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DR. MUMTAZ HOSAIN KAZI

AN OBITUARY

Dr. Mumtaz Hosain Kazi was born on First December, 1938. He came from a family with a long tradition of academic excellence. He received his M.A. from the University of the Punjab in 1958. M.S. from Harvard in 1964 and Ph.D. from Cambridge in 1974. He started teaching at the University of the Punjab in 1958 and served as Chairman, Department of Mathematics for brief period from 1977 to 1978. He was the Editor of the Punjab University Journal of Mathematics from 1976 till his death.

In the late sixties he took active part in the movement to modernize the syllabi and courses of mathematics in the Pakistan Universities. He was highly, proficient in various branches of the subject and gave courses on topics which spanned pure as well as applied mathematics. He was a brilliant teacher and even the most indolent student in his class could not fail to feel interested in the subject. He inspired many of his students to become research mathematicians, some of them are now well-known at the international level. As a man, Dr. Kazi was an extremely nice person. He left a lasting impression on anybody who ever happened to meet him. He was humane, kind and considerate. He would go to any length to help his students.

In 1978, Dr. Kazi joined the Faculty of Science of the University of Petroleum and Minerals at Dahran (Saudi Arabia). Soon after he became ill. His disease was diagnosed as Multiple Mycloma (a form of blood cancer) at Ash-Sharq Hospital at Al-Khobar, Saudia Arabia. The diagnoses was confirmed by Prof. Bridges of Royal Victoria Hospital, Belfast, U.K. In 1980, eversince he remained under regular treatment at King Faisal Specialist Hospital and Research Centre at Riyadh, Saudi Arabia. Despite the treatment the disease continued. Although it was clear that he would never be well again, the awareness of his impending end did not, in any way, dull his spirits. Rather he became more active as if to make the best use of every minute he was left with. Inspite of his bad health and other persuits, he continued to edit the Punjab University Journal of Mathematics with dedication and selflessness. Towards the end of his life, he published profusely. He and his collaborators applied the Wiener-Hopf technique to the problem of Love Waves in the Earth's crust and were able to solve some outstanding problems. His name is still considered to be an authority in the field. He published in international journals of high standard on topics which include seismology, functional analysis and mathematical statistics.

Dr. Kazi was recommended on 12th July, 1979 for appointment as Professor of Applied Mathematics by the Punjab University Selection Board. However anticipating better medical care at Riyadh, Saudi Arabia, he could not join the Mathematics Department of Punjab University. The Punjab University granted his request for retirement from service on medical grounds with effect from September 1, 1983.

Dr. Kazi died at Riyadh on 19th June, 1987. He was burried at Janat-ul-Baqii, Madina Munawwarah. He left behind a widow, a son, two daughters and countless colleagues and pupils who will always cherish his memory for his forbearance, gracefulness cheerfulness, pursuit of excellence and most of all, for his fortitude in the face of awful suffering and inevitable death.

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CONCERNING THE CONVERGENCE OF NEWTON'S METHOD

By

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Abstract

We assume the existence of a simple zero of a nonlinear operator equation in a Banach space. Under the assumption that the Frechet-derivative of the nonlinear operator involved is only Holder continuous, we answer to the following question : given that the equation has a simple zero, when is it true that the Newton iterates converge to that solution. An example is also provided.

Key words and phrases :

Banach space, Newton-Kantorovich method, Holder continuity.
(1980) A.M.S. classification codes : 65J15, 65H10.

Introduction

Consider the equation

$$F(x)=0, \tag{1}$$

where F is a nonlinear operator from a Banach space E into itself. The most popular iteration for solving (1) is given by the Newton-Kantorovich iteration, namely :

$$x_{n+1}=x_n-(F'(x_n))^{-1} F(x_n),\ n=0,\ 1,\ 2,\ldots\ldots \tag{2}$$

for some initial guess $x_0 \in E$.

The main theorem proved here answers to the following question : given that F has a simple zero $x^* \in E$, when is it true that the interates given by (2) of nearby points converge to x^*? Such a question is clearly of interest in numerical analysis since many numerical problems can be reduced to the problem of locating solutions of (1).

Sufficient conditions for the convergence of (2) to a simple zero x^* of F are given by the famous theorem of L. V. Kantorovich [1], [2], [7].

An extensive literature on the Newton-Kantorovich method can be found in [6].

One of the basic assumptions for the convergence of (2) is that the Frechet-derivative F' of F is Lipschitz continuous. The question raised here has been answered in [4].

Here we only assume that F' is Holder (c, p) continuous (to be precised later). Our results can be reduced to the ones in [4] for $p=1$.

Finally wo provide an example where the Lipschitz continuity of F' is not satisfied whereas the Holder continuity is.

We will need the following :

I. Preliminaries

We assume that F is once Frechet-differentiable [2] and $F'(x)$ is the first Frechet-derivative at a point $x \in E$. It is well known that $F'(x) \in L(E)$, the space of bounded linear operators from E to E. We say that the Frechet-derivative $F'(x)$ is Holder continuous over a domain $E_1 \subset E$ if for some $c>0$, $p \in [\,0,1\,]$, and all $x, y \in E_1$,

$$\|F'(x)-F'(y)\| \leqslant c\|x-y\|^p. \tag{3}$$

In this case, we say that $F'(\cdot) \in H_{E_1}(c,p)$.

We will need the following result whose proof can be found in [2].

Lemma 1.

Let $F : E \rightarrow E$ and $E_1 \subset E$. Assume E_1 is open and that $F'(\cdot)$ exists at each point of E_1. If for some convex set $E_2 \subset E_1$, we have

$F'(\cdot) \in H_{E_2}\ (c, p)$, then for all $x, y \in E_2$

$$\|F(x)-F(y)-F'(x)(x-y)\| \leqslant \frac{c}{1+p}\|x-y\|^{1+p}. \tag{4}$$

II. Main Results

Theorem 1. Let $E_1 \subset E$ and $F : E_1 \to E$. Assume $F'(\cdot) \in H_{E_2}\ (c, p)$ on a convex set $E_2 \subset E_1$. Let x_0 be such that :

$$\|F'(x)-F'(x_0)\| \leqslant c_1\|x-x_0\|^p \tag{5}$$

for all $x \in E_2$ and some $c_1 > 0$.

Assume that $F'(x_0)$ has a bounded inverse $F'(x_0)^{-1} \in L(E)$ with

$$\|(F'(x_0))^{-1}\| \leqslant d, \tag{6}$$

$$\|(F'(x_0))^{-1} F(x_0)\| \leq r_0 \tag{7}$$

and that the function g defined by

$$g(r)=dc_1 r^{1+p}+\left[\frac{dcr_0^p}{1+p}-1\right]r-dc_1 r_0 r^p+r_0 \tag{8}$$

has a minimum positive zero $r^* > r_0$.

Moreover, assume that :

$$dc_1(r^*)^p < 1 \tag{9}$$

then

$$q=\frac{dcr_0^p}{(1+p)(1-dc_1(r^*)^p)} < 1. \tag{10}$$

If $\bar{U}(x_0, r^*) \subset E_2$ then the iteration (2) is well defined, remains in $U(x_0, r^*)$ and converges to a solution x^* of (1).

Proof

By the Banach lemma $F'(x)$ has a bounded inverse, since

$$\|F'(x)-F'(x_0)\| \leq c_1\|x-x_0\|^p < c_1(r^*)^p < \frac{1}{d},$$

and

$$\|(F'(x))^{-1}\| \leq \frac{d}{1-dc_1\|x-x_0\|^p} \tag{11}$$

The operator p given by

$$p(x)=x-(F'(x))^{-1} F(x)$$

is well defined on $U(x_0, r^*)$. Assume now that x, $P(x) \in U(x_0, r^*)$ and using (4), we obtain

$$\|P^2(x)-P(x)\| = \|-(F'(P(x)))^{-1}F(P(x))\|$$

$$\leq \frac{d}{1-dc_1\|P(x)-x_0\|^p}\,[\|F(P(x))-F(x)-F'(x)(P(x)-(x)]$$

$$\leq \frac{d}{1-dc_1\|P(x)-x_0\|^p}\cdot\frac{c}{1+p}\|P(x)-x\|^{1+p}$$

$$=\bar{g}(\|P(x)-x\|, \|P(x)-x_0\|),$$

where

$$\bar{g}(v, w) = \frac{1}{1-dc_1v^p}\cdot\frac{dcw^{1+p}}{1+p}.$$

Define the real sequence $\{s_k\}$, $k=0, 1, 2, \ldots\ldots$ by $s_0=0$, $s_1=r_0$ and

$$s_{k+1}-s_k = \frac{1}{1-dc_1s_k^p}\cdot\frac{dc}{1+p}\,(s_k-s_{k-1})^{1+p}.$$

We now have

$$s_2-s_1\leq qr_0<r_0,$$

$$s_2\leq s_1+qr_0=r_0(1+q)<\frac{r_0}{1-q}.$$

Using (10) and $g(r^*)=0$, we get $r_0=(1-q)r^*$. That is

$$s_2<r^*.$$

By induction we can easily get

$$s_{k+1}-s_k\leq q(s_k-s_{k-1}),$$

$$s_{k+1}-s_k<r_0$$

and

$$s_{k+1}<r^*.$$

That is,

$$\lim_{k\to\infty}s_k=s^*, \frac{r_0}{1-q}=r^*.$$

By the basic majorant theorem 2.3 of Rheinboldt [7], there exists an $x^*\in\bar{U}(x_0, r^*)$ such that

$$P(x^*)=x^* \text{ and } \lim_{k\to\infty}x_k=x^*.$$

Finally,

$$\|F(x_k)\| \leq \|F'(x_k)-F'(x_0))(x_{k+1}-x_k)\| + \|F'(x_0)(x_{k+1}-x_k)\|$$
$$\leq [c_1\|x_k-x_0\| + \|F'(x_0)\|]\|x_{k+1}-x_k\|$$
$$\leq (c_1 r^* + \|F'(x_0)\|)\|x_{k+1}-x_k\|,$$

since $x_k \in \bar{U}(x_0, r^*)$. Letting $k \to \infty$, we easily obtain from the above inequality that

$$F(x^*)=0$$

since $\{x_k\}$ is a Cauchy sequence and $(c_1 r^* + \|F'(x_0)\|)$ is a constant.

The proof of the theorem is now complete.

Theorem 2 Let $E_1 \subset E$ and $F : E_1 \to E$. Assume $F'(\cdot) \in H_{E_2}(c,p)$ on a convex sct $E_2 \subset E_1$. Let x^* be a simple zero of F such that :

$$\|F'(x)-F'(x^*)\| \leq c_2\|x-x^*\|^p \tag{12}$$

for all $x \in E_2$ and some $c_2>0$.

Assume :

(*a*) there exists $d_1>0$ such that

$$\|(F'(x^*))^{-1}\| \leqslant d_1, \tag{13}$$

(*b*) let $x_0 \in U(x^*, e)$, where $e=(2c_2d_1)^{-1/p}$ and assume that the function defined by

$$q_1(r)=dc_2r^{1+p}+\left[\frac{dc_2r_0^p}{1+p}-1\right]r-dc_2r_0r^p+r_0 \tag{14}$$

has a minimum positive zero $r^*>r_0$, where,

$$r_0=\frac{p+1-c_2d_1p\|x_0-x^*\|^p}{(1-c_2d_1\|x_0-x^*\|^p)(p+1)}\|x_0-x^*\| \tag{15}$$

and

$$d=\frac{d_1}{1-d_1c_2\|x_0-x^*\|^p}\,. \tag{16}$$

If $\bar{U}(x^*,r^*) \subset E_2$, then the hypotheses of theorem 1 are satisfied for each $x_0 \in U(x^*, r^*)$.

Therefore the Newton sequence $\{x_n\}$, $n=0, 1, 2, \ldots$ exists and remains in $U(x^*, r^*)$ and converges to x^* such that $F(x^*)=0$.

Prroof

By the Banach lemma, $F'(x_0)$ has a bounded inverse, since

$$\|F'(x^*)-F'(x_0)\| \leqslant c_2\|x^*-x_0\|^p < c_2(r^*)^p < \frac{1}{d_1},$$

for $x_0 \in U(x^*, r^*)$ and

$$\|F'(x_0))^{-1}\| \leqslant \frac{d_1}{1-d_1c_2\|x_0-x^*\|^p} = d.$$

We have

$$F(x^*)-F(x_0)=F'(x_0)(x^*-x_0)+\int_0^1 [F'(x_0+t(x^*-x_0))$$

$$-F'(x_0)](x^*-x_0)dt.$$

So,

$$-(F'(x_0))^{-1}F(x_0)=x^*-x_0+(F'(x_0))^{-1}\int_0^1 [F'(x_0+t(x^*-x_0))$$

$$-F'(x_0)](x^*-x_0)dt,$$

axd thus

$$\|(F'(x_0))^{-1}F(x_0)\| \leqslant [1+dc_2\|x^*-x_0\|^p\int_0^1 t^p dt]\|x^*-x_0\|$$

$$\leqslant \frac{p+1-c_2d_1p\|x_0-x^*\|^p}{1-c_2d_1\|x_0-x^*\|^p)(p+1)} = r_0.$$

It follows that for $x_0 \in U(x^*, r^*)$

$$dc_2(r^*)^p < 1,$$

hence (9) holds.

The hypotheses of Theorem 1 are now satisfied for each $x_0 \in U(x^*, r^*)$ and the result follows.

Proposition

Under the hypotheses of Theorem 2, the order of convergence of (2) to a solution x^* of (1) is $1+p$.

That is

$$\|x_{n+1}-x^*\| \leqslant c_3\|x_n-x^*\|^{1+p}$$

where,

$$c_3=\frac{dc_2}{(p+1)^2}\ .$$

Proof

We have

$$x_{n+1}-x^*=x_n-x^*-F'(x_n)^{-1}F(x_n)$$

$$=F(x_n)^{-1}\left\{\int_0^1[\,F'(x_n)-F'(x^*+t(x_n-x^*))\,]\,dt\right\}(x_n-x^*).$$

Hence,

$$\|x_{n+1}-x^*\| \leqslant d\left\{\int_0^1\|F'(x_n)-F'(x^*+t(x_n-x^*))\|dt\right\}\|x_n-x^*\|$$

$$\leqslant\frac{dc_2}{p+1}\left\{\int_0^1\|x_n-x^*-t(x_n-x^*)\|^p dt\right\}\|x_n-x^*\|$$

$$\leqslant\frac{dc_2}{p+1}\|x_n-x^*\|^{1+p}\int_0^1(1-t)^p dt$$

$$\leqslant\frac{dc_2}{p+1}\cdot\frac{1}{p+1}\|x_n-x^*\|^{1+p}$$

$$=c_3\|x_n-x^*\|^{1+p}.$$

III. Applications

To illustrate Theorems 1 and 2, consider the differenrial equation

$$x''+x^{1+p}=0.\ p\in[\,0,\,1\,]$$
$$x(0)=x(1)=0.$$

We divide the interval $[\,0,\,1\,]$ into n subintervals and we set $h=\frac{1}{n}$. Let $\{\,\nu_k\,\}$ be the points of subdivision with

$$0=\nu 0<\nu_1<\ldots<\nu_n=1.$$

A standard approximation for the second derivative is given by

$$x_i'' = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2}, \quad x_i = x(v_i),\ i = 1, 2, \ldots, n-1.$$

Take $x_0 = x_n = 0$ and define the operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(x) = H(x) + h^2 \varphi(x)$$

$$H = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ 0 & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix},$$

$$\varphi(x) = \begin{bmatrix} x_1^{1+p} \\ x_2^{1+p} \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1}^{1+p} \end{bmatrix},$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{bmatrix}$$

Then

$$F'(x) = H + h^2(p+1) \begin{bmatrix} x_1^p & & & & 0 \\ & x_2^p & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & x_{n-1}^p \end{bmatrix}$$

Newton's method cannot be applied to the equation

$$F(x) = 0.$$

We may not be able to evaluate the second Frechet-derivative since it would involve the evaluation of quantities of the form x_i^{-p} and they may not exist.

Let $x \in \mathbb{R}^{n-1}$, $H \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and define the norms of x and H by

$$\|x\| = \max_{1 \leqslant j \leqslant n-1} | x_j |$$

$$\|H\| = \max_{1 \leqslant j \leqslant n-1} \sum_{k=1}^{n-1} | h_{jk} | .$$

For all $x, z \in \mathbb{R}^{n-1}$ for which $| x_i | > 0$, $| z_i | > 0$, $i=1, 2, \ldots, n-1$ we obtain, for $p=\frac{1}{2}$ say,

$$\|F'(x)-F'(z)\| = \|\text{diag} \{ (1+\tfrac{1}{2})h^2(x_j^{\frac{1}{2}})-z_j^{\frac{1}{2}}) \} \|$$

$$= \frac{3}{2} h^2 \max_{1 \leq j \leqslant n-1} | x_j^{\frac{1}{2}} - z_j^{\frac{1}{2}} | \leqslant \frac{3}{2} h^2 [\max | x_j - z_j |]^{\frac{1}{2}}$$

$$= \frac{3}{2} h^2 \|x-z\|^{\frac{1}{2}}.$$

Given $z_0 \in \mathbb{R}^{n-1}$ Nowton's method consists of solving

$$F'(z_n)\,(z_n - z_{n+1}) = F(z_n),\ n=0, 1, 2, \ldots$$

as a system of linear equations.

We choose $n=10$ which gives 9 equations. Since a solution would vanish at the end points and be positive in the interior a reasonable choice of initial approximation seems to be $130 \sin \pi x$. This gives us the following vector :

$$z_0 = \begin{bmatrix} .40172211E+02 \\ .76412079E+02 \\ .10517221E+03 \\ .12363734E+03 \\ .12999998E+03 \\ .12363734E+03 \\ .10517220E+03 \\ .76412071E+02 \\ .40172215E+02 \end{bmatrix}$$

The iterates z_1, z_2 will then be given by

$$z_1 = \begin{bmatrix} .33957119E+02 \\ .65958946E+02 \\ .92615190E+02 \\ .11048508E+03 \\ .11676437E+03 \\ .11048503E+03 \\ .9265187E+03 \\ .65958832E+02 \\ .33957062E+02 \end{bmatrix}$$

and

$$z_2 = \begin{bmatrix} .33577446E+02 \\ .65209305E+02 \\ .91575623E+02 \\ .10917905E+03 \\ .11537511E+03 \\ .10917908E+03 \\ .91575668E+02 \\ .65209358E+02 \\ .33577473E+02 \end{bmatrix}$$

We now use Theorem 2 for $p=\frac{1}{2}$, $c_2=.015$, $x^*=z_2$ and $x_0=z_1$. The choice $x^*=z_2$ is considered reasonable since $\|F(z_2)\|=.005941582$.

Then we easily obtain

$\|x^*-x_0\|=1.389269338$,

$e=1.627003894$,

$d_1=.26132710E+02$,

$r_0=2.184683800$,

and

$$d = 48.576244354.$$

Using Newtan's method in (14) by choosing the first iterate to be $\bar{r}_0 = 9.1E-05$ we obtain

$$r^* = 2.360669374 > r_0.$$

Then the hypotheses of Theorem 2 are satisfied for $x_0 \in U(x^*, r^*)$. Therefore the Newton sequence $\{ x_n \}$, $n=0, 1, 2, \ldots$ exists, remains in $U(x^*, r^*)$ and converges to x^* with order of convergence $\frac{3}{2}$ according to the proposition.

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A NEW ITERATIVE PROCEDURE FOR FINDING "LARGE" SOLUTIONS OF THE QUADRATIC EQUATION IN BANACH SPACE

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Quadratic equation, Banach Space, contraction.

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Abstract

In this paper we introduce a new iterative procedure for finding "Lagre" solutions of the quadratic equation in Banach space, based on the same assumption used to prove existence for the "small" solution.

Introduction

In this paper we introduce the iteration

$$x_{n+1}=B(x_n)^{-1}(x_n-y), n=0, 1, 2, \ldots\ldots \tag{1}$$

for some x_0 in a Banach space X to find solution of the quadratic equation

$$x=y+B(x, x) \tag{2}$$

in X, where $y \in X$ is fixed and B is a bounded symmetric bilinear operator on $X \times X$. Equation (2) has been studied in [1], [2], [6], [7], [9], [10]. It is known that if $1-4\|B\| \cdot \|y\| > o$ then equation (2) has a "small" solution $x \in X$ such that $\|x\| < \frac{1}{2\|B\|}$. In the scalar case though we know that the above condition implies that (2) has two solutions the "large" one being such that $\|x\| \geq \frac{1}{2\|B\|}$. The above observation raises the natural question. Is it true under the same assumption that (2) has a "large" solution ?

The answer is positive under certain assumptions. The basic idea is to introduce a convergent iteration such that if $\|x_0\| \geqq \frac{1}{2\|B\|}$ then $\|x_n\| \geq \frac{1}{2\|B\|}$, $n=0, 1, 2, \ldots\ldots$

It is shown that (1) satisfies the above property. We also study the modified version of (1).

$$x_{n+1}=x_n-B(x_0)^{-1}\,(B\,(x_n, x_n)-x_n+y),\ n=0,\ 1,\ 2,\ \ldots\ldots \tag{3}$$

Finally we provide two simple applications of (1) and (3).

Remark 1

The operator B in (2) is assumed to be symmetric without loss of generality since B can always be replaced by the *mean* $\bar{B}$ of B defined by

$$\bar{B}(x, y)=\tfrac{1}{2}(B\,(x, y)+B(y, x)\,),\ x, y \in X.$$

Note that $\bar{B}(x, x)=B(x, x)$ for all $x \in X$.

From now on we assume that B is a bounded bilinear operator on $X \times X$.

We are now going to introduce an iteration that will guarantee in case of convergence that the solution x is such that $\| x \| \geq \frac{1}{2\|B\|}$.

Proposition 1. Assume :

(*i*) The iteration

$$x_{n+1}=B(x_n)^{-1}\,(x_n-y) \tag{1}$$

is well defined for all $n=0, 1, 2. \ldots\ldots$ for some $x_0 \in X$.

(*ii*) $1-4\|B\|\cdot\|y\| \geq 0$ and

(*iii*) let $p \in [\,P_1, P_2\,]$, where p_1, p_2 are the solutions of the equation

$$\|B\|\,p^2-p+\|y\|=0.$$

If $\|x_0\| \geqslant p$ then $\|x_{n+1}\| \geqslant p$ for all $n=0, 1, 2, \ldots\ldots$

Proof

We have

$$B(x_n, x_{n+1}) = x_n - y$$

so

$$\|x_n - y\| = \|B(x_n, x_{n+1})\|$$
$$\leqslant \|B\| x_n\| \cdot \|x_{n+1}\|$$
$$\leqslant \|B\| \cdot \|x_n\| \cdot \|x_{n+1}\|$$
$$\|x_{n+1}\| \geqq \frac{\|x_n - y\|}{\|B\| \cdot \|x_n\|} .$$

Assume that $\|x_k\| \geqq p$ for all $k=0, 1, 2, \ldots\ldots, n$. Since $\|x_n\| \geqq p \geq \|y\|$, it is enough to show

$$\frac{\|x_n\| - \|y\|}{\|B\| \cdot \|x_n\|} \geqq p$$

or

$$\|x_n\| \geqslant \frac{\|y\|}{1 - p\|B\|} .$$

Finally it suffices to show

$$p \geqq \frac{\|y\|}{1 - p\|B\|}$$

or

$\|B\| p^2 - p + \|y\| \leqslant 0$ but this is true for $p \in [p_1, p_2]$.

Note that

$$p = \frac{1}{2\|B\|} \in [p_1, p_2].$$

We now state the following lemma. The proof can be found in [10].

Lemm 1. Let L_1 and L_2 be bounded linear operators in a Banach space X, where L_1 is invertible, and $\|L_1^{-1}\| . \|L_2\| < 1$. Then $(L_1 + L_2)^{-1}$ exists, and

$$\|(L_1 + L_2)^{-1}\| \leqslant \frac{\|L_1^{-1}\|}{1 - \|L_2\|} . \|L_1^{-1}\|.$$

Lemma 2. Let $z \neq 0$ be fixed in X. Assume that the linear operator $B(z)$ is invertible then $B(x)$ is also invertible for all

$x \in U(z, r) = \{ x \in X \mid \|x-z\| < r \}$, where $r \in (0, r_0)$ and

$$r^0 = [\ \|B\| \cdot \|B(z)^{-1}\|\ ]^{-1}.$$

Proof

We have

$$\|B(x-z)\| \cdot \|B(z)^{-1}\| \leqslant \|B\| \cdot \|x-z\| \cdot \|B(z)^{-1}\|$$
$$\leqslant \|B\| \cdot \|B(z)^{-1}\| \cdot r < 1$$

for $r \in (0, r_0)$. The result now follows from Lemma 1 for $L_1 = B(z)$, $L_2 = B(x-z)$ and $x \in U(z, r)$.

Definition 1

Let $z \neq 0$ be fixed in X. Assume that the linear operator B(z) is invertible.

Define the operators P, T on $U(z, r)$ by

$$P(x) = B(x, x) + y - x, \quad T(x) = (B(x))^{-1}(x-y)$$

and the real polynomials $f(r)$, $g(r)$ on R by

$$f(r) = a' r^2 + b' r + c', \quad g(r) = ar^2 + br + c$$
$$a' = (\|B\| \cdot \|B(z)^{-1}\|)^2$$
$$b' = -2\|B\| \cdot \|B(z)^{-1}\|$$
$$c' = 1 - \|B(z)^{-1}\| - \|B\| \cdot \|B(z)^{-1}\|^2 \cdot \|z-y\|$$
$$a = r_0^{-1}$$
$$b = \|B(z)^{-1}(I - B(z))\| - 1$$
$$c = \|B(z)^{-1} P(z)\|.$$

From now on we assume that B is a bounded symmetric bilinear operator on $X \times X$.

Theorem 1

Let $z \in X$ be such that B(z) is invertible and that the following are true :

(*a*) $c' > 0$;

(*b*) $b > 0$, $b^2 - 4ac > 0$, and

(*c*) there exists $r > 0$ such that $f(r) > 0$ and $g(r) \leqslant 0$.

Then the iteration

$$x_{n+1} = B(x_n)^{-1}(x_n - y), \quad n = 0, 1, 2, \ldots\ldots,$$

is well defined and it converges to a unique solution x of (2) for any $x_0 \in \bar{U}(z, r)$. Moreover, if $1-4\|B\|\cdot\|y\|>0$ and $\|x_0\| \geqq \frac{1}{2\|B\|}$ then $\|x\| \geqq \frac{1}{2\|B\|}$.

Proof

T is well defined by Lemma 2.

Claim 1.

T *maps* $\bar{U}(z, r)$ *into* $\bar{U}(z, r)$.

If $x \in \bar{U}(z, r)$ then

$$T(x)-z=B(x)^{-1}(x-y)-z$$
$$=B(x)^{-1}\left[(I-B(z))(x-z)-P(z)\right]$$

so

$$\|T(x)-z\| \leqslant r \text{ if}$$
$$\frac{1}{1-\|B\|\cdot\|B(z)^{-1}\|r}\left[\|B(z)^{-1}(I-B(z))\|r+\|B(z)^{-1}P(z)\|\right] \leqslant r$$

(using Lemma 1 for $L_1=B(z)$ and $L_2=B(x-z)$) or $g(r) \leqslant 0$ which is true by hypothesis.

Claim 2

T is a contraction operator on $\bar{U}(z, r)$.

If $w, v \in \bar{U}(z, r)$ then

$$\|T(w)-T(v)\|$$
$$=\|B(w)^{-1}(w-y)-B(v)^{-1}(v-y)\|$$
$$=\|B(w)^{-1}\left[I-B(B(v)^{-1}(v-y))\right](w-v)\|$$
$$=\|B(w)^{-1}\left[I-B(B(v)^{-1}(v-z))+B(B(v)^{-1}(z-y))\right](w-v)\|$$
$$\leqslant \frac{1}{1-\|B\|\cdot\|B(z)^{-1}\|\cdot r}\left[\|B(z)^{-1}\| + \frac{\|B\|\cdot\|B(z)^{-1}\|^2 r+\|B\|\cdot\|B(z)^{-1}\|^2\|z-y\|}{1-\|B\|\cdot\|B(z)^{-1}\|\cdot r}\right]\|w-v\|$$
$$=q\cdot\|w-v\|$$

So T is a contraction on $\bar{U}(z, r)$ if $0<q<1$, where

$$q=\frac{1}{1-\|B\|\cdot\|B(z)^{-1}\|\cdot r}\left[\|B(z)^{-1}\| + \frac{\|B\|\cdot B(z)^{-1}\|^2 r+\|B\|\cdot\|B(z)^{-1}\|^2\|z-y\|}{1-\|B\|\cdot\|B(z)^{-1}\|\cdot r}\right]$$

which is true since $f(r)>0$.

Example 1

Consider the equation $x=.2x^2-1$ in $\mathbb{R}$. (4)

Here $B(x, x)=.2x^2$, $y=-1$ and $1-4\,|\,b\,|\cdot|\,y\,|>0$. Then according to definition 1 for $z=5$

$$f(r)=(.04)r^2-(.4)r+9.6,\; g(r)=.2r^2-r+1.$$

Theorem 1 can be applied provided that

$$1.38196601\leqslant r<1.8377225$$

and the iteration becomes

$$x_{n+1}=5\left(1+\frac{1}{x_n}\right)\text{with } x_0=z=5,\; n=0,\,1,\,2,\ldots\ldots$$

Note that $x=x_{12}=5.854101966$ is the "large" solution of (4). This is true since $\|x_n\|\geqq 5$ and $\|x_n\|\geqq\frac{1}{2\,|\,b\,|}=\frac{5}{2}$.

Remark 2

The iteration (1) can be written as

$$x_{n+1}=x_n-B(x_n)^{-1}P(x)_n),\; n=0,\,1,\,2,\ldots\ldots \tag{5}$$

The corresponding Newton-Kantorovich method can be written as

$$z_{n+1}=z_n-(2B(z_n)-I)^{-1}P(z_n),\; n=0,\,1,\,2,\,\ldots\ldots \tag{6}$$

or

$$z_{n+1}=(2B(z_n)-I)^{-1}(B(z_n,\,z_n)-y).$$

The latter iterative procedure is faster and easier to use most of the time (*e.g.* we need 6 iterations to find the solution in example 1 using iteration (6) with the same z), but if we choose an x_0 such that $\|x_0\|\geq\frac{1}{2\|B\|}$, then (6) does not guarantee that the limit

$w=\lim_{n\to\infty} x_n$ is such that $\|w\|<\frac{1}{2\|B\|}$ or $\|w\|\geq\frac{1}{2\|B\|}$. This is exactly the advantage of iteration (5) when compared with (6). The basic defect of (5) is that each step involves the solution of an equation with a different invertible operator $B(x_n)$. For this reason we can study the following modified method

$$x_{n+1}=x_n-B(z)^{-1}P(x_n),\ n=0,\ 1,\ 2,\ \ldots\ldots \tag{7}$$

The proof of the following theorem is omitted which is similar to that of theorem (1).

Theorem 2

Let $z\in X$, assume that the operator $B(z)$ is invertible and that the following are true :

(*a*) $\|B(z)^{-1}(I-B(z))\|<1$

(*b*) $D=(\|B(z)^{-1}(I-B(z)\|-1)^2-4\|B(z)^{-1}\|\cdot\|B\|$
$\|B(z)^{-1}P(z)\|>0.$

Then the iteration (7) is well defined and it converges to a unique solution x of (1) for any $x_0\in\bar{U}(z,r)$, where r is such that

$$c_1\leqslant r<c_2$$

with

$$c_1=\frac{1-\|B(z)^{-1}(I-B(z))\|-\sqrt{D}}{2\|B\|\cdot\|B(z)^{-1}\|}.$$

$$c_2=\frac{1-\|B(z)^{-1}(I-B(z))\|}{2\|B\|\cdot\|B(z)^{-1}\|}$$

We now give an example for Theorem 2.

Example 2

Let $X=\mathbb{R}\times\mathbb{R}$ with max-norm and consider the equation in X $\underline{x}=\underline{y}+\underline{x}^T\underline{M}\,\underline{x}$ where $\underline{Y}=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $Y_1=1.55, Y_2=-.85, \underline{M}=\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ with $M_1=\begin{bmatrix} -.45 & .01 \\ .9 & .02 \end{bmatrix}$, $M_2=\begin{bmatrix} .01 & -.7 \\ .02 & .5 \end{bmatrix}$, the notation

$$\underline{x}^T M \underline{x} = \begin{bmatrix} \underline{x}^T & M_1 \underline{x} \\ \underline{x}^T & M_2 \underline{x} \end{bmatrix} \text{ and}$$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the unknown vector. Equation (8) can be written also as

$$x_1 = -.45x_1^2 + .91x_1x_2 + .02x_2^2 + 1.55$$
$$x_2 = .01x_1^2 - .68x_1x_2 + .5x_2^2 - .85$$

Here $\|B\| = 1.38$. Let $z = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, then $\|B(z) - I\| = .9$, $\|B(z)^{-1}\| = .55555$, $\|P(z)\| = .05$. The requirements of Theorem 2 are satisfied for the above z in the ball $\bar{U}(z, r)$ for some r such that

$$.061321367 \leq r < .326086056 \ .$$

We now use iteration (8) with $x^0 = z$. If we allow an error $\epsilon \leqslant 5.10^{-3}$ then we need 5 iterations. More precisely

$$\underline{x}^{(1)} = \begin{bmatrix} -1.97222223 \\ .97368421 \end{bmatrix}, \quad \underline{x}^{(2)} = \begin{bmatrix} -1.996301957 \\ .976283584 \end{bmatrix}$$

$$\underline{x}^{(3)} = \begin{bmatrix} -1.99715663 \\ .96816564 \end{bmatrix}, \quad \underline{x}^{(4)} = \begin{bmatrix} -2.003524174 \\ .9654191 \end{bmatrix}$$

and

$$x^{(5)} = \begin{bmatrix} -2.00338145 \\ .96224933 \end{bmatrix}.$$

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COMPARISON THEOREMS OF TWO ABSOLUTE WEIGHTED MEAN SUMMABILITY METHODS

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Abstract :

The paper deals with the problem of inclusion and equivalence of two absolute weighted mean summability mothods. Necessary and sufficient conditions concerning (inclusion and equivalence) of these two methods have been established. Examples to show that each of these inclusions may hold in only one way without the other have been given, and an example to show that the equivalence may hold in some non trivial case have been constructed.

1. Let Σa_n be an infinite series with the sequence $\{S_n\}$ of its partial sums. Each sequence $\{q_n\}$ for which $Q_n = q_0 + q_1 + \cdots + q_n \neq 0$, or each n defines the weighted mean method M_q of the sequence $\{ S_n \}$, where

$$t_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k, \quad n=0, 1, 2 \ldots\ldots \tag{1}$$

If $t_n \to s$ as $n \to \infty$, $\{ S_n \}$ is said to be summable M_q to sum s, and if in addition, $\{ t_n \}$ is of bounded variation, then $\{ S_n \}$ is said to be absolutely summable M_q or summable $| M_q |$. We make similar definition with regard to other letters in place of q.

A method of summability is called regular, if it sums every convergent series to its ordinary sum It follows from Toeplitz's

Theorem (see Hardy [6]. Theorem 2) that M_q is regular if, and only if.

$$| Q_n | \to \infty, \text{ as } n \to \infty, \text{ and} \sum_{k=0}^{n} \left| q_k \right| = (| Q_n |).$$

Let $A=(A_{n,k})$ be a sequence-to-sequence transformation given by

$$t_n = \sum_{k=0}^{\infty} A_{n,k} S_k \quad n=0, 1, 2, \ldots\ldots \tag{2}$$

If whenever $\{ S_n \}$ has a bounded variation it follows that $\{ t_n \}$ has a bounded variation, and if the limits are preserved, we shall say that A is absolutely regular.

We shall write throughout (A) $\subseteq$ (B) to mean that any series summable by the method (A) to sum s is necessarily snmmable (B) to the same sum. (A) and (B) are equivalent if, (A) $\supseteq$ (B) and (B) $\subseteq$ (A). In this case we write $A \sim B$. We shall also write for any sequence, $\triangle U_n = U_n - U_{n+1}$

2. On inclusion relations of different summability methods much work has been done already *e.g.* (see [I], [2], [3], [4], [5], [6] and [7]). Dikshit ([4] and [5]) obtained many significant results on inculsion relation concerning absolute (and non-absolute) summability of both Riesz and Norlund means.

3. The object of this paper is to obtain rerults involving an inclusion relation of two absolute weighted mean methods, analogous to those by Dikshit ([4] ; Theorem (3.1) and [5] ; Theorem (2.2)), and to show that even if both M_q and M_r are regular, the inclusion need not hold. These results will be concluded in sections (5) and (6). The last section contains an example to show that the equivalence may holds in some non trivial case.

4. This section is devoted to result that is necessary for our purposes.

Theorem (4.1) (Mears [8]). The sequence-to-sequence transformation given by (2) is absolutely regular if, and only if, the following conditions are satisfied :

$$A_{n,k} \to 0 \text{ as } n \to \infty \text{ for each } k \tag{3}$$

$$\sum_{k=0}^{\infty} A_{n,k} \to 1 \text{ and } n \to \infty. \tag{4}$$

and

$$\sum_{n=0}^{\infty} \left| \sum_{v=k}^{\infty} A_{n,v} - \sum_{v=k}^{\infty} A_{n+1,v} \right| = O(1), \ (k \to \infty). \tag{5}$$

5. In this section we prove our main result :

Theorem (5.1). Suppose that M_r and M_q are regular. $q_n \neq 0$ (all $n \geqslant 0$), then $|M_q| \subseteq |M_r|$ if, and only if,

$$\left| R_{k-1} - \frac{Q_{k-1} r_k}{q_k} \left| \sum_{n=k}^{\infty} \right| \Delta \frac{1}{R_n} \right| = O(1), \text{ for all } k \geqslant 0. \tag{6}$$

Further, if $r_n < 0$ (all $n \geq 0$), then $|M_q| \subseteq |M_n|$ if, and only if,

$$\frac{Q_k r_k}{q_k R_k} = O(1). \tag{7}$$

We remark that a sufficient condition for (7) to be satisfied is that $Q_n = O(q_n)$, where $r_n > Q$. But this condition is not necessary. For this let $\{q_n\}$ and $\{r_n\}$ be any positive constants sequence, then (7) is clearly holds but $Q_n \neq O(q_n)$.

Proof of Theorem (5.1). Let $\{t_n\}$, $\{t_n^*\}$ be respectively the (M_q) and (M_r) transform of $\{S_n\}$. Using this to obtain t_n^* interms of t_n, we have

$$t_n^* = \sum_{v=0}^{n} E_{n,v} \, t_v \tag{8}$$

where

$$E_{n,n} = \frac{r_n Q_n}{R_n q_n} \tag{9}$$

$$E_{n,v} = \frac{Q_v}{R_n} \triangle \frac{r_v}{q_v}, (0 \leqslant V \leqslant n-1) \tag{1}$$

and

$$E_{n,v} = 0 \quad \text{othetwise} \tag{11}$$

To prove the result, it is enough to show that the conditsons of Theorem (4.1) are all satisfied. Condition (3) follows from (10). and the special case in which $S_n = 1$ (all $n \geq 0$) gives $t_n = t_n^* = 1$ (all $n \geq 0$), which by (8) implies (4). Put $E_{n,v}$ given in (8) instead of A_n, given in (2), we have that the left hand side of (5) is equivalent to :

$$\sum_{n=0}^{\infty} \left| \sum_{v=0}^{\infty} E_{n,v} - \sum_{v=0}^{k-1} E_{n,v} - \sum_{v=0}^{\infty} E_{n+1,v} + \sum_{v=0}^{k-1} E_{n+1,v} \right| \tag{12}$$

Since $t_n = t_n^* = 1$. it follows from (8) and (11) that each of the first and third sum inside the absolute of (12) is equal to 1. Hence (12) will reduees to :

$$\sum_{n=k-1}^{\infty} \left| \sum_{v=0}^{k-1} {}_{n,v} - \sum_{v=0}^{k-1} {}_{n+1,v} \right| \tag{13}$$

Using (9) and (10) and observe that

$$\sum_{v=0}^{k-1} Q_v \triangle \frac{r_v}{q_v} = R_{k-1} - Q_{k-1} \frac{r_k}{q_k}$$

we see that (13) will be reduces to :

$$\left| R_{k-1} - \frac{Q_{k-1}\, r_k}{q_k} \right| \sum_{n=k-1}^{\infty} \left| \triangle \frac{1}{R_n} \right| \tag{14}$$

Using (14), we see that (5) is satisfied if, and only if, (6) is valid. This completes the proof of the first part.

Next, if $r_n < 0$, then $\triangle (R_n)^{-1}\, 0$, so that the sum on the left hand side of (6) reduces to $(R_k)^{-1}$, and thus (6) is equivalent to :

$$\frac{R_{k-1}}{R_k} - \frac{Q_{k-1} r_k}{q_k\, R_k} = O(1). \tag{15}$$

so that by regularity of M_q and M_r, (15) is equivalent to (7). This completes the proof.

As a corollary to Theorem (5.1) is the following :

Corollary (5.1). Suppose that M_r and M_q, are regular, and let $\{ q_n \}$ and $\{ r_n \}$ be non-zero sequences. Then $| M_q | \sim | M_r |$ if and only if (6) and its equivalent (obtained by interchanging $R(r)$ and $Q(q)$ are satisfied. Further, if $\{ q_n \}$ and $\{ r_n \}$ are positive, then $| M_q | \sim | M_r |$ if and only if $Q_k r_k \asymp q_k R_k$ $(k \to \infty)$.

6. In this section we will give an example to show that the inclusion may hold in only one way not the other.

Example (6.1) Let $r_n = \dfrac{1}{n+1}$, $(n \geq 0)$, $q_0 = 1$, and $q_n = e^n$, $(n \geq 1)$, (thus M_g and M_r are both regular). Then $| M_q | \subseteq | M_r |$, but the converse is not true.

Proof. Since $\{ q_n \}$ and $\{ r_n \}$ are both positive, the result follows if we show that (7) is satisfied but $r_k Q_k \neq O\,(R_k q_k)$.

Observe that $Q_n \asymp e^n$ and $R_n \asymp \log n$ $(n \to \infty)$. This implies that the left hand side of (7) tends to zero, and that

$$\frac{r_k \, Q_k}{R_k \, q_k} \to \infty \text{ as } k \to \infty$$

This completes the proof.

Example (7.1). Let $Q_n = (n+1)\,!$, and $r_n = d^n$ $(n \geq 0)$, $(d > 1)$. Then $| M_q | \sim | M_k |$.

The proof is similar as before.

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COMPARISON OF METHODS FOR SOLVING ILL-POSED PROBLEMS IN THE FORM OF FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND, WITH AND WITHOUT NON-NEGATIVITY CONSTRAINTS

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Abstract

In many branches of science, problems arise in which it is desired to solve ill-posed problems in the form of Fredholm integral equations of the first Kind.

$$\int_a^b K(x, y) f(y)\, dy = g(x) \qquad c \leqslant x \leqslant d$$

In this paper we shall discuss two different methods to solve only *severely ill-posed problems*, available in the literature. The methods are as follows :

(*i*) Generalized cross-validation regularization method without using non-negativity constraints.

(*ii*) Generalized cross-validation regularization method using non-negativity constraints.

The two methods will be tested on integral equations of first kind of convolution type and graphs have been drawn for comparison purposes.

Introduction

For years ill-posed problems have been considered as mere mathematical anomalies. However, it appeared in the early sixties that this attitude was erroneous and that many ill-posed problems, generally inverse problems, arose from practical situations. Now a days there is no doubt that a systematic study of these problems is of great relevance in many fields of applied physics. For example, problems of image reconstruction and enhancement, X-rays and neutron scattering ; integral equations of the first kind in spectroscopy, chemical analysis queueing theory, astrophysics and photon-correlation, optimal control, seismographic data analysis, calculation of atmospheric temperature profiles, numerical inversion of Laplace transforms ; numerical inversion of radon transforms in computerized tomography ; inverse source problems and inverse scattering problems in optics, meteorology, stereology and other fields.

1. Mothod 1 Cross-validation without non-negativity

Introduction

The concept of a cross-validation criterion is an old one. In its most primitive but nevertheless useful form, it consists of the controlled and uncontrolled division of the data sample into two subsamples, the choice of a statistical predictor, including any necessary estimation of one subsample and then the assessment of its performance by measuring its predictions against the other subsample.

Many authors have refined this technique but the refinement described by Mosteller and Tukey [19], which they term "Simple Cross-validation" is worth mentioning.

Stone [24] brought in the question of choice of predictor and employed the implied cross-validation criterion in a way that integrates the procedures of choice and assessment. Then the method was refined by Wahba [29], Golub, Heath and Wahba [14], but Wahbas' analysis of fredholm integral equations of the first kind is restricted to $L_2(0, 1)$ or more generally, reproducing Kernel Hilbert Spaces (RKHS). The RKHS theory is therefore not directly applicable to convolution type equations on $(-\infty, \infty)$, but is easily modified for this case,

2. Description of the method : [4, 29]

We shall approximate

$$\int_{-\infty}^{\infty} K(x-y)\, f(y)\, dy = g(x), \qquad -\infty < x < \infty \tag{1}$$

by replacing it by

$$\left(K_N\ f_N \right)(X) = \int_0^1 K_N(x-y)\ f_N(y)\ dy = g_N(x) \tag{2}$$

where K_N is periodically continued outside (0, 1).

Consider the integral equation (1)

In tikhonov regularization, the approximate solutions f are defined variationally as

$$C(f;\lambda) = \underset{f \in w}{\text{Min}} \{ \| Kf-g \|^2 + \lambda\, \Omega(f) \} \tag{3}$$

where w is some space of smooth functions and $\lambda > 0$ is a regularization parameter.

Here Ω is some non-negative "stabilizing" functional which controls the sensitivity of the regularized solutions f_λ to perturbations in g.

We shall restrict our attention to pth order regularization of the form

$$C(f;\lambda) = \left\| Kf-g \right\|_2^2 + \lambda \left\| f^{(p)} \right\|_2^2 \tag{4}$$

which is minimized over the subspace $H^p \subset L_2$.

Both norms in (4) are L_2, $f^{(p)}$ denotes the pth derivative of f and λ the regularization parameter.

Pth order Regularization Filters for Convolution Equations.

Consider the smoothing functional $C(f;\lambda)$ of equation (4) with $\Omega(f) = \left\| f^{(p)} \right\|_2^2$. Working in $L_2(R)$ we have in the case of the convolution equation (1).

$$C(f;\lambda) = \left\| k(x)*f(x) - g(x) \right\|_2^2 + \lambda \left\| f^{(p)} \right\|_2^2$$

$$= \frac{1}{2\pi}\left\{ \left\| \hat{K}(w)\hat{f}(w) - \hat{g}(w) \right\|_2^2 \right\} + \lambda \left\| (iw)^p \hat{f}(w) \right\|_2^2 \quad (5)$$

using plancherel's identity, the convolution theorem for FTs and the and propery $(\hat{f^{(p)}}) = (iw)^p \hat{f}$.

Thus

$$C(f:\lambda)\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{ \hat{k}\hat{f}-\hat{g})(\overline{\hat{k}}\overline{\hat{f}}-\overline{\hat{g}})+\lambda w^{2p}\hat{f}\overline{\hat{f}} \right\} dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left| \hat{k} \right|^2 + \lambda w^{2p}) \left| \hat{f} \right|^2 - (\hat{k}\hat{f}\overline{\hat{g}}) + \overline{\hat{k}}\overline{\hat{f}}\hat{g} + \left| \hat{g} \right|^2 \right\} dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\left| \hat{k} \right|^2 + \lambda w^{2p} \right) \left| \hat{f} - \frac{\overline{\hat{k}}\hat{g}}{|\hat{k}|^2 + w^{2p}} \right|^2 dw$$

$$+ \frac{\lambda}{n}\int_{-\infty}^{\infty}\frac{w^{2p}|\hat{g}|^2}{|\hat{k}|^2 + \lambda w^{2p}} dw \quad (6)$$

clearly $C(f;\lambda)$ is minimized w.r.t.f. when

$$\text{when } \hat{f}(w) = \frac{\overline{\hat{k}}\hat{g}}{\left| \hat{k} \right|^2 + \lambda w^{2p}} = z(w;\lambda)\ \frac{\hat{g}(w)}{\hat{k}(w)} \quad (7)$$

$$\text{where } z(w;\lambda) = \frac{\left| \hat{k} \right|^2}{\left| \hat{k} \right|^2 + \lambda w^{2p}} \quad (8)$$

$z(w;\lambda)$ is called the pth order filter or stabilizer

$$\text{(7) can be written as } f_\lambda(y) = \frac{1}{2\pi}\int_{-\infty} z(w;\lambda)\frac{\hat{g}(w)}{\hat{k}(w)} \times \text{Exp}(iwy)\, dw \quad (9)$$

We assume throughout that the support of each function f, g and K is essentially finite and contained within the interval (0, 1) possibly by a change of variable. It is then convenient to adopt the approximating function space T_{N-1} of trigonometric polynomials of degree at most N−1 and period 1, since the discretization error in the convoluion may be made exactly zero at the grid points and FFTs (Fast Fourier Transforms) may be employed in the solution procedure. Let g and K be given at N equally spaced points $x_n = nh$ $n = 0, 1, 2, \ldots\ldots N-1$. With spacing $h = \frac{1}{N}$. Then g and K are interpolated by g_N and $K_N \in T_{N-1}$, where

$$g_N(x) \; \frac{1}{N} \sum_{q=0}^{N-1} \hat{g}_{N,q} \; exp\,(i\, w_q x) \tag{10}$$

$$\hat{g}_{N,q} = \sum_{n=0}^{N-1} g_n \, exp(-iw_q \, x_n) \tag{11}$$

and

$$g\,(x_n) = g_n = g_N\,(x_n),\; w_q = 2\pi q \tag{12}$$

with similar expressions for k_N.

In T_{N-1}, f_λ in (9) is approximated by

$$f_{N;\lambda}(x) = \sum_{q=0}^{N-1} z_{q;\lambda} \frac{\hat{g}_{N,q}}{\hat{k}_{N,q}} \; exp\,(i\, w_q\, x) \tag{13}$$

$$Z_{q;\lambda} = \frac{\left|\hat{K}_{N,q}\right|^2}{\left|\hat{K}_{N,q}\right|^2 + N^2 \lambda \tilde{w}_q^{\,2p}}$$

Where

$$\tilde{W}_q = \begin{cases} W_q, & 0 \leqslant q < \tfrac{1}{2}N \\ W_{N-q}, & \tfrac{1}{2}N \leqslant q \leqslant N-1 \end{cases}$$

The optimal λ in (14) is still to be determined. From equation (13) we know that the filtered solution $f_{N,\lambda}(x) \in T_{N-1}$ which minimizes

$$\sum_{n=0}^{N-1} \left[K_N * f)(x_n) - g_n \right]^2 + \lambda \left\| f^{(p)}(x) \right\|_2^2$$

$$\text{is } f_{N,\lambda}(x) = \frac{1}{N} \sum_{q=0}^{N-1} f_{N,\lambda,q} \exp(2\pi i\, qx),$$

Where

$$f_{N,\lambda,q} = Z_{q;\lambda} \frac{\hat{g}_{N,q}}{\hat{K}_{N,q}}$$

$$\text{with } Z_{q;\lambda} = \frac{\left| \hat{K}_{N,q} \right|^2}{\left| \hat{K}_{N,q} \right|^2 + N^2 \lambda \tilde{w}_q^{2p}}$$

$$\text{Where } \tilde{w}_q = \begin{cases} w_q, & O \leqslant q < \frac{1}{2}N \\ w_{N-q} & \frac{1}{2}N \leqslant q < N-1 \end{cases}$$

The idea of generalized cross-validation (GcV) is quite simple to understand. Suppose we ignore the jth data point g_j and define the filtered solution $f_{N,\lambda}^{(j)}(x) \in T_{N-1}$ as the minimizer of

$$\sum_{n=0}^{N-1} \left[\left(K_N * f \right) \left(x_n \right) - g_n \right]^2 + \lambda \left\| f^{(p)}(x) \right\|_2^2$$

then we get a vector $g_N^{(j)} \in R_N$ defined by

$$g_{N,\lambda}^{(j)} = K f_{N,\lambda}^{(j)} \tag{15}$$

Clearly the jth element $g_{N,\lambda,j}$ of equation (15) should "predict" the missing value g_j. We may thus construct the weighted mean square prediction error over all j.

$$V(\lambda, p)=\frac{1}{N}\sum_{j=0}^{N-1} w_j(\lambda)\left(g^{(j)}_{N.\lambda, j}-g_j\right)^2 \tag{16}$$

The principle of GcV applied to the deconvolution problem then says that the best filtered solution to the problem should minimize the mean equare prediction error in (16). Thus the optimal λ minimizes $V(\lambda, p)$ for given p and does not require knowledge of σ^2.

To minimize $V(\lambda, p)$ in the form given by equation (16) is a time consuming problem. Wahba [29] has suggested an alternative expression which depends on a particular choice of weights, resulting in considerable simplification. Let

$$f_{N,\lambda}=\left(f_{N,\lambda}(x_o)\ldots . f_{N,\lambda}\left(x_{N-1}\right)\right)^T \tag{17}$$

and define $\underline{g}_{N.\lambda}=K\underline{f}_{N,\lambda}$ (18)

then there exists a matrix $A(\lambda)$, called an influence matrix such that

$$\underline{g}_{N,\lambda}=A(\lambda)\underline{g}N \tag{19}$$

Let $K=\text{diag}\left(h\,\hat{K}_{N,q}\right)$ and $\hat{Z}=\text{diag}\left(Z_{q;\lambda}\right)$

then from (13) we see that

$$\underline{f}_{N,}=\psi\,\hat{K}^{-1}\,\hat{Z}\,\hat{\underline{g}}_N \tag{20}$$

where $\hat{\underline{g}}_N=\psi\underline{g}_N$ (21)

and so $A(\lambda)=\psi\,\hat{Z}\,\psi^H$ (22)

since $K=\psi\,\hat{K}\psi^H$ (23)

Wahba (29) has shown in a more general context, that the choice of weights

$$w_j(\lambda)=\left|\frac{1-a_{jj}(\lambda)}{\frac{1}{N}\text{Trace }(I-A(\lambda))}\right|^2, \quad J=0,\ldots\ldots N-1 \qquad (24)$$

where A (λ) is the influence matrix in equation (19), enables the expression (16) to be written in the simpler form

$$\frac{\frac{1}{N}\left\|(I-A(\lambda))g_N\right\|_2^2}{\left[\frac{1}{N}\text{Trace }(1-A(\lambda))\right]^2} \quad [29] \qquad (25)$$

From equation (22) it follows that

$$\frac{\frac{1}{N}\left\|(I-\hat{Z})\,\hat{g}_N\right\|_2^2}{\left[\frac{1}{N}\text{Trace }(I-\hat{Z})\right]^2}$$

i.e.

$$V(\lambda, p)=\frac{\frac{1}{N}\sum_{q=0}^{N-1}\left(1-Z_{q;\lambda}\right)^2\left\|g_{N,q}\right\|^2}{\left[\frac{1}{N}\sum_{q=0}^{N-1}\left(I-Z_{q;\lambda}\right)\right]^2} \qquad (26)$$

Since the matrix A(λ) in (15) is circulant, the weights in (24) are all unity. The expression in (26) is minimized using NAG Routine **EO4 ABA**, which uses a quadratic interpolation technique to obtain a minimum.

3. Cross Validation Regularization Method with Non-negativity Constraints.

In this section we examine an extension of the CV method in T_{N-1} to the case where non-negativity constraints, $f(x)\geq 0$ are also imposed on the solution.

The basic ideas we discuss were proposed by Wahba [30] but the method of computation we adopt differs from hers and is less expensive.

Description of the Method :

We estimate the solution of

$$\int_{-\infty}^{\infty} k(x-y)\, f(y)dy = g(x) \qquad -\infty \leqslant X \leqslant \infty \tag{27}$$

where we know in advance that f is non-negative and hence our estimate f_N is constrained to be non-negative. In contrast to the successive broadness characteristic of unconstrained estimates of narrow functions f (Dawson et all [9]), the use of non-negativity constraints greatly sharpens the estimates (Cooper [7]).

Non-Negative functions are an important class for many physical application *e.g.* Density functions. These constraints have been applied to problems in phisiology by Wagner et all [28] and with prior choice of smoothing by Evans and Wagner [11].

We first describe Wahba's constrained algorithm [30].

$$\text{Let } f_N = \Big(f(x_o), \;\ldots\ldots\ldots, f(x_{N-1}) \Big)^T$$

and consider the pth order regularization functional in T_{N-1}

$$C\Big(f_{N\,;\,\lambda} \Big) = \Big\| \hat{K}\,{}_{\psi}^{H} f_N - {}_{\psi}^{H} g_N \Big\|_2^2 + \lambda \underline{f}_N^{H}\, \psi \hat{J} \psi^{H}\, \underline{f}_N$$

where $\hat{K} = {}_{\psi}^{H} K \psi$

$$\psi \equiv \psi N \text{ and } \psi rs = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi}{N} \text{ irs} \right),\; r, s = 0, 1, 2, \ldots\ldots N-1$$

$$\text{and } \hat{J} = \text{diag}\Big(i\,\tilde{w}_q \Big)^{2p} \tag{29}$$

Let $\underline{f}_\lambda$ be the minimizer of (28) subject to $\underline{f}_N \geqslant \underline{0}$, with components $f_{\lambda,n}$. The indices $n_1, \ldots\ldots\ldots, n_L$ for which $f_{\lambda,n} \geqslant 0$ are first determined. Let E be the $N \times L$ indicator matrix of these indices. That is E has a unit element in the mth row and nth column,

if $m = n_j$, $j = 1, \ldots\ldots, L$ and zeroes elsewhere.

In what follows we denote by I the set of indices $\left(n_1 \ldots\ldots, n_L\right)$ of inactive constraints.

The constrained minimizer of (28) may be written

$$\underline{f}_\lambda = E\left(E^H \psi \hat{K}^H \hat{K} \psi^H E + \lambda E^H \psi \hat{J} \psi^H E\right)^{-1} E^H \psi \hat{K}^H {}^H_\psi \underline{g}_N$$

Defining $\underline{g}_\lambda = K \underline{f}_\lambda$

There exists an $N \times N$ influence matrix $A_L(\lambda)$ satisfying

$$\underline{g}_\lambda = A_L(\lambda) \underline{g}_N.$$

It can be shown that

$$A_L(\lambda) = \psi \hat{K}^H_\psi E\left(\Sigma_K + {}_\lambda\Sigma_J\right)^{-1} E^H \psi \hat{K}^H {}^H_\psi \tag{31}$$

where $\Sigma_K = E^H \psi \hat{K}^H \hat{K} \psi^H E$

$\Sigma_J = E^H \psi \hat{J} \psi^H E$

with the property that

$$\text{Trace}\left(I - A_L(\lambda)\right) = N - L + \lambda \text{ Trace } (B)$$

where $B = \Sigma_J\left(\Sigma_K + \lambda \Sigma_J\right)^{-1}$

Wahba [30] argues that the optimal λ in the constrained setting may be found by minimizing

$$V^C_{\text{approx}}(\lambda) = \frac{\left\|\hat{K}^H_\psi \underline{f}_\lambda - {}^H_\psi \underline{g}_N\right\|^2}{\left[\frac{1}{N}(N - L + \lambda \text{ Trace } (B)\right]^2} \tag{32}$$

clearly $\underline{f}_\lambda$ depends non-linearly on E as well as on λ and so E must be recomputed whenever $\underline{f}_\lambda$ is computed. These iterations can be expensive. Moreever, $V^{C}_{approx}(\lambda)$ need not be a continuous function of λ.

For given λ Wahba uses a quadratic programming algorithm to minimize (28) subject to $\underline{f}_N \geq \underline{0}$. A unique minimnm always exists (see *e.g.* Butler et al [6]).

Having found E and f_λ she then computes B by solving the L linear system defined by

$$\left(\Sigma_K + \lambda\, \Sigma_J\right) B = \Sigma_J \tag{33}$$

using LINPACK (Dongerra et al [10]). She then examines the values $V^{C}_{approx}(\lambda)$, adjusts λ accordingly if a minimum is not found then repeats the process. This is an expensive procedure.

Our Method :

Our method is simpler than hers.

We observed that since $\psi\left(\hat{K}^H \hat{K} + \lambda \hat{J}\right)^H_\psi$ is circulant matrix so is $\Sigma_K + \lambda\, \Sigma_J$ and consequently so is $E\left(\Sigma_K + \lambda\, \Sigma_J\right)^{-1} E^H$ in equation (31). Thus $A_L(\lambda)$ in is clearly circulant. In principle we can use the L-dimensional DFT (Discrete Fourier Transforms) to evaluate $A_L(\lambda)$, thus avoiding the necessity of soving the L-linear system in (33) which is an expensive procedure on computer.

In practice, however, we have used the arproximation.

$$\hat{K}^H_\psi E\left(\Sigma_K + \lambda \Sigma_J\right)^{-1} E^H \psi \hat{K}^H = \text{diag}\left(\tilde{Z}_{q;\lambda}\right) \tag{34}$$

where

$$\tilde{Z}_{q;\lambda} = \begin{cases} Z_{q;\lambda} & , \quad q \in I \\ 0 & , \quad q \notin I \end{cases} \tag{35}$$

and

$$Z_{q;\lambda}=\frac{\left|\hat{K}_{N,q}\right|^2}{\left|\hat{K}_{N,q}\right|^2+N^2\lambda\tilde{w}_q^{2p}}$$

From equation (34) it follows that

$$A_L(\lambda)=\psi \operatorname{diag}\left(\tilde{Z}_{q;\lambda}\right)\psi^H$$

and so from equation (32) we have

$$V^C_{approx}(\lambda)=\frac{\frac{1}{N}\left[\sum_{q\in I}\left(1-Z_{q;\lambda}\right)^2\left|\hat{g}_q\right|^2+\sum_{q\notin I}\left|\hat{g}_q\right|^2\right]}{\frac{1}{N}\left[N-L+\left(\sum_{q\in I}\left(1-Z_{q;\lambda}\right)\right)^2\right]} \qquad (36)$$

We minimize V^C_{approx} in (36) by making a linear search in λ, the function is not always continuous because the index I changes with λ. At each step we minimize $C\left(f_N;\lambda\right)$ in (28) subject to non-negativity constraints using the NAG (U.K.) quadratic programming subroutine **EO4LBF**, which yields the index set I for any given λ. All calculation are done using double precision because the examples tested include the severely ill-posed problems. When a minimizing value of λ is found the corresponding $\underline{f}_\lambda$ is given by NAG subroutine **E04LBF**.

Test Problems :

Problem (1) : This problem is given by Phillips (21) and has a noisy data function g with a maximum abrolute error of about 0.7%. The noise results purely from quadrature errors.

TABLE (1)

x_n	g_n	k_n	f_n
—30.0	0.0100	0.1184	0.0000
—28.0	0.0100	0.1311	0.0000
—26.0	0.0110	0.1464	0.0000
—24.0	0.0170	0.1651	0.0000
—22.0	0.0305	0.1883	0.0006
—20.0	0.0405	0.2179	0.0000
—18.0	0.0585	0.2563	0.0000
—16.0	0.0869	0.3077	0.0000
—14.0	0.1309	0.3788	0.0000
—12.0	0.2018	0.4816	0.0000
—10.0	0.3235	0.6380	0.0000
—8.0	0.5469	0.8914	0.0000
—6.0	0.9621	1.3333	0.0019
—4.0	1.6301	2.1483	0.0345
—2.0	2.4047	3.5103	0.0965
0.0	2.9104	4.3600	0.1321
2.0	2.8912	3.0628	0.1096
4.0	2.4586	1.6329	0.0584
6.0	1.9049	0.8806	0.0349
8.0	1.4144	0.5095	0.0173
10.0	1.0282	0.3137	0.0107
12.0	0.7411	0.2021	0.0028
14.0	0.5409	0.1341	0.0005
16.0	0.4083	0.0906	0.0000
18.0	0.3214	0.0614	0.0000
20.0	0.2623	0.0413	0.0000
22.0	0.2201	0.0269	0.0000
24.0	0.1886	0.0165	0.0000
26.0	0.1580	0.0089	0.9000
28.0	0.1270	0.0031	0.0000
30.0	0.0780	0.0013	0.0000

We have

$$\int_{-30}^{30} K(x-y) f(y) dy = g(x) \tag{37}$$

where K(x), $g(x)$ and $f(x)$ are given in **Table 1** and computer diagrams (1) and (2). The number of grid points is 31.

Problem 2. This problem is given by Turchin [26] and is modified to take the wider Kernal to make the problem severely ill-posed, we have

$$\int_{-3.2}^{3.1} K(x-y) f(y) dy = g(x)$$

where f is the sum of two Gaussian functions

$$f(x) = 0.5 \exp\left[-\frac{(x+04)^2}{0.18}\right] + \exp\left[\frac{(x-0.6)^2}{0.18}\right] \tag{38}$$

with essential support $-1.7 < x < 1.5$

By the essential support of a function of $f(x)$ we mean that part of its domain for which $|f(x)| > \epsilon$ where $\epsilon \geqq 0$ is small, *e.g.* $\epsilon = 1\%$ of $\max[|f(x)|]$.

The Kernal K(x) is given by

$$K(x) = \begin{cases} (5/12)(-x+1.2) & , \quad 0 \leqslant x < 1.2 \\ (5/12)(x+1,2) & ; \quad -1.2 \leqslant x < 0 \\ 0 & , \quad |x| \geqq 1.2 \end{cases}$$

The essential support of $g(x)$ is $-2.5 < x < 2.7$

The functions are displayed in computer graph DIAG(3) with a spacing 0.1.

4. Addition of random noise to the data functions

In solving the problem 2 we have considered the data functions contaminated by varying amounts of random noise. To generate sequence of random errors of the form $|\epsilon_n|$, $n=0, 1, \ldots\ldots\ldots N-1$. We have used the NAG. Algorithm GO5DDA which returns Pseudo-random real numbers taken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental errors we have

A=0

$$B = (X/_{100} \max_{0 \leqslant n \leqslant N-1} \left| g_n \right| \tag{39}$$

where X denotes a choosen percentage, *e.g*, X=0.7, 1.7, 3.3 and 6.7. Thus the random error ϵ_n added to g_n does not exceed $3x\%$ of the maximum value of $g(x)$.

5. Numerical Results

In this section we describe the application ~~of the~~ two methods namely GcV unconstrained and GcV constrained to the test problems 1 and 2. Results are shown in Tables 2 and Table 3.

GcV unconstrained Method

Problem 1. This problem is in fact only mildly ill-posed. Therefore, only a very weak filter is needed to resolve this problem, the filter provided by the method is too strong, the result obtained is not very good as shown in diag (4).

GcV constrained Method

This method worked very well, here N=32, the algorithm was tried and the solution obtained is perfect as shown in Diag (5).

GcV unconstrained Method

Problem 2. This is highly ill-posed problem. For accurate data the method yielded perfect solution resolving the two peaks very clearly, but for 0.7% noise, the method failed to resolve the peaks clearly, however when we used $p=6$, the solution improved a lot and it resolved the two peaks clearly but possessed negative lobes at the end points of the solution as shown in diag (6). Therefore, for such problems, extra information is needed such as non-negativity.

GcV constrained Method

This severely ill-posed problem could not be satisfactorily treated using unconstrained regularization, because large negative lobes were always presesent and the size of these lobes increased with the increase in the noise level. For constrained regularization the results are dramatically superior.

(*i*) With 0.7% noise level, the GcV constrained method gave a very good result as shown in Diag (7).

(*ii*) With a 1.7% noise level again GcV constrained method yieided a very good result as shown in Diag (8)

(*iii*) With 3.3% noise G*c*V constrained method performed reasonably well and result is shown in Diag (9).

(*iv*) With 6.7% noisc the G*c*V constrained method succeeded in resolving the two peaks and the solution seems to be reasonable. The solution is shown in Diag (10).

Concluding Remarks

The overall performance of the methods is quite good.

As for as unconstrained regularization is concerned the method is quite good for low level noise and for mildly and moderated ill-posed problems.

For higher level of noise and for severely ill-posed problems negative lobes at the end points is not a commedable feature. Therefore, for such problems, extra information is needed such as non-negativity.

Constrained regularization method worked very well for severely ill-posed problem (2) even with higher levels of noise. In fact the GcV constrained method can cope with noise level about =10% in severely ill-posed problems and yielded exceedingly excellent results.

Aknowledgement

The author is very much indebted to his supervisor Dr. A.R. Davies for his valuable suggestions and guidance to develope the theory of the algorithms.

TABLE 2

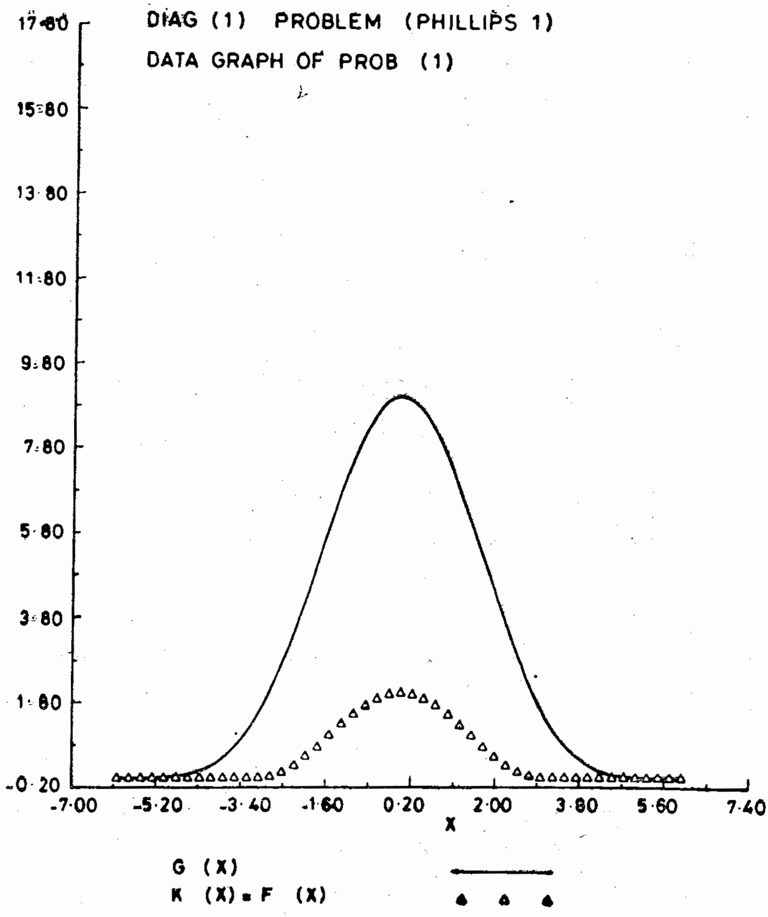
GcV Method (unconstrained)

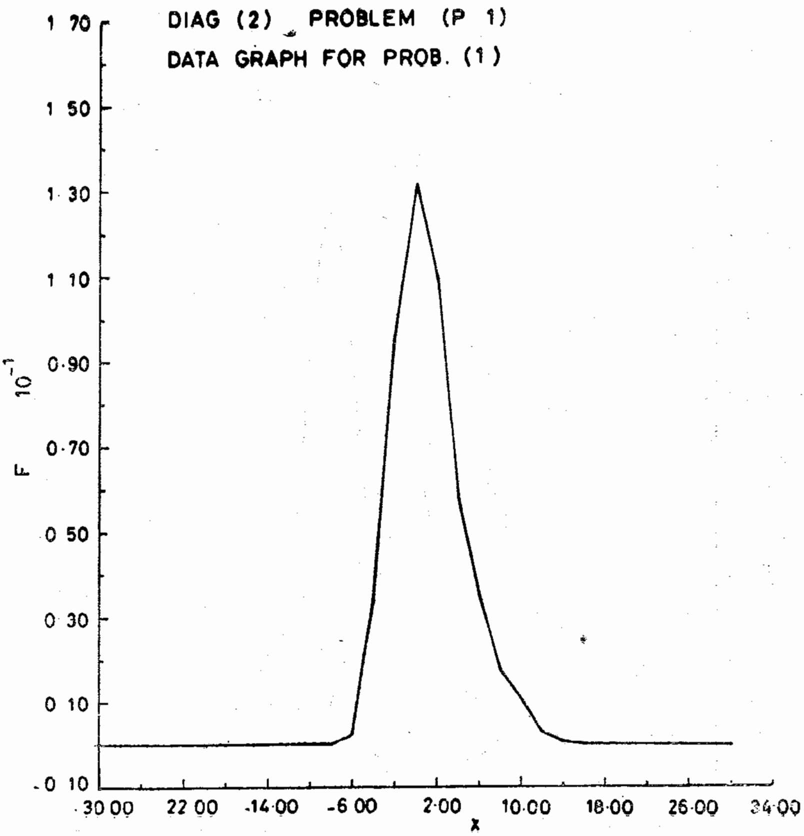
Problem	N	Noise level	λ	$\|f-f_N\|_2$	DIAG
P (1)	32	0.7% already in the data	4.892×10^{0}	8.91×10^{-2}	4
P(2)	64	0.0%	4.30×10^{-14}	2.67×10^{-3}	
P=2		0.7%	1.631×10^{-7}	1.897×10^{-1}	6
P=6		0.7%	3.710×10^{-14}	1.272×10^{-1}	

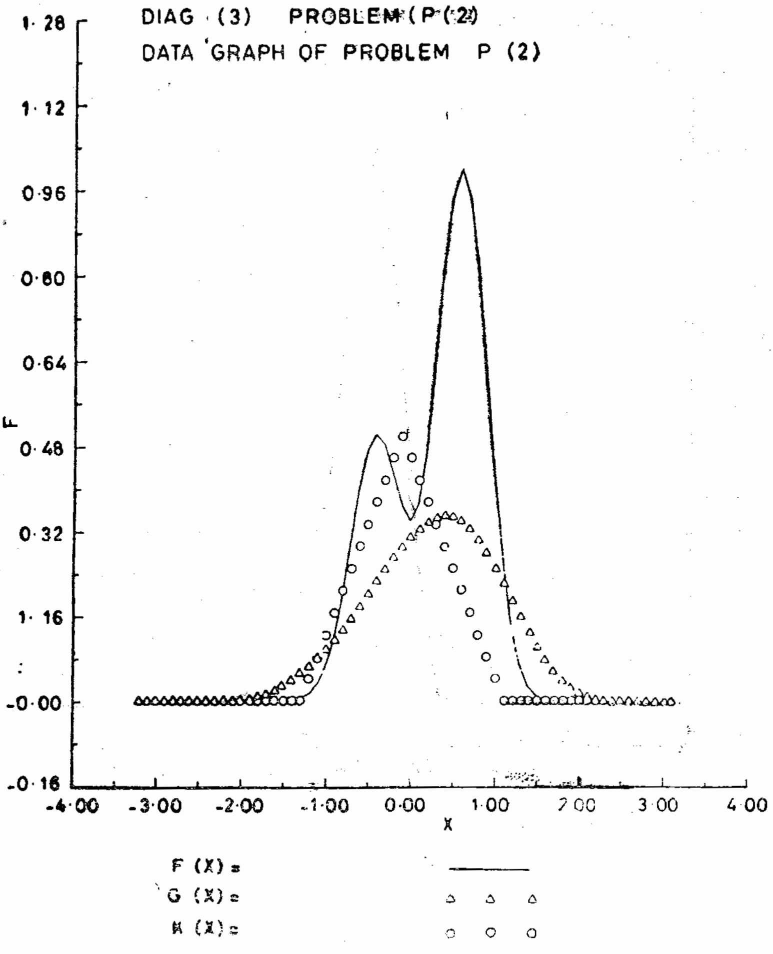
TABLE 3

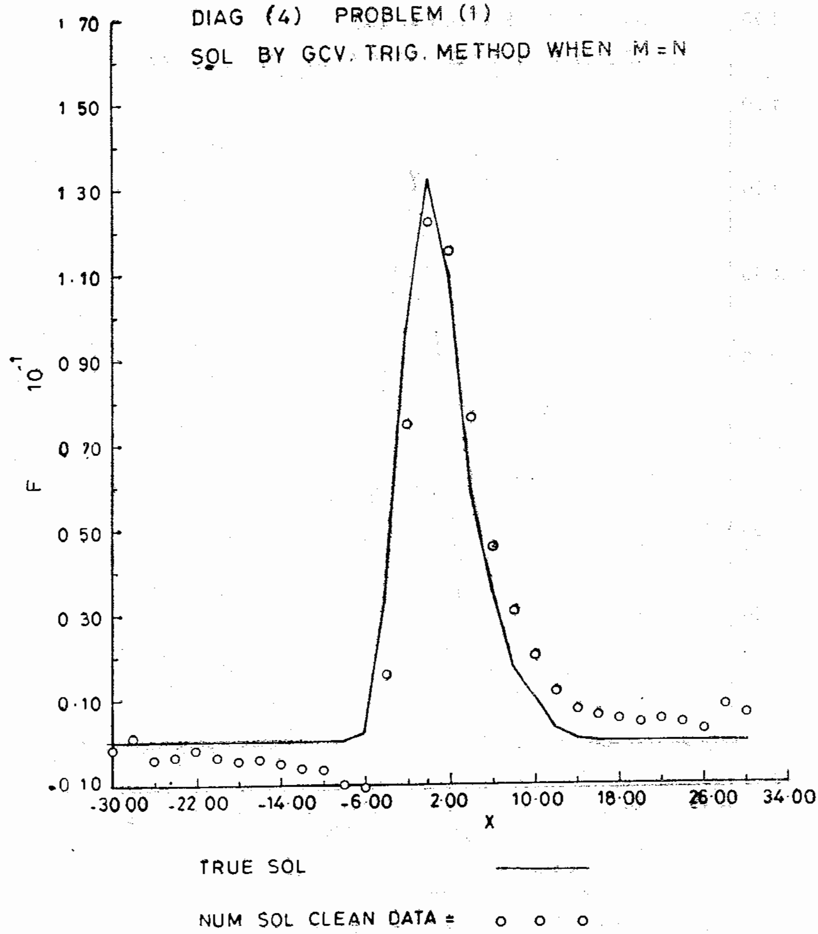
GcV Method (constrained)

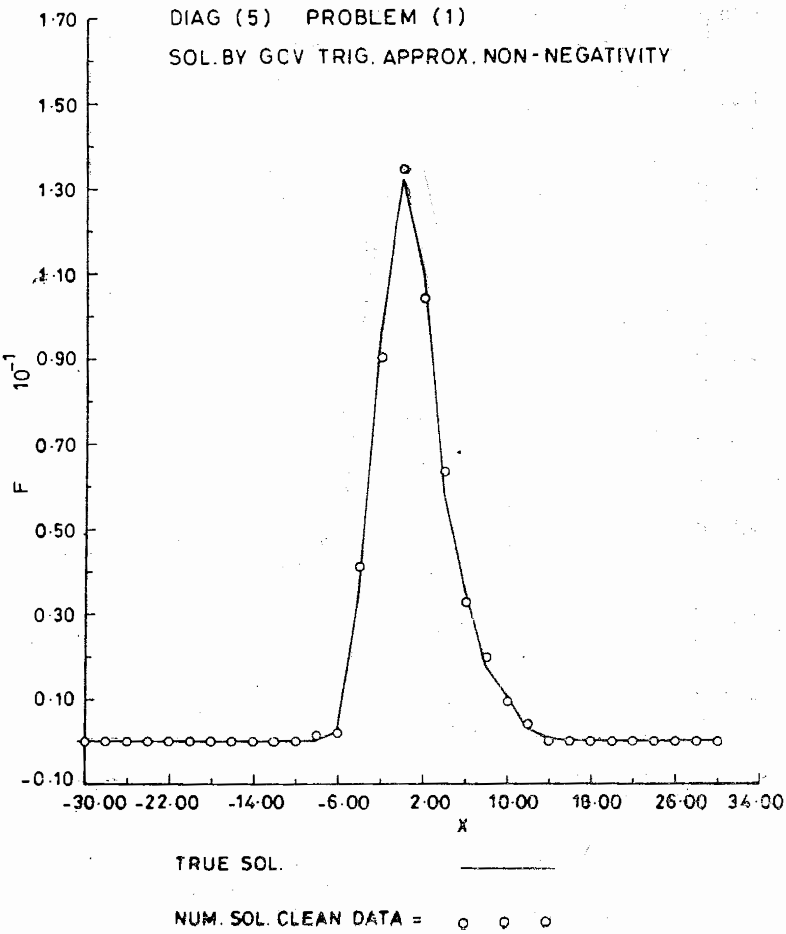
Problem	N	Noise Level	λ	$\|f-f_N\|_2$	DIAG
P(1)	32	0.7% already in the data	1.00×10^{-4}	1.793×10^{-2}	5
P(2)	64	0.0%	2.10×10^{-15}	6.81×10^{-3}	7
,,	,,	0.7%	1.614×10^{-9}	4.79×10^{-2}	7
,,	,,	1.7%	7.402×10^{-9}	7.511×10^{-2}	8
,,	,,	3.3%	1.296×10^{-8}	1.005×10^{-1}	9
,,	,,	6.7%	3.478×10^{-8}	1.415×10^{-1}	10

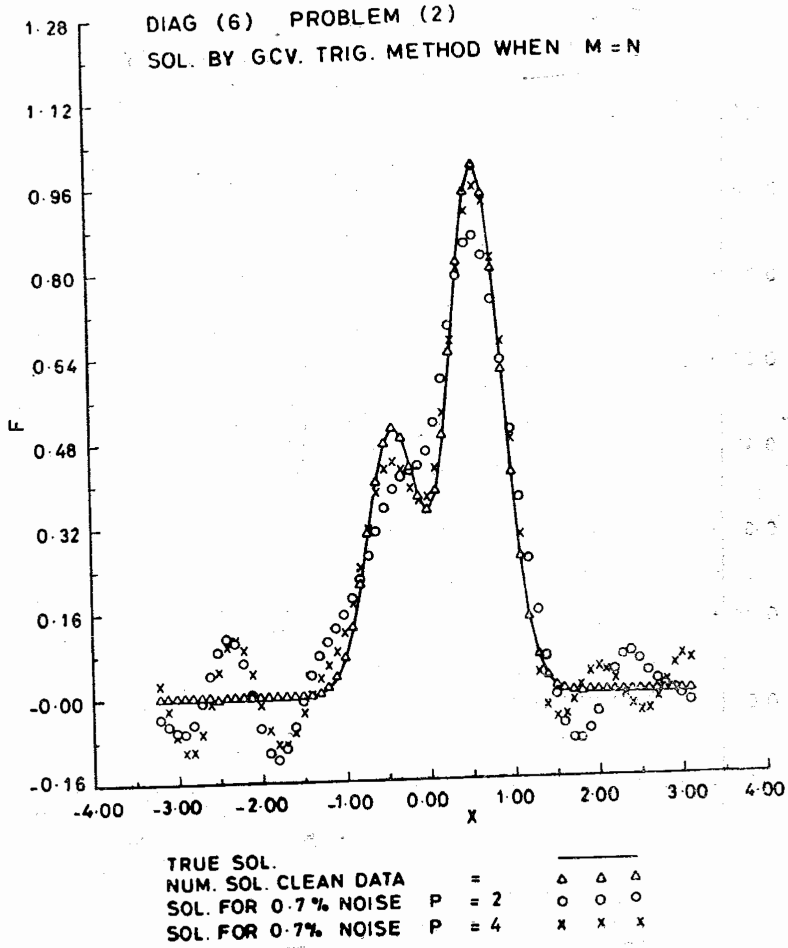

DIAG (1) PROBLEM (PHILLIPS 1)
DATA GRAPH OF PROB (1)
17·80
15·80
13·80
11·80
9·80
7·80
5·80
3·80
1·80
-0·20
-7·00
-5·20
-3·40
-1·60
0·20
2·00
3·80
5·60
7·40
X
G (X)
K (X) = F (X)

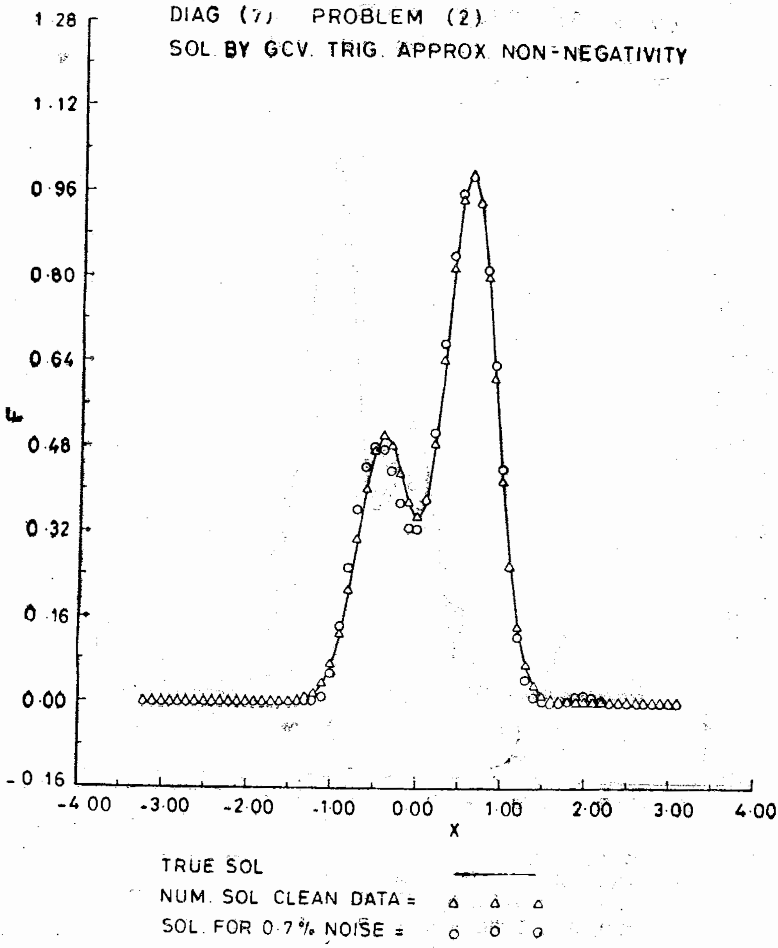

DIAG (2) PROBLEM (P 1)
DATA GRAPH FOR PROB. (1)
1 70
1 50
1 30
1 10
0·90
0·70
0 50
0 30
0 10
-0 10
F 10^{-1}
30 00
22 00
-14·00
-6 00
2·00
10·00
18·00
26·00
34·00
X

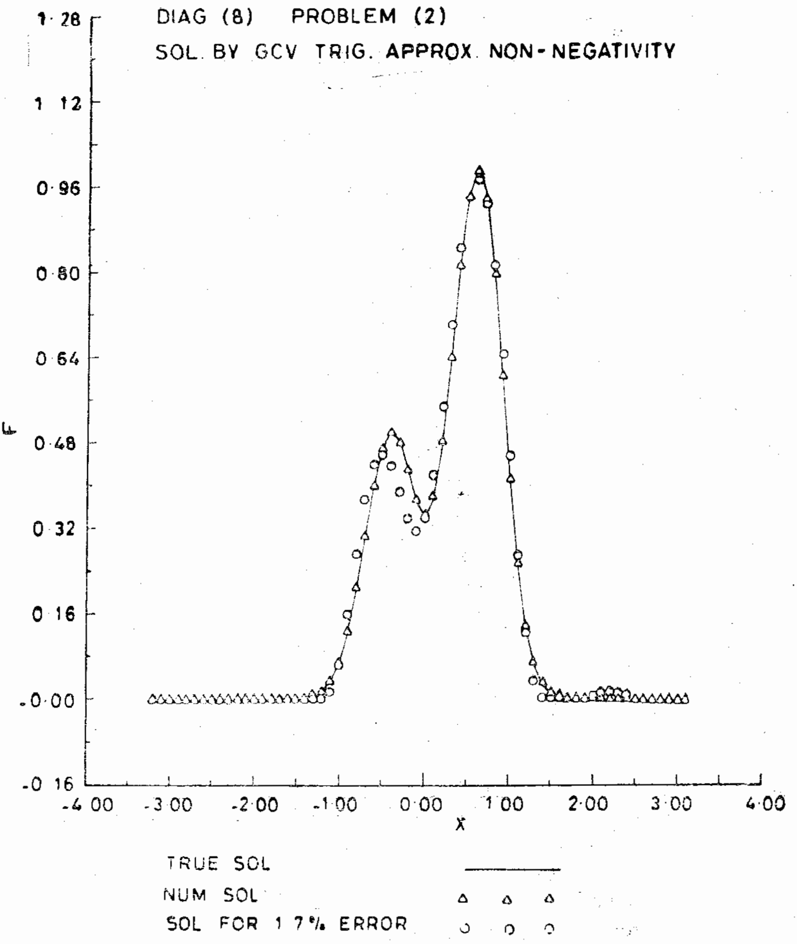

DIAG (3) PROBLEM (P (2))
DATA GRAPH OF PROBLEM P (2)
1·28
1·12
0·96
0·80
0·64
0·48
0·32
1·16
-0·00
-0·16
F
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
F (X) =
G (X) =
H (X) =


DIAG (4) PROBLEM (1)
SOL BY GCV. TRIG. METHOD WHEN M = N
1 70
1 50
1 30
1·10
0 90
0 70
0 50
0 30
0·10
-0 10
F 10^-1
-30 00
-22 00
-14·00
-6·00
2·00
10·00
18·00
26·00
34·00
X
TRUE SOL
NUM SOL CLEAN DATA =


DIAG (5) PROBLEM (1)
SOL. BY GCV TRIG. APPROX. NON-NEGATIVITY
1·70
1·50
1·30
1·10
0·90
0·70
0·50
0·30
0·10
-0·10
F 10^{-1}
-30·00 -22·00 -14·00 -6·00 2·00 10·00 18·00 26·00 34·00
X
TRUE SOL.
NUM. SOL. CLEAN DATA =


DIAG (6) PROBLEM (2)
SOL. BY GCV. TRIG. METHOD WHEN M = N
1·28
1·12
0·96
0·80
0·64
0·48
0·32
0·16
-0·00
-0·16
F
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
TRUE SOL.
NUM. SOL. CLEAN DATA =
SOL. FOR 0·7% NOISE P = 2
SOL. FOR 0·7% NOISE P = 4


DIAG (7) PROBLEM (2)
SOL. BY GCV. TRIG. APPROX. NON-NEGATIVITY
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
X
TRUE SOL
NUM. SOL CLEAN DATA =
SOL. FOR 0.7% NOISE =


DIAG (8) PROBLEM (2)
SOL. BY GCV TRIG. APPROX. NON-NEGATIVITY
1·28
1·12
0·96
0·80
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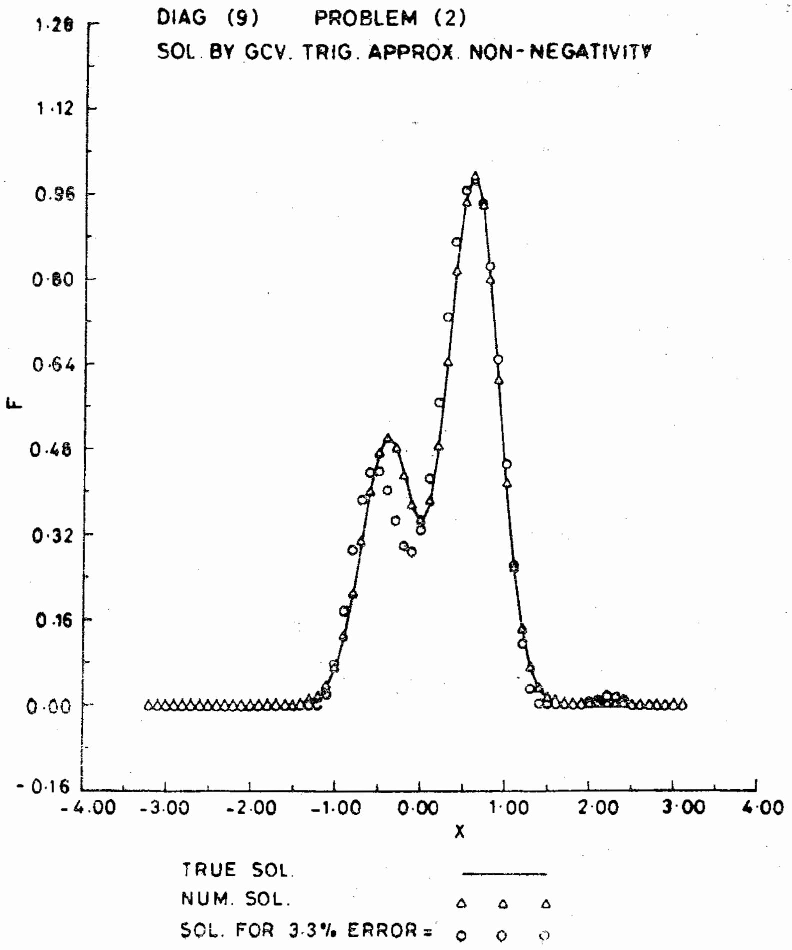
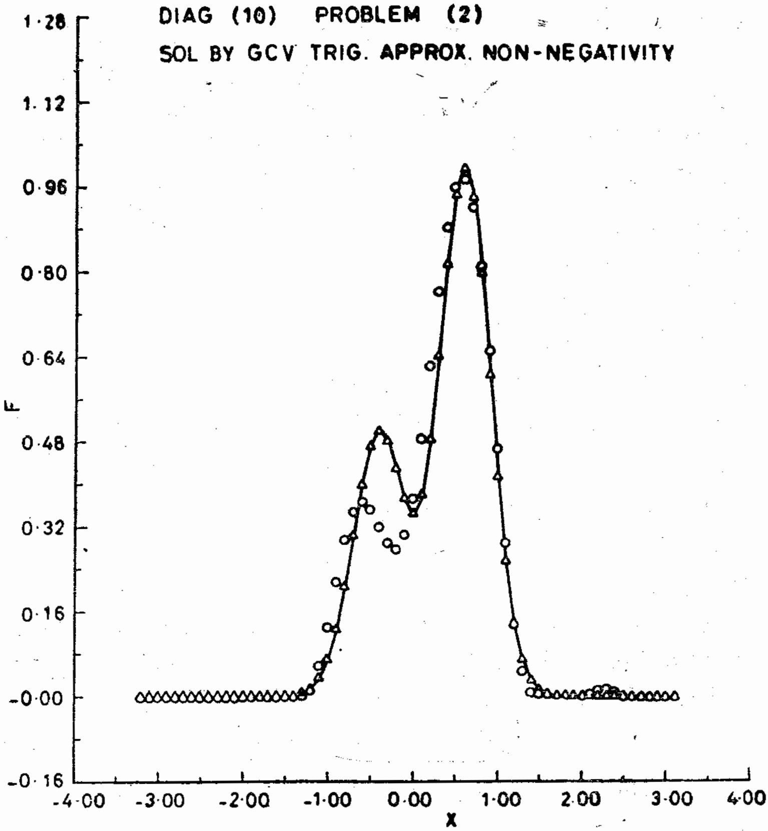
0·48
0·32
0·16
-0·00
-0·16
F
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
TRUE SOL
NUM SOL
SOL FOR 1·7% ERROR


DIAG (9) PROBLEM (2)
SOL. BY GCV. TRIG. APPROX. NON-NEGATIVITY
1·28
1·12
0.96
0·80
0·64
0·48
0·32
0·16
0·00
- 0·16
F
-4.00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
TRUE SOL.
NUM. SOL.
SOL. FOR 3·3% ERROR =



TRUE SOL.

NUM. SOL. CLEAN DATA = △ △ △

SOL. FOR 6·7% NOISE = ○ ○ ○

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UNIPRIMITIVE GROUPS OE DEGREE 120

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Abstract

Symmetric group S_{16} is a uniprimitive group of degree 120 acting on unordered sets. Its sub-degrees and sub-orbits are studied.

Introduction

It is a difficult problem to discover uniprimitive permutation groups of given finite degree in the field of permutation groups (uniprimitive means primitive but not double transitive). The aim of this note is to discover a uniprimitive group of degree 120 and its structure ; sub-degrees and suborbits. We shall prove the following :

Theorem :—S_{16} is a uniprimitive group of degree 120 acting on unordered sets. Its sub-degrees are 1,15,104 and sub-orbits-lengths are 1,28,91.

For basic definitions, notations and preliminary results in permutation groups, we refer to H. Wielandt (3) and for representation theory and group-characters, we refer to M. Burrow (1).

Proof of theorem

Consider $G=S_{16,}$ acting naturally on $\varnothing=\{1, 2, \text{——}, 16\}$. Then G acts on $\Omega=\{A \subseteq \phi \mid |A|=2\}$, the set of $\binom{16}{2}=120$ unordered sets

i.e.

Ω={1, 2}, {1, 3},——, {1, 16}, {2, 3}, {2, 4},——, {2, 16},——, {15, 16}, G is transitive but not doubly transitive on Ω because there is no element in G which takes {1, 2} to {1, 2} and {1, 3,} to {4, 5}.)

We find orbits of $G_{\{1, 2\}}$ *i.e.* suborbits of G on Ω. Let $g \in G$.

Then $g \in G_{\{1, 2\}}$ if and only if $\{1, 2\}^g = \{1, 2\}$; equivalently g either fixes or transposes 1 and 2, and may induce any permutation on {3, 4——, 16}.

Hence

$$\{1, 3\}\, G_{\{1, 2\}} = \Big\{ \{1, 3\}, \{1, 4\}, \text{———}, \{1, 16\}, \{2, 3\}, \{2, 4\}\text{———}. \{2, 16\} \Big\}$$

which is the set of all unordered sets containing exactly one of 1 and 2.

$$\{3, 4\}\, G_{\{1, 2\}} = \Big\{ \{3, 4\}\, \{3, 5\}, \text{——}\, \{3, 16\}, \{4, 5\}, \{4, 6\}, \text{——}\, \{4, 16\} \text{——}\{4, 16\}, \text{——}, \{15, 16\} \Big\}.$$

the set of unordered sets containing neither 1 nor 2. And third orbit is a trivial orbit {{1, 2}}. Hence the orbits of $G_{\{1, 2\}}$ or suborbits of G have lengths 1,28,91

Now we prove that G is primitive.

If G is imprimitive, then G has a block of imprimitivity ψ. The length of ψ divides the order $|\Omega| = 120$ and is a union of some orbits of $G_{\{1, 2\}}$. But this is not possible because orbits of $G_{\{1, 2\}}$ have lengths 1,28,91. Therefore G is primitive and hence uniprimitive.

Again we find the degree of irreducible characters of G *i.e.* subdegrees of G. For this, we go to Higman (2). Let $\{\alpha\}$, $\triangle(\alpha)$, $\Gamma(\alpha)$ be suborbits of G with lengths 1,28,91 respectively ($\alpha \in \Omega$). Then by lemma 2 of Higman (2)

$$\left|\Delta(\alpha)\cap\Delta(\beta)\right|=\begin{cases}\lambda \text{ for } \beta\in\Delta(\alpha)\\ \mu \text{ for } \beta\in\Gamma(\alpha)\end{cases}$$

By lemma 5 of (2).

$$\mu l=k(k-\lambda-1),\text{ where } k=|\Delta(\alpha)|,\ l=|\Gamma(\alpha)|$$

We have

$$91\mu=28(28-\lambda-1)$$

or $13\mu+4\lambda=108$

The solutions are

$$\mu=4,\lambda=14\ ;\ \mu=8,\ \lambda=1.$$

Now $|G|$ is given because $|\Omega|=120$ divides $|G|$. Hence by II lemma 7 of (2),

$$d=(\lambda-\mu)^2+4\,(k-\mu)$$

is a square. So if $\mu=8$, $\lambda=1$, we have $d=129$ which is not square. Therefore the case $\mu=8$, $\lambda=1$ is not possible. If $\mu=4$, $\lambda=14$, then $d=196=14^2$,

By (2, p. 150),

$$f_2\,,f_3=\frac{2k+(\lambda-\mu)\,(k+l)\mp\sqrt{d}\ \ (k+l)}{\mp\sqrt{d}}$$

where f_2, f_3 are the degrees of irreducible characters of G *i.e.* sub-degrees of G. And $f_1=1$ is the degree of principal character of G.

For $\mu=4$, $\lambda=14$, we have

$$f_2=15,\quad f_3=104.$$

Hence we have completed the proof of the theorem.

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NORMAL SUBSPACES OF DIRECTED VECTOR SPACES

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1. INTRODUCTION

Order vector spaces were first considered by F. Riesz, L. Kantorovitch and H. Freudenthal in the middle ninteen thirties. This theory was subsequently enriched by the results of H. Nakano, B.Z. Vulikh. A.G. Pinsker, G, Birkoff. D. Wassilkeff and many others. School of research on vector lattices were then founded in Soviet Union, Japan and United States. Later on H. Freudenthal. F. Riesz, H. Nakano, F. Maeda, S. Bochner, R.S. Phillips studied the notion of normal shbspaces of vector lattices and their applications. For dntaiis see [1], [2] and]3]. The aim of the present paper is to study some properties of normal subspaces of directed vector spaces.

2. DEFINITIONS AND PRELIMINARIES

Let X be a directed vector space and X_+ the set of all positive elements of X.

Definition 2.1

A subset E of X is said to be *solid* if it has the following properties :

(γ_1) If $0 \leqslant x \in E$, then $[-x, x] = y \in X : -x \leqslant y \leq x) \subseteq E$.

(γ_2) If $x \in E$, then there exists a $y \in E \cap X_+$ such that $x \in [-y\ y]$

If we denote $Z_E(x) = \{y : x \in [-y, y], y \in E \cap X_+\}$ then $(\gamma)_1$ and (γ_2) are equivalent to the conditions.

(1) If $x \in E$ and $Z_E(x) \subseteq Z_E(y)$ then $y \in E$.

(2) If $x \in E$ then $E \cap Z_E(x) \neq \varnothing$.

Lemma 2.2

If E is a solid subset of X then $E \subseteq E \cap X_+ - E \cap X_+$.

Definition 2.3

One calls a *normal subspace* of X any vector subspace X_0 which at the same time is a solid subset of X.

3. BASIC RESULTS

Lemma 3.1

Any normal subspace of a directed vector space is a directed vector subspace. The proof is an immediate consequence of lemma 2.2.

Theorem 3.2

A directed vector subspace G of X is a normal subspace if and only if it satisfies the condition : $x \in G$, $y \in X$ and $Z_E(x) \subseteq Z_E(y)$ imply $y \in G$.

Proof

Let G be a normal subspace of X. For any $x \in G$, $y \in X$ with $Z_E(x) \subseteq Z_E(y)$ there exist $z \in G \cap X_+(x)$ such that $z \in Z_E(x) \subseteq Z_E(y)$. It further implies that
$-z \leqslant y \leq z$ Therefore $y \in G$.

Conversely, assume that G satisfies the given condition. Let $x \in G \cap X_+$ and $y \in [-x, x]$. Thus $x \in Z_E(x) \subseteq Z_E(y)$. By our hypothesis $y \in G$. Hence (1) is satisfied. Since G is a directed vector space, $Z_E(x) \neq \phi$ in G and therefore for every $x \in G$ there exists $y \in G \cap X_+$ such that $x \in [-y, y]$. Hence G is a normal subspace.

Remarks 3.3

Sufficient condition in above theorem 3.2 can be relaxed as : $x \in G_+ = G \cap X_+$, $y \in X$ and $Z_E(x) \subseteq Z_E(y)$ imply $y \in G$.

Definition 3.4

Two elements $x_1, x_2 \in X$ are said to be *orthogonal* if

$$inf\left[Z_E\left(x_1\right) \cup Z_E\left(x_2\right)\right]=0.$$ One then writes $x_1 \perp x_2$,

The above definition can be extended to the subsets of X. Two subsets A_1, A_2 are said to be orthogonal if $x_1 \perp x_2$ for every $x_1 \in A_1$, $x_2 \in A_2$. Orthogonal complement of A_1 is the set $A_1^{\perp} = \{x : x \perp A_1\}$

Proposition 3.5

Let G, G_1, G_2 be normal subspaces of X. Then the following assertions are true.

(*i*) If for every $0 \neq x \in X$ there exists $y \in G/\{0\}$ such that $Z_E(x) \subseteq Z_E(y)$ then $G^{\perp} = \{0\}$.

(*ii*) If $G_1 \perp G_2$ then $G_1 \cap G_2 = \{0\}$.

Proof

(*i*) Let $G^{\perp} \neq \{0\}$, then is some none-zero$x$ in $G^{\perp}$. By hypothesis there exists $y \in G/\{0\}$ such that $Z_E(x) \subseteq Z_E(y)$. Since $x \in G^{\perp}$ and $y \in$

G. Therefore $x \perp y$. Thus $\inf\left[Z_E(x) \cup Z_E(y) \right]=0$. Therefore $\inf Z_E(y)=0$. It further implies that $y=0$. Hence a contradiction.

(*ii*) If $G_1 \perp G_2$ and $x \in G_1 \cap G_2$ then $x \perp x$ and therefore $\inf Z_E(x)=0$. Hence $x=0$.

Country to vector lattices, the converse of poppulation 3.5 is not true, in general, for directed vector spaces. We give the following example to illustrate this.

Example 3.6

Let X be the vector space of all functions $x:R_+ \to R$ with usual algebraic operation and natural ordering. Let F be the subspace of X, whose elements are representable in the form

$$x(t) = \sum_{n=p}^{p} c_n t^n$$

where $t \in R_+$ and c_n are scalar. P is a directed vector space under the order relation induced by X, clearly, $Z_E(y) \subseteq Z_E(x)$ if and only if $|x| \leq |y|$ in X.

Let $G_1 = \{x : x \in F ;\ \limsup_{t \to 0} |x(t)| < \infty\}$,

$G_2 = \{x : x \in F ;\ \limsup_{t \to \infty} |x(t)| < \infty\}$,

then G_1 and G_2 satisfy the condition of theorem 3.2. Therefore G_1 and G_2 are normal subspaces of F. Obviously $G_1 \cap G_2 = \{0\}$ and $G_1 \perp G_2$. Hence converse of proposition 3.5 (*ii*) does not hold.

Also $G_1^{\perp} = \{0\}$. On the other hand, let $x \in F \setminus \{0\}$, given by $x(t) = 1/t$. Now if there exists some $y \in G_1 / \{0\}$ such that $Z_E(x) \leq Z_E(y)$.

Then $|y| \leqslant |x|$ therefore $y \in G_1 \cap G_2$. Thus $y=0$. Hence contradiction.

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"SOME RADIUS OF CONVEXITY PROBLEMS FOR ANALYTIC FUNCTIONS RELATED WITH FUNCTIONS OF BOUNDED BOUNDARY ROTATION"

By

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Abstract :

Let $P[A, B], -1 \leqslant B < A \leqslant 1$, be the class of functions p such that $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$. Let f be analytic in E and

$$\frac{(zf'(z))'}{f'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z),\ k \geq 2$$

and $p_1, p_2 \in P[A, B]$. Then we say that $f \in V_k[A, B]$. With $A=1$, $B=-1$, we obtain V_k, the class of functions with bounded boundary rotation. We also define the class $R_k[A, B]$ such that $f \in V_k[A, B]$ if and only if $zf' \in R_k[A, B]$. Also $f \in T_k[A, B; C, D], - \leqslant B < A \leqslant 1$, $-1 \leq D < C \leq 1$, if and only if $\frac{f'(z)}{g'(z)} \in P[A, B]$ and $g \in V_k[C, D]$.

We deal these classes under some integral operators and some radius of convexity problems are also considered.

Key words and Phrases : Subordinate, bounded boundary rotation, starlike, close-to-convex functions, convex domain.

1980 Mathemmatics Subject Classification : Codes : 30 A 32, 30 A 34.

1. Introduction

Let f be analytic in $E = \{z : |z| < 1\}$, and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

A fuunction g, analytic in E, is called subordinate to a function G if there exists a Schwarz function $w(z)$, $w(z)$ analytic in E with $w(0)=0$ and $|w(z)|<1$ in E, such that $g(z)=G(w(z))$.

In [3], Janowski introduced the class P[A, B]. For A and B, $-1\leq B<A\leqslant 1$, and function p, analytic in E with $p(0)=1$ belongs to the class P[A, B] if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$.

Also, given C and D, $-1\leqslant D<C\leqslant 1$, C[C, D] and S* [C, D] denote the classes of functions, analytic in E and given by (1.1), such that $\frac{(zf'(z))'}{f'(z)}\in$ P[C, D] and $\frac{zf'(z)}{f(z)}\in$ P[C, D] respectively. For C=1, and D=−1 we notc that C[1, −1]=C and S*[1, −1]≡S* the class of convex and starlike functions in E.

A function f, analytic in E and given by (1.1), is said to be in the class R_k[C, D], $-1\leqslant D<C\leqslant 1$, if and only if

$$f(z)=\frac{(S_1(z))^{\frac{k}{4}+\frac{1}{2}}}{(S_2(z))^{\frac{k}{4}-\frac{1}{2}}} \tag{1.2}$$

where S_1, $S_2\in$ S*[C, D].

Clearly $k\geq 2$ and R_2 [C, D]=S*[C, D]. Also $R_k[1, -1]=U_k$, the class of functions with bounded radius rotation discussed in [5].

We can also define the following :

Definition 1.1.

Let f be analytic in E and be given by (1.1). Then f belongs to the class V_k[C, D], $k\geq 2$, $-1\leq D<C\leqslant 1$ if and only if

$$f'(z)=\frac{(S_1(z)/z)^{\frac{k}{4}+\frac{1}{2}}}{(S_2(z)/z)^{\frac{k}{4}-\frac{1}{2}}} \tag{1.3}$$

where S_1, $S_2\in$ S*[C, D].

From (1.2) and (1.3), it is clear that

$$f \in V_k[C, D] \quad \text{if, and only if } zf' \in R[C, D] \qquad (1.4)$$

It may be noted that $V_2[C, D] \equiv C[C, D]$ and $V_k[1, -1] = V_k$, the class of funetions of bounded boundary rotation discussed by Paatero [8].

Definition 1.2.

Let f be analytic in E and be given by (1.1). Then f is said to belong to the class $T_k[A, B; C, D]$, $k \geq 2$; $-1 \leqslant B < A \leqslant 1$; $-\leqslant D < C \leqslant 1$ if and only if there exists a function $g \in V_k[C, D]$ such that $\frac{f'(z)}{g'(z)} \in P[A, B]$.

We note that :

(*i*) $T_2[1, -1; 1, -1]$ K, the class of close-to-convex functions introduced and studied by Kaplan [4].

(*ii*) $T_k[1, -1; 1, -1] \equiv T_k$, a class of analytic functions introduced and studied in [7].

(*iii*) $T_2[A, B; C, D] \equiv K[A, B; C, D]$, and this case is discussed by Silvia in [10].

2. Preliminary Results :

Lemma 2.1 [9].

Let $p \in P[A, B]$. Then

$$\frac{1-Ar}{1-Br} \leqslant Re\, p(z) \leqslant |p(z)| \leq \frac{1+Ar}{1+Br}.$$

Lemma 2.2.

Let $f \in V_k[C, D]$. Then f maps $|z| < r_1$ onto a convex domain, where

$$r_1 = \frac{2}{\frac{k}{2}(C-D) - \sqrt{\frac{k^2}{4}(D-C)^2 + 4CD}}$$

This result is sharp.

Proof :

Since $f \in V_k[C, D]$, we have

$$f'(z) = \frac{(S_1(z)/z)^{\frac{k}{4}+\frac{1}{2}}}{(S_2(z)/z)^{\frac{k}{4}-\frac{1}{2}}}, \quad S_1, S_2 \in S^*[C, D].$$

or

$$\frac{(zf'(z))'}{f'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_1(z), \quad (2.2)$$

$$p_1, p_2 \in P[C, D]$$

so

$$\begin{aligned} \text{Re}\, \frac{(zf'(z))'}{f'(z)} &\geqslant \left(\frac{k}{4}+\frac{1}{4}\right) \text{Re}\, p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) |p_2(z)| \\ &\geqq \left(\frac{k}{4}+\frac{1}{2}\right) \frac{1-Cr}{1-Dr} - \left(\frac{k}{4}-\frac{1}{2}\right) \frac{1+Cr}{1+Dr} \\ &= \frac{1+\frac{k}{2}(D-C)r - DCr^2}{1-D^2r^2} \end{aligned}$$

Hence $\text{Re}\,\dfrac{(zf'(z))'}{f'(z)} > 0$ for $|z| < r_1$,

where r_1 is given by (2.1). Sharpness of the result follows if we take $p_1(z)$, and $p_2(z)$ in 2.2) as

$$p_1(z) = \frac{1-Cz}{1-Dz}, \quad p_2(z) = \frac{1+Cz}{1+Dz}.$$

From the relation (1.4) and lemma 2.2, we have :

Lemma 2.3.

Let $f \in R_k[C, D]$. Then f is starlike for $|z| < r_1$, where r_1 is given by (2.1). This resuit is sharp.

The following is the extension of Libera's result [6].

Lemma 2.4 [9].

Let N and D be analytic in E, D maps onto a many-sheeted starlike region. $N(0)=0=D(0)$ and

$$\frac{N'(z)}{D'(z)} \in P[A, B]$$

Then $\frac{N(z)}{D(z)} \in P[A, B]$.

Lemma 2.5 [1].

Let $p \in P[A, B]$. Then, for $z \in E$,

$$\text{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \frac{A-B}{(1-Ar)(1-Br)}, \text{ if } R_1 \leqslant R_2$$

and

$$\text{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geq \frac{(A+B)}{(A-B)} + \frac{2}{(A-B)(1-r^2)}[(L_1 K_1)^{\frac{1}{2}} - (1-ABr^2)], \text{ if } R_2 \leq R_1,$$

where

$$R_1 = \left(\frac{L_1}{K_1}\right)^{\frac{1}{2}}, \quad R_2 = \left(\frac{1-Ar}{1-Br}\right),$$
$$L_1 = (1-A)(1+Ar^2),$$

and

$$K_1 = (1-B)(1+Br^2).$$

This result is sharp.

3. Main Results.

Theerem 3.1

Let α and m be any positive integers and $f \in R_k[C, D]$. Then the function F defined by

$$(F(z))^{\alpha} = \frac{\alpha+m}{z^m} \int_0^z t^{m-1}(f(t))^{\alpha} dt \qquad (3.1)$$

is starlike for $|z| < r_1$, where r_1 is given by (2.1).

Proof :

$$\text{Let } J(z) = \int_0^z t^{m-1}(f(t))^{\alpha} dt$$

so

$$(F(z))^{\alpha}=\frac{\alpha+m}{z^m}\;J(z)$$

and

$$\alpha\frac{zF'(z)}{F(z)}=\frac{zJ'(z)}{J(z)}-m$$

or

$$\frac{zF'(z)}{F(z)}=\frac{1}{\alpha}\left\{\frac{zJ'(z)-mJ(z)}{J(z)}\right\} \qquad 3.2)$$

Let

$$\frac{N(z)}{D(z)}=\frac{\frac{1}{\alpha}\{zJ'(z)-mJ(z)\}}{J(z)}$$

Then $N(0)=0=D(0)$.
By a result due to Bernardi [2] and lemma 2.3, $D(z)$ is a $(m+\alpha-1)$-valent starlike function for $|z|<r_1$, and r_1 is given by (2.1).
Also

$$\frac{N'(z)}{D'(z)}=\frac{\frac{1}{\alpha}\{(zJ'(z))'-mJ'(z)\}}{J'(z)}$$
$$=\frac{zf(z)}{f(z)}$$

Since, by lemma 2.3, f is starlike for $|z|<r_1$, where r_1 is given by (2.1), so $\operatorname{Re}\frac{N'(z)}{D'(z)}>0$ for $|z|<r_1$. This implies that $\operatorname{Re}\frac{N(z)}{D(z)}=\operatorname{Re}\frac{zF'(z)}{F(z)}>0$ for $|z|<r_1$, see [6], and this proves our result.

Corollary 1.

If $k=2$, then $f\in S^*\,]C, D]$ and F is in $S^*\,[C, D[$ for $z\in E$.

Corollary 2.

If $C=1$, $D=-1$, then $f\in R_k\;[1,-1]\equiv U_k$ and F maps $|z|<\frac{2}{-k+\sqrt{k^2-4}}$ onto starshaped domain. The result has been proved in [5].

Using (1.4), we can easily have the following :

Theorem 3.2.

Let $f \in V_k [C, D]$, and F be defined by (3.1). Then F maps $|z|<r_1$ onto a convex domain, and r_1 is given by (2.1).

Theorem 3.3.

Let $f \in T_k [A, B ; C, D]$, and F be defined by

$$F(z)=\frac{m+1}{z^m}\int_0^z t^{m-1} f(t)dt.$$

Then F maps $|z|<r_1$, r_1 given by (2.1), onto a close-to-convex domain.

Proof

Since $f \in T_k [A, B ; C, D]$, there exists a function $g \in V_k [C, D]$ such that $\frac{f'(z)}{g'(z)} \in P[A, B]$.

Let $G(z)=\frac{m+1}{z^m}\int_0^z t^{m-1}g(t)dt.$ Then, by theorem 3.2

with $\alpha=1$, G is convex for $|z|<r_1$.

Now

$$\frac{F'(z)}{G'(z)}=\frac{z^m f(z)-m\int_0^z f(t)\, t^{m-1}\, dt}{z^m g(z)-m\int_0^z t^{m-1}g(t)dt}$$

$$=\frac{\int_0^z t^m f'(t)dt}{\int_0^z t^m g'(t)\, dt}=\frac{N(z)}{D(z)}$$

Thus

$$\frac{N'(z)}{D'(z)}=\frac{f'(z)}{g'(z)} \in P[A, B].$$

This implies that $\frac{N(z)}{D(z)} \in P[A, B]$, for $|z|<r_1$, using lemma 2.4.

Hence $F \in T_2[A, B, 1, -1] \subset T_2[1, -1 : 1, -1) \equiv K$ for $|z| < r_1$ and this proves our result.

We now prove the following :

Theorem 3.4.

Let $f \in V_k[C, D]$, and

$$f_\alpha(z) = \int_0^z (f'(t))^\alpha dt, \qquad 0 \leqslant \alpha \leq 1. \tag{3.3}$$

Then f_α maps $|z| < r_0$ onto a convex domain, where r_0 is the least positive root of

$$(\alpha D^2 - \alpha CD - D^2)x^2 + \frac{\alpha k}{2}(D-C)x + 1 = 0 \tag{3.4}$$

Proof :

We have

$$f'_\alpha(z) = (f'(z))^\alpha$$

so

$$\frac{(zf'_\alpha(z))'}{f'_\alpha(z)} = \frac{\alpha(zf'(z))'}{f'(z)} + (1-\alpha)$$

Using (2.3), we obtain

$$Re\left\{\frac{(zf'_\alpha(z))'}{f'_\alpha(z)}\right\} \geq \frac{\alpha\{1+(k/2)(D-C)r - DCr^2\}}{1-D^2r^2} + (1-\alpha)$$

$$= \frac{1+\alpha(k/2)(D-C)r + ((\alpha-1)D - \alpha C)Dr^2}{1-D^2r^2}$$

and this gives us the required result.

Theorem 3.5.

Let $f \ni T_k[A, B, ; C, D]$, and f_α be defined by (3.3). Then f_α maps $|z| < r_0$ onto a close-to-convex domain, where r_0 is the least positive root of (3.4).

Proof :

We know that $\dfrac{f'(z)}{g'(z)} \in P[A, B]$, where $g \in V_k[C, D]$.

Let

$$g_\alpha(z)=\int_0^z (g'(t))^\alpha dt$$

Then $g_\alpha \in V_k[C, D]$ for $|z|<r_0$, where r_0 is the least positive root of (3.4),

Hence

$$\frac{f'_\alpha(z)}{g'_\alpha(z)}=\left(\frac{f'(z)}{g'(z)}\right)^\alpha=p_1^\alpha(z)\in P[A,B].$$

This means $f_\alpha \in T_2[A, B, 1, -1 \subset T_2[1, -1, 1, -1] = K$ and we have the required result.

Theorem 3.6.

Let $f \in T_k[A, B; C]$. Then

$$Re\ \frac{(zf'(z))'}{f'(z)} \geq \begin{cases} M_1(r), & \text{for } R_1 \leqslant R_2 \\ M_2(r), & \text{for } R_2 \leqslant R_1, \end{cases}$$

where

$$M_1(r)=\frac{1+(k/2)(D-C)r-CDr^2}{1-C^2r^2}-\frac{(A-B)r}{(1-Ar)(1-Br)}$$

and

$$M_2(r)=\frac{1+(k/2)(D-C)r-CDr^2}{1-C^2r^2}+\frac{A+B}{A-B}+\frac{2}{(A-B)(1-r^2)} \times\{(L_1K_1)^{\frac{1}{2}}-(1-ABr^2)\}$$

with R_1, R_2, L_1 and K_1 as defined in lemma 2.5.

Proof ;

Since $f \in T_k[A, B; C, D]$, then

$$\frac{f'(z)}{g'(z)}=\frac{1+Aw(z)}{1+Bw(z)}=p(z), \qquad \text{where } g\in V_k[C, D]$$

Differentiating logarithmically, we obtain

$$\frac{(zf'(z))'}{f'(z)}=\frac{(zg'(z))'}{g'(z)}+\frac{zp'(z)}{p(z)}$$

Using (2.3) and lemma 2.5, we have, for $R_1 \leq R_2$

$$Re \frac{(zf'(z))'}{f'(z)} \geq \frac{1+(k/2)(D-C)r-CDr^2}{1-C^2 r^2} - \frac{(A-B)r}{(1-Ar)(1-Br)}$$
$$= M_1(r),$$

and, for $R_2 \leq R_1$

$$Re \frac{(zf'(z))'}{f'(z)} \leqslant \frac{1+(k/2)(D-C)r-CDr^2}{1-C^2 r^2} + \frac{A+B}{A-B}$$
$$+ \frac{2}{(A-B)(1-r^2)} [(L_1 K_1)^{\frac{1}{2}} - (1-ABr^2)]$$
$$= M_2(r)$$

Sharpness of the bound when $R_1 \leqslant R_2$ follows if we take :

$$p_0(z) = \frac{zf'_0(z)}{f_0(z)} = \frac{1+Az}{1+Bz}, \qquad \text{and}$$

and at $z=-r$

$$\frac{(zg'_0(z))'}{g'_0(z)} = \left(\frac{k}{4}+\frac{1}{2}\right) \frac{1-Cz}{1-Dz} - \left(\frac{k}{4}-\frac{1}{2}\right) \frac{1+Cz}{1+Dz}$$

We note that

$$\frac{zp'_0(z)}{p_0(z)} = \frac{(A-B)z}{(1+Bz)(1+Az)},$$

and at $z=-r$

$$Re \left\{ \frac{zp'_0(z)}{P_0(z)} \right\} = \frac{-(A-B)r}{(1-Ar)(1-Br)}$$

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SEMI-T_0—IDENTIFICATION SPACES, SEMIREGULARIZATION SPACES, FEEBLY INDUCED SPACES, AND PROPERTIES OF TOPOLOGICAL SPACES

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1. Introduction :

In 1937 M.H. Stone introduced regular open sets. If (X,T) is a space and $A \subset X$, then A is regular open, denoted by $A \in RO(X, T)$, iff $A = \text{Int}(\bar{A})$ [34]. In the 1937 investigation, Stone showed that for a space (X, T), RO(X, T) is a base for a topology T_S on X coarser than T and called (X, T_S) the semiregularization of (X, T). Since 1937 semiregularization spaces have been used to investigate many properties of topological spaces.

In 1963 semi open sets were defined. If (X, T) is a space and $A \subset X$, then A is semi open, denoted by $A \in SO(X, T)$, iff there exists $0 \in T$ such that $0 \subset A \subset \bar{0}$ [27]. In 1965 semi open sets were further investigated as β—sets and α—sets were introduced. Let (X, T) be a space and let $A \subset X$. Then A is an α—set, denoted by $A \in \alpha(X, T)$, iff $A \subset \text{Int}(\overline{\text{Int}(\bar{A})})$ [33]. In 1970 semi open sets were used to define and investigate semi closed sets and the semi closure of a set. If (X, T) is a space and A, $B \subset X$, then A is semi closed iff X—A is semi open and the semi closure of B, denoted by scl B, is the intersection of all semi closed sets containing B [1]. In 1971 the semi closure operator was used to show that for a space (X, T) there exists a finest topology F(T) in the class of topologies having the same semi open set as (X, T) [3].

In 1978 the semi closure of open sets was used to define feebly open sets, which were used to define feebly closed sets and the feebly closurse of a set. Let (X, T) be a space and a let A, B, C $\subset$X. Then A is feebly open, denoted by A $\in$ F0(X. T), iff there exists 0 $\in$ T such that 0 $\subset$A $\subset$ scl 0, B is feebly closed iff X—B is feebly open, and the feebly closure of C is the intersection of all feebly closed sets containing C [32]. Further investigation of feebly open sets showed that for a space (X, T), F0(X, T) is a topology on X, called the feebly induced topology on (X, T), T$\subset$F0(X, T))=F0(X, F0(X, T) [6], S0(X, T)=S0(F0(X, T) [7], F0(X,T)=α(X, T)=F(T) [8], and for each A $\subset$X and B $\in$ F0(X, T), Int(Int(A=scl $\overline{\text{(Int (A)}}$ and scl B=Int $\overline{\text{(Int (B)}}$)]9] .These results led to new characterizations of regular open sets. If (X, T) is a space, then R0 (X, T) = {Int $\overline{\text{(Int (A))}}$ | A $\subset$X} =(scl A | A $\in$ F0(X, T)} [10] = {Int (0) | 0 $\in$ T} [11] = {scl 0 | 0 $\in$ T} [12]. Many properties of topological spaces have been further investigated using feeble open sets and regular open sets. In this paper additional properties are further investigated using feeble open and regular open sets, and semi-regularization spaces and feebly induced spaces aae further examined.

2. Semi—T_0—Identification Spaces, Semi-regularization Spaces, and Feebly Induced Spaces.

In [12] it was shown that for a space (X, T), $scl_T U = scl_{T_S} U$ for each U $\in$ T, which was used to show that for each space (X, T), $\left(X, T_S\right)$ is semi—R_0. Below the above result is extended to include feebly open sets.

Theorem 2.1. Let (X, T) be a space and let U $\in$ F0(X, T). Then $scl_{F0(X,T)} U = scl_{T_S} U = scl_T U$.

Proof. Since U $\subset$X, then $scl_T U = scl_{F0(X,T)} U$ [6]. Since R0(X, T)=R0(X, F0(X, T)) [10], then $T_S = F0(X, T)_S$ and $scl_{F0(X,T)} U = scl_{F0(X,T)_S} U = scl_{T_S} U$.

In [12] it was shown that for a R_1 space (X, T), the semi-regularization space of the T_0—identification space of (X, T) equals the T_0—identification space of the semi-regularization space of (X, T), which used so show that the semi-regularization of a R_1 space is R_1. Examples can be given to show that for a R_1 space (X, T), the feebly induced space of the T_0—identification space of (X, T) need not be homeomorphic to the T_0—identification space of the feebly induced space of (X, T). In [13] semi-T_0—identification spaces were introduced. Let (X, T) be a space and let R be the equivalent relation on X defined by xRy if $\mathrm{scl}\{x\}=\mathrm{scl}\{y\}$. Then the semi-$T_0$—identification space of (X, T) is $\left(X_S, Q_S(X, T)\right)$, where X_S is the set of equivalence classes of R and $Q_S(X, T)$ is the decomposition topology on X_S. The result above questions about what would happen if T_0—identification space in the result above was replaced by semi-T_0—identification space. The investigation of these questions led to the following discoveries.

In [14] R_1 spaces were generalized to s-weakly Hausdorff spaces by replacing convergence in the definition of R_1 by semi convergence. Let (X, T) be a space, let $\left\{X_\alpha\right\}_{\alpha \in A}$ be a net in X, and let $x \in X$. Then $\{x_\alpha\}_{\alpha \in A}$ semi converges to x iff $\{x_\alpha\}_{\alpha \in A}$ eventually in every semi open set containing x [15]. A space (X, T) is s-weakly Hausdroff iff $\overline{\{x\}}=\overline{\{y\}}$, whenever there exists a net semi converging to both x and y [14].

Theorem 2.2. Let (X, T) be a space and let $P:(X, T)\rightarrow(X_S, Q_S(X, T))$ the natural map. Then for each $\mu \in Q_S(X, T)$. $P^{-1}(\mathrm{scl}\ \mu)=\mathrm{scl}\ (P^{-1}(\mu))$.

Proof: Let $\mu \in Q_S(X, T)$. Since $\mathrm{scl}\ \mu=\mathrm{Int}(\overline{\mu})$ and P is continuous and open [13], then $P\ (\mathrm{scl}\ \mu)=P^{-1}(\mathrm{Int}(\overline{\mu})))=\mathrm{Int}(P^{-1}(\overline{\mu})=\mathrm{Int}(P^{-1}(\mu)=\mathrm{scl}\ P^{-1}(\mu)$.

Theorem 2.3. Let (X, T) be a space. Then (X, T) is s-weakly Hausdorff iff $(X_S, Q_S(X, T)$ is s—weakly Hausdorff.

Proof : For each $z \in X$ let C_Z be the element of X_S containing z. Suppose (X, T) is s—weakly Hausdorff. Let $x, y \in X$ such that $\overline{\{C_x\}} \neq \overline{\{C_y\}}$. Then $\overline{\{x\}} \neq \overline{\{y\}}$, for suppuse not. Since P is continuous and closed [13], then $\overline{\{C_x\}} = \overline{\{P(x)\}} = P(\overline{\{x\}}) = P(\overline{\{y\}}) = \overline{\{P(y)\}} = \overline{\{C_y\}}$, which is a contradiction. Thus $\overline{\{x\}} \neq \overline{(y)}$ and there exist disjoint semi open sets U and V such that $x \in U$ and $y \in V$ [14]. Since P is continuous and open then P(U) and P(V) are semi open sets [27] and since $P^{-1}(P(W)) = W$ for all $W \in SO(X, T\ T)$ [131], then P(U) and P(V) are disjoint semi open sets containing C_x and C_y, respectively. Thus $(X_S, Q_S(X, T))$ is s—weekly Hausdorff [14].

Conversely, suppose $(X_S, Q\,(X, T)$ is s—weekly Hausdorff. Let $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. Then $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$, say $x \notin \overline{\{y\}}$. Then $x \in X - \overline{\{y\}}$, $C_x \in P(X - \overline{\{y\}}) \in Q_S(X, T)$, and $C_y \notin P(X - \overline{\{y\}}$, which implies $C_x \notin \overline{\{C_y\}}$. Thus $\overline{\{C_x\}} \neq \overline{\{C_y\}}$ and there exist disjoint semi open sets μ and $\surd$ such that $C_x \in \mu$ and $C_y \in \surd$. Since P is continuous and open, then $P^{-1}(\mu)$ and $P^{-1}(\surd)$ are disjoint semi open sets [4] containing x and y, respectively. Thus (X, T) is s—weekly Hausdorff.

Theorem 2.4. Let (X, T) be a s—weakly Hausdorff space. Then (X, T_S) is s—weakly Hausdroff.

Proof : Let $x, y \in X$ such that $\overline{\{x\}}_{T_S} \neq \overline{\{y\}}_{T_S}$. Since $\overline{\{z\}}_T \subset \overline{\{z\}}_{T_S}$ for each $z \in X$ and $\overline{\{x\}}_{T_S}, \neq \overline{\{y\}}_{T_S}$, then $\overline{\{x\}}_T \neq \overline{\{y\}}_T$. Then there exist disjoint T open sets U and V such that $\overline{\{x\}}_T \subset \overline{U}_T$ and $\overline{\{y\}} \subset \overline{V}_T$ [14]. Then $scl_T U$ and $scl_T V$ are disjoint T_S open sets, $x \in \overline{U}_T =$

$\overline{\mathrm{scl}_T U_T} = \overline{\mathrm{scl}_T U_T}_{T_S}$ [2], and similarly, $y \in \overline{\mathrm{scl}_T V_T}_{T_S}$, which implies $\overline{\{x\}}_{T_S} \subset \overline{\mathrm{scl}_T U_T}_{T_S}$ and $\overline{\{y\}}_{T_S} \subset \overline{\mathrm{scl}_T}_{T_S}$. Thus (X, T_S) is s—weakly Hausdorff [14].

Theorem 2.5. If (X, FO(X, T)) is s-weakly Hausdorff, then (X, T) is s-weakly Hausdorff.

Proof : Let $x, y \in X$ such that $\overline{\{x\}}_T \neq \overline{\{y\}}_T$. Since $\overline{\{z\}}_{FO(X,T)} \subset \overline{\{z\}}_T$ for each $z \in X$ and $\overline{\{x\}}_T \neq \overline{\{y\}}_T$, then $\overline{\{x\}}_{FO(X,T)} \neq \overline{\{y\}}_{FO(X,T)}$ and there exist disjoint FO(X, T) semi open sets U and V such that $x \in U$ and $y \in V$. Then U and V are disjoint T semi open sets containing x and y, respectively, and (X, T) is s—weakly Hausdroff.

In 1972 [4] semi homeomorphisms were defined by replacing open in the definition of homeomorphisms by semi open and properties preserved by semi homeomorphisms were called semi topological properties. In investigations of feeble open sets, it has been shown that certain properties are simultaneously shared by both a space and its feebly induced space. In [8] a topological property simultaneously shared by both a space and the feebly induced space was called a feeble property and it was shown that a property is a feeble property iff it is a semi topological property. In [10] r-properties, topological properties simultaneously shared by both a space and its semiregularization space, were investigated and it was shown that every r-topological property is a semi topological property. Since s-weakly Hausdorff is not a semi topological property [16], then the converse of Theorem 2.4 and Theorem 2.5 is false.

Theorem 2.6. Let (X, T) be a s-weakly Hausdorff space, let $P : (X, T) \rightarrow (X_S, Q_S(X, T))$ be the natural map, let $(X^*_S, Q_S(X,T)_S))$ be the semi—T_0—identification space of (X, T_S), and let $P_S : (X,T_S)$

$\rightarrow (X^*_S, Q_S(X. T_S))$ be the natural map. Then $(X_S, Q_S(X,T)_S) = (X^*_S, Q_S(X, T_S))$, which is s-weakly Hausdorff semi$-T_1$.

Proof : Let $x \in X$. Since (X, T) is s-weakly Hausdorff, then (X, T) is semi$-R_0$ [14] and $X_S = \{scl_T\{z\} \mid z \in X\}$ [13], and since (X, T_S) is semi$-R_0$, then $X^*_S = \{scl_{T_S}\{z\} \mid z \in X\}$. Let $x \in X$ and let $y \in X - scl_T\{x\}$. Consider the case that $scl_T\{x\} \in T$. Then $scl_T\{x\} = scl_T(scl_T\{x\}) = scl_{T_S}(scl_T\{x\}) \in T_S$. Since $y \notin scl_T\{x\}$, then $x \notin scl_{T_S}\{y\}$ and $scl_{T_S}\{x\} \cap scl_{T_S}\{y\} = \varnothing$, which implies $y \notin scl_{T_S}\{x\}$. Consider the case that $scl_T\{x\} \notin T$. Let $z \in X - \overline{\{x\}}_T$. Since (X, T) is s-weakly Hausdorff and $\overline{\{z\}}_T \neq \overline{\{x\}}_T$, then there exist disjoint T open sets U and V such that $z \in \overline{U}_T$ and $x \in \overline{V}_T$. For each $z \in X - \overline{\{x\}}_T$ let U_z and V_z be disjoint T open sets such that $z \in \overline{U_z}_T$ and $x \in \overline{V_z}_T$. Then $scl_T U_z$ and $scl_T V_z$ are disjoint T_S open sets, $z \in \overline{U_z}_T = \overline{scl_T U_z}_{T_S} = \overline{scl_T U_z}_{T_S}$, and similarly, $x \in \overline{scl_T V_z}_{T_S}$. Thus $W = \bigcup_{z \in X - \overline{\{x\}}_T} scl_T U_z \in T_S$, $x \notin W$ and $X - \overline{\{x\}}_T \subset \overline{W}_{T_S} = \overline{W}_T$. Since $scl_T\{x\} \notin T$ and (X, T) is semi$-R_0$, then $x \in \overline{X - \{x\}_T}_T$ [17]. Thus $\overline{W}_{T_S} = X$, $Y = W \cup \{y\} \in SO(X, T_S)$, and $x \notin Y$, which implies $y \notin scl_{T_S}\{x\}$. Hence $scl_{T_S}\{x\} \subset scl_T\{x\}$ and since $SO(X, T_S) \subset SO(X, T)$

[12], then $\text{scl}_T\{x\} \subset \text{scl}_{T_S}\{x\}$, which implies $\text{scl}_T\{x\} = \text{scl}_{T_S}\{x\}$ and $X^*_S = X_S$.

A base for $Q_S(X, T)$ is $\beta = \{\text{scl}\ \mu \mid \mu \in Q_S(X, T)\}$. Since P_r is continuous, open, and onto ; and $\{\text{scl}\ U \mid U \in T\}$ is a base for T_S, then $\beta_r = \{P_r(\text{scl}\ U) \mid U \in T\}$ is a base for $Q_S(X, T_S)$. If $\mu \in Q_S(X, T)$, then $P^{-1}(\mu) \in T$, $P^{-1}(\text{scl}\ \mu) = \text{scl}\ P^{-1}(\mu)$, and $\text{scl}\ \mu = P(P^{-1}(\text{scl}\ \mu)) = P_r(\text{scl}\ P^{-1}(\mu)) \in \beta_r$, which implies $\beta \subset \beta_r$. If $U \in T$, then, since $P^{-1}(P(U)) = U$ and $P(U) \in Q_S(X, T)$, $P_r(\text{scl}\ U) = P_r(\text{scl}\ P^{-1}P(U))) = P(P^{-1}(\text{scl}\ P(U))) = \text{scl}\ P(U) \in \beta$, which implies $\beta_r \subset \beta$. Thus $\beta_r = \beta$ and $Q_S(X, T_S) = Q_S(X, T)_S$. Since (X, T) is s-weakly Hausdorff, then (X, T_S) is s-weakly Hausdorff and $(X^*_S, Q_S(X, T_S))$ is s-weakly Hausdorff. Since (X, T_S) is semi$-R_0$, then $(X^*_S, Q_S(X, T_S))$ is semi$-T_1$ [13].

Examples can be given to show that s-weakly Hausdorff in Theorem 2.6 cannot be replaced by T_1.

Theorem 2.7. Let (X, T) be a space. Then (X, T) is R_1 iff $(X_S, Q_S(X, T))$ is R_1.

Proof : For each $z \in X$ let C_z be the element of X_S containing z. Suppose (X, T) is R_1. Let $x, y \in X$ such that $\{\overline{C_x}\} \neq \{\overline{C_y}\}$. Then $\overline{\{x\}} \neq \overline{\{y\}}$ and there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [5]. Then $P(U)$ and $P(V)$ are disjoint open sets such that $\overline{\{C_x\}} \subset P(U)$ and $\overline{\{C_y\}} \subset P(V)$. Thus $(X_S, Q_S(X, T))$ is R_1 [15].

Conversely, suppose $(X_S, Q, _S(X, T))$ is R_1. Let $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. Since $P(\overline{\{z\}}) = \overline{\{C_z\}}$ and $P^{-1}(P(\overline{\{z\}})) = \overline{\{z\}}$ for all $z \in X$ and $\overline{\{x\}} \neq \overline{\{y\}}$, then $\overline{\{C_x\}} \neq \overline{\{C_y\}}$ and there exist disjoint open sets u and

V such that $\{\overline{C_x}\} \subset U$ and $\{\overline{C_y}\} \subset V$. Then $P^{-1}(U)$ and $P^{-1}(V)$ are disjoint open sets such that $\{x\} \subset P^{-1}(U)$ and $\{y\} \subset P^{-1}(V)$. Thus (X, T) is R_1.

Combining the results above give the following result.

Corollary 2.1. Let (X, T) be R_1. Then $(X_S, Q_S(X, T)_S) = (X^*_S, Q_S(X, T_S))$, which is R_1 and semi $-T_1$.

The result above raised questions about what would happen if the semi-regularization space in Theorem 2.6 or Corollary 2.1 was replaced by the feebly induced space. Results in [18] show that for any space (X, T) the feebly induced space of the semi$-T_0-$indentification space of (X, T) equals the semi$-T_0-$identification space of the feebly induced space of (X, T). Also, examples can be given to show that the T_0—identification space of the semi-regularization of a semi$-T_2$ space, which is stronger than $s-$weakly Hausdorff, need not be homeomorphic to the semiregularization of the T_0 —identification space of tne space.

3. Additional Properties and Semiregularization and Feebly Induced Spaces :

In 1975 [28] regular was generalized to s-regular, in 1978 [29] normal was generalised to s-normal, and in 1981 [15] compactness was strengthened to semi compactness by replacing the word open in the definition of regular, normal, and compact by semi open, respectively. In 1982 [19|| s-regular was strengthened to semi-regular and in [20] s-normal was strengthened to semi-normal by replacing closed in the definition of s-regular and s-normal by semi closed, respectively.

In 1961 [5] A.S. Davis was interested in obtaining properties weaker than T_i, which together with T_{i-1} would be equivalent to T_i. $i=1, 2$. Davis' 1961 investigation led to the definition of R_0 and R_1 spaces. In 1975 [30] T_i was generalized to semi$-T_i$ by replacing the word open in the definition of T_i by semi open, $i=0, 1, 2$. These new definitions raised questions about properties weaker than semi$-T_i$, which together with semi$-T_{i-1}$ would be equivalent to semi$-T_i$, $i=1, 2$, which led to the definition of semi$-R_0$ spaces in 1975 [31] and semi$-R_1$ spaces in 1978 [21]. Also, the introduction of semi$-T_i$ spaces raised questions about properties weaker than T_i, which together with semi$-T_i$ would be equivalent to T_i, $i=0, 1, 2$, which led to the definition of s-essentially T_i spaces, $i=0$ [22], $i=1$ [23], and $i=2$ [24]. A space (X, T) is s-essentially T_i iff $(X_S Q, {}_S(X,T))$ is T_i, $i=0, 1, 2$. In this section the properties given above are further investigated using semi-regularization and feebly induced spaces.

Theorem 3.1. Let (X, T) be s-regular. Then $(X, T)_S$ is s-regular.

Proof : Let C be T_S closed and let $x \notin C$. Then C is T closed and $x \notin C$ and there exist disjoint T semi open sets U and V such that $x \in U$ and $C \subset V$. Let $A, B \in T$ such that $A \subset U \subset \bar{A}_T$ and $B \subset V \subset \bar{B}_T$. Then $scl_T A$ and $scl_T B$ are disjoint T_S open sets. Let $D=\{x\} \cup scl_T A$ and $E = C \cup scl_T B$. Since $scl_T A \subset D \subset \bar{A}_T$ and $=\overline{scl_T A}_T = \overline{scl_T A}_{T_S}$, then $D \in SO(X, T_S)$. Similarly, $E \in SO(X, T_S)$.

Theorem 3.2. Let (X, T) be s-normal. Then (X, T_S) is s-normal.

The proof is similar to that for Theorem 3.1 and is omitted.

Theorem 3.3. Let (X, T) be a space such that (X, F0(X, T)) is s-regular (s-normal). Then (X, T) is s-regular (s-normal).

Proof : Consider the case that (X, F0(X, T)) is s-regular. Let C be T closed and let $x \notin C$. Since $T \subset F0(X, T)$ then C is F0(X, T) closed and there exist disjoint F0(X, T) semi open sets U and V such that $x \in U$ and $C \subset V$. Then U and V are disjoint T semi open sets such that $x \in U$ and $C \subset V$. The proof is similar for s-normal and is omitted.

Since s-regular and s-normal are not semi topological properties [25], then the converses of Theorems 3.1, 3.2, and 3 3 are false.

Since semi-regular [19] and semi-normal [20] are semi topological properties, then a space (X, T) is semi-regular (semi-normal) iff (X, F0(X, T)) is semi-regular (semi-normal)

Theorem 3.4. If (X, T) is semi-regular (semi-normal then), (X, T_S) is semi-regular (semi-normal).

Proof: Let C be T_S semi closed and let $x \notin C$. Since $SO(X,T_S) \subset SO(X, T)$, then C is T semi closed and there exist disjoint T semi open sets U and V such that $x \in U$ and $C \subset V$. Let $A, B \in T$ such that $A \subset U \subset \bar{A}_T$ and $B \subset V \subset \bar{B}_T$. Then $D = \{x\} \cup scl_T A$ and $E = C \cup scl_T B$ are disjoint T_S semi open sets such that $x \in D$ and $C \subset E$. The proof for semi-normal is similar and omitted.

Since semi compactness is a semi topological property [26], then a space (X, T) is semi compact iff (X.F0(X, T)) is semi compact.

Theorem 3.5. Let (X, T) be semi compact. Then (X, T_S) is semi compact.

The straight forward proof is omitted.

Examples can be given showing that the converses of Theorem 3.4 and Theorem 3.5 are false.

In [18] it was shown that for any space (X, T), (X, F0(X, T)) is s-essentially T_0; that for a R_0 space (X, T), which is weaker than s-essentially T_1, (X, F0(X, T)) is s-essentially T_1, and that the following statements are equivalent: (*a*) (X, T) is s-essentially T_2, (*b*) (X, F0(X, T)) is R_1, and (*c*) (X, F0(X, T)) is s-essentially T_2. Examples can be given of T_1 spaces whose semiregularizations are not s-essentially T_0 and of non s-essentially T_0 spaces whose semiregularizations are s-essentially T_2.

Theorem 3.6. Let (X, T) be s-essentially T_2. Then (X, T_S) is s-essentially T_2.

Proof: Since (X, T) is s-essentially T_2, then (X, T) is R_1 [24]. $(X_S, Q_S(X, T))$ is T_2, and $(X_S, Q_S(X, T)_S)$ is T_2 [2]. Since (X, T) is R_1, then $(X^*_S, Q_S(X, T_S)) = (X_S, Q_S(X, T)_S)$ is T_2 and (X, T_S) is s-essentially T_2.

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ON SELF-MAPS OF BCI-ALGEBRAS

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Abstract :

In this paper, we define a mapping $f : X \to I$, the centre of BCI-algebra X, and show that this self-map is a homomorphism. Further, using the concept of the centre, a homomorphism $f : X \to Y$ has been factorized into two factors such that one is an onto homomorphism and the other is one-one homomorphism.

Preliminaries : This section includes background material that is needed in the sequel.

Definition 1. [4]. A non-empty set X together with a binary operation * and a special element 0 is said to be BCI-algebra provided the foilowing axioms are satisfied for all $x\ y, z \in X$.

(1) $((x*y)*(x*z)*(z*y)=0$,

(2) $(x*(x*y))*y=0$,

(3) $x*x=0$

(5) $x*y=0=y*x$ implies $x=y$,

(5) $x*0=0$ implies $x=0$,

where $x*y=0$ if and only if $x \leqslant y$.

Definition 2 [1]. Let X be a BCI-algebra and $x, y \in X$. Then x, y are called comparable if and only if $x*y=0$ or $y*x=0$. We choose an element $x_0 \in X$ such that there does exist any $y \neq x_0$ with $y*x_0=0$ and define

$A(x_0)=\{x \in X : x_0 * x=0\}$.

$A(x_0)$ is said to comparable if each pair $x, y \in A(x_0)$ is comparable.

We call it a comparable branch. Clearly each $A(x_0)$ is non-empty, because $x_0 * x_0 = 0$ implies $x_0 \in A(x_0)$. It follows that $A(x_0)=\{x_0\}$, a singleton set is comparable, we will call it uniary comparable. The point $x_0 \in A(x_0)$ is called an initial element of $A(x_0)$. Let I denote the set of all initial elements. We call it the centre of X. A BCI-Algebra X in which each $A(x_0)$ is uniary comparable is known as S_4-algebra. Note that $A(0)=\{0\}$, a singleton set, that is. BCK-part of X is $\{0\}$, hence it is a p-semisimple BCI-algebra. Moreover, in S_4 algebra $x*y=0$ implies $x=y$ for all $x, y \in X$.

We will denote the BCI-part of X by M and it is given as

$$M=\{x \in X : 0*x=0\}$$

We refer to [5], [7], for relevant definitions and other informations about the homorphisms and quotient BCI-algebra X/A., where A is an ideal in X. In a BCI-algebra X, the following hold ([5]).

(6) $(x*y)*z=(x*z)*y$

(7) $x*0=x$

(8) $x \leqslant y$ implies $x*z \leqslant y*z$ and $z*y \leqslant z*x$ for all $x, y, z \in X$.

(9) Let X be a BCI-algebra and M its BCK-part, then M is the maximal BCK-algebra ([4]).

(10) Let X be a BCI-algebra with I as the centre. Then $\bigcup_{x \in I} A(x)=X$ and $\bigcap_{x \in I} A(x)=\varnothing$ ([1])

(11) Let X be a BCI-algebra with M as its BCK-part. Then for $m \in M$, $x \in X - M$, $x*m$, $m*x \in X$ ([5]).

(12) Let X be a BCI-algebra with M as its BCK-part. Let $A(x_0) \subseteq X$ and $A(y_0) \subseteq X$. If $0*x \in A(y_0)$, then $0*x=y_0$ for all $x \in A(x_0)$ ([2]).

(13) Let X be a BCI-algebra with M as its BCK-part. Let $A(x_0) \subseteq X$. Then $x, y \in A(x_0)$ implies $x*y$, $y*x \in M$. ([1]).

(14) Let X be a BCI-algebra with M as its BCK-part. Let $A(x_0) \subseteq X$ and $A(y_0) \subseteq X$. Then $x \in A(x_0)$ and $y \in A(y_0)$ imply $x*$, $y*x \in X-M$ ([1]).

15) Let X be a BCI-algebra with I as its centre. Then I is an S_4—algebra ([2]).

(16) Let X be a BCI-algebra and H a strong ideal in X. Then X/H is an S_4—algebra ([3]).

(17) Let X be an S_4—algebra (p-semisimple), then X is an abelian group ([7]).

Theorem 1. Let X be a BCI-algebra with I as its centre. Define $f: X \to X$ by $f(x) = x_0 \in I$, for $x \in A(x_0)$. Then f is a $*$—homomorphism.

Proof. Case (i). Let $x, y \in M$, Then by (9) M being a maximal BCK-algebra in X implies $x*y \in M$, obviously $A(0) = M$ implies $f(x*y) = 0 = 0*0 = f(x)*f(y)$.

Case (ii), Let $x \in M$, $y \in X-M$. By (10), $\bigcup_{x \in I} A(x_0) = X$ and $\bigcap_{x_0 \in I} A(x_0) = \phi$ implies y is contained in a unique $A(y_0) \subseteq X-M$. By (11), $x*y \in X-M$ and by (10) $x*y \in A(z_0) \subseteq X-M$ for a unique $z_0 \in I \cap (X-M)$. By (8), $0 \leqslant x$, gives $0*y \leqslant x*y$. By (10) $0*y$ and $x*y$ are contained in the same $A(z_0)$. By (12), $0*y \in A(z_0)$ implies $0*y = z_0 = 0*y_0$. It follows that $f(x*y) = z_0 = 0*y_0 = f(x)*f(y)$.

Case (iii) $x \in X-M$, $y \in M$. By (11), $x*y \in X-M$. By (10), $x*y$ is contained in a unique $A(z_0) \subseteq X-M$ for $z_0 \in I \cap (X-M)$. Now $0 \leqslant y$ implies $x*y \leqslant x$. Again by (10), $x \in X-M$ implies x is contained in a unique $A(x_0) \subseteq X-M$. Therefore, $x*y \leq x$ implies $x*y \in A(x_o)$ and $f(x*y) = x_0 = x_0*0 = f(x)(y)$.

Case (iv) $x, y \in X-M$. There are two possibilities.

Case (iv) (a), x, $y \in A(x_0) \subseteq X-M$. Then by (13), $x*y \in M$ implies $f(x*y) = 0 = x_0*x_0 = f(x)*f(y)$.

Case (iv) (b) $x \in A(x_0) \subseteq X-M$. $y \in A(y_0) \subseteq X-M$. Then by (14), $x*y \in X-M$. implies $x*y \in A(z_0) \subseteq X-M$ for some $z_0 \in I \cap (X-M)$. Since $x_0 \leqslant x$, therefore, $x_0*y_0 \leqslant x*y_0$. (1.1)

Again $y_0 \leqslant y$ implies $x*y \leqslant x*y_0$ (1.2)

By (10), x_0*y_0, $x*y_0$ and $x*y$ are contained in the same $A(z_0)$, because $x*y \in A(z_0)$. Since by (15), I is an S_4—algebra therefore, $x_0*y_0 \in I$ and $x_0*y_0 \in A(z_0)$ both imply $x_0*y_0=z_0$. Now by definition $f(x*y)=z_0=x_0*y_0=f(x) * f(y)$. This completes the proof.

Theorem 2. Let X be a BCI-algebra with I and M as its centre and BCK-part respectively. If X/M is the quotient BCI-algebra of X by M. Then for $x \in A(x_0)$, $C_{x_0}=A(x_0)=C_x$.

Proof. By (10), each $x_0 \in I$ gives an $A(x_0)\subseteq X$, therefore it is sufficient to show that $C_{x_0}=A(x_0)$. Let $x \in A(x_0)$. By (13), x_0*x, $x*x_0 \in M$, that is $x\sim x_0$ and $x \in C_{x_0}$ which gives $A(x_0)\subseteq C_{x_0}$.

Next we show that $C_{x_0} \subseteq A(x_0)$. Suppose $C_{x_0} \not\subseteq A(x_0)$. It means that there eaists an element $y \in C_{x_0}$ which is not contained in $A(x_0)$. In other words, there exists a $y \in A(y_0)\subseteq X-A(x_0)$ such that x_0*y, $y*x_0 \in M$, a contradiction of (14). Thus $C_{x_0} \subseteq A(x_0)$. (3.2)

From (3.1) and (3.2), $C_{x_0}=A(x_0)$. This completes rhe proof.

Remark 1. We note that in X/M, $C_{x_0}=C_x$ for $x \in A(x_0\subseteq X$, that is $C_{x_0}=C_x$, for $x \in X$ and we can write $X/M=\{C_x : x \in X\}=\left\{C_{x_0} : x_0 \in I\right\}$. It follows that $0(X/M)=0(I)$.

Let X, Y be BCI-algebras with I, I' being their centres, respectively. We write $I=\{x_0 : x_0$ is initial element in X$\}$ and $I'=\{y_0 : y_0$ is initial element in Y$\}$. $A(x_0)=\{x \in X=x_0*x=0\}$ and $A(y_0)=(y \in Y : y_0*y=0\}$.

Theorem 3. Let X, Y be BCI-algebras with I and I′ as centres respectively. Let $f: x \rightarrow y$ be a homomorphism. If $x_0 \varepsilon$ I and $f(x_0) \varepsilon A(y_0) \subseteq Y$, then $f[A(x_0)] \subseteq A y_0)$.

Proof. Let $x \varepsilon A(x_0) \subseteq X$. Then $x_0 * x = 0$ or $0 = x_0 * x$ gives that $f(0) = f(x_0 * x) = f(x_0) * f(x)$. $0 = f(x_0) * f(x)$ implies $f(x_0) \leqslant f(x)$.

Since $f(x_0) \varepsilon A(y_0) \subseteq Y$. therefore $y_0 \leqslant f(x_0)$ implies $y_0 \leqslant f(x)$ or $f(x) \varepsilon A(y_0)$. Hence $f[A(x_0)] \subseteq A(y_0)$. This completes the proof.

We know that given a homomorphism $f; X \rightarrow Y$, there exist onto and one-one homomorphisms $f_1 : X \rightarrow X/\ker(f)$ and $f_2 = X/\mathrm{Ker}(f) \rightarrow Y$, respectively such that $f = f_2 \, 0 \, f_1$. A question arises, does there exist an ideal of X other than Ker (f) which gives similar result. The following theorem gives a partial answer to this problem.

Theorem 4. Let X and Y be BCI-algebras with I and I′ as the centres respectively. Let M be BCK-part of X. Let $f: X \rightarrow Y$ be a homomorphism such that $f(x) = 0$, for $x \varepsilon$ M. Then there exists two homomorphism $g : X \rightarrow X/M$ and $h : X/M \rightarrow Y$ such that $f = hog$.

Proof. By remarks 1 of theorem 2, $X/M = \{C_x : x \varepsilon X\} = \{c_{x_0} : x_0 \varepsilon : I\}$ and $c_x = c_{x_0} = A(x_0)$ for $x \varepsilon A(x_0)$. We define $g : X \rightarrow X/M$ by $g(x) = C_x$ for $x \varepsilon X$.

Obviously g is onto. By (10), $\bigcap_{x_0 \varepsilon I} A(x_0) = \varnothing$, therefore $x \varepsilon$ X is contained in a unique $A(x_0) \subseteq X$. By theorem 2, $C_x = C_{x_0}$ for $x \varepsilon A(x_0)$. Thus $g(x) = C_{x_0}$ for $x \varepsilon A(x_0) \subseteq X$.

Case (*i*) Let $x, y \varepsilon$ M. By theorem 2, $C_x = C_y = C_0$. Thus $x * y$ ε M implies $g(x*y) = C_{x*y} = C_0 = C_x * C_x = C_x * C_y = g(x) * g(y)$.

Case (*ii*) Let $x \varepsilon$ M, $y \varepsilon$ X$-$M, By theorem 2, $C_x = C_0$ and by

(10) $y \in A(y_0) \subseteq X-M$ for a unique $y_0 \in I \cap (X-M)$, where $C_{y_0} = C_y$. By (11), $x*y \in X-M$ and by (10), $x*y$ is contained in a unique $A(z_0) \subseteq X-M$. Since $x \in M$, therefore $0 \leqslant x$ implies $0*y \leqslant x*y$. By (10), $0*y$ and $x*y$ are contained in $A(z_0)$. By (12), $0*y \in A(z_0)$ implied $0*y = 0*y_0 = z_0$.

By theorem 2, $C_{z_0} = C_{0*y_0} = C_{x*y}$ Thus $g(x*y) = C_{x*y} = C_{z_0} = C_{0*y_0} = C_0 * C_{y_0} = C_x * C_y = g(x)*g(y)$

Case (*iii*) Let $x \in X-M$, $y \in M$. By (10), $x \in X-M$ implies x is contained in a uniqne $A(x_0) \subseteq X-M$ and by theorem 2, $C_x = C_{x_0}$. Similiarly $C_y = C_0$. Now $x*y \in X-M$ imply $x*y \in A(z_0)$, for a unique $z_0 \in I \cap X-M$. Since $0 \leq y$, therefore $x*y \leqslant x$ which imply $x*y \in A(x_0)$ and $C_{x*y} = C_{x_0}$. Thus $g(x*y) = C_{x*y} = C_{x_0} = C_{x_0*0} = C_{x_0} * C_0 = C_x * C_y = g(x)*g(y)$.

Case (*iv*) (*a*) Let $x, y \in A(x_0) \subseteq X-M$. Then $C_x = C_y = C_{x_0}$. By (13), $x*y \in M$ implies $C_{x*y} = C_0$ and $g(x*y) = C_{x*y} = C_0 = C_x * C_x = C_x * C_y = g(x)*g(y)$.

Case (*iv*) (*b*), Let $x \in A(x_0) \subseteq X-M$, $y \in A(y_0) \subseteq X-M$. Then $C_x = C_{x_0}$ and $C_y = C_{y_0}$. By (14), $x*y \in X-M$. By (10), $x*y$ $X-M$ implies that $x*y \in A(z_0) \subseteq X-M$ for a unique $z_0 \in I \cap X-M$.

Since $x_0 \leqslant x$, therefore $x_0 * y_0 \leqslant x*y_0 \ldots\ldots$ (5.1)

Since $y_0 \leqslant y$, thereforefore $x*y \leqslant x*y_0$ (5.2)

From (5.1), 5.2) and (10), it follows that x_0*y_0, $x*y_0$ and $x*y$ are contained in $A(z_0)$ and we can write $C_{x_0*y_0}=C_{x*y_0}=C_{x*y}=C_{z_0}$.

Thus $g(x*y)=C_{x*y}=C_{x*y_0}=C_{x_0}*C_{y_0}=C_x*C_y= g(x)*g(y)$, hence g is homomorphism.

Next we show that $h: X/M\rightarrow Y$ defined by $h[C_x]=f(x_0)$ for all $x \in A(x_0)$ is a homomorphism. Let $C_{x_0}, C_{y_0} \in X/M$. Then $C_{x_0}*C_{y_0}= C_{x_0*y_0} \in X/M$. Since I is a S_4—algebra therefore $x_0*y_0 \in$ I. Let $x_0*y_0=z_0$, then $C_{x_0*y_0}=C_{z_0} \in X/M$. $h(C_{x_0}*C_{y_0})=h(C_{x_0*y_0})=h(C_{z_0})=f(z_0)= f(x_0*y_0)=f(x_0)*f(y_0)=g(C_{x_0})*g(C_{y_0})$. Further $hog(x)= h(C_x)=f(x_0)$ for all $x \in X$. This completes the proof.

Corollary 1. Let X and Y be BCI-algebra with I, I′ as their centres respectively. Let X/M be the quotient BCI-algebra, where M is the BCK-part of X. Let $f: X\rightarrow Y$ be a homomorphism such that $f(x)=0$, for all $x \in M$ and $f/I : I\rightarrow I'$ is one-one, then $h: X/M\rightarrow Y$ defined by $h(c_x)=f(x_0)$ for $x \in A(x_0)$ is one-one homomorphism.

Proof. By above theorem we have h is a homomorphism." Let $x \in A(x_0)\subseteq X$, then by theorem 2$C_x=C_{x_0}=A(x_0)$ and by remark 1, $X/M=\{C_x ; x \in X\}=\{C_{x_0} : x_0 \in I\}$. Let $C_{x_0}, C_{y_0} \in X/M$, then $h(C_{x_0})=h(C_{y_0})$ implies $f(x_0)=f(y_0)$. Since $f: I\rightarrow I'$ is one-one, therefore $x_0=y_0$ and $C_{x_0}=C_{y_0}$ which implies h is one-one. This complete the proof.

Remarks 2. Since $g : X \to X/M$ defined by $g(x) = C_x$, for $x \varepsilon X$ is onto and $h : X \to Y$ defined by $h(C_x) = f(x_0)$ for $x \varepsilon A(x_0)$, is one-one, therefore, $f ; X \to Y$ given by $f = h \, o \, g$. Note that hog is an empimono factorization of f.

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ON THE CATEGORY OF BCI-ALGEBRAS

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Abstract ;

We show that the category BCI of BCI—algebras and BCI—homomorphisms is complete. Further we show that it has coequalizers, kernel pairs and image factorization system. It is also proved that onto homomophisms and co-equalizers coinside in BCI and monomorphisms restricted to BCI—G parts are one one. Two problems have been posed. It has been proved that MBCI is a reflexive subcategory of BCI. Further the existence of a functor from MBCI into BCI is also proved.

1. Introduction :

In [9] and [11], K. Iseki emphasized the importance of the category theoretic approach to BCK—algebras, which were introduced by him. He further proved in [11] that the category BCK of BCK—algebras and BCK—homomorphisms has limits. Further H. Yutani in his paper [18] proved that this category has co-limits. The existence of co-equalizers, characterisation of onto homomorphism and monomorphisms in this category has also been studied by K. Iseki and H. Yutani in [11], [16], [17] aud [18].

BCI—algebras, which are generalization of BCK—algebras, were introduced by K. Iseki [8] and since then have been studied extensively by various researchers. Recently in [7], C.S Hoo in-

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vestigated injectives in the category BCI of BCI—algebras and BCI—homomorphisms. But this category is still uninvestigated. The purpose of this paper is to investigate certain categorical notions in this category.

We shall follow standard definitions. Our categorical concepts shall be those of standards text, like [1] and [14], to which we refer the reader for definitions of standard categorical terms. Our notions of BCK—algebras shall be as developed in [10] and [12] and those of BCI—algebras shall be as in [4], [6], [13] and [15].

We denote by BCI the category of BCI—algebras and BCI—homomorphisms. Recall that in both categories, a homomorphism $f: X \to Y$ means that $f(x_1 * x_2) = f(x_2) * f(x_2)$, and hence $f(0)=0$. This also means that if $x \leqslant y$, then $(x) \leqslant f(y)$. We shall denote a general category by K. its objects by | K | and the set of morphisms from an object A into object B by K(A, B).

2. Limits in BCI :

We now show that the category BCI has arbitrary products and equalizers.

Theorem 2.1. The category BCI has arbitrary products.

Proof. Let X_α, $\alpha \in J$ be a family of BCI objects. Let $\pi_{\alpha \in J} X_\alpha = \{f : f : J \longrightarrow \bigcup_{\alpha \in J} X_\alpha$, is a function and $f(\alpha) \in X_\alpha$ for all $\alpha \in J\}$.

We define the binary operation * in $\pi_{\alpha \in J} X_\alpha$ by :

$(f*g)(\alpha) = f(\alpha)*g(\alpha)$ for all $\alpha \in J$. Then routine calculations give that $\pi_{\alpha \in J} X_\alpha$ is a BCI—algebra under this operation with zero as zero function given by $0(\alpha)=0_\alpha$, 0_α being the zero of X_α.

Further the mapping $pr_\alpha : \pi_{\alpha \in J} X_\alpha \to X_\alpha$, $\alpha \in J$, defined by $pr_\alpha(f) = f(\alpha)$, for all $\alpha \in J$, is a BCI—homomorphism. Further for all X ε | BCI |, f_α ε BCI (X, X_α), α ε J, the mapping ψ : X→ $\underset{\alpha\, \varepsilon\, J}{\pi}$ X_α defined by ($\psi(x)$) (α)=f_α (x) for all x ε X, α ε J, is the unique BCI=homomorphism making the diagram commutative.

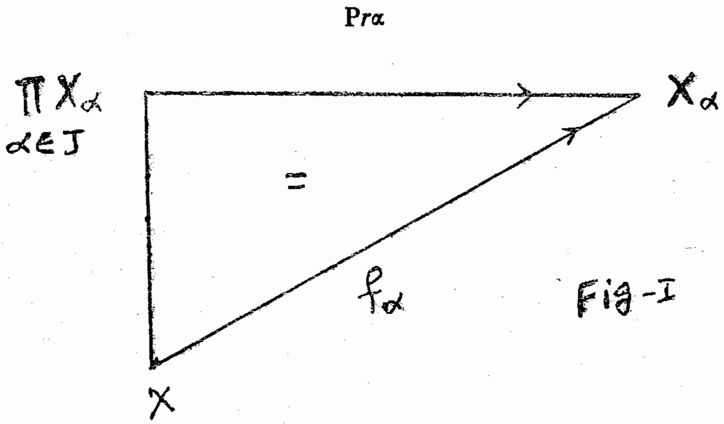

Fig-I

This completes the proof.

Theorem 2.2. The category BCI has equalizers.

Proof. Let f, g ε BCI (X, Y). Define Z={x : x ε X and $f(x)$= $g(x)$}. It is easy to verify that Z is a sub-algebra of X. Let i : Z→X be imbedding of Z into X, given by, $i(z)=z$ for all z ε Z. Obviously i ε BCI (Z, X) and satisfies foi=goi. Further let C ε | BCI |, h ε BCI (C, X) be such that foh=goh. Then the function ψ : C→Z given by $\psi(c)=h(c)$ is well-defined because $f(h(c))=g(h(c))$ implies $h(c)$, ε Z for all c ε C. Further ψ ε BCI (C, Z) and is the unique such BCI–morphism making the diagram commutative.

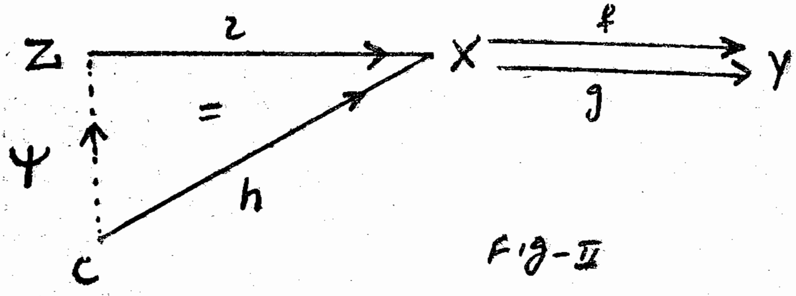

Fig-II

Thus $i=eq$ (f, g). Hence the theorem.

Since category K with arbitrary products and equalizers has arbitrary limits [14], therefore we have the following proposition.

Proposition 2.1. The category BCI has arbitrary limits and thus is complete.

3. Monomorphisms, epimorphisms and co-equalizers in BCI.

In this section we show that BCI has kernel pairs, co-equalizers and onto homormorphism coincide in it. One-one homomorphisms are monomorphisms, but, contrary to BCK [11], it is not obvious that monomorphisms are one-one homomorphisms. Further onto homomorphisms are epimorphisms but converse is not obvious. However, in MBCI epimorphisms are onto homomorphisms and monomorphisms restricted to BCI—G parts are one-one homomorphisms in BCI. We now state the following results proved by C.S.Hoo in [7].

Proposition 3.1. In both BCK and BCI, onto homomorphism are epimorphisms.

Proposition 3.2. In BCI, one-one homomorphisms are monomorphisms. We now prove the following :

Theorem 3.1. Let $f \in$ BCI (X, Y) be onto, then f is a co equalizer.

Proof. Let $f: X \to Y$ be onto and let X x X be the product BCI—algebra of X with itself. Let $Z = \{(x_1, x_2) : x_1, x_2 \in X \text{ and } f(x_1) = f(x_2)\}$. Obviously Z is a sub-algebra of X x X and thus Z $\in$ | BCI |. Let $p_1, p_2 : Z \to X$ be defined by :

$p_1(x_1, x_2) = x_1$, $p_2(x_1\ x_2) = x_2$ for all $(x_1, x_2) \in Z$. It is easy to verify that p_1 and p_2 are BCI—homomorphisms and satisfy $fop_1 = fop_2$.

Let $g \in$ BCI (X, C) be such that $gop_1 = gop_2$. Since $f: X \to Y$ is onto, so far any $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. We define $h : Y \to C$ by $h(y) = g(x)$. where $f(x) = y$, for all $y \in Y$. To show that h is well-defined, we consider $f(x_1) = f(x_2) = y$ (say). This implies $(x_1, x_2) \in Z$ and $gop_1 = gop_2$ gives $g(x_1) = g(x_2)$. Thus $h(y) = g(x_1) = g(x_2)$. Further let $y_1, y_2 \in Y$, then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$ and hence $g(x_1) = h(y_1)$ and $g(x_2) = h(y_2)$.

Further $y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$. Hence $h(y_1 * y_2) = g(x_1 * x_2) = g(x_1) * g(x_2) = h(y_1) * h(y_2)$. Thus $h \in$ BCI (Y, C) and obviously satisfies hof$=g$. The uniqueness of h follows from the fact that f is onto homomorphism and propostion 3.1. Thus f=Coeq (p_1, p_2).

Theorem 3.2. The category BCI has kernel pairs.

Proof. Let $f \in$ BCI (X, Y). Let $Z = \{(x_1. x_2) : x_1, x_2 \in X$ and $f(x_1) = f(x_2)\}$ be the sub-algebra of the product algebra $X \times X$ and let $p_1, p_2 : Z \to X$ be the same as in theorem 3.1. Then $p_1, p_2 \in$ BCI (Z, X) and fop$_1$=fop$_2$. We claim that the pair (p_1, p_2) is a kernel pair of f.

Let $g_1, g_2 \in$ BCI (D, X) be such that fog$_1$=fog$_2$. We take $g : D \to Z$ as

$$g(d) = (g_1(d), g_2(d)) \text{ for all } d \in D.$$

g is well-defined because fog$_1$=fog$_2$ gives $f(g_1(d)) = f(g_2(d))$, which gives $(g_1(d), g_2(d)) \in Z$. Further

$$\begin{aligned} g(d_1 * d_2) &= (g_1(d_1 * d_2), g_2(d_1 * d_2)) \\ &= g_1(d_1) * g_1(d_2), g_2(d_1) * g_2(d_2)) \\ &= g_1(d_1), g_2(d_1),) * (g_1(d_2), g_2(d_2)) \\ &= g(d_1) * g(d_2). \end{aligned}$$

Thus $g \in$ BCI (D, Z) and satisfies $p_1 0 g = g_1$ and $p_2 0 g = g_2$. The uniqueness of g is the consequence of its definition. Hence the theorem.

We now state the following theorem which will be used in the sequel. The proof of this theorem can be carried out on the lines of the proof given by K. Iseki for the similar result for BCK–algebras.

Theorem 3.3 [12]. Let X, Y. Z be BCI–algebras and let $f : X \to Y$ be onto homomorphism. Let $g : X \to Z$ be homomorphism such that Ker $f \subset$ Ker g. Then there exists unique homomorphism $h : Y \to Z$ satisfying hof$=g$.

Theorem 3.4. Every co-equalizer in BCI is an onto homomorphism.

Proof. Let $f \in$ BCI (X, Y) be a co-equalizer. Since every co-equalizer in a category K is a co-equalizer of its kernel pairs, therefore the BCI—algebra Z together with the projections defined in Theorem 3.1 is a kernel pair of f. Thus f=coeq (p_1, p_2). We further note that Z is the ideal congruence on X generated by Ker f=K because

$$\begin{aligned} Z &= \{(x_1, x_2) : x_1, x_2 \in X \text{ and } f(x_1)=f(x_2)\} \\ &= \{(x_1, x_2) : f(x_1) * f(x_2)=0=f(x_2) * f(x_1)\} \\ &= \{(x_1, x_2) : f(x_1 * x_2)=0=f(x_2 * x_1)\} \\ &= \{(x_1, x_2) : x_1 * x_2 \in K \text{ and } x_2 * x_1 \in K\} \end{aligned}$$

Let X/K be the corresponding quotient BCI—algebra and nat : X→X/K be defined by nat$(x)=[x]_K$. We consider nat $(x_1 * x_2)$

$= [x_1 * x_2]_K = [x_1]_K * [x_2]_K$ =nat (x_1)* nat (x_2), where $[x]_K$ denotes the class determined by x. Thus nat $\in$ BCI (X, X/K). Further nat op$_1$ (x_1, x_2)=nat (x_1)= $[x_1]_K = [x_2]_K$ = nat (x_2)=nat op$_2$ (x_1, x_2) for all $(x_1\ x_2) \in$ Z. Thus nat op$_1$=nat op$_2$.

Since f=coeq (p_1, p_2), therefore there exists unique $\psi \in$ BCI (Y, X/K) such that ψ of=nat, that is the following diagram commutes,

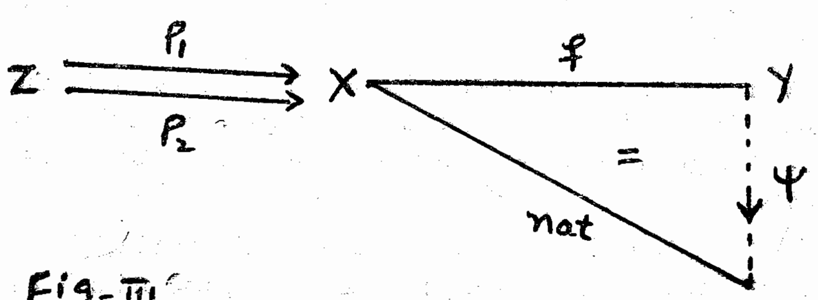

Fig-III

Let $x \in$ Ker nat. Then nat $(x)=[0]_K$, but nat $(x)=[x]_K$. Thus $(0, x) \in$ Z, which gives $x*0=x \in$ Ker f. Thus nat $\subseteq$ Ker f. Obviously nat is onto. Thus by Theorem 3.3 there exists unique $g \in$ BCI

(X/K, Y) such that go nat$=f$, that is, the following diagram commutes.

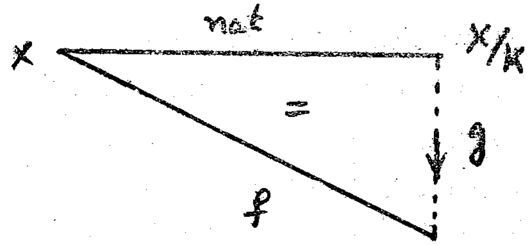

Fig-IV

Now $\psi \circ g \circ \text{nat} = \psi \circ f = \text{nat} = 1_{X/K} \circ \text{nat}$ ....(1)

and $g \circ \psi \circ f = g \circ \text{nat} = f = 1_Y \circ f$..............................(2)

Since nat is onto and f is co-equalizer, therefore both are epimorphisms. Hence (1) and (2) give that

$$\psi \circ g = 1_{X/K} \text{ and } g \circ \psi = 1_Y$$

Thus $\psi \in$ BCI (Y, X/K) is an isomorphism. Since nat is onto, therefore f is onto.

Remark 3, 1. Combining Theorem 3.1 and 3.4 we conclude that in the category BCI co-equalizers and onto homomorphisms coincide. Proposition 3.1 tells us that in the category BCI one-one homomorphisms are monomorphisms. The converse of this is not yet obvious. Thus we pose the following problem :

Problem 1. Monomorphisms in BCI are one-one homomorphisms or not ?

The following two results about monomorphisms in BCI are interesting.

Proposition 3.3. [7]. If $u : X \to Y$ is a monomorphisms in BCI, then the restriction $u_+ : X_+ \to Y_+$, X_+ and Y_+ are BCK parts of X and Y, is a monomorphism and hence one-one homorphism in BCK.

In [2], M.A. Chaudhry and B. Ahmad defined BCI$-$G part, which we denote by X_G, of a BCI$-$algebra X by : $X_G = \{x : x \in X$

and $0*x=x\}$ and proved that it is a proper BCI-sub-algebra of X. Let $f: X \to Y$ be a homomorphism in BCI and let $x \in X_G$, then $f(x)=f(0*x)=f(0)*f(x)=0*f(x)$ gives $f(x) \in Y_G$. Thus it is possible to have the restriction $f_G: x_G \to Y_G$ of f. This restriction has the following interesting.

Theorem 3.5. Let $f: X \to Y$ be monomorphism in BCI, then $f_G: X_G \to Y_G$ is a monomorphism as well as one-one in BCI.

Proof. Let $h, g \in$ BCI (C, X_G) be such that $f_G 0\, h = f_G 0\, g$. We denote both the imbeddings of X_G into X and Y_G into Y by i. Then $f\,0\,i\,0\,h = i\,0\,f_G\,0\,h = i\,0\,f_G\,g = f\,0\,i\,0\,g$. Since f is a monomorphism in BCI, we have ioh = iog and hence $h=g$. Thus $f_G \in$ BCI (X_G, Y_G) is mono. To prove that is one-one, we suppose that f_G is not one-one. Then there are distinct elements $x_1, x_2 \in X_G$ such that $f_G(x_1)=f_G(x_2)$. We take Z={0, 1}, the two element BCI—algebra defined by : $0*0=0=1*1$, $0*1=1*0=1$. We define ϕ, ψ : Z - —> X_G by : $\phi(0)=0$, $\phi(1)=x_1$ and $\psi(0)=0$, $\psi(1)=x_2$. Easy calculations give that $\phi, \psi \in$ BCI (Z, X_G) and $f_G\,0\,\psi = f_G\,0\,\psi$. Thus $\psi = \varnothing$ because f_G is a monomorphism. Hence $\psi(1) = \varnothing(1)$. Thus $x_1 = x_2$, which ls false. Hence f_G: X_G is one-one.

Proposition 2.3 tells us that in BCI onto homomorphism are emimorphisms but the converse is yet univestigated. Thus we pose the following problem.

Problem 2. Epimorphisms in BCI are onto or not ?

We now give a partial solution of this problem.

A BCI—algebra X is called p semisimple if $X_+=\{0\}$. It is called medial if $(x*y*(w*z)=(x*w)*(y*z)$. In [3], W.A. Dhdek studied BCI—algebras satisfying $x*(x*y)=y$. C.S.Hoo proved in [5] that all the three algebras are equivalent. He also proved in [6] that every sub-algebra of medial BCI—algebra is a closed ideal in it.

Let MBCI be the category of medial BCI—algebras and BCI—homomorphism. Obviously MBCI is a full sub-category of BCI. We now prove the following result.

Theorem 3.6. Epimorphism is MBCI are onto homomorphisms.

Proof. Let $f \in$ MBCI (X, Y). Then f is a BCI—homomorphism. Now let $f(x_1)$, $f(x_2) \in f(X)$. Thus x_1, $x_2 \in X$ and $f(x_1) * f(x_2) = f(x_1 * x_2) \in f(X)$. Thus $f(X)$ is a sub algebra of Y. Since Y is medial, therefore $f(X)$ is closed ideal in Y. Let $Y/f(X)$ be the corresponding quotient algehra, which is also medial. We define $g, h : Y/f(X)$ by $g(y) = [y]_{f(X)}$ and $h(y) = [0]_{f(X)}$. Easy calculations give that g and h are BCI—homomorphisms. Further go $f(x) = g(f(x)) = [f(x)]_{f(X)}$ and ho $f(x) = h(f(x)) = [0]_{f(X)}$. Since $f(x) * 0 = f(x) \in f(X)$ and $0 * f(x) = f(0) * f(x) = f(0 * x) \in f(X)$. Thus $[f(x)]_{f(X)} = [0]_{f(X)}$. Hence go $f =$ ho f, which gives $g = h$ because f is epimorphism. Hence $g(y) = h(y)$ for all $y \in Y$, which gives $[y]_{f(X)} = [0]_{f(X)}$. Thus $y * 0 = y \in f(X)$. Hence $Y \subseteq f(X)$, but $f(X) \subseteq Y$. Thus $f(X) = Y$, which gives f is onto.

We now state and prove the foilowing theorem. The proof may be carried out exactly on the similar lines as given by H. Yutani [17] for the category BCK. However for completion we give a simpler proof.

Theorem 3.7. The category BCI has co-equalizers.

Proof. Let f, $g \in$ BCI (X, Y). Let R be the minimum ideal conqruence on containing $\bar{R} = \{(f(x), g(x)) : x \in X\}$. Such an ideal conqruence exists and is the intersection of all ideal conqruences on Y containing $\bar{R}$. The quotient algebra Y/R is a BCI—algebra and the canonical mapping nat : $Y \rightarrow Y/R$ is an onto BCI—homomorphism. We show that nat=coeq (f, g).

Obviously y nat 0 $f(x)$ = nat $(f(x))_R$ = $[f(x)]_R$ = $[g(x)]_R$ = nat 0 $g(x)$ for all $x \in X$, because $R \subseteq \bar{R}$ and $(f(x), g(x)) \in \bar{R}$ for all $x \in X$. Thus nat 0 f = nat 0 g. Let $h \in$ BCI (Y, Z) be such that hog=hof. We now define ψ : Y/R→Z by $\psi([y]_R) = h(y)$. To show that ψ is well-defined, we suppose that $[y_1]_R = [y_2]_R$, Thus $(y_1, y_2) \in R$. Further $R_1 = (y, y') \, y, y' \in Y$ and $h(y')\}$ is the ideal conqruence on Y generated by Ker h. We note that $h(f(x) = h(g(x))$; for all $x \in X$. Thus $(f(x), g(x)) \in R_1$ and hence $\bar{R} \subseteq R$. But R is the minimum such congruence. Hence $(y_1, y_2) \in R$ gives $(y_1, y_2) \in R_1$. Thus $h(y_1) = h(y_2)$, which gives ψ is well-defined. Further $\psi([y_1]_R * [y_2]_R)$ $= \psi([y_1 * y_2]_R) = h(y_1 * y_2) = h(y_1) * h(y_2) = \psi([y_1]_R) * \psi([y_2]_R)$. Thus $\psi \in$ BCI (Y/R, Z) and obviously it satisfies ψ 0 nat $= h$, that is, it makes the diagram commutative. The uniqueness of ψ follows

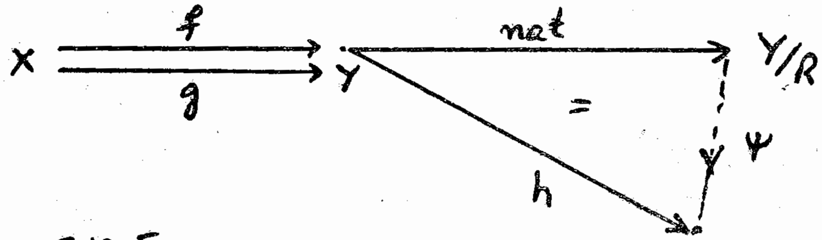

Fig-V

from the fact that nat is onto and hence epimorphisms in BCI.

4. Image Factorization System and Certain other Categorical Notions.

In this section we show that BCI has image factorization system and MBCI is reflexive sub-category of BCI. Further we shall define a functor from BCI into MBCI. For the definition of image fectorization system and a functor, we refer the reader to [1] and [14]. Further we state the following result which will be used in the sequel.

Proposition 4.1 [1, page 40]. Let K be a category in which every morphism factors as a co-equaiizer followed by a monomorphism, then (co-equalizers. monomorphisms) yields an image factorization system in K.

We now use this result to prove the following :

Theorem 4.1. (Co-equalizers, Monomorphisms) from an image factorization system in BCI.

Proof. In view of proposition 4.1 we need to show that every morphism in BCI factors as a co-equalizer followed by monomorphism.

Let $f \in$ BCI (X, Y). Then K=Ker f is closed ideal in X and the quotient algebra X/K is well-defined. Let nat : X→X/K be the canonical onto mapping defined by nat (x)=[X]. Then nat $\in$ BCI (X, X/K). Further we define $m([x_K])=f(x)$. First of all we show that m is well-defined.

Let $[x_1{}_K]=[x_2{}_K]$. Thus $x_1 * x_2 \in$ K and $x_2 * x_1 \in$ K. Hence $f(x_1 * x_2)=0$ and $f(x_2 * x_1)=0$, which gives $f(x_1 * f(x_2)=0$ and $f(x_2) * f(x_1)=0$. Thus $f(x_1)=f(x_2)$. Hence $m([x_1]_K)=m([x_2{}_K])$. Further $m([x_1]_K * [x_2]_K)=m([x_1 * x_2]_K)=f(x_1 * x_2)=f(x_1) * f(x_2)$ $=m([x_1{}_K]) * m([x_2]_K)$. Thus $m \in$ BCI (X/K, Y). Obviously $f=$ mo nat. To show that m is one-one, we take $m([x_1{}_K])=m([x_2{}_K])$. Thus $f(x_1)=f(x_2)$ which gives $f(x_1 * x_2)=0-f(x_2 * x_1)$. Hence $x_1 * x_2 \in$ K and $x_2 * x_1 \in$ K. Thus $[x_1{}_K]=[x_2{}_K]$. Since nat is onto BCI—homomorphism, so it is co-equalizer in BCI and m being one-one BCI—homomorphism is a monomorphism in BCI. Hence the theorem.

Let X be a BCI—Algebra and $X_+=\{x : x \in X$ and $0 * x=0\}$ be its BCK-part. Then it is well-known that X_+ is a closed ideal in X.

Further L. Triande and X. Changchang [15] have proved the quotient algebra X/X_+ is medial. Thus $X/X_+ \in |MBCI|$. Now we prove the following :

Theorem 4.2. The category MBCI is reflexive subcategory of BCI.

Proof. Let $X \in |BCI|$, then $X/X_+ \in |MBCI|$. We define $nat_X : X \rightarrow X/X_+$ by $nat_X(x)=[x]_{X_+}$. Obviously nat_X is a BCI—homomorphism. Thus $nat_X \in BCI(X, X/X_+)$. Further let $Y \in |MBCI|$, $f \in BCI(X, Y)$. Let $x \in X_+$, then $0 * x=0$ gives $f(0 * x)=f(0)=0$. Thus $f(0) * f(x)=0$, which gives $0 * f(x)=0$. Hence $f(x) \in Y_+$ we now define $g: X/X^+ \rightarrow Y$ by ; $g([x]_{X_+})=f(x)$. First of all we show that g is well-defined. Let $[x_1]_{X_+}=[x_2]_{X_+}$. Thus $x_1 * x_2 \in X_+$ and $x_2 * x_1 \in X_+$, which gives $f(x_1 * x_2) \in Y_+$ and $f(x_2 * x_1) \in Y_+$.

Since Y is medial so $Y_+=\{0\}$. Hence $f(x_1)=f(x_2)$, which gives $g([x_1]_{X_+})=g([x_2]_{X_+})$. Obviously g is a BCI—homomorphism.

Thus $g \in MBCI(X/X_+, Y)$ and makes the diagram

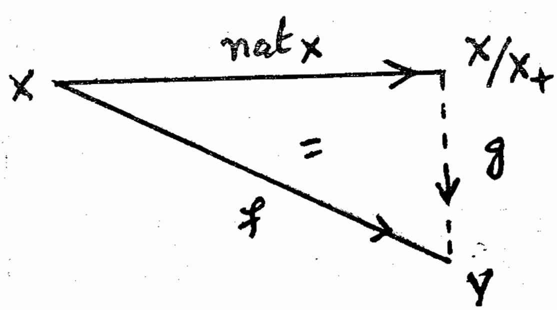

Fig-VI

commutative. The uniqueness of g follows from the fact that nat_X is onto homomorphism and hence an epimorphism. This completes the proof.

We now investigate a functor M from the category BCI into

MBCI, Let f BCI (X, Y), then we define $\bar{f}$: $X/X_+ \to Y/Y_+$ by : $\bar{f}([x]_{X_+}) = [f(x)]_{Y_+}$. Easy calculations give that $\bar{f}$ is well-defined and is a BCI—homomorphism. Thus $\bar{f} \varepsilon$ MBCI $(X/X_+, Y/Y_+)$. We further note that it makes the diagram

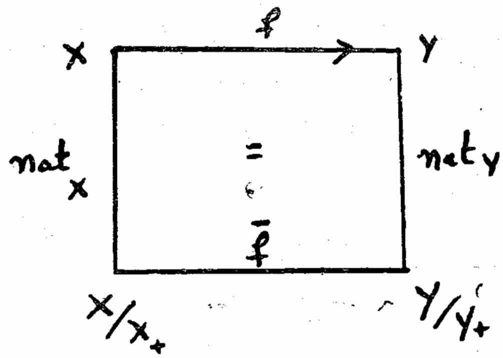

Fig-VII

commutative. The uniqueness of $\bar{f}$ follows from the fact that nat_X is onto and hence an epimorphism. In the sequel we shall denote $\bar{f}$ by M (f).

We now define a function M which maps | BCI | → | MBCI |; $X \to M(X) = X/X_+$, and for each pair of objects X and Y of BCI maps BCI (X, Y)→MBCI $(M(X) = X/X_+, M(y) = Y/Y^+)$: f→ $M(f) = \bar{f}$, where M(f) is the unique BCI—homomorphism making the diagram in Fig. VII commutative.

We note that $M(id_X)$ and $id_{M(X)}$ make the diagram

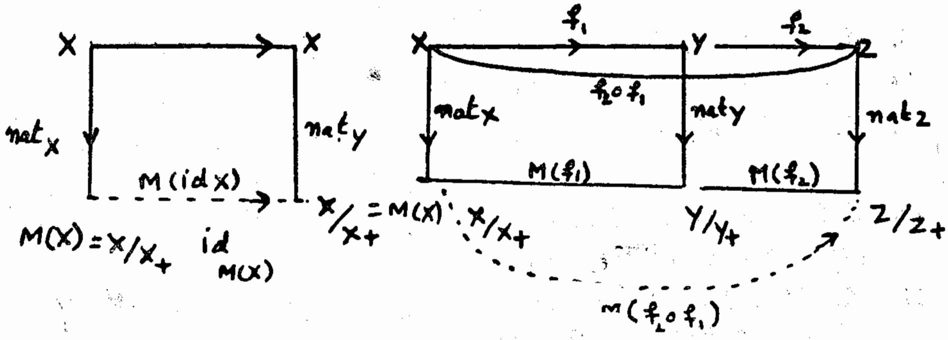

Fig-VIII

(2) commutative. The uniqueness of $M(id_X)$ gives $M(id_X = id_{M(X)}$.
Further $M(f_2) 0 M(f_1)$ and $M(f_2 0 f_1)$ make the outer part of diagram (3) commutative. The uniquencss of $M(f_2 0 f_1)$ gives $M(f_2 0 f_1) = M(f_2) 0 M(f_1)$. Thus we have the following theorem :

Theorem 4.3. The assignment M is a functor from BCI into MBCI.

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