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MORSE COVERS AND TIGHT IMMERSIONS

by

B.A, SALEEMI

Associate Professor
Mathematics Department P.O. Box-9208
King Abdul Aziz University Jeddah, Saudi Arabia

Abstract :

If M is a compact, connected, smooth manifold of demension n, then it is shown that a Morse function on M defines a Morse Cover of M. Using the notion of Morse Cover it is established that the order of the Morse Cover given by a tight Morse function equals the minimal total absolute curvature of M.

1. Introduction :

Let M be a closed, connected, $\overset{\infty}{C}$, n-manifold.

Let Φ be the class of all $\overset{\infty}{C}$ real-valued Morse functions on M. Let C_k (M , $\varnothing$) be the number of critical points of index k of $\phi \in \Phi$.

We write.

$$C(M, \phi) = \sum_{k=0}^{n} C_k (M, \phi) \tag{1}$$

and

$$C(M) = \min_{\varnothing \in \Phi} (C(M, \phi)). \tag{2}$$

If $x : M \to E^{n+N}$, $N \geq 1$, be a smooth immersion of M into Euclidean space E^{n+N} and τ (M, x, N) be its total absolute curva-

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ture [1 , 2] , then we write

$$\tau(M) = \inf_{(x,\, N)} \tau(M, x, N), \qquad (3$$

where the infimum is taken over all smooth immersoins x with variable N. Kuiper has proved [4] that.

$$\tau(M) = C(M).$$

An immersion $x : M \to E^{n+N}$ is called *tight* if $\tau(M, x, N) = \tau(M)$.

2. Morse Covers :

Let $\varnothing \in \Phi$ and let p be a critical point of ϕ of index k. Then Morse lemma [5] guarantees the existence of a co-ordinate neighbourhood U of p with co-ordinates $x^1, x^2, \ldots\ldots\ldots, x^n$ such that the following conditions hold :

(*i*) $x^i(p) = 0, \quad 1 \leqslant i \leq n,$

(*ii*) $\varnothing = \varnothing(p) - (x)^2 - \ldots\ldots - (x^k)^2 + (x^{k+1})^2 + \ldots\ldots + (x^n)^2$ on U.

We call U a *Morse co-ordinate neighbourhood of p* and $x^1, x^2, \ldots\ldots, x^n$ are called *Morse co-ordinates.*

Let N_p be the family of all Morse neighbourhoods of $p \in M$. By Morse lemma, N_p is non-empty. Furthermore if the partial order on N_p is defined by the set-theoretic inclusion $\leqslant$ then every linearly ordered subset of N_p, has an upper bound. Therefore, by Zorn's lemma, N_p has a maximal element W_p, say.

Definition 1. The neighbourhood W_p is called a *Maximal Morse neighbourhood of p* relative to $\varnothing$.

Let $W_{p1}, W_{p2}, \ldots\ldots\ldots W_{pm}$, $m = C(M, \varnothing)$, be the maximal Morse neighbourhoods of the critical points $p_1, p_2, \ldots\ldots\ldots, P_m$. Then we have the following.

Lemma1. $W_{p_1}, W_{p_2}, \ldots\ldots\ldots, W_{p_m}$ is an open cover of M.

Proof. Let $W = \bigcup_{j=1}^{m} W_{p_j}$. Let $q \in M - W$. If every neighbourhood of q contains a critical point of ϕ , then necessarily q belongs to some W_{p_i} and there is nothing to prove. If q has a neighbourhood which does not contain a critical point of ϕ, then we may choose a coordinate neighbourhood V_q with co-ordinates $z^1, z^2, \ldots\ldots, z^n$ such that $z^1 = \phi$. Let $W \cap V_q$ be non-empty. Then for some critical point p_i, $W_{p_i} \cap V_q$ is non-empty. Let $y^1, \ldots\ldots y^n$ be the Morse co-ordinates in W_{p_i}. Then, by definition of W_{p_i},

$$\varnothing = \varnothing(p_i) - (y')^2 - \ldots\ldots - (y^k)^2 + (y^{k+1})^2 + \ldots\ldots + (y^n)^2$$

on W_{p_j}, where k the index of $\varnothing$ at p_i. By a common abuse, we may regard $W_{p_j} \cap V_q$ as an open subset of $\mathbb{R}^n$ with $y^1, \ldots\ldots, y^n$ as coordinates. Since $p_i \in W_{p_j} \cap V_q$, at least one of the $y^{i's}$ is different from zero. By using flip map, if necessary, we may take this non-zero co-ordinate as y^1. If we define the map.

$$F : W_{p_i} \cap V_q \to \mathbb{R}^n$$

by

$$F(y^1, \ldots\ldots, y^n) = (\varnothing(p_j) - (y^1)^2 - \ldots\ldots - (y^k)^2 + (y^{k+1})^2 + \ldots\ldots + (y^n)^2, y^2 \ldots\ldots, y^n),$$

then F is invertible. Therefore we may regard

$$\begin{aligned} U &= \varnothing(p_i) - (y^1)^2 - \ldots\ldots - (y^k)^2 + \ldots\ldots + (y^n)^2, \\ u^j &= y^j, \ 2 \leqslant j \leqslant n \end{aligned} \qquad (1)$$

as some new co-ordinates in $W_{p_j} \cap V_q$,

To extend y^i,s to $V_q - W_{p_i} \cap V_q$, we define the transformation.

$$T: \quad \begin{aligned} y^1 &= \sqrt{\phi(p_j) - z^1 - (z^2)^2 - \ldots - (z^k)^2 + (z^{k+1})^2 + \ldots + (z^n)^2}, \\ y^j &= z^j, \ 2 \leqslant j \leqslant n. \end{aligned} \qquad (2)$$

The Jacobian matrix $\left(\frac{\partial y^i}{\partial z^j}\right)$ of T has rank equal to the rank of the matrix.

$$\gamma = \begin{Bmatrix} 1 & 0\ldots\ldots & \ldots\ldots 0 \\ z^2 & 1\ldots\ldots & \ldots\ldots 0 \\ z^3 & 0\ldots\ldots & \ldots\ldots 1 \\ , & & \\ , & & \\ . & & \\ z^n & 0 & 1 \end{Bmatrix}$$

The rank of γ is clearly n and consequently the transformation T is admissible. Combining (1) and (2), we conclude that the transformation T is well defined on the whole of V_q. Noting that $z^1 = \phi$ on V_q, we have

$$\varnothing = \varnothing(p_i) - (y^1)^2 - \ldots\ldots - (y^k)^2 + (y^{k+1}) + \ldots\ldots + (y^n)^2$$

on the whole of $W_{p_j} \cup V_q$. Thus $y^1, \ldots\ldots, y^n$ are Morse co-ordinates in $V_q \cup W_{p_i}$. Since $W_{p_i}\{y^i\}$ is a maximal Morse neighbourhood of p_i, it follows that

$$V_q \subseteq W_{p_i}$$

Therefore, under the assumption that for all $q \in M - W$, the neighbourhood V_q of q meets W, we have

$$M - W = \Box$$

or

$$M = \bigcup_{i=1}^{m} W_{p_j}$$

Now assume that there exist points $q \in M - W$ which have neighbourhoods disjoint from W. Let V be the union of all such neighbourhoods which have no point in common with W. Then V is an open set and

$$M = W \cup V,\ W \cap V = \Box$$

This contradicts our hypothesis that M is connected. Hence $W_{p_1}, \ldots\ldots\ldots, W_{p_m}$ is an open cover of M.

Definition 2.

The cover $W_{p_1}, \ldots, W_{p_m}$ of M is called *Morse cover* of M relative to ϕ.

Let O (M , $\varnothing$) be the order of Morse cover of M relative to $\varnothing$. Then 0 (M, $\varnothing$)=C (M, $\varnothing$).

Definition 3.

The integer $0\ (M) = \min\limits_{\Phi \in \varnothing} O\ (M, \varnothing)$ is called the *minimal order of M.*

Note that thare exists $\varnothing \in \Phi$ such that

$$0\ (M, \varnothing) = 0\ (M) = C\ (M).$$

Such a Morse function is called a *tight function* and the corresponding Morse cover is called a *tight Morse* cover of M. It seems that a tight function generates a Morse cover of M by <<maximal neighbourhoods>> in the sense that if there is any other Morse function whose set of critical points includes the critical points of the tight function, then the Morse neighbourhoods of the common critical points determined by the latter function are subsets of the neighbourhoods given by the tight function.

We may now restate (4) in the following form :

Theorem 2 :

Let M be a conneccted, closed smooth *n*-manifold.

Then

$$\tau\ (m) = \operatorname*{Inf}_{(x,\ N)} \tau\ (M, x, N) = 0\ (M)$$

Note. Let P^2 be obtained after identifying the diametrically opposite points on the 2-sphere $S^2 : x_0^2 + x_1^2 + x_2^2 = 1.$

Define $f : P^2 \to \mathbb{R}$ by $f(x_0, x_1, x_2) = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2$, where $\lambda_0, \lambda_1, \lambda_2$ are distinct real numbers. Then one immediately verifies that f has precisely the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) as its non-degenerate critical points. By Theorem 2 the maximal Morse neighbourhoods W_0 , W_1 , W_2 of these points cover P^2. Since any manifold admitting a real-valued function with two non-degenerate critical points is homeomorphic to a sphere, it follows that

$$\tau(P^2) = 0(P^2) = 3.$$

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Address :

Faculty of Science,
P.O. Box-9028,
King Abdulaziz University,
Jeddah, Saudt Arabia.

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MACKEY SPACE PROBLEM FOR DOUBLE CENTRALIZER ALGEBRAS

by

LIAQAT ALI KHAN*

Faculty of Science, University of Garyounis,
P.O. Box 9480, Benghazi, Libya

Abstract.

We define semiwell-behaved approximate identity for a B*-algebra A and show that the double centralizer algebra M(A) endowed with the strict topology is a strong Mackey space if A has such an approximate identity. This gives us an improvement of a result of D.C. Taylor ([6], [7]).

1. Introduction.

In [1], Buck introduced the notion of strict topology β on $C_b(X)$, the space of all bounded continuous scalar-valued functions on a locally compact space X, and raised the question as to whether or not $(C_b(X), \beta)$ is a Mackey space. This question was answered by Conway [4] in affirmative in the case when X is locally compact and paracompact. Taylor ([6], [7]) generalized Conway's result to a non-commutative setting. In particular, he considered the strict topology β on the double centralizer algebra M(A) of a B*-algebra A and proved that $(M(A), \beta)$ is a strong Mackey space if A has a countable or, more generally, a well-behaved approximate identity. In [3], Collins and Fontenot studied several types of approximate identities and conjectured that Taylor's result holds if A has a canonical chain β totally bounded approvimate identity. In view of this we define a semiwell-behaved approximate identity and show that $(M(A), \beta)$ is a strong

*On leave from : Department of Mathematics, Federal Government College, H-8, Islamabad, Pakistan.

Mackey space if A has such an approximate identity. This gives us a partial answer to the above conjecture as well as an improvement of Taylor's result.

2. Preliminaries.

Throughout this paper A denotes a B*-algbra, and let M (A) denote the double centralizer algebra of A as introduced by Busby [2]. Then A may be viewed as a closed two-sided ideal in $(M(A), \|.\|)$. The *strict topology* β on M(A) is the locally convex topolgy generated by the seminorms $x \to \max\{\|ax\|, \|xa\|\}$ for $x \in M(A)$ and $a \in A$. Some basic propertirs of β are : (1) $\beta \leqslant \|\,.\,\|$; (2) β and $\|\,.\,\|$ have the same bounded sets ; (3) A is dense in $(M(A), \beta)$; (4) $(M(A), \beta)$ is complete ; (5) A has an identity if $A = M(A)$ and $\beta = \|\,.\,\|$.

The following two theorems, due to Taylor [6], are stated for reference purpose.

Theorem 2.1. *Let A* denote the norm dual of A. Then.*

(1) $A^* = \{F.a : a \in A, F \in A^*\} = \{a.F : a \in A, F \in A^*\}$,
where $F.a\,(b) = F\,(ab)$ *and* $a.F\,(b) = F\,(ba)$ *for all* $b \in A$.

(2) $(M(A), \beta)^*$, *with the strong topology, is a Banach space and is isomatrically isomorphic to* A^*.

Theorem 2.2. *Let* $\{e_\lambda : \lambda \in I\}$ *be an approximate identity for A. Then a subset H of* $(M(A), \beta)^*$ *is equicontinuous iff the following conditions hold :*

(*i*) *H is uniformly bounded* ;

(*ii*) $(e_\lambda.F + F.e_\lambda - e_\lambda.F.e_\lambda) \to F$ *uniformly on* H.

Definition 2.3. (cf. [3] , *p*. 76) An approximate identity $\{e_\lambda : \lambda \in I\}$ for A is said to be *semiwell-behaved* if.

(*i*) $\{e_\lambda\}$ is canonical, *i.e.*, $e_\lambda \geq 0$ for all $\lambda \in I$ and $\lambda_2 > \lambda_1$ implies that $e_{\lambda_2} e_{\lambda_1} = e_{\lambda_1}$;

(*ii*) for a strictly increasing sequence $\{\lambda_n\} \subseteq I$, a sequence $\{c_n\}$ of positive real numbers such that $\sum_{n=1}^{\infty} c_n$ is convergent and

$\lambda \in I$, there exists an integer N such that $m \geq n > N$ implies that

$\| e_\lambda (e_{\lambda_m} - e_{\lambda_n}) \| \leqslant c_n$.

If, in (*it*), we take each $c_n = 0$, $\{e_\lambda\}$ is called *wcll-behaved* [7]. It is shown in ([7], Prop. 3.1) that, if A has a countable approximate identity. Then it has also a well behaved approximate Identity. Clearly, a well-behaved approximate identity is semiwell-behaved. In view of ([3], prop. 7.5), a semiwell-behazed approximate identity is slightly restrictive than a canonical chain β totally bounded approximate identity. ($\{e_\lambda\}$ is *chain* β *totally bounded* [3] if, for any increasing sequence $\{ \lambda_n \} \subseteq I$, $\{ e_{\lambda_n} \}$ is β totally bounded in A.)

3. The main result.

Recall that a locally convex space E is a *Mackey space* ([5], P. 173) if every weak*-compact convex balanced subset of E* is equicontinuous ; E is said to be a *strong Mackey space* if every weak*-countably compact subset of E* is cquicontinuous [4].

We now state our main reseult. This was proved by Taylor in [6] (resp. [7]) in the case when A has a countable (resp. well-behaved) approximate identity. We prove it under the weaker assumption that A has a semiwell-behaved approximate identity.

Theorem 3.1. *Suppose A has a samiwell-beheved approximate identity. Then* (*M*(*A*), β) is a strong Mackey space.

Proof. Let H be weak*-countably compact subset of (M(A),β)*, and let $\{ e_\lambda : \lambda \in I \}$ be a semiwell-behaved approximate identity for A. Since H is pointwise bounded, it follows from the principle of uniform boundedness that H is uniformly bounded. Without loss of generality, we may assume that $\| F \| \leq 1$ for all $F \in H$.

Suppose H is not β-equicontinuous. Then, by Theorem 2.2, there exists an $\epsilon > 0$ such that, for each $\lambda_0 \in I$.

$$\| F - e_\lambda . F - F . e_\lambda + e_\lambda . F . e_\lambda \| \geq 4 \epsilon$$

for some $F \in H$ and $\lambda > \lambda_0$. Using the fact that $((M(A), \beta)^*$, strong. top.) $\approx (A^*, \|\cdot\|)$ (Theorem 2.1), there exists by induction a sequence $\{(F_n, a_n, x_n, \lambda_{2n-1}, \lambda_{2n})\}$ with the following properties (see [6, p. 642] or [7, p. 482]).

(a) $F_n \in H$, a_n is a Hermition element in A with $\|a_n\| \leq 1$, $\lambda_{2n-1} < \lambda_{2n}$;

(b) $x_n = (e_{\lambda_{2n}} - e_{\lambda_{2n-1}}) a_n (e_{\lambda_{2n}} - e_{\lambda_{2n-1}})$ with $\|F_n(x_n)\| \leq \epsilon$.

Since $\{e_\lambda\}$ is canonical, it is easy to see that $x_n x_m = 0$ for $n \neq m$. Then it follows by induction that if $t = \{t_n\} \in l_\infty$, $\|\sum_{n=1}^{m} t_n x_n\| \leq 2 \|t\|_\infty$ for any $m \geq 1$.

We now show that the sequence $\{\sum_{n=1}^{m} t_n x_n\}$ of partial sums is β-Cauchy in $M(A)$.

Let $a \in A$ and $r > 0$. Choose a $\lambda_0 \in I$ such that $\|a - ae_{\lambda_0}\| < r/4\|t\|_\infty$.

Since $\{\lambda_{2n}\}$ is a stricly increasing sequence in I, there exists by hypothesis an integer N_1 such that $\|e_{\lambda_0}(e_{\lambda_{2n}} - e_{\lambda_{2n-1}})\| < 1/2^n$ for all $n \leq N_1$. Choose

$N_2 \geq N_1$ so that $\sum_{n \geq N_1} 1/2^n > r/2\|t\|_\infty \|a\|$. Then, for $q > p \geq N_2$,

$$\|a(\sum_{n=1}^{q} t_n x_n - \sum_{n=1}^{p} t_n x_n)\| \leq \|a - ae_{\lambda_0}\| \, \|\sum_{n=p+1}^{q} t_n x_n\| + \|ae_{\lambda_0} \sum_{n=p+1}^{q} t_n x_n\|$$

$$< r/2 + 2\|a\| \, \|t\|_\infty \sum_{n=p+1}^{q} \|e_{\lambda_0}(e_{\lambda_{2n}} - e_{\lambda_{2n-1}})\|$$

$$< r/2 + \| a \| \| t \|_\infty \Sigma 1/2^n < r,$$

which implies that $\{ \sum_{n=1}^{m} t_n x_n \}$ is β-Cauchy in M (A). Since M(A) is β-complete,

$\beta\text{-}\lim_m (\sum_{n=1}^{m} t_n x_n) \in$ M (A). The mapping S : $(l_\infty, \beta) \to (M(A), \beta)$, given by,

$S(t) = \sum_{n=1}^{\infty} t_n x_n$, is then well-defined and continuous. Thus the adjoint mapping $S^* : (M(A), \beta)^* - (l_\infty, \beta)^*$ is continuous when both spaces are given their respective weak*—topologies. Consequently, S* (H) is a week*-countably compact subset of $(l_\infty, \beta)^*$ and hence equicontinnous in it [4]. Since $l(_\infty, \beta)^* \cong l_1$ (see [1] , [4]) and $S^*F(t) = F(S(t)) = \sum_{n=1}^{\infty} t_n F(x_n)$ $(t \in l_\infty)$, S* (H) may be identified with the sequence $\{ F(x_n) \}$ in l_1.Hence there exists an integer N such that $\sum_{n \geq N} \| F(x_n) \| < \epsilon$ for all F ∊ H. In particular, $\|F_n(x_n)\| < \epsilon$ for all $n \geq$ N which contradicts (*b*). Thus H is equicontinuous in (M(A), β)*. This completes the proof.

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ON ASYMPTOTIC PROPERTIES OF AN ESTIMATE OF A FUNCTIONAL OF A PROBABILITY DENSITY

by

KHALED I. ABDUL-AL

Department of Mathematical Sciences
University of Petroleum & Minerals
Dhahran, Saudi Arabia

Abstract.

Bhattacharyya & Roussas (1969) proposed an estimate of the functional $\Delta = \int f^2(x)\,dx$ by $\tilde{\Delta} = \int f_n^2(x)dx$ where $f_n(x)$ is a kernel estimatc of the probability density $f(x)$. Schuster (1974) proposed an alternative estimate $\hat{\Delta} = \int f_n(x) dF_n(x)$ of Δ, where $F_n(x)$ is the sample distribution function, and showed that the two estimates attain the same rate of strong convergence to Δ. Ahmad (1976) presented two large sample properties of $\hat{\Delta}$; first being the strong convergence of $\hat{\Delta}$ to Δ, and second is the asymptotic normality of $\hat{\Delta}$. In this note, it is proposed to estimate $\theta = E[\gamma(x)] = \int \gamma(x) f(x)\,dx$ by $\theta_n = \int \gamma(x) f_n(x)\,dx$, and show the weak and strong convergence of θ_n to θ and establish the asymptotic normality of θ_n.

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1. Introduction.

Let X be a random variable with distribution function (*d. f.*) F (x) and probability density function (*p. d. f.*) $f(x)$, and let the functional be defined as

$$\theta = \int \gamma(x) f(x)\, dx \tag{1. 1}$$

where γ (X) is real measurable function of random variable X.

The functional θ is important in many estimation problems as the estimate of the characteristic function $\varnothing(t)$, momonts of any order and any mathematical expectations of the form E [g (X)] when $f(x)$ is unkown.

Let $X_1, \ldots\ldots\ldots, X_n$ be identically independent distributed (*iid*) random variables with *d. f.* (F (X) and *p. d. f.* $f(x)$. Let $k(u)$ be a known symmetric *p. d. f.* satisfying the following condition :

$$\sup_{-\infty<u<\infty} k(u) < \infty \quad \text{and} \quad \lim_{|u| \to \infty} |u|\, k(u) = 0 \tag{1, 2}$$

Also let $\{ a_n \}$ be a sequence of real positive numbers such that

$$a_n \to 0 \quad \text{as} \quad n \to \infty \tag{1.3}$$

The kernel estimate of $f(x)$ using $k(u)$ is given by

$$f_n(x) = \frac{1}{a_n} \int k\left(\frac{x-u}{a_n} \right) dF_n(u)$$

$$= \frac{1}{na_n} \sum_{j=1}^{n} k\left(\frac{x-X_j}{a_n} \right) \tag{1.4}$$

where $F_n(x)$ is the sample distribution function.

In this paper, we examine the conditions under which

$$\theta_n = \int \gamma(x) f_n(x)\, dx \tag{1.5}$$

is consistent (weak as well as strong) and asymptotically normal.

All integrals in this paper will be understood to be Lebesgue integrals. Where the limits of integrations over the entire line is considered, they will be omitted.

2. Consistency.

We first examine the conditions under which θ_n is asymptotically unbiased in the sense if $a_n \to 0$ as $n \to \infty$, then

$$\lim_{n \to \infty} E(\theta_n) = \theta \tag{2.1}$$

Now

$$E(\theta_n) = E \frac{1}{a_n} \int\int \gamma(x)\, k\left(\frac{x-u}{a_n}\right) d\,F_n(u)\, dx$$

$$= \frac{1}{a_n} \int\int \gamma(x)\, k\left(\frac{x-u}{a_n}\right) dF(u)\, dx \tag{2.2}$$

In order for (2.1) to hold, the last expression for (2.2) must tend to $\int \gamma(x) f(x)\ dx$. Conditions under which this happens are given by the following theorem.

Theorem 1. Suppose $k(u)$ is a Borel function satisfying the condition (1. 2) and

(*i*) $\int |k(y)|\, dy < \infty$ and (*ii*) $\int k(y)\, dy = 1$

Let $\gamma(y)$ and $f(y)$ satisfy

$$\int |\gamma(y) f(y)/dy < \infty \tag{2.2a}$$

Let $\{a_n\}$ be a sequence of positive constants satisfying (1. 3). Define

$$g_n(x) = \frac{1}{a_n} \int\int \gamma(x)\, k\left(\frac{y}{a_n}\right) f(x-y)\, dy\, dx.$$

Then, at every point x of continuity of (.),

$$\lim_{n \to \infty} g_n(x) = \int \gamma(x) f(x)\, dx) \tag{2.3}$$

Proof. In view of Theorem 1A, Pursen (1962) and (2.2*a*)

$$\lim_{n \to \infty} g_n(x) = \int \gamma(x) f(x)\, dx$$

The equation (2.3) implies that

$$\lim_{n \to \infty} E(\theta_n) = \theta$$

i.e. θ_n is asymptotically unbiased.

Theorem 2. Assume that a_n satisfy (1.3), and $\text{Var}[\gamma(x)]<\infty$,

$$E\mid \theta_n-\theta \mid \to 0 \quad \text{as} \quad n\to\infty \tag{2.4}$$

Proof.

$$E\mid \theta_n-\theta \mid \leqslant E\mid \theta_n - E\theta_n \mid + \mid E\theta_n-\theta \mid$$
$$= I_{n1} + I_{n2}$$

By Theorem 1.

$$I_{n2}\to 0 \quad \text{as} \quad n\to\infty \tag{2.5}$$

and

$$I_{n1}=E\mid \theta_n-E\theta_n \mid \leqslant [E\mid \theta_n-E\theta_n|^2]^{\frac{1}{2}} = [\text{Var}\ \theta_n]^{\frac{1}{2}} \to 0 \tag{2.6}$$

by Theorem 4 (proved later).

From (2.5) and (2.6) we have that

$$E\mid \theta_n-\theta \mid \to 0 \quad \text{as} \quad n\to\infty$$

i.e.,

$$\theta_n \xrightarrow{\ p\ } \theta \quad \text{as} \quad n\to\infty$$

Theorem 3. Assume that a_n satisfies (1.3) and suppose that $\gamma(x)$ is absolutely continuous, and $\text{Var}[\gamma(x)]<\infty$, then

$$\theta_n \xrightarrow{\ W.P.1\ } \theta \quad \text{as} \quad n\to\infty \tag{2.7}$$

Proof. $\mid \theta_n-\theta \mid \leqslant \mid \theta_n-E\theta_n \mid + \mid E\theta_n-\theta \mid$

By Theorem 1

$$\mid E\theta_n-\theta \mid \to 0 \quad \text{as} \quad n\to\infty \tag{2.8}$$

Now

$$\theta_n-E\theta = \frac{1}{a_n}\Big[\int\int\gamma(x)\, k\left(\frac{(x-u)}{a_n}\right) d\,F_n(u)\,dx$$
$$-\int\int\gamma(x)\, k\left(\frac{x-u}{a_n}\right) dF(u)\,dx\Big]$$
$$=\int\int\gamma(a_n z+u)\, k(z)\, dF_n(u)dz-\int\int\gamma(a_n z+u)k(z)f(u)\,du\,dz$$
$$= \frac{1}{n}\int\sum_{i=1}^{n}\gamma(a_n z+x_i)\, k(z)\, dz$$
$$-\int\int\gamma(a_n z+u)\, k(z)\, f(u)\,du\,dz$$

$$= E\left[\frac{1}{n} \sum_{i=1}^{n} \gamma\,(a_n Z+X_i)-E_x E_z \gamma\,(a_n Z\times X)\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E_z\gamma\,(a_n Z+X_i)-E_x E_z\gamma\,(a_n Z+X)$$

Let

$$g_n\,(x)=E_Z\,\gamma\,(a_n Z+x)$$

So

$$\theta_n-E\theta_n = \frac{1}{n} \sum_{i=1}^{n} [g_n\,(x_i)-Eg_n(X_i)]$$

$$= \frac{1}{n} \sum_{i=1}^{n} V_{ni}, \text{ say.}$$

Note $V_{n1}, \ldots, V_{nn}$ are iid random variables and that $EV_{ni}=0$. Since $\gamma\,(x)$ is absolutely continuous and Var $[\gamma\,(x)]<\infty$,

$$|\,\theta_n-E\theta_n\,| \leq \left|\, \frac{1}{n} \sum_{i=1}^{n} E_Z\,\gamma\,(a_n Z+X_i) - E_x E_Z\,(a_n Z+X)\,\right| \to 0 \qquad (2.9)$$

From (2.8) and (2.9), we have that

$$\theta_n \xrightarrow{\text{W.P.1}} \theta \quad \text{as} \quad n\to\infty.$$

Next, we discuss the asymptotic behavior of the variance of the estimate θ_n. It is given by

$$\text{Var}\,(\theta_n)=E\theta^2{}_n-E^2\theta_n$$

Now,

$$E\theta^2{}_n = \frac{1}{n^2 a_n{}^2}\, E\left[\sum_{i=1}^{n} \left(\int \gamma(x)\, k\left(\frac{x-X_i}{a_n} \right)dx\right)^2\right]$$

$$+ \frac{1}{n^2 a_n{}^2}\, E\left[\sum_{i\neq j} \int \gamma\,(x)\, k\left(\frac{x-X_i}{a_n}\right)dx \int \gamma\,(x)\, k\left(\frac{x-X_j}{a_n}\right) dx \right]$$

$$= A_{n1} + A_{n2}$$

$$A_{n1} = \frac{1}{na^2_n} E\left[\int \gamma(x)\, k\left(\frac{x-X}{a_n}\right) dx\right]^2$$

$$= \frac{1}{na^2_n} E\left[\int\int \gamma(x_1)\, k\left(\frac{x_1-X}{a_n}\right) \gamma(x_2)\, k\left(\frac{x_2-X}{a_n}\right) dx_1\, dx_2\right]$$

$$= \frac{1}{na^2_n} \int\int\int \gamma(x_1)\, \gamma(x_2)\, k\left(\frac{x_1-u}{a_n}\right) k\left(\frac{x_2-u}{a_n}\right) f(u)\, dx_1 dx_2 du$$

$$= \frac{1}{n} \int\int\int \gamma(a_n z_1+u)\, \gamma(a_n z_2+u)\, f(u)\, k(z_1)\, k(z_2)\, dz_1\, dz_2$$

and

$$A_{n2} = \frac{n(n-1)}{n^2 a^2_n} E\left[\int \gamma(x)\, k\left(\frac{x-X}{a_n}\right) dx\right]^2$$

$$\frac{(n-1)}{n}\left[\int\int \gamma(a_n z+u)\, k(z)\, f(u)\, dz\, du\right]^2$$

which imply that

$$\text{Var}\ \theta_n = \frac{1}{n}\Big[\int\int\int \gamma(a_n z_1+u)\, \gamma(a_n z_2+u)\, k(z_1)\, k(z_2)\, f(u)\, dz_1\, dz_2\, du$$

$$-\Big[\int\int \gamma(a_n z+u)\, k(z)\, f(u)\, dz\, du\Big]^2$$

then

$$n\ \text{Var}\ \theta_n \to \int \gamma^2(u)\, f(u)\, du - \Big[\int \gamma(u)\, f(u)\, du\Big]^2$$

$$= E[\gamma^2(X)] - [E\gamma(X)]^2$$

In view of the above we have proved the following theorem.

Theorem 4. The estimates θ_n have variance satisfying

$$\lim_{n\to\infty} n\ \text{Var}\ \theta_n = E[\gamma^2(X)] - [E\gamma(X)]^2 = \text{Var}\ [\gamma(X)]$$

and if Var $[\gamma(X)] < \infty$, then

$$\lim_{n\to\infty} \text{Var}\ \theta_n = 0$$

at all points x of continuity of $f(.)$ if $a_n \to 0$.

From Theorem 4 one can state conditions under which the estimates θ_n are consistent in quadratic mean in the sense that $E\,|\,\theta_n - \theta^2 \to 0$ as $n \to \infty$.

The mean square error may be written as

$$E\,|\,\theta_n-\theta\,|^2=\text{Var}\,\theta_n+|\,E\theta_n-\theta|^2$$

Consequently, if $a_n\to 0$ as $n\to\infty$, it then follows that θ_n is a consistent estimate of θ.

3. Asymptotic Normality

Since the estimate θ_n may be written as $\theta_n = \frac{1}{n}\sum_{j=1}^{n} V_{nj}$,

where $V_{nj}=\frac{1}{a_n}\int \gamma(x)\,k\left(\frac{x-X_j}{a_n}\right)dx$ and are independent and identically distributed random variables for all j' i.e.

$V_n = \frac{1}{a_n}\int \gamma(x)\,k\left(\frac{x-X}{a_n}\right)dx$, it is easy lo state conditions under which sequence θ_n is asymptotically normal, in the sense that

$$\sqrt{n}\,(\theta_n-\theta)\to N(0,\sigma^2) \text{ as } n\to\infty$$

where $\sigma^2=\text{Var}\,[\gamma(X)]$.

Theorem 5. Assume the following conditions :

(*i*) $na_n^4\to 0$ as $n\to\infty$,

(*ii*) $\int zk(z)\,dz=0$ and $\int z^2 k(z)dz<\infty$

(*iii*) $f(x)$ is twice differentiable

(*iv*) $\int\gamma(x)\,f'(x)\,dx<\infty$, $\int\gamma(x)\,f''(x)\,dx<\infty$ and $E\,|\,\gamma(x)|^3<\infty$,

then $\sqrt{n}\,(\theta_n-\theta)\to N(0,\sigma^2)$ where $\sigma^2=\text{Var}\,\gamma(X)$.

Proof. To prove the theorem, we divide the argument into two parts :

(1) $\sqrt{n}\,(\theta_n-E\theta_n)\to N(0,\sigma^2)$ as $n\to\infty$

(2) $\sqrt{n}\,(E\theta_n-\theta)\to 0$ as $n\to\infty$

To show (1), it is enough to show that

$$\frac{E\,|\,V_n-EV_n\,|^3}{n^{\frac{1}{2}}\,\sigma^3}\to 0 \text{ as } n\to\infty$$

where

$$V_n = \frac{1}{a_n} \int \gamma(x) \, k\left(\frac{x-X}{a_n}\right) dx$$

But

$$E \mid V_n - EV_n \mid^3 \leqslant 2^3 (E \mid V_n \mid^3 + E^3 \mid V_n \mid)$$

Now

$$E \mid V_n \mid^3 = \int \mid \int \gamma(a_n z + u) \, k(z) \, dz \mid^3 f(u) \, du$$
$$\leq \int\int\int \mid \gamma(a_n z_1 + u) \, \gamma(a_n z_2 + u) \, \gamma(a_n z_3 + u) \mid$$
$$\times k(z_1) \, k(z_2) \, k(z_3) f(u) \, dz_1 dz_2 dz_3 \to \mid \gamma^3(u) \mid du$$
$$= E \mid \gamma(X) \mid^3$$

So

$$E \mid V_n - EV_n \mid^3 \leqslant z^3 E \mid \gamma(X) \mid^3 + E^3 \mid \gamma(X) \mid < \infty$$

because $E[V_n] \to E[\gamma(X)]$ as $n \to \infty$

and $E \mid \gamma(X) \mid^3 < \infty$

wh:ch shows that

$$\frac{E \mid V_n - EV_n \mid^3}{n^{\frac{1}{2}} \sigma^3} \to 0 \quad \text{as } n \to \infty$$

Then Laypanouff condition is satisfied for $\delta = 1$ and $\sqrt{n}(\theta_n - E\theta_n) \to N(0, \sigma^2)$ as $n \to \infty$. Next, we show part (2)

$$\sqrt{n}(E\theta_n - \theta) = \frac{\sqrt{n}}{a_n} \left[\int\int \gamma(x) \, k\left(\frac{x-u}{a_n}\right) f(u) \, du\right) - \int \gamma(x) f(x) \, dx \right]$$
$$= \sqrt{n} \int\int \gamma(x) \, k(z) \, [f(x - a_n z) - f(x) dx dz].$$

Using Taylor's expansion.

$$f(x - a_n z) - f(x) = -a_n z f'(x) + (a_n z)^2 f''(x) + 0(a_n^2)$$

then

$$\sqrt{n}(E\theta_n - 0) = \sqrt{n} \, a_n^2 \left[\int z^2 k(z) \, dz\right] \left[\int \gamma(x) f''(x) \, dx\right] + \sqrt{n} \, 0(a^2{}_n) \to 0 \text{ as } n \to \infty$$

by conditions (i) (iv).

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TWO FACTOR CENTRAL COMPOSITE DESIGN ROBUST TO A SINGLE MISSING OBSERVATION

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DR. MUNIR AKHTAR

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Summary

A two factor central composite design robust to a single missing observation is developed under minimaxloss criterion. The losses due to a single missing observation and variances of parameter estimates are studied for different distances of axial points from the centre of the design. The minimaxloss design is then compared with other central composite designs of the same size.

Key words and phrases : Central composite design ; robust design ; optimum design ; loss of deficiency.

1. Introduction.

Two factor central composite design consists of.

(*a*) four points of a 2^2 factorial design, *i.e.* $(-1,-1)$, $(1,-1)$, $(-1, 1)$ and $(1, 1)$.

(*b*) four axial points two at each axis, a distance from the centre of the design, *i.e.* $(\alpha, 0)$ $(-\alpha, 0)$ $(0, \alpha)$ and $(0,-\alpha)$ and

(*c*) one or more points at the centre of the design.

Points in part (*a*) and (*b*) may be replicated more than once. Let n_f , n_a and n_c represent number of factorial, axial and centre points in the design. The design points $n=n_f+n_a+n_c$.

Missing observations can occur even in well planned experiments. Kiefer (1959) and Kiefer and Wolfowitz (1959) introduced D— and

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Missing observations can occur even in well planned experiments. Kiefer (1959) and Kiefer and Wolfowitz (1959) introduced D— and

G-optimalities and constructed designs which are optimum according to some specific criterion. But even the optimum design may give poor performance when any one or more observations happen to be missing.

Herzberg and Andrews (1975, 1976, 1978) and Andrews and Herzberg (1979) studied the effects of missing observations on D– and G-optimality measures. Box and Draper (1975) introduced a criterion which minimizes the effects of outlying observations and constructed designs robust to outliers.

Mackee and Kshirsagar (1982) studied the effects of missing observations on the parameter estimates and their variances for central composite designs arranged in orthogonal blocks.

Here effect of a single missing observation on $|X'X|$ for a two factor design with one replication of parts (*a*) and (*b*), is investigated. A design for which the maximum loss in terms of $|X'X|$, due to a missing observation is minimum, has been developed. The variances of parameter estimates are investigated over a range of α, for this complete and reduced central composite design. The minimaxloss design is then compared with the exsisting two factor designs of the same size but with different α.

The response surface model used is a second order polynomial

$$y_i=\beta_0+\beta_1X_{1i}+\beta_2X_{2i}+\beta_{11}X^2_{1i}+\beta_{22}X^2_{2i}+\beta_{12}X_{1i}X_{2i}+\epsilon_i$$

were y_i is ith observation X_{1i} and X_{2i} are predictor variables, β_0, β_1, β_2 β_{11}, β_{22}, and β_{12} are coefficients and ϵ_i is the error assumed to be uncorrelated with mean zero and constant variance. The method of estimation used is least squares.

The above model may also be written as

$$\underline{y}=\underline{X}\underline{\beta}+\underline{\epsilon}$$

where $\underline{y}$ is an $n\times 1$ vector of response at different points, $\underline{\beta}$ is a $p\times 1$ vector of coefficients, $\underline{\epsilon}$ is $n\times 1$ vector of error and $\underline{X}$ is a matrix of predictor variables. Some of the least square estimates are

$$\hat{\beta}=(X'X)^{-1}X'y,$$

$$\hat{y}=X\hat{\beta}=X\,(X'X)^{-1}X'y=Ry$$

and $\quad Var(\hat{\beta})=(X'X)^{-1}\sigma^2$

provided $(X'X)^{-1}$ is non-singular.

2. Losses due to a single missing observation.

This two factor design consists of four factorial, four axial and one or more centre points. The mininmaxloss design is one with α and n_c such that

$$L_f = L_a \geq L_c$$

where L_f, L_a and L_c are losses due to a missing factorial, axial or centre point respectively. Loss of the ith point missing

$$L_i=\underline{x}_i'\,(\underline{X'X})^{-1}\underline{x}_i$$

where $\underline{x}_i'$ is the ith row of $\underline{X}$. L_i is also equal to the ith diagonal element of $\underline{R}$.

For two factor design with $n_f=n_a=4$ the explicit expression for

$$L_f=\{(8+n_c)\alpha^6+6n_c\alpha^4-8\,(12-n_c)\alpha^2+32(4+n_c)\}\,(B)^{-1},$$
$$L_a=4\{(4+n_c)\alpha^6-(12-n_c)\alpha^4+3n_c\alpha^2+2\,(8+n_c)\}\,(B)^{-1}$$

and $\quad L_c=\{n_c+4\,(2-o^2)^2/(4+\alpha^4)\}^{-1}$

where $B=4\{4+n_c)\alpha^6+2\,(n_c-4)\alpha^4-4\,(4-n_c)\alpha^2+8\,(4+n_c)\}$

The equation $L_f=L_a$ after some algebra reduces to

$$(3n_c+8)\,\alpha^6-2\,(n_c+24)\alpha^4-4(n_c+24)\alpha^2-8\,(3n_c+8)=0$$

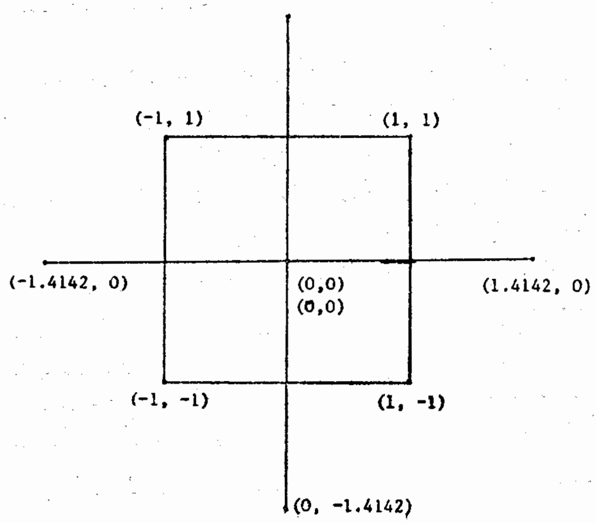
This is a cubic in α^3 and for $n_c \geqq 1$ has positive discriminant which implies that it has one real and two complex conjugate roots. For $n_c \geq 1$ the real root for this equation is $\alpha^2=2\,0$ which gives $\alpha=1.4142$ for which $L_f=L_a=0.625$. For this design with $a=\sqrt{2}$, $L_c=1/n_c$. All designs with $\alpha=1.4142$ and $n_c \geq 2$ satisfy $L_f=L_a \gtrsim L_c$ and thus are minimaxloss designs.

The design matrix for 10 point design with $n_f=n_a=4$, $n_c=2$ and $\alpha=1.4142$ is the following matrix $\underline{D}$:

$$D = \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \\ 1.4142 & 0 \\ -1.4142 & 0 \\ 0 & 1.4142 \\ 0 & -1.4142 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The layout of this design is shown in Figure 1 below.

Figure 1. Central Composite Design with $k=2$, $n_f=n_a=4$ and $n_c=2$ and $\alpha=1.4142$.



L_f, L_a and L_c for two factor design with $n_f = n_a = 4$ and $n_c = 1$ or 2 are plotted against α in figure 2 (a, b).

L_f decreases and L_a increases with the increase of α. L_c has its maximum at $\alpha = 1.4142$. The design with $\alpha = 1.4142$, $n_c = 2$ in figure 2 (b) is minimaxloss design.

The existing two factor central composite design with $n_f = n_a = 4$ and $n_c = 2$ are design with "$\alpha = 1.0$ and orthogonal design with" $\alpha = 1.0781$. The rotatable and outlier robust designs both has $\alpha = 1.4142$.

L_f, L_a, L_c, maximum loss and variance of lasses for two factor designs with $\alpha = 1.0$, 1.0781 and 1.4142 each with one or two centre points are shown in Table 1.

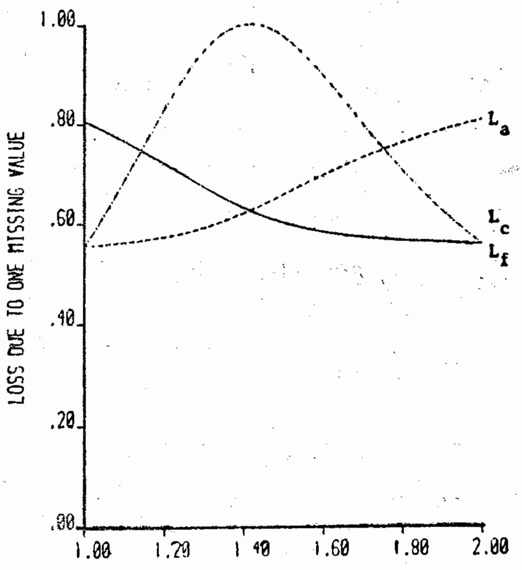


Figure 2(a). Loss due to a single missing observation for c.c.d. with k=2 and n_c=1, plotted against α.

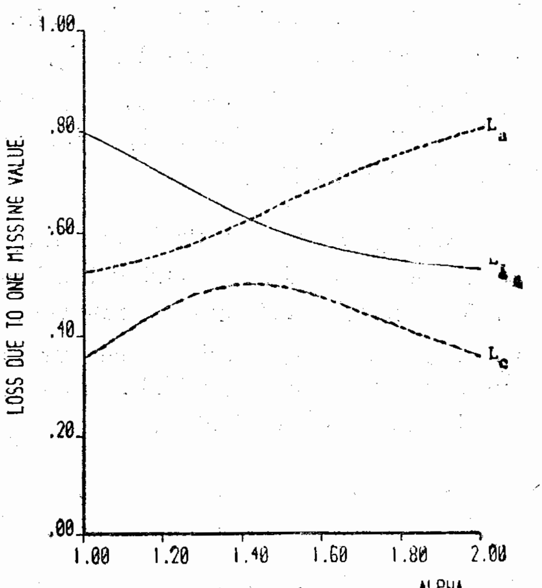


Figure 2(b). Loss due to a single missing observation for c.c.d. with k=2 and n_c=2, plotted against α.

3. Variances of parameter estimates

Variances of parmeter estimates for two factor design with $n_f = n_a = 4$ may be expressed as

$$\text{Var}(\hat{\beta}_0) = \left[n_c + \frac{4(2-\alpha^2)^2}{4+\alpha^4} \right]^{-1}$$

$$\text{Var}(\hat{\beta}_1) = \text{Var}(\hat{\beta}_2) = (4 \times 2\alpha^2)^{-1}$$

$$\text{Var}(\hat{\beta}_{11}) = \text{Var}(\hat{\beta}_{22}) = \frac{1}{2\alpha^4}\left[1 + \frac{2\alpha^4 + 8\alpha^2 - 8 - 2n_c}{(4+n_c)\alpha^4 - 16\alpha^2 + 16 + 4n_c} \right]$$

and $\text{Var}(\hat{\beta}_{12}) = 1/n_f = 1/4$

These variances for design with one or two centre points are plotted against α in figure 3 (*a* , *b*).

Variances of parameter estimates for design with α=1.0,1.0781 and 1.4142 and for these design with a missing factorial, axial or centre point are shown in Table 2.

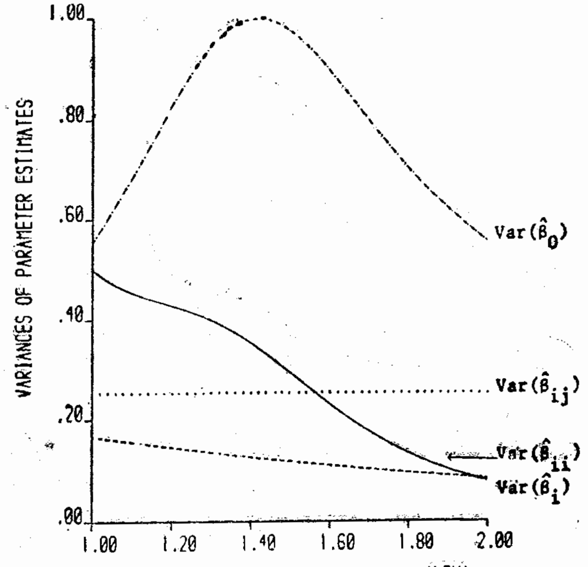


Figure 3(a). Variances of parameter estimates for c.c.d. with k=2 and n_c=1, plotted against α.

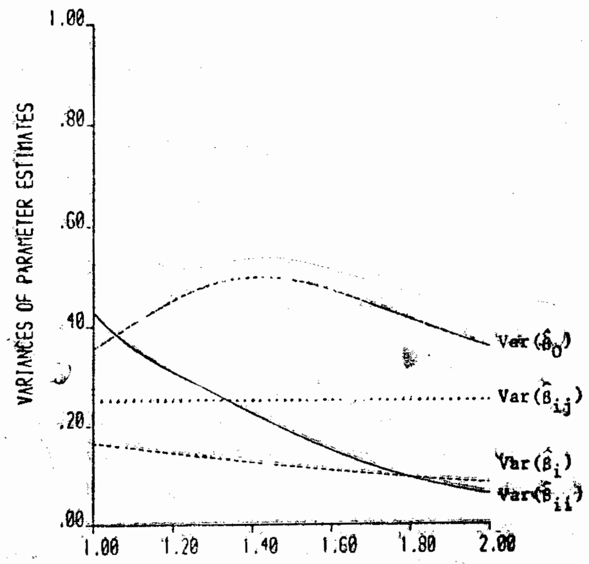


Figure 3(b). Variances of parameter estimates for c.c.d. with k=2 and n_c=2, plotted against α.

4. Discussion and conclusion.

For two factor central composite design, the α values for rotable, outlier robust and minimaxloss designs are same *i.e.* $\alpha=\sqrt{2}=1.4142$. This design has smaller losses due to a single missing observation. The variances of parameter estimates are also comparatively smaller. As the loss due to a missing centre point is maximum for $\alpha=\sqrt{2}$ *i.e.* $L_c=I/n_c$, it is advisable to add few more points at the centre of the design.

It is not possible to have equiloss design, *i.e.* design with $L_f=L_a=L_c$ in central composiie designs with some centre points.

The work on designs robust to one or two missing observations and with different factors is in progress with prominent results.

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TABLE 1

Loss due to a single missing observation at factorial, axial or centre point together with the maximum loss and variance of losses.

No. of variables $k=2$ Total design points $n=10$

No. of parameters $p=6$ No. of centre points $=2$

Alpha	n	Loss due to a single missing observation. Factorial obs.	Axial obs.	Centre obs.	Maximum loss.	Variance of losses.
1.0000	10	0.7976	0.5238	0.3571	0.7976	0.3304E−01
	9	0.8056	0.5556	0.5556	0.8056	0.1736E−01
1.0781	10	0.7662	0.5357	0.3961	0.7662	0.2335E−01
	9	0.7748	0.5612	0.6559	0.7748	0.1143E−01
1.4142	10	0.6250	0.6250	0.5000	0.6250*	0.2778E−02
	9	0.6250	0.6250	1.0000	1.0000	0.1563E−01

* Minimaxloss due to one missing observation.

TABLE 2

Variances of parameter estimates for complete design and for designs with one observation missing,

No. of variables $k=2$ Total design points $n=10$

No. of parameters $p=6$ No. of centre points$=2$

Alpha	n	Variances of parameter estimates. Inter-cept.	Linear (min)	Linear. (max)	Quad-atic. (min)	Quadr-atic. (max)	Inter-action
1.0000	10	0.3571	0.1667	0.1667	0.4286	0.4286	0.2500
	9 f	0.3824	0.3039	0.3039	0.5294	0.5294	0.5588
	9a	0.4000	0.1667	0.2250	0.5250	0.6000	0.2500
	9c	0.5556	0.1667	0.1667	0.5000	0.5000	0.2500

1.0781	10	0.3961	0.1581	0.1581	0.3701	0.3701	0.2500
	9*f*	0.4183	0.2651	0.2651	0.4493	0.4493	0.5173
	9*a*	0.4292	0.1581	0.2207	0.4529	0.4881	0.2500
	9*c*	0.6559	0.1581	0.1581	0.4609	0.4609	0.2500
1.4142	10	0.5000	0.1250	0.1250	0.2188	0.2188	0.2500
	9*f*	0.5000	0.1667	0.1667	0.2292	0.2292	0.4167
	9*a*	0.5000	0.1250	0.2083	0.2292	0.3125	0.2500
	9*c*	1.0000	0.1250	0.1250	0.3438	0.3438	0.2500

f—A factorial observation missing.

a—An axial observation missing.

c—An observation at centre missing.

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ON TRANSLATIVITY OF THE PRODUCT OF NORLUND-WEIGHLED MEAN SUMMABILITY METHODS

by

AZMI K. AL-MADI

Abstract :

In the present paper, necesssary and sufficient conditions for the product of Nörlund-Weighted mean summability methods (N, r) (M_q) to be translative have been established. The paper contains two interesting examples to show that even if both (N, r) and (Mq) are translative, the product (N, r) (Mq) need not be so. Some special cases for which (N, r) (Mq) is translative have been given.

1. Introduction.

Given a series

$$\sum_{n=0}^{\infty} a_n. \tag{1}$$

We will write r, q to denote the sequences $\{r_n\}$, $\{q_n\}$; we shall use throughout for any sequence, $\Delta u_n = u_n - u_{n+1}$. We define the sequence $\{c_n\}$ formally by means of the identity

$$\left(\sum_{n=0}^{\infty} r_n z^n\right)^{-1} = \sum_{n=0}^{\infty} c_n z^n \text{ ; } c_{-n}=0 \ (n>0)$$

and will write $C(z)$ for $\sum_{n=0}^{\infty} c_n z^n$.

Let (N, r) denote the Nörlund method in which the sequence $\{S_n\}$ is transformed into the sequence $\{H_n\}$ where

$$H_n = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} S_k \text{ ; } R_n = r_0 + r_1 + \ldots\ldots + r_n \neq 0 \ (n \geq 0),$$

$$R_{-1} = r_{-1} - 0. \tag{2}$$

Each sequence $\{q_n\}$ for which $Q_n = q_0 + q_1 + \ldots + q_n \neq 0$ for each n defines the weighted mean method (M_q) of the sequence $\{S_n\}$, where

$$U_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k \ , \ n = 0, 1, 2, \ldots \tag{3}$$

It follows from Toeplitz's Theorem (Hardy, 1949, Theorem 2) that the necessary and sufficient conditions for (N, r) to be regular are that

$$\frac{r_n}{R_n} \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{4}$$

and

$$\sum_{k=0}^{n} | r_k | = O (| R_n |). \tag{5}$$

For (M_q) to be regular are that

$$| Q_n | \rightarrow \infty \text{ as } n \rightarrow \infty, \tag{6}$$

and

$$\sum_{k=0}^{n} | q_k | = O (| Q_n |). \tag{7}$$

The product of Nörland-weighted mean methods (N, r) (M_q) may be expressed as the (N, r) transform of (M_q) transform of $\{S_n\}$ and is given by the sequence-to-sequence transformation

$$t_n = \sum_{v=0}^{n} w_{n,v} S_v \ , \tag{8}$$

where

$$w_{n,v} = \frac{q_v}{R_n} \sum_{k=v}^{n} \frac{r_{n-k}}{Q_k} \qquad 0 \leqslant v \leqslant n \tag{9}$$

$$= 0 \qquad v > n. \tag{10}$$

A sequence-to-sequence method A is called translative to the left, if the limitability of S_0, S_1,.........,S_n,......implies the limitability of 0, S0, S_1......,S_{n-1}, to the same limit. A is translative to thc right if the converse holds, A is translative, if it is translative to the left and right.

It is essy to show that every regular (N, r) method is translative. Garabedian and Randels [9 ; Theorem 4] obtained necessary and sufficient conditions for (Mq) to be translative to the right. The author [1 ; Lemma (3.2) obtained necessary and sufficient conditions for (Mq) to be translative to the left.

On translativity of summability methods much work has been done already *e.g*, see [1] , [2] , [3], [4], [5], [6] and [9]. Further, Das [7] has studied the product method for two Nörlund means and obtained many significant results concerning the problem of inclusion and equivalence of the method (N, r) (N, q) with that of Nörland method. Das's results contain special cases of some of the previous results obtained bo Silverman [10] and Silverman and Sza'az [11||. The author [1] and [2] obtained the neeessary and sufficient conditions for (N, r) (N, q) and (M_r) (M_q) to be translative.

2. Object of the paper.

The object of this paper is to obtain the necessary and sufficient conditions for (N, r) (Mq) to be translative. and to show that even if both (N, r) and (M_q) are translative, the product (N, r) (Mq) need not be so, some special non trival cases for (N, r) (M_q) being translative are given. These results will be concluded in sections (4), (5) and (6).

3. Preliminary results :

This section is devoted to results that are necessary for our purposes.

Lemma (3.1) [1 ; Corrolary (2.1)] Suppose that a normal regular summability method (C) is given by the sequence-to-sequence transformation :

$$U_n = \sum_{k=0}^{n} c_{n,k} S_k , \tag{11}$$

such that

$$\sum_{k=0}^{n} c_{n,k} = 1 \qquad (\text{all } n \geq 0). \tag{12}$$

Let $\bar{U}_n$ denote the C-transform of $\{S_{k-1}\}$, and let U_n , $\bar{U}_n$ be obtained from (11) in terms of each other by

$$\bar{U}_{n+1} = \sum_{k=0}^{n} a_{n+1,k} U_k, \quad a_{n+1,k} = 0 \ (k \geq n+1), \tag{13}$$

and

$$U_n = \sum_{k=0}^{n} b_{n,k} \bar{U}_k \ ; \quad b_{n,n+1} = \frac{c_{n,n}}{c_{n+1,n+1}} = \frac{1}{a_{n+1,n}}$$

then (C) is translative to the left if and only if (14)

$$\sum_{k=0}^{n} | a_{n+1,k} | = O(1), \tag{15}$$

and for every fixed k,

$$a_{n+1,k} \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{16}$$

(c) is translative to the right if and only if

$$\sum_{k=0}^{n+1} | b_{n,k} | = O(1), \tag{17}$$

and for every fixed k.

$$b_{n,k} \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{18}$$

Lemma (3.2) [1] ; Theorem (4.2) Let $q_n > 0$ (all $n \geq 0$), and $Q_n \to \infty$ as $n \to \infty$. Then a sufficient condition for (M_q) to be translative is that $\left(\frac{q_{n+1}}{q_n} \right)$ be ultimately monotonic.

4. Main results :

In this section we prove the following two results :

Theorem (4.1) Let (N, r) and (M_q) are both regular, then (N, r) (M_q) is translative to the left if and only if

$$\sum_{u=0}^{n} | A_{n,u} | = O(1), \tag{19}$$

and

$$A_{n,u} \longrightarrow 0 \text{ as } n \rightarrow \infty \tag{20}$$

where
$$A_{n,n} = \frac{q_{n+1} Q_n R_n}{q_n Q_{n+1} R_{n+1}} \tag{21}$$

$$A_{n,u} = \frac{R_n}{R_{n+1}} \sum_{v=u}^{n} Q_v C_{v-u} \Delta_v \left(\frac{q_{v+1}}{q_u} \sum_{k=v}^{n} \frac{r_{n-k}}{Q_{k+1}} \right);$$

$$0 \leq u \leq n-1 \tag{22}$$

where $\{C_n\}$ has to be defined interms of $\{r_n\}$ as in section (1).

and

$$A_{n,u} = 0 \quad u > n. \tag{23}$$

Theorem (4.2) Let (N, r) and (M_q) be both regular, then (N, r) (M_q) is translative to right if and only if

$$\sum_{u=0}^{n} B_{n,u} = O(1), \tag{24}$$

and

$$B_{n\,u}, \longrightarrow 0 \text{ as } n \rightarrow \infty \text{ for every fixed } u, \tag{25}$$

where

$$B_{n,n} = \frac{1}{A_{n,n}}, \tag{26}$$

$$B_{n,u} = \frac{R_{u+1}}{R_n} \sum_{v=u}^{n} Q_{v+1} C_{v-u} \Delta_v \left(\frac{q_v}{q_{v+1}} \sum_{k=v}^{n} \frac{r_{n-k}}{Q_k} \right),$$

$$0 \leqslant v \leq n-1 \tag{27}$$

and

$$B_{n,u} = 0 \quad u > n \tag{28}$$

Proof of Theorem (4.1) Let $\{U_n\}$, $\{\bar{U}_n\}$ be respectively the (M_q) transform of $\{S_n\}$, $\{S_{n-1}\}$. Let $\{t^n\}$, $\{\bar{t}^n\}$ be respectively the (N, r) (M_q) transform of $\{S_n\}$, $\{S_{n-1}\}$. Then

$$U_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k, \tag{29}$$

This gives

$$\bar{U}_{n+1} = \frac{1}{Q_{n+1}} \sum_{k=0}^{n} q_{k+1} S_k. \tag{30}$$

Also

$$t_n = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} U_k, \tag{31}$$

and so

$$\bar{t}_{n+1} = \frac{1}{R_{n+1}} \sum_{k=0}^{n} r_{n-k} \bar{U}_{k+1} \Leftrightarrow \bar{t}^n = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} \bar{U}_k. \tag{32}$$

From (29) obtain S_n in terms of U_n and substituting this in (30) to obtain $\bar{U}_n$ in terms of U_n, the result is

$$\bar{U}_{n+1} = \frac{1}{Q_{n+1}} \sum_{k=0}^{n} \frac{q_{k+1}}{q_k} \left(U_k O_k - U_{k-1} Q_{k-1}\right). \tag{33}$$

The inversion formula of (31) gives

$$U_n = \sum_{k=0}^{n} t_k R_k C_{n-k}, \tag{34}$$

where $\{C_n\}$ is defined in section (1).

Using (33) and (34) to obtain $\bar{U}_{n+1}$ in terms of t_n and substitute this in (32) to obtain $\bar{t}_{n+1}$ in terms of t_n, the result is

$$t_{n+1} = \sum_{u=0}^{n} A_{n,\,u} \bar{t}_{n-1} \tag{35}$$

where $A_{n,\,u}$ is given by (21), (22) and (23).

Using (35) together with Lemma (3.1), the result follows at once.

Proof of Theorem (4.2) Using (29) and (30) to obtain U_n in terms of $\bar{U}_n$, and from (32) obtain $\bar{U}_n$ in terms of $\bar{t}_n$. Substituting this in (31) to get $\bar{t}_n$ in terms of t_n the result is

$$\bar{t}_n = \sum_{u=0}^{n} B_{n,\,v} \bar{t}_{n-1}' \tag{36}$$

where $B_{n,\,u}$ is given by (26), (27) and (28).

Now the result follows on applying Lemma (3.1) to the transformation given by (36).

5. Examples.

In this section we will give two examples to show that even if (N, r) and (M_q) are both translative, the product need not be so.

Example (5.1) Define q_n as follows :

$$q_{2k}=(k+1)^{-1}, \; q_{2k+1}=[(k+1)(k+2)]^{-\frac{1}{2}}, \text{ and } q_{2k+2}=(k+2)^{-1}$$

Then $\left\{ \frac{q_{n+1}}{q_n} \right\}$ is ultimately monotonic, thus by Lemma (3.1). (M_q) is translative. In this case, the author [1 ; section 6] have shown that the method (C, 1) (M_q) is not translative neither to the left nor to the right. As (C, 1) is an (N, r) transform with $r_n=1$ (all $n \geq 1$), this shows that (N, r) (M_q) is not translative.

Example (5.2) Let $q_n=n!$ (all $n \geq 0$), and let $r_0=r_1=1$, $r_n=0$ ($n \geq 1$). Then $\left\{ \frac{q_n+1}{q_n} \right\}$ is monotonic, and thus by Lemm (3.2) (M_q) is translative. Also (N, r) is clearly translative. We will show that neither of the conditions (19), (20), (24) and (25) ars satisfied, Observe that $[\, r(z)\,]^{-1} = C(z)$, we have

$$\sum_{v=u}^{n} r_{n-v} C_{v-u} = 0 \qquad \text{for } n > u \tag{37}$$

$$= 1 \qquad \text{for } n = u. \tag{38}$$

This implies that

$$C_n = (-1)^n \qquad n \geq 0 \tag{39}$$

Write $A_{n,u}$ given in (22) in the form

$$A_{n,u} = \frac{R_u}{R_{n+1}} \Bigg[\sum_{v=u}^{n-2} Q_v C_{v-u} \Bigg(\frac{q_{v+1}}{q_v} \sum_{k=v}^{n} \frac{r_{n-k}}{Q_{k+1}} - \frac{q_{v+2}}{q_{v+1}} \sum_{k=v+1}^{n} \frac{r_{n-k}}{Q_{k+1}} \Bigg)$$

$$+ Q_{n-1} C_{n-1-u} \Big[\frac{q_n}{q_{n-1}} \Big(\frac{r_1}{Q_n} + \frac{r_0}{Q_{n+1}} \Big) - \frac{q_{n+1}}{q_n} \frac{r_0}{Q_{n+1}} \Big]$$

$$+ \frac{q_{n+1}}{q_n} Q_n C_{n-u} \frac{r_0}{Q_{n+1}} \Bigg] \; 0 \leq u \leqslant n-1. \tag{40}$$

Using the hypothesis and (39), it follows from (40) that

$$A_{n,u} = -\frac{R_u}{2} \sum_{v=u}^{n-2} (-1)^{v-u} \Big(\frac{1}{Q_n} + \frac{1}{Q_{n+1}} \Big) Q_v$$

$$+ (-1)^{n-u-1} \frac{R_u}{2} Q_{n-1} \Big(\frac{n}{Q_n} - \frac{1}{Q_{n+1}} \Big)$$

$$+ \frac{R_u}{2} (-1)^{n-u} \frac{n+1}{Q_{n+1}}, \; (0 \leq u \leq n-1).$$

Observe that $Q_n \sim n\,!$ we have that the first term of the right hand side of the latter equation is $\sim R_u\,(2n^2)^{-1}$, and the third term is equivulent to $((n\,!)^{-1})\ \frac{R_n}{2}$. This implies that

$$\frac{A_{n,n}}{(-1)^{n-1-u}} \sim \frac{R_n}{2}.$$

This shows that (19) and (20) are not satisfied. Similarly, (24) and (25) are not satisfied. This completes the proof.

6. Some special cases.

In this section we will give two special cases in which (N, r) (M_q) is translative.

Theorem (6.1). $r_n = q_n = 1$ (all $n \geq 0$). Then (N, r) (M_q) is translative.

Proof. Using the assumption, it may be easily seen that (N, r) (M_q) reduces to Hölder method (H, 2), which is known to be translative.

Theorem (6.2) Let $r_0 = r_1 = 1$, $r_n = 0$ (all $n \geqq 3$), and $q_n = c^n$ (all $n \geq 0$) ; ($c > 1$). Then (N, r) (M_q) is translative.

Proof. The hypothesis shows that (39) is satisfied. Using this, we have from (40) that for $0 \leqslant u \leqslant n-1$,

$$A_{n,\,u} = \frac{R_u}{2}\Bigg[\sum_{v=u}^{n-2} (-1)^{v-u}\,(c^{v+1}-1)(c-1)^{-1}$$

$$\Bigg[\,c\left(\frac{1}{Q_n}+\frac{1}{Q_{n+1}}\right)-c\left(\frac{1}{Q_n}+\frac{1}{Q_{n+1}}\right)\Bigg]$$

$$(-1)^{n-1-u}\,\frac{c^n-1}{c-1}\Bigg[\,c\left(\frac{1}{Q_n}+\frac{1}{Q_{n+1}}\right)-c.\frac{1}{Q_{.+1}}\Bigg)\Bigg]$$

$$+(-1)^{n-u}\,(c^{n+1}-1)\,(c-1)^{-1}.\,c\,\frac{1}{Q_{n+1}}\Bigg]$$

$$=(-1)^{n-u}\,\frac{c^{n+1}\,(c-1)^2\,R_u}{(c^{n+1}-1)(c^{n+2}-1)}$$

This shows that both (19) and (20) are satisfied, and similarly, we can show that (24) and (25) are satisfied. Therefore (N, r) (M_q) is translative.

Lastly, the author would like to express his sincerest thanks to Professor B. Kuttner (Birmingham Universily, Birmingham, U.K.) for his kind encouragement and valuable suggestions which improved the paper.

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Department of Mathematics,
U.A.E. University.
Al-Ain P.O. Box 15551.
United Arab Emirates

The Punjab University
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AN IMPROVED CONDITION FOR SOLVING MULTILINEAR EQUATIONS

by

IOANNIS K. ARGYROS

Department of Mathematics
The University of Iowa
Iowa City, IA, 52242

Abstract. In this paper we improve existing conditions for finding solutions of multilinear equations in Banach space using the contraction mapping principle.

Introduction. We consider the multilinear equation

$$x=y+M(x, x, x, \ldots\ldots\ldots, x) \tag{1}$$

of order k, $k=2, 3, \ldots\ldots$ in a Banach space X, where M is a bounded k-linear operator on X and $y \in X$ is fixed. It is known [2], [4] that if

$$k(k-1).2^{k-1}.\|y\|^{k-1}.\|M\| \leq 1 \tag{2}$$

then a solution x of equation (1) exists and is unique in a certain ball centered at 0, and Newton's literation (or others) converges to such an x.

The purpose of this paper is to improve condition (2). In fact, using the contraction mapping principle we prove existence and uniqueness for a solution x of equation (1) provided that

$$k\left[\frac{k}{k-1}\right]^{k-1} . \|y\|^{k-1} . \|M\| < 1 \tag{3}$$

which improves condition (2) when $k \geq 3$.

As in [3], there is no loss of generality to assume until the end of this paper that M is a bounded *symmetric* k-linear operator on X.

We now prove the theorem.

Theorem. Assume that condition (3) is satisfied and there exists $r > 0$ such that

$$\frac{k\|y\|}{k-1} \leqslant r \leq \sqrt[k-1]{\frac{1}{k\|M\|}}, \; k > 1. \tag{4}$$

Then eqnation (1) has a unique solution x in $\bar{U}(r)$ $=\{x \in X \mid \|x\| \leqslant r\}$.

Proof. Let $r > 0$ be such that (4) is satisfied, then :

Claim 1. The operator T given by

$$T(x) = y + M(x, x, \ldots\ldots, x)$$

is a contraction on $\bar{U}(r)$.

Let $z, \omega \in \bar{U}(r)$.

$$\begin{aligned}\|T(\omega) - T(z)\| &= \|M(\omega, \omega, \ldots\ldots, \omega) - M(z, z, \ldots\ldots, z)\| \\ &= \|M(\omega - z, \omega, \omega, \ldots, \omega) + M(\omega - z, z, \omega, \omega, \ldots, \omega) \\ &\quad + M(\omega - z, z, z, z, \omega, \omega, \ldots, \omega) + \ldots + M(\omega - z, z, z, \ldots, z)\| \\ &\leqslant k\|M\| \, . \, r^{k-1} \, . \, \|\omega - z\|.\end{aligned}$$

Now T is a contraction on $\bar{U}(r)$ by condition (4) and the claim is proved.

Claim 2. T maps $u(r)$ into $\bar{U}(r)$.

we have

$$\begin{aligned}\|T(x)\| &= \|y + M(x, x, \ldots\ldots, x)\| \\ &\leq \|y\| + \|M\| . r^k.\end{aligned}$$

It is enough to show

$$\|y\| + \|M\| . r^k \leqslant r$$

or

$$\|y\|+\|M\|.r\frac{1}{z\|M\|} \leqslant r;$$

the last inequatily is true by condition (4) and the claim is proved.

The result now follows from the contraction mapping priaciple.

Note : For $k \geqslant 3$

$$\left[\frac{k}{k-1}\right]^{k-1} < (k-1)\, 2^{k-1} \qquad (5)$$

Proof. We have for $k \geqslant 3$

$k^{k-1} < (k-1)\,(k+((k-2)^{k-1} = (k-1)\,(2k-2)^{k-1}$

$= (k-1)\,(k-1)^{k-1}\, 2^{k-1}$

Now divide by $(k-1)^{k-1}$ the above inequality to obtain (5).

We now provide a simple example when X=R.

Example. Consider the real equation.

$$x = .4 + 2x^5$$

Here $\|y\| = .4$ $\|M\| = 2$ and $k = 5$. Now condition (2) becomes

$$5.4.2^4.(.4)^4.2 = 16.384 > 1.$$

whereas condition (3) becomes

$$5\left[\frac{5}{4}\right]^4 . |.4|^4.2 = .625 < 1$$

Therefore the iteration schemes in [6] do not apply. whereas Theorem 1 can be applied for $r \in (.5, .562341)$ and the solution obtained is $x = .429093$.

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A NOEE ON QUADRATIC EQUATION IN BANACH SPACE

by

IOANNIS K. ARGYROS

Department of Mathematics
University of Iowa
Iowa City, IA 52242

Abstract. We obtain new lower and upper bounds for the solutions of the quadratic equation in Banach space. We then combine the new results with the already existing to extend the applicability of the existence results.

Introduction. Consider the quadratic equation

$$x=y+B(x,x) \tag{1}$$

in a Banach space X, where B is a bounded bilinear operator on X and $y \in X$ is fixed. In this paper we prove that if x is a solution of (1) then,

$$\|x\| \geq p \tag{2}$$

where $p = \dfrac{-1+\sqrt{1+4\|B\|.\|y\|}}{2\|B\|}$, Moreover, if

$$1-4\|B\|.]y\| > 0 \tag{3}$$

then

$$p \leqslant \|x\| \leq s_1 \text{ or } \|x\| \gtrsim s_2 \tag{4}$$

where

$$s_1 = \frac{1-\sqrt{1-4\|B\|\,.\,\|y\|}}{2\|B\|} \qquad s_2 = \frac{1+\sqrt{1-4\|B.\|y\|}}{2\|B\|}$$

Finally we discuss the effect of these results on known results [1] , [3] for the existence and uniqueness of solution x of (1).

We now state two well known theorems

Theorem 1. If

$$1-4\,\|B\|\,.\,\|y\| > 0 \tag{3}$$

then (1) has a solution x given by

$$x = x_0 + x_1 + \dots\dots + x_n + \dots\dots$$

where

$$x_0 = y$$
$$x_1 = B\,(x_0, x_0)$$
$$x_2 = B\,(x_0, x_1) + B\,(x_1, x_0)$$
$$\vdots$$
$$x_n = \sum_{j=0}^{n-1} B\,(x_j\,,\, x_{n-j-1})$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Moreover x is unique in $U\,(x, r) = \{z \in X \mid \|z-x\| < r)$ where

$$r = \frac{\sqrt{1-4\,\|B\|.\|y\|}}{2\,\|B\|}$$

Theorem 2. If

$$1-4\,\|B\|\,,\,\|y\| > 0 \tag{3}$$

then (1) has a solution x given by

$$x = \lim_{n\to\infty} (y + B\,(x_n\,,\,x_n))$$

for any $x_0 \in U\,(t)$, where

$$s_1 \leqslant t < s_2 - r.$$

Moreover x is unique in $U\,(t)$.

We now prove the theorem.

Theorem 3. Any solution x of (1) is such that

$$\|\,x\,\| > p. \tag{2}$$

Moreover, if

$$1-4\|B\|.\|y\| > 0 \tag{3}$$

then

$$p \leqslant \|x\| \leqslant s_1 \text{ or } \|x\| \geqslant s_2$$

Proof. If x is a solution of (1) then

$$y = B(x, x) - x \Rightarrow$$
$$\|y\| = \|B(x, x) - x\| \leqslant \|B\|.\|x\|^2 + \|x\| \Rightarrow$$
$$\|B\|.\|x\|^2 + \|x\| - \|y\| \geqslant 0 \Rightarrow \|x\| \geqslant p$$

Now

$$x - y = B(x, x)$$
$$\Rightarrow \|B(x. x)\| = \|x - y\| \geqslant \|x\| - \|y\|$$
$$\Rightarrow \|B\| \|x\|^2 - \|x\| + \|y\| \geqslant 0 \Rightarrow \text{(using (2))}$$
$$p \leqslant \|x\| \leqslant s_1 \text{ or } \|x\| \geq s_2.$$

By comparing theorems 1 and 2 with theorem 3 we see on the one hand that (2) is a new bound on the norm of the solution x, on the other hand if (3) holds, theorem 3 extends tne uniqueness of x in U (q) where

$$p \leqslant q < S_2$$

We obtain similar results if we compare the bounds given in theorem 3 with the one's given by Newton's method [3], [4].

We now provide an example. Consider Chandrasekhar's integral equation [1], [3], [4].

$$x(y) = 1 + \lambda\, x(y) \int_0^1 \frac{y}{y+\omega}\, x(\omega)\, d\omega, \ \lambda > 0 \tag{5}$$

in the space C [0, 1] of continuous functions on [0, 1]. Here $\|y\| = 1$ and

$$\|B\| = \lambda \max_{0 \leqslant y \leqslant 1} \int_0^1 \left| \frac{y}{y+\omega} \right| d\omega = \lambda \ 1\, n\ 2.$$

Choose λ=.25, then

$$p = .8691$$
$$s_1 = 1.2870$$
$$s_2 = 4.4837$$
$$r = 1.5983$$

Let $\bar{X} = U(s_2)$, then theorem 3 guarantees uniqueness in $\bar{X}$ whereas theorems, 1 and 2 guarantee uniqueness in the smaller balls $U(x, r)$, $U(t) \subset \bar{X}$ respectively.

However. it can be proved [2] that (5) has a unique solution in X for any $\lambda > 0$. So our example serves only as a comparison between theorem 1 and 2 with theorem 3 and not as a new uniqueness result for (5).

The results obtained here can easily be extended to include equations of the form.

$$x = y + L(x) + B(x, x)$$

where y, B are as before and L is a bounded linear operator on X.

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A NOTE ON MAXIMAL FUNCTION

by

G. M. HABIBULLAH

Departmcnt of Mathematics
Islamia University
Bahawalpur

The 'maximal function' $M(f)$ associated with a measurable fnnction f is defined by

$$(1) \qquad Mf(x) = \sup_{0 \leqslant t \leqslant x} \left| (x-t)^{-1} \int_t^x f(y)\, dy \right|, \; x > 0.$$

The well-known inequalitly of Hardy-Littlewood states

$$(2) \qquad \|M(f)\|_p \leqslant p' \, \|f\|_p, \; (1 < p \leqslant \infty)$$

where $1/p + 1/p' = 1$, and $\|f\|_p$ is the usual norm on L^p.

In this note we consider an extension of the operator in (1) by defining

$$(3) \qquad M^{\alpha}_{\lambda}(f)(x) = \left(\int_0^x \left| (x-t)^{\alpha-1} \int_t^x f(y)\,(dy) \right|^{\lambda} \right)^{1/\lambda}$$

$M(f)$ corresponds to $M^{\alpha}_{\lambda}(f)$ with $\alpha = 0$, $1/\lambda = 0$. Sadosky (5) considered the case $1/\lambda = 0$, $\alpha < 0$ and when $\alpha = 0$, $M^{\alpha}_{\lambda}(f)$ reduces to an operator studied by okikiolu [3, pp. 264].

We also study n-diamensional form of (3) defined by

$$(4) \qquad N^{\alpha}_{\lambda}(f)(x) = \int_0^{\infty} \left| t^{(a-1)n + (n-1)/\lambda} \times \int_{|y| \leqslant t} f(x-t)\, dy \right|^{\lambda} dt)^{1/\lambda}, f \in L^p(E_n).$$

When $\alpha=0$, $1/\lambda=0$, $N^{\alpha}_{\lambda}(f)$ reduces to the n-dimensional maximal function considered by Calderon and Zygmund [2]. We use this function to determine certain estimates for Poisson operator. As in the case of $M(f)$, the main results involving $M^{\alpha}_{\lambda}(f)$ and $N^{\alpha}_{\lambda}(f)$ can be proved on more general measure spaces simply by replacing Euclidean spheres by suitable spheres in the metric space concerned.

We need the following result due to Okikiolu [4].

Lemma 1. Let $f \in L^{p}(0, \infty)$, $p > 1$, $0 \leqslant \beta \leq 1/p$, $1/q=(1/p)-\beta$.

(5) Let $A(f)(x)=x^{\beta-1}\int_{0}^{x} f(y)\, dy$.

Then there is a constant $k(p, \beta)$ indepenent of f such that

(6) $\|A(f)\|_{q} \leq k \|f\|_{p}$.

Throughout, $k-k(\alpha, \beta \ldots\ldots)$ denotes positive constant depending upon parameters involved, not necessarily the same at each occurrence.

Lemma 2. For $0 < \lambda_1 \leq \lambda_2$, we have

(7) $$x^{-\lambda_1} M^{\alpha}_{\lambda_1}(f)(x) \leq \bar{x}^{\lambda_2} M^{\alpha}_{\lambda_2}(f)(x),\ x>0.$$

The result is easily verified by applying Holder's inequality to the expression.

$$M^{\alpha}_{\lambda_1} f(x) = \int_{0}^{x} |(x-t)^{\alpha-1} \int_{t}^{x} f(y)\, dy|^{\lambda_2 (\lambda_1/\lambda_2)}\, dt.$$

Lemma 3. Let $0 \leqslant \nu \leqslant 1$, $1/\lambda = \nu/\lambda_1 + (1-\nu)/\lambda_2$, $0 < \lambda_1$, $\lambda_2 \leq \infty$. Then

$$M_{\lambda}^{\alpha}(f) \leqslant M_{\lambda_1}^{\alpha}(f)^{\nu}\, M_{\lambda_2}(f)^{1-\nu}. \tag{8}$$

Proof. The result is obtained using Holder's inequality on the expression.

$$M_{\lambda}^{\alpha}(f)(x)^{\lambda} = \int_0^x |(x-t)^{\alpha-1} \int_t^x f(y)|^{\beta\lambda_1} \; |(x-t)^{\alpha-1} \times \int_t^x f(y)\, dy|^{(1-\beta)\lambda_2}\, dt.$$

where $\beta = (\nu\lambda)/\lambda_1$, $(1-\beta) = (1-\nu)\lambda/\lambda_2$.

Theorem 1. Let $f \in L^p(0, \infty)$, $0 \leq \alpha \leqslant 1/p < 1$, $\lambda > -1/(1-a)$, $1/r = 1/p - \alpha$. Then

$$\|x^{-1/\lambda}\, M_{\lambda}^{\alpha}\, f(x)\|_q \leq k(\lambda, \alpha, p)\, \|f\|_p. \tag{9}$$

Proof. Since, by definition.

$$M_{\lambda}^{\alpha}(f)(x) \leqslant (\alpha\lambda - \lambda + 1)^{-1}\, x^{\alpha-1+1/\lambda} \int_0^x |f(y)|\, dy$$

the result is obtained by using Lemma 1 with

$$k = \{(1-\alpha)p'\}^{1-\alpha}.$$

Theorem 2. Let $f \in L^p(0, \infty)$, $p > 1$. $0 \leqslant \alpha + 1/\lambda \leqslant 1/p < 1$, $1/q = (1/p) - 1/\lambda - \alpha$. Then

$$\|M_{\lambda}^{\alpha}(f)\|_q \leqslant k(\alpha, p, \lambda)\, \|f\|_p. \tag{10}$$

Proof. If f^* is non-increasing rearrangement of $-f$ [8],

$\|f\|_p = \|f^*\|_p$, $f^*(\tau) \leq \tau^{-1} \int_0^{\tau} f^*(s)\, ds$, $(fg)^*(\tau_1 + \tau_2) \leqslant f^*(\tau_1) g^*(\tau_2)$, so that

$$(M^{\alpha}_{\infty}(f))^{*}(\tau) \leq (\tau/2)^{\alpha}\ (M(f))^{*}(\tau/2)$$

$$\leqslant k\,\tau^{\alpha-1}\int_{0}^{\tau}\ (M(f))^{*}(s)\,ds.$$

Hence by Lemma I we have, for $1/r=1/p-\alpha>0$,

$$\|M^{\alpha}_{\infty}(f)\|_{r}=\|(M^{\alpha}_{\infty}(f))^{*}\|_{r} \leq k(\alpha, p)\|(M(f))^{*}\|_{p}$$

$$=k(\alpha, p)\|M(f)\|_{p}$$

$$\leqslant k(\alpha, p)\|f\|_{p}$$

Also,

$$M^{\alpha}_{r}(f)(x)=\Big(\int_{0}^{x}\ \Big|(x-t)^{\alpha-1}\int_{t}^{x} f(y)\,dy\Big|^{r}\ dt\Big)^{1/r}$$

$$=\Big(\int_{0}^{x}\ \Big|\ u^{\alpha-1}\int_{0}^{u} f(x-u)\,dy\Big|^{r}\ \Big)^{1/r}.$$

Again, if $1/r=1/p-\alpha$, use of Lemma 1 yields.

$$\|M_{r}(f)\|_{\infty} \leq k(\alpha, p)\|f\|_{p}.$$

Now with $1/q=1/r-1/\lambda$, we use Lemma 3 to get

$$|M^{\alpha}_{\lambda}(f)| = |M^{\alpha}_{r}(f)^{r/\lambda}\ M^{\alpha}_{\infty}(f)^{1-r/\lambda}|$$

$$\leqslant k\,\|M^{\alpha}_{r}(f)\|_{\infty}^{r/\lambda}\ \ |M^{\alpha}_{\infty}(f)|^{r/q}$$

Hence

$$\|M^{\alpha}_{\lambda}(f)\|_{q} \leq k\,\|f\|_{p}^{r/\lambda}\ \|M^{\alpha}_{\infty}(f)\|_{q}^{r/q}$$

$$\leqslant k\,\|f\|_{p}$$

Theorem 3. Let $f \in L^{1}(0, \infty)$, $0 < p < 1$ and let S be any measurable set in $(0, \infty)$, then $\lambda > -1/\alpha$, we have

$$\int_S |x^{-\alpha-1/\lambda} M_\lambda^\alpha (f)(x)|^p \, dx \leq k(\alpha, p) (m(S))^{1-p} \times$$

$$(\int_0^\infty | f(x) | \, dx)^p.$$

Proof. Since $x^{-\alpha-1/\lambda} M_\lambda^\alpha (f)(x) \leqslant M(f)(x)$, we get the result by using a similar known result on M (f).

Theorem 4. Let $f \in L^p(-\infty, \infty)$ $p > 1$, $0 < \alpha \leqslant 1$, $1/q = 1/p - 1/\lambda - \alpha > 0$. Then there is a constant $k = k(p, \lambda, \alpha)$ subh that $\|T_\lambda^\alpha (f)\| \leqslant k \|f\|_p$, where $T_\lambda^\alpha (f)$ is defined by

(11) $$T_\lambda^\alpha (f)(x) = \left(\int_{-\infty}^{\infty} \left| \frac{F(x) - F(x-t)}{t^{-\alpha+1}} \right| di\right)^{1-\lambda}$$

and $F(x) = \int_0^x f(t) \, dt.$

Proof. It is clearly sufficient to prove.

$$(\int_0^\infty (T_\lambda^\alpha (f)(x)^q dx)^{1/q} \leqslant k \|f\|_p$$

A similar result involving $(\int_{-\infty}^0 (T_\lambda^\alpha (f)(x))^q (dx)^{1/q}$ can be obtained by changing variables and considering $f(-t)$ in place of $f(t)$. Then

$$\lambda \geq p \geq 1, T_\lambda^\alpha (f)(x)^\lambda = \int_0^\infty (t^{\alpha-1} \int_{x-t}^x f(y) \, dy)^\lambda \, dt +$$

$$\int_0^\infty (t^{\alpha-1} \int_x^{x-t} f(y) \, dy)^\lambda \, dt$$

$$= I_1^\lambda + I_2^\lambda,$$

and using the inequality $(a+b)^{1/\lambda} \leqslant a^{1/\lambda} + b^{1/\lambda}$, $a > 0$, $b > 0$, we have $T^{\alpha}_{\lambda}(f) \leqslant I_1+I_2$. Estimates for I_1 and I_2 can be obtained as for $N^{\alpha}_{\lambda}(f)$ given in the next theorem.

Theorem 5. Let $f \in L^p(E^n)$, $n \geq 1$, $p \geqslant 1$, $0 \leqslant \alpha < 1$, $1/q=(1/p)-(1/\lambda)-\alpha > 0$. Then there is a constant $k=k(n, p, \lambda, \alpha)$ such that

$$(12) \qquad \|N^{\alpha}_{\lambda}(f)\|_q \leqslant k \, \|f\|_p$$

Proof. Since $\int_{|y| \leqslant t} f(x-y)\, dy = \int_0^{t^n \omega n} f^*(s)\, ds$ where ω_n represents the voiume of the unit sphere in E_n, f^* is the non-increasing rearrangement of f, it follows that

$$N^{\alpha}_{\lambda}(f)(x) \leqslant \left(\int_0^{\infty} \left(t^{(\alpha-1)n+(n-1)/\lambda} \int_0^{t^n \omega_n} f^*(s)\, ds\right)^{\lambda} dt\right)^{1/\lambda}$$

$$= \omega_n^{1-1/\lambda_n - 1} \left(\int_0^{\infty} \left(u^{a-1} \int_0^{n} f^*(s)\, ds\right)^{\lambda} du\right)^{1/\lambda},$$

and if $1/r = 1/p - \alpha$, Lemma 1 yields

$$\|N^{\alpha}_{r}(f)\|_{\infty} \leqslant \omega_n^{1-\lambda_n - 1/\lambda} \, k(\alpha, q) \, \|f^*\|_p$$

$$\leqslant k(\alpha, p, n, \lambda) \, \|f\|_p.$$

Also, as in Theorem 2,

$$\|N^{\alpha}_{\infty}(f)\|_r \leqslant k(\alpha, p, n) \|f\|_p$$

Now for $r < \lambda < \infty$, we have

$$N^{\alpha}_{\lambda}(f)(x)^{\lambda} \leqslant \left(\sup \left| t^{(\alpha-1)n} \int_{|y| \leqslant t} f(x-y)dy\right|^{\lambda-r}\right)$$

$$\int_0^{\infty} \left| t^{(\alpha-1)n+(n-1)/r} \int_{|y| \leqslant t} f(x-y) \right|^{r} dt)$$

$$\leqslant N^{r}_{\infty}(f)(x)^{\lambda r/q}\, N^{\lambda}_{r}(f)(x)^{r},$$

and

$$\|N^{\alpha}_{\lambda}(f)\|_q \leqslant k(p, \lambda, \alpha, n)\|f\|_{p}.$$

We now prove some results involving the extension of the maximal function. These applications involving Poisson operator are analogous to the estimates majorized by M (f). [Stein 5, pp.62]. Similar results involving the Weierstrass can also be proved.

Theorem 6. Let the function $\varnothing_a(t)$, $a > 0$, $t \in (-\infty, \infty)$ be measurable on $(0, \infty) x (-\infty, \infty)$ and be absolutely continuous in t. Assume that for each fixed number a, we have

(*i*) $\varnothing_a \in L^{p'}$,

(*ii*) $|t|^{1/p'} \varnothing_a(t) \to 0$ as $|t| \to \infty$,

(*iii*) $t^{1-\alpha}\varnothing'_a \in L^{\lambda'}$, where $p \geqslant 1$, $\lambda \geqslant 1$.

Let the eperator $\varnothing_a$ be defined by

(13) $$\varnothing_a(f)(x) = \int_{-\infty}^{\infty} \varnothing_a(t) f(x-t)\, dt.$$

and

$$t(a) = \left(\int_{-\infty}^{\infty} |t^{1-\alpha} \varnothing'_a(t)|^{\lambda'} dt\right)^{1/\lambda'}.$$

Then

(14) $$\sup_{a>0} \tau(a)^{-1} |\varnothing_a(f)(x)| \leqslant T^{\alpha}_{\lambda}(f)(x).$$

F rmore, for $p > 1$, $\lambda \geq p$, $1/q = 1/p - 1/\lambda - \alpha > 0$, $0 \leqslant \alpha < 1$, there is a constant $k=k(p, r, \alpha, \lambda)$ such that

(15) $$\|\sup_{a>0} \tau(a)^{-1} \mid \phi_a(f)\|_q \leqslant k \| f \|_p.$$

Proof. Since $\varnothing_a \in L^{p'}$, it is clear that $\varnothing_a(f)$ is defined for $f \in L^p$, in fact $\varnothing_a(f)$ is continuous on $(-\infty, \infty)$. Tbus integrating by parts we have

$$\Phi_a(f)(x) = \int_{-\infty}^{\infty} \varnothing_a(t)\,(d/dt)\int_{x-t}^{x} f(y)\,dy)dt$$

$$= \int_{-\infty}^{\infty} d/dt\,(\varnothing_a(t)\int_{x-t}^{x} f\,(y\,dy) - \int_{-\infty}^{\infty} \varnothing'_a(t)$$

$$(\int_{-\infty}^{x} f(y)\,dy)\,dt.$$

Applying the Hõlder inequality, it follows

$$\mid \varnothing_a(t) \int_{x-t}^{x} f(y)\,dy \mid \leqslant t^{1/p'} \mid \varnothing_a(t) \mid \| f \|_p \to 0 \text{ as } t \to \infty,$$

so that

$$\mid \varnothing_a(f)(x) \mid = \mid -\int_{-\infty}^{\infty} \phi'_a(t) \int_{x-t}^{t} f(y)\,dy \mid$$

$$\leqslant \tau(a)\,(\int_{-\infty}^{\infty} \mid t^{\alpha-1} \int_{x-t}^{x} f(y)\,dy \mid^{\lambda} dt)^{1/\lambda}$$

This clearly proves the first result of the theorem. The second part follows from Theorem 4.

Corollary. If $f \in L^p(-\infty,\infty)$, $p > 1$, $\lambda \geq p$, $0 \leqslant \alpha < 1$, $1/q = 1/p - 1/\lambda - \alpha > 0$, then there is a constant $k=k(p, \lambda, \alpha)$ such that

(16) $$\|\sup_{a>0} a^{\alpha+1/\lambda} p_a(f)\|_q \leqslant k \| f \|_p ,$$

where

$$P_a(f)(x) = 1/\pi \int_{-\infty}^{\infty} \frac{a}{a^2+(t-x)^2} f(t)\,dt,\ a>0.$$

It can easily be verified the kernel of $P_a(f)$ satifies the conditions of Theorem 6 and

$$\left\{ \int_{-\infty}^{\infty} \left| t^{1-\alpha}\, d/dt \frac{a}{(a^2+t^2)} \right|^{\lambda'} dt \right\}^{1/\lambda'}$$

$$= 2a^{-\alpha-1/\lambda} \left(\int_{-\infty}^{\infty} \left| \frac{t^{2-\alpha}}{(1+t^2)^2} \right|^{\lambda'} dt \right)^{\lambda'}$$

$$= a^{-\alpha-1/\lambda} k(\alpha, \lambda),$$

since the last integral is convergent.

We now prove the estimate involving the n-dimensional form of the Poisscn operator defined by

$$P_a(f)(x) = c_n^{-1} \int_{E_n} a\,(a^2+|y|^2)^{-(n+1)/2} f(x-y)\,dy,$$

where

$$c_n = \pi^{(n+1)/2}\,\Gamma_{(n+1)/2}$$

Lemma 4. Let $f \in L^p(E_n)$, $n \geq 1$, $n > 1$, $p > 1$, $0 < \alpha < 1$, $\lambda \geqslant p$, then

$$\left(\int_0^{\infty} | a^{n\alpha+(n-1)\lambda} P_a(f)(x)|^{\lambda}\, da \right)^{1/\lambda}$$

$$\leq k(n, \lambda, \alpha)\, N_{\lambda}^{\alpha}(f)(x),$$

where

$$k(n, \lambda, \alpha) = 1 + 2^{n+1}(2^{1+n\alpha+n/\lambda} - 1)^{-1} .$$

Proof. Using the argument as given in [7], pp.44] (for $f \geq 0$ which we may assume, we have

$$c_n P_a(f)(x) = a\left(\int_{|y| \geq a} + \int_{|y| > a}\right) f(x-y)(a^2 + |y|^2)^{(n+1)/2}$$

$$\leq a^{-n} \int_{|t| \leqslant a} f(xt)\, dt + a \int_{|t| > a} f(x-t)\, t^{-n-1} dt$$

and

$$a \int_{|t| > a} f(x-y)\, |y|^{-n-1}\, dy$$

$$= a \sum_{m=1}^{\infty} \int_{a < |y| < 2^m a}^{2^{m-1}} f(x-y)\, |y|^{-n-1} dy$$

$$\leqslant \sum_{m=1}^{\infty} a(2^{m-1} a)^{-n-1} \int_{|y| \leqslant 2^{m} a} f(x-y)\, dy.$$

Hence for $1 < \lambda < \infty$, $0 \leqslant \alpha < 1$

$$c_n \left(\int_0^{\infty} | a^{n\alpha+(n-1)/\lambda} P_a(f)(x)|^{\lambda}\right)^{1/\lambda}$$

$$\leqslant c_n \left(\int_0^{\infty} | a^{(\alpha-1)n+(n-1)/\lambda} \int_{|y| \leqslant a} (f(x-y)\, dy|^{\lambda}\, da\right)^{1/\lambda}$$

$$+ 2^n \sum_{m=1}^{\infty} 2^{1-m-nm\alpha} 2^{-m(n-1)/\lambda} \left\{\int_0^{\infty} | (2^m a)^{(\alpha-1)n+(n-1)/\lambda} \int_{|t| \leq 2^m a} f(x-t)\, dt|^{\lambda}\, da\right\}^{1/\lambda}$$

$$< N_{\lambda}^{\alpha}(f)(x) + 2^{n+1} \sum_{m=1}^{\infty} (1/2)^{m(1+n\alpha+n/\lambda)} N_{\lambda}^{\alpha}(f)(x).$$

Theorem 7. Let $f \in L^p(E_n)$, $n \geq 1$. $p \geqslant 1$, $\lambda \geqslant p$, $0 \leq \alpha < 1$,

$1/q = 1/p - 1/\lambda - \alpha > 0$. Then there are constants $k_1 = k_1(n, \lambda, \alpha)$ and $k_3 = k_2(n, \lambda, \alpha, p)$ such that

(17) $$\sup_{a>0} a^{n(\alpha+1/\lambda)} | p_a(f)(x) | \leq k_1 N_\lambda(f)(x),$$

and

(18) $$\| \sup_{a>0} a^{n(\alpha+1/\lambda)} p_a(f)(x) \|_q \leq k_2 \| f \|_p \, .$$

Proof. We can clearly assume $f \geq 0$. Since the function

$$a^{n+1} / (a^2+y^2)^{(n+1)/2} = (1+(y/a)^2)^{-(n+1)/2}$$

is increasing in a, it follows that $a^n \, p_a(x)$ is an increasing function on $(0, \infty)$. Thus for

$$(v-1)^{-1} a^{n+1-v} p_a(f) = a^n p_a(f) \int_a^\infty y^{-v} \, dy$$

$$\leqslant \int_a^\infty y^{n-v} p_y(f) \, dy$$

and using the Holder's inequality we have

$$(v-1) a^{n+1-v} p_a(f) \leqslant \left(\int_0^\infty y^{(n-n\alpha-n-1/\lambda-v)\lambda'} dy \right)$$

$$\left(\int_0^\infty (y^{n+(n-1)/\lambda} p_y f)^\lambda \, dy \right)^{1/\lambda} .$$

If we choose v such that $n - n(\alpha+1/\lambda)+1-v < 0$, then an application of Lemma 4 proves

(*i*) the second result Is an immediate consequence of Theorem 5.

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ON EXACT AND ASYMPTOTIC MOMENTS OF INVERSE OF MEAN OF A NORMAL POPULATION

By

MUNIR AHMAD AND M. H. KAZI

University of Petroleum and Minerals
Dhahran, Saudi Arabia

Abstract.

In this paper, we obtain the exact expressions for the higher moments of the Srivastava and Bhatnagar (1981) estimator of the inverse of mean of a normal population with mean μ and variance σ^2. We also derive an asymptotic expression for the higher moments of the maximum likelihood estimator of the inverse of mean of the normal population using saddle point method and compare the results with those of Srivastava and Bhatnagar estimator.

1. Introduction.

Recently Srivastava and Bhatnagar (1981) proposed a class of estimators of the inverse of mean. Exact expressions for the first two moments were derived in case of normal population and large sample expressions for non-normal populations. In this note, we obtain exact expressions for the higher moments of the Srivastava and Bhatnagar (1981) estimator which will subsequently be called the S-B estimator and compute its efficiency and relative bias for some values of parameters. We also derive an asymptotic expression for the higher moments of the maximum likelihood estimator of the

Keywords : Moments of inverse mean, efficiency, relative bias, measures of skewness, normal population, saddle point method.

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inverse of mean using saddle point method for large n and compare the results with those of the S-B estimator.

2. Exact Expression for Higher Moments of S-B Estimator.

Srivastava and Bhatnagar (1981) proposed an estimator $t_k = n\bar{x}/(n\bar{x}^2+ks^2)$ for $k>0$ of inverse of mean μ of a normal population with unknown variance σ^2, where $\bar{x}$ and s^2 are unbiased estimators of μ and σ^2 respectively. When σ^2 is known, they considered $t_g = n\bar{x}/(n\bar{x}^2+g\sigma^2)$ for $g>0$ as an estimator of μ^{-1}. When σ^2 is unknown Srivastava and Bhatnagar (1981) take

$$t_k = \mu^{-1}\left(\frac{n}{\theta}\right)^{\frac{1}{2}}\frac{z}{z^2+w}\left[1-\left(1-\frac{k}{n-1}\right)\frac{w}{z^2+w}\right]^{-1} \quad (1)$$

where $z=\sqrt{n}\,\bar{x}/\sigma$, $\theta=\sigma^2/\mu^2$ and $w=(n-1)\,s^2/\sigma^2$. The random variable Z follows a normal distribution with mean $\sqrt{n/\theta}$ and variance 1 and the random variable W is a x^2—distribution with $(n-1)$ degrees of freedom. Srivastava and Bhatnagar (1981) obtained the exact expression for the mean and variance of their estimator. In this section, we obtain exact expression for higher moments of their estimator.

By definition, the rth moment about origin is

$$\mu_r' = E(t_k^r) = \int_{-\infty}^{\infty}\int_{0}^{\infty} t_k^r f(z, w)\, dw\, dz \quad (2)$$

where t_k is given (1) and

$$f(w, z) = \left[2^{n/2}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\right]^{-1} w^{(n-3)/2}\exp\left[\left(-\tfrac{1}{2}(z-\sqrt{n/\theta})^2+w\right]\right.$$

We now write

$$t_k^r = \mu^{-r}(n/\theta)^{r/2}\left(\frac{z}{z^2+w}\right)^r\sum_{\alpha=0}^{\infty}\binom{r+\alpha-1}{\alpha}\left(1-\frac{k}{n-1}\right)^{\alpha}\left(\frac{w}{z^2+w}\right)^{\alpha}$$

and

$$\exp\left[-\tfrac{1}{2}(z-\sqrt{n/\theta})^2+w\right]=e^{-\frac{1}{2}(z^2+w)}\sum_{j=0}^{\infty}\frac{(\sqrt{n/\theta})^{j}}{j\,!}$$

We substitute these values in (2) and obtain

$$\mu_r'=\sum_{\alpha=0}^{\infty}\sum_{j=0}^{\infty}K_\alpha\frac{(\sqrt{n/\theta})^{j}}{j\,!}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{z^{r+j}w^{\alpha+\frac{1}{2}(n-3)}}{(z^2+w)^{r+\alpha}}\times e^{-\frac{1}{2}(z^2+w)}\,dw\,dz \qquad (3)$$

where

$$K_\alpha=\left[2^{n/2}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}e^{-n/2\theta}\,(n/\theta)^{r/2}\binom{r+\alpha-1}{a}\left(1-\frac{k}{n-1}\right)^{\alpha}$$

The integral in (3) will vanish when $r+j$ is odd. There are two cases namely (*i*) r is odd and (*ii*) r is even.

Case (i) : *r is odd.* Let $r=2m+1$. The integral (3) will vanish for even integral values of j. Thus the equation (3) reduces to

$$\mu'_{2m+1}=\sum_{\alpha j}\sum K_\alpha\frac{(\sqrt{n/\theta})2n+1}{(2j+1)\,!}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{z^{2m+1+2j}w^{\alpha+\frac{1}{2}(n-3)}}{(z^2+w)^{2m+\alpha+1}}\times e^{-\frac{1}{2}(z^2+w)}dwdz$$

Apylying the transformation

$$\begin{aligned} z^2&=y_1y_2 \qquad & 0\le y_1\le\infty \\ w&=y_1(1-y_2) \qquad & 0\le y_2<1, \end{aligned} \qquad (4)$$

and simplifying, we have

$$\mu'_{2m+1}=\sum_{\alpha=0}^{\infty}\sum_{j=0}^{\infty}K_\alpha\frac{(\sqrt{n/\theta})2j+1}{(2j+1)\,!}\,2^{a}j\;\Gamma(a)\,\beta\,(b_j,\,c_a) \qquad (5)$$

where $\Gamma(a)$ and $\beta(b, c)$ are the gamma and beta functions respectively, $a_j = j - m + n/2$, $b_j = j + m + 3/2$ and $c_\alpha = \alpha + n/2 - \frac{1}{2}$. If $m = 0$, we obtain the S-B expression for $E(t_k)$.

Case (ii) : *r is even.* Let $r = 2m$, The integral in (3) vanishes for odd integral values of j we obtain

$$\mu'_{2m} = \sum_\alpha \sum_j K_\alpha \frac{(\sqrt{n/\theta})^{2j}}{(2j)\,!} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{(z^2)^{j+m} w^{\alpha+\frac{1}{2}(n-3)}}{(z^2+w)^{2m+\alpha}} \times e^{-\frac{1}{2}(z^2+w)} dw dz$$

Applying the transformation (4), we get

$$\mu'_{2m} = \sum_\alpha \sum_j K_\alpha \frac{(n/\theta)^{2j}}{(2j)\,!} \; 2^a \; \Gamma(a_j)\, \beta\,(b_j - 1,\, c_\alpha) \qquad (6)$$

If $m = 1$, we obtain the S-B expression for $E(t_k^{\,2})$.

Using these expressions, we can derive exact expressions for relative bias, relative mean square error and measures of skewness and kurtosis.

2. Asymptotic Expressious for higher moments of Estimator of Inverse of Mean by Sadddle-point Method.

We know that $(\bar{X})^{-1}$ is the maximum likelihood estimator of $(\mu)^{-1}$ but finite moments of $(\bar{X})^{-1}$ do not exist. However, we can find asymptotic expression for moments of $(\bar{X})^{-1}$ for large values of n by using saddle point method, (See Daniel (1958) and Copson (1976) for the details of the saddle point method).

By definition, the rth moment of the reciprocal of mean is

$$\frac{1}{\sqrt{n}} E(\bar{X})^{-r} = -K(\bar{x})\, d\bar{x} \qquad (7)$$

where $K(\bar{x}) = (\sqrt{2\pi}\,\sigma)^{-1} (\bar{x})^{-r} \exp\left[-\frac{n}{2}\left(\frac{\bar{x}-\mu}{\sigma}\right)^2\right]$. $K(\bar{x})$ appears singularity at $\bar{x}=0$. However, it is easy to see that $\lim_{\bar{x}\to 0} K(\bar{x})=0$ and if we assume that $K(\bar{x})=0$ at $\bar{x}=0$ the function becomes constinuous at $\bar{x}=0$ and it is possible to evaluate the integral asymptotically, using the saddle point method.

Consider the integral

$$I= \int_c g(z) \exp [ph(z) \, dz \tag{8}$$

where c is the path of integration in the z-plane along the real axis and the functions $g(z)$ and $h(z)$ are functions of the complex variable z which in a special case may involve only real values of z. In order to evaluate the integral asymptotically for large values of ρ, the path of integration is deformed to satisfy the following conditions :

(*i*) the path passes through the root z_0 (called saddle point) of $h'(z)=0$.

(*ii*) the imaginary part of $h(z)$ is constant on the path.

If we write $h(z)=h_1+ih_2$ where h_1 and h_2 are real, h_2 is constant on a path of steepest descent, then the dominant part of the asymptotic expansion arises from the part of the path near the highest saddle point. If the path c is deformed to pass through the saddle point, then the integral will be obtained in the neighbourhood of the saddle point. The saddle point is obtained by solving $h'(z)=0$ and the path of integration (3) will be the locus of the points determined by the equation

$$h(z)=h(z_0)-s^2, \qquad -\infty < s < +\infty \tag{9}$$

The saddle point corresponds to the value $s=0$. The integral (8) taken over c is now replaced by the integral of the same integrand over the new path of integration given by the equation (9) which transforms z to s given by $\phi(s)\equiv g(z)\,(dz/ds)$ and the dominant contribution to the integral now stems from the vicinity of saddle point. The integral (8) is written as

$$I=\int_{-\infty}^{\infty} \exp\,[\rho(h(z_0)-s^2)]\,\phi(s)ds$$

$$=\exp\,[\rho\, h(z_0)]\int_{-\infty}^{\infty} e^{-\rho s^2}\,\phi(s)\,ds \qquad (10)$$

For large value of ρ, only small values of s will contribute significantly to the integral. Expanding $\phi(s)$ in a series of powers of s, substituting in (10) and integrating over s, and using the formulae $\int_{-\infty}^{\infty} s^m e^{-\rho s^2} ds=0$ when m is odd and $\int_{-\infty}^{\infty} s^m e^{-\rho s^2} ds=\dfrac{\sqrt{2\pi}\; m\,!\,(\sqrt{2\rho})^{-m-1}}{2^{m/2}\,(m/2)\,!}$ when m is even, we obtain the following asymptotic expansion of the integral for large values of ρ :

$$I=e^{\rho h(z_0)}\,\frac{\pi}{\rho}\,[\phi(0)+\sum_{K=1}^{\infty}\phi^{(2K)}(0)/[2^{(2K)}\rho^K K\,!]\,] \qquad (11)$$

where

$$\phi^{(K)}(0)=\frac{d^K}{ds^K}\,[\phi(s)]|_{s=0},\; K=0,1,2,\ldots\ldots\ldots.$$

In case of the integral (7), we have $g(z)=\dfrac{(z)^{-r}}{\sqrt{2\pi}\,\sigma}$, $h(z)=-\dfrac{1}{2\sigma^2}(z-\mu)^2$ and $\rho=n$. The saddle point is $\bar{z}_0=\mu$ and also $h(z_0)=0$.

The transformation is

$$\bar{z}=(\mu+\sqrt{2}\;\sigma s)$$

and

$$\phi(s) = \frac{1}{\sqrt{n}} (\mu + \sqrt{2}\ \sigma s)^{-r}$$

Substituting these values in (11) we obtain for large n

$$\mu_r' = \mu^{-r}\left[1 + \sum_{j=1}^{\infty} \frac{(r)_{2j}}{(2^j)\, n^j} \left(\frac{\sigma}{\mu}\right)^{2j}\right]$$

where $(a)_K = \alpha(\alpha+1)\ldots\ldots\ldots\ldots(a+K-1)$.

Using the first two terms of the summation, we obtain

$$\mu_r' \sim \mu^{-r}\left[1 + \frac{r(r+1)}{2n}\left(\frac{\sigma}{\mu}\right)^2 + \frac{r(r+1)(r+2)(r+3)}{8n^2}\left(\frac{\sigma}{\mu}\right)^4\right]$$

and if $r=1$ and 2, we get

$$\mu_1' \sim \mu^{-1}\left[\left(1 + \frac{1}{n}\left(\frac{\sigma}{\mu}\right)^2 + \frac{3}{n^2}\left(\frac{\sigma}{\mu}\right)^4\right]\right.$$

and

$$\mu_2' \sim \mu^{-2}\left[1 \times \frac{3}{n}\left(\frac{\sigma}{\mu}\right)^2 + \frac{1}{n^2}\left(\frac{\sigma}{\mu}\right)^4\right]$$

If μ and σ^2 are unknown, these can be replaced by their unbiased estimators or $\frac{\sigma}{\mu}$ is replaced by its consistant estimator $z\sqrt{x}$. It may be noted that the asymptotic relative bias is identical to that given by Srivastava and Bhatnagar (1981) and the asymptotic relative mean square error is $\theta/n + 8\theta^2/n^2$ where $\theta = (\sigma/\mu)^2$ whereas the relative mean square given by Srivastava and Bhatnagar (1981) reduces to $\theta/n + 9\theta^2/n^2$ when $K=0$. The discrepancy in the second terms of the asymptotic formulae of relative mean squared error of t_K for large n obtained by Srivastava and Bhatnagar (1981) and us stems from the fact that a and b in the formula from Copson (1948, p. 265).

$$e^{-n/2\theta} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(b+j)} \frac{(n/2\theta)^j}{j\,!} = \left(\frac{2\theta}{n}\right)^{b-a}\left[1 - \frac{2(b-a)(a-1)\,\theta}{n} + \frac{2(b-a)(b-a+1)(a-1)(a-2)\,\theta^2}{n^2} \ldots\ldots(1)\right]$$

are independent of n, whereas in the case of Srivastava and Bhatnagar (1981) are dependent upon n.

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NUMERICAL LAPLACE TRANSFORM INVERSION BY A REGULARIZATION METHOD

By

M. IQBAL

Department of Mathematics
Punjab University, New Campus,
Lahore, Pakistan

1. Introduction :

In the terminology of ill-posed problems the Laplace transform inversion is a severely ill-posed problem. Unfortunately, many problems of physical interest lead to Laplace transforms whose inverses are not readlly expressed in terms of tabulated functions, although there exist extensive tables of transforms and their inverses. It is highly desirable, therefore, to have methods for approximate numerical inversion.

Numerous methods have been described in the Lierature for the numerical evaluation of the Laplace inversion integral. They fall essentially into two main categories :

(*i*) Quadrature approximation of the complex integral.

(*ii*) Basis expansion methods. A third approach is to treat the problem as an integral equation of the first kind.

(*i*) Quadrature Methods :

Schmittroth [14] has described a method in which the inverse transform is obtained from the complex inversion integral by use of numerical quadrature. This method gives good results but may become time consuming if the inverse transform is required for a large number of values of the independent variable ; the quadrature procedure must be repeated for each value of the independent variable.

Norden [9] Salzer [13] and Sbritliffe and Stephenson [15] attempt an approximate evaluation of the inversion integral using orthogonal polynomials and employing Gaussian quadrature in the complex plane. The main disadvantage of this method is the necessity of finding all roots, real and complex of a polyhomial of high degree.

(*ii*) **Basic Expansion Methods :**

In case where the inverse is required for many values of the independent variable, it is convenient to obtain the inverse as a series expansion in terms of a set of linearly independent functions. The inversion procedure then consists of determining the expansion co-efficients once and for all from the given Laplace transform. The inverse then can be obtained at any value of the independent variable by means of a simple series summation.

Lanczos [6] and Papoulis [10] have described methods in which the inverse transform is obtained as series expansians in terms of trigonometric functions, legendre polynomials and Lagurre polynomials. For a detailed bibliography the reader is referred to Piessen [11] and Piessen and Branders [12]. McWhirter and Pike [7, 8] used Eigen functions expansion for Laplace transform inversion. Recently, de Hoog et al have also discussed two improved methods for numerical inversion of Laplace transforms.

Finally Davies and Martin [5] have given a fairly compreheesive survey of methods of numerical Laplace transfosm inversion.

(*iii*) **Laplace transform inversion as first kind Equation :**

Varah [17, 18] has discussed four methods for dealing with linear discrete ill-posed problems including Laplace transform inversion. In some of his methods he has converted the ill-posed problem to well-posed problem by means of regularization. We shall compare our method with McWhirter and Pike's method and Varah's Methods on the same test examples.

The following terminology will remain standard throughout the paper. The Laplace transform under consideration is denoted by

$g(s)$ and is related to the (unknown) original function $f(t)$ by

$$\int_0^{\infty} e^{-st} f(t)\, dt = g(s) \tag{1}$$

Given $g(s)$, $s \geq 0$ we wish to find $f(t)$, $t \geq 0$, so that (1) holds. Frequently $g(s)$ is only measured at certain points ; however, to test our numerical method, we assume $g(s)$ is given analytically with known $j(t)$, so that we can measure the error in the numerical solution.

We shall employ Maximum Likelihood method to evaluate the solution of (1) Through deconvolution technique and determine the regularization parameter by means of this method [4].

2. Method :

We make the following substitutions in equation (1)

$s = \alpha^x$ and $t = \alpha^{-y}$, $\alpha > 1$ (2)

$$\text{then } g(\alpha^x) = \int_{-\infty}^{\infty} \log \alpha \; e^{-\alpha^{x-y}} f(\alpha^{-y})\, \alpha^{-y}\, dy \tag{3}$$

multipiying both sides by α^x we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)\; F(y(dy) = G(x) \tag{4}$$

where

$$\left.\begin{aligned} G(x) &= \alpha^x g(\alpha^x) = s g(s) \\ K(x) &= \text{Log}\, \alpha \;\; \alpha^x\, e^{-\alpha^x} = \log \alpha \; s e^{-s} \\ F(y) &= f(\alpha^{-y}) = f(t) \end{aligned}\right\} \tag{5}$$

In order that we can apply our deconvolution method to equation (1), it is necessary that $G(x)$ has essentially compact support, *i.e.* $G(x) \to 0$ as $x \to \pm\infty$.

This is clearly the cause if $g(S) = O(S^{-1})$ as $S \to \infty$.

which is a property we shall demand of our data function.

We need to choose two numbers x_{min}, x_{max} such that

$|G(x)| < \epsilon$ whenever $x < x_{min}$ and $x > x_{max}$.

In what follows we choose $\epsilon = 10^{-4} \max |G(x)|$. We find X_{min} and X_{max} as the smallest and largest solution of Non-Linear Equation $|G(x)| = \epsilon$. We then pose the deconvolution problem (4) on the interval (O, T) where $T = X_{max} - X_{min}$, with a further linear substitution onto [0, 1].

We shall use Maximum Likelihood unconstrained Method of 2nd order regularization in T_{N-1} to solve equation (4). The fourier transforms of $F(x)$, $G(x)$ and $K(x)$ in (5) clearly must depend on the parameter α in (2). It turns out that α plays the role of second smoothing parameter in the Numerical solution of (4), in addition to the usual regularization parameter λ.

Since the size of the essential support of $G(x)$ depends upon α, we may write $T = T_\alpha$. For a fixed number N of equidistant data points $\{X_n\}$, we have spacing $h = h_\alpha = \frac{T\alpha}{N}$.

Let $G_{\alpha, n} = G(x_n) = G(nh_\alpha)$, $n = 0, \ldots, N-1$

denote the data on (O, T_α). Then we have the DFT

$$\hat{G}_{\alpha, q} = \sum_{n=0}^{N-1} G_{\alpha, n} \exp\left(-\frac{2\pi i_{nq}}{N}\right), \quad q = 0, \ldots, N-1 \tag{6}$$

Similarly for the Kernel co-efficients.

$$\hat{K}_{\alpha,q} = \sum_{n=0}^{N-1} K_{\alpha, n} \exp\left(\frac{2\pi i}{N} nq\right), \quad q = 0, \ldots, N-1 \tag{7}$$

where $K_{\alpha, n} = \log \alpha \exp\left(-\alpha^{x_n}\right)\alpha^{x_n}$.

Now consider the functional

$$C(f;\lambda) = \left\| Kf - g \right\| + \lambda \left\| f^{(2)} \right\|_2^2 \tag{8}$$

which is minimized over the subspace $H^p C L_2$. Both norms in (8) are L_2, $f^{(2)}$ denotes second derivative of f and λ the regularization parameter. Ths minimizer of (8) in H^p is given by

$$f_\lambda(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(w;\lambda) \frac{\hat{g}(w)}{\hat{k}(w)} \exp(iwy)\, dw \tag{9}$$

f_λ in (9) is approximated by

$$f_{N,\lambda}(x) = \sum_{q=0}^{N-1} Z_{q;\lambda} \frac{\hat{g}_{N,q}}{\hat{K}_{N,q}}, \exp(iw_q x) \tag{10}$$

($\hat{}$ Stands for Fourier Transforms)
where $Z_{q;\lambda}$ is the discrete 2nd order filter given by

$$Z_{q;\lambda} = \frac{\left|\hat{K}_{N,q}\right|^2}{\left|\hat{K}_{N,q}\right|^2 + N^2 \lambda \overset{\frown}{W}{}^4_q} \tag{11}$$

where

$$\overset{\frown}{w}_q = \begin{cases} w_q, & 0 \leq q < \frac{1}{2}N \\ w_{N-q}, & \frac{1}{2}N \leqslant q \leqslant N-1 \end{cases} \tag{12}$$

From equation (10) we know that the filtered solution

$f_{N,\lambda}(x) \in T_{N-1}$ which minimizes

$$\sum_{n=0}^{N-1} \left[(K_N * f)(x_n) - g_n \right]^2 + \lambda \left\| f^{(2)}(x) \right\|_2^2$$

is $f_{N,\lambda}(x) = \frac{1}{N}\sum_{q=0}^{N-1} \hat{f}_{N,\lambda,q} \exp(2\pi iqx)$,

$$\text{where } \hat{f}_{N,\lambda,q} = Z_{q;\lambda} \frac{\hat{g}_{N,q}}{\hat{K}_{N,q}} \tag{13}$$

using (6) and (7) in (13) we get

$$\hat{F}_{\alpha,\lambda,q} = Z_{\alpha,\lambda,q} \frac{\hat{G}_{\alpha,q}}{\hat{K}_{\alpha,q}} \tag{14}$$

where

$$Z_{\alpha,\lambda,q} = \frac{\left|\hat{K}_{\alpha,q}\right|^2}{\left|\hat{K}_{\alpha,q}\right|^2 + \lambda h_\alpha^{-2} \tilde{w}^4_{\alpha,q}} \tag{15}$$

and

$$\tilde{w}_{\alpha,q} = \begin{cases} w_{\alpha,q} & q = 0, \ldots, \tfrac{1}{2}N \\ w_{\alpha,N-q} & q = \tfrac{1}{2}N, \ldots, N-1 \end{cases}$$

where $w_{\alpha,q} = \frac{2\pi}{T_\alpha} q \qquad (T_\alpha = X_{max} - X_{min})$

The optimal λ in (15) is still to be determined by maximum Likelihood Method.

3. Determination of Optimal λ.

Maximum Likelihood Method (ML).

Here we relate the 2nd order convolution filter (11) to certain spectral densities which they play a role in the ML optimization of λ. Assume that the data g_n are noisy, and that there is an underlying function $U_N \in T_{N-1}$ such that

$$g_n = U_N(x_n) + \epsilon_n = U_n + \epsilon_n.$$

We identlfy both $\{U_n\}$ and $\{\epsilon_n\}$ with independent stationary stochastic processes. Since in general, the expectation $E(U_n)$ is not zero, it is suggested by Anderssen and Bloomfield [1, 2] that the data $\{g_n\}$ be detrended so that U_n becomes weakly stationary, this would involve subtracting from data the values of a smooth function of roughly the same shape as U_N. Now consider $f_N \in T_{N-1}$ with f

$=(f_n)=(f_N(x_n))$ defined by $(Kf)_n=U_n$, $n=0, 1,\ldots, N-1$.

where $K=\psi \operatorname{diag}(h\,\hat{K}_{N,q})\,\psi^H$, where ψ is the unitary matrix with elements.

$$\psi_{rs}=\frac{1}{\sqrt{N}}\exp\left(\frac{2\pi}{N} irs\right) \qquad r, s=0,\ldots\ldots, N-1$$

$$f_n=\sum_{m=0}^{N-1}\left\{(K^{-1})_{mn}\int_0^1 \exp(2\pi\, imn)\, ds_n(w)\right\} \quad [4]$$

$$=\int_0^1\left[\hat{K}(w)\right]^{-1}\exp(2\pi i\, w_n)\, d\,S_n(w)$$

$$\text{where } \hat{K}_N(w)=\sum_{n=0}^{N-1} K_n \exp(-2\pi iwn) \tag{16}$$

Assume that f_n is estimated by $\sum_{m=0}^{N-1} l_m \quad g_{n-m}$ where $\{l_m\}$ is a filter which we shall relate to $Z_{q\,;\,\lambda}$ and $\{g_n\}$ is periodically continued for $n \notin [0, N]$. Then the error.

$$f_n-\sum_{m=0}^{N-1} l_m\, g_{n-m} \tag{17}$$

$$\int^{1} \exp(2\pi i\, wn) \left(\left[\hat{K}_N(w) \right]^{-1} - \hat{l}_N(w) \right) d\,S_n(w) - \int^{1} \exp(2\pi\, iwn)\, \hat{l}_N\, dS_{\epsilon}(w) \qquad (18)$$

where $l_N(w)$ is defined as in equation (16). The variance of this error is clearly

$$\int_0^1 \left| \left[\hat{K}_N(w) \right]^{-1} - \hat{l}(w) \right|^2 P_U(w)\, dw \left\{ \int_0^1 \left| \hat{l}_N(w) \right| P\epsilon\,(w)\, dw \qquad (19)$$

which is minimized when

$$\hat{l}_N(w)\hat{K}_N(w) = P_U(w)/[(P_U(w)+P_{\epsilon}(w)]. \qquad (20)$$

Since the discrete Faurier Co-efficients of the filtered solution must satisfy

$$f_{N,\,q\,;\,\lambda} = h\,\hat{l}_{N,\,q}\hat{g}_{N,\,q} = Z_{q\,;\,\lambda}\hat{g}_{N,\,q}[h\,\hat{K}_{N,\,q}]^{-1}$$

we find $Z_{q\,;\,\lambda} = h^2\,\hat{l}_{N,\,q}\hat{K}_{N,\,q}$. Thus from the observation

$h\hat{l}_{N,\,q} = \hat{l}_{N,\,(qh)},\; h\,\hat{K}_{N,\,q} = \hat{K}\,(qh)$, we have from (20) :

Theorem :

In the limit $N \to \infty$, $h \to 0$, the variance of the error $f_N(x_n) - f_{N\,;\,\lambda}(x_n)$ is minimized at x_n by the choice of filter

$$Z_{q\,;\,\lambda} = \frac{P_U(qh)}{P_U(qh)+P_{\epsilon}(qh)} \qquad (21)$$

We now simply relate the filter (21) to the 2nd order filter (11). Assuming that the erors are uncorrelated, $P_{\epsilon}(w)$ has the form $P_{\epsilon}(w) = \sigma^2 = \text{constant}$, where σ^2 is the unknown variance of the noise in the data. Choosing

$$P_U(w)=\frac{\sigma^2\left|\hat{K}_N(w)\right|^2}{\lambda\,\overset{\frown}{w}^4} \tag{22}$$

where

$$\overset{\frown}{w}=\begin{cases} 2\pi\, Nw & 0\le w<\frac{1}{2} \\ 2\pi\, N(1-w) & \frac{1}{2}\leqslant w<1, \end{cases}$$

where $w=\dfrac{2\pi}{Nh}$

then yields (11) from (21). Moreover the spectral density for $\{g_n\}$ is then

$$P_g(w)=P(w)+P_\epsilon(w)=\sigma^2\left[1+\frac{\left|\hat{K}_N(w)\right|^2}{\lambda\overset{\frown}{w}^4}\right]$$

whence $P_g(qh)=\sigma^2(1-Z_{q\,:\,\lambda})^{-1}$ (23)

The statistical likelihood of any suggested values of σ^2 and may now be estimated from the data. Following whittle [19] the logarithm of the likelihood function of P_g is given approximately by

$$\text{constant } -\frac{1}{2}\sum_{q=0}^{N-1}\left[\log P_g(qh)+I(qh)/P_g(qh)\right] \tag{24}$$

where $I(w)=\left|\sum_{n=0}^{N-1} g_n \exp(-2\pi\, i\, wn)\right|^2$ is the periodogram of the data with $I(qh)=\left|\hat{g}_{N,\,q}\right|^2$

We now maximize (24) with respect to σ^2 and λ. The partial maximum with respect to σ^2 may be found exactly (in terms of λ)

with the maximizing value of σ^2 given by

$$\sigma^2 = \frac{1}{N}\sum_{q=1}^{N-1} \left| \hat{g}_{N,q} \right|^2 (1 - Z_{q\,;\,\lambda}) \tag{25}$$

The maximum with respect to λ may then be found by minimizing

$$V_{ML}(\lambda) = \tfrac{1}{2}N \log \left[\sum_{q=1}^{N-1} \left| \hat{g}_{N,q} \right|^2 (1 - Z_{q\,;\,\lambda}) \right] - \tfrac{1}{2} \sum_{q=1}^{N-1} \log (1 - z_{q\,;\,\lambda})$$

Looking in the perspective of equation (6) and (7). The likelihood function can be rewritten as

$$V_{ML}(\lambda, \alpha) = \tfrac{1}{2}N \log \left[\sum_{q=1}^{N-1} \hat{G}_{\alpha,q} \right|^2 (1 - Z_{\alpha,\lambda,q}) \Big]$$
$$- \tfrac{1}{2} \sum_{q=1}^{N-1} \log (1 - Z_{\alpha,\lambda,q}) \tag{26}$$

Thus the optimal regularization parameter is given by the minimizer of a simple function of λ and α depending on the known Fourier Co-efficients $\hat{G}_{\alpha,q}$ and $\hat{K}_{\alpha,q}$. No prior knowledge of σ^2 is assumed but an a *posteriori* estimate is given by equation (25),

In the numerical examples we give in the next section, we have minimized equation (26) with respect to λ for a range of values of $\alpha \geq e$ and compared the L_∞-error of the resulting solution with the values of V (λ, α) for optimal λ. We find that the over all maximum of $V(\lambda, \alpha)$ (over both variables) gives the value of α for which the L_∞-error of the regularized solution is least.

(4) Numerical Results :

In this section we tabulate the results of the above method applied to four test examples. All data functions have the property $g(S)=O(S^{-1})$ and no noise is added apart from machine rounding error. In all cases we have taken N=64 data points.

Example 1. (MeWhirter and Pike [7. 8])

$$g(S)=\frac{1}{(1+S)^2},\ f(t)=t\ e^{-t}$$

The optimal result compared with McWhirter's solution are shown in DIAGS (1, 2) and table 1.

Example 2. (Varah [17, 18])

$$g(s)=\frac{1}{S+\frac{1}{2}},\ f(t)=e^{-t/2}$$

The optimal result compared witn Varah's solution are shown in DIAGS (3, 4) and Table 2.

Example 3. (Varah [17, 18])

$$g(S)=\frac{1}{S(S+\frac{1}{2})},\ f(t)=1-e^{-t/2}$$

The optimal solution compared with Varah's solution are shown in DIAGS (5, 6) Table 3.

Example 4. (Varah [17, 18])

$$G\ (S)=\frac{2}{(S+\frac{1}{2})^3},\ f(t)=t^2\ e^{-t/2}$$

The optimal solution compared with Varah's solution shown in DIAGS (7, 8) and Table 4.

TABLE 1

α	T_α	h_α	λ	$V_{ML}(\lambda, \alpha)$	$\|\|f-f_{\lambda,\alpha}\|\|_\infty$
e	2.096×10^{1}	3.275×10^{-1}	4.51×10^{-14}	1.325×10^{0}	6.9×10^{-3}
6.0	1.20×10^{1}	1.875×10^{-1}	3.64×10^{-14}	9.723×10^{-1}	7.8×10^{-4}
11.0	9.0×10^{0}	1.406×10^{-1}	2.51×10^{-14}	7.5175×10^{-1}	7.2×10^{-4}
16.0	7.60×10^{0}	1.1875×10^{-1}	3.41×10^{-15}	8.673×10^{-1}	6.71×50^{-3}
21.0	6.96×10^{0}	1.0875×10^{-1}	4.27×10^{-14}	1.101×10^{0}	5.63×10^{-3}
26.0	6.40×10^{0}	1.0×10^{-1}	6.57×10^{-14}	2.251×10^{-1}	5.73×10^{-3}

TABLE 3

α	T_α	h_α	λ	$V_{ML}(\lambda, \alpha)$	$\|\|f-f_{\lambda,\alpha}\|\|_\infty$
e	1.840×10^{3}	2.875×10^{-1}	2.56×10^{-3}	2.689×10^{3}	6.31×10^{-1}
6.0	1.020×10^{1}	1.5938×10^{-1}	6.75×10^{-3}	1.893×10^{3}	2.091×10^{-1}
11.0	9.60×10^{0}	1.50×10^{-1}	5.46×10^{-3}	9.821×10^{2}	4.69×10^{-1}
16.0	6.60×10^{0}	1.0313×10^{-1}	4.98×10^{-2}	7.267×10^{2}	3.24×10^{-1}
21.0	5.80×10^{0}	9.063×10^{-2}	1.451×10^{-2}	1.467×10^{2}	7.98×10^{-2}
26.0	5.60×10^{0}	8.750×10^{-2}	1.752×10^{-2}	1.789×10^{2}	6.91×10^{-1}

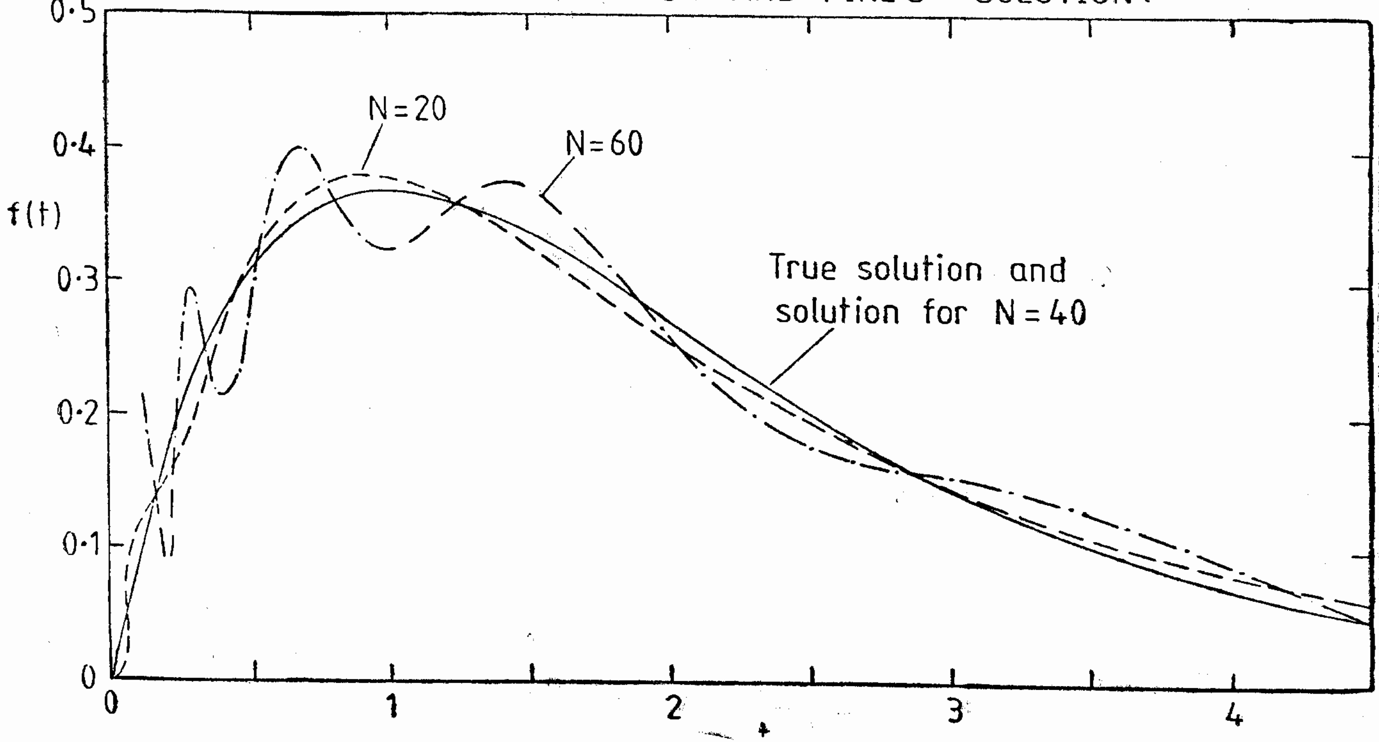
TABLE 4

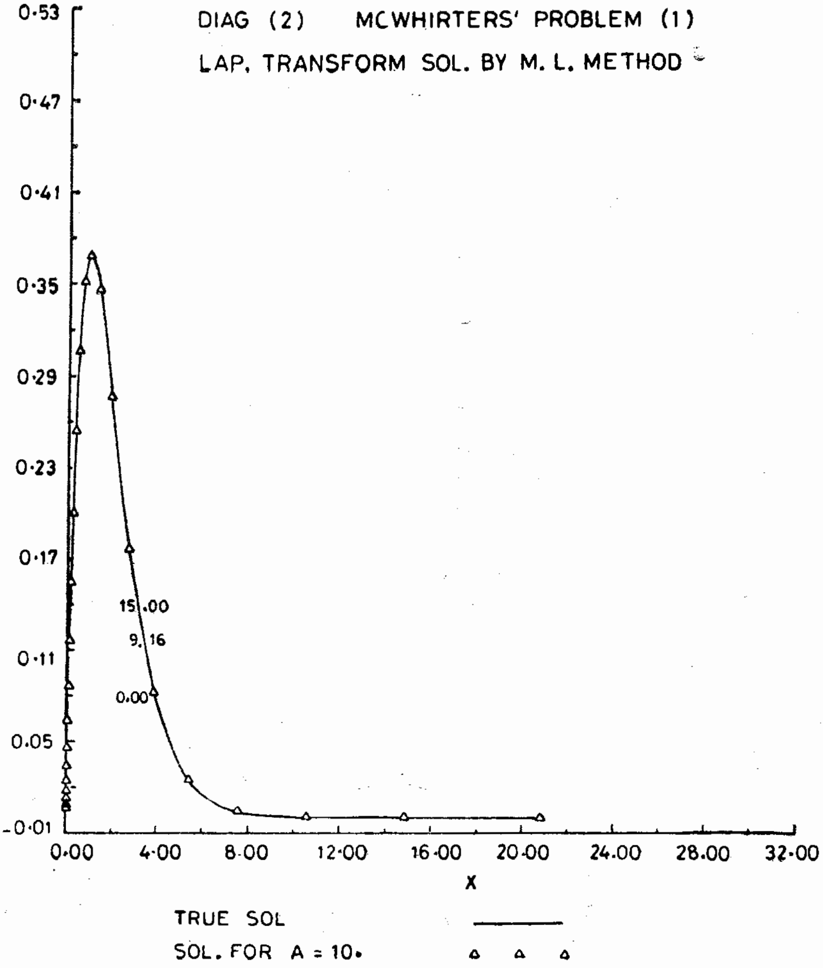
α	T_α	h_α	λ	$V_{ML}(\lambda, \alpha)$	$\|\|f-f_{\lambda,\alpha}\|\|_\infty$
e	2.040×10^{1}	3.1875×10^{-1}	3.11×10^{-11}	2.71×10^{1}	1.31×10^{-2}
6.0	1.140×10^{1}	1.7812×10^{-1}	5.27×10^{-13}	4.68×10^{1}	5.67×10^{-2}
11.0	8.50×10^{0}	1.3281×10^{-1}	6.31×10^{-12}	3.71×10^{1}	6.76×10^{-2}
16.0	7.36×10^{0}	1.150×10^{-1}	7.92×10^{-12}	2.19×10^{1}	4.39×10^{-3}
21.0	6.70×10^{0}	1.0469×10^{-1}	4.21×10^{-12}	1.546×10^{1}	3.0×10^{-3}
26.0	6.26×10^{0}	9.781×10^{-2}	6.33×10^{-12}	1.897×10^{1}	6.9×10^{-2}

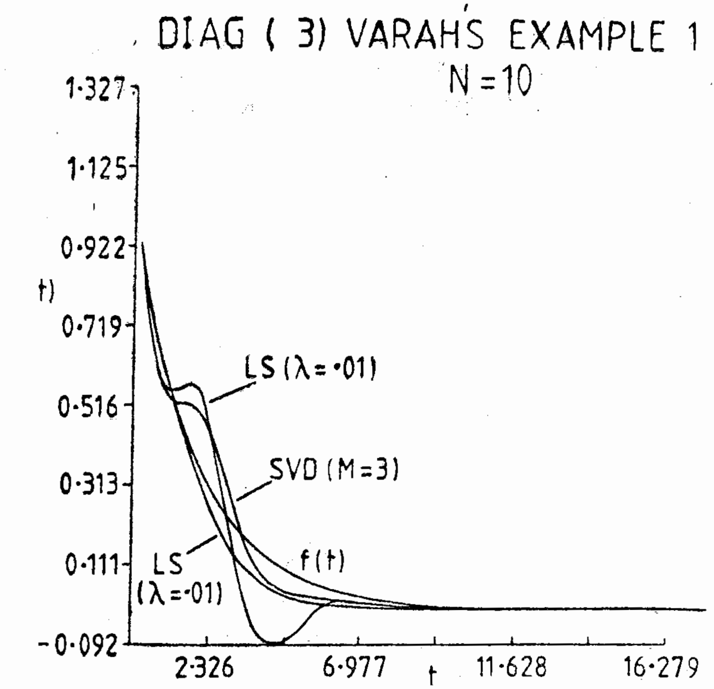
Conclusion

Our method worked very well over all the four test examples and results obtained are perfect as shown in DIAGS (1-8). As regards comparison with McWhirter's solution, we have also obtained a perfect result but over a wider range of the values of t. As far as comparison with Varah's solutions is concerned our solutions are exceedingly better than his as shown in the respective diagrams.

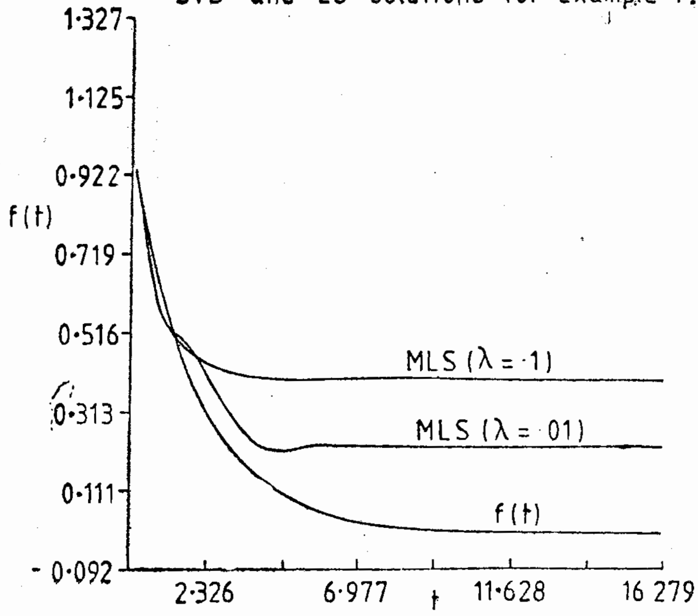
DIAG (1) McWHIRTER AND PIKE'S SOLUTION



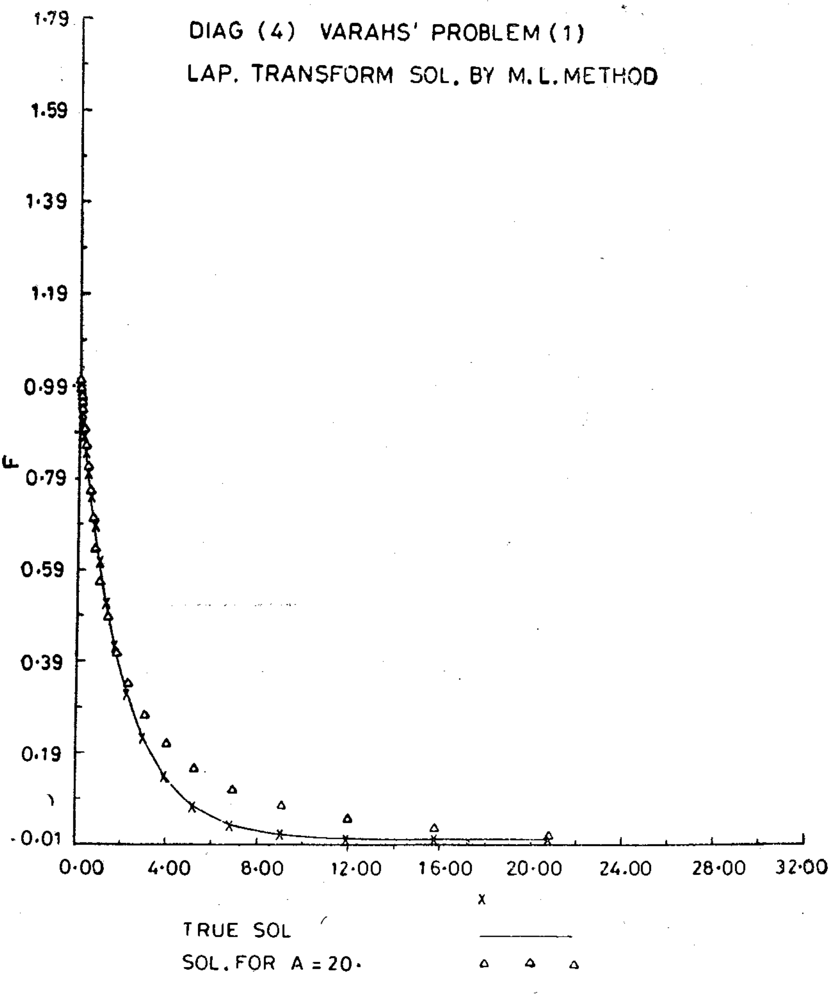

DIAG (2) MCWHIRTERS' PROBLEM (1)
LAP. TRANSFORM SOL. BY M. L. METHOD
0·53
0·47
0·41
0·35
0·29
0·23
0·17
0·11
0.05
-0·01
0·00
4·00
8·00
12·00
16·00
20·00
24·00
28·00
32·00
X
15.00
9.16
0.00
TRUE SOL
SOL. FOR A = 10.



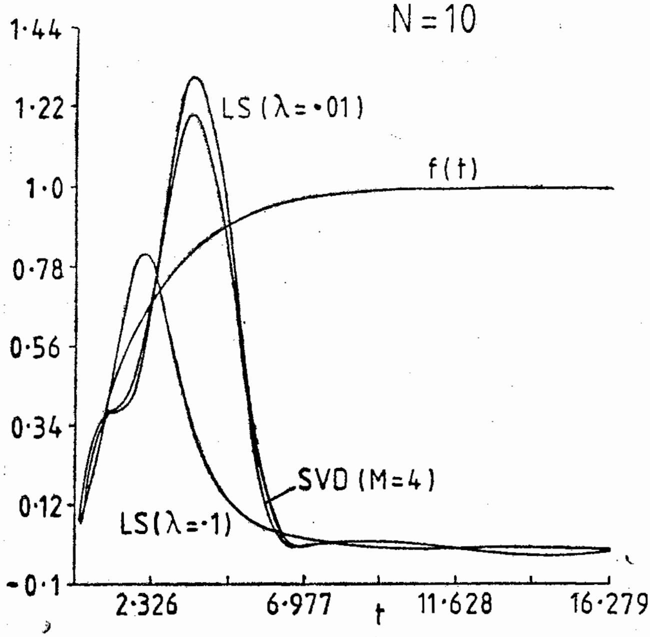
SVD and LS solutions for Example 1.



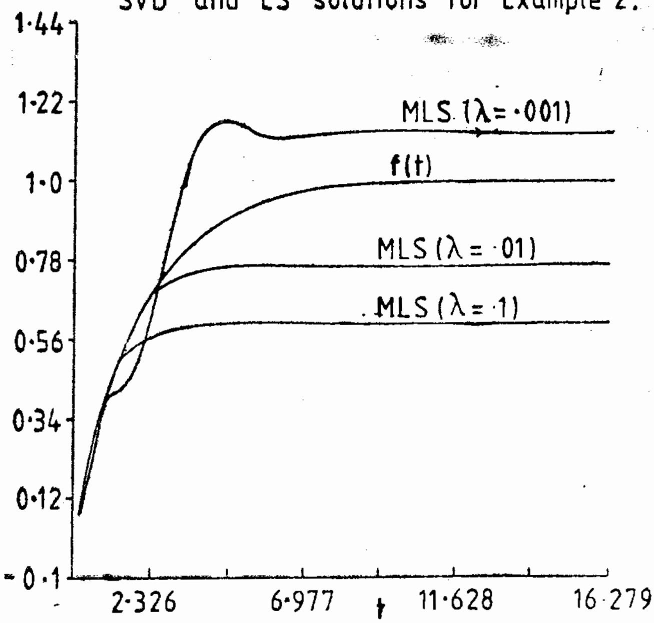
MLS solutions for Example 1,


DIAG (4) VARAHS' PROBLEM (1)
LAP. TRANSFORM SOL. BY M.L. METHOD
1·79
1·59
1·39
1·19
0·99
0·79
0·59
0·39
0·19
-0·01
F
0·00
4·00
8·00
12·00
16·00
20·00
24.00
28·00
32·00
X
TRUE SOL
SOL. FOR A = 20·

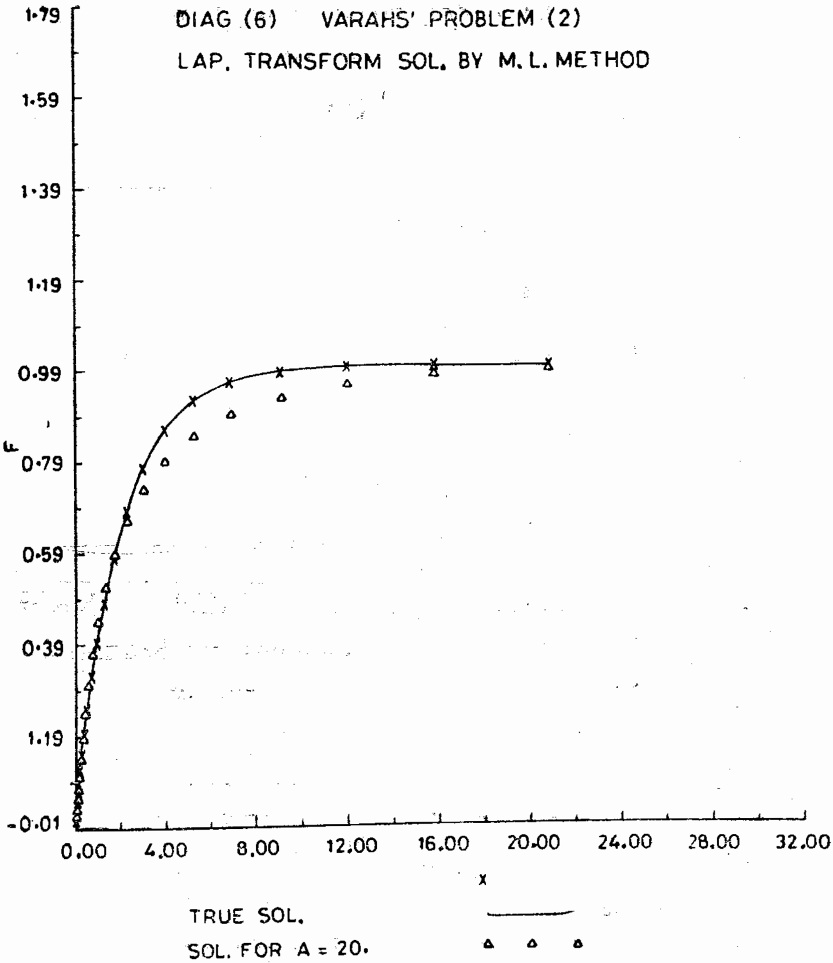
DIAG (5) VARAHS EXAMPLE 2
N = 10



SVD and LS solutions for Example 2.

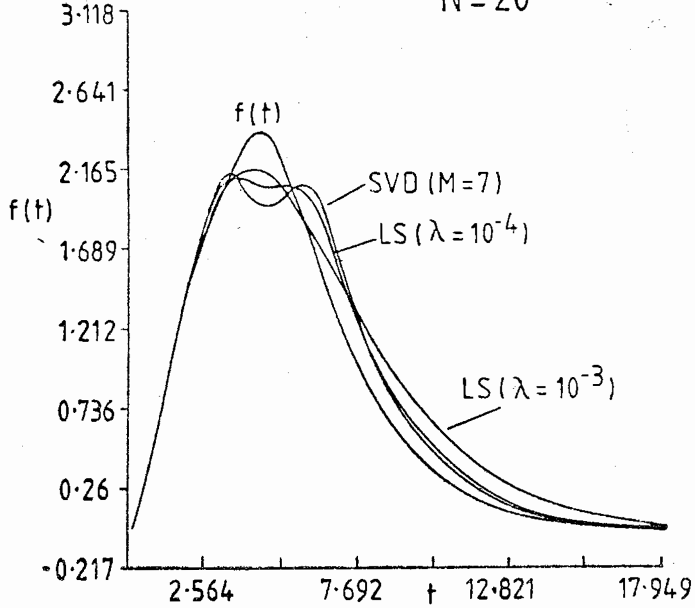


MLS solutions for Example 2

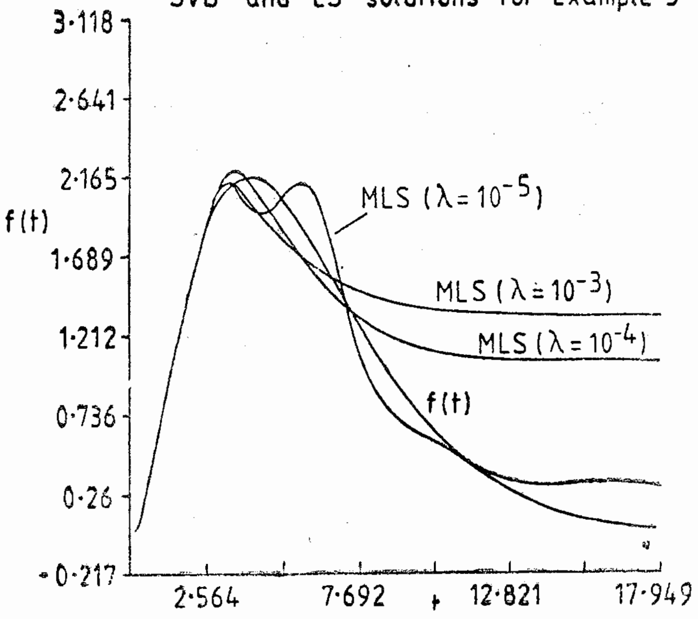

DIAG (6) VARAHS' PROBLEM (2)
LAP. TRANSFORM SOL. BY M.L. METHOD
1·79
1·59
1·39
1·19
0·99
0·79
0·59
0·39
1.19
-0·01
F
0.00
4.00
8.00
12.00
16.00
20.00
24.00
28.00
32.00
X
TRUE SOL.
SOL. FOR A = 20.

DIAG (7) VARAH'S EXAMPLE 3

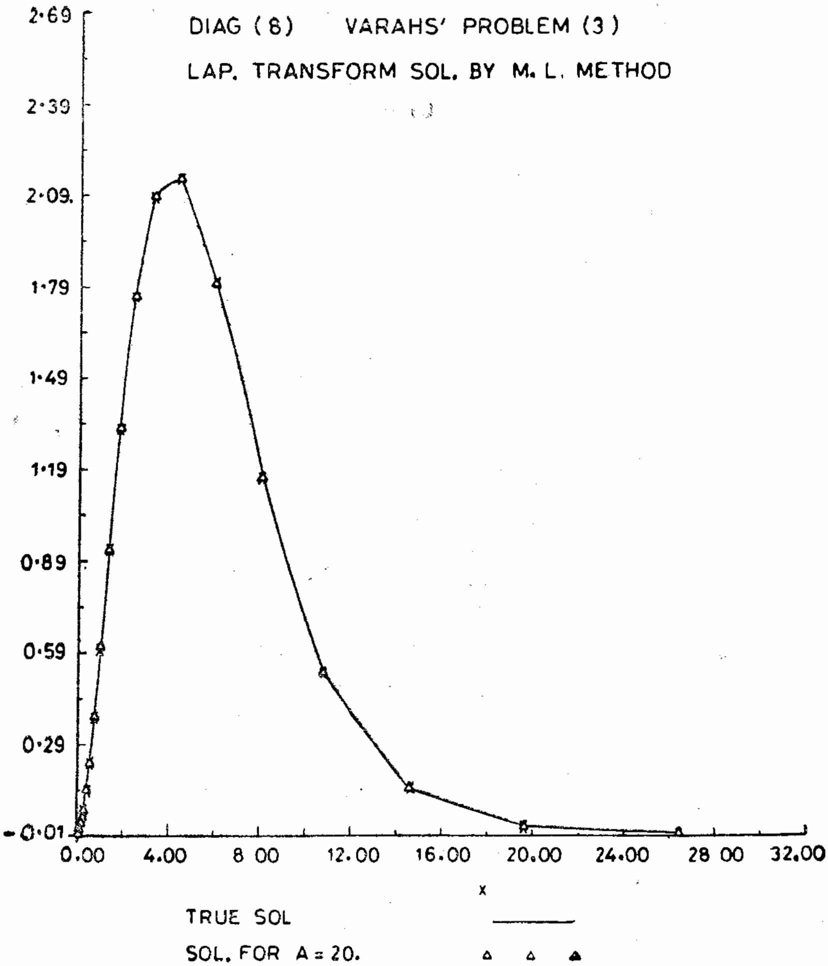
N = 20



SVD and LS solutions for Example 3



MLS solutions for Example 3


DIAG (8) VARAHS' PROBLEM (3)
LAP. TRANSFORM SOL. BY M. L. METHOD
2·69
2·39
2·09
1·79
1·49
1·19
0·89
0·59
0·29
-0·01
0.00
4.00
8 00
12.00
16.00
20.00
24.00
28 00
32.00
X
TRUE SOL
SOL. FOR A = 20.

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