 s_3^k}}, \qquad (16)$$

$$s_3^k=\left(\frac{\omega^2}{\beta_3^2}-\lambda\right)^{\frac{1}{2}} \text{ (real and positive)} \qquad (17)$$

$$\theta^k=\tan^{-1}\left(\frac{q}{p}\right), \qquad (18)$$

$$p=\mu_1\sigma_1^k\mu_2\sigma_2^k\tan(\sigma_1^k H_1)+\mu_2^2(\sigma_2^k)^2\tan(\sigma_2^k H_2), \qquad (19)$$

$$q=\mu_1\sigma_1^k\mu_2 s_3^k\tan(\sigma_2^k H_2)\tan(\sigma_1^k H_1)-\mu_2\sigma_2^k\mu_3 s_3^k \qquad (20)$$

$$\sigma_1^k=\left(\frac{\omega^2}{\beta_1^2}-k^2\right)^{\frac{1}{2}},\ \sigma_2^k=\left(\frac{\omega^2}{\beta_2^2}-k^2\right)^{\frac{1}{2}} \qquad (21)$$

Owing to the factor $e^{-k|x|}$ in the integral containing ψ, these represent nonpropagated modes.

$\varnothing(z, k)$, the improper eigenfunctions belonging to the improper eigenvalues $\lambda=k^2$, $0<k\leqslant\omega/\beta_3$, have expressions similar to those for $\psi(z, k)$. Owing to the form of x–dependence in the integral containing $\varnothing$, these represent waves travelling in the x direction.

The orthonormality relations amongst various proper and improper eigenfunctions are given by (cf. Kazi and Abu-Safiya (1982)).

$$\int_{-H_1}^{\infty}\mu(z)x_m(z)x_n(z)dz=\delta_{mn},\ 1\leqslant m,\ n\leqslant r, \qquad (22a)$$

$$\int_{-H_1}^{\infty}\mu(z)x_m(z)\varnothing(z,k)\,dz=0,\ 1\leqslant m\leqslant r,\ 0<k\leqslant\omega/\beta_3 \qquad (22b)$$

$$\int_{-H_1}^{\infty}\mu(z)x_m(z)\psi(z,k)\,dz=0,\ 1\leqslant m\leqslant r,\ 0<k<\infty \qquad (22c)$$

$$\int_{-H_1}^{\infty} \mu(z)\,\psi(z,k)\,\psi(z,l)\,dz=\delta(k-l),\ 0<k,\ l<\infty \tag{22d}$$

$$\int_{-H_1}^{\infty} \mu(z)\,\varnothing(z,k)\,\psi(z,k)\,dz=0 \tag{22e}$$

$$\int_{-H_1}^{\infty} \mu(z)\,\varnothing(z,k)\,\varnothing(z,l)\,dz=\delta(k-l),\ 0\leqslant k,\ l\leqslant \omega/\beta_3 \tag{22f}$$

The corresponding expressions for $x'_m(z)$, $\psi'(z, k')$ and $\varnothing'(z, k')$ and the orthonormality relations amongst these are the same as for $x_m(z)$, $\psi(z, k)$ and $\varnothing(z, k)$, given above, but in primed notation.

Integral Equation Formulation

Let $\tau(z)$ denote the component of stress at any point of the vertical plane $x=0$:

$$\tau(z)=\tau_{xy}\Big/_{x=0}=\mu(z)\,\frac{\partial v}{\partial x}\Big/_{x=0^-}=\mu'(z)\frac{\partial v'}{\partial x}\Big/_{x=0^+},\quad z>-H_1 \tag{23}$$

we have both

$$\tau(z)=\mu(z)\,\frac{\partial v}{\partial x}\Big/_{x\to 0^-}=-\mu(z)\Big[\sum_{m=1}^{r} ik_m(A_m-B_m)x_m(z)$$
$$+\int_0^{\omega/\beta_3} ik\{C(k)-D(k)\}\varnothing(z,k)dk+\int_0^{\infty} kE(k)\,\psi(z,k)dk\Big] \tag{24}$$

and

$$\tau(z)=\mu'(z)\,\frac{\partial v'}{\partial x}\Big/_{x\to 0^+}=-\mu'(z)\Big[\sum_{m=1}^{s} ik'_m(A'_m-B'_m)x'_m(z)$$
$$+\int_0^{\omega/\beta'_3} ik'\{C'(k')-D'(k')\}\varnothing'(z,k')dk'+\int_0^{\infty} k'E'(k')\psi'(z,k')dk'\Big] \tag{25}$$

On multiplying equation (23) separately by $x_m(z)$ $(m=1, 2...r)$, $\varnothing(z, k)$ $\left(0<k<\frac{\omega}{\beta_3}\right)$ and $\psi(z, k)(0<k\leqslant\infty)$, and integrating with respect to z from $-H_1$ to ∞, we obtain [using orthonormality relations (22a) to (22f)]

$$-ik_m(A_m-B_m)=\int_{-H_1}^{\infty}\tau(n)x_m(n)\,dn,\ m=1,2,...r, \tag{26a}$$

$$-ik\{C(k)-D(k)\}=\int_{-H_1}^{\infty}\tau(n)\varnothing(n, k)\,dn, \tag{26b}$$

and

$$-kE(k)=\int_{-H_1}\tau(n)\,\psi(n, k)\,dn \tag{26c}$$

Proceeding similarly, equation (25) leads to the following

$$-ik'_m(A'_m-B'_m)=\int_{-H_1}^{\infty}\tau(n)x'_m(n)dn,\ m=1, 2,...s, \tag{27a}$$

$$-ik'\{C'(k')-D'(k')\}=\int_{-H_1}^{\infty}\tau(n)\varnothing'(n, k')dn, \tag{27b}$$

$$-k'E'(k')=\int_{-H_1}^{\infty}\tau(n)\,\psi'(n, k')\,dn. \tag{27c}$$

Eliminating $D(k)$, $D'(k')$, $E(k)$, $E'(k')$ [assuming $C(k)=C'(k')=0$ and applying the matching condition (1c)], we obtain

$$\sum_{m=1}^{r}(A_m+B_m)x_m(z)+\sum_{m=1}^{s}(A'_m+B'_m)x'_m(z)=\int_{-H_1}^{\infty}\tau(n)G^*(z,n)dn \tag{28}$$

where

$$G^*(z, n)=G(z, n)+ig(z, n), \tag{29}$$

$$G(z, n)=\int_0^\infty \frac{\psi(z, k)\,\psi(n, k)}{k}\,dk+\int_0^\infty \frac{\psi'(z, k')\,\psi'(n, k')}{k'}\,dk' \qquad (30)$$

and

$$g(z, n)=\int_0^{\omega/\beta_3} \frac{\varnothing(z, k)\,\varnothing(n, k)}{k}\,dk+\int_0^{\omega/\beta'_3} \frac{\varnothing'(z, k')\,\varnothing'(n, k')}{k'}\,dk' \qquad (31)$$

It may be noted that $G^*(z, n)$ is a Green's function type symmetric kernel, whose real and imaginary parts correspond to non-propagated and propagated modes (respectively) arising out of the continuous part of the spectrum.

The integral equation formulation of the problem is given by equations (26*a*), (27*a*) and (28). If the amplitudes (A_m, $m=1, 2, \ldots, r$, A'_m, $m=1, 2, \ldots, s$) of incident Love waves are specified, we have to find the amplitudes of the transmitted and reflected waves from the above mentioned $(r+s+1)$ integral equations. Using matrix formalism, we recast the problem in terms of a 'scattering matrix' in the next section.

The Scattering Matrix Formulation

Introducing $n \times 1$ vectors

$$A=\begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_r \\ A'_1 \\ \cdot \\ \cdot \\ \cdot \\ A'_s \end{bmatrix}, \quad B=\begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_r \\ B'_1 \\ \cdot \\ \cdot \\ \cdot \\ B'_s \end{bmatrix} \quad X(z)=\begin{bmatrix} x_1(z) \\ x_2(z) \\ \cdot \\ \cdot \\ \cdot \\ x_r(z) \\ x'_1(z) \\ \cdot \\ \cdot \\ \cdot \\ x'_s(z) \end{bmatrix} \qquad (32)$$

and the $n \times n$ diagonal matrix

$$\mathbf{K} = \begin{bmatrix} k_1 & & & & & & & & \\ & k_2 & & & & & & \mathrm{O} & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & k_r & & & \\ & & & & & & k'_1 & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ & \mathrm{O} & & & & & & & \cdot \\ & & & & & & & & k'_s \end{bmatrix}, \quad n = r + s, \tag{33}$$

equations (26*a*), (27*a*) can be rewritten

$$-i\mathbf{k}.(\mathbf{A}-\mathbf{B}) = \int_{-\mathrm{H}_1}^{\infty} \mathbf{X}(n)\tau(n)dn \tag{34}$$

and equation (28) as

$$(\mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}})\mathbf{X}(z) = \int_{-\mathrm{H}_1}^{\infty} \mathrm{G}^*(z, n)\tau(n)dn, \quad z > -\mathrm{H}_1 \tag{35}$$

where the superscript T denotes the transpose.

Equations (34) and (35) imply that both $\mathbf{A}-\mathbf{B}$ and the unknown stress $\tau(z)$ on the vertical plane $x=0$ must be linearly related to $\mathbf{A}+\mathbf{B}$. Consequently there exists an $n \times n$ matrix $\mathrm{S} = \| s_{ij} \|$ and an

$n \times 1$ vector

$$\underline{\tau}(z) = \begin{bmatrix} \tau_1(z) \\ \cdot \\ \cdot \\ \cdot \\ \tau_r(z) \\ \tau'_1(z) \\ \cdot \\ \cdot \\ \cdot \\ \tau'_s(z) \end{bmatrix} \tag{36}$$

such that

$$\mathbf{K}.(\mathbf{A}-\mathbf{B}) = i\mathbf{S}.(\mathbf{A}+\mathbf{B}), \tag{37}$$

and

$$\tau(z) = (\mathbf{A}^T + \mathbf{B}^T).\underline{\tau}(z). \tag{38}$$

The matrix $\mathbf{S} = \| s_{ij} \|$ is the SCATTERING MATRIX. Equation (37) can be rewritten

$$\mathbf{B} = \mathbf{T}.\mathbf{A} \tag{39}$$

where

$$\mathbf{T} = (\mathbf{K}+i\mathbf{S})^{-1}.(\mathbf{K}-i\mathbf{S}) \tag{40}$$

provided $\mathbf{K}+i\mathbf{S}$ is non-singular. Substituting (38) into (35), we get

$$(\mathbf{A}^T + \mathbf{B}^T).\{\mathbf{X}(z) - \int_{-H_1}^{\infty} G^*(z, n)\tau(n)dn\} = 0, \quad -H_1 < z, \tag{41}$$

whence

$$X(z) = \int_{-H_1}^{\infty} G^*(z, n)\underline{\tau}(n)dn, \quad -H_1 < z, \tag{42}$$

on account of the arbitrary choice and linear independence of the components of $\mathbf{A}+\mathbf{B}$. From (42), we obtain the following n uncoupled integral equations for the determination of $\tau(n)$:

$$x_m(z)=\int_0^\infty G^*(z,n)\tau_m(n)dn,\ m=1,2,\ldots,r,\ z>-H_1, \qquad (43)$$

$$x'_m(z)=\int_0^\infty G^*(z,n)\tau'_m(n)dn,\ m=1,2,\ldots,s,\ z>-H_1. \qquad (44)$$

Substituting (38) into (34), we get

$$\mathbf{K}.(\mathbf{A}-\mathbf{B})=i\int_0^\infty \mathbf{X}(n)\{(\mathbf{A}^T+\mathbf{B}^T).\ \tau(n)\}dn,$$

whence from (37)

$$\mathbf{S}.(\mathbf{A}+\mathbf{B})=\int_0^\infty \mathbf{X}(n)\{\mathbf{A}^T+\mathbf{B}^T).\ \underline{\tau}(n)\}dn,$$

and so

$$s_{ij}=\int_0^\infty x_i(n)\tau_j(n)dn,\ (i,\ j=1,2,\ldots,r,\ r+1,\ldots,\ n=r+s), \qquad (45)$$

where

$$x_{r+t}=x'_t \quad \text{and} \quad \tau_{r+t}=\tau'_t,\ 1\le t\leqslant s.$$

The problem has thus been reduced to the solution of the integral equations (43) and (44) and the subsequent determination of the scattering matrix **S** from (45) and the related transmission matrix **T** in (40) which yields the required complex reflection and transmission coefficients (after appropriate normalization) through equation (39). The formulation of the problem is exact at this stage. Unfortunately, it is not possible to solve the problem exactly and we must resort to construct approximate solutions. In the next section we shall proceed to the plane wave approximation neglecting the propagated and non-propagated modes arising out of the continuous spectrum. In the subsequent section, we shall construct expressions for the elements s_{ij} (of the scattering matrix) to which variational principle of Schwinger and Levine applies and then improve the earlier approximation in such a way as to incorporate the effects of propagated modes (which correspond to body waves) and non-propagated modes indirectly.

Plane Wave Approximation

If we neglect the propagated modes $\varnothing(z, k)$, $\varnothing'(z, k')$ and the non-propagated modes $\psi(z, k)$, $\psi'(z, k')$ corresponding to the continuous part of the spectrum, then we can set $G^*(z, n)=0$ [see equations (29) to (31)] in the preceding formulation and assume the following expansion for $\tau(z)$ in terms of the whole set of propagated discrete modes in the left-hand domain :

$$\tau(z)=\mu(z)\left\{\sum_{m=1}^{r} D_m x_m(z)\right\} \tag{46}$$

Substituting this into equation (34) we obtain

$$-iK\cdot(A\text{-}B)=\int_{-H_1}^{\infty}\begin{bmatrix} x_1(n) \\ x_2(n) \\ \cdot \\ \cdot \\ \cdot \\ x_r(n) \\ x'_1(n) \\ \cdot \\ \cdot \\ \cdot \\ x'_s(n) \end{bmatrix}\mu(n)\left\{\sum_{m=1}^{r} D_m x_m(n)\right\}dn \tag{47}$$

or

$$-i \begin{bmatrix} k_1(A_1-B_1) \\ k_2(A_2-T_2) \\ \cdot \\ \cdot \\ \cdot \\ k_r(A_r-B_r) \\ k'_1(A'_1-B'_1) \\ \cdot \\ \cdot \\ \cdot \\ k'_s(A'_s-B'_s) \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ \cdot \\ D_r \\ \sum_{m=1}^{r} D_m P_{1m} \lambda_{1m} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{m=1}^{r} D_m P_{sm} \lambda_{sm} \end{bmatrix} \tag{48}$$

because of the orthonormality relation (22*a*), where

$$\lambda_{im} = \left(\frac{k'_i}{k_m}\right)^{\frac{1}{2}}, \; i=1, 2,\dots s \; ; \; m=1, 2,\dots, r, \tag{49}$$

and $$\lambda_{im} P_{im} = \int_{-H_1}^{\infty} \mu(n)\, x'_i(n)\, x_m(n)\, dn, \quad i=1, 2,\dots s \; ; \; m=1, 2,\dots, t. \tag{50}$$

Substituting $G^*=0$ in (35), we get

$$(A^T+B^T).X(z)=0, \; z>-H_1 \tag{51}$$

Eliminating $D_1, D_2,\dots, D_r$ from (48) and simplifying, we obtain :

$$\mathbf{R}.(A-B)=0, \tag{52}$$

where the $s \times n$ matrix **R** is given by :

$$
\mathbf{R}=\left[\begin{array}{cccc:cccc}
\frac{P_{11}}{\lambda_{11}} & \frac{P_{12}}{\lambda_{12}} & \cdots & \frac{P_{1r}}{\lambda_{1r}} & -1 & & & \mathbf{O} \\
\frac{P_{21}}{\lambda_{21}} & \frac{P_{22}}{\lambda_{22}} & \cdots & \frac{P_{2r}}{\lambda_{2r}} & & & & \\
 & & & & & -1 & & \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
\cdot & & & & \mathbf{O} & & & \\
\frac{P_{s1}}{\lambda_{s1}} & \frac{P_{s2}}{\lambda_{s2}} & \cdots & \frac{P_{sr}}{\lambda_{sr}} & & & & -1
\end{array}\right]
\begin{array}{l} \\ \end{array}
\qquad \underbrace{}_{s\times r}\;\underbrace{}_{s\times s}
\tag{53}
$$

Calculating the first moments of (51) with respect to $\mu(z)\ x_i(z)$, $i=1, 2,\cdots, r$, we obtain a set of r simultaneous, linear, algebraic equatons equivalent to the matrix equation :

$$
\mathbf{Q}.(\mathbf{A}+\mathbf{B})=0, \tag{54}
$$

where the $r\times n$ matrix $\mathbf{Q}$ is given by

$$
\mathbf{Q}=\left[\begin{array}{cccc:cccc}
 & & & & \lambda_{11}\,P_{11} & \lambda_{21}\,P_{21} & \cdots & \lambda_{s1}\,P_{s1} \\
1 & & & & \lambda_{12}\,P_{12} & \lambda_{21}\,P_{22} & \cdots & \lambda_{s2}\,P_{s2} \\
 & & & & \cdot & & & \\
 & 1 & \mathbf{O} & & \cdot & & & \\
 & & & & \cdot & & & \\
\mathbf{O} & & & & \cdot & & & \\
 & & & & \cdot & & & \\
 & & & 1 & \cdot & & & \\
 & & & & \lambda_{1r}\,P_{1r} & \lambda_{2r}\,P_{2r} & \cdots & \lambda_{sr}\,P_{sr}
\end{array}\right]
\qquad \underbrace{}_{r\times r}\;\underbrace{}_{r\times s}
\tag{55}
$$

Combining (52) and (54) into a single matrix equation, we get

$$\begin{pmatrix} Q \\ R \end{pmatrix} . A = \begin{pmatrix} -Q \\ R \end{pmatrix} . B \tag{56}$$

or

$$B = T.A \text{ where } T = \begin{pmatrix} -Q \\ R \end{pmatrix}^{-1} \begin{pmatrix} Q \\ R \end{pmatrix} \tag{57}$$

The matrix gives the reflection and transmission coefficients. We now proceed to compute the integral equation (50) to find $\lambda_{lm} P_{lm}$.

Rewrite equation (50) as

$$I = \lambda_{lm} P_{lm} = \int_{-H_1}^{\infty} \mu(z)\, x'_l(z)\, x_m(z)\, dz,$$

$$I = \mu_1 \int_{-H_1}^{0} \varnothing'^{l}_1(z)\, \varnothing^{m}_1(z)\, dz + \mu_2 \int_{0}^{H_2} \varnothing'^{l}_2(z)\, \varnothing^{m}_2(z)\, dz$$

$$+ \mu_3 \int_{H_2}^{\infty} \varnothing'^{l}_3(z) \varnothing^{m}_3(z)\, dz = I_1 + I_2 + I_3, \tag{58}$$

with

$$I_1 = \mu_1 \int_{-H_1}^{0} \varnothing'^{l}_1(z) . \varnothing^{m}_1(z)\, dz$$

$$= \frac{\mu_1 F'_l F_m}{\cos(\sigma'^{l}_1 H_1) \cos(\sigma^{m}_1 H_1)} \int_{-H_1}^{0} \cos\{\sigma'^{l}_1(z+H_1)\} \cos\{\sigma^{m}_1(z+H_1)\} dz$$

[using (7*a*)]

$$= \frac{\mu_1 F'_l F_m}{(\sigma'^{l}_1)^2 - (\sigma^{m}_1)^2} [\sigma'^{l}_1 \tan(\sigma'^{l}_1 H_1) - \sigma^{m}_1 \tan(\sigma^{m}_1 H_1)], \tag{59}$$

$$I_2 = \mu_2 \int_{0}^{H_2} \phi'^{l}_2(z)\, \varnothing^{m}_2(z)\, dz$$

$$= \frac{\mu_2 G_i' G_m}{\cos(\sigma_2^m H_2)\cos(\sigma_2'^i H_2)} \int_0^{H_2} [\mu_2\sigma_2^m \cos\{\sigma_2^m(z-H_2)\} - \mu_3\sigma_3^m \sin\{\sigma_2^m(z-H_2)\}] . \mu_2'\sigma_2'^i \cos\{\sigma_2'^i(z-H_2)\} - \mu_3'\sigma_3'^i \sin\{\sigma_2'^i(z-H_2)\}]dz$$

[using (7b)]

which yields after somewhat lengthy but straightforward calculation:

$$I_2 = \frac{\mu_2 G_i' G_m}{(\sigma_2^m)^2-(\sigma_2'^i)^2}[\sigma_2'^i\{\mu_2\mu_2'(\sigma_2^m)^2+\mu_3\mu_3'\sigma_3^m\sigma_3'^i\}\tan(\sigma_2^m H_2) - \sigma_2^m\{\mu_2\mu_2'(\sigma_2'^i)^2+\mu_3\mu_3'\sigma_3^m\sigma_3'^i\}\tan(\sigma_2'^i H_2) + \{\mu_2\mu_3'\sigma_3'^i(\sigma_2^m)^2-\mu_3\mu_2'\sigma_3^m(\sigma_2'^i)^2\}\tan(\sigma_2^m H_2)\tan(\sigma_2'^i H_2) + \frac{\sigma_2^m\sigma_2'^i(\mu_3\mu_2'\sigma_3^m-\mu_2\mu_3'\sigma_3'^i)}{\cos(\sigma_2^m H_2)\cos(\sigma_2'^i H_2)} + \sigma_2^m\sigma_2'^i(-\mu_2'\mu_3\sigma_3^m+\mu_2\mu_3'\sigma_3'^i\}], \tag{60}$$

$$I_3 = \mu_3 \int_{H_2}^{\infty} \phi_3'^i(z)\,\phi_3^m(z)\,dz$$

$$= \mu_3 \int_{H_2}^{\infty} \frac{G_m G_i'}{\cos(\sigma_2^m H_2)\cos(\sigma_2'^i H_2)} \mu_2\mu_2'\sigma_2^m\sigma_2'^i e^{-(\sigma_3^m+\sigma_3'^i)(z-H_2)} . dz$$

[using (7c)]

$$= \frac{G_m G_i'\mu_2\mu_2'\mu_3\sigma_2^m\sigma_2'^i(\sigma_3^m-\sigma_3'^i)}{\cos(\sigma_2^m H_2)\cos(\sigma_2'^i H_2)\{(\sigma_3^m)^2-(\sigma_3'^i)^2\}} \tag{61}$$

whence from equations (58) to (61) we obtain

$$\lambda_{im} P_{im} = \frac{\mu_1 F_i' F_m}{\left(k_m^2-k_i'^2\right)+\omega^2\left(\frac{1}{\beta_1'^2}-\frac{1}{\beta_1^2}\right)}[\sigma_1'^i\tan(\sigma_1'^i H_1) - \sigma_1^m\tan(\sigma_1^m H_1)]$$

$$+ \frac{\mu_2 G_i' G_m}{\left(k_m^2-k_i'^2\right)+\omega^2\left(\frac{1}{\beta_2'^2}-\frac{1}{\beta_2^2}\right)}[\sigma_2^m\{\mu_2\mu_2'(\sigma_2'^i)^2 + \mu_3\mu_3'\sigma_3^m\sigma_3'^i\}\tan(\sigma_2'^i H_2) - \sigma_2'^i\{\mu_2\mu_2'(\sigma_2^m)^2 + \mu_3\mu_3'\sigma_3^m\sigma_3'^i\}\tan(\sigma_2^m H_2) + \{\mu_3\mu_2'\sigma_3^m(\sigma_2'^i)^2 - \mu_2\mu_3'\sigma_3'^i(\sigma_2^m)^2\}\tan(\sigma_2^m H_2)\tan(\sigma_2'^i H_2)$$

$$+\frac{\sigma_2{}^m\sigma'_2{}^i}{\cos(\sigma_2{}^m H_2)\cos(\sigma'_2{}^i H_2)}(\mu'_3\mu_2\sigma'_3{}^i-\mu_3\mu'_2\sigma_3{}^m)$$
$$-\sigma^m{}_2\sigma'_2{}^i(\mu_2\mu'_3\sigma'_3{}^i-\mu'_2\mu_3\sigma^m{}_3)]$$
$$+\frac{\mu_2\mu'_2\mu_3\sigma_2\sigma'_2{}^i G_m G'_i(\sigma'_3{}^i-\sigma^m{}_3)}{\cos(\sigma_2{}^m H_2)\cos(\sigma'_2{}^i H_2)\left[\left(k'_i{}^2-k_m{}^2\right)+\omega^2\left(\frac{1}{\beta_3{}^2}-\frac{1}{\beta'_3{}^2}\right)\right]} \quad (62)$$

A check on the validity of these formulae is provided by the fact that in the limit as $H_2 \to O$, $\mu_3 \to \mu_2$, $\mu'_3 \to \mu'_2$, $\sigma_2 \to \sigma_3$, $\sigma'_2 \to \sigma'_3$, the expression for $\lambda_{im} P_{im}$ in (62) reduces to corresponding expression found in Niazi and Kazi {(1980) [eq. (41)]} for the welded quarter-spaces problem involving single upper layers.

The form of the transmission matrix T in (57) in the following special cases can be shown to be :

(I) $r=1, s=1$:

$$T=\frac{1}{1+P^2_{11}}\begin{bmatrix} -1+P^2_{11} & -2\lambda_{11}P_{11} \\ \dfrac{-2P_{11}}{\lambda_{11}} & 1-P^2_{11} \end{bmatrix} \quad (63)$$

(II) $r=1, s=2$:

$$T=\frac{1}{1+P^2_{11}+P^2_{11}}\begin{bmatrix} -1+P^2_{11}+P^2_{21} & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} \\ \dfrac{-2P_{11}}{\lambda_{11}} & -P^2_{11}+1+P^2_{21} & \dfrac{-2P_{11}\lambda_{21}P_{21}}{\lambda_{11}} \\ \dfrac{-2P_{21}}{\lambda_{21}} & \dfrac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}} & -P^2_{21}+1+P^2_{11} \end{bmatrix}$$

Variational Formulation and Direct Approximation

Returning to the scattering matrix formulation of the problem, we shall construct expressions for the elements of the matrix in such a way that the variational principle of Schwinger and Levine becomes applicable.

Multiplying the equations

$$x_i(z)=\int_{-H_1}^{\infty} G^*(z,n)\tau_i(n)dn, \quad i=1,2,\dots, n=r+s$$

[(43, (44)]

by $\tau_j(z)$, $j=1, 2, \ldots, n$ and integrating with respect to z over the interval $(-H_1, \infty)$, we obtain

$$S_{ij}=\int_{-H_1}^{\infty} x_i(n)\tau_j(n)\, dn=\int_{-H_1}^{\infty}\int_{-H_1}^{\infty} \tau_i(z)G^*(z, n)\, \tau_j(n)\, dz\, dn \tag{65}$$

by equation (45). Since the kernel $G^*(z, n)$ is symmetric [see equations (29)-(31)] it follows from (65) that $s_{ij}=s_{ji}$ and so the scattering matrix $S=\| s_{ij} \|$ is symmetric. Thus we may write

$$S_{ij}=\frac{\int_{-H_1}^{\infty} x_i(z)\, \tau_j(z)\, dz.\int_{-H_1}^{\infty} x_j(n)\, \tau_i(n) dn}{\int_{-H_1}^{\infty}\int_{-H_1}^{\infty} \tau_i(z)\, G^*(z, n)\, \tau_j(n)\, dz\, dn} \tag{66}$$

If we introduce the notations

$$< f, u>=\int_{-H_1}^{\infty} fu\, dz,\ G^*u=\int_{-H_1}^{\infty} G^*(z, n)\, u\,(n)\, dn,$$

then

$$<G^*u, v> = <u, G^*v> \qquad \forall \quad u, v$$

and we may rewrite (66) as

$$s_{ij}=(<x_i, \tau_j>.<x_j,\tau_i>)/(<G^*\tau_i,\tau_j>) \tag{67}$$

As in Kazi (1978), we have the following

Theorem ; Let $F(u, v)=<x_j, v>+<x_j, u>-<G^*u, v>$. Then F is stationary for variations of u, v about $u=\tau_i$, $v=\tau_j$ where τ_i, τ_j are the solutions of the integral equations

$$x_i(z)=G^*\tau_i=\int_{-H_1}^{\infty} G^*(z, n)\, \tau_i\,(n)\, dn$$

and

$$x_j(z)=G^*\tau_j=\int_{-H_1}^{\infty} G^*(z, n)\, \tau_j(n)\, dn,$$

respectively. Moreover, the stationary value of F is $s_{ij}(\tau_i, \tau_j)$

Corollary (Schwinger-Levine Variational Principle) : Let $R(u, v)=(\langle x_j, u\rangle\langle x_i, v\rangle)/(\langle G^*u, v\rangle)$. Then R is stationary about $u=\alpha\tau_i$, $v=\beta\tau_j$ where α, β are arbitrary non-zero constants. Moreover, $R(\alpha\tau_i, \beta\tau_j)=s_{ij}(\tau_i, \tau_j)$.

By invoking the above theorem we obtain variational improvement of the plane wave approximation used in the previous section by assuming the expansions for $\tau_i(z)$:

$$\tau_i(z)=\sum_{p=1}^{r} D_{ip}\mu(z)\, x_p(z), \quad i=1, 2,\ldots, n \tag{68}$$

and considering

$$\begin{aligned} F(\tau_i,\tau_j)= & \langle x_j, \sum_{p=1}^{r} D_{ip}\mu(z)\, x_p(z)\rangle+\langle x_i, \sum_{q=1}^{r} D_{jq}\mu(z)x_q(z)\rangle \\ & -\langle G^*\tau_i,\tau_j\rangle=\sum_{p=1}^{r} D_{ip}\langle x_j, \mu(z)x_p(z)\rangle+\sum_{q=1}^{r} D_{jq}\langle x_i, \\ & \mu(z)x_q(z)\rangle-\sum_{q=1}^{r}\sum_{p=1}^{r} D_{ip}D_{jq}I_{pq}, \end{aligned} \tag{69}$$

where

$$I_{pq}=\int_{-H_1}^{\infty}\left\{\int_{-H_1}^{\infty} G^*(z, n)x_p(n)\mu(n)dn\right\}\mu(z)x_q(z)\, dz, \tag{70}$$

The requirement that the co-efficients D_{ip} and D_{jq} in (69) make $F(\tau_i, \tau_j)$ stationary implies

$$\frac{\partial F}{\partial D_{ip}}=0, \quad p=1, 2,\ldots, r,$$

and

$$\frac{\partial F}{\partial D_{jq}}=0, \quad q=1, 2,\ldots, r,$$

which lead to a set of r linear algebraic equations for D_{ip}, $p=1$, $2,\ldots, r$ and another set of r linear algebraic equations for D_{jq}, $q=1$, 2,..., r. Solving for D_{ip}'s and D_{jq}'s and substituting in (69), we get the entry s_{ij} of the scattering matrix. Suitable expressions for the integrals I_{pq} are constructed in the appendix.

In the special cases

(*i*) when $r=1$, $s=1$:

$$S=\begin{bmatrix} \dfrac{1}{I_{11}} & \dfrac{\lambda_{11}P_{11}}{I_{11}} \\ \dfrac{\lambda_{11}P_{11}}{I_{11}} & \dfrac{\lambda_{11}^2 P_{11}^2}{I_{11}} \end{bmatrix} \tag{71}$$

and

$$T=(K+iS)^{-1}.(K-iS)$$

$$=\frac{1}{(1+P_{11}^2-i\,I'_{11})}\begin{bmatrix} P_{11}^2-1-i\,I'_{11} & -2P_{11}\lambda_{11} \\ \dfrac{-2P_{11}}{\lambda_{11}} & 1-P_{11}^2-i\,I'_{11} \end{bmatrix} \tag{72}$$

where $I'_{11}=k_1 I_{11}$ and I_{11} is given by (A10) when $m=n=1$.

(*ii*) when $r=1$, $s=2$:

The scattering matrix S and the transmission matrix T are given by

$$S=\begin{bmatrix} \dfrac{1}{I_{11}} & \dfrac{\lambda_{11}P_{11}}{I_{11}} & \dfrac{\lambda_{21}P_{21}}{I_{11}} \\ \dfrac{\lambda_{11}P_{11}}{I_{11}} & \dfrac{\lambda_{11}^2P_{11}^2}{I_{11}} & \dfrac{\lambda_{11}P_{11}\lambda_{21}P_{21}}{I_{11}} \\ \dfrac{\lambda_{21}P_{21}}{I_{11}} & \dfrac{\lambda_{11}P_{11}\lambda_{21}P_{21}}{I_{11}} & \dfrac{\lambda^2_{21}P^2_{21}}{I_{11}} \end{bmatrix}$$

and

$$T=\frac{1}{1+P^2_{11}+P^2_{21}-i\,I'_{11}}\begin{bmatrix} -1+P^2_{11}+P^2_{21}-i\,I'_{11} & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} \\ \dfrac{-2P_{11}}{\lambda_{11}} & -P^2_{11}+1+P^2_{21}-i\,I'_{11} & \dfrac{-2P_{11}\lambda_{21}P_{21}}{\lambda_{11}} \\ \dfrac{-2P_{11}}{\lambda_{21}} & \dfrac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}} & -P^2_{21}+1+P^2_{11}-i\,I'_{11} \end{bmatrix} \tag{73}$$

On comparing the forms of the transmission matrix **T** for the special cases discussed above [see equations (72) and (73)] with those under the plane wave approximation [see equations (63) and (64)], we find that the latter can be recovered from the former on substituting $I'_{11}=0$ and so it follows that the parameter I'_{11} incorporates the effects of propagated and non-propagated modes which arise out of the continuous part of the spectrum.

Numerical computation of our results under both approximations, and for several special laterally discontinuous structures involving double surface layers will be presented in another paper.

Appendix

Substituting

$$G^*(z, n)=G(z, n)+ig(z, n) \tag{29}$$

$$G(z, n)=\int_0^\infty [\psi(z, k)\, \psi(n, k)dk]/k+\int_0^\infty \frac{[\psi'(z, k')\, \psi'(n, k')]dk'}{k'} \tag{30}$$

$$g(z, n)=\int_0^{\omega/\beta_3} \frac{[\varnothing(z, k)\varnothing(n, k)]dk}{k}+\int_0^{\omega/\beta'_3} \frac{\varnothing'(z, k')\varnothing'(n, k')\, dk'}{k'} \tag{31}$$

in (70) and using the orthonormality relations [see (22*a*)-(22*f*)] we obtain

$$I_{mn}=\int_0^\infty \frac{dk'}{k'}\int_{-H_1}^\infty \mu(n)\, \psi'(n, k')x_m(n)dn\int_{-H_1}^\infty \mu(z)\, \psi'(z, k')\, x_n(z)dz$$

$$+i\int_0^{\omega/\beta'_3} \frac{dk'}{k'}\int_{-H_1}^\infty\mu(n)\, \varnothing'(n, k')x_m(n)dn\int_{-H_1}^\infty\mu(z)\varnothing'(z, k')\, x_n(z)\, dz \tag{A1}$$

Next, we evaluate integrals of the form

$$I(k', m)=\int_{-H_1}^\infty \mu(z)x_m(z)\varnothing'(z, k')\, dz$$

and

$$I'(k', m)=\int_{-H_1}^{\infty}\mu(z)x_m(z)\,\psi'(z, k')\,dz, \quad m=1, 2, \ldots, r,$$

which occur in (A1).

$$\text{Let } I(k', m)=\int_{-H_1}^{\infty}\mu(z)x_m(z)\varnothing'(z, k')\,dz$$

$$=\mu_1\int_{-H_1}^{0}\varnothing'_1(z, k')\varnothing^m{}_1(z)dz+\mu_2\int_{0}^{H_2}\varnothing'_2(z, k')\varnothing^m{}_2(z)dz$$

$$+\mu_3\int_{H_2}^{\infty}\varnothing'_3(z, k')\varnothing^m{}_3(z)\,dz$$

$$=I_1+I_2+I_3, \qquad (\lambda'=k'^2, \quad 0<k'<\omega/\beta'_3) \tag{A2}$$

where

$$I_1=\mu_1\int_{-H_1}^{0}\varnothing'_1(z, k')\;\varnothing^m{}_1(z)\,dz$$

$$=\frac{\mu_1F_m}{\cos(\sigma_1{}^mH_1)}\cdot\frac{G^{k'}\mu'_2\sigma_2^{\prime k'}}{\cos(\sigma'_1{}^{k'}H_1)\cos(\sigma_2^{\prime k'}H_2)}\int_{-H_1}^{0}\cos\{\sigma^m{}_1(z+H_1)\}\times\cos\{\sigma'_1{}^{k'}(z+H_1)\}dz$$

[using expressions for $\varnothing^m{}_1(z)$ from equation (7*a*) and for $\varnothing'_1(z, k')$ similar to $\psi_1(z, \lambda)$ in equation (15*a*)]

$$=\frac{\mu_1F_mG^{k'}\mu'_2\sigma_2^{\prime k'}}{\cos(\sigma'_2{}^{k'}H_2)}\cdot\frac{1}{(\sigma'_1{}^{k'})^2-(\sigma^m{}_1)^2}[\sigma'_1{}^{k'}\tan(\sigma'_1{}^{k'}H_1)-\sigma^m{}_1\tan(\sigma^m{}_1H_1)], \tag{A3}$$

$$I_2=\mu_2\int_{0}^{H_2}\varnothing'_2(z, k')\varnothing^m{}_2(z)\,dz$$

$$=\frac{\mu_2G'^{k'}G_m}{\cos(\sigma^m{}_2H_2)\cos(\sigma_2^{\prime k'}H_2)}\int_{0}^{H_2}[\mu_2\sigma^m{}_2\cos\{\sigma^m{}_2(z-H_2)\}]$$

$$-\mu_3\sigma^m{}_3 \sin \{\sigma_2{}^m(z-H_2)\}].$$
$$[\mu_2'\sigma_2'^{k'} \cos (\sigma_2'^{k'}z)-\mu_1'\sigma_1'^{k'}\sin (\sigma_1'{}_2{}^{k'}z) \tan (\sigma_1'^{k'}H_1)]dz$$

[using expressions for $\varnothing_2{}^m(z)$ from (7*b*) and for $\varnothing_2'(z, k')$ similar to $\psi_2(z, \lambda)$ in (15*b*)]

$$= \frac{\mu_2 G'^{k'} G_m}{\cos (\sigma_2{}^m H_2) \cos (\sigma_2'^{k'} H_2)} \cdot \frac{1}{(\sigma_2{}^m)^2-(\sigma_2'^{k'})^2} [\{\mu_2\mu'_2(\sigma_2{}^m)^2\sigma_2'^{k'}$$
$$+\mu_3\sigma_3{}^m\mu_1'\sigma_2'^{k'}\sigma_1'^{k'} \tan (\sigma_1'^{k'}H_1) . \sin (\sigma_2{}^m H_2)-\{\mu_2\mu_2'\sigma_2{}^m(\sigma_2'^{k'})^2$$
$$+\mu_3\sigma_3{}^m\mu_1'\sigma_1'^{k'}\sigma_2{}^m \tan (\sigma_1'^{k'}H_1)\} \sin (\sigma_2'^{k'}H_2)+\{\sigma_2{}^m\mu_3\mu_2'\sigma_3{}^{mk'}\sigma_2'$$
$$-\mu_1'\mu_2\sigma_2{}^{mk'}\sigma_1'\sigma_2'^{k'} \tan (\sigma_1'^{k'}H_1)\}\{ \cos (\sigma_2'^{k'}H_2)-\cos (\sigma_2{}^m H_2)\}]$$

(obtained after considerable simplification), (A4)

$$I_3=\mu_3 \int_{H_2}^{\infty} \varnothing_3'(z, k') \varnothing_3{}^m(z)dz$$

$$= \frac{-\mu_3 G_m \mu_2 \sigma_2{}^m}{\cos (\sigma_2{}^m H_2)} \cdot \frac{\int_{H_2}^{\infty} \sin \{\theta'^{k'}+s_3'^{k'}(z-H_2)\}.e^{-\sigma_3{}^m(z-H_2)}dz}{\sqrt{\pi\mu_3 s_3'^{k'}}}$$

[using expressions for $\varnothing_3{}^m(z)$ from (7*c*) and for $\varnothing'_3(z, k')$ similar to $\psi_3(z, \lambda)$ in (15*c*)].

$$= \frac{-\mu_3 G_m \sigma_2{}^m}{\cos(\sigma_2{}^m H_2)} \cdot \frac{1}{\sqrt{\pi\mu_3's_3'^{k'}}} \cdot \frac{1}{(\sigma_3{}^m)^2+(s_3'^{k'})^2} [(\cos \theta'^{k'})s_3'^{k'}$$
$$+(\sin \theta'^{k'})\sigma_3{}^m],$$
$$= \frac{-\mu_3 G_m G'^{k'}}{\cos (\sigma_2{}^m H_2)} \cdot \frac{p'}{\mu_3's_3'^{k'}} \cdot \frac{\sigma_2{}^m}{(\sigma_3{}^m)^2+(s_3'^{k'})^2} [s_3'^{k'}+(\tan \theta'^{k'})\sigma_3{}^m],$$

(A5)

obtained on using the relation

$$G'_{k'}\, p'/(\mu_3's_3'^{k'}) \cos \theta'^{k'}=\frac{1}{\sqrt{\pi s_3'^{k'}\mu_3}}, \qquad \text{[see (16)]}$$

From (A2)–(A5) we get

$$I(k', m)= \int_{-H_1}^{\infty} \mu(z)x_m(z) \varnothing'(z, k')dz$$

$$=\frac{\mu_1 F_m G'^{k'} \mu_2' \sigma_2'^{k'}}{\left[\left(k^2_m - k'^2\right)+\omega^2\left(\frac{1}{\beta'^2_1}-\frac{1}{\beta_1^2}\right)\right]\cos(\sigma_2'^{k'} H_2)}$$

$$[\sigma_1'^{k'} \tan(\sigma_1'^{k'} H_1) - \sigma_1^m \tan(\sigma_1^m H_1)]$$

$$+\frac{\mu_2 G'^{k'} G_m}{\cos(\sigma_2^m H_2)\cos(\sigma_2'^{k'} H_2)} \cdot \frac{1}{(k'^2 - k_m^2)+\omega^2\left(\frac{1}{\beta_2^2}-\frac{1}{\beta_2'^2}\right)}.$$

$$[\{\mu_2 \mu_2'(\sigma_2^m)^2 \sigma_2'^{k'} + \mu_3 \sigma_3^m \mu_1' \sigma_2'^{k'} \sigma_1'^{k'} \, . \tan(\sigma_1'^{k'} H_1)\} \sin(\sigma_2^m H_2)$$
$$-\{\mu_2 \mu_2' \sigma_2^m (\sigma_2'^{k'})^2 + \mu_3 \sigma_3^m \mu_1' \sigma_1'^{k'} \sigma_2^m \tan(\sigma_1'^{k'} H_1)\} \sin(\sigma_2'^{k'} H_2)$$
$$+\{\mu_3 \sigma_2^m \mu_2' \sigma_3^{mk'} \sigma_2' - \mu_1' \mu_2 \sigma_2^{mk'} \sigma_1' \sigma_2'^{k'} \tan(\sigma_1'^{k'} H_1)\}.$$
$$\{\cos(\sigma_2'^{k'} H_2) - \cos(\sigma_2^m H_2\}]$$

$$\frac{-\mu_3 G_m G'^{k'}}{\cos(\sigma_2^m H_2)} \cdot \frac{p'}{\mu_3' s_3'^{k'}} \cdot \frac{\sigma_2^m}{(k_m^2 - k'^2)+\omega^2\left(\frac{1}{\beta_3'^2}-\frac{1}{\beta_3^2}\right)}$$

$$[s_3'^{k'} + (\tan\theta'^{k'})\sigma_3^m] \qquad \text{(A6)}$$

[on using relations of the type given in equations (12), (13), (17) and (21)].

Thus

$$\int_{-H_1}^{\infty} \mu(n)\varnothing'(n,k')x_m(n)\,dn \int_{-H_1}^{\infty} \mu(z)\varnothing'(z,k')x_n(z)dz = I(k',m)\,I(k',n) \qquad \text{(A7)}$$

Likewise

$$\int_{-H_1}^{\infty} \mu(n)\,\psi'(n,k')x_m(n)dn \int_{-H_1}^{\infty} \mu(z)\,\psi'(z,k')x_n(z)\,dz = I'(k',m)\,I'(k',n) \qquad \text{(A8)}$$

where the expression for $I'(k',m)$ can be obtained from (A6) on replacing k'^2 by $-k'^2$ *i.e.*,

$$I'(k',m) = I(ik',m) \qquad \text{(A9)}$$

From (A1), (A7) and (A8), we finally obtain :

$$I_{mn} = \int_0^{\infty} \frac{I'(k',m)\,I'(k,n)}{k'}\,dk' + i\int_0^{\omega/\beta'_3} \frac{I(k',m)\,I(k',n)}{k'}\,dk', \qquad \text{(A10)}$$

where $I(k',m)$ and $I'(k',m)$ are given by (A6) and A(9) respectively.

The real and imaginary parts of I_{mn} correspond to the non-propagated and propagated modes arising from the continuous part of the spectrum. The integrands in the integrals occuring in (A10) are regular. These integrals are convergent. However, the integrals will have to be evaluated numerically because of the complicated forms of the integrands.

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ON A THEOREM FOR FINDING "LARGE" SOLUTIONS OF MULTILINEAR EQUATIONS IN BANACH SPACE

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Abstract. A new iteration for finding "large" solutions of the multilinear equation in Banach space is introduced based on the same assumption used to prove existence for the "small" solution.

Introduction. We introduce the literation.

$$x_{n+1}=\left[M_k\,(x_n)^{k-1}\right]^{-1}(x_n-y)\,,\; n=0\,,1\,,2\,,\ldots\ldots k\geq 2 \quad (1)$$

for some x_0 in a Banach space x to find solutions of the multilinear equation,

$$x=y+M_k\,(x)^k\,,\quad k\geq 2 \quad (2)$$

in x, where $y\in x$ is fixed and M_k is a bounded symmetric k-linear operator on x [4]. It is well known [3], [4] that if

$$\|y\|\leqslant \tfrac{1}{2}p,$$

where

$$p=\sqrt[k-1]{\frac{\|M_k\|}{k(k-1)}}$$

then equation (2) has a "small" solution $x\in X$ such that

$$\|x\|\leq p$$

The above estimate raises the natural question. Is it true under the same assumption that equation (1) has a "large" solution $x \in X$ such that $\| x \| \geq p$.

The answer is positive under certain assumptions. The basic idea is to introduce a convergent literation such that if

$$\| x_0 \| \geq p \text{ then } \| x_n \| \geq p, n=0, 1, 2, \dots\dots\dots$$

It is shown that (1) satisfies the above property.

We now state a well-known lemma [4].

Lemma 1. Let L_1 and L_2 be bounded linear operators in a Banach space X, where L_1 is invertible, and $\| L_1^{-1} \| . \| L_2 \| \leq 1$. Then the linear operator $(L_1+L_2)^{-1}$ exists, and (3)

$$\| (L_1+L_2)^{-1} \| \leq \frac{\| L_1^{-1} \|}{1 - \| L_2 \| . \| L_1^{-1} \|}$$

Definition 1. Assume that the linear operator $M_k\ (z)^{k-1}$ is invertible for some $z \in x$. Define the real polynomials $a(r)$, $\bar{a}(r)$ by

$$a\ (r)=a_{k-1}r^{k-1}+a_{k-2}r^{k-2}+\dots\dots\dots+a_2r^2+a_1r+a_0$$

and

$$\bar{a}\ (r)=a\ (r)-1$$

where $a_m = \| M_k \| \, \| (M_k\ (z)\left(k-1\right)^{-2} \| \, \| z \|^{m-1} . \begin{pmatrix} k-1 \\ m-1 \end{pmatrix}$,

$m=0, 1, 2 \dots\dots, k-1.$

By Descartes rule of signs [2], the equation

$$\bar{a}\ (r) = 0 \tag{4}$$

has two positive solutions s_1, s_2 or none.

Lemma 2. Let $z \in x$ be such that :

(*a*) the linear operator $M_k\ (z)^{k-1}$ is invertible ;

(*b*) the equation (4) has two positive solutions s_1, s_2 with $s_1<s_2$ If $k>2$ (one positive solution s_2 if $k=2$).

Then the linear operator $M_k\ (x)^{k-1}$ is invertible for $x \in U\ (z\ ,\ r)=$ $\{x \in x \mid \|\ x-z\ \| < r\}$ and for some $r \in (s_1\ ,\ s_2)$ if $k>2$ ($r \in (0\ ,\ s_2)$ if $k=2$) and

$$\|\ M_k\ x^{k-1}\| \leq \frac{\|\ (M_k\ (z^{k-1}))^{-1}\ \|}{\bar{a}\ (r)} \tag{5}$$

Proof. We have

$$M_k\ (x)^{k-1} = M_k\ [(x-z)+z]^{k-1}$$

$$= M_k\ (x-z)^{k-1} + \begin{bmatrix} k-1 \\ 1 \end{bmatrix} M_k\ (x-z)^{k-2}\ z + \ldots + \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} M_k\ (x-z) z^{k-2} + M_k\ z^{k-1}.$$

The hypotheses of Lemma 1 for

$$L_1 = M_k\ (z)^{k-1}.$$

$$L_2 = M_k\ (x-z)^{k+1} + \begin{bmatrix} k-1 \\ 1 \end{bmatrix} M_k\ (x-z) z^{k-2} + \ldots\ldots + \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} M_k\ (x-z) z^{k-2}$$ are satisfied if

$$a\ (r) < 1 \text{ and } M_k\ (z)^{k-1} \text{ is invertible.}$$

which are true by (a) and (b).

Finally (5) follows from (3).

Definition 2. Let $z \in x$ be such that $M_k\ (z)^{k-1}$ is invertible.

Define the real functions $c\ (r)$, $\bar{c}\ (r)$, $b\ (r)$ and $d\ (r)$ by

$$c\ (r) = \frac{1}{\bar{a}\ (r)} \Big\{ \|\ (M_k\ (z)^{k-1})^{-1} (I - M_k\ (z)^{k-1})\ \|\ r +$$

$$+ \| (M_k(z)^{k-1})^{-1} M_k \| \, \|z\| \left[(r+\|z\|)^{k-2} + (r+\|z\|)^{k-3} + \dots + (r+\|z\|)^{k-2} \|z\| \right] r + \| (M_k(z)^{k-1})^{-1} P(z) \| \Big]$$

where

$$P(z) = M_k(z)^k + y - z,$$

$$\bar{c}(r) = c(r) - r, \; b(r) = r + \|z\|,$$

and

$$d(r) = \frac{1}{\bar{a}(r)} \left\{ \frac{k-1}{\bar{a}(r)} (r + \|z-y\|)(r+\|z\|)^{k-2} \| (M_k(z)^{k-1})^{-1} M^k \| - \| (M_k(z)^{k-1})^{-1} \| \right\}.$$

Finally, define the operator T on X by

$$T(x) = [M_k(x)^{k-1}]^{-1} (x-y).$$

We now state the main result.

Theorem 1. Assume that there exist $x \in X$ and $r > 0$ such that :

(*i*) $\bar{a}(r) < 0$, $\bar{c}(r) \leqslant 0$, $d(r) < 1$;

(*ii*) the linear operator $M_k(z)^{k-1}$ is invertible on $\bar{U}(z, r)$.

Then the literation (1) converges to a unique solution x of (2) in $\bar{U}(z, r)$.

Proof. T is well defined on $\bar{U}(z, r)$.

Claim 1. T *maps* $\bar{U}(z, r)$ *into* $\bar{U}(z, r)$.

If $x \in \bar{U}(z, r)$ then

$$T(x) - z = (M_k x^{k-1})^{-1} (x-y) - z$$

$$=(M_k(x)^{k-1})^{-1}[x-z-M_k z(x^{k-1}-z^{k-1})-P(z)$$

$$=(M_k(x^{k-1})^{-1}[((I-M_k(z)^{k-1})$$

$$-M_k z(x^{k-2}+x^{k-3}z+\ldots+z^{n-3}))(x-z)-P(z)].$$

Now using (5) and the estimate.

$$\| x \| = \| (x-z)+z \| \leqslant \| x-z \| + \| z \| \leq r+ \| z \| = b(r)$$
it is enough to show that

$$\| T(x)-z \| \leq c(r) \leqslant r \text{ or } \bar{c}(r) \leq 0,$$
which is true by (*i*).

Claim 2. T *is a cantraction operator on* $\bar{U}(z, r)$.

If w, $U \in \bar{U}(z, r)$ then

$$T(w)-T(v)=(M_k(w)^{k-1})^{-1}(w-y)-(M_k(v)^{k-1})^{-1}(v-y)$$

$$=(M_k(w)^{k-1})^{-1}(w-y)-(M_k(w)^{k-1})^{-1}(v-y)$$

$$+(M_k(w)^{k-1})^{-1}(v-y)-(M_k(v)^{k-1})^{-1}(v-y)$$

$$=(M_k(w)^{k-1})^{-1}[(w-v)-M_k(w)^{k-2}+w^{k-3}r+\ldots\ldots$$

$$+v^{k-2})(w-v)(M_k(v)^{k-1})^{-1}(v-z+z-y)].$$

As in claim 1,

$$\| T(w)-T(v) \| \leqslant d(r) \| w-v \|$$

So T is a contraction on $\bar{U}(z, r)$ if $d(r)<1$, which is true by hypothesis (*i*). The result now follows from the contraction mapping principle.

Therem 2. Assume :

(a) The hypotheses of Theorem 1 are satisfied for some $r > 0$;

(b) The real equation $j(t)=0$, where

$$f(t)=p.\|M_k\|.t^{k-1}-t+\|y\|$$

has two positive solutions t_1 , t_2 with $t_1 < t_2$ if k > 2 (one positive solution if k=2) ;

(c) $[p, b(r)] c [t_1, t_2]$ if k > 2 ($p \in [t_1, t_2]$ if $k=2$), ; and

(d) $\|y\| \leqslant p.$

Then if $p \leqslant \|x_0\| \leq b(r)$,

(i) $p \leq \|x_n\| \leq b, c)$, $n=0, 1, 2, \ldots$;

(ii) the solution x of (1) is such that

$$p \leq \|x\| \leq b(r).$$

Proof. We have,

$$\|x_n - y\| = \|M_k (x_n)^{k-1} x_{n+1}\| \leq \|M_k\| . \|x_n\|^{k-1} . \|x_{n+1}\|$$

or

$$\|x_{n+1}\| \geqq \frac{\|x_n - y\|}{\|M_k\| . \|x_n\|^{k-1}} \geq \frac{|\,\|x_n\| - \|y\|\,|}{\|M_k\| . \|x_n\|^{k-1}}$$

Assume that $p \leq \|x_n\| \leq b(r)$ for all $k=0, 1, 2, \ldots\ldots, n.$

Since $\|x_n\| \geqq p \geq \|y\|$, it is enough to show

$$\frac{\|x_n\| - \|y\|}{\|M_k\| . \|x_n\|^{k-1}} \geq p$$

or $f(\|x_n\|) \leq 0$ which is true by (c), so

$$p \leq \|x_n\| \text{ for all } n=0\ ,1\ ,2\ ,\ldots$$

Now the x_n 's $\in \bar{U}(z\ ,r)$, $n=0, 1, 2, \ldots$, so

$$\|x_n\| \leq b(r),$$

which completes the proof of (i).

Finally (ii) follows from (a) and (i).

Remark. The literation (1) can be written as

$$x_{n+1}=x_n - \left[M_k\ (x_n)^{k-1}\right]^{-1} \left[M_k\ (x_n)^{k-1}(x_n)+y-x_n\right],$$
$$n=1\ ,2\ ,\ldots\ (6)$$

The Newton-Kantorovich method corresponding to (2) can be written as,

$$z_{n+1}=z_n - \left[kM_k\ (z_n)^{k-1}-1\right]^{-1} \left[M_k\ (z_n)^{k-1}(z_n)+y-z_n\right],$$
$$n=0\ ,1\ ,2\ ,\ldots\quad (7)$$

The latter literation is faster and easir to use most of the time, but if we choose an x_0 such that $\|x_0\| \geq p$, then (7) does not guarantee that the limit $w=\lim\limits_{n\to\infty} z_n$ is such that $\|w\| \leq p$ or $\|w\| \leq p$. This is exactly the advantage of iteration (6) when compared with (7). The basic defect of (6) is that each step involves the solution of an equation with a different invertible operator $M_k\ (x)^{k-1}$. For this reason one can easily prove theorem for the modified method.

$$x_{n+1}=x_n - \left[M_k\ (x_0)^{k-1}\right]^{-1} \left[M_k\ (x_n)^{k-1}(x_n+y-x_n)\right],$$
$$n=0\ ,1\ ,2\ ,\ldots\quad (8)$$

for some $x_0 \in x$ such that the linear opeartar $M_k\,(x_0)^{k-1}$ is invertible.

We now provide a simple example for Theorem 1.

Example. Consider the quadratic equation.

$$x = .2\,x^2 - 1 \text{ in } X = R \tag{9}$$

Here $M_2 x^2 = .2x^2$, $y = -1$ and $1 - 4\|M_2\|.\|y\| \angle 0$. Then according to Definitions 1 and 2 for $z=5$,

$$\bar{a}(r) = 5\,r - 1$$

$$\bar{c}(r) = .2\,r^2 - r + 1$$

and

$$d(r) = (.04)\,r^2 - (.4)\,r + 1.96$$

Theorem 1 can be applied provided that

$$1.38196601 \leq r < 1.8377225$$

and (1) becomes

$$x_{n+1} = 5\left\{1 + \frac{1}{x_n}\right\},\ n = 0, 1, 2, \ldots$$

Choose $x_0 = z = 5$. Then $x = x_{12} = 5.854101966$ is the large solution of (9). This is true since $\|x_n\| \geq 5$ and $\|x_n\| \geq p = \frac{1}{2\|M_2\|} = \frac{5}{2}$

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UNIQUENESS-EXISTENCE THEOREMS FOR THE SOLUTIONS OF POLYNOMIAL EQUATIONS IN BANACH SPACE

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Abstract. In this paper we classify the polynmial equations in Banach space in three distinct kinds by use of the Frechet derivative. For the two more general kinds, necessary and sufficient conditions will be given for their solution by means of formulas involving the nth root of linear operators. Some uniqueness results are also obtained.

Introduction. Let X and Y be real or complex linear spaces over the field F of real or complex numbers and consider the abstract polynomial equation of degree n on X.

$$P_n(x) = 0 \tag{1}$$

where

$$P_n(x) = M_n x^n + M_{n-1} x^{n-1} + \dots + M_2 x^2 + M_1 x + M_0 \tag{2}$$

or

$$P_n(x) = P_n(x_0) + P_n'(x_0)(x-x_0) + \frac{1}{2} P_n''(x_0)(x-x_0)^2 + \dots\dots + \frac{1}{n!} P_n^{(n)}(x_0)(n-x_0)^n \tag{3}$$

for any $x_0 \in x$, where the M_k 's are k-linear operators on X,

$k=1, 2, \ldots, n$, M_0 is fixed in X and $P_n^{(n)}(x_0)$ denotes the nth Frechet derivative of P_n at $x_0 \in X$.

Obviously (1) is a natural generalization of the scalar polynomial equation to the more abstract setting of a linear space. This class of abstract polynomial equations includes a number of interesting differential and integral equations [1], [3], [4], [5], which contain nonlinearities consisting of powers or products of the unknown functions, mingled with linear or integral operators.

In this paper, we classify equation (3) by use of the Frechet derivative, in three distinct kinds. For the two more general kinds, necessary and sufficient conditions are given for their solution by means of formulas involving the nth root of linear operators. Some uniqueness results are also obtained.

Definition 1. Denote by L (X , Y) the linear space over the field F of the linear operators from a linear space X into a linear space Y. For $k=2, 3, \ldots$ a linear operator from X into the space $L(x^{k-1}, y)$ of $(k-1)$—linear operators from X into Y is called *k-linear* operator from X into Y. For example, if an k-linear operator M_k from X into Y and k points $x_1, x_2, \ldots, x_k \in X$ are given, then

$$z=M_k\, x_1\, x_2 \ldots x_k$$

will be a point of Y the convention being that M_k operates on x_1, the $(k-1)$-linear operator $M_k\, x_1$ operators on x_2, and so on. The order of operation is important. Finally denote L (X , Y) by L (X) if X=Y.

Notation 1. Given a k-linear operator M_k from X into Y and a permutation $i=(i_1, i_2 \ldots . i_k)$ of the integers $1, 2, \ldots k$, the notation $M_k\,(i)$ can be used for the k-linear operator from X into Y such that

$$M_k\,(i)\,x_1\,x_2 \ldots x_k = M_k\,x_{i_1}\,x_{i_2} \ldots x_{i_k}$$

for all $x_1, x_2, \ldots, x_k \in X$.

Thus, there are $k!$ k-linear operators $M_k\,(i)$ associated with a given k-linear operator M_k .

Definition 2. A k-linear operator M_k from X into Y is said to be *symmetric* if,

$$M_k = M_k\,(i)$$

for all $i \in R_k$, where R_k denotes the set of all permutations of the integers 1 , 2 , ... , k. The symmetric k-linear operator

$$\bar{M}_k = \frac{1}{k!} \sum_{i \in R_k} M_k\,(i)$$

is called the *mean* of M_k .

Notation 2. The notation.

$$M_x\,x^p = M_k\,\overbrace{xx \ldots\ldots x}^{p}\,, \tag{2}$$

$p \leqslant k$, $M_k \in L\,(X^k\,, Y)$, for the result of applying M_k to $x \in X$ p-times will be used. If $p < k$, then (2) will represent a $(k-p)$-linear operator from X into Y. For $p=k$, note that

$$M_k\,x^k = \bar{M}_k\,x^k = M_k\,(i)\,x^k \tag{3}$$

for all $i \in R_k$,$x \in X$. It follows from (3) that the multilinear operators M_1 , M_2 , ... , M_k in (1) may be assumed to be symmetric without loss of generality, since each M_i in (1) may be replaced by $\bar{M}_i$, $i=2, 3, \ldots, k$, without changing the value of $P_k(x)$. Unless the contrary is explicity stated, the multilinear operators M_i, $i=2, 3, \ldots, k$ will be assumed to by symmetric.

Assume from now on that x, y are normed spaces.

Definition 3. A linear operator L from X into Y is said to be bounded if.

$$\|L\| = \sup_{\|x\|=1} \|L x\| \tag{4}$$

is finite. The quantity $\|L\|$ is called the bound (or norm) of L.

Definition 4. For $k \geq 2$, a k-linear operator M_k from X into Y is said to be *bounded* if it is a bounded linear operator from X into $L(X^{k-1}, Y)$, the Banach space of bounded $(k-1)$-linear operators from X into Y, The *bound* (or Norm) $\|M_k\|$ of M_k is defined by (4), with M_k being considered to be an element of $L(X, L(X^{k-1}, Y))$.

Notation 3. The space of bounded k-linear operators from X into Y will be denoted henceforth by $L(X^k, Y)$. Note that by Definitions (3) and (4) if $M_k \in L(X^k\ Y,)$ and $p \leq k$ then

$$\|M_k x^p\| \leq \|M_k\| \cdot \|x\|^p$$

Definition 5. An abstract polynomial operator P_k from X into Y of degree k defined by

$$P_k(x)=M_k x^k + M_{k-1}x^{k-1} + \ldots + M_2x^2 + M_1x + M_0$$

is said to be *bounded* if its co-efficients Mi, $i = 1, 2, \ldots\ldots, k$ are bounded multilinear operators from X into Y. From now on we assume p is bounded.

Definition 6. If T is an operator from X to Y, and for some $x_0 \in X$ there exists a linear operator L from X to Y such that

$$\lim_{\|\triangle x\| \to 0} \frac{\|T(x_0 + \triangle x) - T(x_0) - L(\triangle x)\|}{\|\triangle x\|} = 0,$$

L is called the *Frechet derivative of T at* x_0, denoted by $T'(x_0)$, and *T is said to be differentiable once at* x_0.

Definition 7. If for some $\delta > 0$, T is differentiable once at all x for which $\| x - x_0 \| < \delta$, and a bilinear operator B from X to Y exists such that

$$\lim_{\|\triangle x\| \to 0} \frac{\|T'(x_0 + \triangle x) - T'(x_0) - B(\triangle x)\|}{\|\triangle x\|} = 0,$$

B is called the *second Frechet derivative of T at* x_0, denoted by $T''(x_0)$, and *T is said to be differentiable twice at* x_0.

Definition 8. If for some $\delta > 0$, T is differentiable $k-1$ times at all x for which $\|x - x_0\| < \delta$, and a k-linear operator M from X to Y exists such that

$$\lim_{\|\triangle x\| \to 0} \frac{\|T^{(k-1)}(x_0 + \triangle x) - T^{(k-1)}(x) - M(\triangle x)\|}{\|\triangle x\|}$$

M is called the k^{th} *derivative of T* at x_0, denoted by $T^{(k)}(x_0)$ and *T is said to be differentiable k times at* x_0.

Definition 9. The operator

$$P_n^{(k)}(x) = n(n-1)\ldots(n-k-l)M_n x^{n-k} +$$

$$(n-1)(n-2)\ldots(n-k-2)M_{n-1}x^{n-k-1} + \ldots k!\, M_k$$

is *called the* k^{th} *derivative of the abstract polynomial operator* P_n , $k = 1 , 2 , \dots , n.$

Note that $P_n^{(k)}(x) \in L(X^k , Y)$ for $k=1 , 2 , \dots , n$, and that

$p'_n(x) , p''_n(x) , \dots , P_n^n(x)$ are symmetric

multilinear operators. The computation of $p_n(x)$ and its derivatives at point $x=x_0$ may be accomplished by adapting *Horner's algorithm* for scalar polynomials to this purpose. An algebraic formulation of this algorithm may be obtained by setting.

$$M_i^{(0)} = M_i , \quad i=0 , 1 , \dots , n ,.$$

$$M_n^{(j)}=M_n , j = 1 , 2 , \dots , n+1,$$

and calculating

$$M_{n-k}^{(j+1)} = M_{n-k+1}^{(j+1)} x_0 + M_{n-k}^{(j)}$$

$$j=0 , 1 , \dots , n-1 ; k=1 , 2 , \dots , n-j$$

The results of this calculation are

$$M_j^{(j+1)} = \frac{1}{j!} p_n^{(j)}(x_0) ,$$

$j=0 , 1 , \dots n$, notation $p_n^{(0)}(x_0)$ being used for $p(x_0)$.

Note that *Taylor's identity*

$$P_n(x)=p_n(x_0)+p'_n(x_0)(x-x_0)+\frac{1}{2} p_n''(x_0)(x-x_0)^2 +$$

$$+\dots+\frac{1}{n!} p_n^{(n)}(x_0)(x-x_0)^n$$

holds at any $x_0 \in X$.

From now on we assume that p_n is differentiable n-times on X.

Preposition 1. If x^* is a solution of the equation $p_n(x)=0$, then the equation has a second solution $x \neq x^*$ if and only if the equation.

$$p'_n(x^*)h+\frac{1}{2}p''_n(x^*)h^2+\ldots+\frac{1}{n!}p^{(n)}_n(x^*)h^n=0$$

has a nonzero solution h.

Proof. Let $x=x^*+h$, $h \neq 0$ then x is a solution if and only if.

$$0=p_n(x)=p_n(x^*+h)$$

$$=p_n(x^*)+p'_n(x^*)h+\frac{1}{2}p''_n(x^*)h^2+\ldots+\frac{1}{n!}p^{(n)}_n(x^*)h^n$$

$$=p'_n(x^*)h+\frac{1}{2}p''_n(x^*)h^2+\ldots+\frac{1}{n!}p^{(n)}_n(x^*)h^n$$

since $p_n(x^*)=0$.

Preposition 2. Assume :

(a) There exists $x^* \in X$ such that $p_n(x^*)=0$ and $p'_n(x^*)$ invertible :

(b) There exists $h \neq 0$ such that $p'_n(x^*+h)$ is not invertible and

$$\|P'_n(x^*)^{-1}\| \left[2\|M_2\|.\|h\|+\ldots+n\|M_n\|.\|h\|(\|x^*\|^{n-2}+\|x^*\|^{n-3}\|h\|+\ldots+\|h\|^{n-2})\right] \geq 1$$

then x^* is unique in the ball

$U(r)=\{x \in X \mid \|x^*-x\|<r,\ r=\|h\|\}$.

Proof. We have by (a) and (b)

$$\|p'_n(x^*+h)-p'_n(x^*)\| \geq \frac{1}{\|p'_n(x^*)^{-1}\|}$$

or

$$\|(2M_2 (x^*+h)-2M_2x^*)+\ldots+(nM_n (x^*+h)^{n-1}-nM_n x^{*n-1})\|$$

$$\geq \frac{1}{\|p'_n (x^*)^{-1}\|}$$

or

$$\|P'_n (x^*)^{-1}\| \left\{ 2\|M_2\| \|h\|+\ldots+n\|M_n\|.\|h\|(\|x^*\|^{n-2}+\|x^*\|^{n-3}+ \|h\|+\ldots+\|h\|^{n-2}) \right\} \geq 1.$$

Definition 10. The equation $p_n (x) = 0$ is said to be of

(*a*) *First kind*, if there exists $x_0 \in X$ such that

$$P_n^{(k)} (x_0) = 0_k \text{ , } k=1 \text{ , } 2 \text{ , } \ldots \text{ , } n-1$$

where 0_k is the 0 k-linear operator on X.

(*b*) *Second kind*, if $P'_n (x_0) \neq 0$, for all $x_0 \in X$ and there exists x_0 such that $P'_n (x_0)$ is invertible.

(*c*) *Third kind*, if the equation is not of first or second kind.

Example 1.

(*a*) The polynomial equations of ordinary algebra are of *first kind*, *e.g.* let $x_0 = \frac{1}{4}$ and

$$P (x)=16x^4-16x^3+6x^2-x-\frac{263}{16}.$$

It is easy to verify that

$$P' (x_0)=p'' (x_0)=p'''(x_0)=0.$$

(*b*) *Second kind*, Let $n=2$ for simplicity and consider the quadratic equation.

$$P (x)=M_2x^2+M_1x+M_0 \text{ , } x \in I R^2$$

where M_2 is defined by

$$\left[\begin{array}{cc} \frac{1}{2} C_1 & 0 \\ 0 & 0 \end{array} \middle/ \begin{array}{cc} \frac{1}{2} C_2 & 0 \\ 0 & 0 \end{array}\right], C_1\, C_2 \in \mathrm{IR}$$

and M_1 is defined by the matrix.

$$\begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \text{with } s_1\, s_4 \neq s_2\, s_3.$$

Then P_2 is a second segree polynomial on IR^2 and

$$P_2'(x_0) = 2M_2 x_0 + M_1 \ , \ x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1\, x_1 & 0 \\ c_2\, x_1 & 0 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \begin{bmatrix} c_1 x_1 + s_1 & s_2 \\ c_2 x_1 + s_3 & s_4 \end{bmatrix}$$

is nonzero for any $x_0 \in \mathrm{IR}^2$ and is invertible for some $x_0 \in \mathrm{IR}^2$; $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, say.

(*c*) *Third kind.* Again, let

$c_1 = c_2 \neq 0 \ , \ s_1 = s_2 = s_3 = s_4 = s \neq 0$

then $P_2'(x_0) \neq 0$ for any $x_0 \in \mathrm{IR}^2$, but $P_2'(x_0)$ is not invertible for any $x_0 \in \mathrm{IR}^2$.

Definition 11. If the equation $P_n(x) = 0$ is of first kind then it obviously reduces to.

$$M_n\, h^n = z \tag{7}$$

or

$$\frac{1}{n!} P_n^{(n)}(x_0)\, h^n = z$$

where $z = -P_n(x_0)$ and $P_n^{(k)}(x_0) = 0_k$, $k = 1, 2, \dots, n-1$. Equation (7) is then called the *normal form* of (3).

If the equation $P_n(x) = 0$ is of second kind, then by composing through both sides by $p'_n\ (x_0)^{-1}$ we obtain

$$\tilde{P}_n\ (x) = 0$$

where

$$\tilde{P}_n\ (x) = \tilde{M}_0 + I\ (x) + \tilde{M}_2 x^2 + \ldots + \tilde{M}_n\ x^n$$

with $\tilde{M}_0 = P'_n\ (x_0)^{-1} \quad P_n\ (x_0)$ and

$$\tilde{M}_k = p'_k\ (x_0) M_k\ , \quad k=1\ ,2\ , \ldots\ , n.$$

Finally denote by rad (M_k) the sets satisfying.

$$M_k\ (x+h)^k = M_k\ h^k\ , \text{ for all } h \in X\ , x \in \text{rad}\ (M_k)\ k=1,2,\ldots,n.$$

If $k=1$, rad $(M_1)=$Ker (M_1). Denote by $R=\bigcap_{k=1}^{n} \text{rad}\ (M_k)$

and note that $R \neq \varnothing$ since $0 \in R$.

Theorem 1. If the equation $P_n\ (x)=0$ is of first kind and $x=x^*$ is a solution , then $x=x^*+w$ is also a solution for any $w \in R$.

Proof. Let $x_1=x^*-x_0$, then

$$0=P_n\ (x_0+x_1)$$

$$=P_n\ (x_0)+P'_n\ (x_0)x_1 + \ldots + \frac{1}{n!} P_n^{(n)}\ (x_0)\ x_1{}^n$$

$$=P_n\ (x_0) + \frac{1}{n!}\ P_n^{(n)}\ (x_0)x_1{}^n$$

and

$$P_n\ (x_0+x_1+w)=P_n\ (x_0)+P'_n\ (x_0)\ (x_1+w)+\ldots+$$

$$\frac{1}{n!} P_n^{(n)} (x_0)(x_1+w)^n$$

$$= P_n(x_0) + \frac{1}{n!} P_n^{(n)} (x_0) x_1^n$$

$$= 0$$

Since $w \in R$, so $x^* + w$ is a solution of $P_n(x) = 0$.

Theorem 2. Assume (3) is of second kind for some $x_0 \in X$, then

(*a*) If n is even, then $x=x_0+h$ is a solution of (3) if and only if $x=x_0-h$ is solution.

(*b*) If n is odd and $P_n(x_0)=0$, then $x=x_0+h$ is a solution of (3) if and only if $x=x_0-h$ is solution.

Proof. As before if $x=x_0+h$ is a solution of (3), then

$$0 = P_n(x)$$

$$= P_n(x_0+h)$$

$$= P_n(x_0) + \frac{1}{n!} P_n^{(n)} (x_0) h^n \tag{9}$$

now,

$$P_n(x_0-h) = P_n(x_0) + \frac{1}{n!} P_n^{(n)} (x_0)(-h)^n \tag{10}$$

if n is even then $P_p^{(n)} (x_0)(-h)^n = P_n^{(n)} (x_0) h^n$ and then by (9) and (10), $P_n(x_0-h)=0$, *i.e.* $x=x_0-h$ is a solution of $P_n(x)=0$.

If n is odd $P_n^{(n)} (x_0)(-h)^n = -P_n^{(n)} (x_0) h^n$, using again (9), (10) and the fact that $P_n(x_0)=0$, we obtain $P_n(x_0 - h) = 0$, *i.e.* $x=x_0-h$ is a solution of $P_n(x) = 0$.

Theorem 3. If $P_n(x)=0$ is of second kind and for any $u, v \in X$, there exists $x=x(u, v)$ such that,

$$P'_n(u) + P'_n(v) = P'_n(x) \tag{11}$$

then

$$P'_n(u) + P'_n(v) \neq \theta_1 \text{ for any } u, v \in X.$$

Proof. Since the equation $P_n(x)=0$ is of second kind $P'_n(x) \neq 0$ for all $x \in X$, therefore,

$$P'_n(u) + P'_n(v) \neq 0 \text{ for all } u, v \in X.$$

Note that (11) is a strong hypothesis, however it is sometimes true, for example take $n=2$ and $x = \frac{u+v}{2}$ in (3).

Theorem 4 . Let $P_n(x)=0$ be of second kind and $x=x^*$ be a solution. Then $x=x^*+w$ cannot be a solution for any non zero $w \in R$.

Proof. Since $P_n(x)=0$ is of second kind, there exists $x_0 \in X$ such that $P'_n(x_0)$ is invertible. Set $h=x^*-x_0$; then

$P_n(x_0+h)=P_n(x^*)=0$, so

$$P'_n(x_0)^{-1}P_n(x_0)+h+\frac{1}{2}P'_n(x_0)^{-1}P''_n(x_0)h^2+\ldots$$

$$+\frac{1}{n!}P'_n(x_0)^{-1}P^{(n)}_n(x_0)h^n=0.$$

Suppose $x=x^*+w$ were a solution. Then
$P_n(x_0+h+w)=0$, *i.e.*,

$$0=P'_n(x_0)^{-1}P_n(x_0)+h+w+\frac{1}{2}P'_n(x_0)^{-1}P''_n(x_0)h^2+\ldots+$$

$$\frac{1}{n!}P'_n(x_0)^{-1}P^{(n)}_n(x_0)h^n$$

So $w=0$, contrary to hypothesis. The theorem now follows.

Definition 12. Let E, L be linear operators on X; then E *is called the nth root of* L if $E^n x = Lx$ for all $x \in X$. We also write

$$E^n = L \quad \text{or} \quad E = L^{\frac{1}{n}}.$$

Definition 13. The set

$F(M_n) = \{x \in X \mid M_n x^{n-1} = [M_n(M_n x)^{n-1}]^{\frac{1}{n}}\}$ is called *the factor set of* M_n.

Example 2. Let $X = \mathbb{R}^2$, then $x \in F(M_2)$ if $(M_2(x))^2 = M_2(M_2 x^2)$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$; i.e.

$M_2 x M_2 x y = M_2(M_2 x^2)\, y$ for all $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$.

Therefore

$$M_2(x, M_2 xy) = M_2(M_2 x^2, y). \tag{12}$$

Let us choose the array

$$\begin{pmatrix} 1 & 0 & / & 0 & 0 \\ 0 & 0 & / & 0 & 1 \end{pmatrix}, \text{ then}$$

$M_2 xy = M_2 yx = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$, therefore M_2 is a symmetric bilinear operator on $\mathbb{R}^2$.

Now (12) becomes

$$M_2\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_1 y_1 \\ x_2 xy \end{pmatrix}\right] = M_2\left[\begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right]$$

i.e., $x_1^2 y_1 = x_1^2 y_1$

$x_2^2 y_2 = x_2^2 y_2$.

therefore $F(M_2) = \mathbb{R}^2$.

Note that the equation $M_2 x^2 = z$, where $x = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $z_1, z_2 \in \mathbb{R}^+$ has a solution $u = \begin{pmatrix} \sqrt{z_1} \\ \sqrt{z_2} \end{pmatrix}$, since $M_2 u^2 = \begin{pmatrix} (\sqrt{z_1})^2 \\ (\sqrt{z_2})^2 \end{pmatrix} = z$.

Theorem 5. Assume that the equation $P_n(x)=0$ is of first kind for some $x_0 \in X$, then $x=u \in F(M_n)$ is solution if and only if $(M_n z^{n-1})^{\frac{1}{n}}$ exists and satisfies $(M_n z^{n-1})^{\frac{1}{n}}=M_n u^{n-1})$ and $(M_n z^{n-1})^{\frac{1}{n}} u=z$, where $z=-P_n(x_0)$.

Proof. Let $u \in F(M_n)$ be a solution of $P_n(x)=0$. According to definition, 11 we have

$$z=M_n u^n.$$

Now, $M_n z^{n-1}=M_n(M_n u^n)^{n-1}$

$$=(M_n u^{n-1})^n, \text{ since } u \in F(M_n) \quad \text{so} \quad (M_n z^{n-1})^{\frac{1}{n}} \text{ exists,}$$

$(M_n z^{n-1})^{\frac{1}{n}}=M_n u^{n-1}$ and $(M_n z^{n-1})^{\frac{1}{n}} u=M_n u^{n-1}u=M_n u^n=z.$

Conversely, if $(M_n z^{n-1})^{\frac{1}{n}}$ exists, $(M_n z^{n-1})^{\frac{1}{n}}=M_n u^{n-1}$ and $z=(M_n z^{n-1})^{\frac{1}{n}} u$ then

$$z=(M_n z^{n-1})^{\frac{1}{n}} u=M_n u^{n-1}u=M_n u^n$$

so u is a solution. Finally, we have

$$M_n u^{n-1}=(M_n z^{n-1})^{\frac{1}{n}}=(M_n(M_n u^n)^{n-1})^{\frac{1}{n}},$$

therefore $u \in F(M_n)$ and the theorem is proved.

A similar theorem has been proved [9] for (8), when $n=2$.

Definition 14. Dofine the linear operators L_n, $\tilde{L}_n$, $n=1, 2, \ldots\ldots$ on X by

$$L_n=L_n(x_1, x_2)=(M_n x_1^{n-1}+M_n x_1^{n-2} x_2+\ldots+M_n x_2^{n-1})+\ldots+M_1,$$

for all $n=1, 2, \ldots$ and all $x_1, x_2 \in X$, (13)

$$\overset{\frown}{L}_n=\overset{\frown}{L}_n(z, z_1,\ldots z_{n-1})=\begin{cases}\overset{\frown}{P}'_1 & \text{if } n=1\\ \overset{\frown}{P}''_2(z) & \text{if } n=2\\ \overset{\frown}{P}_n^{(n)} z_1 z_2\ldots z_{n-1}+M_1 & \text{if } n=3, 4\ldots\end{cases} \tag{14}$$

for $z, z_1,\ldots, z_{n-1}\in X$.

Using (3) L_n can also defined by

$$L_n=P'_n(x_0)+\tfrac{1}{2}[P''_n(x_0)(x_1-x_0)+P''_n(x_0)(x_2-x_0)]+\ldots+$$
$$+\frac{1}{n!}[P_n^{(n)}(x_0)(x_1-x_0)^{n-1}+P_n^{(n)}(x_0)(x_1-x_0)^{n-2}(x_2-x_0)+$$
$$+\ldots+P_n^{(n)}(x_0)(x_2-x_0)^{n-1}],$$

for $x_0, x_1, x_2\in X, n=1, 2,\ldots\ldots$ (15)

Note that

$P_n(x_1)-P_n(x_2)=L_n(x_1-x_2), n=1, 2,\ldots$

If L_n is non-singular for all $x_1, x_2\in X$ with $x_1\neq x_2$ and $P_n(x)=0$ for some $x\in X$ then it is obvious by (15) that (1) has a unique solution x in X.

The condition L_n being non-singular imposes severe restrictions on X, but never the less it may sometimes be true. The following theorem which is trivially true for $n=1$ is a positive result in this direction.

Theorem 6. Assume :

(*a*) the linear operators $\overset{\frown}{L}_n, n=2, 3,\ldots$ are non-singular for all $z, z_1,\ldots, z_{n-1}\in X$.

(*b*) there exist $C_1^1, C_1^2,\ldots, C_1^{n-1}, C_2^1, C_2^2,\ldots, C_2^{n-1}\in F$ such that

$$M_n(C_1^1x_1+C_2^1x_2)(C_1^2x_1+C_2^2x_2)\ldots(C_1^{n-1}x_1+C_2^{n-1}x_2)(x_1-x_2)$$
$$=M_n(x_1^n-x_2^n), n=2, 3\ldots \tag{16}$$

for all $x_1, x_2\in X$ and

(*c*) there exists $x\in X$ such that $\overset{\frown}{P}_n(x)=0$ then

(A) $\overset{\hookrightarrow}{P}_n(x_1)-\overset{\hookrightarrow}{P}_n(x_2)=\overset{\hookrightarrow}{L}_n(x_1-x_2)$, $n=1, 2,\ldots$ for all $x_1, x_2 \in X$ (17)

(B) the solution x of (1) is unique in X.

Proof. We have

$$\begin{aligned}\overset{\hookrightarrow}{P}_n(x_1)-\overset{\hookrightarrow}{P}_n(x_2)&=M_n x_1^n+M_1 x_1-(M_n x_2^n+M_n x_2)\\&=M_n(x_1^n-x_2^n)+M_1(x_1-x_2)\\&=\overset{\hookrightarrow}{L}_n(x_1-x_2) \qquad \text{by (16)}\end{aligned}$$

with, $z_k=C_1^k x_1+C_2^k x_2$, $k=1, 2,\ldots, n-1$ and $z=z_1$

If x' is another solution of (1) then by (16)

$$\overset{\hookrightarrow}{L}_n(x-x')=0$$

and by (*a*) $x-x'=0$ or $x=x'$.

Remark 1. Equation (16) constitutes a system of $2(n-1)$ unknows with $n+1$ equations with respect to $C_1^1, C_1^2,\ldots, C_1^{n-1}$, $C_2^1, C_2^2,\ldots, C_2^{n-1}$. The number of unknowns $U(n)$ is greater than the number of equations $E(n)$ if $n>3$. This implies that (16) may not hold for $n>3$.

However for $n=2$ or $n=3$, (16) becomes respectively

$$\left.\begin{aligned}2C_1^1-1&=0\\2C_2^1-1&=0\\C_1^1&=C_2^1\end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned}C_1^1C_1^2&=1\\C_2^1C_2^2&=1\\C_1^1C_2^2+C_2^1C_2^2-C_1^1C_1^2&=0\\C_2^1C_2^2-C_1^1C_2^2-C_2^1C_1^2&=0\end{aligned}\right\} \tag{19}$$

System (18) has the solution

$$C_1^1=C_2^1=\tfrac{1}{2} \tag{20}$$

and system (19) has an infinity of solutions given by the equations

$$(C_1^1)^2+(C_2^1)^2=C_1^1C_2^1$$

$C_1^1 \neq 0,\ C_2^1 \neq 0$

$$C_1^2 = \frac{1}{C_1^1},\quad C_2^2 = \frac{1}{C_2^1} \text{ in } F = C.$$

The solution (20) suggests that (1) has a unique solution x in *star-shaped regions.* To prove that we first need the following definitions.

Definition 15. Let w be fixed in X. A set S_w is said to be *star-shaped* with respect to $w \in X$ if

$$\{z : z = w + t(y-z),\ 0 \leqslant t \leq 1,\ y \in S_w\} \subseteq S_w$$

The set nS_w defined by

$$nS_w = \{z : z - w + t(y-z),\ y \in S_w,\ 0 \leqslant t \leqslant n\},\ n = 1, 2, \dots$$

obviously contains S_w and is likewise star-shaped with respect to w.

Note that a special case of star-shaped regions are the convex sets.

Theorem 7. If $\overset{\frown}{L}_2$ is non-singular for all $z \in S_w$, then

(A) $\overset{\frown}{P}_2(x_1) \neq \overset{\frown}{P}_2(x_2)$
for all $x_2 \in 2S_w$

(B) If $\overset{\frown}{P}_2(z) = 0$, $z \in S_w$ then z is the unique solution of the equation
$\overset{\frown}{P}_2(x) = 0$ is S_w.

Proof. If $x_2 \in 2S_w$, then $x = \frac{1}{2}(x_1 + x_2) \in S_w$ and (A), (B) now follow from theorem 6.

Note that a similar theorem can be stated if S_w is replaced by a eonvex set $C \subset X$ in theorem 7.

Also note the fact that $\overset{\frown}{P}_2(x_1) = \overset{\frown}{P}_2(x_2)$ implies that $\overset{\frown}{L}_2$ is singular at $x = \frac{1}{2}\ (x_1 \times x_2)$ is analgous to Rolle's theorem for real scalar functions.

An illustration of this situation will be given.

Example 3. Consider the differential equation

$$\frac{d^2x}{dt^2}+t+1=0, \quad t\in[0, \infty). \tag{21}$$

$$x(0)=0$$

$$x'(0)=0$$

As X take C_0'' $[0, \infty)$, the space of all continuously differentiable (twice) real fnnctions $x=x(t)$, $0\leqslant t<\infty$, such that $x(0)=0$, $x'(0)=0$, and as Y take the space $C[0, \infty)$ of all continuous real functions $M_0=M(t)$ on $0\leqslant t<\infty$. Equation (21) is a quadratic equation of the form (1) with $n=2$, with

$$M_2x^2=\frac{d^2x}{dt^2}, \; M_1x=t, \; M_0=1.$$

The derivative

$$\overset{\frown}{P}_2'(x)=2M_2x+M_1\neq 0 \text{ for all } x\in X.$$

It is easy to verify that

$$x=x(t)=-\frac{1}{2}\left(\frac{t^3}{3}+t^2\right)$$

is the unique solution of (21) in X.

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NUMERICAL SOLUTION OF ILL-POSED PROBLEMS BY REGULARIZATION METHODS

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Abstract

In many branches of science, problems arise in which it is desired to solve ill-posed problems in the form of integral equations of the first kind.

$$\int_a^b k(x, y) \quad f(y)dy = g(x) \qquad c \leqslant x \leqslant d.$$

In this paper we shall employ two different methods to solve mildy, moderately and severely ill-posed problems, available in the literature, the Methods are as follows :

(*i*) Generalized cross-validation regularization method using trigonometric polynomials.

(ii) Maximum Likelihood regularization method using trigonometric polynomials.

The two methods will be tested on integral equations of first kind of convolution type and graphs will be drawn for comparison purposes.

In this paper we discuss the use of Trigonometric polynomials approximating spaces with the different values of p, which is the order of regularization.

We shall approximate

$$\int_{-\infty}^{\infty} K(x-y) \; f(y)\, dy = g(x), \quad -\infty < x < \infty \tag{1}$$

by replacing it by

$$(K_N f_N)(x) = \int_0^1 K_N(x-y)\, f_N(y) dy = g_N(x) \tag{2}$$

where k_N is periodically continued outside (0, 1), then it can be proved that the discretization error in the convolution is precisely zero at the grid points $\{x_n\}$, [2].

1. Method 1, using Trigonometric polynomials :

Consider the integral equation (1)

In tikhonov regularization, the approximate solutions f_λ are defined variationally by [13].

$$C(f;\lambda) = \underset{f \in w}{\mathrm{Min}} \left\{ \left| \left| Kf - g \right| \right|^2 + \lambda \Omega(f) \right\} \tag{3}$$

where w is some space of smooth functions and $\lambda > 0$ is a regularization parameter.

Here Ω is some non-negative "stabilizing" functional which controls the sensitivity of the regularized solutions f_λ to perturbations in g.

We shall rcstrict our attention to pth order regularization of the form.

$$C(f;\lambda) = \left| \left| Kf - g \right| \right|_2^2 + \lambda \left| \left| f^{(p)} \right| \right|_2^2 \tag{4}$$

which is minimized over the subspace $H^p \subset L_2$.

Both norms in (4) are L_2. $f^{(p)}$ denotes the pth derivative of f and λ the regularization parameter.

2. Pth order Regularization Filters for Convolution Equations.

Consider the smoothing functional $C(f:\lambda)$ of equation (4) with $\Omega(f)=\left\|f^{(p)}\right\|_2^2$. Working in L_2 (ℝ) we have in the case of the convolution equation (1).

$$C(f;\lambda)=\left\|k(x)*f(x)-g(x)\right\|_2^2+\lambda\left\|f^{(p)}\right\|_2^2$$

$$=\frac{1}{2\pi}\left\{\left\|\hat{k}(w)\hat{f}(w)-\hat{g}(w)\right\|_2^2+\lambda\left\|(iw)^p\hat{f}(w)\right\|_2^2\right. \qquad (5)$$

using plancherel's identity, the convolution theorem for FTs and the propery $(f^{(p)})^{\wedge}=(iw)^p\hat{f}$.

Thus

$$C(f:\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{(\hat{k}\hat{f}-\hat{g})(\overline{\hat{k}\hat{f}-\hat{g}})+\lambda w^{2p}\hat{f}\overline{\hat{f}}\right\}dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{|\hat{k}|^2+\lambda w^{2p})|\hat{f}|^2-(\hat{k}\hat{f}\overline{\hat{g}})+\overline{\hat{k}\hat{f}}\hat{g}+|\hat{g}|^2\right\}dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}(|\hat{k}|^2+\lambda w^{2p})\left|\hat{f}-\frac{\overline{\hat{k}}\hat{g}}{|\hat{k}|^2+\lambda w^{2p}}\right|^2 dw$$

$$+\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\frac{w^{2p}|g|^2}{|\hat{k}|^2+\lambda w^{2p}}dw \qquad (6)$$

clearly $C(f;\lambda)$ is minimized w.r.t. f when

$$\hat{f}(w)=\frac{\overline{\hat{k}}\hat{g}}{|\hat{k}|^2+\lambda w^{2p}}=z(w;\lambda)\frac{\hat{g}(w)}{\hat{k}(w)} \qquad (7)$$

$$\text{where } z(w;\lambda)=\frac{|\hat{k}|^2}{|\hat{k}|^2+\lambda w^{2p}} \qquad (8)$$

$z(w;\lambda)$ is called the pth order filter or stabilizer.

(7) can be written as $f_\lambda(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(w\,;\lambda)\, \frac{\hat{g}(w)}{\hat{k}(w)} \,\text{Exp}\,(iwy)\, dw$ (9)

We assume throughout that the support of each function f, g and K is essentially finite and contained within the interval [0, 1) possibly by a change of variable. It is then convenient to adopt the approximating function space T_{N-1} of trigonometric polynomials of degree at most N−1 and period 1, since the discretization error in the convolution may be made exactly zero at the grid points and FFTs (Fast Fourier Transforms) may be employed in the solution procedure. Let g and K be given at N equally spaced points $x_n = nh$ $n = 0, 1, 2, \ldots, N-1.$ with spacing $h = \frac{1}{N}$. Then g and K are interpolated by g_N and $K_N \in T_{N-1}$, where

$$g_N(x) = \frac{1}{N} \sum_{q=0}^{N-1} \hat{g}_{N.\,q} \exp\,(iw_q x) \tag{10}$$

$$\hat{g}_{N,\,q} = \sum_{n=0}^{N-1} g_n \exp\,(-iw_q x_n\,) \tag{11}$$

and

$$g\,(x_n) = g_n = g_N\,(x_n)\;,\; w_q = 2\pi q \tag{12}$$

with similar expressions for k_N.

In T_{N-1}, f_λ in (9) is approximated by

$$f_{N\,;\,\lambda}(x) = \sum_{q=0}^{N=1} z_{q\,;\,\lambda} \frac{\hat{g}_{N\,,\,q}}{\hat{k}_{N,\,q}} \exp\,(iw_q\; x) \tag{13}$$

where $Z_{q\,;\,\lambda}$ is the discrete pth order filter given by

$$Z_{q\;;\,\lambda} = \frac{|\hat{K}_{N\,,\,q}|^2}{|\hat{K}_{N\,,\,q}|^{2} + N^2 \lambda w_q^{2p}} \tag{14}$$

where

$$\overset{\frown}{w}_q = \begin{cases} w_q, & 0 \le q < \frac{1}{2} N \\ w_{N-q}, & \frac{1}{2}N \le q \le N-1 \end{cases}$$

The optimal λ in (14) is still to be determined.

Choice of Regularization parameter in Trigonometric Approximation.

We mention below two specific Methods of choosing λ in the context of trigonometric regularization. The choices are optimal in various senses.

3. Wahba's cross-validation Method (CV) [15]

From equation (13) we know that the filtered solution $f_{N,\lambda}(x) \in T_{N-1}$ which minimizes

$$\sum_{n=0}^{N-1} [(K_N * f)(x_n) - g_n]^2 + \lambda \| f^{(p)}(x) \|_2^2$$

is $f_{N,\lambda}(x) = \frac{1}{N} \sum_{q=0}^{N-1} f_{N,\lambda,q} \exp(2\pi i q x)$,

where

$$f_{N,\lambda,q} = Z_{q;\lambda} \frac{\hat{g}_{N,q}}{\hat{K}_{N,q}}$$

with $Z_{q;\lambda} = \dfrac{|\hat{K}_{N,q}|^2}{|\hat{K}_{N,q}|^2 + N^2 \lambda w_q^{2p}}$

where $w_q = \begin{cases} w_q, & 0 \le q < \frac{1}{2} N \\ w_{N-q}, & \frac{1}{3} N \leqslant q \leqslant N-1 \end{cases}$

The idea of generalized cross-validation (GcV) is quite simple to understand. Suppose we ignore the jth date point g_j and define the filtered solution $f^{(j)}_{N,\lambda}(x) \in T_{N-1}$ as the minimizer of

$$\sum_{\substack{n=0 \\ n \neq j}}^{N-1} [(K_N * f)(x_n) - g_n]^2 + \lambda \| f^{(p)}(x) \|_2^2$$

then we get a vector $g^{(j)}_{N,\lambda} \in R_N$ defined by

$$\underline{g}^{[j]}_{N,\lambda} = K \underline{f}^{[j]}_{N,\lambda} \tag{15}$$

Clearly the jth element $g^{[j]}_{N,\lambda,j}$ of equation (15) should "predict" the missing value g_j. We may thus construct the weighted mean square prediction error over all j.

$$V(\lambda, p) = \frac{1}{N} \sum_{j=0}^{N-1} w_j(\lambda) (g^{[j]}_{N,\lambda,j} - g_j)^2 \tag{16}$$

The principle of GcV applied to the deconvolution problem then says that the best filtered solution to the problem should minimize the mean equare prediciion error in (16). Thus the optimal λ minimizes $V(\lambda, p)$ for given p and does not require a knowledge of σ^2.

To minimize $V(\lambda, p)$ in the form given by equation (16) is a time consuming problem. Wahba [15] has suggested an alternative expression which depends on a particular choice of weights, resulting in considerable simplification. Let

$$\underline{f}_{N,\lambda} = (f_{N,\lambda}(x_o), \ldots f_{N,\lambda}(x_{N-1}))^T \tag{17}$$

and define

$$\underline{g}_{N,\lambda} = K \underline{f}_{N,\lambda} \tag{18}$$

then there exists a matrix $A(\lambda)$, called an influence matrix such that

$$\underline{g}_{N,\lambda} = A(\lambda) \underline{g}_N \tag{19}$$

Let $K = \mathrm{diag}(h \hat{K}_{N,q})$ and $\hat{Z} = \mathrm{diag}(Z_{q;\lambda})$

then from (25) we see that

$$\underline{f}_{N,\lambda} = \psi \hat{K}^{-1} \hat{Z} \hat{\underline{g}}_N \tag{20}$$

where $\hat{\underline{g}}_N , = \psi^H \underline{g}_N ,$ (21)

(22)

and so $A(\lambda) = \psi \hat{Z} \psi^H$

since $K = \psi \hat{K} \psi^H$ (23)

Wahba [15] has shown in a more general contest, that the choice of weights.

$$w_j(\lambda) = \left| \frac{1 - a_{jj}(\lambda)}{\frac{1}{N} \text{Trace}(I - A(\lambda))} \right|^2 \quad J = 0, \ldots\ldots N-1 \tag{24}$$

where $A(\lambda)$ is the influence matrix in equation (19), enables the expression (16) to be written in the simpler form

$$V(\lambda, p) = \frac{\frac{1}{N} \left\| (I - A(\lambda)) g_N \right\|_2^2}{\left[\frac{1}{N} \text{Trace}(I - A(\lambda)) \right]^2} \quad [15] \tag{25}$$

From equation (22) it follows that

$$V(\lambda, p) = \frac{\frac{1}{N} \left\| (I - \hat{Z}) \hat{g}_N \right\|_2^2}{\left[\frac{I}{N} \text{Trace}(I - \hat{Z}) \right]^2}$$

i.e.,

$$\frac{\frac{1}{N} \sum_{q=0}^{N-1} (1 - Z_{q;\lambda})^2 \left| \hat{g}_{N,q} \right|^2}{\left[\frac{1}{N} \sum_{q=0}^{N-1} (I - Z_{q:\lambda}) \right]^2}$$

Since the matrix $A(\lambda)$ in (22) is circulant, the weights in (24) are all unity. The expression in (26) is minimized using NAG Routine E04 ABA, which uses a quadratic interpolation technique to obtain a minimum.

3.1 Maximnm Liklihood Method (ML)

For the details of the Method, the reader is refered to [1] and [2 , 3].

We shall estimate the value of the regularization parameter by minimizing the function,

$$V_{ML}(\lambda , p) = \tfrac{1}{2}N \log \left[\sum_{q=1}^{N-1} \left| g_{N,q} \right|^2 (1 - Z_{q;\lambda}) \right]$$

$$- \tfrac{1}{2} \sum_{q=1}^{N-1} \log (1 - Z_{q;\lambda}) \qquad (28)$$

equation (27) yields the optimal value of the regularization parameter λ , depending on the known Fourier coefficients $\hat{g}_{N,q}$ and $\hat{K}_{N,q}$. No prior knowledge of σ^2 is assumed but an *a posteriori* estimate is given by equation,

$$\sigma^2 = \frac{1}{N} \sum_{q=1}^{N-1} \left| \hat{g}_{N,q} \right|^2 (1 - Z_{q;\lambda}) \qquad (28)$$

4. Test problems.

Problem P (1) This problem has been taken from [5],

and is given by $\int_{-4.2}^{4.4} K(x-y) f(y)\, dy = g(x)$

where $f(x)$ and $K(x)$ are both Gaussian

i.e. $K(x) = \dfrac{1}{\sigma_k \sqrt{2\pi}} \exp\left(- \dfrac{x^2}{2\sigma_k^2} \right)$

and $f(x) = \dfrac{1}{\sqrt{2\pi}} \exp\left(- \dfrac{x^2}{2\sigma_f^2} \right)$

then $g(x)$ is also Gaussian with a standard deviation $= \sqrt{\sigma_f^2 + \sigma_k^2}$, the data of Johnson [8 , 9] listed in Table 1. were obtained for $\sigma_k = 1$, $\sigma_f = 0.7$; $g(x)$ was sampled at the points $x_q = 0.2\,(q-22)$, $q = 1, 2, \ldots\ldots 44$, and normally distributed noise with a standard deviation of 3.3%was added to $g(x_q)$ values. The data graph showing the values of f, K and g in DIAG (1);

Problem P (2) : This example has been taken from Turchin [14] and is given by

$$\int_{-3.2}^{3.1} K(x-y)\, f(y)\, dy = g(x)$$

it is a moderately ill-posed problem. where f is the sum of two Gaussian functions,

$$f(x) = 0.5 \exp\left[\frac{-(x+0.4)^2}{0.18}\right] + \exp\left[\frac{-(x-0.6)^2}{0.18}\right]$$

the Kernel K (x) is Triangular and is given by,

$$K(x) = \begin{cases} (5/8)\ (-x+0.8) & 0 \leqslant x < 0.8 \\ (5/8)\ (x+0.8) & -0.8 \leq x < 0 \\ 0 & |x| \geq 0.8 \end{cases}$$

the essential support of $g(x)$ is $-2.1 < x < 2.3$

The fuuctions are displayed in DIAG (2) with a spacing 0.1.

Proplem P (3). This example has been taken from Medgyessy [10] The solution function is the sum of six Gaussians aud the Kernel is also Gaussian.

we have

$$\int_{-\infty}^{\infty} K(k-y)\, f(y)\, dy = g(x)$$

where $g(x) = \sum_{k=1}^{6} A_k \exp\left[-\frac{(x-\alpha_k)^2}{\beta_k}\right]$

$A_1 = 10$	$\alpha_1 = 0.5$	$\beta_1 = 0.04$
$A_2 = 10$	$\alpha_2 = 0.7$	$\beta_2 = 0.02$
$A_3 = 5$	$\alpha_3 = 0.875$	$\beta_3 = 0.02$
$A_4 = 10$	$\alpha_4 = 1.125$	$\beta_4 = 0.04$
$A_5 = 5$	$\alpha_5 = 1.325$	$\beta_5 = 0.02$
$A_6 = 5$	$\alpha_6 = 1.525$	$\beta_6 = 0.02$

The essential support of $g(x)$ is $0 < x < 2.$

the Kernel is

$$K(x) = \frac{1}{\sqrt{\pi \lambda}} \exp\left(-\frac{x^2}{\lambda}\right), \ \lambda = 0.015$$

with essential support (—0.26, 0.26)

The solution is

$$f(x) = \sum_{k=1}^{6} \left\{ \left(\frac{\beta k}{\beta k-\lambda} \right)^{\frac{1}{2}} A_k \exp \left[- \frac{(x-\alpha k^2)}{\beta k-\lambda} \right] \right\}$$

with essential support (0.26, 1.74).

The functions are given in DIAD (3).

Addition of random noise to the data functions.

In solving the problems P (1) to P (3) we have considered the data functious contaminated by varying amounts of random noise. To generate sequences of random errors of the form $\xi_n = 0$, 1...N—1. We have used the NAG Aigorithm GO5DDA which returns pseudo-random real numbers taken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental errors we have have

$$A = 0$$
$$B = \frac{X}{100} \max_{0 \leqslant n \leqslant N-1} \left| g_n \right| \qquad (29)$$

where X denotes a chosen percentage, *e.g.* X=0.7 , 3.3 and 6.7. Thus the random error ξ_n added to g_n does not exceed $3x\%$ of the maximum value of $g(x)$.

4. Numerical Results :

In this section we describe the application of the two methods to the test problems P (1)—P (3), Results are shown in Table 2.

Example P (1).

(*i*) **GCV Trig.** worked very well in the case of clean data, in the case of 3.3% noise the solution is good as shown in DIAG (4).

(*ii*) **ML Trig.** did not yield a good result in case of clean data, though it has resolved the peak quite clearly, but in the case of 3.3% noise the solution is best as shown in DIAG (5).

Example P (2) :

(*i*) **GCV Trig.** For clean data the results are quite good resolving the two peaks very clearly. In the case of 1.7% noise, when we increased the order of regularization from P=2 to P=3 the solution improved as shown in DIAD (6).

(*ii*) **ML Trig.** For Accurate data the results are quite pood resolving two peaks very clearly but for 1.7% noise it could not resolve the peaks very clearly. When P=3 the solution improved and resolved the two peaks as shown in DIAG (7).

Problem P (3) :

(*i*) **GCV Trig.** For clean data the method resolved all the six peaks and the resolution in this case is better, but for 1.7% noise this method could resolve only four peaks clearly as shown in DIAG (8).

(*ii*) **ML Trig.** For clean data the method succeeded in resolving all the six peaks, but for 1.7% noise it has resolved 5 peaks as shown in DIAG (9).

Concluding Remarks.

The overall performance of the methods applied to mildly, moderately and severely ill-posed problems is quite good, both trigonometric methods are equally good for low noise level and for mildly and moderately ill-posed problems. For higher levels of noise and for severely ill-posed problems negative lobes at the end points is not acommendable feature, therefore for such problems extra information is needed such as non-negativity.

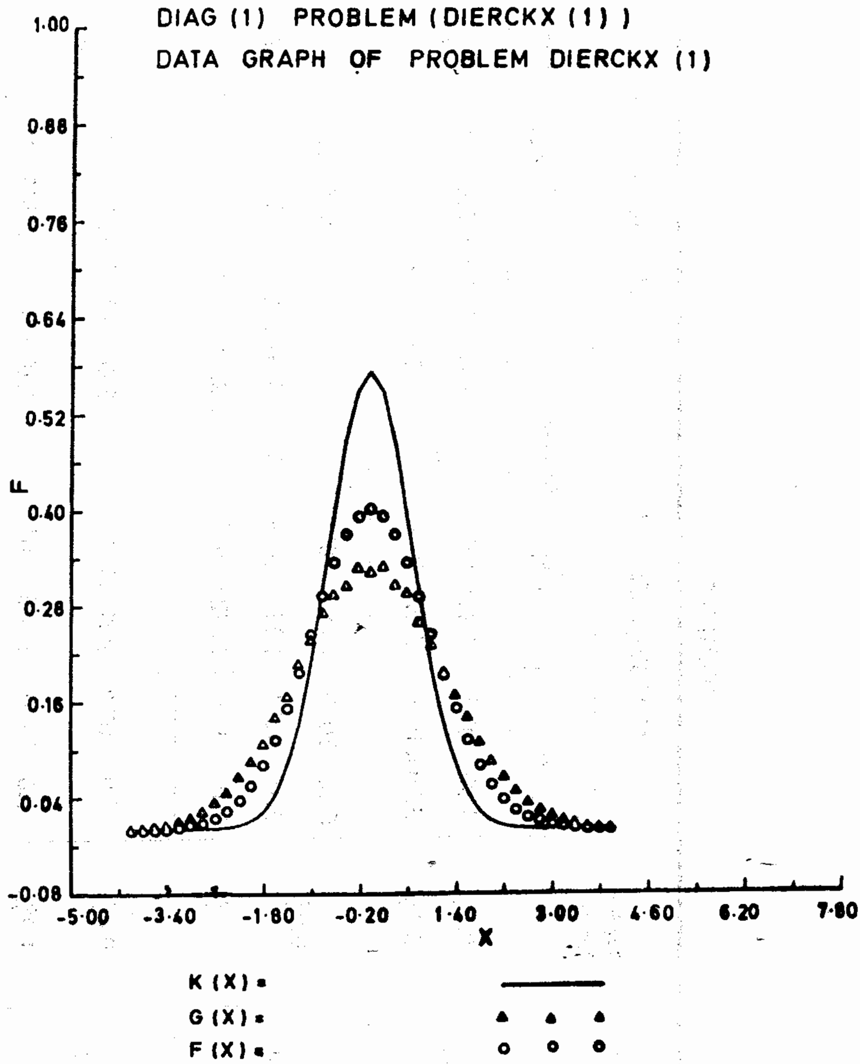
TABLE 1

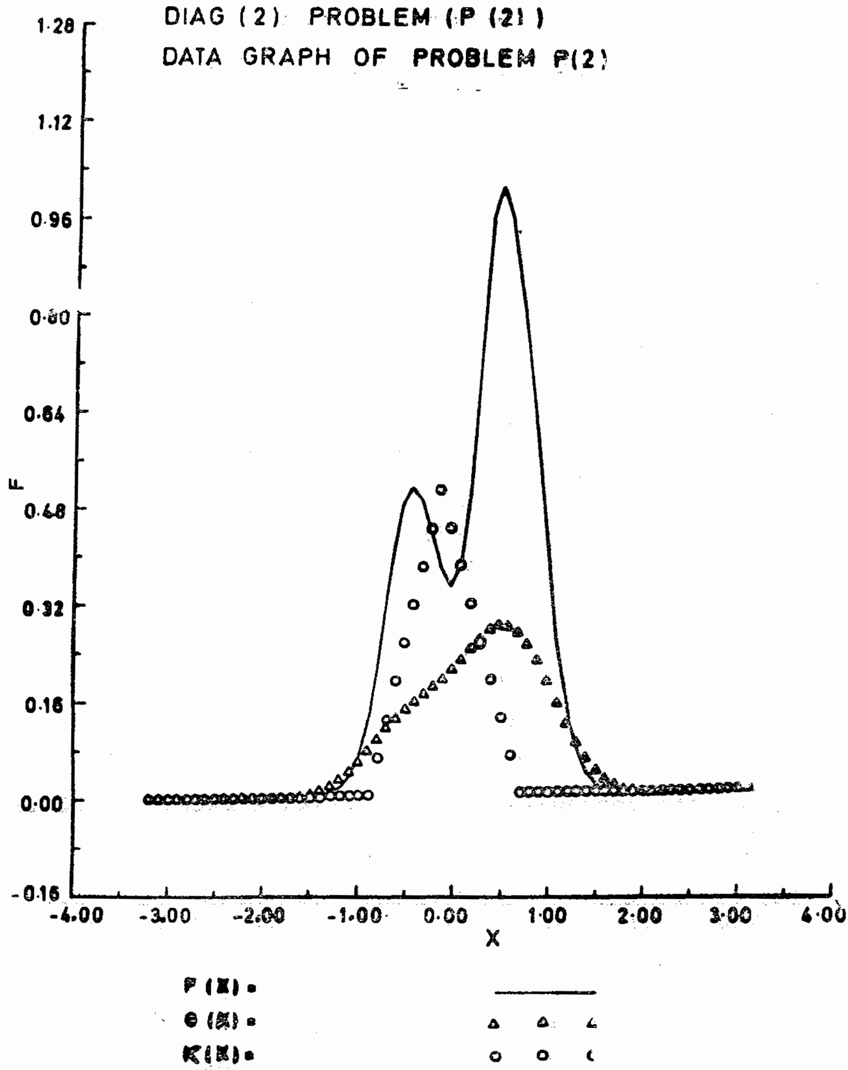
Johson's Test Data

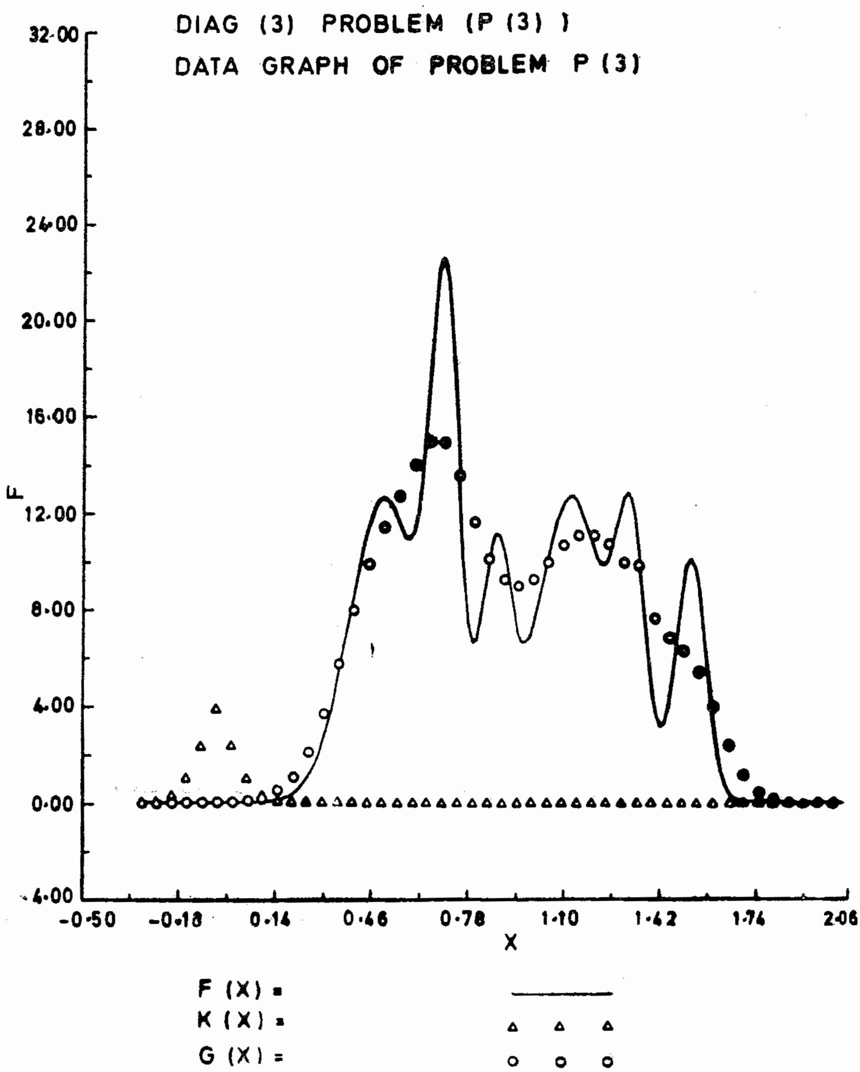
q	x_q	g_q	q	x_q	g_q
1	−4.2	0.001	23	0.2	0.328
2	−4.0	0.002	24	0.4	0.305
3	−3.8	0.003	25	0.6	0.295
4	−3.6	0.004	26	0.8	0.259
5	−3.4	0.007	27	1.0	0.229
6	−3.2	0.010	28	1.2	0.196
7	−3.0	0.016	29	1.4	0.167
8	−2.8	0.023	30	1.6	0.140
9	−2.6	0.034	31	1.8	0.108
10	−2.4	0.046	32	2.0	0.084
11	−2.2	0.066	33	2.2	0.065
12	−2.0	0.085	34	2.4	0.048
13	−1.8	0.107	35	2.6	0.034
14	−1.6	0.140	36	2.8	0.024
15	−1.4	0.166	37	3.0	0.016
16	−1.2	0.206	38	3.2	0.011
17	−1.0	0.236	39	3.4	0.007
18	−0.8	0.269	40	3.6	0.004
19	−0.6	0.293	41	3.8	0.003
20	−0.4	0.303	42	4.0	0.002
21	−0.2	0.226	43	4.2	0.001
22	+0.0	0.321	44	4.4	0.000

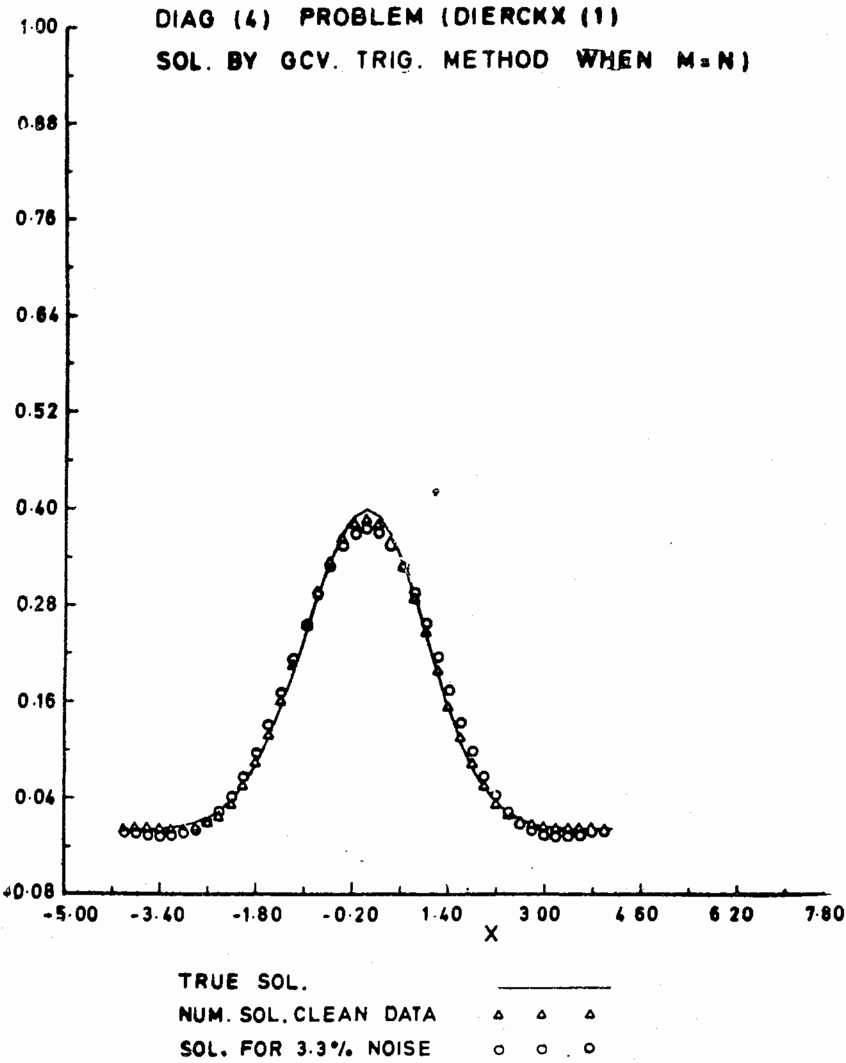
TABLE 2

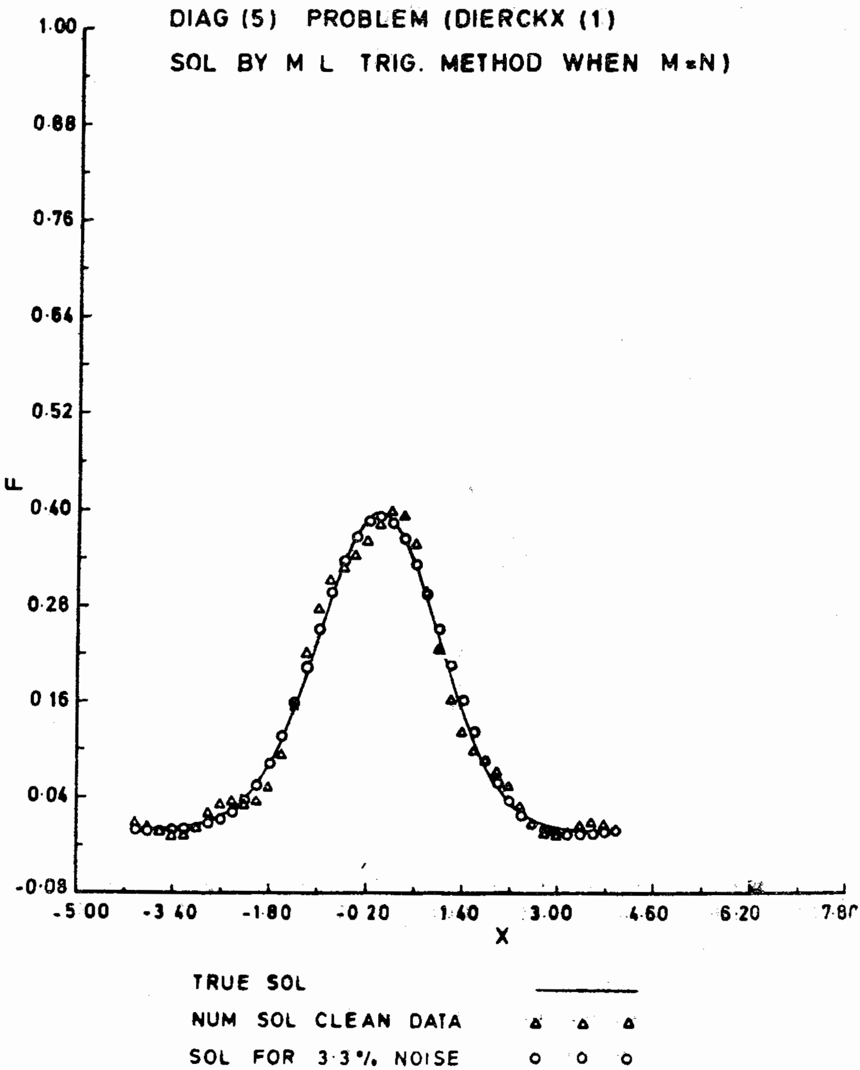
Problem	N	Noise level	GcV Trig. Method			ML Trig. Method		
			λ	$\|f-f_N\|_2$	DIAG	λ	$\|f-f_N\|_2$	DIAG
P (1)	44	0.0%	1.90×10^{-12}	3.20×10^{-3}	4	9.40×10^{-25}	8.760×10^{-3}	5
		3.3%	5.70×10^{-8}	5.29×10^{-2}		3.90×10^{-8}	7.0×10^{-2}	
P (2)	64	0.0%	6.80×10^{-12}	1.186×10^{-2}	6	1.50×10^{-15}	2.975×10^{-2}	7
P=2			3.10×10^{-9}	4.257×10^{-2}		3.90×10^{-2}	1.305×10^{-1}	
P=3		1.7%	8.0×10^{-10}	4.082×10^{-2}		2.70×10^{-9}	1.223×10^{-1}	
P (3)	64	0.0%	1.0×10^{-19}	9.30×10^{-2}	8	3.40×10^{-15}	6.260×10^{-1}	9
		1.7%	1.6×10^{-11}	1.406×10^{0}		1.10×10^{-11}	3.545×10^{0}	

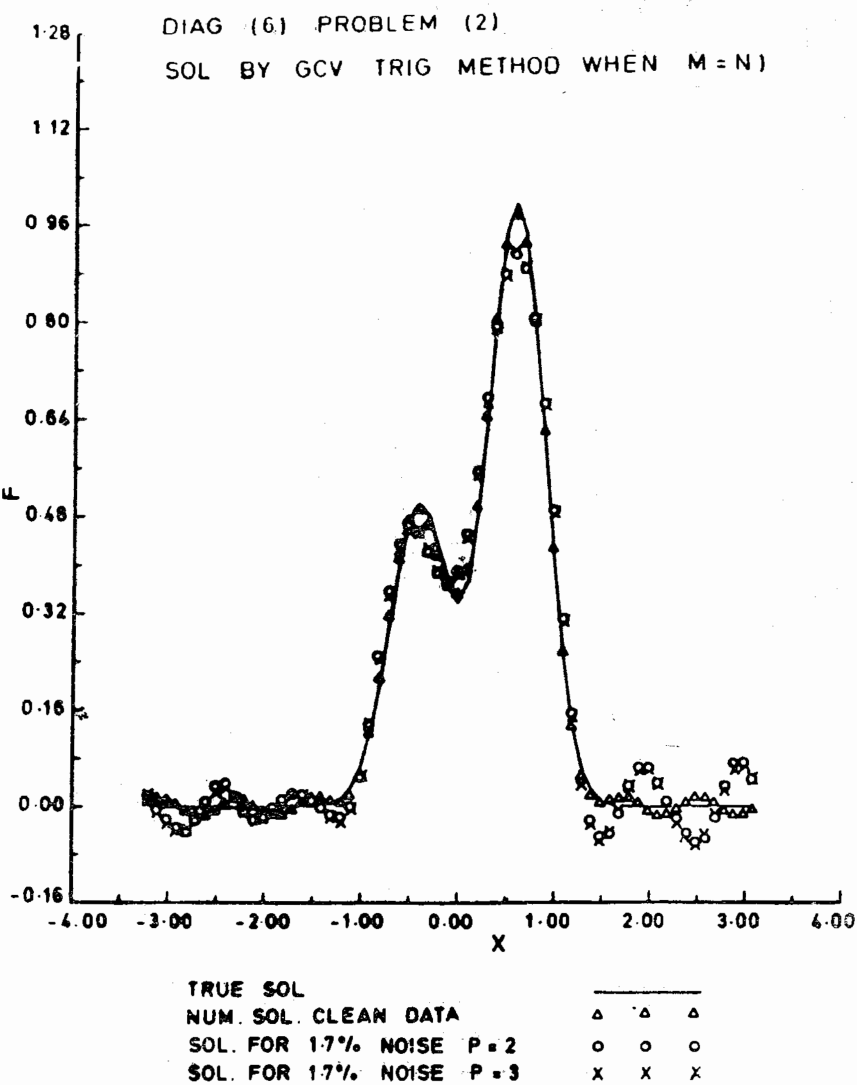

DIAG (1) PROBLEM (DIERCKX (1))
DATA GRAPH OF PROBLEM DIERCKX (1)
1.00
0.88
0.76
0.64
0.52
0.40
0.28
0.16
0.04
-0.08
F
-5.00
-3.40
-1.80
-0.20
1.40
3.00
4.60
6.20
7.80
X
K (X) =
G (X) =
F (X) =

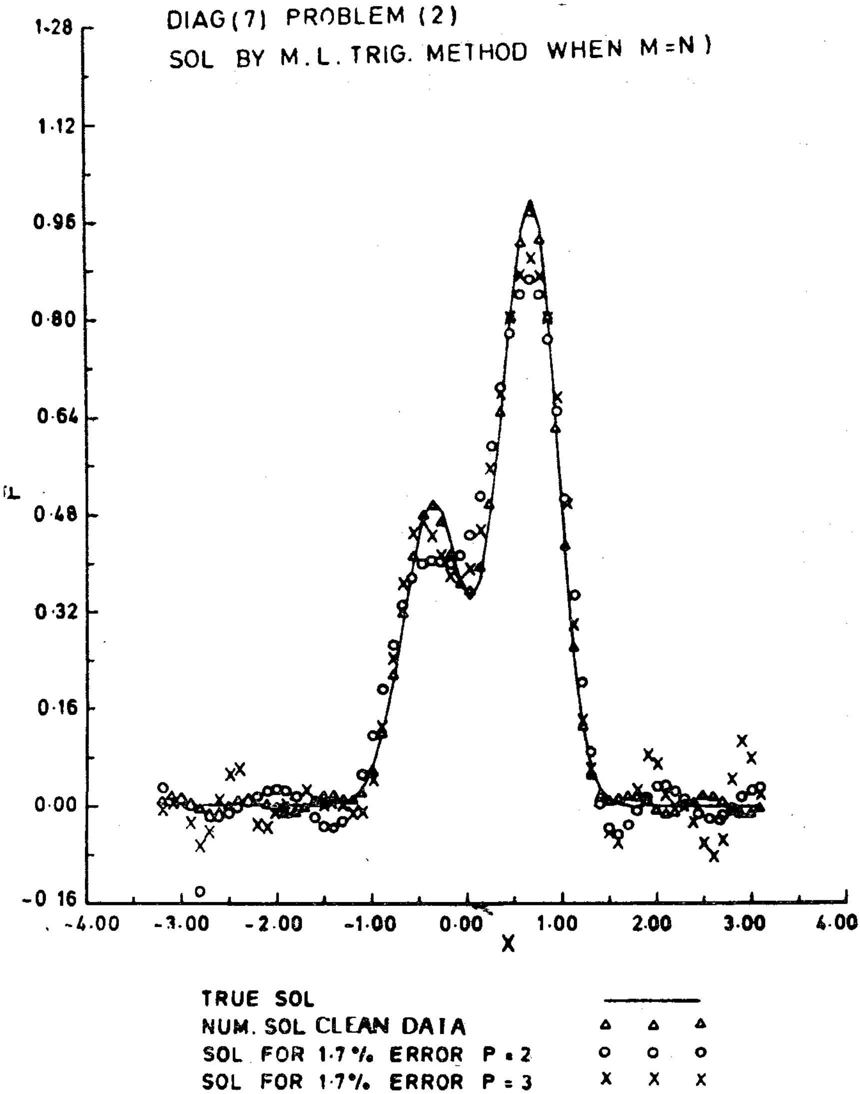

DIAG (2) PROBLEM (P (2))
DATA GRAPH OF PROBLEM P(2)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
F
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
X
F (X) =
G (X) =
K(X) =

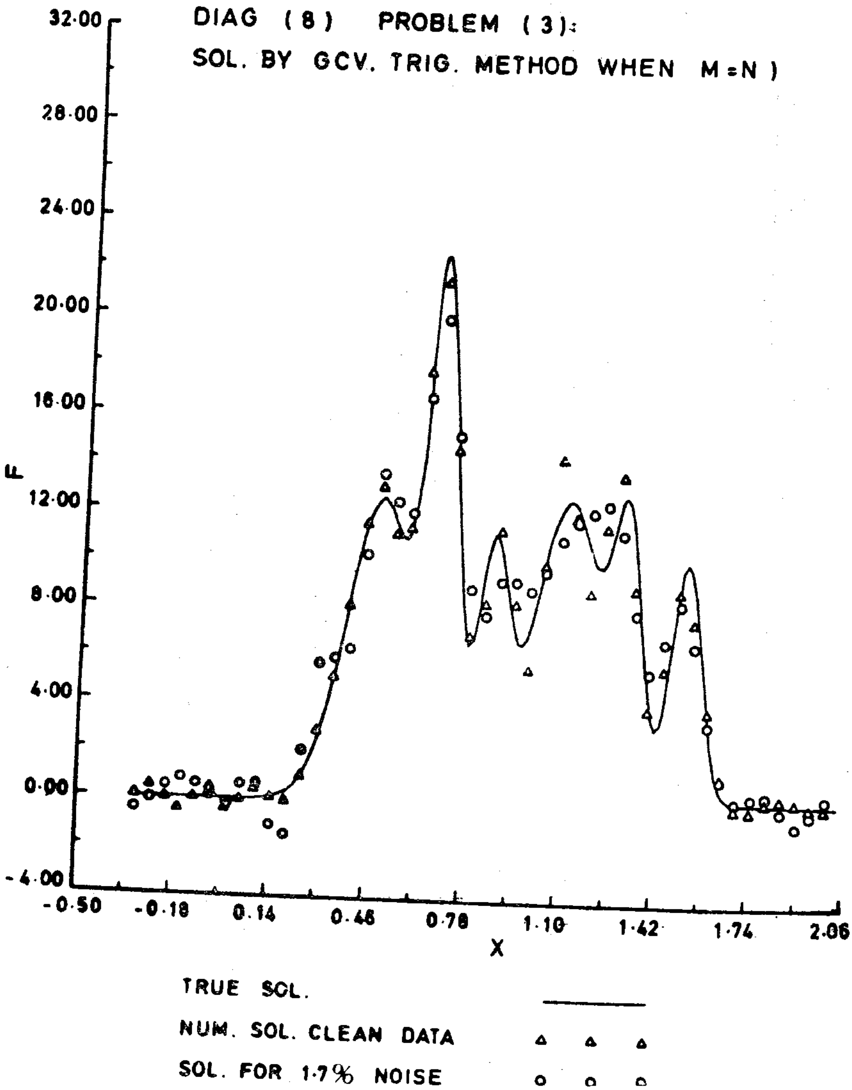

DIAG (3) PROBLEM (P (3))
DATA GRAPH OF PROBLEM P (3)
32.00
28.00
24.00
20.00
16.00
12.00
8.00
4.00
0.00
-4.00
F
-0.50
-0.18
0.14
0.46
0.78
1.10
1.42
1.74
2.06
X
F (X) =
K (X) =
G (X) =

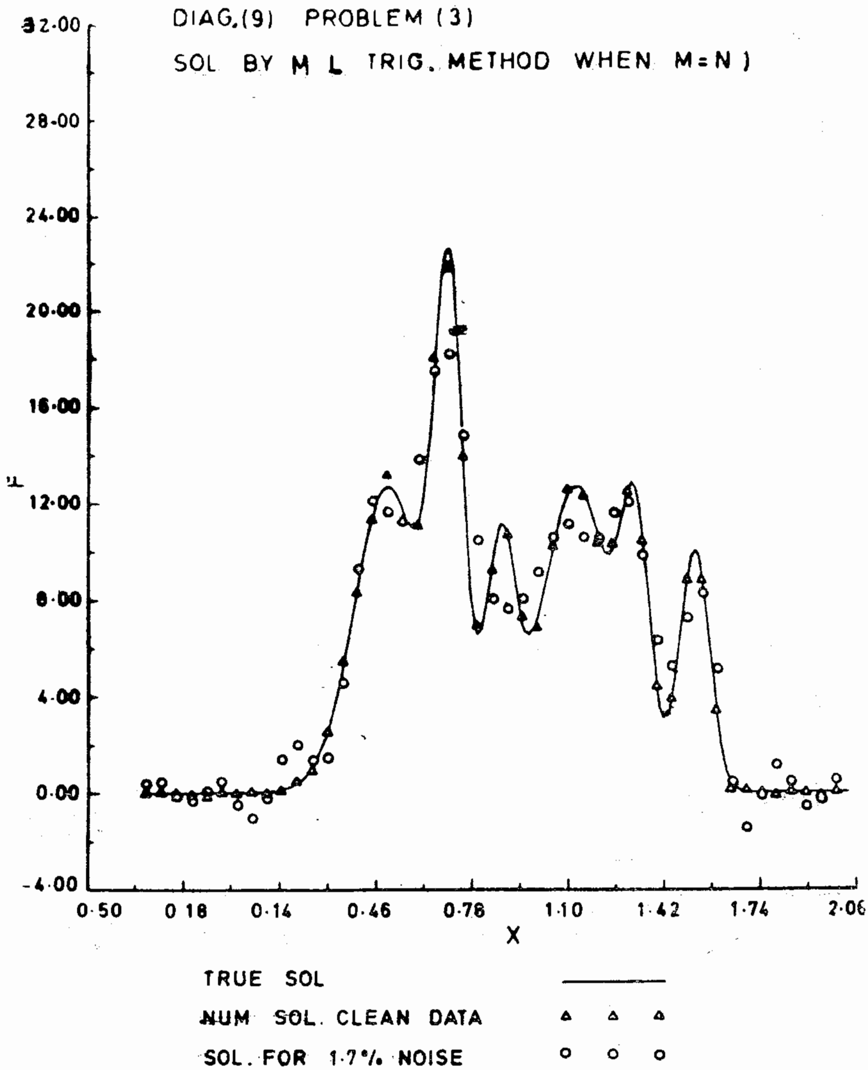

DIAG (4) PROBLEM (DIERCKX (1)
SOL. BY GCV. TRIG. METHOD WHEN M=N)
1.00
0.88
0.76
0.64
0.52
0.40
0.28
0.16
0.04
-0.08
-5.00
-3.40
-1.80
-0.20
1.40
3.00
4.60
6.20
7.80
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 3.3% NOISE


DIAG (5) PROBLEM (DIERCKX (1)
SOL BY M L TRIG. METHOD WHEN M=N)
1.00
0.88
0.76
0.64
0.52
0.40
0.28
0.16
0.04
-0.08
F
-5.00
-3.40
-1.80
-0.20
1.40
3.00
4.60
6.20
7.80
X
TRUE SOL
NUM SOL CLEAN DATA
SOL FOR 3.3% NOISE


DIAG (6) PROBLEM (2)
SOL BY GCV TRIG METHOD WHEN M = N)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
F
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
X
TRUE SOL
NUM. SOL. CLEAN DATA Δ Δ Δ
SOL. FOR 1.7% NOISE P = 2 o o o
SOL. FOR 1.7% NOISE P = 3 x x x


DIAG(7) PROBLEM (2)
SOL BY M.L.TRIG. METHOD WHEN M=N)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
X
TRUE SOL
NUM. SOL CLEAN DATA
SOL FOR 1.7% ERROR P = 2
SOL FOR 1.7% ERROR P = 3


DIAG (8) PROBLEM (3):
SOL. BY GCV. TRIG. METHOD WHEN M=N)
32.00
28.00
24.00
20.00
16.00
12.00
8.00
4.00
0.00
-4.00
F
-0.50
-0.18
0.14
0.46
0.78
1.10
1.42
1.74
2.06
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 1.7% NOISE


DIAG.(9) PROBLEM (3)
SOL BY M L TRIG. METHOD WHEN M=N)
32.00
28.00
24.00
20.00
16.00
12.00
8.00
4.00
0.00
-4.00
U
0.50
0.18
0.14
0.46
0.78
1.10
1.42
1.74
2.06
X
TRUE SOL
NUM SOL. CLEAN DATA
SOL. FOR 1.7% NOISE

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COINCIDENCE THEOREMS, FIXED POINT THEOREMS AND CONVERGENCE OF THE SEQUENCES OF COINCIDENCE VALUES

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ABSTRACT

Coincidence theorems generalizing the coincidence theorems of Goebel and Park are proved for a pair of maps on an arbitrary set having values in a metric space. Apart from giving a few applications on normed spaces, some known fixed point theorems for a pair of commuting maps on a metric space are improved. Finally, new kind of convergence theorems for a pair of sequences of maps on an arbitrary set having values in a metric space are proved.*

1. Introduction and Definitions.

Throughout this paper, let A be an arbitrary set, (X, d) a metric space and S, T maps on A with values in X. A point Z in A is said to be a *coincidence point* of S and T if $Sz=Tz$. $Sz=Tz$ may be called *coincidence value* of S and T at a coincidence point z.

Consides the following conditions :

(1.1) $S(A) \subset T(A)$;

(1.2) $d(Sx, Sy) \leqslant k\, d(Tx, Ty)$
for every x, y in A and some k in (0, 1);

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(1.3) $d(Sx, Sy) \leqslant k.\max\{d(Tx, Ty), d(Sx, Tx),$
$d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$
for every x, y in A and some k in $(0, 1)$;

(1.4) for a given $\epsilon > 0$, there exists a $\delta < 0$ such that for x, y in A, $\epsilon \leqslant d(Tx, Ty) < \epsilon + \delta$ implies $d(Sx, Sy) < \epsilon$, and $Tx = Ty$ implies $Sx = Sy$;

(1.5) for a given $\epsilon > 0$, there exists a $\delta > 0$ such that for x, y in A,
$\epsilon \leqslant \max\{d(Tx, Ty), [d(Sx, Tx) + d(Sy, Ty)]/2,$
$[d(Sx, Ty) + d(Sy, Tx)]/2\}$
$< \epsilon + \delta$ implies $d(Sx, Sy) < \epsilon$.

Note that (1.2)⇒(1.3) and (1.2)⇒(1.4)⇒(1.5).

The following coincidence theorem of K. Goebel, proved in 1963, has recently drawn some attention (see, for instance, [11], [13], [18]).

Theorem 1.1 ([6]). If T (A) is a complete subspace of X then S and T satisfying (1.1) and (1.2) have a coincidence point in A.

The well-known Meir-Keeler [9] type contractive condition (1.4) with A=X and T the identity map on X has been extensively studied, among others, by Ciric [3], Maiti and Pal [8], Park [13], Park and Bae [14] and Park and Rhoades [15]. Using the fixed point theorem of Meir and Keeler [9], Park has established the following coincidence theorem.

Theorem 1.2 ([13]). If T (A) is a complete subspace of X then S and T satisfying (1.1) and (1.4) have a coincidence point Z in A and $(ST^{-1})^n x_0$ converges to Tz for all x_0 in T (A).

Theorems 1.1 and 1.2 were proved using known fixed point theorems. In 2, without using any fixed point theorems, we prove coincidence theorems (Theorems 2.1 and 2.6 below) for maps S and T satisfying (1.3) and (1.5), which generalize Theorem 1.1 and 1.2. These results, apart from yielding significant variants of known fixed point theorems for a pair of commuting maps on a matric space, are applied to establish new fixed point theorems in Banach spaces, which improve considerably the corresponding results in [6] and [13].

Let P_n be a sequence of maps on X with a fixed point z_n and P the limit (pointwise or uniform) map of P_n with a fixed point z. Several mathematicians have investigated the conditions under which z_n may tend to z (see, for instance, Bonsall [1], Nadler [10], Singh [19], Rhoades [17] and Istratescu [7]). Let S_n, T_n be two sequences of maps from A to X and S, T their limit (pointwise or uniform) maps from A to X. Further, let z_n be a coincidence point of S_n and T_n for each $n=1, 2, \ldots$ and z be a coincidence point of S and T. Since S and T (or S_n and T_n) may have more than one coincidence point, even under the condition (1.2), it is not safe to talk of z_n tending to z, when A is also a metric space. So the question is whether $S_n z_n = T_n z_n$ will tend to $Sz = Tz$. In §3, we investigate the conditions under which $S_n z_n$ tends to Sz. Such results (Cf. Theorems 3.1–3.3) may be called *convergence theorems for coincidence values*. It may be mentioned that the coincidence value of S and T satisfying (1.3) is always unique if coincidence points exist.

I will denote the identity map on a space under consideration, and, in § 3, k satisfying (1.2) or (1.3) or similar contractive conditions will be called *control constant*. C (ST) will denote the set of all coincidence points of S and T, *i.e.* $C(ST) = \{z \in A ; Sz = Tz\}$.

If for a point $x_0 \in A$ there exists a sequence $\{x_n\}$ of points of A such that $Tx_{n+1} = Sx_n$, $n=0, 1, 2, \ldots$, then

$$0(S, x_0, T) = \{Tx_n : n=1, 2, \ldots\}$$

will be called the orbit for (S, T) at x_0. We shall use $0(S, x_0, T)$ as a set and as a sequence as the situation demands. Note that (1.1) always guarantees the existence of an orbit $0(S, x_0, T)$ for every $x_0 \in A$.

We shall call X to be (S, x_0, T)-orbitally complete if $\overline{0(S, x_0, T)}$, the closure of $0(S, x_0, T)$, is complete. It is obvious that the completeness of X implies the orbital completeness, and the space may be (S, x_0, T)—orbitally complete without being complete (see Example 2.4 below).

2. Coincidence and fixed point theorems.

Theorem. 2.1. Let S and T satisfy (1.3). If there exists a point x_0 in A such that T (A) is (S, x_0, T)—orbitally complete then.

(*i*) S and T have a coincidence point z,

(*ii*) 0 (S, x_0, T) converges to $Sz = Tz$,

(*iii*) for $z_i \in C\,(ST)$, $i = 1, 2, \ldots\ldots$,
$Sz_i = Tz_i = Sz_j = Tz_j$, $i, j = 1, 2, \ldots\ldots$

Proof. Following Ranganathan [16] (see also [5]), it can be shown that $\{Tx_n\}$ is a Cauchy sequence. Then $\{Tx_n\}$ has a limit in T (A). Call it p. So there exists a point z in A such that $z \in T^{-1}p$, that is $Tz = p$. Clearly $Sx_n \to p$. Now replacing x by x_n and y by z in (1.3) yields, in the limit, $Sz = Tz$. This proves (*i*) and (*ii*).

To establish (*iii*). let z_i and z_j $(i \neq j)$ be coincidence points of S and T. Then for $i \neq j$,

$$d\,(Sz_j, Sz_j) \leq k \max \{ d\,(Tz_i, Tz_j), d\,(Sz_i, Tz_i), d\,Sz_j, Tz_j), d\,(Sz_i, Tz_j), d\,(Sz_j, Tz_i) \}$$

$$= k\, d\,(Sz_i, Sz_j),$$

yielding $Sz_i = Sz_j$. This ends the proof.

Since S (A) ⊂ T (A) implies the existence of an orbit 0 (S, x_0,T) for every $x_0 \in A$, we have :

Corrollary 2.2. If T (A) is a complete subspace of X then S and T satisfying (1.1) and (1.3) have a coincidence. Furthermore, the coincidence value is unique.

Corollary 2.3. Let S and T be maps on a metric space. X and satisfy (1.3) with A=X. If, for some $x_0 \in X$, T (X) is (S. x_0, T)—orbitally complete and S, T commute on C (ST) then S, T have a unique common fixed point and 0 (S, x_0, T) converges to the fixed point.

Proof. In view of theorem 2.1 there exist points p in T (X) and z in X such that $0(S, x_0, T)$ converges to $p=Sz=Tz$. Since $z \in C(ST)$,

$$Sp=STz=TSz=Tp.$$

So

$$d(Tx_{n+1}, Tp)=d(Sx_n, Sp)$$
$$\leqslant k. \max\{d(Tx_n, Tp), d(Sx_n, Tx_n), d(Sp, Tp), d(Sx_n, Tp), d(Sp, Tx_n)\}$$

yields, in the limit, $d(p, Tp) \leqslant k\, d(p, Tp)$, proving $p=Tp=Sp$.

The uniqueness of the common fixed point follows easily from (1.3) or from the fact (Cf. Theorem 2 1 (*iii*) that the coincidence value $p=Sz$ is unique.

Ranganathan [16] and Das and Naik [5] have independently obtained the conclusion of Corollary 2.3 in a complete metric space X with some additional conditions, namely, $S(X) \subset T(X)$, T continuous and S, T commuting on X. Example 2.4 (below) shows that Corollary 2.3 is indeed superior to their result. Ciric's result [4, Th. 1], one of the important generalizations of the Bancah contraction principle is exactly obtained from Corollary 2.3 by taking T=I which however cannot be obtained from [16] or [5] as the space therein is complete.

Expmple 2.4. Let X be the set of nonnegative rationals and d the absolute value metric on X. Let S be the identity map on X and T be defined over X by

$$Tx = \begin{cases} 5x, & x \leqslant 1, \\ 10x, & x > 1. \end{cases}$$

Clearly for any $x, y \in X$,

$$d(Sx, Sy) \leqslant (1/5)\, d(Tx, Ty),$$

and all other hypotheses of Corollary 2.3 are satisfied for $x_0 = 1/5$. Thus Corollary 2.3 guarantees the unique common fixed point, namely 0. Note that $X=S(X) \not\subset T(X)$. T is not continuous and X not complete.

The following example illustrates Theorem 2.1 and shows that Theorem 2.1 is indeed a generalization of Theorem 1.1.

Example 2.5. Let $A=\{a, b, c\}$, $X=\{1, 2, 3\}$ and
$d(1, 1)=d(2, 2)=d(3, 3)=0$,
$d(1, 2)=d(2, 1)=d(1, 3)=(3, 1)=3/2$,
$d(2, 3)=d(3, 2)=2$.

Further, let $S, T : A \to X$ be defined by

$$Sa=Sb=1, \; Sc=2$$

and $Ta=1, Tb=2, Tc=3$,

Then (1.1) and (1.3) are satisfied for any $k \in [3/4, 1)$. So Theorem 2.1 applies. However, theorem 1.1 is not applicable as S and T do not satisfy (1.2), since

$$d(Sa, Sc)=d(Ta, Tc).$$

Now we present a generalization of Theorems 1.1 and 1.2.

Theorem 2.6. Let S and T satisfy (1.5). If there exists a point x_0 in A such that T (A) is (S, x_0, T)—orbitally complete then the conclusions (*i*)—(*iii*) of Theorem 2.1 are true.

Proof. If for some n, $Sx_n=Tx_{n+1}=Tx_n$, then we are done. So assume that $Tx_{n+1} \neq Tx_n$ for each n. Now following the technique used in the proof of Theorem 4 of Park and Rhoades [15] it can be shown that $\{Tx_n\}$ is a Cauchy sequence. So $\{Tx_n\}$ converges to some point p in T (A). and there exists a point z in A such that $Tz=p$. Assume that $Sz \neq Tz$. Then

$$0 \neq \max \{ d(Tx_n, Tz), [d(Sx_n, Tx_n)+d(Sz, Tz)]/2, [d(Sx_n, Tz)+d(Sz, Tx_n)]/2 \}=K(x_n, z), \text{ say.}$$

So from (1.5),

$$d(Sx_n, Sz) < K(x_n, z),$$

and one of the following must hold :

$$d(Sx_n, Sz) < d(Tx_n, Tz)$$
$$2\, d(Sx_n, Sz) < d(Sx_n, Tx_n)+d(Sz, Tz)$$
$$2\, d(Sx_n, Sz) < d(Sx_n, Tz)+d(Sz, Tx_n).$$

Making $n \to \infty$ these relations yield

$$d(Sz, Tz)=0.$$

Consequently $Sz=Tz$. This ends the proof of (*i*) and (*ii*). The proof of (*iii*) follows easily.

Corollary 2.7. Let S and T be maps on a metric space X and satisfy (1.5) with $A=X$. If, for some $x_0 \in X$, T (X) is (S, x_0, T)—orbitally complete and S, T commute on C (ST) then S, T have a unique common fixed point and 0 (S, x_0, T) converges to the fixed point.

Proof. An appropriate blend of the proof of Corollary 2.3 yields the result.

Corollary 2.7 includes several fixed point theorems from [2, 8, 9, 14]. Indeed. as noted in [14], Boyd-Wong's fixed point theorem for nonlinear contractions [2] is included in Corollary 2.7.

Let Y be a Benach space and F an operator on Y. Further, let α, β be numbers, $|\alpha| \neq |\beta|$ and $F_{\alpha\beta}=\alpha I+\beta F$, wherein I is the identity operator on Y. Note that if

$$F_{\alpha\beta} z=z=F_{\beta\alpha} z$$

then

$$\alpha z+\beta Fz=\beta z+\alpha Fz,$$

and z is a fixed point of F. Hence applying Theorems 2.1 and 2.6, we have the following results.

Theorem 2.8, If $F_{\alpha\beta}(Y) \subset F_{\beta\alpha}(Y)$ and $F_{\beta\alpha}(Y)$ is a closed subset of Y and

$$\|F_{\alpha\beta}x-F_{\alpha\beta}y\| \leqslant k.\max\{\|F_{\beta\alpha}x-F_{\beta\alpha}y\|, \|F_{\alpha\beta}x-F_{\beta\alpha}x\|, \|F_{\alpha\beta}y-F_{\beta\alpha}y\|, \|F_{\alpha\beta}x-F_{\beta\alpha}y\|, \|F_{\alpha\beta}y-F_{\beta\alpha}x\|\}$$

for all x, y in Y and for some k in (0, 1), then F has a unique fixed point.

Theorem 2.9. Suppose that $F_{\alpha\beta}(Y) \subset F_{\beta\alpha}(Y)$, $F_{\beta\alpha}(Y)$ is a closed subset of Y, and that for a given $\epsilon > 0$, there exists a $\delta > 0$ such that for x, y in Y,

$$\epsilon \leqslant \max \{ \|F_{\beta\alpha}x - F_{\beta\alpha}y\|,$$

$$[\ \|F_{\alpha\beta}x - F_{\beta\alpha}x\| + \|F_{\alpha\beta}y - F_{\beta\alpha}y\|\ ]/2,$$

$$[\ \|F_{\alpha\beta}x - F_{\beta\alpha}y\| + \|F_{\alpha\beta}y - F_{\beta\alpha}x\|\ ]/2 \}$$

$$< \epsilon + \delta \text{ implies } \|F_{\alpha\beta}x - F_{\alpha\beta}y\| > \epsilon.$$

Then F has a unique fixed point.

Therem 2.8 and 2.9 generalize the corresponding results in [6] and [13].

The following result is evidently a special case of Theorem 2.8.

Corollary 2.10. If $F(Y) = Y$ and

$$\max \{ \|x-y\|, \|x-Fx\|, \|y-Fy\|,$$

$$\|x-Fy\|, \|y-Fx\|\}$$

$$\geq \|x-y\|/k$$

for all x, y in Y and some k in $(0, 1)$, then F has a unique fixed point.

It is interesting to observe that Corollary 2.10 is the Banach space version of Corollary 2.3 with S as the identiry map on the space.

§ **3.** In this section we consider theorems concerning uniform and pointwise convergence of sequences of coincidence values.

Theorem 3.1. *Let S_n and T_n be maps from A to X with z_n as their coincidence point for each $n = 1, 2, \ldots\ldots$ Let $S, T : A \to X$ with z as their coincidennce point satisfy (1.3) for every x, y in A and some k in $(0, 1)$. If the sequences $\{S_n\}$ and $\{T_n\}$ converge uniformly to S and T respectively on $\{Z_n\}$, then*

$S_n z_n = T_n z_n \longrightarrow Sz = Tz$.

Proof. We have for any n,

$$d(S_n z_n, Sz) \leqslant d(S_n z_n, Sz_n) + d(Sz_n, Sz)$$
$$\leqslant d(S_n z_n, Sz_n) + k.\max\{d(Tz_n, Tz), d(Sz_n, Tz_n), d(Sz, Tz), d(Sz_n, Tz), d(Sz, Tz_n)\}$$
$$= d(S_n z_n, Sz_n) + k.\max\{d(Tz_n, Sz), d(Sz_n, Tz_n), d(Sz_n, Sz)\}$$

So one of the following must hold :

$$d(S_n z_n, Sz) \leqslant d(S_n z_n, Sz_n) + k\,[\,d(Tz_n, T_n z_n) + d(S_n z_n, Sz)]$$

i.e. $(1-k)\,d(S_n z_n, Sz) \leqslant d(S_n z_n, Sz_n) + k\,d(Tz_n, T_n z_n)$;

or $(d(S_n z_n, Sz) \leqslant d(S_n z_n, Sz_n) + k\,[d(Sz_n, S_n z_n) + d(T_n z_n, Tz_n)]$
$$= (1+k)\,d(S_n z_n, Sz_n) + k\,(T_n z_n, Tz_n)\ ;$$

or $d(S_n z_n, Sz) \leqslant d(S_n z_n, Sz_n) + k\,[d(S_n z_n, Sz_n) + d(S_n z_n, Sz)]$

i.e. $(1-k)\,d(S_n z_n, Sz) \leqslant (1+k)\,d(S_n z_n, Sz_n)$.

These inequalities together imply

$$d(S_n z_n, Sz) \leqslant \max\{1/(1-k), 1+k, (1+k)/(1-k)\}\,d(S_n z_n, Sz_n) + \max\{k/(1-k), k\}\,d(T_n z_n, Tz_n)$$
$$= (1+k)/(1-k)\,d(S_n z_n, Sz_n) + k/(1-k)\,d(T_n z_n, Tz_n).$$

Since $S_n \rightarrow S$ and $T_n \rightarrow T$ uniformly on $\{z_n\}$, for fixed $\epsilon_i > 0$, $i=1, 2$, we can choose positive integers N_1 and N_2 such that

$$d(S_n z_n, S_n z) < \frac{1-k}{2(1+k)}\,\epsilon_1 \text{ for all } n \geq N_1 \text{ and}$$

$$d(T_n z_n, Tz_n) < \frac{1-k}{2k}\,\epsilon_2 \text{ for all } n \geq N_2.$$

Choose $N = \max\{N_1, N_2\}$ and $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. Then for all $n \geqslant N$ we have

$$d(S_n z_n, Sz) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $S_n z_n \rightarrow Sz$.

Theorem 3.2. *Let S_n and T_n be maps from A to X with z_n as their coincidence point for each $n=1, 2, \ldots\ldots\ldots$ Suppose that*

(3.2) $d(S_n x, S_n y) \leqslant k.\text{may}\{d(T_n x, T_n y), d(S_n x, T_n x), d(S_n y. T_n y)$
$$d(S_n x, T_n y), d(S_n y, T_n x)\}$$

for every x, y in A, for each $n=1, 2, \ldots\ldots$ and for the same control constant k. If $S, T : A \rightarrow X$ are pointwise limits of $\{S_n\}$ and $\{T_n\}$ respectively. then S and T satisfy (1.3) with k as control constant. Further, if z is a coincidence point of S and T then $S_n z_n = T_n z_n \rightarrow Sz = Tz$.

The restriction that the pair (S_n, T_n) satisfying (3.2) has the same control constant k seems to be a trifle severe. We relax this restriction in the following :

Theorem 3 3 *Let S_n and T_n be maps from A to X with z_n as their coincidence point for each $n=1, 2, \ldots\ldots$ Suppose that the pair (S_n, T_n) satisfies (3.2) with control constant k_n for every x, y in A and for each $n=1, 2, \ldots\ldots$ If $S, T : \rightarrow X$ are pointwise limits of $\{S_n\}$ and $\{T_n\}$ respectively, and if $k_n \rightarrow k \in (0, 1)$, then S and T satisfy (1.3) with k as control constant. Further, if z is a coincidence point of S and T then $S_n z_n = T_n z_n \rightarrow Sz = Tz$.*

In the absence of the condition "$k_n \rightarrow k \in (0, 1)$", Theorem 3.3 breaks down in general (see Example 3.0 below). In case $A=X$ and $T=I$, Theorem 3.3 without "$k_n \rightarrow k \in (0, 1)$" becomes false, in general, and $S_n z_n = z_n$ need not converge to $Sz = z$ as can also be seen using Example 1 of Nadler [10].

Proof of Theorem 3.3. For any $x. y$ in A, we have

$$d(Sx, Sy) \leqslant d(Sx, S_n x) + d(S_n y, Sy) + d(S_n x, S_n y)$$
$$\leqslant d(S_n, S_n x) + d(S_n y, Sy) + k_n . \max \{ d(T_n x, T_n y), d(S_n x, T_n x), d(S_n y, T_n y), d(S_n x, T_n y), d(S_n y, T_n x)\}.$$

This yields (1.3), since $S_n x \rightarrow Sx$, $T_n x \rightarrow Tx$ for every x in A and $k_n \rightarrow k$ as $n \rightarrow \infty$.

We now show that $S_n z_n \rightarrow Sz$. For each $\in$, $0 < \in 1-k$, there exists $N > 0$ such that for all $n \geq N$,

$$d(S_n z, Sz) < \frac{1-k-\in}{2(1+k+\in)} \in,$$

$$d(T_n z, Tz) < \frac{1-k-\in}{2(1+k+\in)} \in,$$

and $\quad k_n \leqslant k + \in$

so for $n \geqslant N$

$$d(S_n z_n, Sz) \leqslant d(S_n z_n, S_n z) + d(S_n z, Sz)$$
$$\leqslant k_n . \max \{ (d\, T_n z_n, T_n z), d(S_n z_n, T_n z_n),$$
$$d(S_n z. T_n z), d(S_n z_n, T_n z),$$
$$d(S_n z, T_n z_n) \} + d(S_n z, Sz),$$
$$\leqslant (k+\epsilon) . \max \{ d(S_n z_n, T_n z), d(S_n z, T_n z),$$
$$d(S_n z, S_n z_n) \} + d(S_n z, Sz).$$

So one of the following relations must hold for all $n \geqslant N$:

$$d(S_n z_n, Sz) \leqslant (k+\epsilon) [\, d(S_n z_n, Sz) + d(Tz, T_n z) \,] + d(S_n z, Sz)$$

i.e. $d(S_n z_n, Sz) \leqslant (1-k-\epsilon)^{-1} [\, d(S_n z, Sz) + (k+\epsilon)\, d(T_n z, Tz) \,],$

or $d(S_n z_n, Sz) \leqslant (k+\epsilon) [\, d(S_n z, Sz) + d(Tz, T_n z) \,] + d(S_n z, Sz)$

$$= (1+k+\epsilon)\, d(S_n z, Sz) + (k+\epsilon)\, d(Tz, T_n z)\, ;$$

or $d(S_n z_n, Sz) \leqslant (k+\epsilon) [\, d(S_n z, Sz) + d(Sz, S_n z_n) \,] + d(S_n z, Sz)$

i.e. $d(S_n z_n, Sz) \leqslant (1+k+\epsilon)(1-k-\epsilon)^{-1} d(S_n z, Sz).$

Hence for all $n \geq N$.

$$d(S_n z_n, Sz) \leqslant \max \{ (1-k-\epsilon)^{-1}, (1+k+\epsilon), (1+k+\epsilon) \times (1-k-\epsilon)^{-1} \}\, d(S_n z, Sz)$$
$$+ \max \{ (k+\epsilon)(1-k-\epsilon)^{-1}, k+\epsilon \}\, d(T_n z, Tz)$$
$$< \frac{1+k+\epsilon}{1-k-\epsilon} [\, d\, S_n z, Sz) + d(T_n z, Tz) \,]$$
$$< \epsilon.$$

This ends the proof.

Now we illustrate the results of § 3. In all that follows Q denotes the set of rational numbers.

Example 3.4. Let A, X and $S_n, T_n : A \to X$ be such that

$A = \{ x \in Q : x \geq 1 \}$, $X = \{ x \in Q : x < 0 \}$,

$$S_n x = -\tfrac{1}{2} - \frac{2n+5}{4n+8} x$$

$$T_n x = \frac{1}{4} - \frac{6n+13}{8n+16} x ,$$

and d be the usual metric on X.

Then $Sx = -(1+x)/2$, $Tx = (1-3x)/4$, $z = 3$,

$$S_n z_n = T_n z_n = -(4n+9)/(2n+3),\ z_n = (6n+12)/(2n+3).$$

Clearly,

$$d(Sx, Sy) \leqslant k\, d(Tx, Ty) \text{ for any } k \in \left(\frac{2}{3}, 1\right), \text{ and}$$

$S_n z_n \to -2 = Sz$. This illustrates Theorem 3.1.

Moreover, since

$$d(S_n x, S_n y) = d(T_n x, T_n y),\ k_n = \frac{4n+10}{6n+13} \to \frac{2}{3},$$

$$\leqslant \frac{3}{4} d(T_n x, T_n y),$$

this example illustrates Theorems 3.2 and 3.3,

The following example shows that Theorem 3.3 breaks down in the absence of the hypothesis "$k_n \to k \in (0, 1)$".

Example 3.5. Let $A = \{ x \in Q : x > 1 \}$, $X = \{ x : x < 0 \}$, and $S_n x = -\frac{1}{2} - \frac{2n+1}{4} x$, $T_n x = \frac{1}{2} - \frac{(n+1)(2n+1)}{2n^2} x$,

$x \in A$. Then

$$d(S_n x, S_n y) = k_n\, d(T_n x, T_n y),\ k_n = \frac{n}{n+1} \to 1,$$

and for $z_n = 4n^2/(2n+1)$.

$$S_n z_n = T_n z_n = -\tfrac{1}{2} - n \to -\infty.$$

Moreover, $Sx = -(1+x)/2$, $Tx = (1-x)/2$, $d(Sx, Sy) \leqslant d(Tx, Ty)$ and there exists no point z in A such that $Sz = Tz$.

If we take $A = X$, $T = I$ in Theorems 3.1–3.3, then a multitude of results regarding the convergence of sequences of maps and the sequence of their fixed points are obtained. Theorems 25 & 23 of Rhoades [17], for example, are obtained from Theorems 3.1 and 3.2.

Since Example 3.5 shows that S and T satisfying (1 1) and

(1.3) with $k=1$ need not have a coincidence, we pose the following problems.

Problem 1. Under what additional condition (s), S and T satisfying (1.2) or (1.3) (with $k=1$) will have a coincidence ?

Problem 2. If S and T satisfying (1.2) or (1.3) [with $k=1$ and A=X] have a coincidence, and if the maps are commuting on C (ST) then under what additional condition (s) the maps will have a common fixed point ?

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FIXED POINTS OF MAPPINGS WITH DIMINISHING PROBABILISTIC ORBITAL DIAMETERS

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1. Introduction.

Balluce and Kirk [1] introduced the concept of diminishing orbital diameters (d.o.d) and established that a non-expansive self-mapping on a metric space with d.o.d. has a fixed point. Subsequently a number of results related to such mappings have appeared in the literature. The notions of non-expansive mappings and mappings with diminishing probabilistic orbital diameters (d.p.o.d.) on probabilistic metric spaces (PM-spaces) have been introduced by Istratescu and sacuiu [5].

In this paper we show that a non-expansive mapping on PM-space having d.p.o.d. has a fixed point. Next we investigate that the condition of non-expensiveness on the mapping may be relaxed to the condition of the mapping being with relatively compact orbits. Finally, the notion of joint d.p.o.d. is introduced and a common fixed point theorem for a pair of mappings having joint d.p.o.d. is included. Our results are indeed the extensions, to PM-spaces, of some of the results of Belluce and Kirk [1, 2], Kannan

[6], Kirk [7], Park [9] and Ranganathan, Srivastava and Gupta [10].

2. Preliminaries.

A Menger space is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space [11] and t-norm t is such that the inequality.

$$F_{u,v}(x+y) \geqq t\{F_{u,w}(x), F_{w,v}(y)\}$$

holds for all $u, v, w \in X$ and for all $x \geqq 0, y \geq 0$, where $F_{u,v}$ denotes the value of $\mathcal{F}$ at $(u, v) \in X x X$. For detailed study of Menger space and the topological preliminaries on it we refer to Schweizer and Sklar [11].

A self-mapping T on a PM-space $(X, \mathcal{F})$ is called non-expansive if

$$F_{Tu, Tv}(x) \leqslant F_{u,v}(x)$$

for all $x \geq 0$ and every $u, v \in X$.

For each $u \in X$, let $0(T_u^n)$ denote the sequence of iterates of T, that is,

$$0(T_u^n) = \bigcup_{i=n}^{\infty} \{T^i(u)\}$$

where $T_u^0 = u$.

Let A be a nonempty subset of X. The function $D_A(.)$ defined by

$$D_A(x) = \sup_{t<x} \{ \inf_{u, v \in A} F_{u,v}(l)\}$$

is called the probabilistic diameter of A [4]. Since for every n we have the inclusions

$0(T_u^0 \supseteq \ldots 0(T_u^n) \supseteq 0(T_u^{n+1}) \supseteq \ldots$ and, thus, for the probabilistic diameters, we have

$$D_0(T_u^0) \leqslant D_0(T_u^1) \leq \ldots D_0(T_u^n) \leqslant \ldots$$

We set

$$\delta_u(x)=\lim\{D_0(T_u^n)\}$$

and call the number $\delta_u(x)$ the limiting probabilistic orbital diameter of T at u.

Let T be a self-mapping on a PM-space X then T is said to have d.p.o.d. at u if for $\delta_0(T_u) \neq H$, where H is a distribution function [11, page 314],

$$\delta_u(x) > \delta_{0(T_u)}(x).$$

3. Fixed point theorems.

First we establish the following result :

Lemma. Let $(X, \mathcal{F}, t)$ be a Menger space where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. If $T : X \to X$ be a non-expensive mapping such that for a sequence $\{n(k)\}$ of positive integers, $\lim_k T^{n(k)}(u)=z$ then z generates an isometric sequence.

Proof. Let, for $\lambda > 0$,

$$F_{T^m(u),T^n(u)}(x)-F_{T^{m+k}(u),T^{n+k}(u)}(x)=1-\lambda \neq 0.$$

Then, as T is nonexpansive,

$$(1)\quad F_{T^m(u),T^n(u)}(x)-F_{T^{m+1}(u),T^{n+1}(u)}(x) \leqslant 1-\lambda,$$

$$(1=k, k+1,\ldots),$$

Also $\lim_j T^{n(j)}(T^1(u))=\lim_j T^{n(j)+1}(u)=T^1(u)$.

Hence a positive integer i exists such that $j \geq i$ implies

$$F_{T^{m+n(j)}(u),T^m(u)}\left(\frac{1-2h}{3}x\right)>1-\lambda \text{ and}$$

$$F_{T^{n+m(j)}(u),T^n(u)}\left(\frac{1-2h)}{3}x\right) > 1-\lambda.$$

But

$$F_{T^m(u),T^n(u)}(x) > t\left\{F_{T^m(u),T^{m+n(j)}(u)}\left(\frac{1-2h}{3}x\right),\right.$$

$$F_{T^{m+n(j)}(u),T^{n+n(j)}(u)}\left(\frac{1+4h}{3}x\right),$$

$$F_{T^{n+n(j)}(u),T^{n}(u)}\left(\frac{1-2h}{3}x\right)\Big\}$$

$$> t\Big\{1-\lambda,\ F_{T^{m+n(j)}(u),\ T^{n+n(j)}(u)}\left(\frac{1+4h}{3}x\right)\Big\}$$

$$> t\{1-\lambda,\ F_{T^{m+n(j)}(u),T^{n+n(j)}(u)}(x)\}$$

which is a contradiction to (1) for $n(j) \geqslant \max(n(i), k)$. Hence the result.

Theorem 1. Let $(X, \mathcal{F}, t)$ be a Menger space where t is continuous and satisfies $t(x, x) \geqslant x$ for every $x \in [0, 1]$ and let $T : X \to X$ be a nonexpensive mapping having d.p.o.d. If for some $u \in X$, $\lim_k T^{n(k)}(u) = z$ then $\lim_n T^n(u) = z$ and $T(z) = z$.

Proof. In view of the lemma, z generates an isometric sequence since $\lim_k T^{n(k)}(u) = z$. Thus, for given positive integers m and n,

$$F_{T^{m}(z),T^{n}(z)}(x) = F_{T^{m+k}(z),T^{n+k}(z)}(x),\ k = 1, 2, \ldots$$

Therefore for a positive integer k.

$$D_{0\,(T(z))}(x) = \sup \inf F_{T(z),\,T^{n}(z)}(t)$$

$$= \sup \inf F_{T^{k}(z),\,T^{n+k-1}(z)}(t)$$

$$= D_{0\,(T^{k}(z))},$$

implying

$$\lim_k D_{0(T^{k}(z))}(x) = \delta_z(x) = D_{0(T(z))}(x)$$

As $\delta_z(x) = \delta_{T(z)}(x)$ and T has d.p.o.d., we get $D_{0(T(z))}(x) = H$.

Thus Tz is a fixed point of T. The continuity of T implies $\lim_k T^{n(k)+1}(u) = T(z)$.

Thus if $\epsilon > 0$, there is an integer k such that

$$F_{T^{n(k)+1}(u),\ T(z)}(\epsilon) > 1-\lambda \text{ for } \lambda > 0.$$

Since $T(z)$ is a fixed point of T and T is non-expensive, we have

$$F_{T^{n}(u),\ T(z)}(\epsilon) > 1-\lambda \text{ for } n \geq n(k)+1.$$

Thus $\lim_n T^n(u) = T(z)$. But $\lim_n T^n(u) = z$ as well. Hence z is a fixed point of T.

The condition of non-expensiveness on T in above theorem may be relaxed to the condition of T being with relatively compact orbits. Thus we have the following result.

Theorem 2. Let T be a self-mapping on a Menger space X. If there exists a $u \in X$ such that $\overline{0(u)}$ is compact, T is continuous and has d.p.o.d. on $\overline{0(u)}$ with $\delta_u \neq H$ then $0(u)$ has a cluster point z in $\overline{0(u)}$ and $z = T(z)$.

Proof. We take $u \in X$ with $\delta_u \neq H$. Since T has d.p.o.d. at u, we have for some m.

$$\delta_u(x) > \delta_{0(T^n(u))}(x). \quad \text{Hence } \overline{0(T^m(u))} \neq \overline{0(u)}.$$

It is clear that $u \notin 0(T^m(u))$ and $u \notin T(u)$, otherwise we have $0(T^m(u)) = 0(u)$. Thus we, as in [9], conclude that $u \notin \overline{0(T^2(u))}$ in any case and therefore $T \restriction \overline{0(u)}$ is strongly non-periodic [3]. Now the proof is completely by Theorem 1 of 'Ciric' [3].

Remark. The above result extends some of the results in [1, 2], [6], [7] to PM-space, and is indeed an improved version (in PM-space) of Theorem 3.3 of Belluce and Kirk [2].

As an immediate consequence of Theroem 2, we have the following result :

Throrem 3. Let $T : X \to X$ be compact and for some $k \in N$, T^k is orbitally continuous. If T and T^k are mappings with d.p.o.d. then for each $u \in X$, $\{T^k(u)\}$ has a cluster point which is fixed under T.

Let $\{T_1, T_2\}$ be a pair of mappings from a Menger space X to itself. For $u_0 \in X$, let $u_n = T_1 u_{n-1}$ if n is odd and $u_n = T_2 u_{n-1}$ if n is even ; then the sequence

$$J_{S(u_0)} = \{u_0, T_1 u_0, T_2 T_1 u_0, \ldots\ldots\}$$

is called the joint sequence of literates of S at u_0, [8].

We now introduce the notion of joint d.p.o.d. in a PM-space.

Let $\delta_{S(u_0)}(x) = \lim \{ D_{J_{S(u_n)}} \}$. We call the number $\delta_{S(u_0)}(x)$ the joint limiting p.o.d. of S at u_0. If for every $u_0 \in X$,

$$\delta_{S(u_0)}(x) < \delta_{J_{S(u_0)}}(x) \text{ whenever } \delta_{J_{S(u_0)}}(x) \neq H,$$

then S will be called to have joint d.p.o.d. on X.

Our next result is an extension to PM-spaces of a result in [10]. Its proof may be completed on the lines of [10].

Theorem 4. Let X be a compact Menger space and $S=\{T_1, T_2\}$ be a pair of continuous self-mappings on X such that S has joint d.p.o.d. on X. Then for each $u_0 \in X$ a subsequence of $J_{S(u_0)}$ converges to a common fixed point of T_1 and T_2.

Remark 2. In case $T_1=T_2$, the above theorem is an extension to PM-spaces of a result of Kirk [7].

Remark 3. The superiority of the above result is clear from the fact that this result is applicable even if a continuous self-mapping T on a PM-space does not have d p.o.d., since in such a case it might be possible to obtain a family S of continuous self-mappings on X such that SU {T} has a joint d.p.o.d. (see, for an illustration, [10]).

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GOALS IN THE MATHEMATICS CURRICULUM

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There is considerable debate pertaining to which objectives learners are to attain. The mathematics curriculum is no exception. One hears much about a return to the basis. The basis generally are perceived as emphasizing the three R's (reading, writing, and arithmetic). Thus, the third R—arithmetic—has essential content for all learners to master. Within the framework of essentialism, which objectives, methods of teaching, and appraisal procedures need to be in evidence ?

INSTRUCTIONAL MANAGEMENT SYSTEMS

Instructional Management Systems (IMS) advocate the utilization of precise, measurable ends. Vagueness and ambiguity need to be eliminated from goals of instruction according to LMS tenants. With clarity of intent in objectives, the teacher knows precisely which sequential ends students are to attain. Thus, learning activities may be selected by the teacher to guide pupils to achieve each objective on an individual basis. An objective needs to be attained by the student before progressing to the next sequential end. The teacher can then measure if a learner has/has not achieved a specific goal. Uncertainty on the teacher's part is not in evidence to determine if a student has mastered content necessary in goal attainment.

The Missouri Department of Elementary and Secondary Education [1] listed the following characteristics of IMS.

1. High expectations for learning. Teachers and administra-

tors expect a high level of achievement by all students and communicate their expectations to students and parents. No students are expected to fail, and the school assumes responsibility for seeing that they don't.

2. Strong leadership by building principals. The building principal is an instructional leader who participates in all phases of instruction. The principal is a visible leader of instruction, not just an office-bound administrator.

3. Emphasis on instruction in the basic skills. Since mastery of the basic skills is essential to learning in all other subjects, the effective schools make sure students at least master the basic skills.

4. Clear-cut instructional objectives. Each teacher has specific instructional objectives within the overall curriculum which are communicated to students, parents and the general public. In effective schools, teachers and administrators—not textbooks—are clearly in charge of the curriculum and teaching activities.

5. Mastery learning and testing for mastery. Students are taught, tested, retaught and retested to the extent necessary to assure mastery of important objectives.

6. School Discipline and climate. The effective schools may not be shiny and modern, but they are at least safe, orderly and free of distractions. All teachers and students, as well as parents, know the school's expectations about behavior and discipline.

The following are definitely not emphasized by IMS :

1. Open—ended general objectives in the mathematics curriculum.

2. Leaway in interpretation as to which subject matter should be taught so that students may choose sequential goals to achieve in a flexible mathematics curriculum.

3. Pupil-teacher planning in selecting objectives.

4. learners in a classroom achieving at a similar/same level of progress. Each student progresses as rapidly as possible in achieving objectives.

LEARNING CENTERS AND MATHEMATICS

Educators, advocating humanism as a psychology of learning, believe that students should be involved in decision-making. Thus, the mathematics teacher, alone, does not select objectives, learning activities, and evaluation procedures for students. Rather, within a flexible framework developed by the teacher, the learner may select from among alternatives which sequential activities to pursue. A learning centers approach might then be in evidence. An adequate number of centers and tasks needs to be available so that the involved student may truly choose which activities to pursue and which to omit. Continuous progress must be made by the learner in completing personal suitable tasks. Each student may then achieve at a unique optimal rate of progress. Diverse objectives in mathematics may be achieved when comparing one student with another.

Choices made by learners in tasks pursued depend upon personal interests, abilities, capacity, and motivation. The kinds of tasks chosen may emphasize individual or committee endeavours, an activity centered or subject matter emphasis, inductive or deductive methods, as well as concrete or abstract experiences.

Morris and Pai [2] wrote the following pertaining to the thinking of Carl Rogers :

But what are the conditions for such learning, and what must the teacher do to facilitate them ? Like other humanistic educators, Rogers assumes that human beings have a natural potentiality for learning and curiosity. John Holt argues that this potentiality and desire for knowledge develops spontaneously unless smothered by a repressive and punitive climate. Consequently, humanistic educators seek to remove restrictions from our schools so that the child's capacity for learning can be cultivated. They attempt to provide the child with a more supportive, understanding, and nonthreatening environment for self-discovered learning. For example, if Jimmy is having serious difficulty in reading, he should not be forced to recite or read aloud in front of his peers, whose reactions may strengthen his own perception of himself as a failure. Rogers believes that significant learning can be promoted by allowing children to confront various problematic situations directly. If students choose their own direction, discover their own resources, formulate their own problems, decide their own course of action, and accept the consequences of their choice, significant learning ean be maximized. This suggests that significant learning is not possible unless the learner's feelings and the intellect are both involved in the learning process.

Advocates of learning centers do not emphasize :

1. precise, measurable objectives for student attainment. What is specific to measure in pupil progress may be relevant. Interests and purposes of learners are significant, but can not by any means be precisely measured.

2. teachers selecting objectives, learning activities, and evaluation techniques for students.

3. a rigid, formal curriculum. Rather, input for students in curriculum development is important.

4. each pupil being assigned the same/similar tasks as compared to other learners in the classroom,

Structure of Knowledge and the Mathematics Curriculum.

Mathematics may be perceived as having considerable structure. There are selected concepts and generalizations which hold true consistently. Thus, concepts, such as the following may be stressed in teaching and learning :

1. The commutative property of addition and multiplication.
2. The associative property of addition and multiplication.

3. The distributive property of multiplication over addition.
4. The identity elements for addition and multiplication.
5. The property of closure for addition and multiplication.

Key concepts and generalizations, as advocated by mathematicians on the higher education level, then become objectives for students to attain on the elementary, junior high school or middle school, and senior high school years.

To achieve these structural ideas, the teacher of mathematics needs to have students utilize inductive methods of learning. Lecture and heavy use of explanations is not recommended. Rather, the teacher identifies problems and questions. To secure content in answer to the questions and problems, a variety of reference sources need to be utilized. Answers to problematic situations come from students. Methods of learning used by students should be similar to those emphasized by professional mathematicians.

Woolfolk and Nicolich [3] wrote :

Jerome Burner is a well-known modern cognitive theorist......... Burner has been especially interested in instruction based upon a cognitive learning perspective. He believes that teachers should provide problem situations that stimulate students to discover for themselves the structure of the subject matter. *Structure* is made up of the fundamental ideas, relationships, or patterns of the subjcct matter, that is, the essential information. Specific facts and details are not part of the basic structure. However, if students really understand the basic structure they should be able to figure out many of these details on their own Thus Burner believes that classroom learning should take place inductively, moving from specific examples presented by the teacher to generalizations, about the structure of the subject, that are discovered by the students.

Structure of knowledge advocates in mathematics do not believe in :

1. Student-teacher planning as to objective the former is to

attain. Rather, structural ideas need to be achieved as identified by subject matter specialists.

2. Teachers presenting subject matter deductively for learners to acquire.

3. Content for student attainment being chosen by others than professionals in the mathematics curriculum.

4. Emphasizing abstract experiences for students as compared to the concrete and semi-concrete. Sequence in learning activities must progress from manipulative (real objects and items), to the iconic (pictures, films, film-strips, slides, and transparencies) to the symbolic (abstract words, letters, and numerals).

THE MATHEMATICS LABORATORY

Mathematics laboratories philosophy in teaching and learning believe that students are active, not passive beings. Learners need to choose and select, rather than to listen to lectures and lengthy explanations of subject matter. Concrete experiences need to be at the heart of the mathematics curriculum. An adequate number of real objects need to be in the offing to stimulate student achievement. Thus, for example, objects and materials need to be in evidence from which learners may select to weigh, measure lengths and widths, determine the volume, as well as find areas, perimeters, and circumferences.

Within the framework of concrete experiences, students use abstract learnings to record weights, measurements, areas, and circumferences.

Involving the mathematics laboratory concepts, Ediger [4] wrote :

Pupils should have ample opportunities to experience the mathematics laboratory concept of working. The mathematics laboratory emphasizes tenets of teaching and learning such as the following :

(*a*) Pupils are actively involved in ongoing learning activities.

(*b*) A variety of experiences is in evidence so that pupils may select materials and aids necessary for problem solving.

(*c*) Practical experiences are emphasized for learners in that they actually measure the length, and/or height of selected people and things ; weigh real objects and record their findings ; find the volume of important containers ; as well as determine areas of selected geometric figures.

(*d*) Pupils become interested in mathematics due to reality being involved in ongoing learning activities.

(*e*) Provision is made for individual differences since there is a variety of learning opportunities for pupils from which to select on an individual basis.

(*f*) Meaning is attached to what is being learned since pupils individually and in committees work on tasks adjusted to their present achievement levels.

A mathematics laboratory philosophy does not advocate :

1. A textbook methodology in teaching and learning situations.

2. Students being recipients of facts, concepts, and generalizations from teachers.

3. Lecture and extensive explanation approaches in teaching mathematics.

4. Abstract, symbolic learnings to the exclusion of using realia in the mathematics curriculum.

A Miniature Society Concept in the Mathematics Curriculum.

There are selected mathematics educators who believe strongly in guiding students to acquire and apply facts, concepts, and generalizations useful in society. The community becomes an ideal place then in having learners attain understandings, skills, and attitudinal goals. Thus, for example, students with appropriate readiness experiences and with teacher stimulation might engage in finding unit prices for soap. cereal, flour, and cake mixes. How much then does each brand name and generic brand cost per ounce or gram ? Other factors also need to be evaluated, in addition to unit pricing, and that is quality within each item.

Students in a miniature society context, might determine the cost of :

1. A given number of items from a supermarket.
2. Selected items purchased from a hardware store.
3. Items of clothing from a clothing store.
4. Cost of gasoline, after buying a certain number of liters or gallons.

A miniature supermarket may be developed in the classroom. Empty cereal, fruit and vegetable, as well as other containers may be placed on shelves in the classroom setting. Appropriate clearly labeled prices need to be attached to each food item. Play money may be used by learners in shopping for needed items. Paper and pencil, as well as the hand held calculator may be used to determine cost of a given set of items purchased, as well as change to be received from money given in payment.

John Dewey [5] wrote :

The development within the young of the attitudes and dispositions necessary to the continuous and progressive life of a society cannot take place by direct conveyance of beliefs, emotions, and knowledge. It takes place through the intermediary of the environment. The environment consists of the sum total of conditions which are concerned in the execution of the activity characteristic of a living being, The social environment consists of all the activities of fellow beings that are bound up in the carrying on of the activities of any one of its members. It is truly educative in its effect in the degree in which an individual shares or participates in some conjoint activity. By doing his share in the associoted activity, the individual appropriates the purpose which actuates it, becomes familiar with its methods and subject matters, acquires needed skill, and is saturated with its emotional spirit.

The deeper and more intimate educative formation of disposition comes, without conscious intent, as the young gradually partake of the activities of the various groups to which they belong. As a society becomes more complex, however, it is found necessary to provide a special social environment which shall especially look after nurturing the capacities of the immature. Three of the more important functions of this special environment are ; simplifying and ordering the factors of the disposition it is wished to develop ; purifying and idealizing the existing social customs ; creating a wider and better balanced environment than that by which the young would be likely, if left to themselves, to be influenced.

A miniature society mathematics curriculum does not emphasize :

1. A textbook centered method of teaching mathematics.

2. A teacher initiated curriculum whereby the instructor selects objectives, learning activities, and appraisal procedures for pupils.

3. Minimizing concrete, life-like experiences for students.

4. Students being recipients of content in a highly structured mathematics curriculum.

IN CLOSING

Numerous philosophies are in evidence pertaining to goals in mathematics for learners to attain. These include :

1. IMS with its emphasis upon precise, measurable ends for learner attainment.

2. Learning centers with its stress placed up students becoming quality decision makers in ongoing experiences,

3. Structure of knowledge with its advocacy of students acquiring major concepts and generalizations as identified by professional mathematicians.

4. A mathematics laboratory with emphasis placed up students using concrete materials in mathematics achievement.

5. A miniature society philosophy in which learners use mathematics in the functional real world.

Teachers and supervisors need to study and evoluate each philosophy. Ultimately, those philosophies which guide each pupil to achieve optimally should be emphasized in the mathematics curriculum.

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