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ON THE SCATTERING DATA FOR A BOUNDARY VALUE PROBLEM ON THE HALF LINE

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Abstract

In this paper we are concerned with a boundary value problem generated on the half line by a generalized Stürm Liouville differential equation and the condition at zero. This problem is studied for the important special case when the potential is real function. The solutions of the differential equation of the considered boundary value problem are given. The scattering function for the considered equation is obtained and their properties are studied. The spectrum of the boundary value problem is investigated and its resolvent is constructed. The spectral expansion of a certain function in $L_2(o, \infty)$ is obtained and whence Parseval's equality is formulated. Furthermore, the scattering data for the boundary value problem is induced.

Introduction

In classical quantum mechanics, the stationary state of a system consisting of two particles of masses m_1 and m_2 and energy E is described by the ψ-function, which satisfies the Schrödinger equation

$$\frac{-\hbar^2}{2M} \triangle\psi + V(x)\,\psi = E\psi$$

where $\hbar$ is Planch's constant, $M = \dfrac{m_1 m_2}{m_1 + m_2}$, V (x) is the interaction potential, and $x = |\vec{x}|$ is distance between the two particles. Since the potential $V(x)$ depends only on the distance $|\vec{x}|$, the variables in the previous equation seprate upon setting

$$\psi(\vec{x}) = x^{-1} u_l(E, x) v_l^m(\theta, \psi)$$

where $v_l^m(\theta, \phi)$ are spherical harmonics. The function $u_l(E, x)$ satisfies the equation

$$\frac{-\hbar^2}{2M}\{u_l'' - l(l+1)x^{-2}u_l\} + V(x)u_l = E u_l$$

and the boundary condition $u_l(E, X) = o$. Introducing the notations

$$q(x) = \frac{2M}{h^2} V(x), \lambda = \frac{2ME}{h^2}, u_l(\lambda, x) = u_l(E, x)$$

for the sake of brevity, we are led to the boundary value problem

$$-u_l'' + q(x)u_l + l(l+1)x^{-2}u_l = \lambda u_l), 0<x<\infty), \quad (0.1)$$

$$u_l(\lambda, o) = o. \quad (0.2)$$

The solutions of this boundary value problem which are bounded at infinity will be referred to as radial wave functions.

It is well known that the problem of recovering the potential from the experimental data is known as the inverse problem of quantum scattering theory [5]. Therefore, it is interest to define the collection of quantities $\{S(k); k_n; m_n (n = \overline{1, l})\}$ by the so called the scattering data of the problem, where $S(k)$ is the scattering function, k_n are the eigenvalues and m_n^{-1} are the norm of the eigenfunctions of the considered problem. This data specify the behaviour of the radial wave functions at infinity.

Throughout this paper, we will consider a particular form of the boundary value problem (0.1)–(0.2). Let us consider the boundary value problem generated on the half line $o \leqslant x < \infty$ by the generalized Stürm-Liouville differential equation

$$-y'' + q(x)\,y = \lambda\,\rho(x)\,y \qquad (1)$$

and the boundary condition

$$y(o) = o, \qquad (2)$$

where the function $q(x)$ is real and satisfies the condition

$$\int_{o}^{\infty} x\,|\,q(x)\,|\,dx < \infty\,, \qquad (3)$$

which is assumed to hold throughout the paper. In addition the function $\rho(x)$ is a discontinuous function at $x = 1$:

$$\rho(x) = \begin{cases} a^2, & o \leqslant x \leqslant 1 \\ 1, & 1 < x < \infty \end{cases}, \quad a \neq 1.$$

It should be mentioned that the inverse problem of scattering theory for the boundary value problem (1)–(2) was completely investigated in many works when $\rho(x) \equiv 1$ (see [1, 2, 5]).

In § 1, we obtain certain solutions of the equation (1) and investigate the scattering function for that equation. We study the spactrum of the boundary value problem (1)–(2) in § 2 and we cernstruct the resolvent. In § 3 we obtain the spectral expansion for a certain function in $L_2(o, \infty)$ by eigenfunctions of the boundary value problem (1)–(2) and whence we formulate Parseval's equalities. Finally, we define the scattering data of the problem (1)–(2).

§ 1. Certain solutions of the equation (1) and its scattering function

Now, we deal with solutions of the equation (1) which satisfy specific initial conditions at $x = o$ or which have a specific asymptotic behaviour at $x \to \infty$. It is convenient to use the same solutions of the equation (1) on the interval $(1, \infty)$ as [6].

From condition (3) it is clear that (1) reduces to the simpler equation $-y'' = \lambda\rho y$ as $x \to \infty$ (see [5. p. 295]). This permits us a complete investigation of the properties of the solution to equation (1). We shall use the following notation :

$$\sigma(x) = \int_x^\infty |q(t)|\,dt, \qquad \sigma_1(x) = \int_x^\infty t\,|q(t)|\,dt, \tag{4}$$

$\lambda^{\frac{1}{2}} = k = \alpha + i\tau$ such that $o \leqslant \arg k \leqslant \pi$.

Let $\phi(x, k)$ and $\theta(x, k)$ denote the solutions to the equation (1) on the interval [o, 1] which satisfy the initial conditions

$$\left.\begin{aligned} \phi(o, k) &= o, & \phi'(o, k) &= 1 \\ \theta(o, k) &= 1, & \theta'(o, k) &= o \end{aligned}\right\} \tag{5}$$

We note that, in the case when $q(x) = o$, $\phi(x, k) = \dfrac{\sin kax}{ka}$ and $\theta(x, k) = \cos kax$. Since the initial conditions (5) do not depend on k, it follows that, for every $x \in [o, 1]$, $\phi(x, k)$ and $\theta(x, k)$ are entire functions.

Lemma 1. The solution $\phi(x, k)$ of the equation (1) on the interval [o, 1] may be expressed on the form

$$\phi(x, k) = \frac{\sin kax}{ka} \int_0^x A(x, t)\frac{\sin kat}{ka}\,dt, \quad o \leqslant t \leqslant x \leqslant 1 \tag{6}$$

where the kernel $A(x, t)$ has summable dervatives A'_x, A'_t and satisfies the conditions

$$\frac{dA(x, x)}{dx} = \frac{1}{2}q(x), \quad A(x, o) = o.$$

For the proof of this lemma see [4.p. 18].

Lemma 2. The solution $\theta(x, k)$ can be written on the form

$$\theta(x, k) = \cos kax + \int_0^x B(x, t)\cos kat\,dt, \quad o \leqslant t \leqslant x \leqslant 1 \tag{7}$$

where the kernel B (x, t) has summable dervatives B'_x , B'_t and satisfies the conditions

$$\frac{dB(x, x)}{dx} = \frac{1}{2} q(x) , \left[\frac{\partial B(x, t)}{\partial t} \right]_{t = o} = o$$

See [6.p. 261—263].

The following lemma can be proved by making use of [6, p. 295].

Lemma 3. If the condition (3) is satisfied, then as x > 1 and $\tau \geq o$ the equation (1) has the solution F (x, k) which may be expressed in the form

$$F(x, k) = \exp(ixk) + \int_x^\infty K(x, t) \exp(itk)\, dt, \tag{8}$$

where the kernel K (x, t) has continuous dervaties with respect to x and t and satisfies the inequalities

$$| K(x, t) | \leq \frac{1}{2} \exp\{\sigma_1(x)\}. \sigma\left(\frac{x+t}{2}\right),$$

$$| K'_x(x, t) | , | K'_t(x, t) | \leq \frac{1}{2} \left| q\left(\frac{x+t}{2}\right) \right|$$

$$+ \frac{1}{2} \exp\{\sigma(x)\} \sigma\left(\frac{x+t}{2}\right),$$

where $\sigma(x)$ and $\sigma_1(x)$ are defined by (4). The solution F (x, k) is an analytic function of k in the upper half plane $\tau > o$ and is continuous on the real line. This solution has the following asymptotic behaviour

$$F(x, k) = \exp(ikx)(1 + o(1)), \; F'(x, k), = \exp(ikx)(ik + o(1)) \tag{9}$$

as $x \to \infty$ for all $\tau \geq o$, $k \neq o$ and

$$F(x, k) = \exp(ikx)\left(1 + 0\left(\frac{1}{k}\right)\right), \; F(x, k) = ik \exp(ikx)\left(1 + 0\left(\frac{1}{k}\right)\right) \tag{10}$$

as $| k | \to \infty$ for all x and $\tau \geq o$.

Further, for real $k \neq o$, the functions (F (x, k) and F (x, − k) form a fundamental system of solutions of equation (1) on the interval $(1, \infty)$ and their Wronskian equal to $-2ik$:

$$W\{F(x, k), F(x,-k)\} = F(x, k)\, F'(x,-k) - F'(x, k)\, F(x-k) = -2ik \tag{11}$$

Theorem 1. The identity

$$\frac{-2ik\,\phi(x, k)}{F(o, k)} = F(x,-k) - S(k)\, F(x, k) \tag{12}$$

holds for all $k \neq o$, where (S (k) is called the scattering function for equation (1) such that

$$S(k) = \frac{F(o,-k)}{F(o, k)} = \overline{S(-k)} = [S(-k)]^{-1}.$$

Proof. Since the two functions, F (x, k) (from (11)), form a fundamental system of solutions to equation (1) for all $k \neq o$, we can write

$$\phi(x, k) = a^{+}(k)\, F(x, k) + a^{-}(k)\, F(x,-k),$$

where $a^{+}(k)$ and $a^{-}(k)$ are determined by the condition (5). Thus letting x approach o to get

$$a^{+}(k)\, F(o, k) + a^{-}(k)\, F(o,-k) = o,$$

$$a^{+}(k)\, F'(o, k) + a^{-}(k)\, F'(o,-k) = 1.$$

Then, we find

$$a^{+}(k) = \frac{1}{2ik} F(o,-k) \text{ and } a^{-}(k) = -\frac{1}{2ik} F(o, k),$$

whence

$$\phi(x, k) = (2ik)^{-1} [F(x, k)\, F(o,-k) - F(x,-k)\, (F(o, k)].$$

Since (q (x) is real, it follows that $F(o,-k) = \overline{F(o, k)}$ and hence that $F(o, k) \neq o$ for all real $k \neq o$. Therefore

$$\frac{-2ik\,\phi(x, k)}{F(o, k)} = F(x,-k) - S(k)\, F(x, k)$$

with

$$S(k) = \frac{F(o,-k)}{F(o,k)} = \left[\frac{\overline{F(o,k)}}{F(o,-k)} \right] = \left[\frac{(F(o,k)}{F(o,-k)} \right]^{-1}$$

as claimed.

Next let us examine the scattering function S (k) of the equation (1).

Theorem 2. For larges real $k \neq o$ the following asymptotic holds

$$S(k) - S_o(k) = 0\left(\frac{1}{k}\right),$$

where

$$S_o(k) = \{\cos ka + \frac{i}{a} \sin ka\} \{\cos ka - \frac{i}{a} \sin ka\}^{-1} \exp(-2ik). \quad (13)$$

Proof. Since the functions $\phi(x, k)$ and $\theta(x, k)$ are construct a fundamental system of solutions of equation (1) on the interval [o, 1], thus we have

$$F(x, k) = c_1(k)\,\phi(x, k) + d_1(k)\,\theta(x, k). \quad (14)$$

where

$$c_1(k) = F'(o, k) \quad \text{and} \quad d_1(k) = F(o, k).$$

Now, we find an expression for the function (F (o, k).

From (14) we have

$$F(1, k) = F'(o, k)\,\phi(1, k) + F(o, k)\,\theta(1, k)$$

and

$$F'(1, k) = F'(o, k)\,\phi'(1, k) + F(o, k)\,\theta'(1, k).$$

Thus, we find

$$F(o, k) = F(1, k)\,\phi'(1, k) - F'(1, k)\,\phi(1, k)$$

Taking into account (6) and (8) to obtain

$$F(o, k) = \{\exp(ik) + \int_1^{\infty} K(1, t) \exp(ikt)\,dt\} \{\cos ka + A(1, 1) \frac{\sin ka}{ka}$$

$$+ \int_0^1 A'_x (1, t) \frac{\sin kat}{ka} dt\} - \{ik \exp(ik) - K(1, 1) \exp(ik)$$

$$+ \int_1^\infty K'_x (1, t) \exp(ikt) dt\} \left\{ \frac{\sin ka}{ka} + \int_0^1 A(1, t) \frac{\sin ka}{ka} dt \right\}$$

$$= \{ \cos ka - \frac{i}{a} \sin ka\} \exp(ik) + 0 \left(\frac{1}{k} \right).$$

Therefore

$$S(k) = F(o, -k) [F(o, k)]^{-1}$$

$$= \{\cos ka + \frac{i}{a} \sin ka\} \{\cos ka - \frac{i}{a} \sin ka\}^{-1} \exp(-2ik) + 0 \left(\frac{1}{k} \right)$$

$$= S_o(k) + 0 \left(\frac{1}{k} \right),$$

where $S_o(k)$ is the required results (13). Hence the theorem is proved.

§ 2. The spectrum of the boundary value problem (1)—(2)

In this section we investigate the spectrum and obtain the resolvent of the boundary value problem (1)–(2).

Theorem 3. The boundary value problem (1)–(2) does not have eigenvalues on the positive semi axis.

The proof of this theorem follows immediately from the work of Naimark [5. p. 301].

Theorem 4. The necessary and sufficient conditions that $\lambda \neq o$ be an eigenvalue of the boundary value problem (1)—(2) are $\lambda = k^2$, $\tau > o$, $F(o, k) = o$. They are countable in number and its limit points can lie only on the real axis.

Proof. Let $\lambda = k^2$ be an eigenvalue of the boundary value problem (1)—(2). Thus there is a non trivial solution $y(x, k) \in L_2$ (o, ∞) of the equation (1). Consequently $y(x, k)$ must be satisfies (1)–(2). Since the general solution of (1) can be written on the form

$$y(x, k) = c_1 F(x, k) + c_2 F_1(x, k),$$

where $F_1(x, k)$ is the solution of the integral equation

$$F_1(x, k) = \exp(-ikx) + \frac{1}{2ik} \int_b^x \exp\{ik(x-t)\}\{q(t)$$

$$+ k^2 (1-\rho(t))\} F_1(t, k)\, dt + \frac{1}{2ik} \int_x^\infty \exp\{ik(t-x)\}\{q(t)$$

$$+ k^2 (1-\rho(t))\} F_1(t, k)\, dt$$

and c_1, c_2 are constants. Since $F(x, k) \in L_2(o, \infty)$ and $F_1(x, k)$ $\bar{\in} L_2(o, \infty)$ as $\tau > o$, then it is necessary to satisfy the condition $c_2 = o$. Thus $y(x, k) = c_1 F(x, k)$. Meanwhile since $y(x, k)$ must be satisfies the boundary value problem (1)–(2) thus it satisfies the condition (2) and whence $F(o, k) = o$.

We now prove the sufficient condition of the theorem. For this purpose let $F(o, k) = o$ and $\tau > o$. Accordingly the function $F(x, k)$ satisfies the boundary value problem (1)–(2). Since $F(x, k) \in L_2(o, \infty)$ as $\tau > o$ then it appears an eigenfunction of (1)–(2) and $\lambda = k^2$ being an eigenvalue of one.

Now, since $F(o, k) \neq o$ for all real $k \neq o$, then the point $k = o$ is the only possible real zero of the function $F(o, k)$. Formula (10) implies that $F(o, k) \to 1$ as $|k| \to \infty$, which shows that the zeros of $F(o, k)$ form an at most countable set having o as the only possible limit point and this completes the proof of the theorem.

Theorem 5. The eigenvalues of the boundary value problem (1)–(2) are lie on the imaginary axis. They are all simple.

Proof. Let $k = k_o$ ($\tau_o > o$ or $k_o = o$) be one of the zeros of $F(o, k)$. Then by virtue of (5) and (10),

$$W\{\phi(x, k_o), F(x, k_o)\} = F(o. k_o) = o, \tag{15}$$

whence

$$F(x, k_o) = c\,\phi(x, k_o).$$

Therefore, the limit $\lim\limits_{x \to o} F'(x, k_o) = F'(o, k_o) = c$, and

$$F(x. k_o) = F'(o, k_o)\,\phi(x, k_o) \tag{16}$$

This formula yields

$$\lim_{x \to o} W\{F(x, k_1), \overline{F(x, k_2)}\} = o \tag{17}$$

for two arbitrary zeros k_1 and k_2 of the function $F(o, k)$. Since $q(x)$ is real, the function $\overline{F(x, k_2)}$ satisfies the same differential equation as $F(x, k_1)$ if we replace k_1^2 by $\bar{k}_2^2$. Therefore

$$W\{F(x, k_1), \overline{F(x, k_2)}\}\Big|_x^\infty = (k_1^2 - \bar{k}_2^2)\int_x^\infty F(x, k_1)\,\rho(x)\,\bar{F}(x, k_2)\,dx$$

which upon using (17) and taking into account that the functions $F(x, k)$ and $F'(x, k)$ are approach o as $x \to \infty$, implies that

$$(k_1^2 - \bar{k}_2^2)\int_o^\infty F(x, k_1)\,\rho(x)\,\bar{F}(x, k_2)\,dx = o$$

whenever k_1 and k_2 are zeros of the function $F(o, k)$. In particular, the choice $k_1 = k_2$ implies that $k_1^2 - \bar{k}_1^2 = o$ or $k_1 = i\tau$ where $\tau \geq o$. Therefore the zeros of the function $F(o, k)$ can lie only on the imaginary axis. Now differentiating the equation

$$-F''(x, k) + q(x)\,F(x, k) = k^2\,\rho(x)\,F(x, k)$$

with respect to k, one obtains the following equation for F (o, k) :

$$-F''(x, k)+q(x) F(x, k)=k^2 \rho(x) F(x, k)+2k\rho(x) F(x, k)$$

Therefore,

$$W\{F'(x, k), F(x, k)\}\Big|_x^{\infty} = 2k \int_x^{o} |F(x, k)|^2 \rho(x)\, dx.$$

Let $k = k_o = i\tau_o$, $\tau_o > o$ be a zero of the function $F(o, k)$. Then the function $F(x, ik_o)$ is realvalued). On the other hand, by virtue of the formulae (16) and (9) and (10), we get

$$\lim_{x \to o} W\{F(x, i\tau_o). F(x, i\tau_o)\} = F(o, i\tau_o) F'(o, i\tau_o)$$

and

$$\lim_{x \to \infty} W F\{x, i\tau_o), F(x, i\tau_o)\} = o.$$

Consequently,

$$F(o, i\tau_o) F'(o, i\tau_o) = 2i\tau_o \int_o^{\infty} |F(x, i\tau_o)|^2 \rho(x)\, dx \qquad (18)$$

Since $\int_o^{\infty} |F(x, i\tau_o)|^2 \rho(x)\, dx > o$, we see that $F(o, i\tau_o) \neq o$, i.e., the zeros of the function F (o, k) are all simple. Hence the theorem is proved.

Theorem 6. If λ is not an eigenvalue of the boundary value problem (1)-(2), and $F(o, \lambda^{\frac{1}{2}}) \neq o$, then the Green's function (G (x, t, λ) for $-y' + q(x) y - \lambda\rho(x) y = \rho f$ on the interval $o \leqslant x < \infty$ is $R(x, t, k^2)$, where

$$R(x, t, k^2) = \frac{1}{F(o, k)} \begin{cases} F(x, k)\,\phi(t, k), & t \leqslant x \\ F(t, k)\,\phi(x, k), & t \geqslant x \end{cases} \qquad (19)$$

that is, if εL_2 (o, ∞), then

$$y = \int_o^\infty R(x, t, k^2)\, \rho(t)\, f(t)\, dt$$

Proof. By variation of parameters we can find Green's function and thus the resolvent of the boundary value problem (1)—(2) on the form

$$R(x, t, k^2) = \frac{1}{F(o, k)} \begin{cases} F(x, k)\, \phi(t, k), & t \leqslant z \\ F(t, k)\, \phi(x, k), & t \geq x. \end{cases}$$

and whence (19) is proved.

Theorem 7. Every point of the positive semi axis $\lambda > o$ is in the continuous spectrum of the boundary valve problm (1)−(2).

Taking into account the formula (19) and by virtue of the work [3. p. 355] we can prove this theorem.

§ 3. The spectral expansion by eigenfunctions, Parseval's eqnality and the scattering data of (1)−(2)

The section is devoted to obtain the spectral expansion of a certain function in terms of eigenfunctions and whence Parseval's equality for the boundary value problem (1)−(2). In addition, the scattering data of the problem is induced.

Here, we find the expansion by eigenfunctions of the problem (1)−(2) by Titchmarsh's method.

Theorem 8. If the function $f(x) \varepsilon L_2$ (o, ∞), f (x) is finite in a neighbourhood of the points x=o, x=∞, and has a continuous second derivative in L_2 (o, ∞), then

$$\int_o^\infty R(x, t, k)\, \rho(t)\, f(t)\, dt = -\frac{(f\, x)}{k^2} + \frac{1}{k^2} \int_o^\infty R(x, t, k)\, h(t)\, dt, \qquad 20$$

where $$h(t) = - f''(t) + q(t) f(t).$$

Furthermore, if $\tau > o$, $|k| \to \infty$, then

$$\int_o^\infty R(x, t, k)\, \rho(t) f(t)\, dt = -\frac{f(x)}{k^2} + 0\left(\frac{2}{k^2}\right) \tag{21}$$

Proof. By virtue of the formula (19), we have

$$\int_o^\infty R(x, t, k)\, \rho(t) f(t)\, dt$$

$$= \frac{F(x, k)}{F(o, k)} \int_o^x \phi(t, k)\, \rho(t) f(t)\, dt$$

$$+ \frac{\phi(x, k)}{F(o, k)} \int_x^\infty F(t, k)\, \rho(t) f(t)\, dt$$

$$= \frac{F(x, k)}{F(o, k)} \int_o^x \left\{ -\frac{1}{k^2} \phi''(t, k) + \frac{1}{k^2} q(t) \phi(t, k) \right\} f(t)\, dt$$

$$+ \frac{\phi(x, k)}{F(o, k)} \int_x^\infty \left\{ -\frac{1}{k^2} F''(t, k) + \frac{1}{k^2} q(t) F(t, k) \right\} f(t)\, dt$$

By integrating this equality by parts twice we can get the equality (20). From formula (19) it is easily seen that

$$\int_o^\infty R(x, t, k)\, h(t)\, dt = o(1)$$

and whence (21), is follows atonce.

The following lemma is well known.

Lemma 4. $\bar{R}_\lambda = R_{\bar{\lambda}}$.

With the help of Theorem 8 and Lemma 4 we prove the following theorem,

Theorem 9. If the function f (x) satisfies the conditions of Theorem 8, then the expansion in eigenfunctions of (1)—(2) can be written on the form

$$f(x) = \frac{1}{2\pi} \int_0^{\infty} dk \int_0^{\infty} u(x, k)\, \bar{u}(x, k)\, \rho(t)\, f(t)\, dt$$

$$+ \sum_{n=1}^{l} m_n^2 \int_0^{\infty} F(x, i\tau_n)\, F(t, i\tau_n)\, \rho(t)\, f(t)\, dt. \qquad (22)$$

Proof. Suppose that f (x) satisfies the conditions of Theorem 8. Then (21) holds. We integrate both sides of (21) with respect to λ over the circle Γ_r of radius r and center at zero. As a result we have

$$-f(x) + \oint_{\Gamma_r} 0 \left(\frac{1}{\lambda}\right) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_r} d\lambda \oint_0^{\infty} R(x, t, \lambda)\, \rho(t)\, f(t)\, dt \qquad (23)$$

It is evident that the function R (x, t, λ) is analytic fudction in the supper half plane $\tau > o$. Therefore

$$\frac{1}{2\pi i} \oint_{\Gamma_r} d\lambda \int_0^{\infty} R(x. t, \lambda)\, \rho(t)\, f(t)\, dt = I_r^1 + I_r^2 + I_r^3, \qquad (24)$$

From (19) and by virtue of the formula (13) we get

where

$$I_r^1 = \frac{1}{2\pi i} \int_{-r-i\delta}^{r-i\delta} d\lambda \int_0^{\infty} R(x, t, \lambda)\, \rho(t)\, f(t)\, dt \qquad (25)$$

$$I_r^2 = \frac{1}{2\pi i} \int_{r+i\delta}^{-r+i\delta} d\lambda \int_0^{\infty} R(x, t, \lambda)\, \rho(t)\, f(t)\, dt \qquad (26)$$

$$I_r^3 = \frac{1}{2\pi i} \left\{ \int_{r-i\delta}^{r+i\delta} d\lambda \int_0^{\infty} R(x, t, \lambda)\, \rho(t)\, f(t)\, dt \right.$$

$$+\int_{-r+i\delta}^{-r-i\delta} d\lambda \int_{o}^{\infty} R\,(x,\,t,\,\lambda)\,\rho\,(t)\;f\,(t)\,dt\Big\} \qquad (27)$$

where δ is any positive number. From Lemma 4 and formula (21) it follows that $I_r^3 \to o$ as $r \to \infty$. Therefore, by going over in (23) to the limit as $r \to \infty$ and using (25) and (26) we find $f\,(x) = \frac{1}{2\pi i}$

$$\int_{-\infty}^{\infty} d\lambda \int_{o}^{\infty} \{R\,(x,\,t,\,\lambda+i\delta) - R\,(x,\,t,\,\lambda-i\delta)\}\,\rho\,(t)\,f\,(t)\,dt \qquad (28)$$

From (19) it follows that the limit

$$\lim_{\delta\to o} \int_{o}^{\infty} R\,(x,\,t,\,\lambda \pm i\delta)\,\rho\,(t)\,f\,(t)\,dt$$

$$= \int_{o}^{\infty} R\,(x,\,t,\,\lambda \pm io)\,\rho\,(t)\,f\,(t)\,dt$$

exists for values of λ in $(-\infty,\,b)$ and $(b,\,\infty)$ and for $\lambda \,\varepsilon\, (-b,\,b)$ the integral $\int_{o}^{\infty} R\,(x,\,t,\,\lambda)\,\rho\,(t)\,f\,(t)\,dt$ is an analytic function except at the zeros $\lambda = -\tau_n^2$, $n = \overline{1,\,l}$ of the function $F\,(o,\,\lambda)$. Hence from (28) we have

$$f\,(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\lambda \int_{o}^{\infty} \{R\,(x,\,t,\,\lambda+io) - R(x,\,t,\,\lambda-io)\}\,\rho(t)\,f(t)\,dt$$

$$+ \sum_{n=1}^{l} \operatorname{Res}\Big(\int_{o}^{\infty} R\,(x,\,t,\,\lambda)\,\rho\,(t)\,f\,(t)\,dt\,\Big|_{\lambda=-\tau_n^2} \qquad (29)$$

Here, let us compute the first quantity in the right hand side of equation (29).

$$\frac{1}{2\pi i} \int_0^\infty \int_0^\infty \{R(x, t, \lambda+io) - R(x, t, \lambda-io)\} \rho(t) f(t) \, dt \, d\lambda$$

$$= \frac{1}{\pi i} \int_0^\infty k \, dk \int_0^x \{F(x, k) F^{-1}(o, k) - \overline{F(x, k)} \; \overline{F^{-1}(o, k)}\} \phi(t, k) \rho(t) f(t) \, dt$$

$$+ \frac{1}{\pi i} \int_0^\infty k \, dk \int_x^\infty \{F(t, k) F^{-1}(o, k) - \overline{F(t, k)} \; \overline{F^{-1}(o, k)}\} \phi(x, k) \rho(t) f(t) \, dt$$

$$= \frac{1}{\pi i} \int_0^\infty k \, dk \int_0^\infty \{\phi(x, k) [F(t, k) F^{-1}(o, k) - \overline{F(t, k)} \; \overline{F^{-1}(o, k)}]\} \rho(t) f(t) \, dt$$

It follows from the identity (12) that

$$-2ik \, \phi(x, k) = F(x, -k) F(o, k) - F(x, k) F(o, -k).$$

Therefore, the first quantity in (29) as $\tau = o$ equals to

$$\frac{1}{\pi} \int_0^\infty 2k^2 \int_0^\infty \phi(x, k) F^{-1}(o, k) \; \overline{F^{-1}(o, k)} \; \overline{\phi(t, k)} \rho(t) f(t) \, dt.$$

Let's take $u(x, k) = -2ik \, \phi(x, k) F^{-1}(o, k)$

$$= F(x, k) - F(x, k) S(k)$$

Thus we have

$$\frac{1}{\pi i} \int_0^\infty k \int_0^\infty \{R(x, t, k+io) - R(x, t, k-io)\} \rho(t) f(t) \, dt$$

$$= \frac{1}{2\pi} \int_0^\infty dh \int_0^\infty u(x, k) \, \bar{u}(t, k) \rho(t) f(t) \, dt \qquad (30)$$

Now, by using the formula of estimating residues in the case of simple zeros (see Theorem 5) the second quantity in the right hand side of (29) can be determined as follows :

$$\sum_{n=1}^{l} \operatorname{Res} \left(\int_0^\infty (R\, x, t, \lambda)\, \rho(t)\, f(t)\, dt \right) \Big|_{\lambda = -\tau_n^2} =$$

$$= \sum_{n=1}^{l} \operatorname{Res} \left(2k \int_0^\infty R(x, t, k)\, \rho(t)\, f(t)\, dt \right) \Big|_{k = i\tau_n}$$

$$= \sum_{n=1}^{l} \operatorname{Res} \left\{ 2k \int_0^x \left[R(x, t, k) + \int_0^\infty R(x, t, k) \right] \rho(t)\, f(t)\, dt \right\} \Big|_{k = i\tau_n}$$

$$= \sum_{n=1}^{l} \operatorname{Res} \left[2k \left\{ \frac{F(x, k)}{F(o, k)} \int_0^x \phi(t, k)\, \rho(t)\, f(t)\, dt \right. \right.$$

$$\left. \left. + \frac{\phi(x, k)}{F(o, k)} \int_0^\infty F(t, k)\, \rho(t)\, f(t)\, dt \right\} \right] \Big|_{k = i\tau_n}$$

Now, let m_n^{-1} be the norm of the function $F(x, i\tau_n)$ in $L_2(o, \infty)$. According to formula (18), we have

$$m_n^{-2} = \int_0^\infty F(x, i\tau_n)\,|^2 \rho(x)\, dx = \frac{F'(o, i\tau_n)\, F(o, i\tau_n)}{2\, i\tau_n}$$

therefore by virtue of the formula (16), we obtain

$$\sum_{n=1}^{l} \operatorname{Res} \left(2k \int_0^\infty R(x, t, k)\, \rho(t)\, f(t)\, dt \right) \Big|_{k = i\tau_n}$$

$$= \sum_{n=1}^{l} m_n^2 \int_0^\infty F(x, i\tau_n)\, F(t, i\tau_n)\, \rho(t)\, f(t)\, dt \qquad (31)$$

Hence, from (30) and (31) we obtain the following expansion by eigenfunctions of the boundary value problem (1)–(2)

$$f(x) = \frac{1}{2\pi} \int_0^\infty \rho(t)\, f(t) \int_0^\infty u(x, k)\, \bar{u}(t, k)\, dk\, dt$$

$$+ \sum_{n=1}^{l} m_n^2 \int_0^\infty F(x, i\tau_n)\, F(t, i\tau_n)\, \rho(t)\, f(t)\, dt.$$

This is the required result (22) and hence the theorem is completely proved.

Theorem 10. The following Parseval's equality holds

$$\sum_{n=1}^{l} u(x, i\tau_n)\, \overline{u(t, i\tau_n)} + \frac{1}{2\pi} \int_0^\infty u(x, k)\, \bar{u}(t, k)\, dk = \delta(x-t)^{-1} \rho(x)$$

where

$$u(x, k) = F(x, -k) - F(x, k)\, S(k)$$

and

$$u(x, i\tau_n) = F(x, i\tau_n)\, m_n, \qquad n = \overline{1, l}.$$

We claim that the collection of quantities $\{S(k), i\tau_n \cdot m_n$ $(n=1, 2, \ldots, l)\}$ is the scattering data of the boundary value problem (1)—(2) with a real-valued potential $q(x)$ which subject to the constraint (3) and with a real discontinuous function $\rho(x)$. The inverse scattering problem for this problem is to recover $q(x)$ and $\rho(x)$ given by the scattering data and we shall study this problem in a forthcoming paper.

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HIGHLY ACCURATE METHODS FOR SOLVING UNILATERAL PROBLEMS

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Abstract

Variational inequality theory has proved to be immensely useful in the study of many branches of mathematical and engineering sciences. In this paper, we develop pade approximant methods of order 6 and 8 to solve the variational inequalities associated with unilateral problems. In the case of a known obstacle, these problems can be alternately formulated as nonlinear boundary value problems without constraints for which the technique of pade approximants can be successfully employed.

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1. Introduction and Formulation

Variational inequality theory provides a rich spectrum of new ideas in mathematical and engineering sciences with interesting results in elasticity, operations research, general equilibrium theory, etc. This theory was developed simultaneously not only to study the fundamental facts on the qualitative behaviour of solutions to a wide class of linear and nonlinear problems, but also to solve them numerically more efficiently. In this paper. we use the penalty function technique of Lewy and Stampacchia [1] to characterize the variational inequalities by a sequence of equalities. This system of equations is then solved by using the pade approximant method. Our results indicate that our methods of order 6 and 8 are much better than the previous methods of Jain [2] and Usmani [3].

For the purpose of aome numerical experience, we consider the simple example of an elastic string lying over an elastic obstacle. The formulation and the approximation of the elastic string is very simple, however, it should be emphasized that the kind of numerical problems which occur for more complicated system will be the same. Our approach to these problems is to consider in a general manner seemingly independent of the nonlinear problems in terms of variational inequalities and are later specialized.

We are concerned with the numerical solution of the unilateral problem of the type :

$$\left.\begin{aligned} L[u(x)] &\geq f(x), & x &\in \Omega \\ u(x) &\geq \psi(x), & x &\in \Omega \\ u(x) &= 0, & x &\in \partial\Omega \end{aligned}\right\} \qquad (1.1)$$

where Ω is a polygonal domain with boundary $\partial\Omega$ and closure $\overline{\Omega} = \Omega \cup \partial\Omega$, L is a linear, self-adjoint coercive differential operator, f is a given function and ψ is a given obstacle function. A large number of problems arising in elasticity, fluid flow through porous media and general equilibrium theory of economics and transportation can be written in the form (1.1).

The problem (1.1) is studied via the variational inequality formulation in the Sobolev space $W_2^1(\Omega) \equiv H$, which is a Hilbert space, see Oden and Kikuchi [4], for notations and definition. To do this, we define

$$M \equiv \{v : v \in H, \ v \geq \psi \ \text{ on } \partial\Omega\},$$

which is a closed convex set in H. The problem (1.1) is thus equivalent to finding $u \in M$ such that

$$a(n, v-u) \geqslant \langle f, v-u \rangle \quad \text{for all } v \in M, \tag{1.2}$$

where the bilinear form a (u, v) is associated with the operator L and is, in fact $\langle Lu, v \rangle$ after the integration by parts has been performed. Also the bilinear form a (u, v) associated with L is coercive continuous, so there exists a unique solution $u \in M$ satisfying (1.2), see Noor [5] and Oden and Kikuchi [4].

In order to apply the pade approximants, we must have equations instead of inequalities. Thus using the idea of Lewy and Stampacchia [1], the inaquality (1.2) can be characterized by a sequence of equations as follows :

$$a(u, v) + (\nu(u-\psi)(u-\psi), v) = \langle f, v \rangle, \tag{1.3}$$

for all $v \in H$, where $\nu(t)$ is a discontinuous function defined by

$$\nu(t) = \begin{cases} 1, & \text{for } t \geqslant 0 \\ 0, & \text{for } t < 0, \end{cases} \tag{1.4}$$

which is known as the penalty function and $\psi < 0$ on $\partial\Omega$ is an elastic obstacle.

2. Numerical Methods

In order to solve system (1.3) by the methods based on pade approximants, we first consider the linear two-point boundary value problem

$$y''(x) = \lambda^2 y(x) + g \,;\; y(a) = A,\; y(b) = B, \tag{2.1}$$

where $\lambda^2 \geqslant 0$ and g are constants. This equation will be the basis for obtaining the numerical solution of (1,3). The system (2.1) belongs to a general class of boundary value problems of the type

$$y''(x) = f(x)\,y(x) + g(x), \qquad y(a) = A,\; y(b) = B \tag{2.2}$$

where f (x) and g (x) are continuous functions with $f(x) \geqslant 0$ on [a, b] and a, b, A, B are arbitrary real finite constants. Since the analytical solution of (2.2) cannot be obtained in general for arbitrary choices of f (x) and g (x), hence the numerical techniques are always used to solve such systems. Various authors including Fox [6], Aziz and Hubbard [7], Usmani [3], Jain [2], and Varga [8] have used finite difference methods to solve this system. Recently Tirmizi [9] and Tirmizi and Twizell [10] used multiderivative methods based on pade approximants. Noor and Tirmizi [11] developed two methods of order 2 and 4 for unilateral problems. In this paper we report two higher order methods of order six and eight.

For our purpose, it is enough to consider the system (2.1). It is known that the analytical solution of (2.1) can be written as :

$$y(x) = A_1 e^{\lambda x} + A_2 e^{-\lambda x} + p \tag{2.3}$$

where $p = -g/\lambda^2$ is a particular integral, A_1 and A_2 are constants to be determined by the given boundary conditions in (2.1).

Suppose x is incremented using a constant stepsize $h=(b-a)/(N+1)$, where N is a positive integer. The solution of (2.1) will be computed at the N points $x_i = ih$, $i = 1\,(1)\,N$. We also define x_n such that $x_n = a + nh$, $n = 1\,(1)\,N+1$. It is easy to show, (see Tirmizi [9]), that the solution of y (x) of (2.1) satisfies the following recurrence relation :

$$y(x-h) - R\,y(x) + y(x+h) = S \tag{2.4}$$

where

$R = \exp(\lambda h) + \exp(-\lambda h)$ and $S = p\,(2-h)$. Using this relation, the methods based on Pade approximant will determine the solution $y = u(x_n)$, $n = 1\,(1)\,N$, the accuracy depending on the approximation to $\exp(\pm\lambda h)$ used in (2.4). We apply (m, k) Pade approximants to $\exp(\theta)$ of the form

$$\exp(\theta) = R_{m,k}(\theta) = P_k(\theta)\,(Q_m(\theta))^{-1} + \theta(h^{m+k+1}) \tag{2.5}$$

where

$$P_k(\theta) = \sum_{j=0}^{k} \frac{(m+k-j)!\,k!}{(m+k)!\,j!\,(k-j)!}\,(\theta)^j \tag{2.6}$$

and

$$Q_m(\theta) = \sum_{j=0}^{m} \frac{(m+k-j)!\ m!}{(m+k)!\ j!\ m-j)!} (-\theta)^j , \qquad (2.7)$$

see Varga [8].

Using (2.5), (2.6) and (2.7), we develope two highly accurate algorithms based on the (1, 6) and the (1, 8) Pade approximants from the novel relation (2.4). We obtain the local truncation error of these methods by using the Taylor series expansion to y_{n+1} about y_n.

Method 1 : (1, 6) Pade approximant

$$e^{\theta} = (1 + \frac{6}{7}\theta + \frac{5}{14}\theta^2 + \frac{2}{21}\theta^3 + \frac{1}{56}\theta^4 + \frac{1}{420}\theta^5$$

$$+ \frac{1}{5040}\theta^6) / (1 - \frac{1}{7}\theta) - Cy_{n-1} + Dy_n - Cy_{n+1}$$

$$= Q + T \qquad (2.8)$$

where

$$C = 1 - \frac{1}{49}\lambda^2 h^2 ; \ D = 2 + \frac{47}{49}\lambda^2 h^2 + \frac{37}{588}\lambda^4 h^4$$

$$+ \frac{19}{17640}\lambda^6 h^6 ,$$

$$Q = gh^2 (1 + \frac{37}{588}\lambda^2 h^2 + \frac{19}{17640}\lambda^4 h^4) \text{ and}$$

$$T = \text{Local error} = -\frac{1}{141,120} h^8 y^{(8)}(x) + 0(h^{10})$$

Method 2 : (1, 8) Pade Approximant

$$e^{\theta} = (1 + \frac{8}{9}\theta + \frac{7}{18}\theta^2 + \frac{1}{9}\theta^3 + \frac{5}{216}\theta^4 + \frac{1}{270}\theta^5$$

$$+ \frac{1}{2160}\theta^6 + \frac{1}{22680}\theta^7 + \frac{1}{362880}\theta^8) / (1 - \frac{1}{9}\theta),$$

$$-Cy_{n-1} + Dy_n - Cy_{n+1} = Q + T \qquad (2.9)$$

where $$C = 1 - \frac{1}{81}\lambda^2 h^2,$$

$$D = 2 + \frac{79}{81}\lambda^2 h^2 + \frac{23}{324}\lambda^4 h^4 + \frac{17}{9720}\lambda^6 h^6 + \frac{5}{326592}\lambda^8 h^8$$

$$Q = gh^2 \left(1 + \frac{23}{324}\lambda^2 h^2 + \frac{17}{9720}\lambda^4 h^4 + \frac{5}{326592}\lambda^6 h^6\right)$$

$$T = \text{Local error} = \frac{1}{16,329,600} h^{10} y^{(10)}(x) + 0(h^{12})$$

Examination of their local error expressions shows that Methods 1 and 2 are of order Six and Eight respectively which will be confirmed in the next section, and that these are consistent in the sense of Henrici [12]. In the following section, we consider the convergence criteria of our methods.

3. Convergence

We now consider the convergence of Method 1 based on (1, 6) Pade approximant. To do so, we define the discretization error $e_n = y_n - z_n$, where z_n is the numerical approximation to y_n and is obtained by neglecting the truncation error in (2.8). Let $Y = (y_n)$, $Z = (z_n)$, $C = (c_n)$, $T = (t_n)$, $E = (e_n)$, be n dimensional vectors. We also define $\|E\| = \max_n |e_n|$, where $\|.\|$ represents the ∞-norm of a matrix vector. Using these notations, we can rewrite equation (2.8) in matrix form as follows :

(i) $MY = C + T$

(ii) $MZ = C$ (2.10)

(iii) $ME = T$

where M is a tridiagonal matrix and

$$M = A + h^2 BD_1 + h^4 D_2 + h^6 D_3 ;$$

$A = (a_{ij})$, with $a_{ii} = 2$, $a_{ij} = -1$, $|i-j| = 1$, $A^{-1} = (a^*_{ij})$;

$B = (b_{ij})$ with $b_{ii} = \frac{47}{49}$, $b_{ij} = \frac{1}{49}$; $|i-j| = 1$;

$D_1 = \text{Diag}(\lambda^2)$, $D_2 = \text{Diag}\left(\frac{37}{588}\lambda^4\right)$ and $D_3 = \text{diag}\left(\frac{19}{17640}\lambda^6\right)$.

It is well known that A is monotone and $\| A^{-1} \| \leqslant (b-a)/8h^2$.

Our main purpose here is to derive a bound on E.

For this, we need the following :

From (2.10) (iii), we obtain

$$E = M^{-1} T = (A+Q)^{-1} T = (I+A)^{-1} T = (I+A^{-1} Q) A^{-1} T$$

where

$$Q = h^2 BD_1 + h^4 D_2 + h^6 D_3 .$$

Taking norm of both sides we can have,

$\| E \| = \| A^{-1} \| . \| T \| / (1 - \| A^{-1} \| . \| Q \|)$ provided $\| A^{-1} \| . \| Q \| < 1$.

It is clear for $\| Q \|$ that $\| B \| = 1$, $\| D_2 \| < 1$ and $\| D_3 \| < 1$. Now from (10) it seen that $\| T \| = 0(h^8)$. Also as given above $\| A^{-1} \| = 0(h^{-2})$, therefore, $\| E \| = 0(h^6)$.

So it follows that our method 1 based on the (1, 6) pade approximant is a sixth order convergent method. On the similar lines it is not difficult to prove that the method 2 based on the (1, 8) pade approximant is an eighth order convergent method. The numerical experiment in the next section will confirm their order of convergence.

4. An Example

As an example, we consider the boundary value problem describing the equilibrium configuration of an elastic string pulled at the ends and lying over an elastic step of constant height 1 and unit rigidity of the type :

$$\left.\begin{aligned} -u'' &\geqslant 0 \quad \text{on } \Omega = (0, \pi) \\ u &\geqq \psi \quad \text{on } \Omega = (0, \pi) \\ u(0) &= 0 = u(\pi) \end{aligned}\right\}, \tag{4.1}$$

where ψ is the given obstacle function defined by

$$\psi(x) = \begin{cases} -1, & \text{for } 0 \leqslant x \leqslant \pi/4 \\ 1 & \text{for } \frac{\pi}{4} \leqslant x \leqslant \frac{3\pi}{4} \\ -1 & \text{for } \frac{3\pi}{4} \leqslant x \leqslant \pi. \end{cases} \tag{4.2}$$

It can be shown easily [4] that the solution of (4.1) can be characterized by the variational inequality

$$a(u, v-u) \geqq 0, \qquad \text{for all } v \in M, \tag{4.3}$$

where $\quad M = \{v ; v \in H, v \geqq \psi \text{ on } \Omega\}.$

and

$$a(u. v) = \int_0^\pi \frac{du}{dx} \cdot \frac{dv}{dx} \, d\Omega \text{ , for all } u, v \in H.$$

In this case, $a(v, v) = \| v \|^2$ and $a(u, v) \leqslant \| u \| \| v \|$, that is $a(u, v)$ is a coercive continuous bilinear form and $f \equiv 0$, so that there does exist a unique solution of (4.3).

Now using the penalty function technique of Lewy and Stampacchia [1], the problem (4.3) can be written as

$$\left.\begin{aligned} u'' - \nu(u-\phi)(u-\psi) &= 0, \qquad 0 < x < \pi \\ u(0) &= 0 = u(\pi). \end{aligned}\right\} \tag{4.4}$$

where ψ and ν are as defined by (4.2) and (1.4) respectively. Since the obstacle function ψ is known, so it is possible to find the exact solution of the problem (4.1) coinciding with the interval $\frac{\pi}{4} \leqslant x \leqslant \frac{3\pi}{4}$. Consequently from (1.4), (4.2) and (4.4), we obtain the following equations

$$\left.\begin{array}{lll} \text{(i)} & u'' = 0 & \text{for } 0 < x < \frac{\pi}{4}, \ \frac{3\pi}{4} < x < \pi \\ \text{(ii)} & u'' - u = -1 & \text{for } \frac{\pi}{4} < x < \frac{3\pi}{4}, \end{array}\right\} \quad (4.5)$$

with the boundary conditions at $x=0$, $x=\pi$ and the conditions of continuity of u and u′ at $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$ with analytical solution

$$u(x) = \begin{cases} \dfrac{4x}{\pi + 4 \operatorname{Coth} \frac{\pi}{4}}, & \text{for } 0 \leqslant x < \frac{\pi}{4} \\[2ex] 1 - \dfrac{4 \operatorname{Cosh}\left(\frac{\pi}{2} - x\right)}{\pi \operatorname{Sinh} \frac{\pi}{4} + 4 \operatorname{Cosh} \frac{\pi}{4}}, & \text{for } \frac{\pi}{4} \leqslant x \leqslant \frac{3\pi}{4} \\[2ex] 4(\pi - x) / [\pi + 4 \operatorname{Coth} \frac{\pi}{4}]. & \text{for } \frac{3\pi}{4} < x < \pi. \end{cases}$$

The numerical solution between the intervals $0 < x \leqslant \frac{1}{4}\pi$ and $\frac{3}{4}\pi < x \leqslant \pi$ are obtained by taking λ^2, λ^4, λ^6 and g zero in methods 1 and 2 which are then reduced to a standard central finite difference scheme. The numerical calculations are exact, for example $\| E \|$ at mid points takes volues 0.11×10^{-15} and 0.23×10^{-15} for the stepsize $h = \frac{\pi}{16}$ and $h = \frac{\pi}{24}$ respectively. Further reduction in stepsize does not alter the exactness. These value are not incorporated in the table.

In table 1, we give the numerical results for problem (4.5 (ii) and the error $\| E \|$ is calculated at $x = \frac{\pi}{2}$. The computed values of the Sixth order method 1 are compared with the Sixth order Lobatto method (with approximation I) of Jain [2]. Our Method 1 performs well and gives better results than those of Jain. The results of eighth order method 2 are compared to the eighth order method of Usmani [3]. The superiority of the method is obvious. It is also confirmed that if the stepsize is halved, then $\| E \|$ is approximately reduced by a factor 2^{-p} where p is the order of the method. The numerical calculations are done on IBM 3083 computer at King Saud University, Riyadh.

TABLE 1

Observed $\| E \|$.

h	Methods of order 6		Methods of order 8	
	Method 1	Jain	Method 2	Usmani
$\pi/12$	0.291—09	0.394—08	0.173—12	0.218—11
$\pi/24$	0.457—11	0.619—10	0.470—14	0.686—14
$\pi/48$	0.902—13	0.964—12	0.134—13	0.105—13

'—' denotes round off regain.

5. Conclusion

In this paper, we have shown that if the abstacle function is known, then variational inequalities can be characterized by a sequence of equations, which are then solved by using the higher order numerical methods based on the pade approximant. The results obtained in the paper are much better than the previous methods. A detailed analysis of such methods both analytically and numerically will constitute an immediate and interesting subject of future study. Pade approximants are relatively new methods and an alternate ways of tackling the unilateral problems coupled with the theory of variational inequalities.

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ON CERTAIN FORMULAS FOR THE STIRLING NUMBERS OF THE FIRST AND THE SECOND KIND

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Abstract

The aim of this paper is to derive certain formulas for the Stirling numbers of the first and the second kind from corresponding modifications of our formula for the generalised Bernoulli numbers.

The Stirling numbers of the first kind s (n, k) and the second kind S (n, k) are generated by the Taylor expansions

$$(x)_n = \sum_{k=o}^{n} S(n, k)\, x^k, \quad n = o, 1, 2, \dots, \tag{1}$$

and the Newton expansions

$$x^n = \sum_{k=o}^{n} S(n, k)\, (x)_k, \quad n = o, 1, 2, \dots, \tag{2}$$

respectively, where $(x)_\nu$ for an arbitrary x denotes the product

$$(x)_\nu = x(x-1)\dots(x-\nu+1), \quad \nu = 1, 2, \dots; \quad (x)_o = 1. \tag{3}$$

The Schlömilch formula

$$S(m, m-n) = \sum_{\nu=o}^{n} (-1)^\nu \binom{m+n}{n-\nu}\binom{m+\nu-1}{n+\nu} S(n+\nu, \nu) \tag{4}$$

for integers $o \leqq n \leqq m-1$ $(m \geqq 1)$ gives the Stirling numbers of the first kind in terms of those of the second kind. (For $n \geqq 1$ the summation in (4) is taken over $1 \leqq v \leqq n$). The formula

$$S(n, k) = \frac{1}{k!} \sum_{v=o}^{k} (-1)^{k-v} \binom{k}{v} v^n \tag{5}$$

for integers $o \leqq k \geqq n$ ($n \geqq o$) gives the Stirling numbers of the second kind. See, for example, the formulas (1)—(5) in [1], pp. 204, 207, 213 and 216 or in [2], pp. 22, 168—169 and 219 or in [3], pp. 310 and 319.

Remark 1. We can write the Schlömilch formula (4) in the form

$$S(m, m-n) = \binom{m+n}{2n} \sum_{v=o}^{n} (-1)^v \frac{m-n}{m+v} \binom{2n}{n-v} S(n+v, v) \tag{6}$$

for integers $o \leqq n \leqq m-1$ ($m \geqq 1$). For $n \geqq 1$ the summation in (6) is taken over $1 \leqq v \leqq n$).

The aim of this paper is to derive three formulas for the Stirling numbers of the first and the second kind from our formula (see [4], p. 665, Formula (3))

$$B_n^{(c)} = \sum_{v=o}^{n} (-1)^v \frac{\binom{\tau+n}{n-v}\binom{\tau+v-1}{v}}{\binom{n+v}{v}} S(n+v, v),\ n=o, 1, \ldots \tag{7}$$

(for $n \geqq 1$ the summation in (7) is taken over $1 \leqq v \leqq n$) for the gneeralized Bernoulli numbers $B_n^{(\tau)}$, generated by the Taylor expansion

$$\left(\frac{t}{e^t - 1}\right)^{\tau} = \sum_{n=o}^{\infty} B_n^{(\tau)} \frac{t^n}{n!},\ |t| < 2\pi,\ 1^{\tau} = 1, \tag{8}$$

for every complex number τ (see [1], p. 227, section (18).

Remark 2. We can write our formula (7) in the form

$$B_n^{(\tau)} = \binom{\tau-1}{n}^{-1} \sum_{v=o}^{n} (-1)^v \binom{\tau+n}{n-v}\binom{\tau+v-1}{n+v} S(n+v, v) \tag{9}$$

for $n = 0, 1, 2, \dots$ if the complex number $\tau \neq 1, 2, \dots$, and for $n = 0, 1, 2, \dots, \tau - 1$ if $\tau = 1, 2, \dots$. (For $n \geqq 1$ the summation in (9) is taken over $1 \leqq v \leqq n$).

With the help of our formula (9) we can immediately obtain the Schlömilch (4) as follows : For all pairs of integers (n, m) such that $0 < n < m-1$ we have the formula

$$B_n^{(m)} = \frac{S(m, m-n)}{\binom{m-1}{n}} \tag{10}$$

(see in [1], p. 228 or in [2], p. 218, Formula (10)). From (10) and (9) (for $\tau = m$) we immediately obtain the Schlömilch formula (4).

Remark 3. We can write our formula (7) in the form

$$B_n^{(\tau)} = \frac{\binom{\tau+n}{n}}{\binom{2n}{n}} \sum_{v=0}^{n} (-1)^v \frac{\tau}{\tau+v} \binom{2n}{n-v} S(n+v, v) \tag{11}$$

for $n = 0, 1, 2, \dots$ if the complex number $\tau \neq 0, -1, -2, \dots$ and for $n = 0, 1, 2, \dots, |\tau| - 1$ if $\tau = -1, -2, \dots$. (For $n \geqq 1$ the summation in (11) is taken over $1 \leqq v \leqq n$).

In particular, for a positive integer $\tau = m = 1, 2, \dots$, our formula (11) for $n = 0, 1, 2, \dots$ can be found in [2], p. 217, Formula (7).

Now, with the help of (11) for $\tau = m = 1, 2, \dots$ and (10), we directly obtain our version (6) of the Schlömilch formula (4) for the Stiring numbers of the first kind.

Further, we shall derive a formula for the Stirling numbers of the second kind which is analogous to the formula (6).

Theorem. Let m ($m \geqq 1$) and n ($0 \leqq n \leqq m-1$) be integers. Then

$$S(m+n, m) = \binom{m+n}{2n} \sum_{v=0}^{n} (-1)^{n-v} \frac{m-n}{m-v} \binom{2n}{n-v} S(n+v, v), \tag{12}$$

(For $n \geqq 1$ the summation in (12) is taken over $1 \leqq v \leqq n$).

Remark 4. The formula (12) can be found in [5], pp. IV–V and p. 3, Formula (1.17), without any proof, with other notations and in an implicit form.

Proof. We have the expansions

$$(e^t - 1)^m = m! \sum_{n=m}^{\infty} S(n, m) \frac{t^n}{n!}, \quad m = 1, 2, \ldots, \tag{13}$$

(see, for example, in [1], p. 206 or in [2], p. 202 or in [3], p. 313). From (13) we obtain the expansions

$$\left(\frac{t}{e^t}\right)^{-m} = m! \sum_{h=o}^{\infty} S(n+m, m) \frac{t^n}{(n+m)!}, \quad m = 1, 2, \ldots, \tag{14}$$

in the finite t-plane. From the comparison of (14) with (8) and (11) for $\tau = -m$ we obtain the identities (12) for $0 \leqq n \leqq m-1$ (for $v = -m$ and $n > m$, our formula (7) is reduced to

$$B_n^{(-m} = \frac{S(n+m, m)}{\binom{n+m}{m}},$$

i.e. to the corresponding coefficients in (14)).

This completes the proof of the Theorem.

The formulas (6) and (12) are very convenient for calculating $S(m, m)$, $S(m, m-1)$, $S(m, m-2)$, ... and $S(m, m)$, $S(m+1, m)$, $S(m+2, m)$, ..., respectively.

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ON ISOMORPHISM THEOREMS IN BCI–ALGEBRAS

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Abstract

In this paper, we state and prove four isomorphism theorems for BCI-algebras.

1. Introduction

Isomorphism theorems are found in the literature for various algebraic structures. But for BCI-algebras such theorems have not been investigated yet, except for the first isomorphism theorem about BCK-algebras, proved by K. Iseki ([81]). The purpose of this paper is to develop isomorphism theorems for BCI-algebras, naturally applicable to BCK-algebras which is a subclass of BCI-algebras.

2. Preliminaries

A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions :

(1) $(x*y)*(x*z) \leqslant z*y$,

(2) $x*(x*y) \leqslant y$,

(3) $x \leqslant x$

(4) $x \leqslant y$, $y \leqslant x$ imply $x = y$,

(5) $x \leqslant o$, imples $x = 0$,

where $x \leqslant y$ is defined by $x*y = 0$.

If (5) is replaced by $o \leqslant x$, then the algebra is called a BCK-algebra.

In any BCI-algebra X, the following hold :

(6) $(x_*y)_*z = (x_*z)_*y$,

(7) $x_*o = x$, for x, y, z, in X (see Iseki [6]).

A non-empty subset A of a BCI-algebra X is called a sub-algebra if the binary operation * is closed in A. Further a non-empty subset A of a BCI-algebra X is called an ideal if (i) $o \varepsilon A$ (ii) $y_*x \varepsilon A$, $x \varepsilon A$ imply $y \varepsilon A$. An ideal in a BCI-algebra is not necessarily closed. To avoid this difficulty the concept of a closed ideal is defined as under :

Definition 1 [5]. A non-empty subset A of a BCI-algebra X is called a closed ideal in X if

(i) $o_* x \varepsilon A$, for all $x \varepsilon A$,

(ii) $y_* x \varepsilon A$, $x \varepsilon A$ imply $y \varepsilon A$.

In the sequel we shall simply call a closed ideal as ideal.

Remark 1. It is easy to verify that $o \varepsilon A$. Further, if $x \varepsilon A$ and $y \leqslant x$, then $y_*x = o \varepsilon A$ gives that $y \varepsilon A$. Moreover, we note that every ideal is a sub-algebra, the converse is not always true ([2]).

K. Iseki and S. Tanaka ([7], [1]), defined quotient algebras and proved first isomorphism theorem for BCK-algebra. In the sequel, we define BCI-quotient algebras and prove four isomorphism theorems in the setting of BCI-algebras.

Let A be an ideal in a BCI-algebra X. Following K. Iseki and S. Tanoka [7], we define $\sim$ in X as : $x \sim y$ if and only if $x*y \varepsilon A$, $y_*x \varepsilon A$. Then it is easily verified that $\sim$ is an equivalence relation and satisfies $x_1 \sim y_1$, $x_2 \sim y_2$ imply $x_1 * x_2 \sim y_1 * y_2$. Let $X/A = \{ C_x^A : x \varepsilon X \}$. We define $_*$ in X/A as $C_x^A * C_y^A = C_{x*y}^A$, where $C_x^A = \{ y \varepsilon X : x \sim y \}$. Obviously $C_o^A = A$. Routine calculations give that $_*$ is well-defined in X/A. Next, we define $C_x^A * C_y^A = C_o^A$ if and only if $C_x^A \leqslant C_y^A$. Then X/A becomesa BCI-algebra.

A mapping $\phi : X \to Y$ is called homomorphism ([8]) if $\phi(x_* y) = \phi(x) * \phi(y)$, for all $x, y \in X$. If, in addition, ϕ is one-one and onto, then ϕ is called an isomorphism. Obviously $\phi(o) = o$. Then Kernel of ϕ is given by Ker $\phi = \{x \in X : \phi(x) = o\}$.

3. Three Isomorphism Theorems

We now state and prove the first isomorphism theorem. The proof of this theorem can be given with minor modifications on the lines of the proof of first isomorphism theorem for BCK-algebras given by K. Iseki and S. Tanaka ([7], [8]). However for completion we give the proof.

Theorem 1. (First Isomorphism Theorem) Let $\phi : X \to Y$ be an onto homomorphism. Then Ker ϕ is an ideal in X and X/Ker ϕ is isomorphic to Y.

Proof. First, we show that Ker ϕ = K is an ideal in X. For all $x \in K$, $o * x \in K$, because $\phi(o * x) = \phi(o) * \phi(x) = o * o = o$. Now, let $x_* y \in K$, $y \in K$. Then $\phi(x) = \phi(x) * o = \phi(x) * \phi(y) = \phi(x_* y) = o$ gives $x \in K$. This proves that Ker ϕ is an ideal in X.

Next, we define a mapping $\psi : X/K \to Y$ by $\psi(C_x^K) = \phi(x)$. It is easy to see that ψ is well-defined and onto. To show that ψ is one-one, let $\phi(x) = \phi(y)$. Then $o = \phi(x) * \phi(y) = \phi(x * y) = \phi(y_* x)$ implies $x * y$, $y * x \in K$ and thus $x \sim y$. Consequently $C_x^K = C_y^K$. Finally, consider $\psi(C_x^K * C_y^K) = \psi(C_{x_* y}^K) = \phi(x_* y) = \phi(x) * \phi(y) = \psi(C_x^K) * \psi(C_y^K)$, which shows that ψ is a homorphism. This completes the proof.

Before proceeding further we give two examples which provide us the background for our second and third isomorphism theorems and facilitate their understanding.

Example 1. Consider the BCI-algebra $x=\{o, a, b, c, d, e, f, g, h\}$ in which $*$ is defined as

*	o	a	b	c	d	e	f	g	h
o	o	o	o	o	d	d	f	f	h
a	a	o	a	o	d	d	g	f	h
b	b	b	o	o	e	d	f	f	h
c	c	b	a	o	e	d	g	f	h
d	d	d	d	d	o	o	h	h	f
e	e	e	d	d	b	o	h	h	f
f	f	f	f	f	h	h	o	o	d
g	g	f	g	f	h	h	a	o	d
h	h	h	h	h	f	f	d	d	o

We take $H = \{o, a, d\}$ an ideal in X and $K = \{o, d, f, h\}$ a sub-algebra of X. We note that $H \cap K = \{o, d\}$ is an ideal in **K**. Thus $K/_{H \cap K}$ is well defined. Easy calculations give that $K/_{H \cap K}$

$= \{ C_o^{H \cap K} = \{o,d\}, C_f^{H \cap K} = \{f, h\} \}$ is a two element BCI-algebra under the operation $C_x^{H \cap K} * C_y^{H \cap K} = C_{x*y}^{H \cap K}$.

$$\text{Let } Y = \bigcup_{k \in K} C_k^H = C_o^H \cup C_d^H \cup C_f^H \cup C_h^H = C_o^H \cup C_f^H$$

$= \{o, a, d\} \cup \{f, g, h\} = \{o, a, d, f, g, h\}$, which is a sub-algebra of X containing H and **K**. We further note that Y is not an ideal in X. The mapping $\phi : Y \rightarrow K/_{H \cap K}$ defined by $\phi(y) = C_k^{H \cap K}$ iff $y \in C_k^H$ for some $k \in K$, that is, $\phi(o) = C_o^{H \cap K}$, $\varphi(a) = C_o^{H \cap K}$, $(d) = C_o^{H \cap K}$, $\phi(f) = C_f^{H \cap K}$, $\phi(g) = C_f^{H \cap K}$, $\phi(h) = C_f^{H \cap K}$ if an onto homomorphism with kernel $H = \{o, a, d\}$. Thus $Y/H \cong K/_{H \cap K}$.

Example 2. We consider the same BCI-algebra X as in example 1 and take the ideals $H = \{o, a\}$ and $K = \{o, a, b, c, f, g\}$ in X. We note that $H \subseteq K$ and is also an ideal in K. Easy calculations give that

$X/H = \{C_o^H = H, C_b^H = \{b, c\}, C_d^H = \{d\}, C_e^H = \{e\}, C_g^H = \{g, f\}, C_h^H = \{h\}\}$ $X/K = C_o^K = K, C_d^K = \{d, e, h\}\}$ are BCI-algebras under the operation $C_x * C_y = C_{x*y}$. Further the mapping f: $X/H \to X/K$ defined by $f(C_x^H) = C_x^K$, that is,

$f(C_o^H) = C_o^K = K, f(C_b^H) = C_b^K = C_o^K = K, f(C_d^H) = C_d^K,$

$f(C_e^H) = C_e^K = C_d^K, f(C_g^H) = C_g^K = C_o^K = K, f(C_h^H) = C_h^K$

$= C_d^K$, is an onto homomorphism with kernel $\{C_o^H = H, C_b^H = \{b,c\},$

$C_g^H = \{g, f\}\}$, We also note that $K/H = \{H, C_b^H = b, c\},$

$C_g^H = \{g, f\}\}$ is an ideal in X/H. This can be verified directly as well as from the observation Ker (f) = K/H. Thus $X/H/_{K/H} \cong X/K$.

We now state and prove second and third isomorphism theorems in BCI-algebras.

Theorem 2. (Second Isomorphism Theorem)

Let H be an ideal and K a sub-algebra of a BCI-algebra X. Let $Y = \bigcup_{k \in K} C_k^H$. Then Y is a sub-algebra of X containing H and K, $H \cap K$ is an ideal in K and $Y/H \cong K/H \cap K$.

Proof. First, we show that Y is a sub-algebra of X. Clearly $o \in Y$. Let $y_1, y_2 \in Y$. Then there exist $k_1, k_2 \in K$ such that $y_1 \in C_{k_1}^H$ and $y_2 \in C_{k_2}^H$, which gives $y_1 \sim k_1$ and $y_2 \sim k_2$ and therefore $y_1 * y_2 \sim k_1 * k_2$. Thus $y_1 * y_2 \in C_{k_1 * k_2}^H \subseteq Y$.

This proves that Y is sub-algebra of X. Further if $k \in K$, then $k \in C_k^H \subseteq Y$. Thus $K \subseteq Y$. Also $o \in K$ implies $H = C_o^H \subseteq Y$.

Since H and K are sub-algebras, we have $o * x \in H \cap K$, for all $x \in H \cap K$. Let $y \in K$, $y * x \in H \cap K$, $x \in H \cap K$. H is ideal in X gives $y \in H$ and hence $y \in H \cap K$. This gives that $H \cap K$ is ideal in K. Now we define $\phi : Y \rightarrow K/H \cap K$ by $\phi(y) = C_k^{H \cap K}$ if and only if $y \in C_k^H$, for some $k \in k$. It is easy to see that ϕ is wel-defined. Let $\phi(y_1) = C_{k_1}^{H \cap K}$ and $\phi(y_2) = C_{k_2}^{H \cap K}$. Then $y_1 \in C_{k_1}^H$ and $y_2 \in C_{k_2}^H$.

Thus $y_1 * y_2 \in C_{k_1 * k_2}^H$ which implies that $\phi(y_1 * y_2)$

$$= C_{k_1 * k_2}^{H \cap K}.$$

This gives $\phi(y_1 * y_2) = C_{k_1 * k_2}^{H \cap K} = C_{k_1}^{H \cap K} * C_{k_2}^{H \cap K}$

$$= \phi(y_1) * \phi(y_2).$$

Hence ϕ is a BCI-homomorphism. For each $C_k^{H \cap K} \in K/H \cup K$, there exists $k \in C_k^H \subseteq Y$ such that $\phi(k) = C_k^{H \cap K}$. Hence ϕ is onto. By first isomorphism theorem, we have $Y/\mathrm{Ker}\ \phi \cong K/H \cap K$.

Finally, we show that $\mathrm{Ker}\ \phi = H$. Let $h \in H$. Then $h \in H = C_o^H$ implies $\phi(h) = C_o^{H \cap K}$. Hence $h \in \mathrm{Ker}\ \phi$ and $H \subseteq \mathrm{Ker}(\phi)$. Conversely let $h \in \mathrm{Ker}(\phi)$. Then $\phi(h) = C_o^{H \cap K}$ implies $h \in C_o^H = H$. Therefore $\mathrm{Ker}\ \phi \subseteq H$. Hence $\mathrm{Ker}(\phi) = H$, which gives

$$Y/H \cong K/H \cap K.$$

This completes the proof.

Definition 2 ([4]). A BCI-algebra X is called medial if it satisfies the following :

(7) $(x * y) * (z * u) = (x * z) * (y * u)$.

In ([5]) Dudek proved that medial BCI-algebras satisfy

(8) $x * (x * y) = y$.

M. Anwar Chaudhry and B. Ahmad ([2]), proved several results about BCI-algebras satisfying (7) and (8). C.S. Hoo eventually established in ([5]) that they are same in BCI-algebras. It is also shown in ([1]) that (7) is equivalent to

(9) $x * y = o * (y * x)$.

Further it is interesting to note that the medial BCI-algebras have the property : $x * y$, $x \varepsilon A$ imply $y \varepsilon A$. The proof of which is directly obtained from (8). In view of the remark 1, and following lemma, it follows that ideals and sub-algebras coincide in medial sub-algebras.

Lemma. Let A be a sub-algebra of a medial BCI-algebra X. Then A is an ideal in X.

Proof. Obviously $o * x \varepsilon A$, for all $x \varepsilon A$. Next, Let $x * y$, $y \varepsilon A$. Then (9) gives $y * x = o * (x * y) \varepsilon A$. Now $y * x$, $y \varepsilon A$ timply $x \varepsilon A$. This proves that A is ideal in X.

Remark 2. If A and B are sub-algebras in a medial BCI-algebra X, then it is easy to see that $AB = \{a * b = a \varepsilon A, b \varepsilon B\}$ is also a sub-algebra of X. Clearly, it is not true for BCI-algebras. Moreover, we have the following :

Theorem 3. Let A and B be sub-algebras of a medial BCI-algebra X.

Then

(i) $AB = BA$,

(ii) $AB = \bigcup_{b \varepsilon B} C_b^A$.

Proof. (i) Let $a * b \varepsilon AB$. Consider

$a * b = o * (b * a) = (o * o) * (b * a) = (o * b) * (o * a) = b_1 * a_1 \varepsilon BA$

and conversely. This proves (i).

(ii) It is obvious that $A, B \subseteq \bigcup_{b \varepsilon B} C_b^A$. Since $\bigcup_{b \varepsilon B} C_b^A$ is sub-algebra, we have $AB \subseteq \bigcup_{b \varepsilon B} C_b^A$. Conversely, Let $y \varepsilon \bigcup_{b \varepsilon B} C_b^A$. Then there exists $b \varepsilon B$ such that $y \varepsilon C_b^A$. This implies $y_* b$, $b_* y \varepsilon A$. Further $y = o * (o_* y)$. Bnt $o_* y = (b_* b) _* y = (b_* y) _* b \varepsilon AB$. Thus $\exists\ p \varepsilon A$, $q \varepsilon B$ such that $o_* y = p_* q$, Now $y = o * (o_* y) = o _* (p_* q) = q_* p \varepsilon BA = AB$. Hence the converse. This completes the proof of (ii).

The following is the analog of the well-known second isomorphism theorem of group theory.

Corollary. Let H and K be sub-algebras of medial BCI-algebra X.

$$HK/H = K/H \cap K.$$

In the proof of third isomorphism theorem we shall use the following resut.

Theorem 4. Let A and A_1 be ideals in a BCI-algebra X such that $A \subseteq A_1 \subseteq X$. Then $a \varepsilon A_1$, $C_a^A \varepsilon X/A$ implies $C_a^A \subseteq A_1$ and $A_1 / A \subseteq X/A$ ([10]).

We now prove third isomorphism theorem for BCI-algebra.

Theorem 5. **(Third isomorphism theorem)**

Let H and K be ideals in a BCI-algebra X and $H \subseteq K$. Then

$$X/H \ / \ K/H = X/K$$

Proof. **Clearly H is ideal in K and X/H, X/K and K/H are well-defined. We define $\phi = X/H \to X/Y$ by**

$$\phi (C_x^H) = C_x^K .$$

Easy calculations give that ϕ is well-defined and onto. Further

$$\phi (C_x^H {}_* C_y^H) = \phi (C_{x_* y}^H) = (C_{x_* y}^H = C_x^K {}_* C_y^K = \phi (C_x^H)_* \phi (C_y^H)$$

gives that ϕ is a homomorphism. By the first isomorphism theorem, we have

$$X/H/\text{Ker } \phi \cong X/K$$

Next, we show that Ker ϕ = K/H.

Let $C_K^H \varepsilon K/H$. Since K and H are ideals in X such that $H \subseteq K$. Therefore by theorem 4, $C_k^H \varepsilon X/H$. Further $\phi(C_k^H) = C_k^K = K = C_o^K$ implies $C_k^H \varepsilon$ Ker ϕ. Thus $K/H \subseteq$ Ker ϕ. Conversely, Let $C_x^H \varepsilon$ Ker ϕ, then $\phi, (C_x^H) = C_o^K = K$. But $\phi(C_x^H) = C_x^K$ implies $C_x^K = K$. Thus $x \varepsilon K$ and hence $C_x^H \varepsilon K/H$ Consequently,

Ker ϕ = K/H. This completes the proof.

4. Fourth Isomorphism Theorem

Before proving the theorem we give the necessary back-ground material and an example which provides the motivation and facilitates the understanding of the theorem. The concepts of congreuences and regular congruences have been defined for BCK-algebras in ([9]). We adopt the same definition for BCI-algebras.

Definition 2. ([9]). An equivalence relation q on a BCI—algebra X is said to be a congruence if $(x_1, y_1), (x_2, y_2) \varepsilon q$ imply $(x_1 * x_2, y_1 * y_2) \varepsilon q$.

We note that if q is a congruence on X, then it partitions X into disjoint classes $\{C_x^q : x \varepsilon X\}$, where $C_x^q = \{y \varepsilon X : (y, x) \varepsilon q\}$. It is easy to verify that C_o^q is an ideal in X. Define $*$ in X/q, the set of disjoint class, by: $C_x^q * C_y^q = C_{x*y}^q$. K. Iseki ([9]) and Andrzej wronski [11] have shown that in case of BCK-algebras the operation $*$ is well-defined and X/q satisfies all the axioms of a BCK-algebra except (4). The same is true for BCI-algebras. Such congruences have been called non-regular congruences and in ([11]) it has been shown that there exists BCK-algebras having regular as well as non-regular congruences. In order to make X/q, a BCI-algebra, we define the concept of a regular congruence for BCI-algebras.

Definition 3. A congruence q on X is said to be a regular congruence if X/q, as defined above, is a BCI-algebra.

We note that every congruence generated by an ideal A in X is regular. In the sequel we shall denote it by $q_A = \{(x, y) : x * y \in A$ $y * x \in A\}$. Further every congruence on a medial BCI-algebra is reqular.

Definition 4. Let q_1 and q_2 be two congruences on a BCI-algebra X. The composition or product of q_1 and q_2 denoted by $q_1 {}^o q_2$ is defined as :

$$q_1 \circ q_2 = \{(x, y) : (x, z) \in q_1 \text{ and } (z, y) \in q_2 \text{ for some } z \in X\}.$$

The congrueuces q_1 and q_2 are said to be commutative if and only if $q_1 {}^o q_2 = q_2 {}^o q_1$. It is interesting to note that composition of two regular congruence is not necessarily commutative.

In the proof of the fourth isomorphism theorem, we shall be needing following results, which have been proved in ([10]).

Theorem 6. ([10]). Let q_1 and q_2 be two congruences on a BCI-algebra X. Then $q_1 \circ q_2$ is a congruence on X if and only if and only if $q_1 \circ q_2 = q_2 \circ q_1$.

Theorem 7. ([10]). Let A and B be ideals in a BCI-algebra X and q_A, q_B be congruences generated by the idaals A and B, respectively. If $q_A {}^o q_B$ is a congruence on X i.e; if $q_A {}^o q_B = q_B {}^o q_A$, then $\bigcup_{a \in A} C_a^B = \bigcup_{a \in B} C_b^A$ and is an ideal in X.

Example 3. We consider the BCK-algebra X = {o, a, b, c, d, e f, g, h, i, j, k}, which is of course a BCI-algebra. The binary operation $*$ in X is given by : ([9]),

*	o	a	b	c	d	e	f	g	h	i	j	k
o	o	o	o	o	o	o	o	o	o	o	o	o
a	a	o	o	a	a	o	o	a	a	o	o	a
b	b	a	o	b	b	a	o	b	b	a	o	b
c	c	c	c	o	c	c	c	o	c	c	o	o
d	d	d	d	d	o	o	o	o	o	o	o	o
e	e	d	d	e	a	o	o	a	a	o	o	a
f	f	e	d	f	b	a	o	b	b	a	o	b
g	g	c	c	d	c	c	c	o	c	c	o	o
h	h	h	h	h	h	h	h	h	o	o	o	o
i	i	h	h	i	i	h	h	i	a	o	o	a
j	j	i	h	j	j	i	h	j	b	a	o	b
k	k	k	k	h	k	k	k	h	c	c	c	o

Take H = { o, a, b, d, e, f, h} and K = {o, a, b, c, h, i, j, k}, two sub-algebras of X. Further take H′ = {o, d, h} and K′ = {o, c} ideals in H and K, respectively. Let U = H ∩ K. Then H ∩ K = {o, a, b, h}. Further H ∩ K′ = {o} and H′ ∩ K = {o, h}. Easy calculations give that

$$\bigcup_{x \in H \cap K} C_X^{H'} = \{o, a, b, d, e, f, h, i, j\}, \text{ a sub-algebra of X}$$

containing H and H ∩ K, $\bigcup_{x \in H \cap K'} C_x^{H'} = \{o, d, h,\}$, which is an ideal in $\bigcup_{x \in H \cap K} C_x^{H'}$. Thus $\bigcup_{x \in H \cap K} C_x^{H'} / \bigcup_{x \in H \cup K'} C_x^{H'}$ is well-defined and routine calculations give that it equals

$$\{C_o^{H'}, C_a^{H'}, C_b^{H'}\} \qquad (1)$$

Similarly $\bigcup_{x \in H \cap K} C_x^{K'} = \bigcup_{x \in \{o,a,b,h\}} C_x^{K'} = \{o, c\} \cup \{a\} \cup \{b\}$ $\cup \{h, k\} = \{o, a, b, c, h, k\}$ is a sub-algebra of X containing $H \cap K$ and K′. Moreover $K'' = \bigcup_{x \in H' \cap K} C_x^{K'} = \{o, c, h, k\}$ is an ideal in $\bigcup_{x \in H \cap K} C_x^{K'}$, Thus $\bigcup_{x \in H \cap K} C_x^{K'} / \bigcup_{x \in H' \cap K} C_x^{K'}$ is well defined and routine calculations give that it equals

$$\{ C_o^{K''}, C_a^{K''}, C_b^{K''} \} \qquad (2)$$

Further we note that $H \cap K' = \{o\}$ and $H' \cap K = \{o, h\}$ are ideals in $H \cap K = U$ and it can be easily checked that all regular congruences on $H \cap K$ commute.

Now

$$\bigcup_{x \in H \cap K} C_x^{H \cap K'} = \bigcup_{x \in \{o, h\}} C_x^{\{o\}} = \{o, h\},$$

and

$$\bigcup_{x \in H \cap K'} C_x^{H' \cap K} = \bigcup_{x \in \{o\}} C_x^{\{o, h\}} = \{o, h\}.$$

Let

$$V = \bigcup_{x \in H' \cap K} C_x^{H \cap K'} = \bigcup_{x \in H \cap K'} C_x^{H' \cap K} = \{o, h\}.$$

Since

V is an ideal in U, thus U/V is well defined and easy calculations give that it equals $\{ C_o^V, C_a^V, C_b^V \}$ (3)

We define $\sigma : \bigcup_{x \in H \cap K} C_x^{H'} = \bigcup_{u \in U} C_u^{H'}$ (o, a, b, d, e, f, h, i, j) $\to U/V$ by $\sigma(x) = C_u^V$ if and only if $x \in C_u^{H'}$ for $u \in U = H \cap K$, that is, $\sigma(o) = C_o^V$, $\sigma(a) = C_a^V$, $\sigma(b) = C_b^V$, $\sigma(d) = C_o^V$, $\sigma(e) = C_a^V$, $\sigma(f) = C_b^V$ $\sigma(h) = C_o^V$; $\sigma(i) = C_a^V$. $\sigma(j) = C_b^V$.

Obviously σ is onto and easy calculations give that σ is a homomorphism.

Further Ker $\sigma = \{a, d, h\} = \bigcup_{x \in H \cap K'} C_x^{H'}$. Thus $\bigcup_{x \in H \cup K} C_x^{H'}$ /

Ker $\sigma = \bigcup_{x \in H \cap K} C_x^{H'} / \bigcup_{x \in H \cap K'} C_x^{H'} \cong U/V$. Similarly

$\bigcup_{x \in H \cap K} C_x^{K'} / \bigcup_{x \in H' \cap K} C_x^{K'} \cong U/V$. Hence the algebras (1), (2) and (3) are isomorphic.

We now state and prove the fourth isomorphism theorem in BCI-algebras.

Theorem 8. (Fourth isomorphism theorem)

Let H and K be subalgebras of a BCI-algbera H. Let H′ and K′ be ideals in H and K, respectively. Then

$$\bigcup_{x \in H \cap K} C_x^{H'} / \bigcup_{x \in H \cap K'} C_x^{H'} \cong \bigcup_{x \in H \cap K} C_x^{K'} / \bigcup_{x \in H' \cap K} C_x^{K'}$$

provided any two regular congruences on $H \cap K$ commute.

Proof. Let $U = H \cap K$. Then U is subalgebra of X. But $H \geqslant H \cap K \leqslant K$. Therefore U is a subalgebra of H as well as K. Since $H' \leqslant H, K' \leqslant K$ and each of H, H′, K. K′ is a subalgebra of X, so $H \cap K'$ and $H' \cap K$ are subalgebras of $H \cap K$.

Now, we show that $H \cap K'$ is an ideal in the BCI-algebra U. Since $H \cap K'$ is a subalgebra of U, so o, $o * x \in H \cap K'$. $\forall x \in H \cap K'$. Let $u*x$, $x \in H \cap K'$ with $u \in U$. Then $u \in U = H \cap K$ and $u*x$, $x \in K'$. But K′ is an ideal in K. Therefore, $u \in H$ and $u \in K'$, so that $u \in H \cap K'$. Hence it follows that $H \cap K'$ is an ideal in U. Similarly $H' \cap K$ is an ideal in U.

Since any two congruences on $H \cap K = U$ commute, therefore, by theorem 7.

$$\bigcup_{x \in H \cap K} C_x^{H \cap K'} = \bigcup_{x \in H \cap K} C_x^{H' \cap K} = V \text{ (say)} \qquad (1)$$

Obviously σ is onto and easy calculations give that σ is a homomorphism.

Further Ker $\sigma = \{a, d, h\} = \bigcup_{x\varepsilon H\cap K'} C_x^{H'}$. Thus $\bigcup_{x\varepsilon H\cup K} C_x^{H'}$ /

Ker $\sigma = \bigcup_{x\varepsilon H\cap K} C_x^{H'} / \bigcup_{x\varepsilon H\cap K'} C_x^{H'} \cong U/V$. Similarly

$\bigcup_{x\varepsilon H\cap K} C_x^{K'} / \bigcup_{x\varepsilon H'\cap K} C_x^{K'} \cong U/V$. Hence the algebras (1), (2) and (3) are isomorphic.

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$$\bigcup_{x\varepsilon H\cap K} C_x^{H'} / \bigcup_{x\varepsilon H\cap K'} C_x^{H'} \cong \bigcup_{x\varepsilon H\cap K} C_x^{K'} / \bigcup_{x\varepsilon H'\cap K} C_x^{K'}$$

provided any two regular congruences on $H\cap K$ commute.

Proof. Let $U = H\cap K$. Then U is subalgebra of X. But $H \geqslant H\cap K \leqslant K$. Therefore U is a subalgebra of H as well as K. Since $H' \leqslant H, K' \leqslant K$ and each of H, H′, K, K′ is a subalgebra of X, so $H\cap K'$ and $H'\cap K$ are subalgebras of $H\cap K$.

Now, we show that $H\cap K'$ is an ideal in the BCI-algebra U. Since $H\cap K'$ is a subalgebra of U, so $o, o * x\varepsilon H\cap K'$. $\forall x\varepsilon H\cap K'$. Let $u*x$, $x\varepsilon H\cap K'$ with $u\varepsilon U$. Then $u\varepsilon U = H\cap K$ and $u*x$, $x\varepsilon K'$. But K′ is an ideal in K. Therefore, $u\varepsilon H$ and $u\varepsilon K'$, so that $u\varepsilon H\cap K'$. Hence it follows that $H\cap K'$ is an ideal in U. Similarly $H'\cap K$ is an ideal in U.

Since any two congruences on $H\cap K = U$ commute, therefore, by theorem 7.

$$\bigcup_{x\varepsilon H\cap K} C_x^{H\cap K'} = \bigcup_{x\varepsilon H\cap K} C_x^{H'\cap K} = V \text{ (say)} \qquad (1)$$

and V is an ideal in U. Thus the quotient algebra U/V is well-defined. We define

$$\sigma : \bigcup_{u \varepsilon U} C_u^{H'} \to U/V$$

by

$\sigma(x) = C_{u'}^{V}$, if and only if $x\sigma\, C_u^{H'}$ for some $u\sigma U$; $\forall x \varepsilon \bigcup_{u \varepsilon U} C_u^{H'}$. (2)

Clearly $\sigma(x)\, \varepsilon U/V$, $\forall x \varepsilon \bigcup_{x \varepsilon U} C_u^{H'}$. Let $x \varepsilon C_u^{H'} = C_{u_1}^{H'}$ with $u, u_1\, \varepsilon U$.

Then $(u, u_1)\, \varepsilon q_H!$. Thus $u * u_1 \,.\, u_1 * u \varepsilon H'$. But $u * u_1 , u_1 * u \varepsilon U$ because $U = H \cap K$ is a subalgebra of X such that $u, u_1\, \varepsilon U$. Therefore, $u * u_1 , u_1 * u \varepsilon H' \cap K \leqslant V$ (Second isomorphism theorem).

Thus, $(u, u_1)\, \varepsilon q_V$ in U and hence $C_u^V = C_{u_1}^V$. Hence $\sigma(x)$ defined

by (2) is unique for every $x \varepsilon \bigcup_{u \varepsilon U} C_u^{H'}$. Thus σ is well defined.

σ is onto because if $C_u^V\, \varepsilon U/V$, then $u \varepsilon U$ so $u \varepsilon C_u^{H'} \leqslant \bigcup_{u \varepsilon U} C_u^{H'}$

and hence by (2) $\sigma(u) = C_u^V$. Further let $x_1 , x_2\, \varepsilon \bigcup_{u \varepsilon U} C_u^{H'}$.

Then there exist $u_1 , u_2\, \varepsilon U$ such that $x_1\, \varepsilon\, C_{u_1}^{H'}$ and $x_2\, \varepsilon\, C_{u_2}^{H'}$.

Then $(x_1 , u_1), (x_2 , u_2)\, \varepsilon\, q_{H'}$. But $q_{H'}$ is a congruence on H (generated by H'). Therefore, $(x_1 * x_2 , u_1 * u_2)\, \varepsilon\, q_{H'}$. Thus $x_1 * x_2\, \varepsilon\, C_{u_1 * u_2}^{H'}$. Hence, in view of (2), we have $\sigma(x_1 * x_2)$

$$= C_{u_1 * u_2}^V = C_{u_1}^V * C_{u_2}^V = \sigma(x_1) * \sigma(x_2).$$

Hence σ as defined in (2) is an onto homorphism from $\bigcup_{u \varepsilon U} C_u^{H'}$ to

U/V. Thus, by first isomorphism theorem, we get

$$\bigcup_{u\varepsilon U} C_u^{H'} \;/\; \text{Ker}\,\sigma \cong U/V \tag{3}$$

Now we show that Ker $\sigma = \bigcup_{u\varepsilon H\cap K'} C_u^{H'}$. Let $w\varepsilon$Ker σ.

Then there exists $u\varepsilon U = H\cap K$ such that $w\varepsilon C_u^{H'}$ and $u\varepsilon C_u^V = \sigma(w)$ $= C_o^V = V = \bigcup_{x\varepsilon K\cap H'} C_x^{H\cup K'}$ (by (1)). Then $(w, u)\ \varepsilon q_{H'}$ in H and $u*x = x_1\ \varepsilon H\cap K'$ (say), for some $x\varepsilon K\cap H'$. Note that $u*x_1$ $= u*(u*x) \leqslant x\varepsilon H\cap'K$ imply $u*x_1\ \varepsilon H'\cap\ R \leqslant H'$. Also $x_1 * u$ $=(u*x)*u=(u*u)*x=o*x\varepsilon H'\cap K \leqslant H'$.

Therefore $u*x_1$, $x_1 *u\varepsilon H'$ such that x_1, $u\varepsilon U = H\cap K \leqslant H$. But H′ is an ideal in H. Therefore $(u, x_1)\ \varepsilon q_{H'}$ in H. Hence (w, u). $(u, x_1)\ \varepsilon q_{H'}$ in H. Which gives that $(w, x_1)\ \varepsilon q_{H'}$. But x_1 $\varepsilon H\cap K'$. Therefore, $w\epsilon C_{x_1}^{H'} \leqslant \bigcup_{u\varepsilon H\cap K'} C_u^{H'}$. Hence

$$\text{Ker}\,\sigma \leqslant \bigcup_{u\varepsilon H\cap K'} C_u^{H'} \tag{4}$$

On the other hand let $x\ \varepsilon \bigcup_{u\varepsilon H\cap K'} C_u^{H'}$. Then $x\varepsilon C_{u_1}^{H'}$ for some $u_1\ \varepsilon H\cap K'$. But $H\cap K' \leqslant U$ and $H\cap K' \leqslant \bigcup_{u\varepsilon H\cap K'} C_u^{H'\cap K}$ $= V$. Therefore, $u_1\ \varepsilon U$ such that $u_1\ \varepsilon$ V and $x\ \varepsilon\ C_{u_1}^{H'}$. Hence $\sigma(x)$ $= C_{u_1}^V = V = C_o^V$. Therefore, $x\varepsilon$Ker σ, so that

$$\bigcup_{u\varepsilon H\cap K'} C_u^{H'} \leqslant \text{Ker}\,\sigma \tag{5}$$

Thus, by (4) and (5), we get Ker $\sigma = \bigcup_{u\varepsilon H\cap K'} C_u^{H'}$. So (3) becomes

$$\bigcup_{u\varepsilon H\cap K} C_u^{H'} \Big/ \bigcup_{u\varepsilon H\cap K'} C_u^{H'} \cong U/V \qquad (6)$$

Similarly, we get

$$\bigcup_{u\varepsilon H\cap K} C_u^{K'} \Big/ \bigcup_{u\varepsilon H'\cap K} C_u^{K'} \cong U/V \qquad (7)$$

But the relation '$\cong$' is an equivalence relation. Therefore (6) and (7) give

$$\bigcup_{u\varepsilon H\cap K} C_u^{H'} \Big/ \bigcup_{u\varepsilon H\cap K'} C_u^{H'} \cong \bigcup_{u\varepsilon H\cap K} C_u^{K'} \Big/ \bigcup_{u\varepsilon H\cap K} C_u^{K'} .$$

Remark 2. Let X be a medial BCI-algebra, i.e., a BCI-algebra satisfying $x*(x*y)=y$. Then we have proved that in this case ideals and sub-algebras coincide. Further we have shown that if A and B are sub-algebras of a medial BCI-algebra then AB = BA and $AB = \bigcup_{b\varepsilon B} C_b^A$. Thus in this case $\bigcup_{u\varepsilon H\cap K} C_u^{H'} \cong H'(H\cap K)$,

$\bigcup_{u\varepsilon H\cap K'} C_u^{H'} = H'(H\cap K')$. $\bigcup_{u\varepsilon H\cap K} C_u^{H'} = K'(H\cap K)$ and

$\bigcup_{u\varepsilon H'\cap K} C_u^{K'} = K'(H'\cap K)$. Further on every sub-algebra A of a medial BCI-algebra X all regular congruences commute. Thus we have the following corollary :

Corollary. Let H, K be subalgerbras of a medial BCI-algebra X. Let H′ and K′ be ideals in H and K, respectively. Then

$$H'(H\cap K)/_{H'(H\cap K')} \cong K'(H\cap K)/_{K'(H'\cap K)}$$

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TURCHIN-KLEIN REGULARIZATION METHOD USING B-SPLINES FOR ILL-POSED PROBLEMS

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Abstract

A method is presented in this paper for estimating solutions of ill-posed problems in the form of integral equations of the first kind, given noisy and clean data. Regularization is done by Turchin Klein method using B-splines.

We propose a technique by which an approximately optimal amount of smoothing may be computed based only on the data and the assumed known noise Variance. Numerical examples are given and we have considered mildly, moderately and highly ill-posed problems.

1. Introduction

The concept of ill-posedness was introduced by Hadamard in the field of partial differential equations. For years ill-posed problems have been considered as mere mathematical anomalies, however it appeared in the early sixties that this attitude is erroneous and that many ill-posed problems, generally inverse problems, arose from practical situations. Now-a-days there is no doubt that a systematic study of these problems is of great relevance in many fields of applied physics. For example problems of image reconstruction and enhancement ; x-rays and neutron scattering ; integral equations of the first kind in spectroscopy, chemical analysis, queueing theory,

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astrophysics and photon-correlation, optimal control, seismographic data analysis, calculation of atmosperic temperature profiles ; numerical inversion of Laplace transforms ; numerical inversion of Radon transforms in computerized tomography ; inverse source problems and inverse scattering problems in optics, meteorology, stereology and other fields. For a general account of the theory of ill-posed problems the reader is referred to Lavrentiev [14] Morozov [16]. Tikhonov and Arsenin [23] and Groetsch [12].

In this paper we shall use theory of cardinal B-splines for pth order tikhonov regularization for convolution equations of the first kind. Before we discuss the details of the theory, we give some definitions and elementary properties.

(i) Cardinal B-splines

B-splines with equidistant knots are called cardinal B-splines with knots on the first n natural number we have

$$\mu\,(x\,;\,0,1,2,\ldots,n) = Q_n\,(x) \tag{1}$$

called the forward cardinal B-spline, and

$\mu\,(x\,;\,-\tfrac{1}{2}n,\,-\tfrac{1}{2}n+1,\ldots,\tfrac{1}{2}n) = M_n\,(x)$ called the central cardinal B-spline, it is easily shown that

$$Q_n\,(x) = \frac{1}{(n-1)!}\sum_{j=0}^{n}\,(-1)^j\,(x-j)_+^{n-1} \tag{2}$$

and $M_n\,(x) = Q_n\,(x+\tfrac{1}{2}n)$.

(ii) Periodic cardinal B-splines

A cardinal B-spline may be periodically continued on any interval, of length not less than its natural support. We call the resulting function a periodic cardinal B-spline.

Consider an uniform knot-spacing H. Let $B_j\,(H\,;x)$ be the nth order cardinal B-spline (n even) with knots

$(j-\tfrac{1}{2}n)\,H,\ldots,(j+\tfrac{1}{2}n)\,H$, i.e.

$B_j\,(H\,;x) \equiv Q_n\left(\dfrac{x}{H}-j+\tfrac{1}{2}n\right)$ where $Q_n\,(x)$ is given in equation (2). In addition let $MH = 1$ where $M \leqslant N$ is an integral power of 2.

We define the approximating space $B_M (0, 1)$ to be the span of

$$\left[B_j \ (H : x) \right]_{j=0}^{M-1}$$

We can then approximate a function $\phi \in L_2$ (0, 1) by $\phi_M \in B_M$ (0.1)

of the form $\phi_M (x) = \sum_{j=o}^{M-1} \alpha_j \ B_j \ (H : x),$ (3)

we shall assume that each basis function B_j (H ; x) is periodically continued outside the interval (0, 1), w ith period 1.

Then B_j (H ; x) has a Fourier series

$$B_j \ (H ; x) = \sum_{q=-\infty}^{\infty} \hat{B}_{jq} \exp (i w_q \ x) \ \ (\text{where } w_q = 2\pi q)$$

($\wedge$ denotes Fourier Transform and V represents inverse Fourier Transform).

where $\hat{B}_{jq} = \int_o^1 B_j \ (H ; x) \exp (-i w_q \ x) \, dx.$

since B_j (H ; x) is simply a translation of B_o (H ; x) by an amount jH, we have

$$\hat{B}_{jq} = \hat{B}_{o,q} \exp (-i w_q \ jH) \tag{4}$$

$$\text{where } \hat{B}_{o,q} = H \left[\frac{\sin \frac{w_q H}{2}}{w_q \ H/2} \right]^n \tag{5}$$

Clearly ϕ_M (x) in equation (3) will be periodic on (o, 1).

Let $\phi_M = \left[\phi (x_o), \phi (x_1), \dots , \phi (x_{M-1}) \right]^T$, where $X_j = jH$, If $\phi_M(x) \in B_M(0, 1)$ interpolates $\phi(x)$ at $\left[X_j \right]_{j=o}^{M-1}$, then the exact Fourier coefficients $\hat{\phi}_{M,q}$, defined by

$$\hat{\phi}_M = \left[\hat{\phi}_{M,0}, \quad \ldots, \hat{\phi}_{M,M-1} \right]^T, \text{ where } \hat{\phi}_M = \psi_M^H \phi_M,$$

by means of an attenuation factor τ_q :

$$\hat{\phi}^B_{M,q} = \tau_q \, \hat{\phi}_{M,q} \text{ (mod M)}, \; q=o, \pm 1, \pm 2, \ldots$$

For cubic cardinal splines (n = 4) it is shown by stoer [22], Gautschi [11] that

$$\tau_q = \left[\frac{\sin \frac{\pi q}{M}}{\pi q/M} \right]^4 \times \left[\frac{3}{1+2 \cos^2 \left(\frac{\pi q}{M} \right)} \right] \tag{6}$$

2. Pth order Tikhonov Regularization using cardinal cubic B-splines.
We shall approximate the convolution equation

$$\int_{-\infty}^{\infty} K(x-y) \, f(y) \, dy = g(x) \qquad -\infty < x < \infty$$

by

$$\int_0^1 k_N(x-y) \; f_M(y) \, dy = g_N(x) \tag{7}$$

where we assume that f, g and k have essentially finite support in [o, 1], f_M is a cubic spline of the form

$$f_M(x) = \sum_{j=o}^{M-1} \alpha_j \; B_j(H; x), \qquad M \leqslant N \tag{8}$$

and $k_N, g_N \in T_N$ are trigonometric interpolants given by equation

$$g_N(x) = \frac{1}{N} \times \sum_{q=o}^{N-1} \hat{q}_{N,q} \exp(iw_q x)$$

with similar expression for k_N.

The real M dimensional vector

$$\underline{\alpha} = [\alpha_o, \ldots, \alpha_{M-1}]^T$$

of unknown coefficients will be determined, in what follows, The spline in equation (8) has the Fourier series

$$f_M(x) = \sum_{q=-\infty}^{\infty} \hat{f}^B_{M,q} \exp(iw_q x) \tag{9}$$

where

$$\hat{f}^B_{M,q} = \sum_{j=o}^{M-1} \alpha_j \hat{B}_{jq}$$

$$= \hat{B}_{o,q} \sum_{j=o}^{M-1} \alpha_j \exp\left(\frac{-2\pi i}{M} jq\right) \tag{10}$$

$$= \sqrt{M}\, \hat{B}_{o,q}\, \hat{\alpha}_s, \quad S \equiv q \pmod{M} \tag{11}$$

where $\hat{\underline{\alpha}} = \psi^H_M \underline{\alpha}$ (12)

In our work we find it advantageous to determine $\hat{\underline{\alpha}}$ rather than $\underline{\alpha}$, because of the simple properties available in discrete Fourier space. The vector $\underline{\alpha}$ in equation (8) may then be determined from the inverse M-dimensional FFT

$$\underline{\alpha} = \psi_M \hat{\underline{\alpha}} \tag{13}$$

Now following Klein [9] consider the smoothing functional for pth order Tikhonov regularization in the form

$$C(f_M : \lambda) = C(\underline{\alpha}, \lambda) = \left\| \frac{1}{\sigma} \left[k_N(x) * f_M(x) - g_N(x) \right] \right\|_2^2 + \left\| f_M^{(p)} \right\|_2^2 \qquad (14)$$

where $\|\cdot\|$ denotes the inner product norm on L_2 (0.1). Equation (14) demands an explicit knowledge of the variance σ^2 of the noise in the data. In this paper therefore, we assume that σ^2 is known *a priori*. Turchin [24, 25] suggests that σ^2 may be estimated by

$$\sigma_T^2 = \frac{1}{N(N-2l)} \sum_{q=l}^{N-(l+1)} \left| \hat{g}_{N,q} \right|^2, \; l = \frac{N}{4} \qquad (15)$$

and we sometimes use this approximation in our practical examples later.

Since $k_N * f_M \in T_N$ for any square integrable periodic function f_M of period unity, plancherel's theorem gives

$$\left\| \frac{1}{\sigma} (k_N * f_M - g_N) \right\|_2^2 = \frac{1}{N^2 \sigma^2} \sum_{q=-\frac{1}{2}N}^{\frac{1}{2}N}{}'' \left| \hat{k}_{N,q} \hat{f}^B_{M,q} - \hat{g}_{N,q} \right|^2$$

Hance using equation (11)

$$\left\| \frac{1}{\sigma} (k_N * f_M - g_N) \right\|_2^2 = \frac{1}{N^2 \sigma^2} \sum_{q=-N/2}^{N/2} \times$$

$$\left[\left(\sqrt{M}\, \hat{B}_{o,q} \hat{k}_{N;q} \hat{\alpha}_S - \hat{g}_{N,q} \right) \times \left(\sqrt{M}\, \bar{\hat{B}}_{o,q} \bar{\hat{k}}_{N,q} \bar{\hat{\alpha}}_s - \bar{\hat{g}}_{N,q} \right) \right] \qquad (16)$$

where $s \equiv q \pmod{M}$

Also, Plancherel's theorem applied to the regularizing functional in equation (16) gives

$$\left\| f_M^{(P)} \right\|^2 = \sum_{q=-\infty}^{\infty} w_q^{2p} \left| \hat{f}_{M,q}^{B} \right|^2$$

$$= 2 \sum_{q=1}^{\infty} w_q^{2p} \left| \hat{f}_{M,q}^{B} \right|^2$$

$$= 2M \sum_{q=1}^{\infty} w_q^{2p} \hat{B}_{o,q} \left| \hat{\alpha}_s \right|^2 \qquad (17)$$

where $s \equiv q \pmod{M}$

We observe that the expressions (16) and (17) may be reduced to simple expressions involving M terms only.

2.1 The smoothing function when $M = \frac{1}{2} N$

In expressions (16) and (17) we wish to arrange the summations over q to summations over s, where $s \equiv q \pmod{M}$.

$$\hat{P} = \frac{1}{N\sigma} I, \text{ of order } N \times N \qquad (18)$$

$$W^{(1)} = \frac{\operatorname{diag}\left(\sqrt{M}\, \hat{B}_{o,S}\, \hat{K}_{N,S}\right)}{\operatorname{diag}\left(\sqrt{M}\, \hat{B}_{o,M-s}\, \hat{K}_{N,M-S}\right)} \text{ order } N \times M \qquad (19)$$

where $S = o, \ldots, M-1$

From the simple property $\hat{K}_{N,q} = \hat{K}_{N,N-q}$ of discrete FTS it then follows that expression (16) simplifies to

$$\left\| \frac{1}{\sigma} \left(k_N * f_M - g_N \right) \right\|_2^2 = \left\| \hat{P} \left(W^{(1)} \underline{\hat{\alpha}} - \hat{g}_N \right) \right\|_2^2 \qquad (20)$$

The simplification of expression (17) requires the use of an attenuation factor. Since $S \equiv q \pmod{M}$ we may write

$$\left\| f_M^{(P)} \right\|^2 = 2M \sum_{S=1}^{M-1} \left(\left| \hat{\alpha}_S \right| \sum_{n=0}^{\infty} w_{Mn+S}^{2p} \hat{B}_{O,\,Mn+S}^2 \right)$$

$$= 2M \sum_{S=1}^{M-1} \tau_S \left| \hat{\alpha}_S \right|^2 \tag{21}$$

where $$\tau_S = \sum_{n=0}^{\infty} W_{Mn+S}^{2p} \hat{B}_{0,\,Mn+S} \tag{22}$$

$$= (2\pi)^{2p} \sum_{n=0}^{\infty} (Mn+S)^{2p} H^2 \left[\frac{\operatorname{Sin} \frac{\pi(Mn+S)}{M}}{\frac{\pi\,(Mn+S)}{M}} \right]^8$$

$$= (2\pi)^{2P} S^8 \hat{B}_{0,\,S}^2 \sum_{n=0}^{\infty} (Mn+S)^{2p-8} \tag{23}$$

Since $\hat{\alpha}_S = \bar{\hat{\alpha}}_{M-S}$, Equation (21) further simplifies to

$$\left\| f_M^{(p)} \right\|_2^2 = 2M \sum_{S=1}^{M}{}' \left(\tau_S + \tau_{M-S} \right) \left| \hat{\alpha}_S \right|^2 \tag{24}$$

In particular when $p = 2$, from (23) it follows that

$$\tau_S = (2\pi)^4 S^4 \hat{B}_{0,\,S}^2 \sum_{n=0}^{\infty} \left[\frac{S}{Mn+S} \right]^4$$

while $$\tau_{M-S} = (2\pi)^4 S^4 \hat{B}_{0,S}^2 \sum_{n=1}^{\infty} \left[\frac{S}{Mn-S} \right]^4$$

so that $$\tau_s + \tau_{M+S} = (2\pi)^4 S^4 \hat{B}_{0,S}^2 \sum_{n=-\infty}^{\infty} \left(\frac{S}{Mn+S} \right)^4$$

$$= \frac{(2\pi)^4 S^4 \hat{B}_{0,S}^2 \left[1 + 2\cos^2\left(\frac{\pi S}{M}\right)\right.}{3\left(\frac{\sin \frac{\pi S}{M}}{\frac{\pi S}{M}} \right)^4}$$

(see pennisi [20])

$$\tau_S + \tau_{M-S} = \frac{16}{3} M^2 \sin^4\left(\frac{\pi S}{M}\right)\left[1+2\cos^2\frac{(\pi S)}{M}\right] \quad (25)$$

Defining the $M\times M$ matrix

$$W^{(2)} = \text{diag}\{[M(\tau_S + \tau_{M-S})]\} \quad (26)$$

it follows from (24) that

$$\left\| f_M^{(P)} \right\|_2^2 = \left\| W^{(2)} \hat{\underline{\alpha}} \right\|^2 \quad (27)$$

Thus the smoothing functional mey be expressed as

$$C(\underline{\alpha}, \lambda) = \left\| \left(\hat{P} (W^{(1)} \hat{\underline{\alpha}} - \hat{\underline{g}}_N \right) \right\|_2^2 + \lambda \left\| W^{(2)} \hat{\underline{\alpha}} \right\|_2^2 \quad (28)$$

where the first norm is the vector 2-norm in C^N, and the second is in C^M. The minimizer of (28) is clearly

$$\hat{\underline{\alpha}} = (W + \lambda V)^{-1} W^{(1)H} \left(\hat{P}\right)^2 \hat{\underline{g}}_N \quad (29)$$

where

$$\left.\begin{aligned} W &= w^{(1)H} \left(\hat{p}\right)^2 w^{(1)} \\ V &= W^{(2)H} W^{(2)} \end{aligned}\right\} \quad (30)$$

It is not necessary to invert the Matrix $W + \lambda V$ directly since it is diagonal. It follows that

$$\hat{\alpha}_S = \frac{\frac{1}{\sqrt{N}}\left[\hat{B}_{O,S} \hat{k}_{N,S} \hat{g}_{N,S} + \hat{B}_{O,M-S} \hat{k}_{N,M-S} \hat{g}_{N,M+S} \right]}{\left[\hat{B}^2_{O,S} \left| \hat{k}_{N,S} \right|^2 + \hat{B}^2_{O,M-S} \left| \hat{k}_{N,M-S} \right|^2 \right] + N^2 \sigma^2 \lambda \left(\tau_S + \tau_{M-S} \right)} \quad (31)$$

$$\hat{\alpha}_S =$$

$$\frac{\frac{1}{\sqrt{M}}\left[\hat{B}_{O,S}\left[\bar{\hat{k}}_{N,S}\,\hat{g}_{N,S}+\left(\frac{S}{M-S}\right)^4\,\hat{k}_{N,M-S}\,\bar{\hat{g}}_{N,M-S}\right]\right]}{\hat{B}_{O,S}\left[\left|\hat{k}_{N,S}\right|^2+\left(\frac{S}{M-S}\right)^8\left|\hat{k}_{N,M-S}\right]^2+N^2\sigma^2\lambda\left[\tau_S+\tau_{M-S}\right]} \tag{32}$$

Since $\hat{B}_{0,\,M-S} = \left(\frac{S}{M-S}\right)^4 \hat{B}_{0,S}$

we can easily verify that

$$\hat{\alpha}_S = \bar{\hat{\alpha}}_{M-S}$$

we that the inverse FFT $\underline{\alpha} = \psi_M \underline{\hat{\alpha}}$ is a real vector as required.

2.2 The Filter for Cardinal B-spline Regularization

In the trigonometric regularization the discrete Fourier coefficients of the filtered solution $\hat{f}_{N,q,\lambda}$ were related to $\hat{g}_{N,q}$ and $\hat{K}_{N,q}$ by means of a simple filter $Z_{q\,:\,\lambda}$ given by

$$\hat{f}_{N,q,\lambda} = Z_{q\,;\,\lambda}\,\frac{\hat{g}_{N,q}}{\hat{K}_{N,q}},\ q = o, \ldots, N-1 \text{ (see [7])}.$$

A similar filter can be derived for cardinal B-spline regularization.

The Fourier coefficients of the regularized (filtered) solution $f_M(x) \in B_M(0, 1)$ clearly depend on λ through enquations (10), (13) and (32). In equation (32) we denote the dependence of

$\hat{\alpha}_S$ on λ by writing $\hat{\alpha}_S = \hat{\sigma}_S(\lambda)$. Thus the Fourier coefficients of the filtered solution are

$$f^{B}_{M,q}(\lambda) = \sqrt{M}\ \hat{B}_{0,q}\ \hat{\alpha}_S(\lambda), \quad S \equiv q \pmod{M}.$$

whereas those of the unregularized (unfiltered) solution are

$$\hat{f}^{B}_{M,q}(0) = \sqrt{M}\ \hat{B}_{0,q}\ \hat{\alpha}_S(0)$$

clearly the underlying filter $Z_{q;\lambda}$ must satisfy

$\hat{f}^{B}_{M,q}(\lambda) = Z_{q:\lambda}\ \hat{f}^{B}_{M,q}(0)$, so that we can deduce

$$Z_{q:\lambda} = \frac{\hat{\alpha}_S(\lambda)}{\hat{\alpha}_S(0)} \tag{3}$$

$$Z_{q;\lambda} = \frac{\left[\hat{B}^2_{0,S}\left[\left|\hat{k}_{N,S}\right|^2 + \left(\frac{S}{M-S}\right)^8\left|\hat{k}_{N,M-S}\right|^2\right]\right]}{\hat{B}^2_{0,S}\left[\left|\hat{k}_{N,S}\right|^2 + \left(\frac{S}{M-S}\right)^8\left|\hat{k}_{N,M-S}\right|^2\right] + N^2\sigma^2\lambda(\tau_S + \tau_{M-S})} \tag{34}$$

The filter will of course apply to every Fourier coefficient, $q = 0, \pm 1, \pm 2, \ldots$ but will have only M possible values depending on q modulo M.

3. Calculation of λ by the TK method

Klein [9] has uased ntural splines to solve integral equations of the first kind and it is possible to modify this theory for the approximating space $B_M(0, 1)$.

Klein considers the *a priori* c.d.f.

$$P(\underline{g}/\lambda) = \int_{R^N} P(\underline{g} \mid \underline{\alpha})\, P_\lambda(\underline{\alpha})\, d\,\underline{\alpha} \tag{35}$$

for the data vector $\underline{g}$ given λ. Baye's theorem then gives a *posteriori* c.d.f.

$$P(\lambda/\underline{g}) = \text{const. } P(\underline{g}/\lambda)\, P(\lambda) \tag{36}$$

in terms of an unknown *a priori* p.d.f. P (λ) for λ.

Following Klein it can be shown that

$$P(\underline{g}/\lambda = \left[\det \frac{1}{2\pi} \hat{P}^2 \quad \det \left\{ \lambda W \, (W + \lambda v)^{-1} \right\} \right] \times$$

$$\exp\left[-\tfrac{1}{2} C(\hat{\underline{\alpha}}, \lambda) \right] \tag{37}$$

where $\hat{P}$, W and V are defined by equations (18) and (30) respectively and $C(\hat{\underline{\alpha}} : \lambda)$ is given by equation (28). Substituting into equation (35), we find that a condition for a stationary point of $P(\lambda / \underline{g})$ is

$$\frac{d}{d\lambda} [\log P(\lambda)] + \text{Trace}\,[W (W + \lambda V)^{-1}] - \lambda \hat{\underline{\alpha}}^H \underline{v}\, \hat{\underline{\alpha}} = 0 \tag{38}$$

where $\hat{\underline{\alpha}}$ is defined in equations (29) and (32). An optimal value of λ minimizes $P(\lambda / \underline{g})$ and is therefore a root of the non-linear equation (38) in λ,

Klenin argues that if the unknown distribution P (λ) is sufficiently "narrow" then the effect of the first term in equation (38) on the root will be small. In practice, therefore, we must neglect the first term and determine λ by solving.

$$\text{Trace}\,[W (W + \lambda v)^{-1}] - \lambda \hat{\underline{\alpha}}^H \underline{v}\, \hat{\underline{\alpha}} = 0 \tag{39}$$

From equations (29—32) and (34) we can rewrite equation (39) in the from $F(\lambda) = S_1(\lambda) - \lambda S_2(\lambda) = 0$ (40)

where $S_1(\lambda) = \sum_{S=0}^{M-1} Z_{S:\lambda}$ and

$$S_2(\lambda) = \sum_{S=0}^{M-1} (\tau_S + \tau_{M-S}) Z^2_{S;\lambda} \times$$

$$\left[\frac{\bar{\hat{K}}_{N,S}\ \hat{g}_{N,S} + \left(\frac{S}{M-S}\right)^4 \hat{K}_{N,M-S}\ \bar{\hat{g}}_{N,M-S}}{\left|\hat{K}_{N,S}\right|^2 + \left(\frac{S}{M-S}\right)^8 \left|\hat{K}_{N,M-S}\right|^2} \right] \qquad (41)$$

where $\tau_S + \tau_{M-S}$ is given by equation (25).

In all our numberical experiments (next section) we have found that the non-linear function F (λ) has the form given in figure 1,

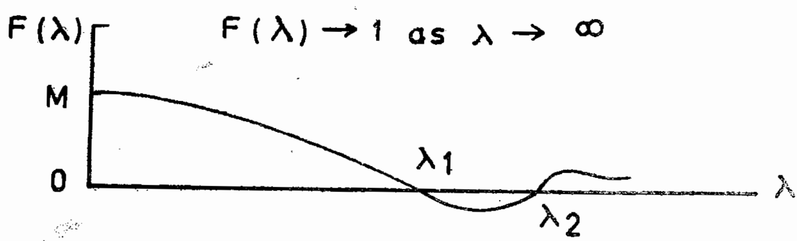


Figure 1.

with the properties $F(0) = M$ and $F(\lambda) \to 1$ as $\lambda \to \infty$. since $S_1(\lambda) \to 1$ and $\lambda S_2(\lambda) \to 0$ of the two roots λ_1, λ_2 we have found empirically that the smallest root λ_1 is the most appropriate root to use for the regularization. The roots were computed using the successive approximation.

$$\lambda^{(r+1)} = \frac{S_1(\lambda^{(r)})}{S_2(\lambda^{(r)})} \qquad (42)$$

with $\lambda^{(0)} = 0$ for the first root and $\lambda^{(0)} >> \lambda_1$ for the second root.

4. Some Test Problems

In this section we introduce three test problems :

P (1A), P (1B) and P (1C).

Problem P (1A): This example is given by Turchin [24, 25], i.e.

$$\int_{-2}^{2} k(x-y)\, f(y)\, dy = g(x),$$ where f is the sum of two Gaussian functions

$$f(x) = (0.5) \exp\left[-\frac{(x+0.4)^2}{0.18}\right] + \exp\left[-\frac{(x+0.6)^2}{0.18}\right]$$

with essential support $-1.3 < x < 1.5$.

By the essential support of a function f (x) we mean that part of its domain for which $|f(x)| < \epsilon$ where $\epsilon > 0$ and small, e.g.=1% of $\max\{|f(x)|\}$. K (x) is triangular with eqnation

$$K(x) = \begin{cases} -x + 0.5, & 0 \leqslant x < 0.5 \\ x + 0.5, & -0.5 \leqslant x < 0 \\ 0 & |x| \geqslant 0.5 \end{cases}$$

We calculated the values of g (x) by the NAG Algorithm DOIACA with accuracy 10^{-7} using extended precision. The essential support of g (x) is $-1.8 < x < 2.0$. In diag (1) the functions f, g and K′ are ploted with grid spacing 0.1.

Problem P (1B). This example is the same as P (1A) except the Triangular Kernel is made wider.

$$K(x) = \begin{cases} (5/8)(-x + 0.8), & 0 \leqslant x < 0.8 \\ (5/8)(x + 0.8), & -0.8 \leqslant x < 0 \\ 0, & |x| \geqslant 0.8 \end{cases}$$

The wider Kernel makes the problem more ill-posed. The essential support of g (x) is $-2.1 < x < 2.3$. The functions are displayed in DIAG (2) with a spacing 0.1.

Problem P (IC) : The problem is mede highly ill-posed by choosing an even wider Kernel.

$$K(x) = \begin{cases} (5/12)(-x + 1.2), & 0 \leqslant x < 1.2 \\ (5/12)(x + 1.2), & -1.2 \leqslant x < 0 \\ 0 , & |x| \geq 1.2 \end{cases}$$

The essential support of g (x) is $-2.5 < x \leqslant 2.7$.

The functions are displayed in DIAG (3) with a spacing 0.1.

4.1 Addition of Random Noise to the Data Functions

In solving the test problems we have considered the data functions contaminated by varying amounts of random noise. To generate sequences of random errors of the from $\{\epsilon_n\}$, for $n=0, 1, \ldots, N-1$.

We have used the NAG Algorithm G05DDA which returns pseudo-random real numbers taken from a normal distribution, of prescribed mean A and standard deviation B. To mimic experimental errors we have A = 0

$$B = \frac{x}{100} \quad \max_{0 \leqslant n \leqslant N-1} |g_n| \tag{43}$$

where x denotes a chosen percentage, e.g.

$x = 0.7, 1.7, 3.3$ and 6.7

Thus the random error ϵ_n added to g_n does not exceed 3x% of the maximum value of g (x).

4.2 Numerical Results

In this section we describe the numerical results. We have obtained for each of the above test problems using the cardinal B–spline theory developed in section 2. We calculate optimal λ in each case using the TK method of Section 3. For completeness we consider the two cases M (number of splines)=N (Number of data points), and $M=\frac{1}{2}N$, theory for $M=\frac{1}{2}N$ is a trivial modification of that given in section 2. Throughout we use second order regularization exclusively, i.e. p=2, and we compared the numerical results in both cases.

For each problem the number of data points N is chosen to be 64 by introducing zero values of K and g if necessary, then supp (g) is mapped onto (0, 1). In each case the values of σ is estimated by σ_T in equation (15). A summary of the results is given in Table 1.

In each case the parameter $\lambda_1 \; \sigma_T^2$ measures the strength of the corresponding filter and the quality of the regularized solution may be seen from the appropriate diagram.

The error norm $\| f - f_M \|_2$ estimated from

$$\left\{ \frac{1}{N} \sum_{j=0}^{N-1} \left[f(x_j) - f_M(x_j) \right]^2 \right\}^{1/2}$$

also serves as a measure of the quality of the regularized solution,

In P (1A) excellent solutions are obtained for noiseless data for both TK (M=64) and TK (M=32), with similar results for noisy data upto a level of 3.3% as shown in DIAGS (4 AND 5).

(In P (1B) and P (1C) good solutions are obtained for noiseless data. In the highly ill-posed case P (1C), a slightly better solution is obtained for M=32 rather than M=64 with noisy data, but large osicillatory lobes are present in both cases, as shown in DIAGS (6—9).

Conclusion

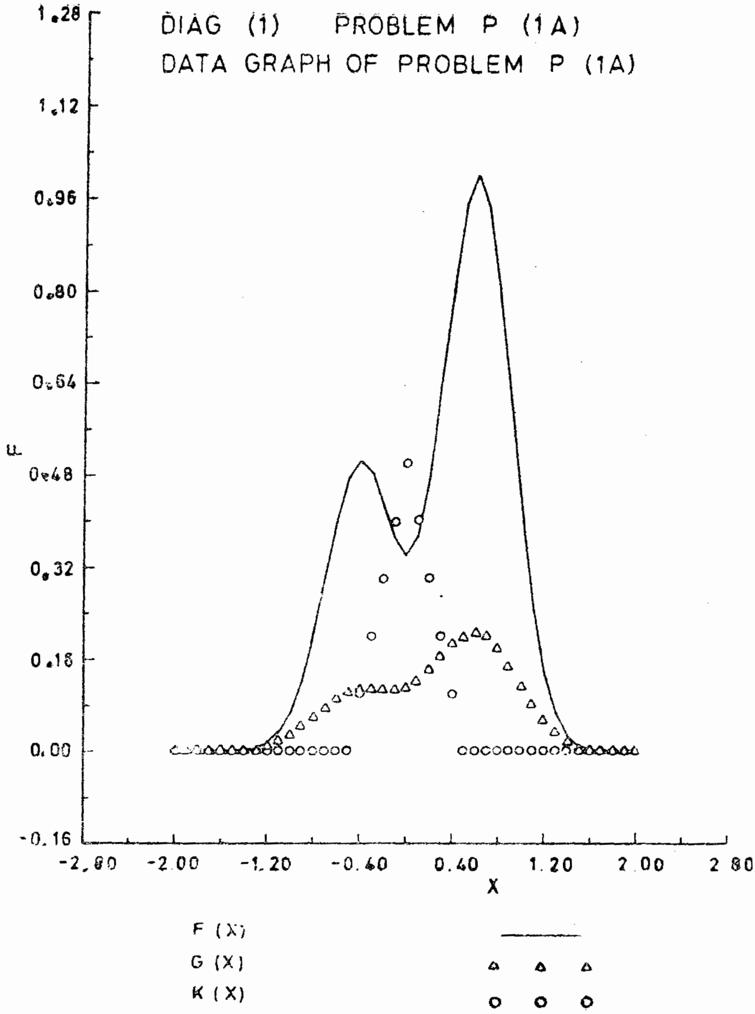
The methods worked very well for Mildly, moderately and severely ill-posed problems with and without random noise. The results can be compared by looking at the respective diagrams and Table 1.

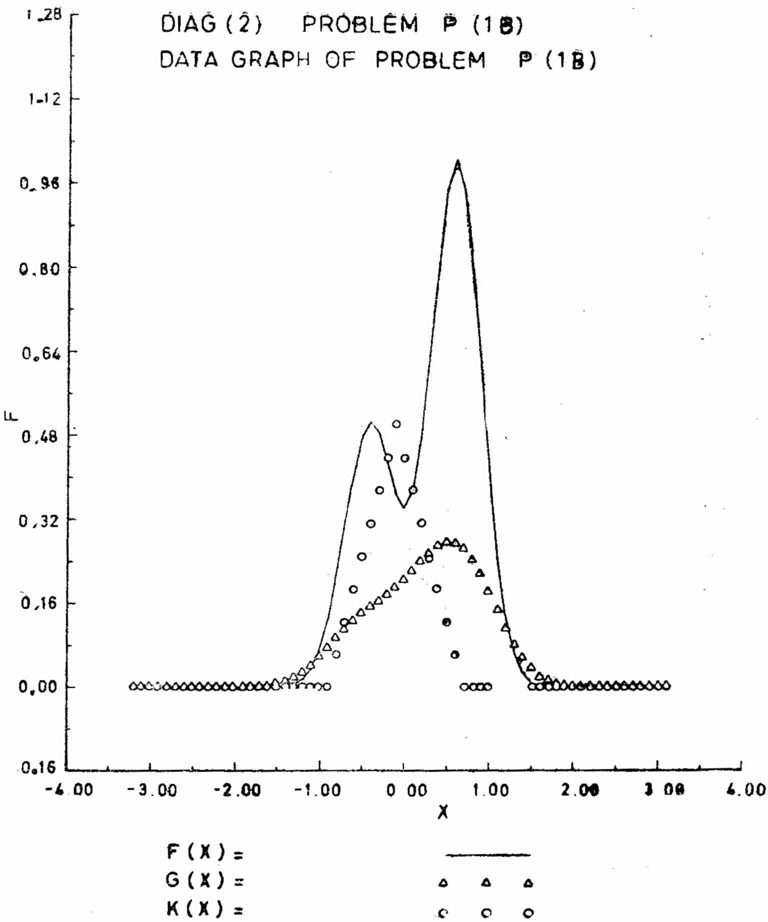
Acknowledgements

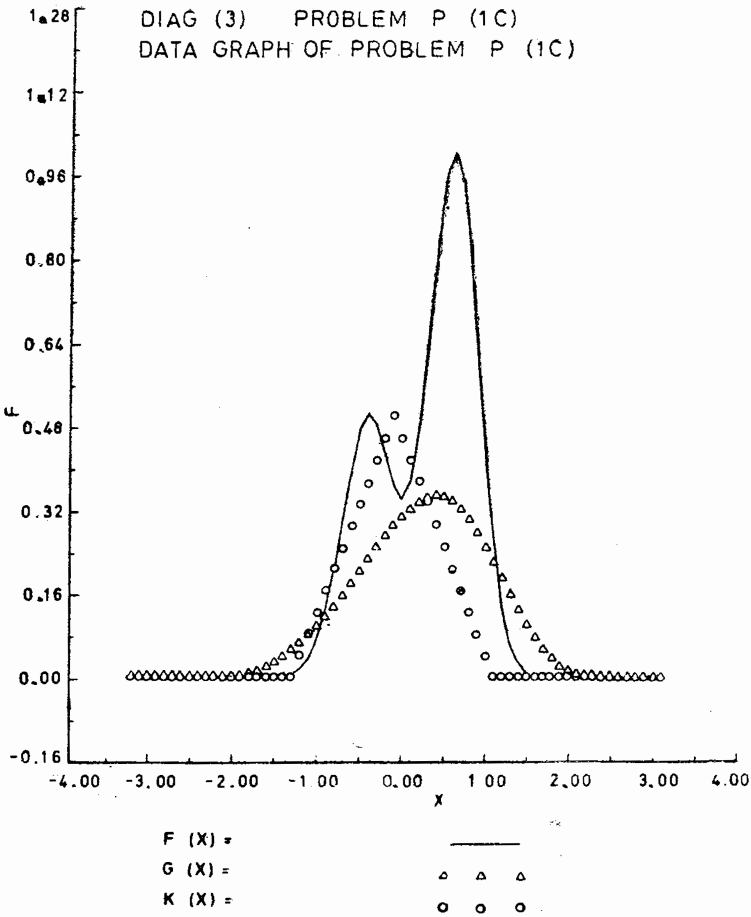
The author is very greatful to his supervisor Dr. A.R. Davies, who has suggested the problems and helped a lot in developing the theory of the method.

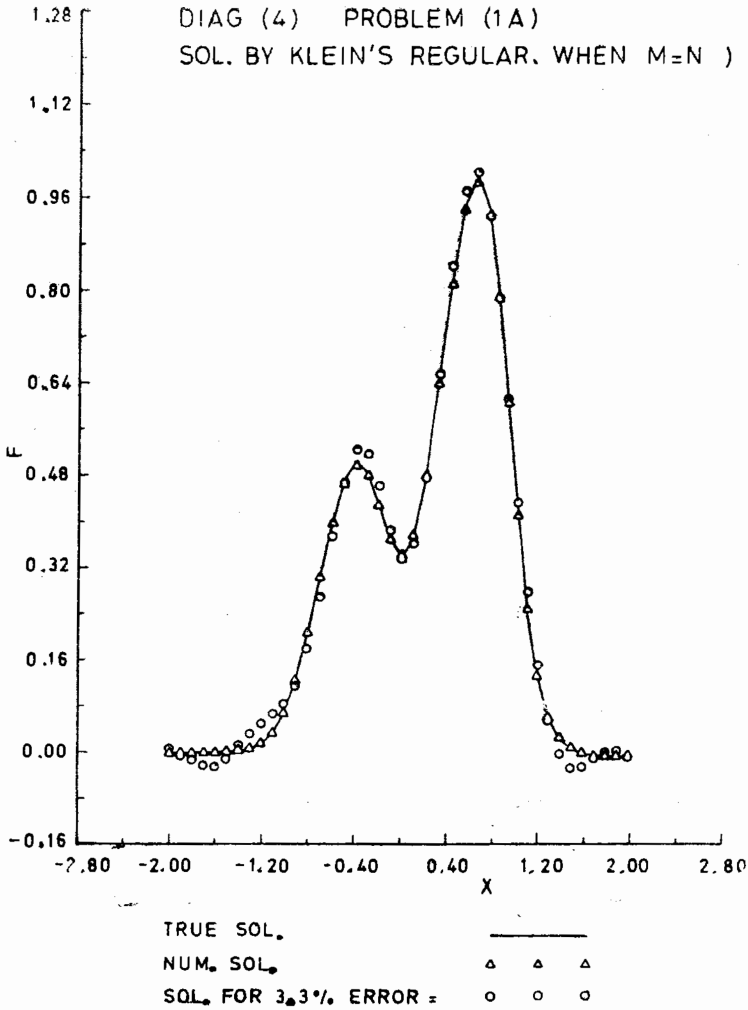
TABLE 1

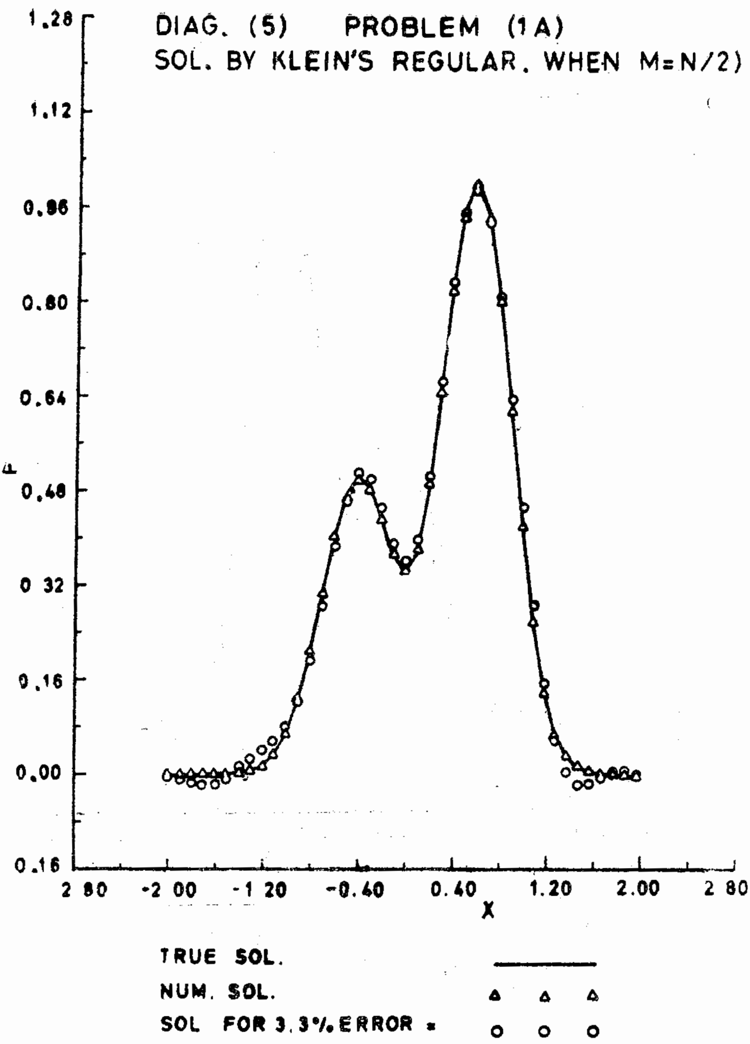
Problem	N	Noise level	σ	σ_T	B_M: $M = N$ $\lambda_1 \sigma_T^2$	B_M: $M = N$ $\|f - f_M\|_2$	B_M: $M = N$ DIAG	B_M: $M = \frac{1}{2} N$ $\lambda_1 \sigma_T^2$	B_M: $M = \frac{1}{2} N$ $\|f - f_M\|_2$	B_M: $M = \frac{1}{2} N$ DIAG
P (IA)	64	0.0%	0.0	2.60 10^{-6}	1.50 10^{-8}	9.29 10^{-3}	4	2.78 10^{-8}	1.01 10^{-2}	5
		3.3%	6.83 10^{-3}	6.09 10^{-3}	2.51 10^{-4}	5.27 10^{-2}		5.85 10^{-4}	4.41 10^{-2}	
P (IB)	64	0.0%	0.0	3.30 10^{-6}	2.55 10^{-8}	3.06 10^{-2}	6	4.92 10^{-8}	3.00 10^{-2}	7
		1.7%	4.68 10^{-3}	5.57 10^{-3}	3.05 10^{-5}	1.04 10^{-1}		6.10 10^{-5}	1.04 10^{-1}	
P (IC)	64	0.0%	0.0	2.20 10^{-6}	1.58 10^{-8}	2.20 10^{-2}	8	3.15 10^{-8}	9.38 10^{-3}	9
		0.7%	2.44 10^{-3}	2.42 10^{-3}	1.54 10^{-5}	1.54 10^{-1}		1.21 10^{-5}	1.63 10^{-1}	


DIAG (1) PROBLEM P (1A)
DATA GRAPH OF PROBLEM P (1A)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
-2.80
-2.00
-1.20
-0.40
0.40
1.20
2.00
2.80
X
F (X)
G (X)
K (X)

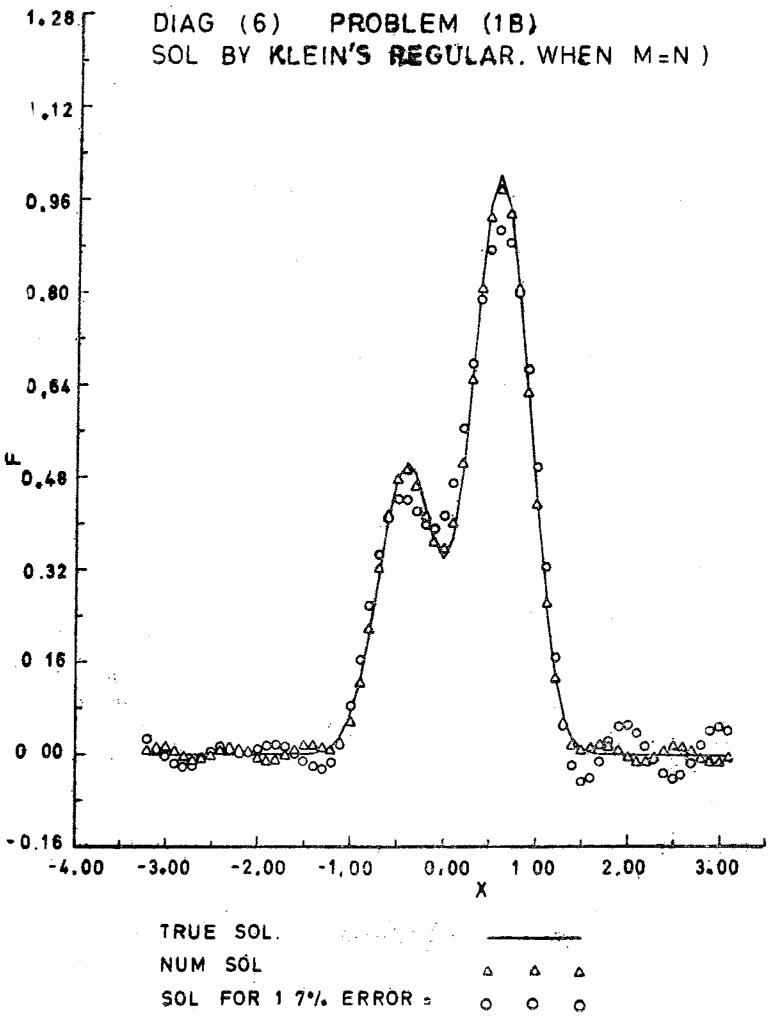

DIAG (2) PROBLEM P (1B)
DATA GRAPH OF PROBLEM P (1B)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
0.16
F
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
X
F(X) =
G(X) =
K(X) =

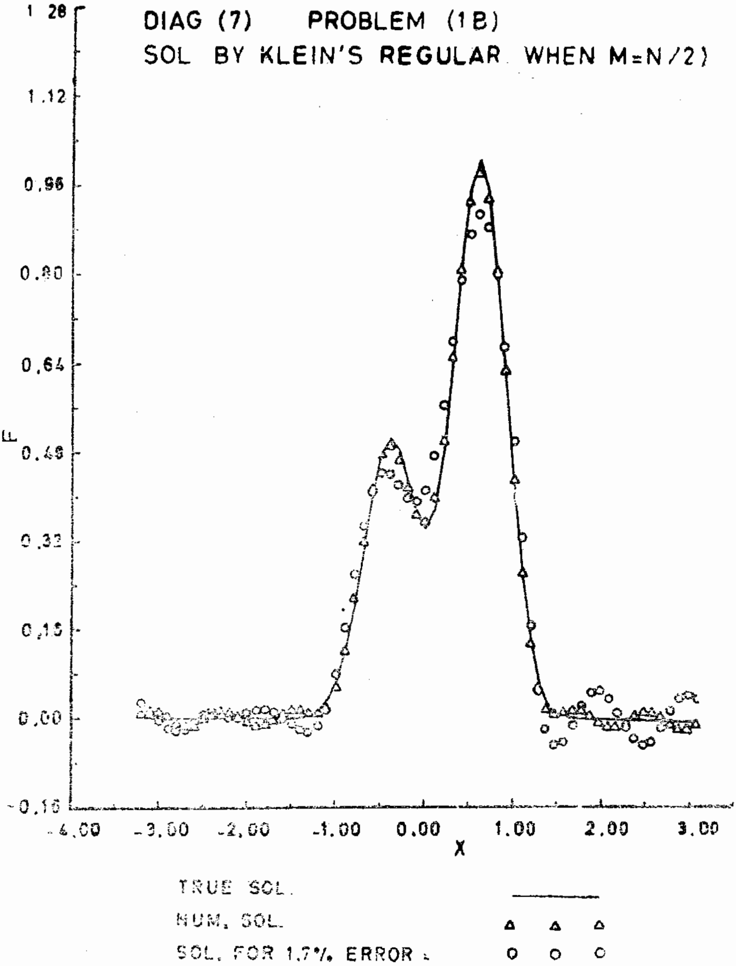

DIAG (3) PROBLEM P (1C)
DATA GRAPH OF PROBLEM P (1C)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
F
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
X
F (X) =
G (X) =
K (X) =

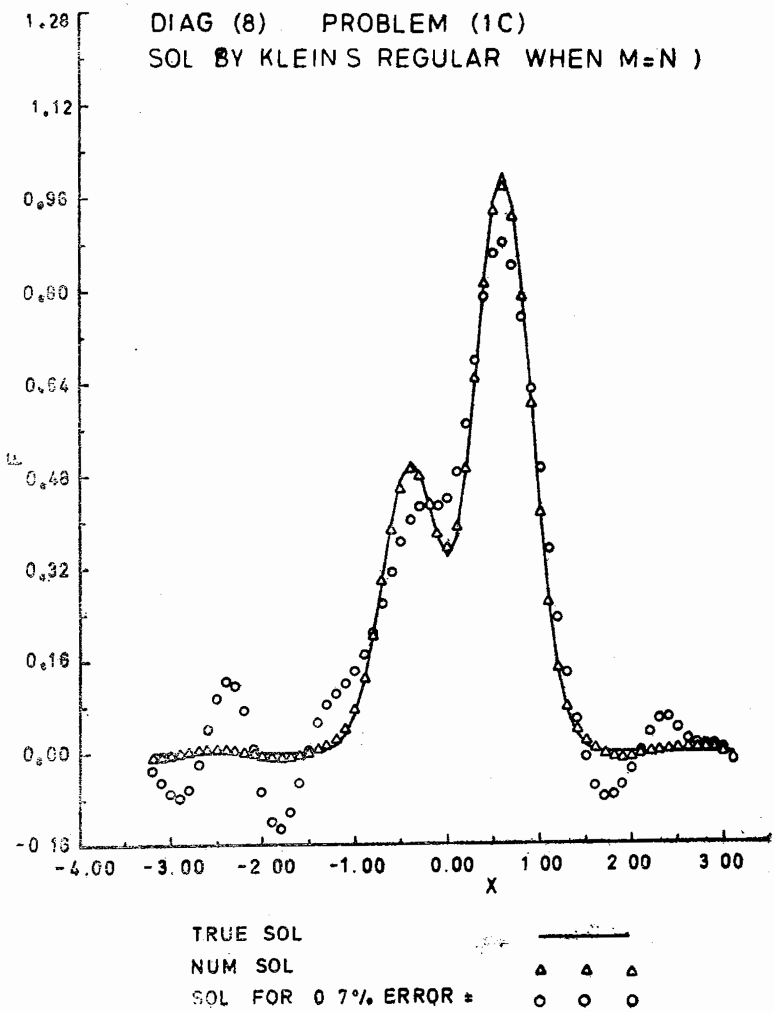

DIAG (4) PROBLEM (1A)
SOL. BY KLEIN'S REGULAR. WHEN M=N)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
F
-2.80
-2.00
-1.20
-0.40
0.40
1.20
2.00
2.80
X
TRUE SOL.
NUM. SOL.
SOL. FOR 3.3% ERROR =

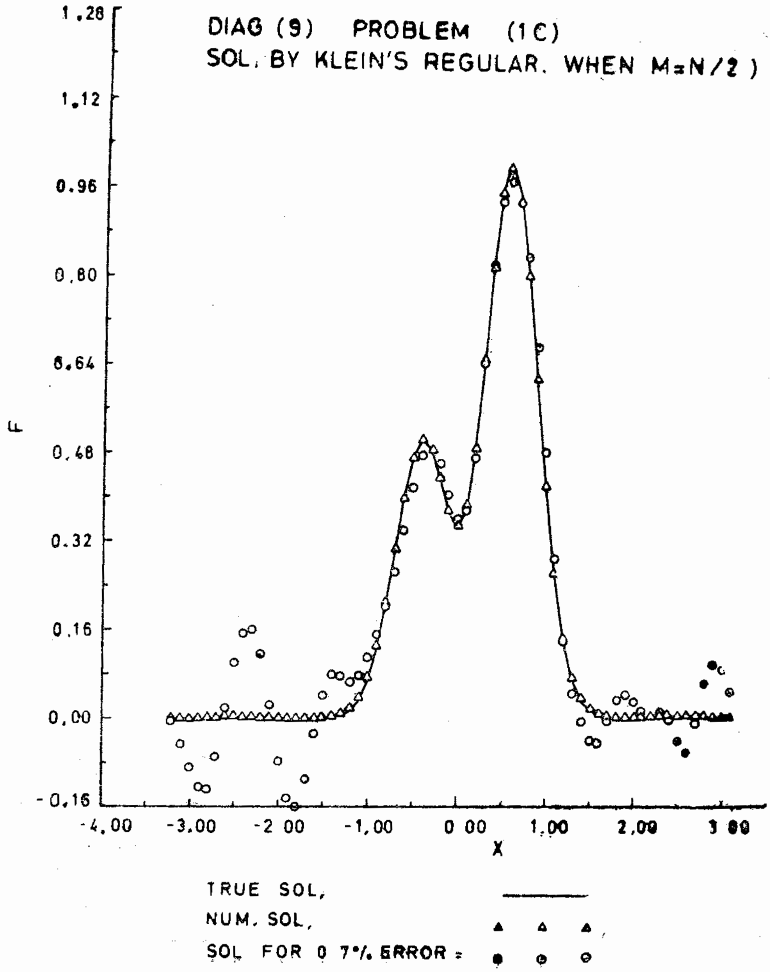


DIAG. (5) PROBLEM (1A)
SOL. BY KLEIN'S REGULAR. WHEN M=N/2)
1.28
1.12
0.96
0.80
0.64
0.48
0 32
0.16
0.00
0.16
F
2 80
-2 00
-1 20
-0.40
0.40
1.20
2.00
2 80
X
TRUE SOL.
NUM. SOL.
SOL FOR 3.3%ERROR =


DIAG (6) PROBLEM (1B)
SOL BY KLEIN'S REGULAR. WHEN M=N)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
F
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
X
TRUE SOL.
NUM SOL
SOL FOR 1 7% ERROR =


DIAG (7) PROBLEM (1B)
SOL BY KLEIN'S REGULAR WHEN M=N/2)
F
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
X
TRUE SOL.
NUM. SOL.
SOL. FOR 1.7% ERROR


DIAG (8) PROBLEM (1C)
SOL BY KLEIN S REGULAR WHEN M=N)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0 16
-4.00
-3.00
-2 00
-1.00
0.00
1 00
2 00
3 00
X
TRUE SOL
NUM SOL
SOL FOR 0 7% ERROR =


DIAG (9) PROBLEM (1C)
SOL. BY KLEIN'S REGULAR. WHEN M=N/2)
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
F
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
X
TRUE SOL,
NUM. SOL,
SOL FOR 0 7% ERROR =

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NUMERICAL COMPUTATION OF TWO-DIMENSIONAL TIDAL FLOWS

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Abstract

A two-dimensional explicit finite difference hydrodynamical numerical model is developed to reproduce the tidal elevation, tidal current and residual current in the Bay of Sonmiani and Indus Delta, In this study, the tidal constituents on the open boundary are based on the results of a coarser grid model. The model solves the phenomena of M_2, S_2, K_1 and O_1–tide separately. Combined effect of tides in terms of tidal current is also studied. The results fill the existing gap of observational measurement in the area and provide the wave structure.

Introduction

Due to non-linearities in the hyperbolic partial differential equations describing the hydrodynamical processes, it is extremely difficult to find their analytical solution. If, at all, a solution is obtained, it requires a lot of simplification. However, with the development of numerical techniques, it became possible to solve these processes in complete form.

In modelling of free-surface flow, numerical models provide a very successful alternative to expensive physical hydraulic models. Numerical models are low cost, efficient and offer almost unlimited flexibility in simulation of various alternatives. They are applied to simulate transport phenomena of heat, waste, sediment, wind-driven circulation, short waves and tidal flow processes.

In the present study, tidal level and tidal currents will be reproduced because they are of the prime importance for problems of coastal protection, navigation and dredging. Tidal current data is used for the design of marine offshore structures, such as jetties, breakwaters and offshore platforms. They also provide the essential information for studies on pollutant movement.

For the offshore coastal areas of Third World countries very poor observational data about tidal levels and tidal current is available, because measurement of tidal levels and tidal current is technically complicated, laborious and costly. The hydrodynamical-numerical method is used to fill this gap by reproduction of the data.

The hydrodynamical-numerical method (Hansen, 1956) consists of the Navier-Stokes equation and equation of continuity. These equations are solved with the explicit finite-difference technique. These equations include the effects of bottom friction, rotation of earth and atmospheric pressure gradient. In the present study, this method is applied to the area with scarce observational data required for running the model. Results of the hydrodynamical-numerical model of the Northern Arabian Sea (Elahi, 1977) are used as the boundary values on the open boundaries. They are used to generate tide from the sea onto the model area, which contains the Sonmiani Bay and the Indus Delta.

The model is a simulation model. Its aim is to compute physically realistic elevations and currents which can subsequently be compared, when possible, with observations.

Discription of the Model

The surface elevation together with the vertically-averaged currents are computed from the Navier-Stokes equation and equation of continuity. The equations are vertically integrated over depth.

The volume transport vector is defined by

$$\mathbb{V} = \langle U, V \rangle$$

$$U = \int_{-h}^{\zeta} u\,dz\,, \quad V = \int_{-h}^{\zeta} v\,dz. \qquad (1)$$

The mean velocity is obtained by averaging the volume transport over depth

$$\bar{\mathbb{V}} = < \bar{U},\bar{V} >, \frac{V}{h+\zeta} = \frac{< U,V >}{h+\zeta} . \tag{2}$$

The vertically-integrated equations are not completely independent of the vertical structure of flow, since the latter enters in the bottom stress terms. The system of equations which has been employed in the model is

$$\frac{\partial \zeta}{\partial t} = - \frac{1}{R \cos \phi} \left[\frac{\partial (HU)}{\partial \lambda} + \frac{\partial (HV \cos \phi)}{\partial \phi} \right] \tag{3}$$

$$\frac{\partial v}{\partial t} = fv + \frac{A_h}{R^2} \left[\frac{1}{\cos^2 \phi} \frac{\partial^2 v}{\partial \lambda^2} \tan \phi \frac{\partial v}{\partial \phi} + \frac{\partial^2 v}{\partial \phi^2} \right] - \frac{g}{R \operatorname{cost} \phi} \frac{\partial \zeta}{\partial \lambda} - \tau_b^\lambda \tag{4}$$

$$\frac{\partial v}{\partial t} = fv + \frac{A_h}{R^2} \left[\frac{1}{\cos^2 \phi} \frac{\partial^2 v}{\partial \lambda^2} - \tan \phi \frac{\partial v}{\partial \phi} + \frac{\partial^2 v}{\partial \phi^2} \right] - \frac{g}{R} \frac{\partial \zeta}{\partial \lambda} - \tau_b^\phi , \tag{5}$$

where λ, ϕ are geographical longitude and latitude, Z is depth below the undisturbed surface, t is time, ζ is elevation of the sea surface above the undisturbed level, h is undisturbed depth of water, R is the radius of the earth, $f=2\omega \sin \phi$ is coriolis parameter, ω is angular speed of the earth's rotation, g is the acceleration due to gravity, U,V are eastward and northward vertically integrated components of velocity at depth 2 and A_h is coefficient of horizontal eddy viscosity.

The stresses due to roughness of bottom $\tau_b = [\tau_b^\lambda , \tau_b^\phi]$ are parametrized empirically by using the Newton-Taylor formulation (G. I. Taylor, 1919), which is a quadratic law relating bottom stress to the depth mean current.

$$\tau_b^\lambda = rH^{-1} U (U^2 + V^2)^{\frac{1}{2}} \tag{6}$$

$$\tau_b^\phi = rH^{-1} U (U^2 + V^2)^{\frac{1}{2}} ,$$

where $r = .003$ is the bottom friction coefficient.

The coefficient of horizontal eddy viscosity is related to grid size and time step of the numerical scheme by the relation

$$A_h = \frac{1-\alpha}{4} \cdot \frac{\Delta l}{\Delta t} \, . \tag{8}$$

α takes value between 0.9 and 1, depending upon the inner viscosity of the fluid (Sündermann, 1966). Through numerical experimentation, its numerical value is estimated 0.99 in shallow water areas and 0.9 in deep water area.

The tides in the Arabian Sea are supported predominently by the semidiurnal constituents M_2 and S_2 with small contributions from the diurnal constituents K_1 and O_1 . The tide in the model is generated by prescribing amplitudes and phases of tidal constituents at open boundaries, Water levels as functions of time for 4 main tidal constituents, M_2 , S_2 , K_2 and O_1 are supposed to be known and calculated by

$$\zeta(t) = \sum_{i=1}^{4} A_i \cos(\sigma_i t - k_i),$$

where A_i is amplitude, K_i is phase of incoming tide and σ_i is frequency.

Partial tide		σ frequency, 10^{-4} / S
	Semidiurnal	
M_2	Principal lunar	1.40519
S_2	Principal solar	1.45444
	Diurnal	
K_1	Declination lunar-solar	0.72921
O_1	Principal lunar	0.67598

Table 1 : Partial tides and their frequencies,

Initial and Boundary Conditions

The system of equations (3 – 5) can be solved only when the initial and boundary conditions are added to the system.

It is assumed that the ocean is initially at rest, so water levels and water currents are assumed to be zero everywhere at time t=0,

$$\zeta = U = N = 0. \tag{10}$$

Along the coastal boundaries no slip condition is applied. Thus at coastal points, the normal and tangential components of velocity (V_n , V_t) are assumed to be zero

$$V_n = V_t = 0. \tag{11}$$

On the open boundaries, the tidal elevation must be given at every time step by using equation (9). Moreover, the velocity gradients in the normal direction are zero, i.e.,

$$\frac{\partial V}{\partial n} = 0. \tag{12}$$

Numerical Model

An explicit finite difference scheme (Hansen, 1956) is used to solve the equations numerically. In equations 3, 4 and 5, partial derivatives with respect to time are replaced by forward difference and partial derivatives with respect to space coordinates are replaced by central difference. The method suffers from certain short comings due to square grid. It requires rather fine mesh size in order to give a satisfactory representation of complicated and distorted boundary lines, this leads to high computational effort. Replacement of the time derivative by forward finite difference imposes an additional condition, a fine mesh size requires small time step for stability of numerical scheme. If the region under study is large, then the method is not appropriate because it needs large computer time and storage capacity. A variable finite difference feature is introduced to allow detailed study of the tidal flows in certain confined subregions. The variable mesh feature provides great flexibility in analyzing the region. If a detailed analysis of the tidal flows in a certain sub-region is required, results can be obtained without significant restrictions imposed by computer time or memory capacity (Lardner, Belen and Celerge, 1982).

Tidal charts for the Arabian Sea were developed based upon observational data. This information was used as the driving force on the open boundaries of the Northern Arabian Sea model with the coarsest mesh size 54 km. The results are given in Figs. 1–4. An

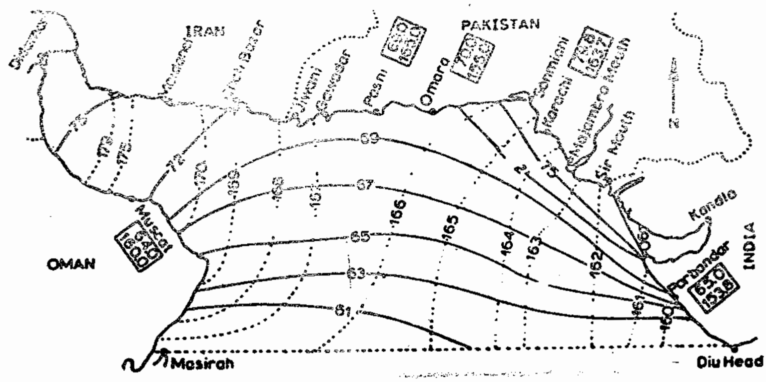

Fig. 1. M_2 —tide in the Northern Arabian Sea. Co-tidal (———) and co-range (……) lines. The boxes show the observed values, amplitudes in cm (upper) and phases in degree (lower).

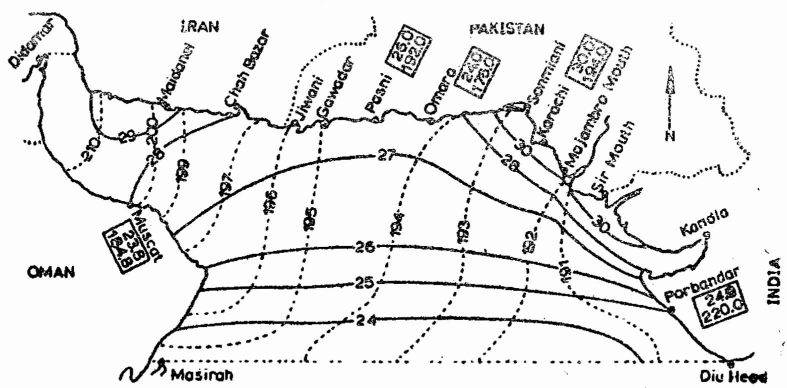

Fig. 2. S_2 —tide in the Northern Arabian Sea.

intermediate mesh size is used for the shallow water region of Pakistan coastal water and the finest mesh size 9 km is used for the Sonmiani

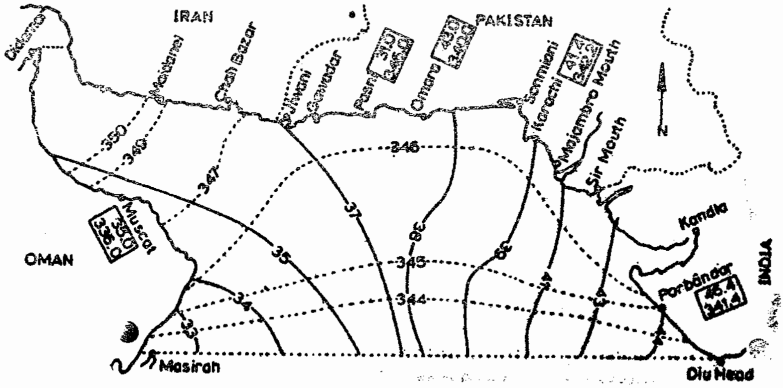

Fig. 3. K_1 – tide in the Northern Arabian Sea.

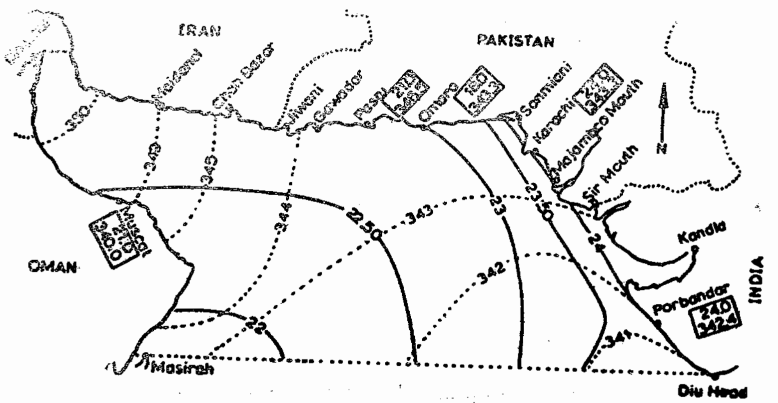

Fig. 4. O_1 – tide in the Northern Arabian Sea.

Bay – Indus Delta area. The driving force, in each confined sub-region, is the previously computed elevation in a coarser model.

Finite Differences Formation

Equations (3 – 5) are discretized in time and space into a set of finite difference equations by using the method of forward time-central

(FTCS) explicit differencing.

$$U^{(t+\triangle t)}(N, M) = \left(1 - r\triangle t \, \frac{U^{(t)}(N,M)^2 + V_u^{(t)}(N,M)^2}{HU^{(t)}(N,M)}\right) U(N,M)$$

$$+ 2\omega\triangle t \sin(\phi(N)) V_u^{(t)}(N, M)$$

$$+ A_h \triangle t \nabla^2 U^{(t)}(N, M) + g\triangle t \, \frac{\zeta^{(t+\triangle t/2)}(N,M) - \zeta^{(t+\triangle t/2)}(N,M+1)}{R\cos(\phi(N))} \tag{13}$$

$$U^{(t+\triangle t)}(N,M) = \left(1 - r\triangle t \, \frac{V^{(t)}(N,M)^2 + U_v^{(t)}(N,M)^2}{HV^{(t)}(N, M)}\right) V(N, M)$$

$$-2\omega\triangle t \sin(\phi(N)) U_v^{(t)}(N, M)$$

$$+ A_h \triangle t \nabla^2 V^{(t)}(N, M) - g\triangle t \, \frac{\zeta^{(t+\triangle t/2)}(N, M) - \zeta^{(t+\triangle t/2)}(N+1, M)}{R\triangle\phi} \tag{14}$$

$$\zeta^{(t+\triangle t/2)}(N, M) = \zeta^{(t-\triangle t/2)}(N, M) + \frac{\triangle t}{R\cos(\phi(N))}$$

$$\left(\frac{HU^{(t)}(N, M)\, U^{(t)}(N, M) - HU^{(t)}(N, M-1)\, U^{(t)}(N, M-1)}{\triangle\lambda}\right.$$

$$HV^{(t)}(N,M)\, V^{(t)}(N, M)\cos\left(\phi(N) - \frac{\triangle\phi}{2}\right)$$

$$\left. - \frac{- HV^{(t)}(N-1,M)\, V^{(t)}(N-1,M)\cos(\phi(N-1))}{\triangle\phi}\right), \tag{15}$$

where

$$HU^{(t)}(N, M) = hU(N, M) + \tfrac{1}{2}\left(\zeta^{(t-\triangle t/2)}(N, M) + \zeta^{(t-\triangle t/2)}(N, M+1)\right) \tag{16}$$

$$\overset{(t)}{HV}(N, M) = hV(N, M) + \tfrac{1}{2}(\overset{(t-\triangle t/2)}{\zeta}(N, M) + \overset{(t-\triangle t/2)}{\zeta}(N, M+1) \tag{17}$$

$$\overset{(t)}{H_v}(N, M) = \tfrac{1}{4}(\overset{(t)}{U}(N, M-1) + \overset{(t)}{U}(N, M) + (U+1, M-1) + \overset{(t)}{U}(N+1, M)) \tag{18}$$

$$\overset{(t)}{H_u}(N,M) = \tfrac{1}{4}(\overset{(t)}{V}(N,M-1) + \overset{(t)}{V}(N,M) + V(N-1, M+1) + \overset{(t)}{V}(N-1, M)) \tag{19}$$

$$V^2\, U = \frac{1}{\triangle l^2}(\overset{(t)}{U}(N-1, M) + \overset{(t)}{U}(N+1, M) + \overset{(t)}{U}(N, M+1) + \overset{(t)}{U}(N, M-1)) \tag{20}$$

$$V^2\, V = \frac{1}{\triangle l^2}(\overset{(t)}{V}(N-1, M) + \overset{(t)}{V}(N+1, M) + \overset{(t)}{V}(N, M+1) + \overset{(t)}{V}(N, M-1)), \tag{21}$$

where

$$\triangle l = R \triangle \phi.$$

For this method, efficient time step is required for numerical stability and is obtained by using the Courant-Friedrichs-Lewy stability condition (Neumann and Richtmyer 1950). The maximum time step be chosen so that any combination of signals can transverse at most one zone per time step [$\triangle t = 90$ sec], mathematically, it is interpreted as

$$\triangle t \leqslant \frac{\triangle l}{\sqrt{2gh_{max}}}, \tag{22}$$

where $\triangle l$ is the grid size and h_{max} is maximum water depth in the area.

The model is covered by the computational grid of 26×30 computational points (Fig. 5). The grid size is .083° $\equiv 10^6$ cm. Position of the water level and velocity and closed boundary are explained in Fig. 6.

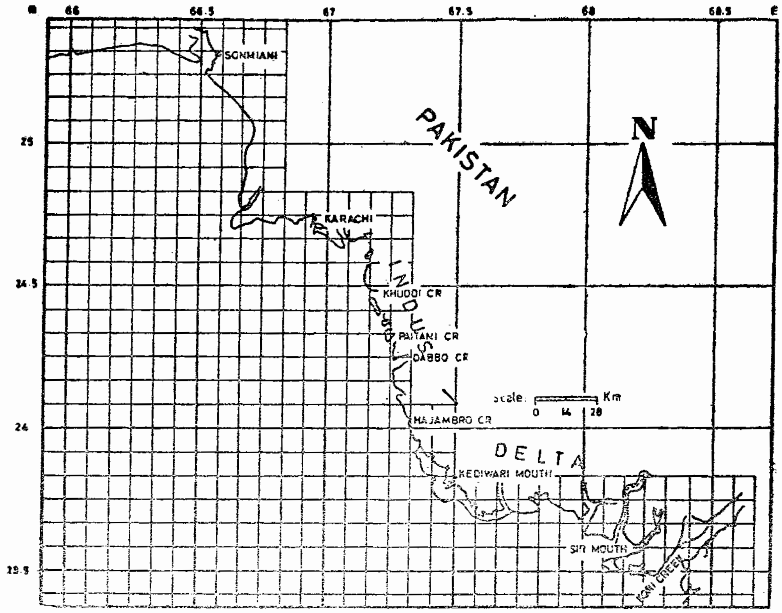

Fig. 5. Computational grid with .083° space step.

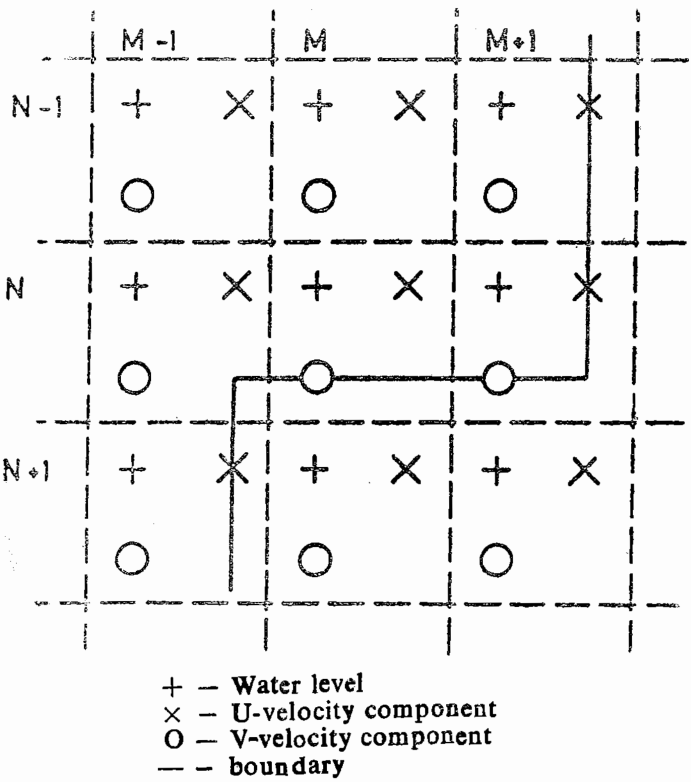

Fig. 6. Hydrodynamical-numerical grid.

Results and Analysis

Tidal Charts

Co-tidal and co-range charts for the four major partial tides M_2, S_2, K_1 and O_1 are shown in Figs. 7, 8, 9 and 10 respectively.

The comparison is made with only tidal gauge, Karachi. The difference is about 1 cm in amplitude and 3° in phase of M_2—tide, 3 cm in amplitude and 1° in phase of S_2—tide, 1 cm in amplitude and 4° in phase of K_1—tide and 0.2 cm in amplitude and 1° in phase O_1—tide. Thus, they have a good agreement as the computational error is negligibly small. In Fig. 7 the results of M_2—tide

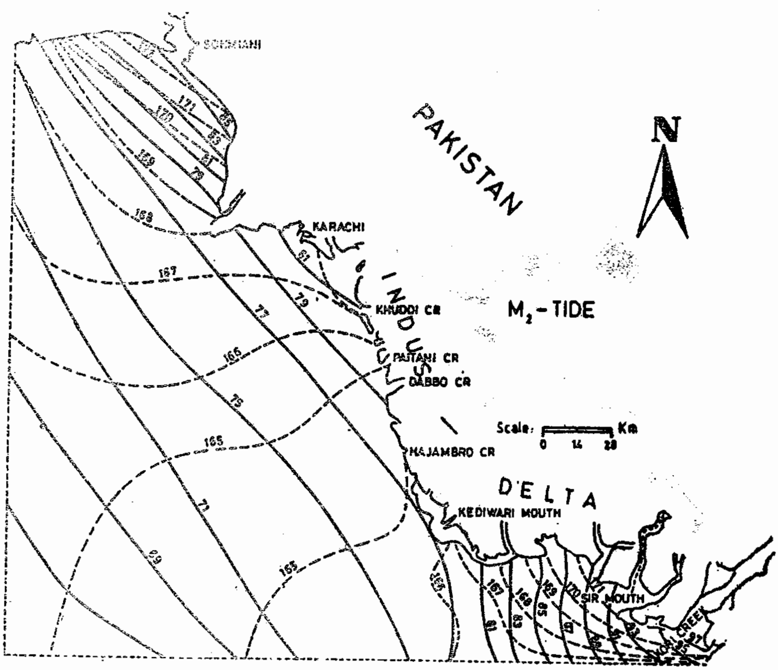

Fig. 7. M_2—tide in the Sonmiani Bay and Indus Delta, co-tidal (——) and co-range (......) lines.

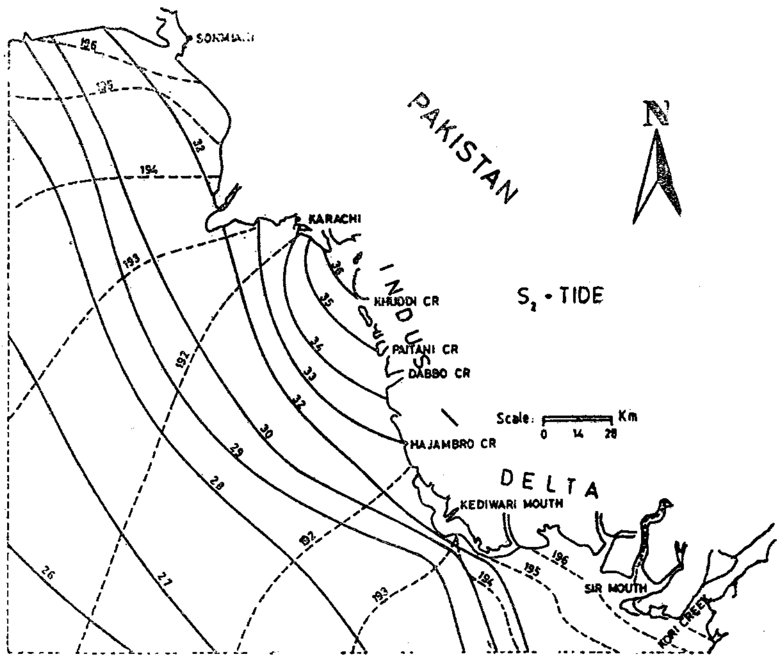

Fig. 8. S_2 – tide in the Sonmiani Bay and Indus-Delta.

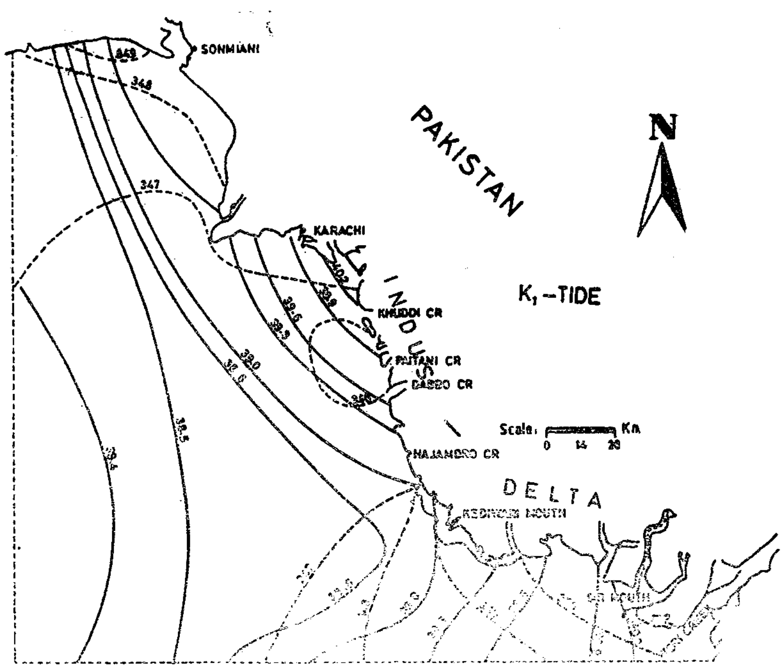

Fig. 9. K_1 – tide in the Sonmiani Bay and Indus Delta.

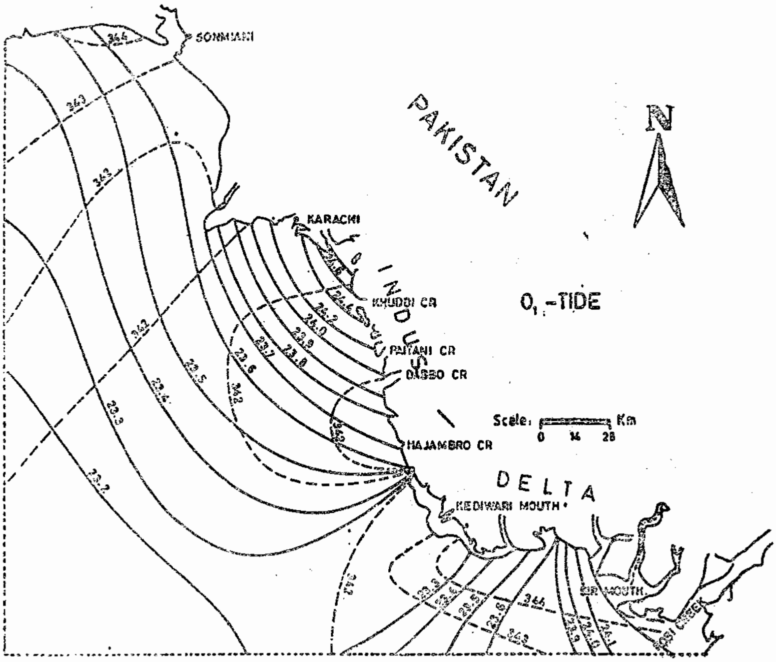

Fig. 10. O_1 —tide in the Sonmiani Bay and Indus Delta.

shows that the greatest tidal amplitude appears near the Kori Creek, whereas, the greatest tide amplitude of S_2, K_1 and O_1 appears near the Port Qasim. The values of amplitude and phases for 25 computational points along the coastal are shown in Table 2.

The tidal current, as the result of superposition of M_2, S_2, K_1 and O_1 —tide, are reproduced is the area. These are plotted for period of 24 hours on August 1, 1973 for the spring tide. The velocity fields are plotted for every two hours. In the velocity field for the flood (Fig. 11) and the ebb (Fig. 12), the flow is directed towards Sonmiani, Port Qasim, Khuddi Creek, Dubbo Creek, Sir Mouth and Kori Creek during the flood. The greatest magnitude of tidal currents occurs around Sir Mouth. Its value during flood is 75 cm/sec and during ebb is 50 cm/sec.

From the tidal current and tidal elevation charts, one knows about the movement of the tidal wave.

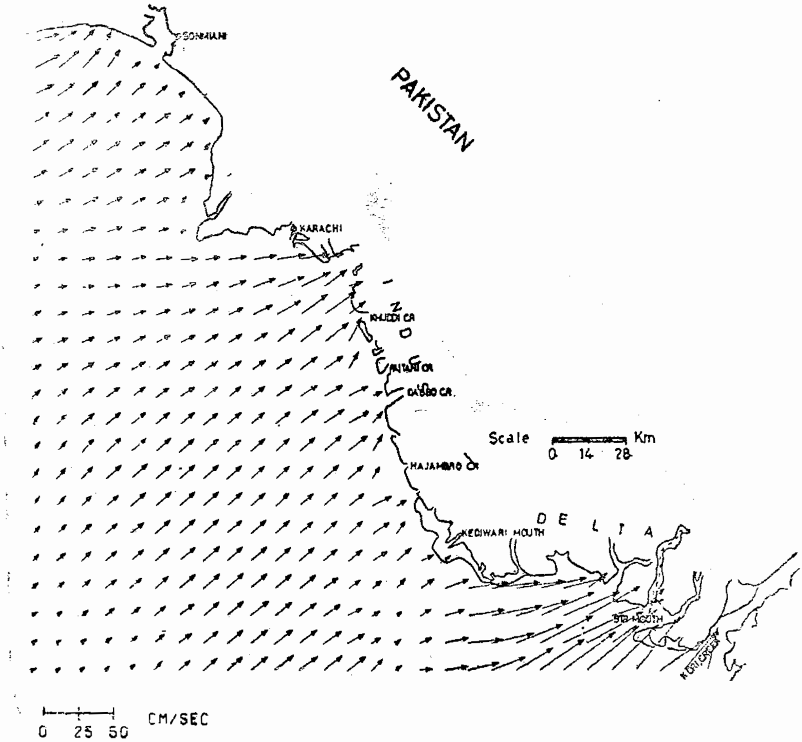

Fig. 11. Flood velocity field on Aug. 1, 1973 at 8.00 a.m.

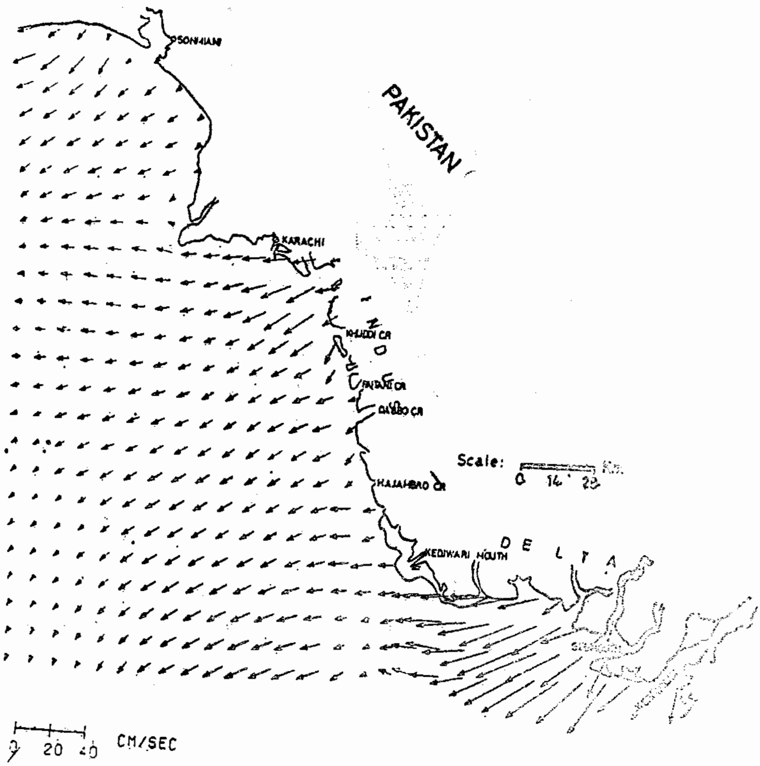

Fig. 12. Ebb velocity field on Aug. 1, 1973 at 4.00 p.m.

No.	PLACE	Position		M_2		S_2		K_1		O_1	
		N	E	a	k	a	k	a	k	a	k
		° '	°								
1	Sonmiani W. 1	25.33	66.33	83.63	172.40	32.32	197.68	39.37	349.29	23.67	344.44
2	Sonmiani W. 2	25.33	66.41	84.89	172.83	32.72	197.56	39.52	349.24	23.72	344.29
3	Sonmiani	25.25	66.50	84.06	172.35	32.37	195.81	39.41	348.38	23.63	343.55
4	Sonmiani East 1	25.25	66.58	85.13	172.84	32.70	195.95	39.53	348.47	23.67	343.54
5	Sonmiani East 2	25.17	66.66	85.11	172.75	32.69	195.62	39.52	348.31	23.68	343.45
6	Habb	24.84	66.66	80.20	169.41	31.44	193.39	39.12	347.06	23.55	342.73
7	Knoff Kod	24.67	66.75	78.61	167.65	31.84	193.19	38.97	347.04	23.68	342.64
8	Manze	24.67	66.23	79.22	167.51	32.44	192.63	39.94	346.86	23.78	342.49
9	Hawkas Bay	24.67	66.91	80.00	167.47	33.17	192.14	39.43	346.73	23.92	342.37
10	Karachi 1	24.67	66.99	81.00	167.56	34.10	191.67	39.73	346.65	24.11	342.25
11	Karachi 2	24.67	67.08	81.94	167.84	35.14	191.13	40.04	346.55	24.33	342.37
12	Karachi 3	24.67	67.16	82.73	168.28	36.05	190.72	40.30	346.57	24.53	342.00
13	Qasim Port	26.59	67.25	82.92	169.43	36.75	191.72	40.57	347.55	24.76	342.30
14	Khuddi 1	24.50	67.25	82.11	167.94	36.34	191.05	40.37	346.88	24.60	342.10
15	Khuddi 2	24.42	67.25	80.90	165.15	35.51	190.08	39.96	345.95	24.24	341.65
16	Paitiani	24.34	67.25	80.07	164.42	34.61	189.97	39.71	345.74	24.05	341.53
17	Dabbo 1	24.26	67.33	80.17	164.16	34.61	191.23	39.61	346.66	24.01	342.15
18	Dabbo 2	24.17	67.33	79.73	163.95	34.26	191.23	39.50	346.67	23.91	342.13
19	Dabbo 3	24.09	67.33	79.01	163.87	33.54	191.29	39.26	346.75	23.71	342.14
20	Hajambro	24.00	67.33	78.03	163.97	32.84	190.98	39.23	346.52	23.62	341.96
21	Kedwari Mouth 1	23.76	67.50	77.51	166.80	30.93	194.01	39.10	349.09	23.21	343.81
22	Kedwari Mouth 2	23.76	67.58	77.97	166.52	31.01	194.34	39.27	349.33	23.26	344.02
23	Sir Mouth 1	23.60	68.16	96.21	174.72	38.01	198.31	41.66	350.92	24.55	345.30
24	Sir Mouth 2	23.60	68.24	99.07	177.08	38.63	200.63	42.80	351.92	24.71	346.07
25	Kori CR	23.43	68.41	94.05	173.38	37.80	200.57	41.65	351.06	24.71	345.30

Table 2. Amplitude a(cm), phase k (deg) of the major tidal constituents

Residual Currents

The residual currents in the absence of wind and density currents are due to the weak non-linearities in the momentum equations. These were computed by averaging the tidal currents for one period of M_2 – tide

$$\overline{U} = \frac{1}{T} \int_0^T U\, dt, \quad \overline{V} = \int_0^T V\, dt \tag{23}$$

where U and V are time averaged velocities and T is the period of the M_2 – tide.

The velocity field due to the residual current is shown in Fig. 13. The residual currents flow eastwards along the Coast of Karachi and flow towards Post Qasim and Khuddi Creek. This predicits the

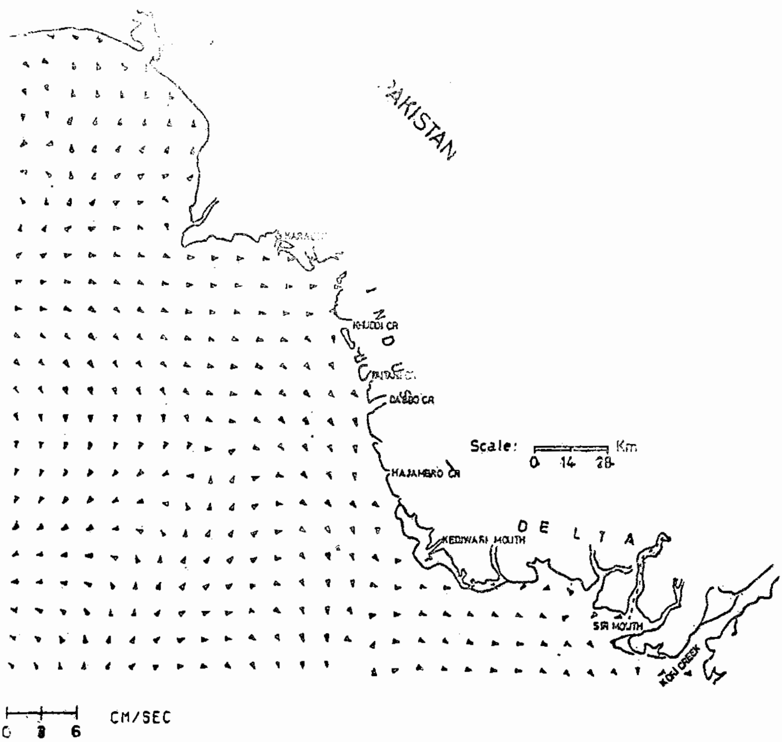

Fig. 13. Residual currents.

tendency of sedimentation of the approach channel of the port. Along rest of the Indus Delta, the flow is along or away from the coast. In Sonmiani Bay, the residual currents flow anticlockwise, with tendency to move inside the channel.

Conclusion

The tidal charts prepared with the help of the hydrodynamical-numerical model had an excellent accuracy. They presented a detailed picture of the equal amplitude, equal phase distribution, tidal current and residual current in the area. Thus, they reproduce the structure of the tidal wave. The tide table may be used to check the accuracy of the results of short sets of observation at temporary tidal gauges.

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IDEALS IN BCI-ALGEBRAS

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Abstract

In this paper we give the notion of the centre of a BCI-algebra and show that it is a p-semisimple sub-algebra. Further various properties of BCI-ideals have been studied and necessary and sufficient conditions for certain ideals to be closed have been investigated.

1. Preliminaries

A BCI-algebra is an algebra $(X,*,o)$ of type $(2,o)$ with the following conditions :

(1) $(x*y)*(x*z) \leqslant z*y$,

(2) $x*(x*y) \leqslant y$,

(3) $x \leqslant x$,

(4) $x \leqslant y$, $y \leqslant x$ imply $x = y$,

(5) $x \leqslant o$ implies $x = o$,

where $x \leqslant y$ if and only if $x*y = o$ ([8]).

Let X be a BCI-algebra and $M = \{x \in X : o \leqslant x\}$ be its BCK-part. Then, X is called proper BCI-algebra if $X\text{-}M \neq \phi$. Note that a BCK-algebra is a trivially BCI-algebra. K. Iseki [8] proved that M is an ideal and a maximal BCK-algebra in X. It is established that $x \in M$,

$y \varepsilon X\text{-}M$ imply $x^*y, y^*x \varepsilon X - M$. The following properties will be useful in the sequel.

(6) $(x^*y)^*z = (x^*z)^*y$.

(7) $x \leqslant y$ implies $x^*z \leqslant y^*z$ and $z^*y \leqslant z^*x$.

(8) $x^*o = x$ ([11]).

(9) $(x^*z)^*(y^*z) \leqslant x^*y$, for all $x, y, z \varepsilon X$ ([8]).

We recall that a subset H of a BCI-algebra X is an ideal of X if

(i) $o \varepsilon H$,

(ii) $x^*y \varepsilon H$, $y \varepsilon H$ imply $x \varepsilon H$.

It is easy to verify that for $y \varepsilon H$, $x \leqslant y$ imply $x \varepsilon H$ and it is not yet proved that every ideal is a sub-algebra. To remove this difficulty C.S. Hoo [6] has given the concept of a closed ideal as under.

A non-empty subset H of a BCI-algebra X is called closed ideal if

(i) $o^*x \varepsilon H$, $x \varepsilon H$,

(ii) x^*y H, $y \varepsilon H$ imply $x \varepsilon H$.

Remark 1. Obviously $o \varepsilon H$. Thus every closed ideal is an ideal. It is interesting to know does there exist an ideal which is not closed ?

Definition 1. An ideal H in a BCI-algebra X is said to be a minimal ideal containing a subset B of X, if there does not exist any ideal $H_1 \subset H$, which also contains B.

Definition 2 [1]. Let X be a BCI-algebra. Then $x, y \varepsilon X$ are called comparable iff $x \leqslant y$ or $y \leqslant x$.

Definition 3[1]. Let X be a BCI-algebra. Choose an element $x_o \varepsilon X$ such that there does not exist $y \neq x_o$ with $y^*x_o = o$. We define

$$A(x_o) = \{x \varepsilon X : x_o \leqslant x\}$$

Clearly, $x_o \varepsilon A(x_o)$ and hence each $A(x_o)$ is non-empty. The point $x_o \varepsilon A(x_o)$ will be called initial element of $A(x_o)$, that is if for some $y \varepsilon X$, $y*x_o = o$, then $y = x_o$. Let I denote the set of all initial elements. We will call it the centre of X.

Note that $M=A(o)$ and if $o \neq x_o \varepsilon I$, then $A(x_o) \subset X-M$.

(10) In (1), we proved that in a BCI-algebra ϕ. $\bigcup_{x_o \in I} A(x_o) = X$ and $\bigcap_{x_o \in I} A(x_o) = \phi$. Further, It is obvious that if $x, y \in X$ are comparable, then they both are contained in same $A(x_o)$, for some $x_o \varepsilon I$.

(11) Let X be a BCI-algebra with M its BCK-part. If $A(x_o) \subseteq X$, then $x, y \varepsilon A(x_o)$ imply $x*y$, $y*x \varepsilon M$. ([1]).

(12) Let X be a BCI-algebra with M its BCK-part and $A(x_o)$, $A(y_o) \subset X$ for $x_o \neq y_o$. If $x \varepsilon A(x_o)$, $y \varepsilon A(y_o)$, then $x*y$, $y*x \varepsilon X-M$. ([1]).

(13) Let X be a p-semisimple algebra. Then every sub-algebra A of X is an ideal in X ([12]).

(14) Let X be a BCI-algebra and $A(x_o)$, $A(y_o) \subset X$. If $o*x_o = y_o$, then $o*y_o = x_o$ for x_o, y_o being the initial element of $A(x_o)$ and $A(y_o)$ respectively ([1]).

(15) Let X be a BCI-algebra, then following are equivalent :

(i) X is medial,

(ii) X is p-semisimple,

(iii) $x*(x*y) = y$ ([4], [6], [12]).

Definition 4 [1]. Let X be a BCI-algebra with M as its Bck-Part. An ideal A of X is called a *proper BCI-Ideal* in X if $A \cap (X-M) \neq \phi$.

2. Ideal in BCI-Algebras

K. Iseki [8] proved that in a BCK-algebra $X, x*y \leqslant x$ is satisfied for all $x, y \varepsilon X$, However, it is interesting to note that :

Lemma 1. Let X be a BCI-algebra with M as its BCK-part then $x*y \leqslant x$ holds for $x \varepsilon X$, $y \varepsilon M$.

Proof. Since $o \leqslant y$, therefore $x*y \leqslant x*o = x$ or $x*y \leqslant x$. This completes the proof.

Theorem 1. The centre I of a BCI-algebra X is p-Semisimple.

Proof. Obviously $o \varepsilon I$. Let $x_o, y_o \varepsilon I$ where $x_o \neq y_o$. By (12), $x_o * y_o \varepsilon X - M$. Let $x_o * y_o \varepsilon A(z_o)$, for $z_o \varepsilon I$. By (11), z_o, $x_o * y_o \varepsilon A(z_o)$ imply $(x_o * y_o) * z_o \varepsilon M$: that is $o \leqslant (x_o * y_o) * z_o$. By (7), $x_o * ((x_o * y_o) * z_o) \leqslant x_o * o$ Or $x_o * ((x_o * y_o) * z_o) \leqslant x_o$ which implies $x_o * ((x_o * y_o) * z_o) = x_o$ because $x_o \varepsilon I$. Now $x_o = x_o * ((x_o * y_o) * z_o))$ gives $x_o * y_o = (x_o * ((x_o * y_a) * z_o)) * y_o$ or $x_o * y_o = (x_o * y_o) * ((x_o * y_o) * z_o) \leqslant z_o$. Or $x_o * y_o \leqslant z_o$ which gives $x_o * y_o = z_o$. Hence I is closed.

Further, For $x_o, y_o \varepsilon I$, $x_o * (x_o * y_o) \leqslant y_o$ implies $x_o * (x_o * y_o) = y_o$. Hence by (15), I is p-semisimple. This completes the proof.

Proposition 1. Let X be a BCI-algebra. Let $x_o \varepsilon I$ and $A(x_o) \subseteq X$. If $o*x_o \varepsilon A(x_o)$, then $o*x_o = x_o$.

Proof. Since x_o is the initial element of $A(x_o)$ therefore, $x_o \varepsilon I$. Now $o, x_o \varepsilon I$ imply $o*x_o \varepsilon I$ because I is closed. Thus $o*x_o \varepsilon I$ and $o*x_o \varepsilon A(x_o)$ both imply $o*x_o \varepsilon I \cap A(x_o) = \{x_o\}$, that is $o*x_o = x_o$. This completes the proof.

Proposition 2. Let X be a BCI-algebra with M and I as its BCK-part and centre respectively. Let $x_o, y_o \in I$ where $x_o \neq y_o$ and $A(x_o)$, $A(y_o) \subset X-M$ such that $o*x_o = x_o$ and $o*y_o = y_o$. Then for $x \in A(x_o)$, $y \in A(y_o)$, $x*y \notin A(x_o) \cup A(y_o)$.

Proof. Let $x_o * y_o \in A(x_o)$. By definition of $A(x_o)$, $x_o \leqslant x_o * y_o$. By (7) $x_o * x \leqslant (x_o * y_o)*x$ or $o \leqslant (x_o * x)*y_o$ or $o \leqslant y_o$ or $y_o \in M$, a contradiction. Hence $x_o * y_o \notin A(x_o)$. Similarly, we can show that $x_o * y_o \notin A(y_o)$. Hence $x_o * y_o \notin A(x_o) \cup A(y_o)$. Further $x_o \leqslant x$ implies $x_o * y \leqslant x*y$. Also $y_o \leqslant y$ implies $x_o * y \leqslant x_o * y_o$. Now by (10) $x_o * y \leqslant x_o * y_o$ and $x_o * y \leqslant x*y$ both imply $x_o * y_o$, $x_o * y$ and $x*y$ are contained in the same $A(z_o) \subseteq X$ for $z_o \in I$. Since $x_o * y_o \notin A(x_o) \cup A(y_o)$, therefore $x*y \notin A(x_o) \cup A(y_o)$. This completes the proof.

Proposition 3. Let I be the centre of X. For $x_o, y_o \in I$, if $o*x_o = x_o$, $o*y_o = y_o$, then $x_o * y_o = y_o * x_o$.

Proof. We consider $x_o * y_o = (o*x_o)*(o*y_o) \leqslant y_o * x_o$. Further $y_o * x_o = (o*y_o)*(o*x_o) \leqslant x_o * y_o$. Thus $x_o * y_o \leqslant y_o * x_o$ and $y_o * x_o \leqslant x_o * y_o$ both imply $x_o * y_o = y_o * x_o$. This completes the proof.

Theorem 2. Let X be a BCI-algebra with M and I as its BCK-part and centre respectively. Let $o \neq x_o$, $o \neq y_o \in I$. If, for $x \in A(x_o)$, $o*x \in A(y_o)$, then $o*x = y_o \in A(y_o)$.

Proof. Let $x \in A(x_o)$ then by definition of $A(x_o)$, we can write $x_o \leqslant x$, which implies $o*x \leqslant o*x_o$. Since $o, x_o \in I$ and I is closed, therefore $o*x_o \in I$. Since $o*x \in A(y_o)$, therefore, by (10), $o*x \leqslant o*x_o$ implies $o*x_o \in A(y_o)$. Now $o*x_o \in I$ and $o*x_o \in A(y_o)$ both imply $o*x_o \in A(y_o) \cap I = \{y_o\}$; that is $o*x_o = y_o$. Thus $o*x \leqslant o*x_o = y_o$ implies $o*x = y_o$, because $y_o \in I$. This completes the proof.

Theorem 3. Let X be a BCI-algebra and $A(x_o) \subseteq X$, $x_o \in I$. For $x, y \in A(x_o)$, $o*x = o*y$.

Proof. Let $x, y \in A(x_o)$. Then $x_o \leqslant x$, $x_o \leqslant y$ and $o*x \leqslant o*x_o$ and $o*y \leqslant o*x_o$. By (10), $o*x_o$, $o*x$ are all contained in a unique $A(z_o) \subset X$ for $z_o \in I$. By Theorem 2, $o*x = z_o = o*y$, which implies $o*x = o*y$. This completes the proof.

Theorem 4. Let X be a BCI-algebra wi.h M and I as its BCK-part and centre respectively. Let $x_o, y_o \in I$ be such that $x_o \neq y_o$ and $A(x_o)$, $A(y_o) \subseteq X$. If $o*x_o = x_o$, then for y $A(y_o)$, $o*y \notin A(x_o)$.

Proof. Suppose $o*y \in A(x_o)$. Since $y_o \leqslant y$, therefore $o*y \leqslant o*y_o$. By theorem 3, $y, y_o \in A(y_o)$ implies $o*y = o*y_o$. By theorem 2, $o*y \in (A(x_o)$ implies $o*y = x_o$. Now $o*y_o = o*y = x_o$ implies $o*y_o = x_o$ or $(o*y_o)*x_o = o$ or $(o*x_o)*y_o = o$. Since I is p-semisimple, therefore $(o*x_o)*y_o = o$ imply $o*x_o = y_o$. But $o*x_o = x_o$. Thus $x_o = o*x_o = y_o$ implies $x_o = y_o$, a contradiction. This completes the proof.

Theorem 5. Let X be a BCI-alyebra with I as its centre. Let $N \subseteq I$ and H $\bigcup_{x_o \in N} A(x_o)$. H is a closed ideal in X if and only if N is a closed ideal in I.

Proof. Suppose H is a closed ideal in X. We show that N is a closed ideal in I. Obviously $I \cap H = N$. Let $A(x_o), A(y_o) \subseteq H$. Then x_o, y_o are initial elements of $A(x_o)$ and $A(y_o)$ respectively; and $x_o, y_o \varepsilon N \subseteq I$. We prove that $x_o * y_o \varepsilon N$. Since $x_o \varepsilon A(x_o)$ and $y_o \varepsilon A(y_o)$. Therefore $x_o, y_o \varepsilon H$. Further H is closed, therefore $x_o * y_o \varepsilon H$. Also $x_o, y_o \varepsilon I$ and I is closed, therefore $x_o * y_o \varepsilon I$. Now $x_o * y_o \varepsilon I$ and $x_o * y_o \varepsilon H$ imply $x_o * y_o \varepsilon H \cap I = N$; that is $x_o * y_o \varepsilon N$. Thus N is a sub-algebra of I. By [12] N is an ideal in I because I is p-semisimple. Since N is sub-algebra and ideal therefore N is a closed ideal in I.

Conversely, since $N \subseteq I$ and N is a closed ideal in I, therefore $o \varepsilon N$ as well as it is a sub-algebra in I; that is $x_o, y_o \varepsilon N$ imply $x_o * y_o \varepsilon N$. We show that

$$H = \bigcup_{x_o \varepsilon N} A(x_o)$$ is a closed ideal in X simply proving :

(i) H is a sub-algebra,

(ii) For $y \varepsilon X - H$, $x \varepsilon H$, $y*x \varepsilon X - H$.

(i) Case 1. Since $o \varepsilon N$, therefore $M = A(o) \subseteq H$. Let $x_o \varepsilon N$ and $x, y \varepsilon A(x_o) \subset H$. By (11) $x*y, y*x \varepsilon M \subset H$. Thus $x*y, y*x \varepsilon H$.

Case 2. For $x_o, y_o \varepsilon N$, $x_o * y_o$, $y_o * x_o \varepsilon N$. Put $x_o * y_o = h_o$ (say) εN. Let $x \varepsilon A(x_o) \subseteq H$, $y \varepsilon A(y_o) \subseteq H$. Then $x_o \leqslant x$ implies $x_o * y_o \leqslant x*y_o$ or $h_o \leqslant x*y_o$ or $x*y_o \varepsilon A(h_o) \subseteq H$ Again, $y_o \leqslant y$ gives $x*y \leqslant x*y_o \varepsilon A(h_o) \subseteq H$. Thus $x*y \varepsilon H$. Similarly, put $y_o * x_o = k_o \varepsilon N$, then $y*x \varepsilon H$. It follows that H is a sub-algebra.

(ii) Suppose $x \in A(x_o) \subseteq H$, $y \in A(y_o) \subseteq X-H$, we show that $y*x \in X-H$. Now $x_o \in N$, $y_o \in I-N$. Since N is a closed ideal, therefore $y_o * x_o \in I-N$. Put $y_o * x_o = t_o$ where $t_o \in I-N$ and $A(t_o) \subseteq X-H$. Now $x_o \leqslant x$ implies $y_o * x \leqslant y_o * x = t_o$ or $y_o * x \leqslant t_o$ or $y_o * x = t_o$, because $t_o \in I$. Thus $y_o * x = t_o \in A(t_o) \subseteq X-H$. Again, $y_o \leqslant y$ gives $y_o * x \leqslant y*x$ or $t_o \leqslant y*x$ or $y*x \in A(t_o) \subseteq X-H$ or $y*x \in X-H$. This completes the proof.

Lemma 2. Let X be a BCI-algebra and H an ideal in X. Let $L \subseteq X-H$. If $K = H \cup L$ is an ideal in X, then for each $x \in L$, $x_o \in K$, where x_o is the initial element of x, i,e. $x \in A(x_o)$.

Proof. Let $x \in L$ and x_o be its initial element, that is $x \in A(x_o)$. Then $x \in K$ and $x_o * x = o \in K$ because K is an ideal, which implies $x_o \in K$. This completes the proof.

Definition 4 [10]. An ideal A of a BCI-algebra X is called a maximal ideal in X if it is not properly contained in any other proper ideal of X.

Theorem 6. Let X be a BCI-algebra with I as its centre. Let $N \subseteq I$, then $H = \bigcup_{x_o \in N} A(x_o)$ is a maximal closed ideal in X if and only if N is a maximal closed ideal in I.

Proof. Necessity, we note that $A(o) = M \subseteq H$. Suppose $H = \bigcup_{x_o \in N} A(x_o)$ is maximal closed ideal In X. Theorem 5 gives that N is a closed ideal in I. We show that N is a maximal closed ideal in I. Suppose $T = N \cup P$, where P is a non-empty subset of $I-N$, is a closed ideal in I. Then by Theorem 5 $K = \bigcup_{y_o \in T} A(y_o) \supset H$ is a closed ideal in X, which contradicts that H is maximal Thus N is a maximal closed ideal in I, which by Theorem 5 gives that $H = \bigcup_{x_0 \in N} A(x_0)$ is a closed ideal in X. Now we show that H is a maximal ideal in X. Let K be a closed ideal of X such that $H \subset K$. Thus $K - H \neq \varnothing$. Let $y \varepsilon K - H$, then by (10). $y \varepsilon A(y_0)$ for a unique $y_0 \varepsilon I$. Lemma 2 gives $y_0 \varepsilon K$ and hence $y_0 \varepsilon I \cap K$. Further we note that $y_0 \notin N$ because otherwise $y_0 \varepsilon N \subseteq H$ and (11) $y^*y_0 \in M \subseteq H$ and H being an ideal will give $y \in H$. Now let $z \in A(y_0)$, then by (11), $z^*y_0 \varepsilon M \subseteq H \subset K$ and $y_0 \varepsilon K$ and K being a close ideal give that $z \varepsilon K$. Thus $A(y_0) \subset K$. Let $T = I \cap K$. Then $K = \bigcup_{z_0 \in T} A(z_0)$. Further we have already shown that at least one $y_0 \varepsilon T - N$. Now Theorem 5 gives T is a closed ideal of I, which contradicts that N is maximal. Thus H is a maximal closed ideal in X. This completes the proof.

Theorem 7. Let X be a proper BCI-algebra. If $H = M \cup A(x_0)$, $x_0 \varepsilon I$, is an ideal, then H is a minimal ideal containing M.

Proof. Suppose $K = M \cup L$, $L \subset A(x_0)$ be an other ideal containing M. Let $x \varepsilon A(x_0) - L$ and $y \varepsilon L$ then by (11) $x^*y \in M \subset K$ implies $x^*y \varepsilon K$. Now $x^*y \in K$, $y \in K$ and K being an ideal implies $x \in K$, a contradiction. Hence K is not an ideal. Therefore there does not exist any proper ideal of the form $M \cup L$. This proves that H is a minimal ideal containing M.

Theorem 8. Let X be a BCI-algebra with M as its BCK-part and order of X be n. Then, there does not exist a closed proper BCI-ideal of order n-1.

Proof. Case 1. Take $M' = M - \{m\}$ and suppose $H = M' \cup (X - M)$ is a closed ideal of order $n - 1$ in X. Since X is a BCI-algebra, therefore for any $x \varepsilon X - M$, $m \varepsilon (M - M')$. $m^*x \varepsilon X$. By K. Iseki [6], $m^*x \varepsilon X - M \subset H$; and hence $m^*x \varepsilon H$. Now $m^*x \varepsilon H$, $x \varepsilon H$ and H being closed ideal implies $m \in H$; a contradiction.

Case 2 (i) Suppose $H=X-\{y\}$ is a closed ideal in X, for some $y \in A(x_0)$, where $A(x_0) \subset X-M$. Choose $x \in (A(x_0)-\{y\})$ then by (11) $y*x \in M \subset H$ implies $y*x \in H$. Thus $y*x \in H$, $x \in H$ being a closed ideal implies $y \in H$, a contradicton.

Case 2 (ii) Suppose $H=X-\{x_0\}$ is closed ideal in X for $x_0 \in A(x_0)$ being the initial element of $A(x_0)$. Choose $x_0 \neq x \in A(x_0)$. Then by (11) $x_0 * x \in M \subset H$ or $x_0 * x \in H$. Now $x_0 * x \in H$, $x \in H$ and H being an ideal implies $x_0 \in H$, contradiction.

Case 2 (iii) Suppose $H=X-\{y_0\}$ is a closed ideal in X for $y_0 \in A(y_0)$ where $A(y_0)=\{y_0\}$. Choose $o \neq x_0 \in H \cap I$, then $y_0 * x_0 \neq y_0$ because otherwise $y_0 * x_0 = y_0$ implies $o = x_0$, contradiction. But $x_0, y_0 \in I$, I is closed, therefore $y_0 * x_0 \in I$; that is $y_0 * x_0 \in H$. Now $y_0 * x_0 \in H$, $x_0 \in H$ and H being an ideal implies $y_0 \in H$, a contradiction. This completes the proof.

Remark. A BCK-ideal of order (n-1) may exist in a BCI-algebra of order n such that $O(X—M)=1$.

Definition 5 [3]. Let X be a BCI-algebra, $G \subseteq X$ is defined as $G=\{x \in X : o*x=x\}$ and is known as BCI-G part of X

Theorem 9. Let X be BCI-algebra with G and I as its BCI-G part and centre respectively, then G is a sub-algebra of I.

Proof. Let $x \in G$. Then $o*x=x$. By (10) $o*x$ is contained in a unique $A(y_0) \subset X$ for some $y_0 \in I$. By Theorem 2, $o*x = y_0 \in I$ or $x = y_0 \in I$ which implies $x \in I$ and $G \subseteq I$. Further by [11], G is a sub-algebra. Hence $G \subset I$ and G being a sub-algebra implies G is a sub-algebra in I. This completes the proof.

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MONOMORPHISMS AND EPIMORPHISMS IN THE CATEGORY OF BCI-ALGEBRAS

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Abstract

In this paper we study some properties of monomorphisms and epi-morphisms of BCI, the category of BCI-algebras and BCI-homomorphisms.

1. Introduction

In [4] we investigated some properties of BCI, the category of BCI-algebras and BCI-homomorphisms. It was not obvious that monomorphisms are one-one or not ? Recently C.S.Hoo [8] has answered this question and has proved that monomorphisms are preciseely one-one homomorphisms. In this paper we use this fact to prove that if $f \in$ BCI (X, Y) is mono, I_X, G_X denote the centre and G. part of the BCI-algebra X, then

(i) $f(X - I_X) \subseteq Y - I_Y$,

(ii) $f(I_X - G_X) \subseteq I_Y - G_Y$.

The second problem posed by us in [4] that epimorphisms in BCI are onto homomorphisms or not is also partially solved in this paper.

2. Preliminaries

We shall follow standard definitions. Our categorical concepts shall be those of the standard text [12], to which we refer the reader for definitions of standard categorical terms. Our notions of BCI-algebras shall be as developed in [3], [10] and [11].

We denote by BCI, the category of BCI-algebras and BCI-homomorphisms. We recall that a homomorphism $f : X \to Y$ means that $f(x_1 * x_2) = (f(x_1) * (f(x_2))$ and hence $f(o) = o$. This also means that if $x \leqslant y$, then $f(x) \leqslant f(y)$. We shall denote a general category by K, its objects by $|K|$ and the set of morphisms from an object X into an object Y by K (X, Y).

We recall $p \in$ BCI (X, Y) is a monorphism if for all $Z \in |BCI|$, $f, g \in$ BCI (Z, X) satisfying $pof = pog$, we have $f = g$. Further $p \in$ BCI (X, Y) is called epimorphism if $\forall Z \in |BCI|$, $f, g \in$ BCI (Y, Z) satisfying $fop = gop$, we have $f = g$.

Definition 2.1. [*1*]. Let X be a BCI-algebra and $x, y \in X$. Then x, y are called comparable iff $x * y = o$ or $y * x = o$. We choose an element $x_o \in X$ such that for all $y \in X$ satisfying $y * x_o = o$, we have $y = x_o$. This is called an initial element of X. Let I_X denotes the set of all initial elements of X, then I_X is called its centre. Further, we define

$$A(x_o) = \{x \in X : x_o * x = o\}.$$

$A(x_o)$ is said to be comparable if each pair $x, y \in A(x_o)$ is comparable. We call it a comparable branch. Clearly each $A(x_o)$ is non-empty, because $x_x * x_o = o$ implies $x_o \in A(x_o)$. Obviously $A(x_o) = \{x_o\}$, a singleton set is comparable, we will call it uniary comparable.

Let $M = \{x \in X : o * x = o\}$, denote the BCK-part of X. Note that $M = A(o)$. If $M = \{o\}$, then the algebra X is called p-semi-simple. A BCI-algebra X in which each $A(x_o)$ is uniary comparable is p-semi-simple. Moreover, in p-semisimple BCI-algebra X, $x*y = o$ implies $x = y$ for $x, y \in X$.

(1) Let X be a BCI-algebra, I_X as its centre. Then

$$\bigcup_{x_o \in I_X} A(x_o) = X \text{ and } \bigcap_{x_o \in I_X} A(x_o) = \phi. \quad ([1]).$$

(2) Let X be a BCI-algebra and $A(x_o) \subseteq X$. Then $x, y \in A(x_o)$ imply $x*x, y*x \in M$. ([1])

(3) Let X be a BCI-algebra with I_X as its centre, then I_X is p-semi-simple. ([2]).

(4) Homomorphic image of a group is a group ([13]).

(5) Every sub-algebra of a p-semi-simple algebra X is an ideal in X ([14]).

(6) Let X be a BCI-algebra with I as its centre. Let $A(x_o)$, $A(y_o) \subseteq X$ such that $o \neq x_o, y_o \in I$. If for $x \in A(x_o)$, $o*x \in A(y_o)$, then $o*x = yo$, $y_o \in A(yo)$.

(7) Let X be a BCI-algebra with I_X as its centre. Let $N \subset I_X$ and $H = \bigcup_{x_o N} A(x_o)$. H is a closed ideal in X iff N is a closed ideal in I_X ([2]).

(8) In a BCI-algebra X the following are equivalent ([1], [6], and [14]).

(1) X is p-semi-simple,

(2) X is medial,

(3) $x*y = o$ implies $x = y$,

(4) $o*(o*x) = x$,

(9) Let X be a BCI-algebra. $A(x_o)$, $A(y_o) \subseteq X$, where $x_o, y_o \in I_X$. Then for all $x \in A(x_o)$, if $o*x \in A(y_o)$, then $o*x = y_o$. ([2]).

In [5], the BCI-G part of X is defined as

$$G_X = \{x : x \in X \text{ and } o*x=x\}.$$

Further it was shown that G_X is a sub-algebra of X and infact a commutative group. In [2] it is shown that G_X is a sub-algebra of I_X. If $f \in BCI(X, Y)$, then in [4] it is shown that $f(G_X) \subseteq G_Y$, where G_X and G_Y are BCI-G parts of X and Y, respectively. We now generalize this result in the form of following theorem.

Theorem 1. Let $f \in BCI(X, Y)$, I_X and I_Y are centres of X and Y, respectively, then $f(I_X) \subseteq I_Y$.

Proof. Let $x \in I_X$. Since (3) gives I_X is p-semisimple therefore (8) gives $x=o*(o*x)$. Thus

$$f(x)=f(o*(o*x))=f(o)*(f(o)*(f(x)) = o*(o*f(x)).$$

We show that $f(x) \in I_y$. Let $y \in Y$ be such that $y \leqslant f(x)$. Thus $o*f(x) \leqslant o*y$, which gives $(o*f(x))*(o*y)=o$ or $(o*f(x))*(o*y))*(o*f(x))=o*(o*f(x))$. Thus $o*(o*f(x))=((o*f(x))*(o*f(x)))*(o*y)=o*(o*y) \leqslant y$ or $f(x) \leqslant y$. Thus $y=f(x)$. Hence $f(x) \in I_y$. This completes the proof.

Remark 1. It is easy to verify that $f(I_x)$ is a sub-algebra of I_y and hence is ideal in I_y.

Lemma 1. Let $f \in BCI(X, Y)$ such that $f(m) = 0$ for all all $m \in M=A(0)$. Then if $x \leqslant y$ in X, we have $f(x) = f(y)$.

Proof. Since $x \leqslant y$, we have $f(x) \leqslant f(y)$. Also since $x*y=0$, we have $0*(y*x) = (y*y)*(y*x) \leqslant x*y=0$. Hence $0 \leqslant y*x$, that is, $y*x \in M$. Thus $0=f(y*x) = f(y)*f(x)$, giving $f(y) \leqslant f(x)$. This proves that $f(x)=f(y)$.

The following result characterizes I_X.

Theorem 2. $I_X = \{x \in X : 0*(0*x) = x\}$.

Proof. If $x \in I_X$, then since $0*(0*x) \leqslant x$. we have $0*(0*x) = x$. conversely, if $0*(0*x) = x$ and $y \leqslant x$, then $0*x \leqslant 0*y$, and hence $(0*x)*(0*y) = 0$. This means that $\{(0*x)*(0*y)\}*(0*x) = 0*(0*x) = x$, that is, $x = \{(0*x)*(0*x)\}*(0*y) = 0*(0*y) \leqslant y$. Thus $x = y$ and hence $x \in I_x$. This completes the proof

Lemma 2. Let X, Y $\in$ | BCI | and I_x, I_y be their centres. If $f \in BCI(X, Y)$ is a monomorphism, then $f(X - I_X) \subseteq Y - I_Y$.

Proof. Suppose that $x \in X - I_X$ and $f(x) \in I_Y$. Then

$$f(x) = 0*\{0*f(x)\} = f\{0*(0*x)\}$$

Since f is a monomorphism, we have $x = 0*(0*x)$ and hence $x \in I_X$; a contradiction.

Thus $f(x) \in I_Y$ which implies $f(x) \in Y - I_Y$; that is $f(x - I_X) \subseteq Y - I_Y$. This completes the proof.

Lemma 3. Let X, Y $\in$ | BCI | and I_X, I_Y be their respective centres. Let G_X, G_Y be their BCI-G parts respectively. If $f \in$ BCI (X, Y) is a monomorphism, then $(f(I_X - G_X) \subseteq I_Y - G_Y$.

Proof. Let $x \in I_X - G_X$ and $f(x) \in G_Y$. Let $f(x) = g \in G_Y$. then $o*g = g$ implies $o*f(x) = f(x)$——(i)

Since $x \in I_X - G_X$, therefore $o*x = y \in I_X - G_X$, where $x \neq y$, Then $f(o*x) = f(y)$. or $o*f(x) = f(y)$——(ii)

From (i) and (ii) $f(x) = f(y)$. Since f is monomorphism, therefore $x = y$, a contradiction. Hence $f(x) \notin G_Y$. But by theorem 1, $f(I_X) \subseteq I_Y$. Thus $f(x) \in I_Y - G_Y$. This completes the proof.

Combining Lemma 2 and Lemma 3 we get the follows result

Theorem 3. Let $X, Y \in |BCI|$ and I_X, I_Y be their centres respectively. Let G_X, G_Y be the BCI-G parts respectively. If $f \in BCI(X, Y)$, is a monomorphism, then

(i) $f(I_X - G_X) \subseteq I_Y - G_Y$

(ii) $f(X - I_X) \subseteq Y - I_Y$.

Remark. Let X, Y be BCI-algebras with I_X, I_Y as their centres respectively. Let $N \subseteq I_X$ be a sub-algebra and $H = \bigcup_{x_0 \in N} A(x_0)$.

By (7). H is closed ideal in X if and only if N is closed ideal in I_X.

Let $f : X \to Y$ be a homomorphism. We construct $K = \bigcup_{y_0 \in f(N)} A(y_0)$.

then K is a ideal in Y because $f(N) \subseteq I_Y$ and $f(N)$ being homomorphie image of a sub-algebra N is a sub-algebra of I_Y. Thus $f(N)$ is ideal in I_Y, which gives K is ideal in Y.

Theorem 4. Let $X, Y \in |BCI|$ and $f \in BCI(X, Y)$ is an epimorphism. If $K = \bigcup_{y_0 \in f(I_X)} A(y_0)$. Then Y=K.

Proof. We know that $X = \bigcup_{x_0 \in I_X} A(x_0)$ and X is an S-ideal in X. Further I_X is a Sub-algebra in I_X. Again $f(I_X) \subseteq I_Y$. Thus $f(I_X)$ being sub-algebra of I_Y is a ideal in I_Y which implies $K = \bigcup_{y_0 \in f(I_X)} A(y_0$ is a ideal in Y. Let Y/K be the the corresponding quotient algebra.

$$X \xrightarrow{f} Y \xrightarrow{g} Y/K$$
$$\xrightarrow[h]{}$$

We define g, h : Y – Y/K by

$g(y) = [y]_k$

$h(y) = [o]_k$.

It can easily be verified that g, h ∈ BCI (Y, Y/K). Further (gof) $(x) = g(f(x) = [f(x)]_k$ and (hof) $(x) = h(f(x) = [o]_k$. We claim that $[o]_k = [f(x)]_k$. To verify this we will show that $f(x)^* o = f(x) \in K$ and $o^* f(x) \in K \ \forall x \in X$. Let $x \in X$, then $\exists x_o \in I_X$ such that $x_o \leqslant x$. This gives $f(x_o) \leqslant f(x)$. But $f(x_o) \in f(I_X)$. Thus $f(x) \in A(f(x_o)) \subseteq \bigcup_{y_o \in f(I_x)} A(y_o) = K$.

Hence $f(X) \subseteq K$. Since $o^*x \in X$: therefore, $f(o^*x) = f(o)^* f(x) = o^* f(x) \in f(X) \subseteq K$. Thus $[o]_k = [f(x)]_k$, which gtves gof=hof. Since f is epimorphism, therefore g=h. Thus g (y)=h (y) for all y ∈ Y, which gives $[y]_k = [o]_k$. Thus $y^*o = y \in k$. Hence $Y \subseteq K$ but $K \subseteq Y$. Thus K=Y.

Theorem 5. Let X, Y ∈ | BCI | and I_X, I_Y be their centres respectively. Let f ∈ BCI (X, Y) be an epimorphism such that f (X) is an ideal in Y, then f is onto.

Proof. The quotient-algebra Y/f (X) is a BCI-algebra and we define g, h : Y ⟶ Y/f (X) by $g(y) = [y]_{f(X)}$ and $h(y) = [o]_{f(X)}$. Further, (gof) $(x) = g(f(x) = [f(x)]_{f(X)}$ and

$$(hof)(x) = h(f(x)) = [o]_{f(X)} .$$

Since f (X) is closed ideal in Y, therefore $[o]_{f(X)} = [f(x)]_{f(x)}$ which gives gof=hof. Since f is epimorphism. Therefore g=h and hence g (y)=h (y) for all y ∈ Y, which gives $[o]_{f(X)} = [y]_{f(X)}$ or $y^*o \in f(X)$ for $y \in Y$ or $y^*o = y \in f(x)$ which implies $Y \subseteq f(X)$. But $f(X) \subseteq Y$. Hence Y=f (X) which implies f is onto. This completes the proof.

The following problem still remains unsolved.

Problem. If f ≤ BCI (X, Y) is epi, then f (x) is idcal in Y or not ?

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A CLASS OF UNIVALENT FUNCTIONS WHOSE DERIVATIVES HAVE A POSITIVE REAL PART

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Abstract

A sharp coefficient estimates and distortion theorems are determined for the class $R^{\lambda}(\alpha, \beta, A, B)$ of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying the condition

$$\left| \frac{f'(z)-1}{(B-A)\beta(f'(z)-1+(1-\alpha)\cos\lambda\, e^{-i\lambda})+A(f'(z)-1)} \right| < 1$$

for some α, β $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1)$ and $-1 \leqslant A < B \leqslant 1, 0 < B \leqslant 1$ and for all z in $U = \{z : |z| < 1\}$. A sufficient condition for a function to belong to $R^{\lambda}(\alpha, \beta, A, B)$ has also been determined. We shall also prove that a subclass of analytic functions is closed under convolution.

1. Introduction

Let N be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the unit disc $U = \{z : |z| < 1\}$. If a function $f(z) \in N$ satisfies the condition $\text{Re}(f'(z)) > 0$ for all $z \in U$, then it is well known that f (z) is univalent in U. We denote the class of such functions by R. This class was introduced by MacGregor [6].

Let R_α denote the class of functions $f(z) \in N$ that satisfy the condition $\text{Re}\,(f'(z)) > \alpha$, for $0 \leqslant \alpha < 1$, $z \in U$. Clearly $R_0 = R$.

In [1] Ahuja introduced the class $R^\lambda(\alpha, \beta)$ of functions $f(z) \in N$, defined as follows :

Definition 1. Let $f(z) \in N$; and let $0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$, and $|\lambda| < \frac{\pi}{2}$, Then f (z) is said to be in $R^\lambda(\alpha, \beta)$ if it satisfies the condition

$$\left| \frac{f'(z)-1}{2\beta\,(f'(z)-1+(1-\alpha)\cos\lambda\, e^{-i\lambda})-(f'(z)-1)} \right| < 1 \tag{1.2}$$

for all z in U. We note that $R^0(0, 1)=R$ and $R^0(\alpha, 1)=R_\alpha$.

In this paper, we introduce the class $R^\lambda(\alpha, \beta, A, B)$ of functions $f(z) \in N$, defined as follows :

Definition 2. Let $f(z) \in N$; and let $0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$, $|\lambda| < \frac{\pi}{2}$ and $-1 \leqslant A < B \leqslant 1$, $0 < B \leqslant 1$. Then f (z) is said to be in $R^\lambda(\alpha, \beta, A, B)$ if it satisfies the condition

$$\left| \frac{f'(z)-1}{(B-A)\,\beta\,(f'(z)-1+(1-\alpha)\cos\lambda\, e^{-i\lambda})+A\,(f'(z)-1)} \right| < 1 \tag{1.3}$$

for all z in U. We note that $R^\lambda(\alpha, \beta, -1, 1) = R^\lambda(\alpha, \beta)$.

By taking different values of the parameters $\alpha, \beta, \lambda, A, B$ ($0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$, $|\lambda| < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1$, $0 < B \leqslant 1$), the class $R^\lambda(\alpha, \beta, A, B)$ reduces to various well known subclasses of R ; for example,

$$R^{\lambda}_{\alpha,\beta} = R^{\lambda}\left(\frac{1-\beta+2\alpha\beta}{1+\beta}, \frac{1+\beta}{2}, -1, 1\right)$$

$$= \left\{ f \in N : \left| \frac{f'(z)-1}{f'(z)-1+2(1-\alpha)\cos\lambda\, e^{-i\lambda}} \right| < \beta, o \leqslant \alpha < 1, o < \beta \leqslant 1, z \in U \right\},$$

$$R(\gamma) = R^{o}\left(\frac{1-\gamma}{1+\gamma}, \frac{1+\gamma}{2}, -1, 1\right)$$

$$= \left\{ f \in N : \left| \frac{f'(z)-1}{f'(z)+1} \right| < \gamma, o < \gamma \leqslant 1, z \in U \right\},$$

$$R^{*} = R^{o}(o, \tfrac{1}{2}, -1, 1) = \{ f \in N : | f'(z)-1 | < 1, z \in U \}.$$

The above classes have been introduced, respectively, by Makowka [7], Caplinger and Causey [2], Padmanabhan [9], and MacGregor [5].

Also by taking different values of the parameters α, β, λ, A, B, the class R^{λ} (a, β, A, B) reduces to the following subclasses of R, introduced by Ahuja [1] :

$$R^{\lambda}_{\alpha} = R^{\lambda}(\alpha, 1, -1, 1) = \{ f \in N : \mathrm{Re}\,(e^{i\lambda} f'(z)) > \alpha \cos\lambda,$$

$$o \leqslant \alpha < 1, | \lambda | < \frac{\pi}{2}, z \in U \},$$

$$R^{*\lambda} = R^{\lambda}\left(o, \frac{2-\cos\lambda}{2}, -1, 1\right)$$

$$= \{ f \in N : | e^{i\lambda} f'(z) - (1+i\sin\lambda) | < 1, | \lambda | < \frac{\pi}{2}, z \in U \},$$

$$R^{*\lambda}_{\delta} = R^{\lambda}\left(o, \frac{2\delta-1}{2\delta}, -1, 1\right)$$

$$= \left\{ f \in N : \left| \frac{e^{i\lambda} f'(z) - i\sin\lambda}{\cos\lambda} - \delta \right| < \delta, \delta > \tfrac{1}{2}, | \lambda | < \frac{\pi}{2}, z \in U \right\},$$

$$R^{*\lambda}(\alpha) = R^{\lambda}(o, 1-\alpha, -1, 1)$$

$$= \left\{ f \in N : \left| \frac{e^{i\lambda} f'(z) - i\sin\lambda}{\cos\lambda} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha},\ 0 \leq \alpha < 1, \right.$$

$$\left. |\lambda| < \frac{\pi}{2},\ z \in U \right\}.$$

We, further, observe that by special choices of α, β, λ, A and B our class $R^{\lambda}(\alpha, \beta, A, B)$ give rise to the following new subclasses of R :

1. $$R^{*\lambda}_{\delta,\alpha} = R^{\lambda}\left(\alpha, \frac{2\delta - 1}{2\delta}, -1, 1\right)$$

$$= \left\{ f \in N : \left| \frac{e^{i\lambda} f'(z) - \alpha\cos\lambda - i\sin\lambda}{(1-\alpha)\cos\lambda} - \delta \right| < \delta,\ \delta > \tfrac{1}{2}, \right.$$

$$\left. 0 \leq \alpha < 1,\ |\lambda| < \frac{\pi}{2},\ z \in U \right\},$$

2. $$R(\gamma, A, B) = R^{0}\left(\frac{-A + A\gamma}{B\gamma - A}, \frac{B\gamma - A}{B - A}, A, B\right)$$

$$= \left\{ f \in N : \left| \frac{f'(z) - 1}{Bf'(z) - A} \right| < \gamma,\ 0 < \gamma \leq 1,\ -1 \leq A < B \leq 1, \right.$$

$$\left. 0 < B \leq 1,\ z \in U \right\},$$

3. $$R^{\lambda}_{\alpha}(A, B) = R^{\lambda}(\alpha, 1, A, B)$$

$$= \left\{ f \in N : \left| \frac{f'(z) - 1}{B f'(z) - [B + (A - B)(1-\alpha)\cos\lambda\, e^{-i\lambda}]} \right| \right.$$

$$\left. < 1,\ z \in U \right\},$$

4. $$R^{\lambda}_{\alpha,\beta}(A, B) = R^{\lambda}\left(\frac{-A + A\beta - (A - B)\alpha\beta}{B\beta - A}, \frac{B\beta - A}{B - A}, A, B\right)$$

$$= \left\{ f \in N : \left| \frac{f'(z) - 1}{B f'(z) - [B + (A - B)(1-\alpha)\cos\lambda\, e^{-i\lambda}]} \right| \right.$$

$$\left. < \beta,\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ z \in U \right\}.$$

Since the class $R^{\gamma}(\alpha, \beta, A, B)$ includes various subclasses of R, a study of its various properties will lead to a unified study of these classes. In this paper, we determine a sufficient condition, coefficient estimates and distortion theorems for $R^{\lambda}(\alpha, \beta, A, B)$. We shall further prove that the subclass $R^{*\lambda}_{\delta,\alpha}$ of R, is closed under convolution.

2. Sufficient Condition

Theorem 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U. Then $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$, if for some α, λ, A and B ($o \leqslant \alpha < 1$, $|\lambda| < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$) the condition

$$\sum_{n=2}^{\infty} n\,|a_n| \leqslant \frac{(B-A)\,\beta\,(1-\alpha)\cos\lambda}{1-A-(B-A)\,\beta}, \text{ whenever}$$

$$o < \beta \leqslant \frac{-A}{(B-A)}, \quad (2.1)$$

and

$$\sum_{n=2}^{\infty} n\,|a_n| \leqslant \frac{(B-A)\,\beta\,(1-\alpha)\cos\lambda}{1+A+(B-A)\,\beta}, \text{ whenever}$$

$$\frac{-A}{(B-A)} \leqslant \beta \leqslant 1, \quad (2.2)$$

holds.

Proof. Let $|z| = r < 1$, and suppose $o < \beta \leqslant \frac{-A}{(B-A)}$. Then

$$|f'(z)-1| - |(B-A)\,\beta\,(f'(z)-1+(1-\alpha)\cos\lambda\, e^{-i\lambda}) + A\,(f'(z)-1)|$$

$$= \left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| - \left|(B-A)\,\beta\,(1-\alpha)\cos\lambda\, e^{-i\lambda} - \sum_{n=2}^{\infty}(-A-(B-A)\,\beta)\, n a_n z^{n-1}\right|$$

$$\leqslant \sum_{n=2}^{\infty}(1+|-A-(B-A)\,\beta|)\, n\,|a_n|\, r^{n-1} - (B-A)\,\beta\,(1-\alpha)\cos\lambda$$

$$< \sum_{n=2}^{\infty} (1-A-(B-A)\beta)\, n\, |a_n| - (B-A)\beta(1-\alpha)\cos\lambda.$$

The last quantity is nonpositive by (2.1), so that $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$.

Next, we assume that (2.2) holds for $\frac{-A}{(B-A)} \leqslant \beta \leqslant 1$. Then

$$|f'(z)-1| - |(B-A)\beta(f'(z)-1+(1-\alpha)\cos\lambda\, e^{-i\lambda}) + A(f'(z)-1)|$$

$$\leqslant \sum_{n=2}^{\infty} (1+A+(B-A)\beta)\, n\, |a_n| - (B-A)\beta(1-\alpha)\cos\lambda \leqslant o.$$

This proves that $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$.

Corollary 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U. Then

$f(z) \in R^{*\lambda}_{\delta, \alpha}$ if for some $o \leqslant \alpha < 1$, $|\lambda| < \frac{\pi}{2}$,

$$\sum_{n=2}^{\infty} n\, |a_n| \leqslant (2\delta-1)(1-\alpha)\cos\lambda, \text{ whenever } \tfrac{1}{2} < \delta \leqslant 1,$$

$$\sum_{n=2}^{\infty} n\, |a_n| \leqslant (1-\alpha)\cos\lambda, \text{ whenever } \delta \leqslant 1,$$

holds.

Corollary 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U. Then $f(z) \in R(\gamma, A, B)$ if for some γ, A, B ($o < \gamma \leqslant 1$, $-1 \leqslant A \leqslant B < 1$, $o < B \leqslant 1$,

$$\sum_{n=2}^{\infty} n\, |a_n| \leqslant \frac{(B-A)\gamma}{(1+B\gamma)}$$

holds,

Corollary 3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U. Then $f(z) \in R_{\alpha}^{\lambda}(A, B)$ if for some α, λ, A and B $(o \leqslant \alpha < 1, |\lambda| < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1, o < B \leqslant 1)$,

$$\sum_{n=2}^{\infty} n |a_n| \leqslant \frac{(B-A)(1-\alpha)\cos\lambda}{(1+B)}$$

holds.

Corollary 4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U. Then $f(z) \in R_{\alpha,\beta}^{\lambda}(A, B)$ if for some α, β, λ, A and B $(o \leqslant \alpha \leqslant 1$, $o < \beta \leqslant 1, |\lambda| < \frac{\pi}{2}, -1 \leqslant A < B \leqslant 1, o < B \leqslant 1)$,

$$\sum_{n=2}^{\infty} n |a_n| \leqslant \frac{(B-A)\beta(1-\alpha)\cos\lambda}{(1+B\beta)}$$

holds.

Remark. By choosing appropriate values of the parameters α, β, λ, A and B in Theorem 1, we obtain the results of Ahuja [1].

Motivated by Theorem 1, we introduce a class $R^{\bar{\lambda}}(\alpha, \beta, A, B)$ of the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, analytic in U, which for some α, λ, A and B $(o \leqslant \alpha < 1, |\lambda| < \frac{\pi}{2}, -1 \leqslant A < B \leqslant 1, o < B \leqslant 1)$ satisfy (2 1) and (2.2). Clearly, $R^{\bar{\lambda}}(\alpha, \beta, A, B) \subset R^{\lambda}(\alpha, \beta, A, B)$. Then the following theorem is in order.

Theorem 2 If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n ,$$

belong to $\bar{R}^{\lambda}(\alpha, \beta, A, B)$, then so does F (z), where F (z) is defined by

$$F(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n .$$

Proof. First suppose that f (z) and g (z) satisfy

(2.1). Then

$$\sum_{n=2}^{\infty} n \mid a_n \mid \leqslant \frac{(B-A)\beta(1-\alpha)\cos\lambda}{1-A-(B-A)\beta} \left(o<\beta\leqslant \frac{-A}{(B-A)} \right), \tag{2.3}$$

$$\sum_{n=2}^{\infty} n \mid b_n \mid \leqslant \frac{(B-A)\beta(1-\alpha)\cos\lambda}{1-A-(B-A)\beta}, \left(o<\beta\leqslant \frac{-A}{(B-A)} \right). \tag{2.4}$$

(2.3) and (2.4) immediately yield

$$\mid a_n \mid \leqslant \frac{(B-A)\beta(1-\alpha)\cos\lambda}{n(1-A-(B-A)\beta)} < 1, (n \geq 2, -1 \leqslant A < B \leqslant 1, o < B \leqslant 1), \tag{2.5}$$

$$\mid b_n \mid \leqslant \frac{(B-A)\beta(1-\alpha)\cos\lambda}{n(1-A-(B-A)\beta)} < 1, (n \geq 2, -1 \leqslant A < B \leqslant 1, o < B \leqslant 1). \tag{2.6}$$

Therefore, using (2.5) we obtain

$$\sum_{n=2}^{\infty} n \mid a_n \mid^2 \leqslant \sum_{n=2}^{\infty} n \mid a_n \mid \leqslant \frac{(B-A)\beta(1-\alpha)\cos\lambda}{(1-A-(B-A)\beta)} \quad \text{(by (2.3)}. \tag{2.7}$$

Similarly, using (2.4) and (2.6) we get

$$\sum_{n=2}^{\infty} n \mid b_n \mid^2 \leqslant \frac{(B-A)\beta(1-\alpha)\cos\lambda}{(1-A-(B-A)\beta)} . \tag{2.8}$$

Now we have

$$\sum_{n=2}^{\infty} n\,|a_n|\,|b_n| \leqslant \left(\sum_{n=2}^{\infty} n\,|a_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty} n\,|b_n|^2\right)^{\frac{1}{2}}$$

$$\leqslant \frac{(B-A)\,\beta\,(1-\alpha)\cos\lambda}{(1-A-(B-A)\,\beta)} \cdot \left(o<\beta\leqslant \frac{-A}{(B-A)}\right),$$

where we have applied Schwarz's inequality [8] and relations (2.7) and (2.8). The last inequality proves that $F(z) \in \overline{R^{\lambda}}(\alpha, \beta, A, B)$. For the case when f, g satisfy (2.2), the proof is similar, and hence is omitted.

3. Coefficient Estimates

Theorem 3. Let $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$, $o<\beta\leqslant\left(\frac{1-A}{B-A}\right)$, and

$f(z) = z + \sum_{m=2}^{\infty} a_m z^m$, $z \in U$. Then

$$|a_n| \leqslant \frac{(B-A)\,\beta\,(1-\alpha)\cos\lambda}{n}, \quad n \geqslant 2. \tag{3.1}$$

The inequality is sharp for all admissible values of α, β, λ, A, B and for each n.

Proof. Since $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$, it follows from Schwarz's Lemma [8]

$$f'(z) = \frac{1+\{[(B-A)\,\beta+A]-(B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda}\}\,w(z)}{1+[(B-A)\,\beta+A]\,w(z)}, \tag{3.2}$$

where $w(z) = \sum_{m=1}^{\infty} t_m z^m = z\,\phi(z)$ is analytic in U and satisfies the condition $w(o) = o$, $|w(z)| < 1$ for $z \in U$. Then (3.2) may be written as

$$\left\{(B-A)\,\beta\,(1-\alpha)\cos\lambda\, e^{-i\lambda} + \sum_{m=2}^{\infty} [(B-A)\,\beta + A]\, m a_m z^{m-1}\right\}.$$

$$\left\{\sum_{m=1}^{\infty} t_m z^m\right\} = -\sum_{m=2}^{\infty} m a_m z^{m-1} . \qquad (3.3)$$

Equating corresponding coefficients on both sides of (3.3) we find that the coefficient a_n on the right of (3.3) depends only on $a_2, a_3 \ldots,$ a_{n-1} on the left of (3.3). Therefore, for $n \geq 2$, (3.3) yields

$$\left\{(B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda} + \sum_{m=2}^{n-1} [(B-A)\beta + A] m\, a_m z^{m-1}\right\} w(z)$$

$$= -\sum_{m=2}^{n} m\, a_m z^{m-1} - \sum_{m=n+1}^{\infty} b_m z^{m-1} ,$$

where $\sum_{m=n+1}^{\infty} b_m z^{m-1}$ converges in U. Then since $|w(z)| < 1$

we get

$$\left|(B-A)\,\beta\,(1-\alpha)\cos\lambda\, e^{-i\lambda} + \sum_{m=2}^{n-1} [(B-A)\,\beta + A]\, m\, a_m z^{m-1}\right|$$

$$\geq \left|\sum_{m=2}^{n} m\, a_m z^{m-1} + \sum_{m=n+1}^{\infty} b_m z^{m-1}\right| . \qquad (3.4)$$

Writing $z = re^{i\theta}$, $r < 1$, squaring both sides of (3.4), and then integrating we obtain

$$(B-A)^2 \beta^2 (1-\alpha)^2 \cos^2\lambda + \sum_{m=2}^{n-1} [(B-A)\,\beta + A]^2 m^2 |a_m|^2 r^{2(m-1)}$$

$$\geq \sum_{m=2}^{n} m^2 |a_m|^2 r^{2(m-1)} + \sum_{m=n+1}^{\infty} |b_m|^2 r^{2(m-1)}.$$

Taking the limit as r approaches 1 we have

$$n^2 \, | a_n |^2 \leqslant (B-A)^2 \beta^2 (1-\alpha)^2 \cos^2 \lambda - \{1-[(B-A)\beta+A]^2\}. \sum_{m=2}^{n-1} m^2 \, | a_m |^2 . \qquad (3.5)$$

Since $o<\beta \leqslant \left(\frac{1-A}{B-A}\right)$, (3.5) yields

$$n^2 \, | a_n |^2 \leqslant (B-A)^2 \beta^2 (1-\alpha)^2 \cos^2 \lambda ,$$

which implies

$$| a_n | \leqslant \frac{(B-A) \beta (1-\alpha) \cos \lambda}{n} \quad (n \geqq 2).$$

The following example shows that the inequality (3.1) is sharp.

Example. Let

$$f(z) = \int_0^z \frac{1-\{[(B-A)\beta+A]-(B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda}\}\, t^{n-1}}{1-[(B-A)\beta+A]\, t^{n-1}} \, dt,$$

where $o \leqslant \alpha < 1$, $o < \beta \leqslant \left(\frac{1-A}{B-A}\right)$, $|\lambda| < \frac{\pi}{2}$, and $-1 \leqslant A < B \leqslant 1$, $o<\beta\leqslant 1$. Then it is easy to see that

$$\left| \frac{f'(z) - 1}{(B-A)\beta\,(f'(z)-1+(1-\alpha)\cos\lambda\, e^{-i\lambda})+A\,(f'(z)-1)} \right| < 1, \; (z \in U),$$

which proves that $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$. The function $f(z)$ has the expansion

$$f(z) = z + \frac{(B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda}}{z} z^n + \ldots, \; (z \in U),$$

which shows that the estimate (3.1) is sharp.

Corollary 5. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $R^{*\lambda}_{\delta,\alpha}$, then

$$| a_n | \leqslant \left(\frac{2\delta - 1}{\delta} \right) \frac{(1-\alpha) \cos \lambda}{n}, \ (n \geq 2).$$

Corollary 6. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $R(\gamma, A, B)$, then

$$| a_n | \leqslant \frac{(B-A)\gamma}{n}, \ (n \geq 2).$$

Corollary 7. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $R_\alpha^\lambda (A, B)$, then

$$| a_n | \leqslant \frac{(B-A)(1-\alpha) \cos \lambda}{n}, \ (n \geq 2).$$

Corollary 8. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $R_{\alpha, \beta}^\lambda (A, B)$, then

$$| a_n | \leqslant \frac{(B-A)\beta(1-\alpha) \cos \lambda}{n}, \ (n \geq 2).$$

Remark. By taking appropriate values of the parameters α, β, λ, A and B in Theorem 3 we obtain the corresponding results established by Makówka [7], Caplineger and Causey [2], Padmanabhan [9], Goel [4], MacGregor [5, 6] and Ahuja [1].

4. Distortion Theorems

We now turn to an investigation of distortion properties of $R^\lambda (\alpha, \beta, A, B)$.

Theorem 4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R^\lambda (\alpha, \beta, A, B)$. Then for $z \in U$,

$| f(z) | \leqslant$

$$\int_0^{|z|} \frac{1+(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,t+[(B-A)\,\beta+A\,\{(B-A)\,\beta\,(1-\alpha)\cos^2\lambda-[(B+A]\}\,t^2}{1-[(B-A)\,\beta+A]^2\,t^2}\,dt, \tag{4.1}$$

and

$$\int_0^{|z|} \frac{1-(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,t+[(B-A)\,\beta+A]\,\{(B-A)\,\beta\,(1-\alpha)\cos^2\lambda-[(B-A)\,\beta+A]\}\,t^2}{1-[(B-A)\,\beta+A]^2\,t^2}\,dt. \tag{4.2}$$

For $\beta=\left(\frac{-A}{B-A}\right)$, the above estimates reduce to

$$|\,f(z)\,| \leqslant r - \frac{A\,(1-\alpha)\cos\lambda\,.\,r^2}{2},$$

$$|\,f(z)\,| \geqslant r + \frac{A\,(1-\alpha)\cos\lambda\,.\,r^2}{2}, \ (\,|\,z\,| = r).$$

The bounds are sharp.

Proof. Since $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$, we observe that the condition (1.3) coupled with an application of Schwarz's Lemma [8], implies

$|\,f'(z) - \xi\,| < R$, where

$$\xi = \frac{1-[(B-A)\,\beta+A]\,\{[(B-A)\,\beta+A]-(B-A)\,\beta\,(1-\alpha)\cos^2\lambda\}\,r^2 - i\beta\left(\frac{B-A}{2}\right)[(B-A)\,\beta+A]\,(1-\alpha)\sin 2\lambda\,r^2}{1-[(B-A)\,\beta+A]^2\,r^2}, \tag{4.3}$$

$$R = \frac{(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,r}{1-[(B-A)\,\beta+A]^2\,r^2}, \ (\,|\,z\,| = r). \tag{4.4}$$

Hence we have

$$\frac{1-(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,r+[(B-A)\,\beta+A]\,\{(B-A)\,\beta\,(1-\alpha)\cos^2\lambda-[(B-A)\,\beta+A]\}\,r^2}{1-[(B-A)\,\beta+A]^2\,r^2}$$

$$\leqslant \mathrm{Re}\,(f'(z)) \leqslant$$

$$\frac{1+(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,r+[(B-A)\,\beta+A]\,\{(B-A)\,\beta\,(1-\alpha)\cos^2\lambda-[(B-A)\,\beta+A]\}\,r^2}{1-[(B-A)\,\beta+A]^2\,r^2}. \quad (4.5)$$

If

$$g\,(z)=\frac{1+(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,z+[(B-A)\,\beta+A]\,\{(B-A)\,\beta\,(1-\alpha)\cos^2\lambda-[(B-A)\,\beta+A]\}\,z^2}{1-[(B-A)\,\beta+A]^2\,z^2},$$

then, since $g\,(o)=1=f'\,(o)$ and g is univalent in U, it follows that f' is subordinate to g. Hence

$$|\,f'(z)\,| \leqslant$$

$$\frac{1+(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,r+[(B-A)\,\beta+A]\,\{(B-A)\,\beta\,(1-\alpha)\cos^2\lambda-[(B-A)\,\beta+A]\}\,r^2}{1-[(B-A)\,\beta+A]^2\,r^2}, \quad (4\,6)$$

In view of

$$|\,f(z)\,|=\left|\int_o^z f'(s)\,ds\,\right| \leqslant \int_o^{|z|}\left|\,f'(t\,e^{i\theta})\,\right|\,dt,$$

and with the aid of (4.6) we may write

$$|\,f(z)\,| \leqslant$$

$$\int_o^{|z|}\frac{1+(B-A)\,\beta\,(1-\alpha)\cos\lambda\,.\,t+[(B-A)\,\beta+A]\{(B-A)\,\beta\,(1-\alpha)\cos^2\lambda-[(B-A)\,\beta+A\}\,t^2}{1-[(B-A)\,\beta+A]^2\,t^2}\,dt.$$

Further, by using (4.5) we obtain

$$|f(z)| \geq \int_{0}^{|z|} \text{Re}\,(f'(te^{i\theta}))\,dt \geq$$

$$\geq \int_{0}^{|z|} \frac{1-(B-A)\beta(1-\alpha)\cos\lambda\,.\,t+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}t^2}{1-[(B-A)\beta+A]^2t^2}\,dt.$$

The following example shows that the inequalities (4.1) and (4.2) are sharp.

Example. Let

$$f(z) = \int_{0}^{z} \frac{1+(B-A)\beta(1-\alpha)\cos\lambda\,.\,t+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}t^2}{1-[(B-A)\beta+A]^2t^2}\,dt, \tag{4.7}$$

where $o \leqslant \alpha < 1$, $o < \beta \leqslant 1$ $|\lambda| < \frac{\pi}{2}$ and $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$. It is easy to verify that $f(z) \in R^{\lambda}(\alpha, \beta, A, B)$, and that the equalities in (4.1) and (4.2) are attained for $z = \pm r$.

Remark 1. Taking appropriate values of the parameters α, β, λ, A and B in Theorem 4 we get the distortion theorems for functions in the classes $R^{*\lambda}_{\delta,\alpha}$, $R(\gamma, A, B)$, $R^{\lambda}_{\alpha}(A, B)$ and $R^{\lambda}_{\alpha,\beta}(A, B)$.

Remark 2. For suitable values of α, β, λ, A and B in Theorem 4, we obtain the results of Ahuja [1], Maköwka [7], Padmanabhan [9], Shaffer [10], and MacGregor [5, 6].

5. Convolution of Functions

In this section, we prove that the class $R^{*\lambda}_{\delta,\alpha}$ is closed under convolution.

Theorem 5. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to $R^{*\lambda}_{\delta,\alpha}$, then

$$F(z) = 1 + \tfrac{1}{2} \sum_{n=2}^{\infty} n\, a_n\, b_n\, z^n$$

is also a member of $R^{*\lambda}_{\delta,\alpha}$.

Proof. Since f and g belong to $R^{*\lambda}_{\delta,\alpha}$, therefore

$$\left| \frac{(1+i\tan\lambda) f'(z) - i\tan\lambda - \alpha}{1-\alpha} - \delta \right| < \delta, \ (z \in U),$$

and

$$\left| \frac{(1+i\tan\lambda) g'(z) - i\tan\lambda - \alpha}{1-\alpha} - \delta \right| < \delta, \ (z \in U).$$

It is well known [8] that if $h(z) = \sum_{n=o}^{\infty} c_n z^n$ is analytic in U and $|h(z)| \leqslant D$, then

$$\sum_{n=o}^{\infty} |c_n|^2 \leqslant D^2 . \tag{5.1}$$

Using the estimate (to the function

$$\frac{(1+i\tan\lambda) f'(z) - i\tan\lambda - \alpha}{1-\alpha} - \delta ,$$

we get

$$(1-\delta)^2 + \left| \frac{1+i\tan\lambda}{1-\alpha} \right|^2 \sum_{n=2}^{\infty} n^2 |a_n|^2 \leqslant \delta^2 ,$$

which yields

$$\sum_{n=2}^{\infty} n^2 \, | a_n |^2 \leqslant (2\delta-1)(1-\alpha)^2 \cos^2 \lambda, \ (\delta > \tfrac{1}{2}). \tag{5.2}$$

Similarly, applying the estimate (5.1) to the function

$$\frac{(1+i \tan \lambda)\, g'(z) - i \tan \lambda - \alpha}{1-\alpha} - \delta ,$$

we obtain

$$\sum_{n=2}^{\infty} n^2 \, | b_n |^2 \leqslant (2\delta-1)(1-\alpha)^2 \cos^2 \lambda, \ (\delta > \tfrac{1}{2}). \tag{5.3}$$

We have :

$$\left| \frac{(1+i\tan \lambda)\, F'(z) - i\tan \lambda - \alpha}{1-\alpha} - \delta \right|^2$$

$$= \left| (1-\delta) + \tfrac{1}{2} \left(\frac{1+i\tan \lambda}{1-\alpha} \right) \sum_{n=2}^{\infty} n^2 a_n b_n z^{n-1} \right|^2$$

$$\leqslant (1-\delta)^2 + \left(\frac{1-\delta}{1-\alpha} \right) \sec \lambda \sum_{n=2}^{\infty} n^2 \left| a_n \right| \left| b_n \right| r^{n-1} +$$

$$\tfrac{1}{4} \frac{\sec^2 \lambda}{(1-\alpha)^2} \sum_{n=2}^{\infty} n^4 \left| a_n \right|^2 \left| b_n \right|^2 . r^{2(n-1)} , \ (| z | = r)$$

$$\leqslant (1-\delta)^2 + \left(\frac{1-\delta}{1-\alpha} \right) \sec \lambda . \sum_{n=2}^{\infty} n^2 \left| a_n \right| \left| b_n \right| +$$

$$\tfrac{1}{4} \frac{\sec^2 \lambda}{(1-\alpha)^2} \sum_{n=2}^{\infty} n^4 \left| a_n \right|^4 \left| b_n \right|^2$$

$$\leqslant (1-\delta)^2 + \left\{ \left(\frac{1-\delta}{1-\alpha} \right) \sec \lambda + \left(\frac{2\delta-1}{2\delta} \right)^2 \right\} \sum_{n=2}^{\infty} n^2 \, | a_n | | b_n |$$

where we have used Corollary 5, that is, the estimates

$$|a_n| \leqslant \left(\frac{2\delta-1}{\delta}\right)\frac{(1-\alpha)\cos\lambda}{n},$$

and

$$|b_n| \leqslant \left(\frac{2\delta-1}{\delta}\right)\frac{(1-\alpha)\cos\lambda}{n}.$$

Applying Cauchy–Schwarz's inequality [8] we get

$$\left|\frac{(1+i\tan\lambda)F'(z)-i\tan\lambda-\alpha}{1-\alpha}-\delta\right|^2$$

$$\leqslant (1-\delta)^2+\left\{\left(\frac{1-\delta}{1-\alpha}\right)\sec\lambda+\left(\frac{2\delta-1}{2\delta}\right)^2\right\}$$

$$\left(\sum_{n=2}^{\infty} n^2\left|a_n\right|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty} n^2\left|b_n\right|^2\right)^{\frac{1}{2}}$$

$$\leqslant (1-\delta)^2+\left\{\left(\frac{1-\delta}{1-\alpha}\right)\sec\lambda+\left(\frac{2\delta-1}{2\delta}\right)^2\right\}$$

$$(2\delta-1)(1-\alpha)^2\cos^2\lambda$$

$$\leqslant (1-\delta)^2+(1-\delta)(2\delta-1)(1-\alpha)+\frac{(2\delta-1)^3}{4\delta^2}(1-\alpha)^2,$$

by using (5.2) and (5.3). Therefore

$$\left|\frac{(1+i\tan\lambda)F'(z)-i\tan\lambda-\alpha}{1-\alpha}-\delta\right|^2<\delta^2,$$

$$\text{if } (1-\delta)^2+(1-\delta)(2\delta-1)(1-\alpha)+\frac{(2\delta-1)^3}{4\delta^2}(1-\alpha)^2<\delta^2$$

i.e., if $(2\delta-1)\left\{-4\delta^2\alpha-4\delta^3(1-\alpha)+(2\delta-1)^2(1-\alpha)^2\right\}<0$, which

is strue for $\delta>\frac{1}{2}$ and $0\leqslant\alpha<1$. Hence $F(z)\in R^{*\lambda}_{\delta,\alpha}$.

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