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ON A SINGULAR BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract

In this paper we are concerned with a singular boundary value problem. This problem is generated on the half line by a differential equation of the second order and a boundary condition including a spectral parameter. The solutions of the considered differential equation are obtained and their properties are given. The discrete spectrum of the problem is investigated and its resolvent is obtained. Furthermore, the resolvent set and continuous spectrum of the singular boundary value problem are studied.

Introduction

It is well known (see pp 144—152 of Ref. 1) that boundary valu problems with spectral parameter in the boundary condition have many interesting applications in mathematical physics. A regular boundary value problem with parameter in the boundary condition was investigated in Ref. 2. In Ref. 3, the case of two-point boundary value problems with eigenvalue parameter in the boundary conditions was discussed. The present paper is devoted to study a singular boundary value problem generated on the half line $0 \leqslant x < \infty$ by the differential equation

$$-y'' + q(x)\, y = \lambda y \qquad (1)$$

and the boundary condition

$$y'(0) - \alpha \lambda y(0) = 0, \qquad (2)$$

where the coefficient $q(x)$ is a complex-valued continuous function on [0, ∞) and satisfies the condition

$$\int_0^\infty x \, | q(x) | \, dx < \infty. \tag{3}$$

Also λ is a complex parameter and α is a real number. In § 1, we obtain some solutions of equation (1) and study their properties. We investigate the discrete spectrum of the boundary value problem (1) - (2) in § 2. In § 3, some theorems on the resolvent set and continuous spectrum of the problem are formulated.

1. Some particular solutions of equation (1).

We shall require solutions of equation (1) which satisfy specific initial conditions at $x = 0$ or which have specific asymptotic behaviour as $x \to \infty$.

Now, from condition 3) it is clear that (1) reduces to the simpler equation $-y'' = \lambda y$ as $x \to \infty$. This permits us a complete investigation of the properties of the solution to equation (1), and this is our aim in this section. We shall use the following notation:

$\sigma(x) = \int_x^\infty | q(t) | \, dt$, $\sigma_1(x) = \int_x^\infty \sigma(t) \, dt$ and $s = \lambda^{\frac{1}{2}}$ such that $0 \leqslant \arg s \leqslant \pi$. One can easily verify that condition (3) is equivalent to the summability of the function $\sigma(x)$ over the entire half line [0, ∞), i.e., to the inequality $\sigma_1(0) < \infty$.

Theorem 1. For any s in the closed upper half plane, equation (1) has a solution $Q(x, s)$ which satisfies as $x \to 0$ the conditions

$$Q(x, s) = 1 + O(x), \qquad Q'(x, s) = \alpha s + O(1) \tag{4}$$

This solution is an analytic function of s for $\operatorname{Im} s > 0$ and continuous in the closed half plane $\operatorname{Im} s \geqslant 0$.

Proof. Let the function $Q(x, s) = O(x)$ for $x \to 0$ satisfy the integral equation

$$Q(x, s) = \cos sx + \alpha \sin sx + \int_0^x \frac{\sin s(x-t)}{s} q(t) . Q(t, s) \, dt \tag{5}$$

Then $Q(x, s)$ is obviously a solution of equation (1). We seek the solution of this integral equation for $\operatorname{Im} s \geq 0$ in the form

$$Q(x, s) = e^{isx} \psi(s, x$$

Then the resulting equation for $\psi(s, x)$ is

$$\psi(x, s) = (\cos sx + \alpha \sin sx)\, e^{isx} + \int_0^x \frac{\sin s(x-t)}{sx} \,|\, q(t) \,|\, e^{-is(x-t)} \psi(t, s)\, dt. \tag{6}$$

which can be solved by successive approximation upon setting

$$\psi(x, s) = \sum_{k=0}^{\infty} \psi_k(x, s), \tag{7}$$

where

$$\psi_o(x, s) = (\cos sx + \alpha \sin sx)\, e^{isx}$$

and

$$\psi_k(x, s) = \int_0^x \frac{\sin s(x-t)}{sx} \,|\, q(t) \,|\, \psi_{k-1}(t, s)\, dt.$$

Since, for any s from the closed upper half plane, i.e. $\operatorname{Im} s \geq 0$, we have

$$|(\cos sx + \alpha \sin sx)\, e^{isx}| \leqslant C$$

and

$$\left| \frac{\sin s(x-t)}{sx} e^{is(x-t)} \right| \leqslant \left| \frac{\sin s(x-t)}{sx} \right| = \left| \frac{\sin sx \cos st - \cos sx \sin st}{sx} \right| \leqslant 1 - \frac{t}{x} \leqslant 1 \quad \text{for} \quad 0 \leqslant t \leqslant x,$$

the series (7) is majorized by the series $\sum_{k=0}^{\infty} Z_k(x)$, where

$$Z_k(x) = x \int_0^x |\, q(t) \,|\, Z_{k-1}(t)\, dt.$$

In fact, for k=0, we have

$$|\psi_0(x,s)| \leqslant C = Z_0(x).$$

Suppose that this is true for k = n, i.e.,

$$|\psi_n(x,s)| \leqslant Z_n(x) = x\int_0^x |q(t)|\, Z_{n-1}(t)\,dt.$$

We shall show that this is true when k = n + 1.

$$|\psi_{n+1}(x\ \ s)| \leqslant x\int_0^x |q(t)|\, Z_n(t)\,dt = Z_{n+1}.$$

The series $\sum_{k=0}^{\infty} Z_k(x)$ is clearly uniformly convergent on each finite interval of the half line [0, ∞). Indeed, a simple induction shows that

$$0 \leqslant Z_k(x) \leqslant \frac{C}{k!}\,[x\int_0^x |q(t)|\,dt]^k$$

It follows that for any $0 < a < \infty$, the series (7) converges uniformly in the domain $0 \leqslant x \leqslant a$, $\operatorname{Im} s \geqslant 0$, and its sum $\psi(x, s)$ satisfies equation (6) and the inequality

$$|\psi(x,s)| \leqq C \exp\{x\int_0^x |q(t)|\,dt\}. \tag{8}$$

Moreover, $\psi(x, s)$ is an analytic function of s for $\operatorname{Im} s > 0$, and is continuous in the closed upper half plane $\operatorname{Im} s \geqslant 0$. But this means that the function $Q(x, s) = e^{-isx}\psi(x, s)$ satisfies both the equations (5) and (1) and the inequality

$$|Q(x,s)\,e^{isx}| \leqslant C \exp\{x\int_0^x |q(t)|\,dt\}. \tag{9}$$

Also, $Q(x, s)$ is an analytic function of s for $\operatorname{Im} s > 0$ and continuous in the closed half plane $\operatorname{Im} s \geqslant 0$.

Now, equation (5) and the one resulting from it upon differentiating with respect to x imply, together with estimate (9), that

$$| Q (x, s) - (\cos sx + \alpha \sin sx) |$$

$$\leqslant \int_0^x \frac{\sin s(x-t)}{x} x | q(t) | . | Q(t, s) | dt$$

$$\leqslant C \int_0^x x | q(t) | \{\exp [| \operatorname{Im} st | + t \int_0^t | q(\zeta) | d\zeta]\}dt$$

$$\leqslant C \int_0^x x | q(t) | dt \exp \{ | \operatorname{Im} sx | + \int_0^x x | q(t) | dt\}$$

and

$$| Q'(x, s) - (\alpha s \cos x - s \sin sx) | \leqslant C \int_0^x | q(t) | dt \exp \{ | \operatorname{Im} sx |$$

$$+ \int_0^x x | q(t) | dt\}.$$

That is to say, Q (x, s) satisfies conditions (4), i.e.,

$$Q(x, s) = 1 + O(x)$$

$$Q'(x, s) = \alpha s + O(1) \quad \text{for } x \to 0.$$

Remark 1. Similarly, we can prove that equation (1) is solvable for Im s $\leqslant$ 0 and its Q (x, s) is analytic in s in the half plane Im s $<$ 0 and continuous for Im s $\leqslant$ 0.

Lemma 1. For any s from the closed upper half plane, equation (1) has a solution F (x, s) that can be represented in the form

$$F(x, s) = e^{isx} + \int_x^\infty k(x, t) e^{ist} dt,$$

where the kernel k (x, t) satisfies the inequality

$$| k(x, t) | \leqslant \frac{1}{2} \sigma\left(\frac{x+t}{2}\right) \exp\left\{\sigma_1(x) - \sigma_1\left(\frac{x+t}{2}\right)\right\}.$$

In addition

$$k(x, x) = \frac{1}{2}\int_x^{\infty} | q(t) | \, dt.$$

For proof of this Lemma see Marchenko see pp. 120 - 124 of Ref. 4

Lemma 2. The solution F (x, s) is an analytic function of s in the upper half plane Im s > 0 and is continuous on the real line. The following estimates hold through the half plane Im s ≥ 0

$$| F(x, s) | \leqslant \exp\{-\text{Im}\, sx + \sigma_1(x)\} \tag{10}$$

$$| F(x, s) - e^{isx} | \leqslant \{\sigma_1(x) - \sigma_1(x + \frac{1}{|s|})\} \exp\{-\text{Im}\, sx + \sigma_1(x)\} \tag{11}$$

and

$$| F'(x, s) - ise^{isx} | \leqslant \sigma(x) \exp\{-\text{Im}\, sx + \sigma_1(x)\} \tag{12}$$

See pp. 126—128 of Ref. 4.

Lemma 3. The solution F (x, s) has the following asymptotic behaviour

$$F(x, s) = e^{isx}(1 + O(1)),\ F'(x, s) = e^{isx}(is + O(1)) \tag{13}$$

as $x \to \infty$ for all Im s ≥ 0, s ≠ 0

and

$$F(x, s) = e^{isx}(1 + O(\frac{1}{s})),\ F'(x, s) = ise^{isx}(1 + O(\frac{1}{s})) \tag{14}$$

as $|x| \to \infty$ for all x and Im s ≥ 0 (see pp. 294—298 of Ref. 5). Further, for real s ≠ 0, the functions F (x, s) and F (x,—s) form a fundamental system of solutions of equation (1) and their Wronskian is equal to −2 is :

$$W[F(x, s), F(x, -s)] = F(x, s) F'(x, -s) - F'(x, s) F(x, -s)$$
$$= -2is, \ \text{Im}\, s = 0.$$

2. Discrete spectrum of the boundary value problem (1)—(2).

In this section, we define the eigenvalues of the boundary value problem (1)—(2).

Theorem 2. The boundary value problem (1)—(2) does not have eigenvalues on the positive semi-axis See Ref. 5.

Theorem 3. The eigenvalues of the boundary value problem (1) – (2) are given by solutions s in the upper half plane of

$$W(s) = F'(0, s) - \alpha s^2 F(0, s) = 0.$$

The eigenvalues are then $\lambda = s^2$ They are bounded, finite or countable in number and accumulate only on the real axis.

Proof. Equation (1) has a solution satisfying the initial conditions.

$$Q(0, s) = 1 \quad \text{and} \quad Q'(0, s) = \alpha s, \tag{15}$$

Since the two functions $F(x, -s)$ and $F(x, s)$, form a fundamental system of solutions to equation (1) for all $s \neq 0$, we can write

$$Q(x, s) = C_1 F(x, -s) + C_2 F(x, s).$$

Letting x approach 0 and taking the initial condition (15) into account, we find

$$C_1 = \frac{F'(0, s) - \alpha s F(0, s)}{2is} \quad \text{and} \quad C_2 = \frac{F'(0, s) - \alpha s F(0, -s)}{-2is}.$$

Whence

$$Q(x, s) = (2is^2)^{-1} \{[F'(0, s) - \alpha s^2 F(0, s)] F(x, -s) - [F'(0, -s) - \alpha s^2 F(0, -s)] F(x, s)\}.$$

Since the general solution of equation (1) which satisfies the initial condition (15) has the form $y = CQ(x, s)$, it follows that $\lambda = s^2$ is an eigenvalue of the boundary value problem (1) – (2) if and only if the function $Q(x, s)$ is in $£_2(0, \infty)$. From relation (13), $F(x, s)$ is in $£_2(0, \infty)$ and $F(x, -s)$ is not as $\operatorname{Im} s > 0$, Consequently $Q(x, s)$ is in $£_2(0, \infty)$ if and only if

$$F'(0, s) - \alpha s^2 F(0, s) = W(s) = 0.$$

Now we shall prove that the zeros of the function W(s) are bounded in the upper half plane Im $s > 0$.

This follows immediately from (14), since

$$F'(0, s) = is\,(1 + O(1)) \quad \text{as} \quad |s| \to \infty.$$

Thus W(s) cannot be zero and hence its zeros are bounded. Now, since in the upper half plane Im $s > 0$ the function W(s) is analytic as are F (0, s) and F′ (0, s), the set of its zeros is no more than countable and can have 0 as the only possible limit point on the real axis. Hence the theorem is completely proved.

Corollary. Theorem 3 conforms to our earlier result in Theorem 2.

3. The resolvent set and continuous spectrum of the boundry value problem (1)–(2).

The objective of the present section is to construct the resolvent and prove some theorems on the resolvent set, continuous spectrum of the boundary value problem (1)–(2).

Theorem 4.

(i) The set of numbers $\{\lambda = s^2,\ W(s) \neq 0,\ \text{Im}\, s > 0\}$ belongs to the resolvent set of the boundary value problem (1)–(2).

(ii) The resolvent of the boundary value problem (1)–(2) is an integral operator $R_\lambda(f) = \int_0^\infty R(x, t, s)\, f(t)\, dt$ with the kernel

$$R(x, t, s) = \frac{-1}{W(s)} \begin{cases} F(x, s)\, Q(t, s), & 0 \leqslant t \leqslant x \\ Q(x, s)\, F(t, s), & x \leqslant t < \infty \end{cases}$$

Proof Statement (i) is evident. Since by Theorem 3, it follows that all numbers $\lambda = s^2$, Im $s > 0$, $W(s) \neq 0$ belong to the resolvent set of the boundary value problem (1)–(2). Now, let $\lambda = s^2$ be not an eigenvalue of the boundary value problem (1)–(2), that is, $W(s) \neq 0$. Then, by variation of parameters we find Green's function and thus the resolvent $R(x, t, s)$ of the boundary value problem (1) — (2). If f is in $\pounds_2(0, -\infty)$, then we obtain

$$y = R_\lambda(f) = -\frac{1}{W(s)}\left[F(x, s)\int_0^x Q(t, s) f(t)\, dt + Q(x, s)\int_x^\infty F(t, s) f(t)\, dt\right],$$

$\operatorname{Im} s > 0$, and hence the proof of (ii) follows at once.

Lemma 4. For every $\tau = \operatorname{Im} s > 0$, the formulae

$$A_o f(x) = \exp(-\tau x)\int_0^\infty \exp(\tau\zeta) f(\zeta)\, d\zeta$$

$$B_o f(x) = \exp(\tau x)\int_x^\infty \exp(-\tau\zeta) f(\zeta)\, d\zeta$$

define in the space $\pounds_2(0, \infty)$ linear continuous operators, and

$$\| A_o \| \leqslant \frac{1}{\tau}, \quad \| B_o \| \leqslant \frac{1}{\tau}$$

For the proof see p. 302 of Ref. 5.

Theorem 5.

(i) For every $\delta > 0$, there is a number C_δ such that

$$\| R_\lambda \| \leqslant \frac{C_\delta}{| W(s) | \tau} \quad \text{for} \quad \tau > 0, | s | \geqslant \delta.$$

(ii) Every point on the non-negative real axis $\lambda \geqslant 0$ is in the continuous spectrum of the boundary value problem (1)—(2).

Proof. We now use the inequalities

$$| Q(x, s) | \leqslant C_\delta \exp(\tau x), \quad | F(x, s) | \leqslant C_\delta \exp(-\tau x),$$
$$\tau > 0, \quad | s | \geqslant \delta.$$

The first of which comes from (8) (We recall that by hypothesis $\int_0^\infty x\,|q(x)|\,dx < \infty$), and the second from relation (10) so that

$$\| R_\lambda f \| \leq \frac{1}{|W(s)|} C_\delta \{ \exp(-\tau x) \int_0^x \exp(\tau t) f(t)\,dt$$

$$+ \exp \tau x \int_x^\infty \exp(-\tau t)\, f(t)\,dt \}, \qquad \tau > 0.$$

Using Lemma 4, we obtain the inequality

$$\| R_\lambda f \| \leq \frac{C_\delta}{|W(s)|\,\tau}, \qquad \tau > 0.$$

Hence (i) is proved.

For statement (ii), let λ be a point not on the positive semi-axis and not an eigenvalue and let b be a positive number. Let

$$U(x, s) = \begin{cases} \overline{Q(x, s)}, & 0 \leq x \leq b \\ 0, & b < x < \infty \end{cases}$$

Then $U(x, s) \in £_2(0, \infty)$. Thus we find, for $x > b$,

$$R_\lambda U(x, s) = \frac{-F(x, s)}{W(s)} \int_0^b Q(t, s)\, \overline{Q(t, s)}\, dt$$

$$= \frac{-F(x, s)}{W(s)} \int_0^b |Q(t, s)|^2\, dt$$

$$= \frac{-F(x, s)}{W(s)} \int_0^b |U'(x, s)|^2\, dt.$$

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$$\| R_\lambda U(x,s) \|^2 = \int_0^\infty | R_\lambda U(x,s) |^2 dx \geq \int_b^\infty | R_\lambda U(x\ s) |^2 dx$$

$$= \frac{1}{| W(s) |^2} \left(\int_0^b | U(x,s) |^2 dx \right)^2$$

$$\times \int_b^\infty | F(x,s) |^2 dx .$$

Thus

$$\| R_\lambda \|^2 \geq \frac{1}{| W(s) |^2} \int_0^b | U(x.s) |^2 dx . \int_b^\infty | F(x,s) |^2 dx$$

Now choose b large enough so that $F(x,s) = e^{isx} (1+O(1))$ $| O(1) | < \frac{1}{2}$. Thus we have $| F(x,s) | \geq \frac{1}{2} \exp(-\tau x)$, as $\tau \geq 0$, and therefore

$$\int_b^\infty | F(x,s) |^2 dx \geq \frac{1}{4} \int_b^\infty \exp(-2\tau x)\, dx = \frac{1}{4} \exp(-2\tau b)/2\tau.$$

Then since on any rectangle in the upper half plane with one side on the positive x-axis the integral $\int_0^b | U(x,s) |^2 dx$ is bounded away from zero and we find

$$\| R_\lambda \| \geq \frac{A \exp(-\tau b)}{| W(s) |\ 2\sqrt{2\tau}} ,$$

where A is a constant.

Hence as s approaches any point on the real axis, $\| R_\lambda \|$ is unbounded and the square of the point is in the spectrum of the boundary value problem (1)—(2).

Let us denote by L the operator generated in the space $\pounds_2(0, \infty)$ by the differential expression $-y'' + q(x)y$ and the boundary condition $y'(0) - \alpha\lambda y(0) = 0$.

Let $R(L - \lambda I)$ be the range of $(L - \lambda I)$ Then we have to show that for $\lambda \geq 0$, $R(L - \lambda I)$ is dense in $\pounds_2(0, \infty)$ so that the inverse can be defined. A condition equivalent to this is that the orthogonal complement of $R(L - \lambda I)$ is the zero element. But since the space of solutions of $L^* z = \lambda z$ coincides with the orthogonal complement, by using Lagrange's formula (see p. 7 of Ref. 5) and the resolvent of the boundary value problem (1)–(2) it follows that the operator L^* adjoint to L is generated by the differential expression $-z'' + \bar{q}(x)z$ and the boundary condition $z'(0) - \alpha\bar{\lambda} z(0) = 0$. Hence, by Theorem 2 the number λ cannot be an eigenvalue of the operator L^* and the assertion follows. Thus the set of numbers $\lambda \geq 0$ constitutes the continuous spectrum of the boundary value problem (1)–(2) and hence the theorem is completely proved.

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NUMERICAL SOLUTION OF SOME HIGHLY IMPROPERLY POSED PROBLEMS

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1. Abstract

The paper is concerned with unconstrained and constrained regularization of higly ill-posed problems in the form of Fredholm integral equations of the first kind. In the first part of the paper unconstrained method of Maximum likelihood is used to find the optimal value of the regularization parameter. In the second part constrained Maximum likelihood is used on the same test problems in order to compare the efficiency of the two methods. Comparison of the methods is established through Tables of results and computer diagrams. Highly improperly posed problems available in the literature are tested by the methods.

2. Introduction

Consider the Fredholm Integral Equation of the first kind of convolution type :

$$(k\ f)(x) = \int_{-\infty}^{\infty} k\,(x-t)\, f\,(t)\, dt = g\,(x), \quad -\infty < x < \infty \tag{2.1}$$

where k and g are known functions in L_2 (R), and $f \in H^p$ (R) is to be found. Denoting $\wedge$ as Fourier Transformation symbol, then from the convolution theorem we have

$$\hat{k}\,(\omega)\,\hat{f}\,(\omega) = \hat{g}\,(\omega) \tag{2.2}$$

whence

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(\omega)}{\hat{k}(\omega)} \exp(i\,\omega\,t)\,d\omega \qquad (2.3)$$

The improperly posedness of (2.1) is reflected by the fact that any small perturbation ϵ in g, whose transform $\hat{\epsilon}(\omega)$ dose not decay faster than $\hat{k}(\omega)$ as $w \to \infty$, will result in a perturbation in $\hat{g}(\omega)/\hat{k}(\omega)$ which will grow without bound, when g is inexact. Therefore, we may seek a stable or filtered approximation to f given by

$$f_\lambda(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega\,;\lambda) \frac{\hat{g}(\omega)}{\hat{k}(\omega)} \exp(i\,\omega\,t)\,d\omega \qquad (2.4)$$

where $Z(\omega\,;\lambda)$ is a filtered function dependent on a parameter λ.

Filters may be constructed in several ways, either directly for the convolution kernel [1] or as a special case of general Fredholm Integral Equation [2, 3] provided in the latter case it is realized that in (2.1), the oprator k is not compact and the Fourier transform (FT) here plays the role of a singular function expansion in the context of compact operators. In this paper we construct a maximum likelihood (ML) method with non-negativity constraints and without non-negativity constraints, which determine the regularization parameter λ optimally.

3. Maximum likelihood Method (without non-negativity)

We assume that the support of each function f, g and k is essentially finite and contained within the interval [o. T.]. Let T_N be the space of trigonometric polynomials of degree at most N. and period T. We shall seek a filtered solution of (2.1).

We assume that $g_N(X)$ and g_n are stationary stochastic processes with zero mean, where

$$g_N(x) = \int_0^T \exp(i\,\omega\,x)\, d\,\xi_{g_N}(\omega)$$

$$= \frac{1}{N} \sum_q \hat{g}_{N,q} \exp(i\,\omega_q\,x) \qquad (3.1)$$

and

$$g_n = \int_0^T \exp(i\,\omega\,x)\, d\,\xi_{\epsilon}(\omega) \qquad (3.2)$$

The relevent features of the fuuction ξ_{g_N} and ξ_{ϵ} is that the variance of an integral

$$\int_0^T \theta(\omega)\; d\,\xi_{g_N}(\omega) \text{ is } \int_0^T |\,\theta(\omega)\,|^2\; P_U(\omega)\, d\omega \qquad (3.3)$$

suppose that we have a filter $\{ S_k \}_{k=-\infty}^{\infty}$ such that $f_N(X_k)$ is estimated by

$$\Sigma\, S_{k-m}\, g_m \qquad (3.4)$$

Since

$$\left.\begin{aligned} f_N(z) &= \int_0^T \exp(i\,\omega\,x)\, d\,\xi_{g_N}(\omega)\,|\,\hat{k}(\omega)\,| \\ &\text{where} \\ \hat{k}_N(\omega) &= \Sigma\, K_n \exp(\,i\,\omega\,x_n\,) \end{aligned}\right\} \qquad (3.5)$$

The error of estimate is

$$f_N(\,x_n\,) - \Sigma\, S_{n-m}\, g_m = \int_0^T \exp(\,i\,\omega\,x_n\,)$$

$$\times \left[\frac{1}{\hat{k}_N(\omega)} - \hat{S}_N(\omega) \right] d\xi_{g_N}(\omega)$$

$$- \int_0^T \exp(i\omega x_n)\ \hat{S}_N(\omega)\ d\xi_\epsilon(\omega) \tag{3.6}$$

where

$$\hat{S}(\omega) = \sum_{r=-\infty}^{\infty} S_r \exp(-i\omega x_r) \tag{3.7}$$

then the variance of (3.6) is

$$\int_0^T \left| \frac{1}{\hat{k}_N(\omega)} - \hat{S}_N(\omega) \right|^2 P_{g_N}(\omega)\, d\omega + \int_0^T \left| \hat{S}_N(\omega) \right|^2 P_\epsilon(\omega)\, d\omega \tag{3.8}$$

which is minimized when

$$\hat{S}_N(\omega)\ \hat{k}_N(\omega) = \frac{Pg_N(\omega)}{Pg_N(\omega) + P\epsilon(\omega)} = \beta(\omega) \tag{3.9}$$

Now we shall find the relationship between the filter $\hat{S}(W)$ given by (3.7) and the filter $Z(W)$ Given by [4].

The filtered solution has Fourier Transform

$$\hat{f}_{N,q;\lambda} = \hat{S}_N(\omega_q)\ \hat{g}_N(\omega_q) = Z_{q,\lambda} \frac{\hat{g}(\omega_q)}{\hat{k}(\omega_q)}$$

where (3.10)

$$\hat{g}_N(\omega) = \sum_n g_n \exp(-i\omega x_n);$$

We can compare this with Fourier Transform of (3.4) to get

$$Z(\omega) \frac{\hat{g}(\omega)}{\hat{k}(\omega)} = \hat{S}(\omega)\, \hat{g}(\omega)$$

or

$$Z(\omega) = \hat{S}(\omega)\, \hat{k}(\omega)$$

Thus in our method the ratio $\beta(W)$ in (3.9) in Anderssen and Bloomfields work [1, 2] is equivalent to our filter $Z(W)$.

4. Optimization by Maximum Likelihood (ML Unconstrained)

To optimize the filter w. r. t. λ, we now modify A and B's work [1, 2] accordingly. This involves choosing an error distribution of the form $P_{\epsilon}(\omega) = b\,\phi(\omega)$, where b is an unknown constant and $\phi(\omega)$ is a known function. Also for second order filter (i.e. $p = 2$), we choose as the distribution for g_N

$$P_{g_N}(\omega) = \frac{b\,\phi(\omega)\, |\hat{k}(\omega)|^2}{\lambda\, \omega^4} \tag{4.1}$$

so that

$$Z(\omega) = \frac{|\hat{k}(\omega)|^2}{|\hat{k}(\omega)|^2 + B\lambda\omega^4}$$

$$= \beta(\omega) \left(\because B = \frac{N^2}{T^2} \right) \tag{4.2}$$

Thus the distribution for g_n is given by

$$P_{g_N} = Pg_n - P_{\epsilon}$$

$$Pg_n(\omega, b, \lambda) = b\phi(\omega) + \frac{b\,\phi(\omega)\, |\hat{k}(\omega)|^2}{B\lambda\omega^4}$$

$$= b\,\phi(\omega) \left[1 + \frac{|\hat{k}(\omega)|^2}{B\lambda\omega^4} \right] \tag{4.3}$$

Now Let $\phi(\omega) = 1$ so $P_g = b$

$$\therefore \quad Pg_n(\omega, b, \lambda) = b\left[1 + \frac{|\hat{k}(\omega)|^2}{B\lambda\omega^4}\right] = \frac{1}{1 - z_q} \tag{4.4}$$

where

$$z_q = \frac{|\hat{k}_q|^2}{|\hat{k}_q|^2 + B\lambda\omega^4} \tag{4.5}$$

and

$$S(\omega_q) = \left|\sum_{k=o}^{N-1} g_k \exp(-i\omega x_k)\right|^2 = |\hat{g}_q|^2 \tag{4.6}$$

Anderssen and Bloomfield show how to eliminate the constant b from the problem.

First they approximate the likelihood function of the parameters λ, b by using a formula due to Whittle [5]. This says that the logarithm of likelihood function of Pg_n is approximately,

$$L = \text{Constant} - \tfrac{1}{2}\sum_{q=o}^{N-1}\left[\log Pg_n(\omega_q) + \frac{S(\omega_q)}{Pg_n(\omega_q)}\right] \tag{4.7}$$

(4.7) can be maximized w. r. t. λ, which is equivalent to minimize

$$V_{ML}(\lambda) = \left(\frac{N}{2}\right) \text{Log}\left[\sum_{q=o}^{N-1} |\hat{g}_q|^2 (1 - z_q)\right] \times \sum_{q=1}^{N-1} \log(1 - z_q) \tag{4.8}$$

(For minimizing we have used NAG routine E04ABA based on quadratic interpolation technique).

Now knowing λ from (4.8) we can have

$$\hat{f}_{\lambda, q} = \sum_{q=o}^{N-1} z_q \frac{|\hat{g}_q|}{|\hat{k}_q|} \tag{4.9}$$

Then by inverse Fourier Transform of (4.9) we can find the desired solution function *f*.

5. Maximum likelihood (ML) Method with Non-negativity

In this section our main interest is to develope a method for choosing optimal λ suitable for non-negatively constrained regularization using maximum likelihood with Trigonometric approximation. We propose an extension of the ML Method of the previous section to the constrained case. The performance of ML regularization in the constrained case is dramatically superior as compared to the unconstrained case and it is not expensive to compute.

From the cross validation (CV) Constraned regularizotion method discussed in Iqbal [6] and Wahba [7], we conclude that the indicator set I, obtained through the quadratic programming subroutine (NAG subroutine E04LBF) plays a key role in the algorithm.

It affects the filter function and ultimately affects the expession for $V_{ML}(\lambda)$. Our pth order unconstrained filter is given by equation (4.5) and our unconstrained $V_{ML}(\lambda)$ by equation (4.8). If I is the indicator set underlying the matrix E (see Iqbal [6]) i.e. the set of inactive constraint indices, we approximate the contrained filter.

by

$$\tilde{Z}_{q;\lambda} = \begin{cases} z_{q,\lambda}, & q \in I \\ 0, & q \notin I \end{cases} \tag{5.1}$$

Then $V_{ML}(\lambda)$ in the constrained case may be approximated by

$$V^{M}_{approx}(\lambda) = \frac{N}{2} \log \Big[\sum_{q \in I} (1 - z_{z;\lambda}) \, | \hat{g}_q |^2 + \sum_{q \notin I} | \hat{g}_q |^2 \Big] - \Sigma \log (1 - z_{q;\lambda}) \tag{5.2}$$

where L is the number of inactive constraints.

To minimize $V^{M}_{approx}(\lambda)$ we used the NAG quodratic programming subroutine E04LBF.

For each λ evaluation in the minimization process the subroutine E04LBF is repeated.

Since $V^M_{approx}(\lambda)$ is not necessarily a continuous function of λ we have made a linear search in order to find the optimal value of λ in the constrained case, correspo ding to the least value of V^M_{approx} λ and noted the corresponding solution Vector $\underline{f}_\lambda$.

6. **Problems discussed**

P (1). This problem is highly improperly posed given by Turchin [8] where f is two gaussian function. With essential support $-1.3 < x < 1.5$, k (x) is Triangular with equations given below

$$\int_{-3.1}^{3.2} k(x-t)\, f(t)\, dt = g(x)$$

$$k(x) = \begin{cases} (5/12)(-x + 1.2), & o \leqslant x < 1.2 \\ (5/12)(\ \ x + 1.2), & -1.2 \leqslant x < o \\ o, & |x| \geq 1.2 \end{cases}$$

We have calculated the values of g (x) by the NAG algorithm D01ABA using Rombergs method with accuracy 10^{-7}, 64 grid values have been considered as shown in DIAG (1).

P (2) This problem has been taken from Medgyessy [9] the solution function is the sum of six Guassians and the kernel is also Guassian, we have

$$\int_{-\infty}^{\infty} k(x-y)\, f(y)\, dy = g(x)$$

$$g(x) = \sum_{k=1}^{6} A_k \exp\left[-\frac{(x-\alpha_k)^2}{\beta_k}\right]$$

where

A1 = 10.0	$\alpha_1 = 0.5$	$\beta_1 = 0.04$
A2 = 10.0	$\alpha_2 = 0.7$	$\beta_2 = 0.02$
A3 = 5.0	$\alpha_3 = 0.875$	$\beta_3 = 0.02$
A4 = 10.0	$\alpha_4 = 1.125$	$\beta_4 = 0.04$
A5 = 5.0	$\alpha_5 = 1.325$	$\beta_5 = 0.02$
A6 = 5.0	$\alpha_6 = 1.525$	$\beta_6 = 0.02$

The essential support of g (x) is $0 < x$ 2 the essential support of k (x) is (−0.26, 0.26).

where

$$k(x) = \frac{1}{\sqrt{\lambda \pi}} \exp\left(-\frac{x^2}{\lambda}\right), \quad \lambda = 0.015$$

The solution is

$$f(x) = \sum_{k=1}^{6} \left(\frac{\beta_k}{\beta_k - \lambda}\right)^{\frac{1}{2}} A_k \exp\left(\frac{(x-\alpha_k)^2}{(\beta_k - \lambda)}\right)$$

The essential support of f (x) is (0.26. 1.74) as shown in DIAG (2).

6. (a) Numerical Results (Without non-negativity)

Random noise in the problems is also used, the results are shown for the clean data and for the noisy data.

P (1). Although this is a severely ill-posed problem, for clean data, the method yielded almost perfect solution. For 0.7% noise the method resolved the two peaks clearly.

When p is increased from 2 to 4 in the unconstrained case the solution improves slightly as shown in DIAG (3) and Table 1.

P (2) For clean data the method succeded in resolving all the six peaks, but for 1.7% noise the method resolved almost 5 peaks as shown in DIAG (4) and Table I.

6. (b) Numerical Results (with non-negativity constraints)

We have employed this algorithm on problems P (1) and P (2) with different noise levels added to the data vector the results are summarized in Table 2.

P (1). This higgiy improperly posed problem could not be satisfactorily treated using unconstrained regularization, because large negative lobes were always there. For constrained regularization, the results are enormously superior.

With 0.7% noise the solution is quite good as shown in DIAG (5) with 1.7 noise again the solution resolved the two peaks clearly as shown in DIAG (6), with 3.3% noise, the method could not succeed in resolving the two peaks very clearly as shown in DIAG (7).

P (2). For clean data ML constrained method yielded a good solution resoving all the six peaks, with 1.7% noise again ML constrained gave a very good result as shown in DIAG (8).

Conclusion

For mildly and moderately ill-posed problems and with low level noise, ML constrained method is comparable with CV constrained method [6].

For highly ill-posed problems with low level noise ML constrained worked very well problem P (2) is best solved by ML constrained methods.

TABLE 1

ML Regularization (Unconstrained)

Problem	No of Grid Points	Noise Level	λ	$\| f-f_n \|_2$	Diag
P (1)	64	0.0%	3.20×10^{-16}	7.25×10^{-3}	3
P=2	64	0.7%	5.50×10^{-11}	2.301×10^{-1}	3
P=4	64	0.7%	3.40×10^{-13}	1.352×10^{-1}	3
P (2)	64	0.0%	3.40×10^{-15}	6.260×10^{-1}	4
	64	1.7%	1.10×10^{-11}	3.545×10^{0}	4

TABLE 2

ML Regularization (Constrained Case)

Problem	No of Grid Points	Noise Level	λ	$\| f-f_N \|_2$	Diag
P (1)	64	0.0%	3.10×10^{-17}	6.80×10^{-3}	5
P (1)	64	0.7%	2.511×10^{-10}	5.86×10^{-2}	5
P (1)	64	1.7%	6.100×10^{-9}	7.463×10^{-2}	6
P (1)	64	3.3%	4.130×10^{-7}	2.110×10^{-1}	7
P (2)	64	0.0%	6.10×10^{-15}	5.208×10^{-1}	8
P (2)	64	1.7%	4.142×10^{-11}	2.121×10^{0}	8

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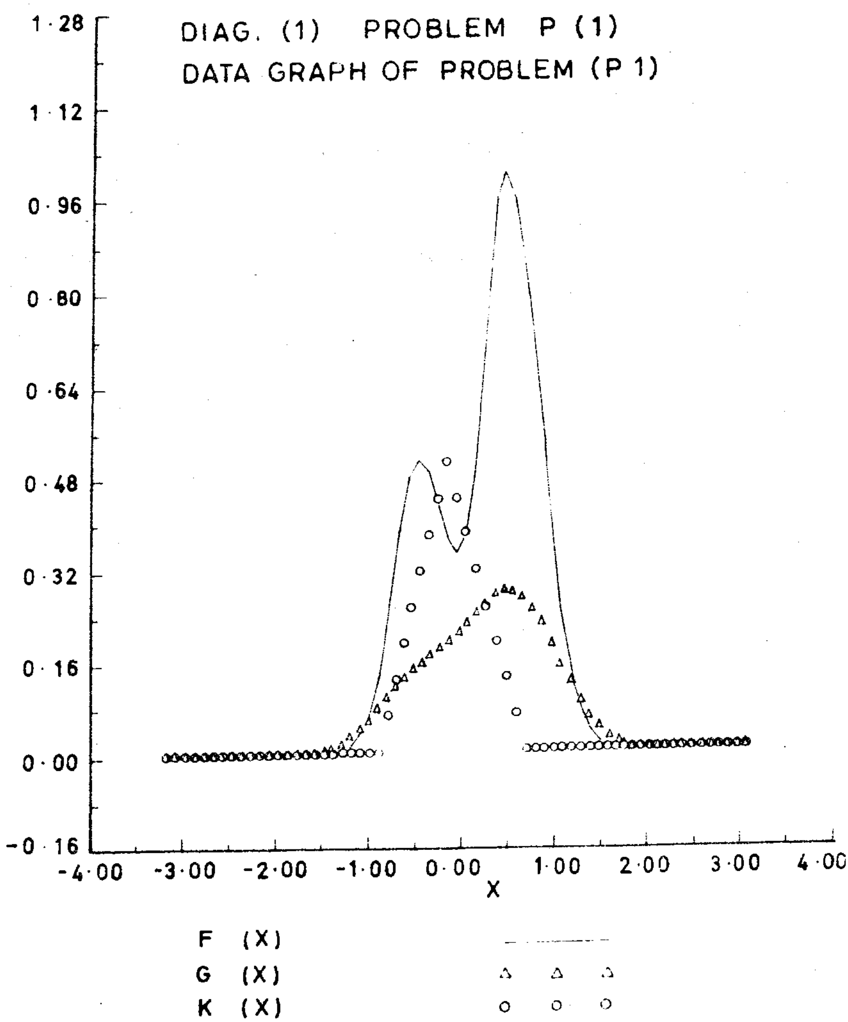
DIAG. (1) PROBLEM P (1)
DATA GRAPH OF PROBLEM (P 1)
1·28
1·12
0·96
0·80
0·64
0·48
0·32
0·16
0·00
-0·16
-4·00
-3·00
-2·00
-1·00
0·00
1·00
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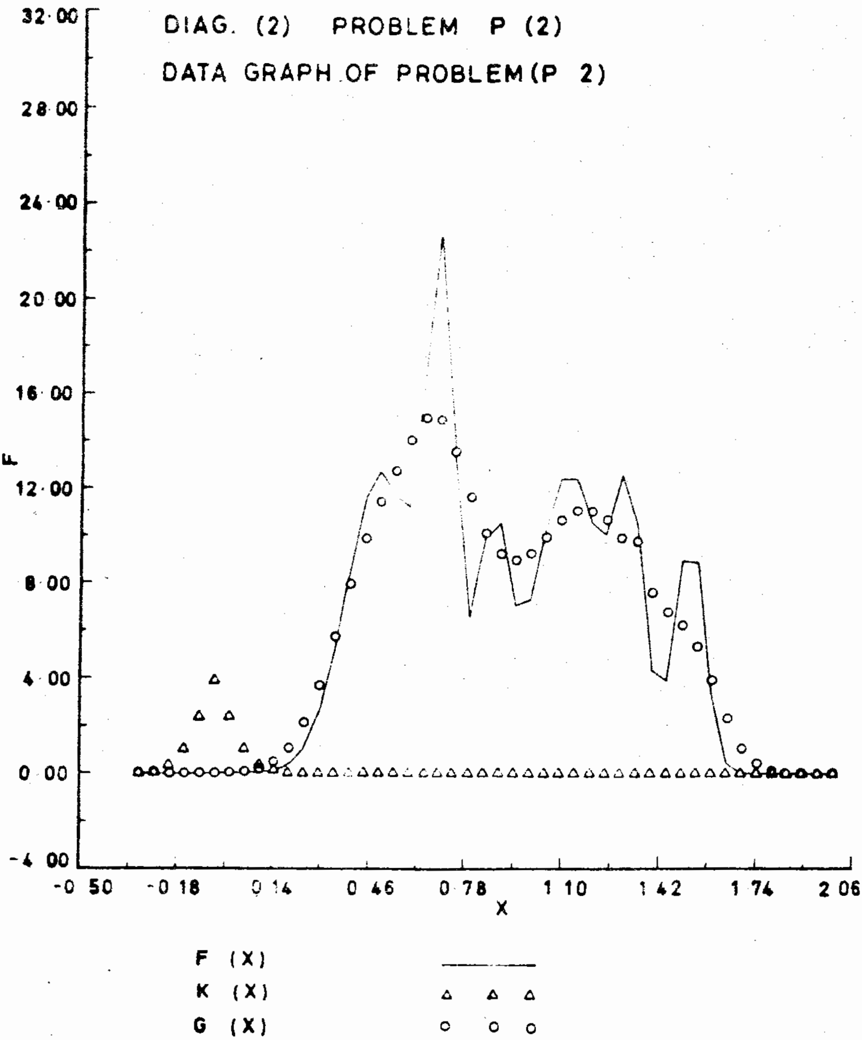
4·00
X
F (X)
G (X)
K (X)


DIAG. (2) PROBLEM P (2)
DATA GRAPH OF PROBLEM (P 2)
32·00
28·00
24·00
20·00
16·00
12·00
8·00
4·00
0·00
-4·00
F
-0·50
-0·18
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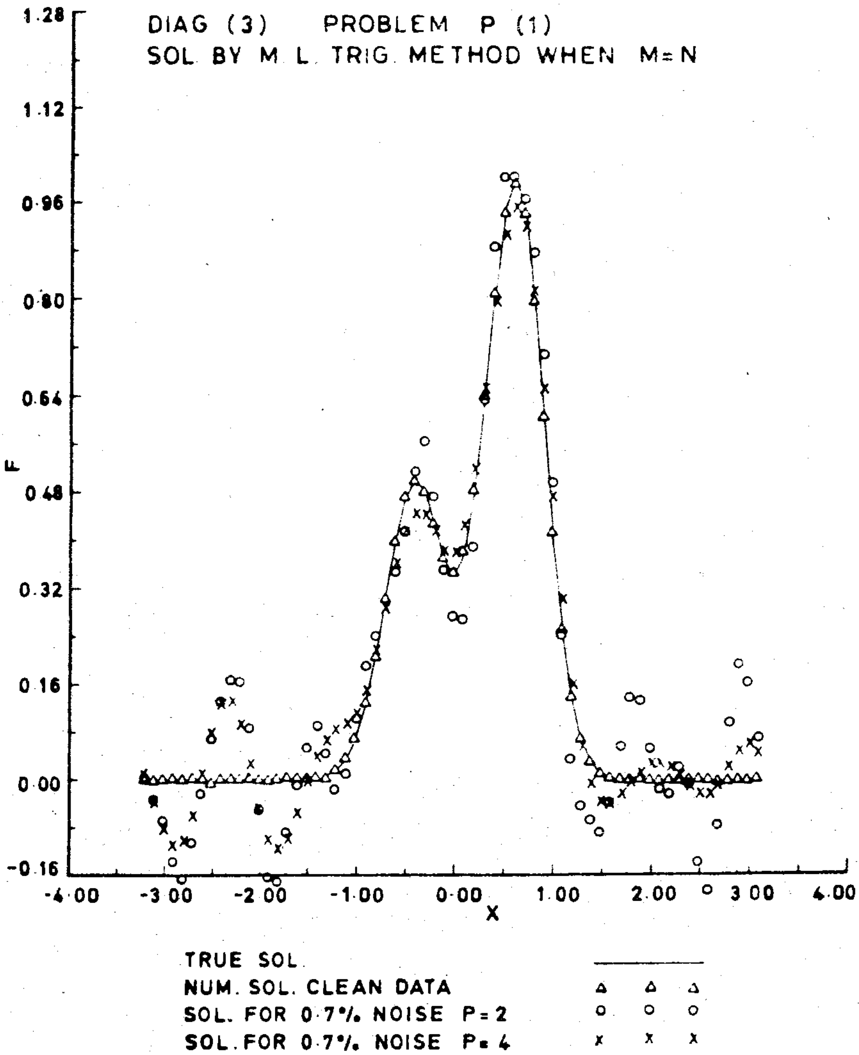
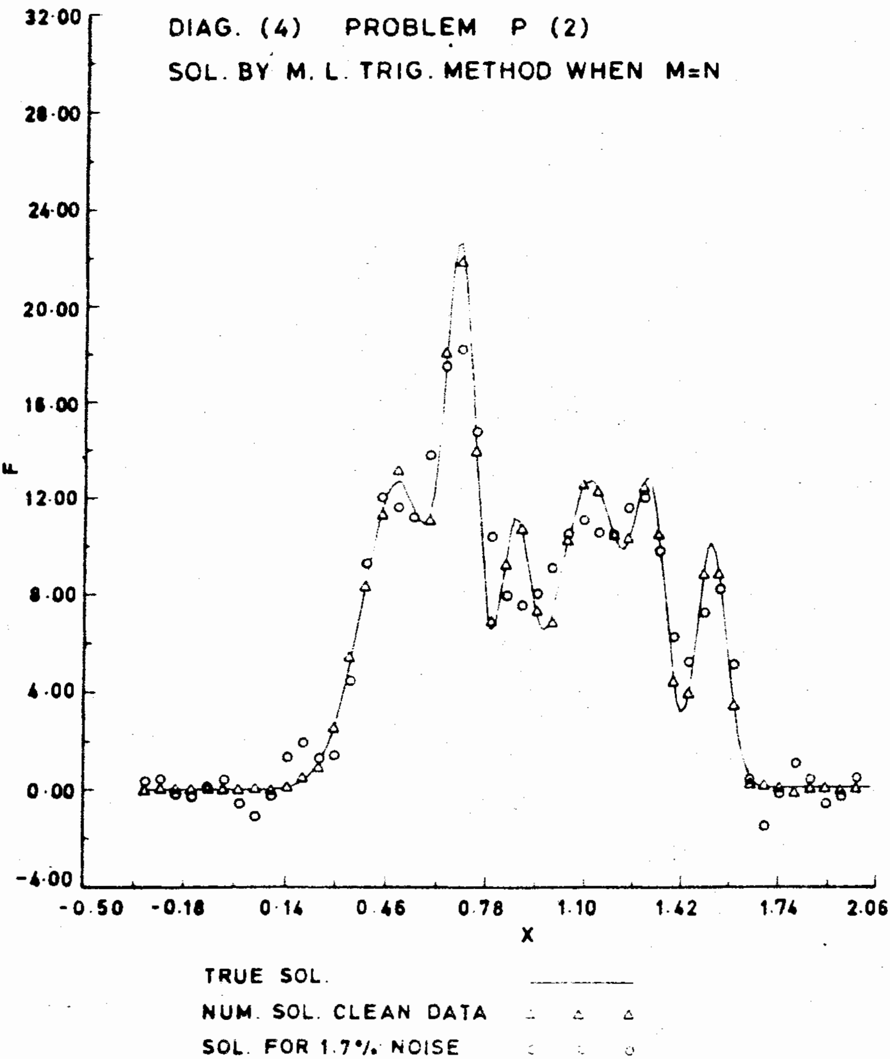
X
F (X)
K (X)
G (X)


DIAG (3) PROBLEM P (1)
SOL. BY M. L. TRIG. METHOD WHEN M=N
1.28
1.12
0.96
0.80
0.64
0.48
0.32
0.16
0.00
-0.16
F
-4.00
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
4.00
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 0.7% NOISE P=2
SOL. FOR 0.7% NOISE P=4



DIAG. (4) PROBLEM P (2)
SOL. BY M. L. TRIG. METHOD WHEN M=N
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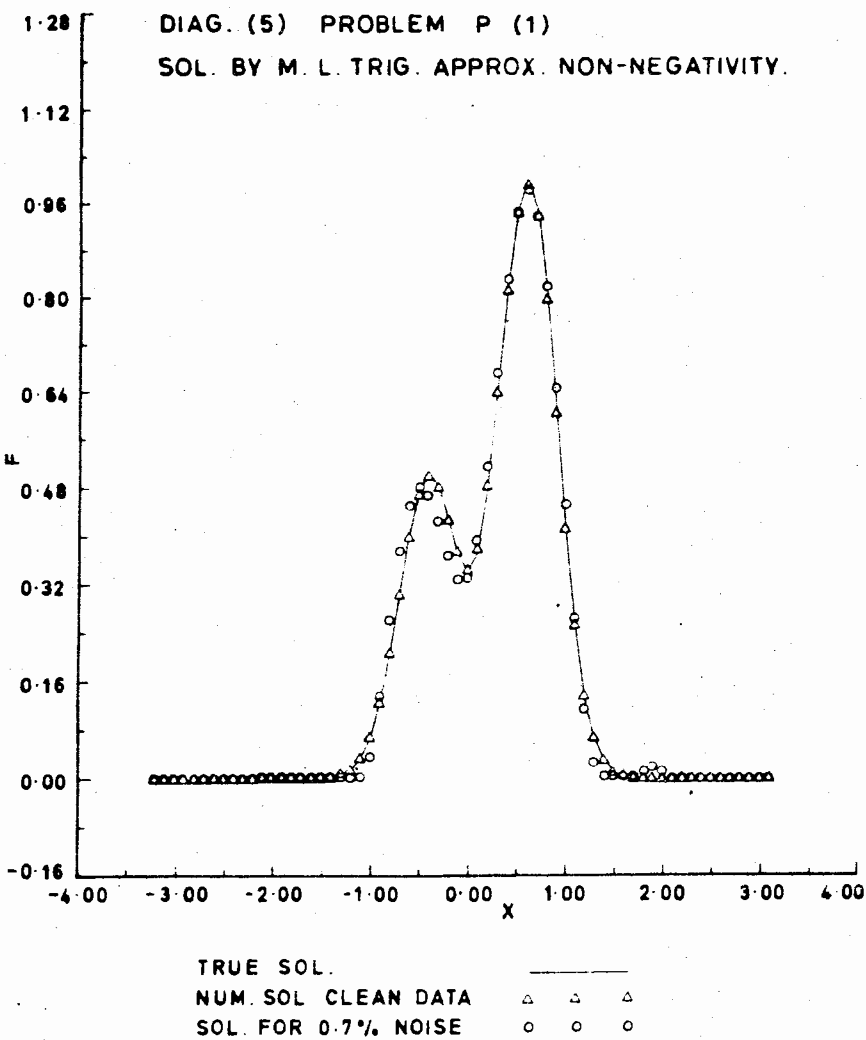
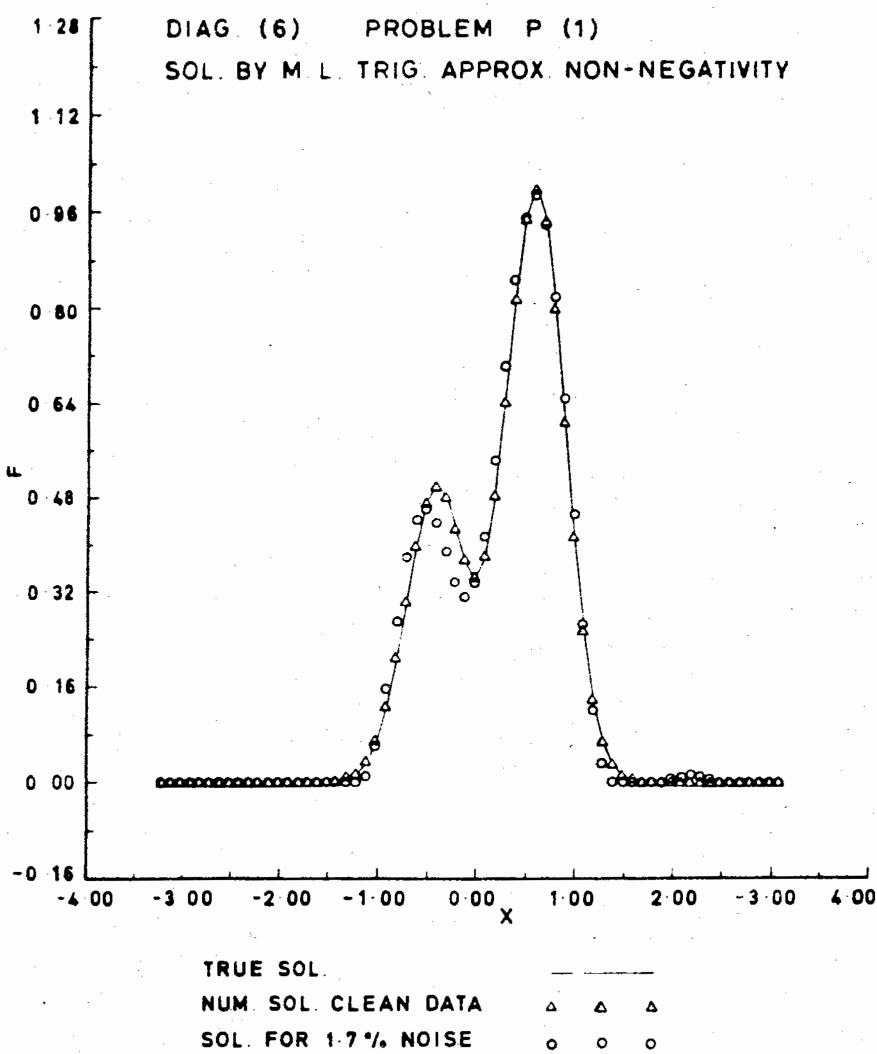
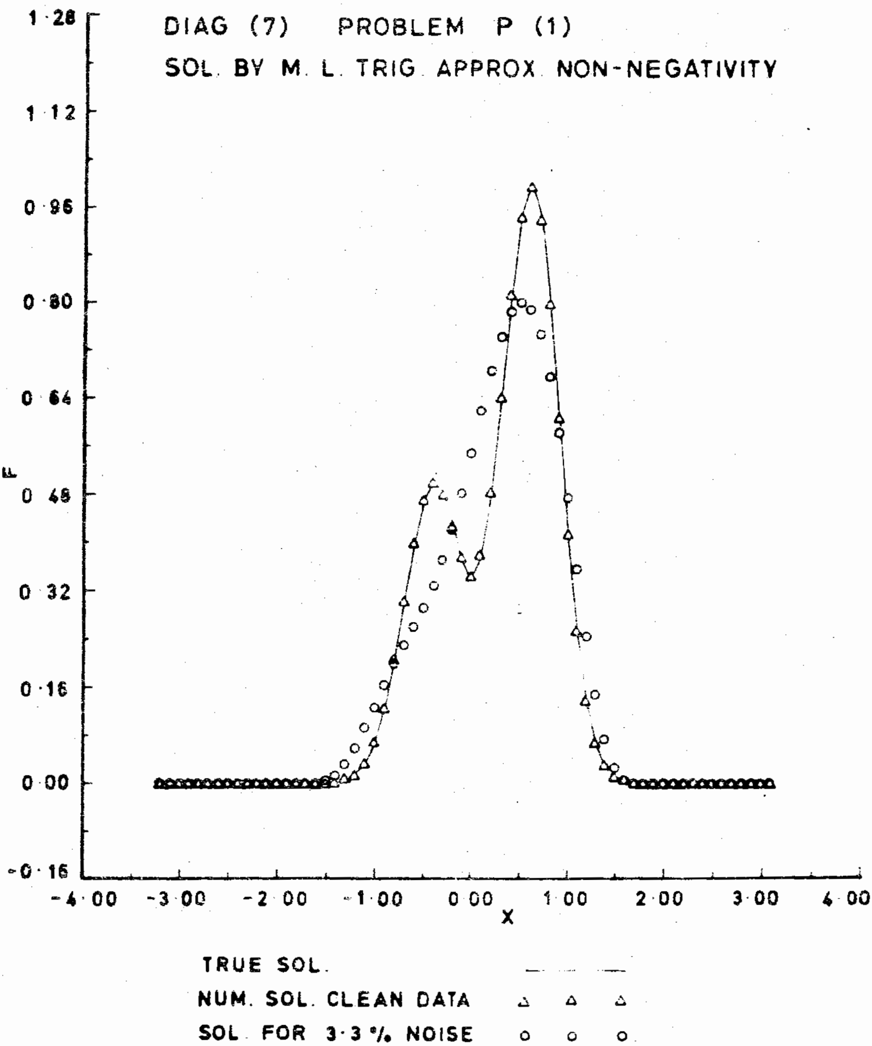
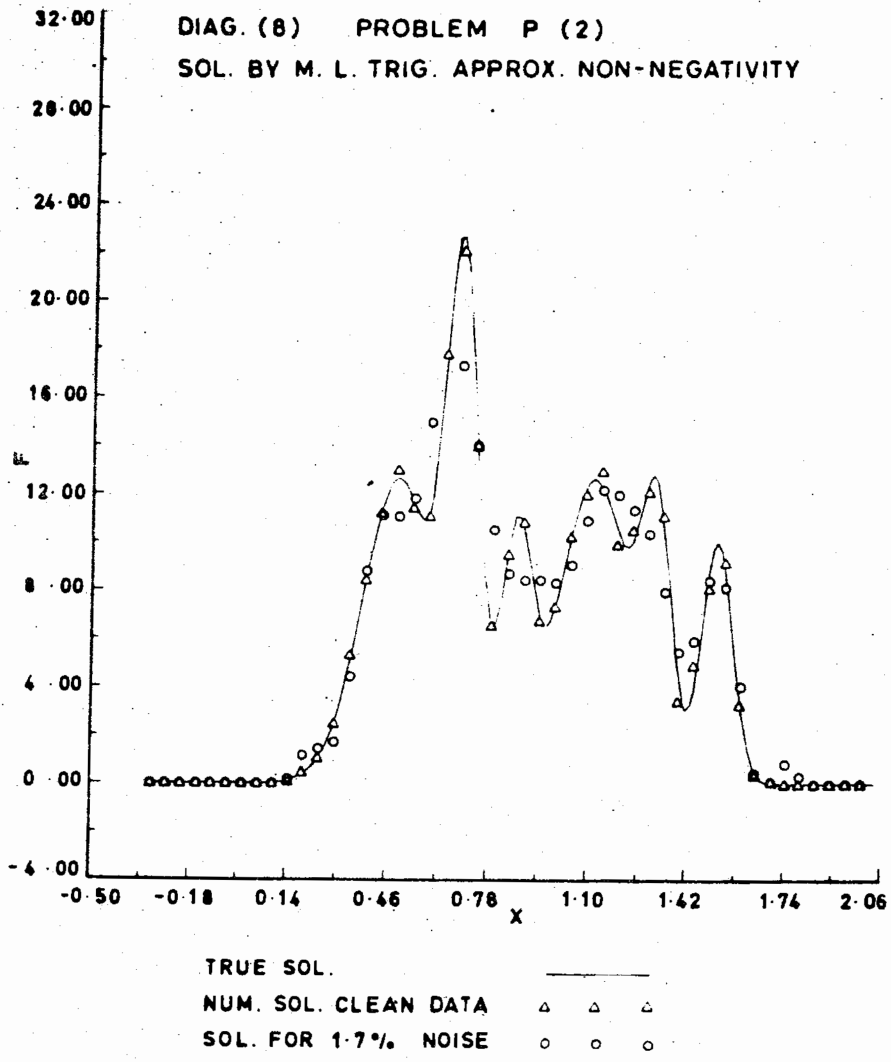
28·00
24·00
20·00
16·00
12·00
8·00
4·00
0·00
-4·00
F
-0.50
-0.18
0.14
0.46
0.78
1.10
1.42
1.74
2.06
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 1.7% NOISE


DIAG. (5) PROBLEM P (1)
SOL. BY M. L. TRIG. APPROX. NON-NEGATIVITY.
1·28
1·12
0·96
0·80
0·64
0·48
0·32
0·16
0·00
-0·16
F
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
TRUE SOL.
NUM. SOL CLEAN DATA
SOL. FOR 0·7% NOISE


DIAG. (6) PROBLEM P (1)
SOL. BY M. L. TRIG. APPROX. NON-NEGATIVITY
1·28
1·12
0·96
0·80
0·64
0·48
0·32
0·16
0·00
-0·16
F
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
-3·00
4·00
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 1·7 % NOISE



DIAG (7) PROBLEM P (1)
SOL. BY M. L. TRIG. APPROX. NON-NEGATIVITY
1·28
1·12
0·96
0·80
0·64
0·48
0·32
0·16
0·00
-0·16
F
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 3·3 % NOISE


DIAG. (8) PROBLEM P (2)
SOL. BY M. L. TRIG. APPROX. NON-NEGATIVITY
32·00
28·00
24·00
20·00
16·00
12·00
8·00
4·00
0·00
-4·00
u
-0·50
-0·18
0·14
0·46
0·78
1·10
1·42
1·74
2·06
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 1·7% NOISE

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ON SOME PROJECTION METHODS FOR ENCLOSING THE ROOT OF A NONLINEAR OPERATOR EQUATION

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Abstract

We provide some sufficient conditions for the monotone convergence of certain iteration projection methods to the solution of a nonlinear operator equation in a Banach space. Our conditions simplify earlier hypotheses.

Key words and phrases : Banach space, monotone convergence.
(1980) AMS classification codes : 47D15, 47H17, 65J15, 65B05.

1. Introduction

We consider the equation

$$L(x) = T(x) \tag{1}$$

where L is a linear operator and T is a nonlinear operator defined on some convex subset D of a linear space E with values in a linear space $\hat{E}$.

We study the convergence of the iterations.

$$L(y_{n+1}) = T(y_n) + A_n(y, x)(y_{n+1} - y_n) \tag{2}$$

and

$$L(x_{n+1}) = T(x_n) + A_n(y, x)(x_{n+1} - x_n) \tag{3}$$

to a solution x^* of equation (1), where $A_n(y, x)$, $n \geq 0$ is a linear operator.

If p is a linear projection operator ($p^2 = p$), that projects the space $\hat{E}$ into $\hat{E}_p \subseteq \hat{E}$, then the operator PT will be assumed to be Frechet differentiable on D and its derivative $PT'_x(x)$ corresponds to the operator $PB(y, x)$, $y, x \in D \times D$, $PT'_x(x) = PB(x, x)$ for all $x \in D$.

We will assume that

$$A_n(y, x) = APB(y_n, x_n), \; n \geq 0.$$

Iterations (2) and (3) have been studied extensively under several assumptions [1]—[3], [6]—[8], when P=L=I, the identity operator on D. However, the iterates $\{x_n\}$ and $\{y_n\}$ can rarely be computed in infinite dimensional spaces in this case. But if the space $\hat{E}_p$ is finite dimensional with $\dim(\hat{E}_p) = N$, then iterations (2) and (3) reduce to systems of linear algebraic equations of order at most N. This case has been studied in [3], [4] and in particular in [5]. The assumptions in [5] involve the positivity of the operators $PB(y, x) - APB(y, x)$, $QT''_x(x)$ with $Q = I - P$ and $L(y) - A_n(y, x)\,y$ on some interval $[y_0, x_0]$, which is difficult to verify.

In this paper we simplify the above assumptions and provide some further conditions for the convergence of iterations (2) and (3) to a solution x^* of equation 1.

We finally illustrate our results with an example.

2. Convergence Results

We assume that E and $\hat{E}$ have been partially ordered "$\leq$" by a cone and we will call them partially ordered topological spaces [4], [6], [8] (POTL-spaces).

Definition 1. A POTL-space is called normal if given a local base μ for the topology, there exists a positive number η so that if $0 \leq z \in \mu$, then $[0, z] = \{x ; 0 \leq x \leq z\} \subset \eta U$.

Definition 2. A POTL-space is called regular if every order bounded increasing sequence has a limit.

If the topology of a POTL-space is given by a norm then this space is called a partially ordered normed space (PON-space). If a PON-space is complete with respect to its topology then it is called a partially ordered Banach space (POB-space). According to Definition 1 a PON-space is normal if and only if there exist a positive number α such that

$$\|x\| \leq \alpha \|y\| \text{ for all } x, y \in E \text{ with } 0 \leq x \leq y. \quad (4)$$

Let us note that any regular POB-space is normal. The reverse is not true. For example, the space C [0, 1] of all continuous real functions defined on [0, 1], ordered by the cone of nonnegative functions, is normal but it is not regular. All finite dimensional POTL-spaces are both normal and regular. Denote by $(E, \hat{E}$ the set of all operators from E to $\hat{E}$. Let $L(E, \hat{E})$ be the set of all linear operators and $B(E, \hat{E})$, the set of all continuous linear operators from E to $\hat{E}$. Let an operator $N \in (E, \hat{E})$. N is called isotone (resp. antitone) if $x \leq y$ implies $N(x) \leq N(y)$ (resp. $N(x) \geq N(y)$). N is called nonnegative if $x \geq 0$ implies $N(x) \geq 0$. N is called inverse nonnegative if $N(x) \geq 0$ implies $x \geq 0$ For linear operators the nonnegativity is clearly equivalent with the isotony. Also, a linear operator is inverse nonnegative if and only if it is invertible and its inverse is nonnegative (see also [4], [6] [8]).

We can now formulate our main result.

Theorem 1. Let $F = D \subset E \rightarrow \hat{E}$, where E is a regular POTL-space and $\hat{E}$ is a POTL-space. Assume

(a) there exist points x_0, y_0, $y_{-1} \in D$ with

$$x_0 \leq y_0 \leq y_{-1}, [x_0, y_{-1}] \subset D, L(x_0) - T(x_0) \leq 0 \leq L(y_0) - T(y_0).$$

Set

$$S_1 = \{(x, y) \in E^2;\ x_0 \leq x \leq y \leq y_0\},$$

$$S_2 = \{(u, y_{-1}) \in E^2;\ x_0 \leq u \leq y_0\}$$

and

$$S_3 = S_1 \cup S_2 .$$

(b) Assume that there exists an operator $A = S_3 \to B(E, \hat{E})$ such that

$$(L(y) - T(y)) - (L(x) - T(x)) \leq A(w, z)(y - x) \qquad (5)$$

for all $(x, y), (y, w) \in S_2$, $(w, z) \in S_3$.

(c) Suppose that for any $(u, v) \in S_3$ the linear operator $A(u, v)$ has a continuous nonsingular nonnegative left subinverse.

Then there exist two sequences $\{x_n\}, \{y_n\}$, $n \geq 1$ satisfying (2), (3),

$$L(x_n) - T(x_n) \leq 0 \leq L(y_n) - T(y_n), \qquad (6)$$

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \quad \leq y_1 \leq y_0 \qquad (7)$$

and

$$\lim_{n \to \infty} x_n = x^*, \quad \lim_{n \to \infty} y_n = y^*, \qquad (8)$$

Moreover, if the operators $A_n = A(y_n, y_{n-1})$ are inverse nonnegative then any solution of the equation (1) from the interval $[x_0, y_0]$ belongs to the interval $[x^*, y^*]$.

Proof. Let us define the operator $M : [0, y_0 - x_0] \to E$ by

$$M(x) = x - L_0 (L(x_0) - T(x_0) + A_0(x))$$

where L_0 is a continuous nonsingular nonnegative left subinverse of A_0. It can easily be seen that M is isotone, continuous with

$$M(0) = - L_0 (L(x_0) - T(x_0)) \geq 0$$

and

$$\begin{aligned} M(y_0 - x_0) &= y_0 - x_0 - L_0 (L(y_0) - T(y_0)) \\ &\quad + L_0 [(L(y_0) - T(y_0)) - (L(x_0) - T(x_0)) \\ &\quad - A_0(y_0 - x_0)] \\ &\leq y_0 - x_0 - L_0 (L(y_0) - T(y_0)) \\ &\leq y_0 - x_0 . \end{aligned}$$

It now follows from the well known theorem of L.V. Kantorovich [4] that the operator M has a fixed point $w \in [0, y_0 - x_0]$. Set $x_1 = x_0 + w$ to get

$$L(x_0) - T(x_0) + A_0(x_1 - x_0) = 0, \; x_0 \leq x_1 \leq y_0 .$$

By (5), we get

$$\begin{aligned} L(x_1) - T(x_1) &= (L(x_1) - T(x_1)) - (L(x_0) - T(x_0)) \\ &\quad + A_0(x_0 - x_1) \leq 0. \end{aligned}$$

Let us now define the operator $M_1 : [0, y_0 - x_1] \to E$ by

$$M_1(x) = x + L_0 (L(y_0) - T(y_0) - A_0(x)).$$

It can easily be seen that M_1 is continuous, isotone with

$$M_1(0) = L_0(L(y_0) - T(y_0)) \geqslant 0$$

and

$$M_1(y_0 - x_1) = y_0 - x_1 + L_0(L(x_1) - T(x_1))$$
$$+ L_0[(L(y_0) - T(y_0)) - (L(x_1 - T(x_1))$$
$$- A_0(y_0 - x_1)]$$
$$\leqslant y_0 - x_1 + L_0(L(x_1) - T(x_1))$$
$$\leqslant y_0 - x_1 .$$

As before, there exists $z \in [0, y_0 - x_1]$ such that $M_1(z) = z$. Set $y_1 = y_0 - z$ to get

$$L(y_0) - T(y_0) + A_0(y_1 - y_0) = 0, x_1 \leqslant y_1 \leqslant y_0 .$$

But from (5) and the above

$$L(y_1) - T(y_1) = (L(y_1) - T(y_1)) - L(y_0) - T(y_0))$$
$$- A_0(y_1 - y_0) \geqslant 0 .$$

Using induction on n we can now show, following the above technique, that there exist sequences $\{x_n\}$, $\{y_n\}$, $n \geqslant 1$ satisfyiug (1), (2), (6) and (7). Since the space E is regular, using (7) we get that there exist $x^*, y^* \in E$ satisfying (8), with $x^* \leqslant y^*$. Let $x_0 \leqslant z \leqslant y_0$ and $L(z) - T(z) = 0$ then we get

$$A_0(y_1 - z) = A_0(y_0) - (L(y_0) - T(y_0)) - A_0(z)$$
$$= A_0(y_0 - z) - [(L(y_0) - T(y_0))$$
$$- (L(z) - T(z))] \geqslant 0$$

and

$$A_0(x_1 - z) = A_0(x_0) - (L(x_0) - T(x_0)) - A_0(z)$$

$$= A_0(x_0 - z) - [(L(x_0) - T(x_0)) - (L(z) - T(z))] \leqslant 0.$$

If A_0 is inverse isotone, then $x_1 \leqslant z \leqslant y_1$ and by induction $x_n \leqslant z \leqslant y_n$. Hence $x^* \leqslant z \leqslant y^*$.

That completes the proof of the theorem.

Using (1), (2), (6), (7) and (8) we can easily prove the following theorem which gives us natural conditions under which the points x^* and y^* are solutions of the equation (1).

Theorem 2. Let $L - T$ be continuous at x^* and y^* and the hypotheses of Theorem 1 be true. Assume that one of the conditions is satisfied :

(a) $x^* = y^*$;

(b) E is normal and there exists an operator $H : E \rightarrow \hat{E}$ $(H(0) = 0)$ which has an isotone inverse continuous at the origin and $A_n \leqslant H$ for sufficiently large n ;

(c) $\hat{E}$ is normal and there exists an operator $G : E \rightarrow \hat{E}$ $(G(0) = 0)$ continuous at the origin and such that $A_n \leqslant G$ for sufficiently large n.

(d) The operators L_n, $n \geqslant 0$ are equicontinuous.

Then $L(x^*) - T(x^*) = L(y^*) - T(y^*) = 0$.

Moreover, assume that there exists an operator $G_1 : S_1 \rightarrow L(E, \hat{E})$ such that $G_1(x, y)$ has a nonnegative left superinverse for each $(x, y) \in S_1$ and

$$L(y) - T(y) - (L(x) - T(x)) \geqslant G_1(x, y)(y - x) \text{ for all } (x, y) \in S_1 .$$

Then if $(x^*, y^*) \in S_1$ and $L(x^*) - T(x^*) = L(y^*) - T(y^*) = 0$ then $x^* = y^*$.

We now complete this paper with an application.

III Applications.

Let $E = \hat{E} = \mathbb{R}^k$ with $k = 2N$. We define a projection operator P_N by

$$P_N(v) = \begin{cases} v_i, & i = 1, 2, \dots, N \\ 0, & i = N+1, \dots, k, \ v = (v_1, v_2, \dots, v_k) \in E. \end{cases}$$

We consider the system of equations

$$v_i = f_i(v_1, \dots, v_k), \ i = 1, 2, \dots, k. \tag{9}$$

Set $T(v) = \{f_i(v_1, \dots, v_k)\}$, $i = 1, 2, \dots, k$, then

$$P_N T(v) = \begin{cases} f_i(v_1, \dots, v_k), & i = 1, \dots, N, \\ 0, & i = N+1, \dots, k. \end{cases}$$

$$P_N T'(v)u = \begin{cases} \sum_{j=1}^{k} f'_{ij}(v_1, \dots, v_k)\, u_j, & i = 1, 2, \dots, N \\ 0, \ i = N+1, \dots, k, & f'_{ij} = \dfrac{\partial f_i}{\partial v_j}, \end{cases}$$

$$P_N B(w, z)u = \left\{ \begin{cases} \sum_{j=1}^{k} F_{ij}(w_1, \dots, w_k, z_1, \dots, z_k)\, u_j, & i = 1, \dots, N \\ 0, \ i = N+1, \dots, k \end{cases} \right\} = C_N^i(w, z)u,$$

where $F_{ij}(v_1, \dots, v_k, v_1, \dots, v_k) = \partial f_i(v_1, \dots, v_k)/\partial v_j$. Choose

$$A_n(y, x) = C_N^i(y, x),$$

then iterations (2) and (3) become

$$y_{i,n+1} = f_i\,(y_{1,n}, \dots, y_{k,n}) + C_N^i(y_{i,n}, x_{i,n})(y_{i,n+1} - y_{i,n}) \qquad (10)$$

$$x_{i,n+1} = f_i\,(x_{1,n}, \dots, x_{k,n}) + C_N^i(y_{i,n}, x_{i,n})(x_{i,n+1} - x_{i,n}), \qquad (11)$$

Let us assume that the determinant D_n of the above N-th order linear systems is nonzero, then (10) and (11) become

$$y_{i,n+1} = \frac{\sum_{m=1}^{N} D_{im}\, F_m^1(y_n, x_n)}{D_n},\ i = 1, \dots, N \qquad (12)$$

$$y_{i,n+1} = f_i\,(y_{1n}, \dots, y_{kn}),\ i = N+1, \dots, k \qquad (13)$$

and

$$x_{i,n+1} = \frac{\sum_{m=1}^{N} D_{im}\, F_m^2(y_n, x_n)}{D_n},\ i = 1, \dots, N, \qquad (14)$$

$$x_{i,n+1} = f_i\,(x_{1,n}, \dots, x_{k,n}),\ i = N+1, \dots, k \qquad (15)$$

respectively. Here D_{im} is the cofactor of the element in the i-th column and m-th row of D_n and $F_m^1(y_n, x_n)$, i=1, 2 are given by

$$F_m^1(y_n, x_n) = f_m(y_{i,n}, \dots, y_{k,n}) + \sum_{i=N+1}^{k} \alpha_{mj}^n\, f_j\,(y_{1,n}, \dots, y_{k,n}) - \sum_{j=1}^{k} \alpha_{mj}^n\, y_{j,n},$$

and

$$F_m^2(y_n, x_n) = f_m(x_{1,n}, \dots, x_{k,n})$$

$$+ \sum_{j=N+1}^{k} \alpha_{mj}^n \; f_j \; (x_{1,n}, \dots, x_{k,n}) - \sum_{j=1}^{k} \alpha_{mj}^n \; x_{j,n} \; ,$$

where $\alpha_{mj}^n = F_{mj} \; (y_n, x_n)$.

If the hypotheses of Theorem 1 and 2 are now satisfied for the equation (9 then the results apply to obtain a solution x^* of equation (4) in $[y_0, x_0]$.

In particular consider the system of differential equations

$$\ddot{q}_i = f_i \; (t, q_1, q_2), \; i = 1, 2, \; 0 \leqslant t \leqslant 1 \qquad (16)$$

subject to the boundary conditions

$$q_i \; (0) = d_i \; , \; q_i \; (1) = e_i \; , \; i = 1, 2. \qquad (17)$$

The functions f_1 and f_2 are assumed to be sufficiently smooth. for the discretization a uniform mesh

$$t_j = jh, \; j = 0, 1, \dots, N+1, \; h = \frac{1}{N+1}$$

and the corresponding central-difference approximation of the second derivatives are used. Then the discretized equations given by

$$x = T\,(x) \qquad (18)$$

with

$$T\,(x) = (B+I)\,(x) + h^2 \; \varphi\,(x) - b, \; x \in \mathbb{R}^{2N}$$

where

$$B = \begin{bmatrix} A+I & 0 \\ 0 & A+I \end{bmatrix}, \; A = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}.$$

$$\varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \end{bmatrix}, \; \varphi_i(x) = (f_i(t, x_j, x_{n+j})), \; j=1,2,\ldots,N),$$

$i=1, 2$, $x \in \mathbb{R}^{2N}$ and $b \in \mathbb{R}^{2N}$ is the vector of boundary values that has zero components except for $b_1 = d_1$, $b_n = e_1$, $b_{n+1} = d_2$, $b_{2n} = e_2$. That is (18) plays the role of (9) (in vector form).

As a numerical example, consider the problem (16)—(17) with

$$f_1(t, q_1, q_2) = q_1^2 + q_1 + .1q_2^2 - 1.2$$

$$f_2(t, q_1, q_2) = .2q_1^2 + q_2^2 + 2q_2 - .6$$

$$d_1 = d_2 = e_1 = e_2 = 0 .$$

Choose $N = 49$ and starting points

$x_{i,0} = 0$, $y_{i,0} = (t_j(1-t_j), j=1,\ldots,N)$, with $t = .1(.1).5$.

It is trivial to check that the hypotheses of Theorem 1 are satisfied with the above values. Furthermore the components of the first two iterates corresponding to the above values using the procedure described above and (12)-(13), (14)-(15) we get the following values.

	t	p=1	p=2
$x_{1,p}$	.1	.0478993317265	.0490944353538
	.2	.0843667040291	.0866974354188
	.3	.1099518629493	.1132355483832
	.4	.1251063839064	.1290273001442
	.5	.1301240123325	.1342691068706
$x_{2,p}$	:1	.0219768501208	.0227479238400
	.2	.0384462112803	.0399528292723
	.3	.0498537074028	.0519796383151
	.4	.0565496306877	.0590905187490
	.5	.0587562590344	.0614432572165
$y_{1,p}$	.1	.0494803602542	.0490951403091
	.2	.0874507511044	.0866988216544
	.3	.1242981809478	.1132375255317
	.4	.1302974325097	.1290296859551
	.5	.1356123753407	.1342716394060
$y_{2,p}$	.1	.0235492475283	.0227486289905
	.2	.0415200498433	.0399542200344
	.3	.0541939935471	.0519816281202
	.4	.0617399319012	.0590929252230
	.5	.0642461600398	.0614458137439

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ON THE SOLUTION OF SOME EQUATIONS SATISFYING CERTAIN DIFFERENTIAL EQUATIONS

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Abstract

We improve the rate of convergence of the modified Newton-Kantorovich iteration. The basic assumption is that an operator satisfies a certain differential equation.

Key words and phrases: Newton-Kantorovich method, Hölder continuity.

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Introduction

Consider the equation

$$F(x) = 0 \tag{1}$$

where F is a nonlinear operator between two Banach spaces X and $\hat{X}$. The most popular methods for approximating solutions x^* of equation (1) are undoubtedly the Newton-Kantorovich method

$$z_{n+1} = z_n - F'(z_n)^{-1} F(z_n), \; n = 0, 1, 2, \ldots ; \tag{2}$$

the modified Newton-Kantorovich method

$$x_{n+1} = x_n - F'(x_0)^{-1} F(x_n), \; n = 0, 1, 2, \ldots \tag{3}$$

or variations of those called Newton-like methods [1], [4].

It is well known [1], [2], [4] that under certain assumptions, one of which is that the Fréchet-derivative F' of F satisfies a Lipschitz condition, equation (1) has a locally unique solution such that

$$\| z_{n+1} - x^* \| \leq \alpha \| z_n - x^* \|^2 , 0 < \alpha < 1 \tag{4}$$

and

$$\| x_{n+1} - x^* \| \leq \beta \| x_n - x^* \| , 0 < \beta < 1, n=0, 1, 2, .. \tag{5}$$

for some x_0 , z_0 sufficiently close to the solution x^*.

Note that to compute the x_n 's, $n = 1, 2, ..$ we calculate only the inverse of the linear operator $F'(x_0)$ but the rate of convergence is 1, whereas if we can calculate all the inverses of $F'(z_n)$ then the rate of convergence is 2.

In the first part of this paper we extend the above results to include the case when the linear operator F' is only (c, p) Hölder continuous (to be precised later) for some $c > 0$ and $p \in [0, 1]$. Our results can be reduced to the ones in [2] for $p = 1$.

In particular, we show that

$$\| z_{n+1} - x^* \| \leq \alpha^* \| z_n - x^* \|^{1+p}, \ 0 < \alpha^* < 1 \tag{6}$$

and

$$\| x_{n+1} - x^* \| \leq \beta^* \| x_n - x^* \| , 0 < \beta^* < 1, n=0, 1, 2... \tag{7}$$

In the second part we show that using an iteration of the form

$$z_{n+1} = z_n - A_0^{-1} F(z_n), n = 0, 1, 2, ... \tag{8}$$

where A_0^{-1} is the inverse of a fixed linear op rator we can achieve order of convergence $1+p$. That is by inverting only one operator we can achieve the same order of convergence as with iteration (2) for $p=1$ and higher order of convergence than iteration (3) for $p \neq 0$.

To prove the above claim we assume that the operator F satisfies a differential equation of the form

$$F'(x) = G(F(x)) \tag{9}$$

where $G(\,.\,)$ is a given operator on X.

Main results. We will need the definition.

Definition. We say that the Fréchet-derivative $F'(x)$ of F is (c, p)-Hölder continuous on $\bar{X} \subset X$ if for some $c > 0$, $p \in [0, 1]$

$$\| F'(x) - F'(y) \| \leq c \| x - y \|^{p} \text{ for all } x, y \in \bar{X}. \tag{10}$$

we then say that $F'(\,.\,) \in H_{\bar{X}}(c, p)$.

It is well known (see, e g. [2]) that if $\bar{X}$ is convex then

$$\| F(x) - F(y) - F'(x)(x-y) \| \leq \frac{c}{1+p} \| x - y \|^{1+p}$$

$$\text{for all } x, y \in \bar{X}, \tag{11}$$

We can now prove the following theorem on the existence of a solution x^* of equation (1).

Theorem 1. Assume:

(a) the point $x^* \in X$ is a solution of the equation

$$F(x) = 0\,;$$

(b) there exists $b > 0$, $x_0 \in X$ such that the inverse of the linear operator $F'(x_0)^{-1}$ exists,

$$\| F'(x_0)^{-1} \| \leq b \tag{12}$$

$$2^p cb \| x_0 - x^* \|^p < 1 \tag{13}$$

(c) the linear operator $F'(x_0) \in H_{U^*}(c, p)$, where $U^* = \bar{U}(x^*, \| x_0 - x^* \|)$, is a sphere centered at x^* and of radius $\| x_0 - x^* \|$.

Then the iteration $\{ x_n \}$ given by (3), $n = 0, 1, 2, \ldots$ remains in U^* and converges to x^* as $n \to \infty$, which is the unique solution of (1) in U^*.

Moreover, the following estimate is true :

$$\| x_n - x^* \| \leqslant d^n \| x_0 - x^* \|, \ n = 1, 2, \ldots \tag{14}$$

where,

$$d = d(r) = 2^p cbr^p < 1$$

for some r such that

$$\| x_0 - x^* \| \leqslant r < \tfrac{1}{2} \cdot (cb)^{-p} \tag{15}$$

Proof. Using the identity

$$x_{n+1} - x^* = F'(x_0)^{-1} \left(\int_0^1 (F'(x_0) - F'(x^* + t(x_n - x^*))) \right.$$

$$(x_n - x^*) \, dt],$$

and assuming that $\| x_k - x^* \| \leqslant r$ for $k = 1, 2, \ldots, n$ we easily obtain by (10)

$$\| x_{n+1} - x^* \| \leqslant cb (\| x_0 - x^* \| + \| x_n - x^* \|)^p \| x_n - x^* \|$$

$$\leqslant cb (2r)^p \, r \leqslant r \tag{16}$$

by the choice or r.

That is, $x_{n+1} \in \bar{U}(x^*, r)$.

Moreover by (16), we get

$$\| x_{n+1} - x^* \| \leqslant d(r) \| x_n - x^* \|$$

$$\leqslant d \cdot d \| x_{n-1} - x^* \|$$

$$\leqslant \ldots$$

$$\leqslant d^{n+1} \| x_0 - x^* \| .$$

Since, $0 < d < 1$ the above inequality shows that the sequence $\{ x_n \}$, $n = 0, 1, 2, \ldots$ converges to x^* in such a way that (14) is satisfied.

That completes the proof of the theorem.

The above theorem shows that the iteration given by (3) converges to x^* only linearly. But we can do even better,

Theorem 2. Assume ;

(a) the point $x^* \in X$ is a solution of the equation

$$F(x) = 0$$

such that the inverse of the linear operator $F'(x^*)^{-1}$ exists and

$$\| F'(x^*)^{-1} \| \leqslant \bar{b}, \text{ for some } \bar{b} > 0.$$

(b) For some $x_0 \in X$, the following estimate is true :

$$q \equiv k^{\frac{1}{p}} \| x_0 - x^* \| < 1 \tag{17}$$

where,

$$k = \frac{c\bar{b}}{(p + 1)^2}.$$

(c) The linear operator $F'(x^*) \in H_{U^*}(c, p)$, where $U^* = \bar{U}(x^*, \| x_0 - x^* \|)$.

Then the iteration given by

$$z_{n+1} = z_n - F'(x^*)^{-1} F(z_n), \text{ with } z_0 = x_0 \tag{18}$$

remains in U^* and converges to x^* as $n \to \infty$.

Moreover,

$$\| z_{n+1} - x^* \| \leqslant q^{(p+1)^n - 1} \| x_0 - x^* \|, \; n = 1, 2, \ldots .$$

Proof. As in Theorem 1, using the identity

$$z_{n+1} - x^* = F'(x^*)^{-1} \left[\int_0^1 (F'(x^*) - F'(x^* + t(z_n - x^*))) (z_n - x^*) \, dt \right]$$

we obtain by (10)

$$\| z_{n+1} - x^* \| \leqslant k \| z_n - x^* \|^{p+1}. \tag{19}$$

The result now follows from (19) and (17) by induction.

The order of convergence of $\{ z_n \}$, $n = 0, 1, 2, \ldots$ to x^* has been improved from 1 to $1+p$.

The order of convergence $1+p$ can easily be proved by repeating a proof similar to the proof of Theorem 2 for the iteration (2).

The computation of the iterates $\{z_n\}$, $n = 1, 2, \ldots$ however requires the additional cost of evaluating the inverses of the operators $F'(z_n)$, $n=0, 1, 2, \ldots$ (which must be uniformly bounded). But for the use of iteration (18) it is only required to compute the inverse of $F'(x^*)$ once and for all.

Note that the operator $F'(x^*)$ cannot be computed in practice since the solution x^* is unknown. However, if the operator F satisfies the differential equation

$$F'(x) = G(F(x))$$

where $G(\,.\,)$ is a known operator on X, then

$$F'(x^*) = G(F(x^*)) = G(0)$$

can be evaluated without knowing the value of x^*.

We can prove a global existence theorem.

Theorem 3. Let $F'(x) = G(F(x))$ and assume :

(a) the operator $G(0)$ is invertible on X and there exist constants b_1, $b_2 > 0$ such that

$$\| F(x) \| \leq b_1 \text{ for all } x \in X,$$

$$\| G(0)^{-1} \| \leq b_2 ;$$

(b) the operator G is (c_1, p_1) Hölder continuous on X with $c_1 > 0$ and $p_1 \in [0, 1]$; and

(c) the following estimate is true

$$d_1 = c_1 \, b_2 \, b_1^{p_1} < 1. \tag{20}$$

Then the equation

$$F(x) = 0$$

has a unique solution $x^* \in X$. Moreover, the iteration generated by

$$x_{n+1} = x_n - G(0)^{-1} F(x_n) \tag{21}$$

converges to x^* with

$$\| x_n - x^* \| \leq \frac{d_1^n}{1-d_1} \| x_1 - x_0 \|, \quad n = 0, 1, 2, \ldots .$$

Proof. Define the operator T on X by

$$T(x) = G(0)(x) - F(x).$$

Then

$$T'(x) = G(0) - F'(x) = G(0) - G(F(x))$$

and

$$\| T'(x) \| \leq c_1 \| F(x) \|^{p_1}$$

by the (c_1, p_1) – Hölder continuity of G.

The theorem now follows from (20) and the contraction mapping principle [2].

Note that if F' is (c, p)-Hölder continuous then the convergence of (21) will be of order $1+p$ as soon as (13) is satisfied with x_0 replaced by an iterate x_n sufficiently close to x^*.

Applications

Example 1. As an application of Theorem 2 (for $p=1$) consider the real function

$$F(x) = e^x + y$$

for some $y < 0$. Then obviously the solution x^* of the equation

$$F(x) = 0$$

is given by $x^* = ln(-y)$.

Here,

$$F'(x) = F(x) - y \equiv G(F(x))$$

and the iteration

$$x_{n+1} = x_n + \frac{1}{y}\left(e^{x_n} + y\right)$$

converges quadratically to the solution x^*.

A more interesting application is given by the following example.

Example 2. Consider the differential equation

$$y'' + y^{1+p} = 0, \; p \in (0, 1] \tag{22}$$

$$y(0) = y(1) = 0.$$

We divide the interval [0, 1] into n subintervals and we set $h = \frac{1}{n}$. Let $\{v_k\}$ be the points of subdivision with

$$0 \leq v_0 < v_1 < \ldots < v_n = 1.$$

A standard approximation for the second derivative is given by

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \quad y_i = y(v_i), \; i = 1, 2, \ldots, n-1.$$

Take $y_0 = y_n = 0$ and define the operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(y) = H(y) + h^2 \varphi(y),$$

$$H = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ \vdots & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}.$$

$$\varphi(y) = \begin{bmatrix} y_1^{1+p} \\ y_2^{1+p} \\ \vdots \\ y_{n-1}^{1+p} \end{bmatrix}$$

and

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix},$$

Then

$$F'(y) = H + h^2\,(p+1) \begin{bmatrix} y_1^p & & & 0 \\ & y_2^p & & \\ & & \ddots & \\ 0 & & & y_{n-1}^p \end{bmatrix}. \qquad (23)$$

The Newton-Kantorovich hypotheses on which the work in [1], [2] and the references there is based for the solution of the equation

$$F(y) = 0 \qquad (24)$$

may not be satisfied.

We may not be able to evaluate the second Frechet-derivative since it would involve the evaluation of quantities y_i^{-p} and they may not exist.

Let $y \in \mathbb{R}^{n-1}$, $M \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and define the norms of y and M by

$$\| y \| = \max_{1 \leq j \leq n-1} | y_j |$$

$$\| M \| = \max_{1 \leq j \leq n-1} \sum_{k=1}^{n-1} | m_{jk} | .$$

For all $y, z \in \mathbb{R}^{n-1}$ for which $| y_i | > 0$, $| z_i | > 0$, $i = 0, 1, 2, \ldots, n-1$ we obtain for $p = \frac{1}{2}$, say

$$\| F'(y) - F'(z) \| = \| \operatorname{diag} \{ \tfrac{3}{2} h^2 (y_j^{\frac{1}{2}} - z_j^{\frac{1}{2}}) \} \|$$

$$= \frac{3}{2} h^2 \max_{1 \leq j \leq n-1} | y_j^{\frac{1}{2}} - z_j^{\frac{1}{2}} |$$

$$\leq \frac{3}{2} h^2 [\max | y_j - z_j |]^{\frac{1}{2}}$$

$$= \frac{3}{2} h^2 \| y - z \|^{\frac{1}{2}} .$$

That is, $c = \frac{3}{2} h^2$ and $p = \frac{1}{2}$. Therefore, the results in [1], [2] and [3] cannot be applied here.

We can choose $n = 10$ which gives (9) equations for iteration (2). Since a solution would vanish at the end points and be positive in the interior a reasonable choice of initial approximation seems to be $130 \sin \pi x$. This gives us the following vector

$$z_0 = \begin{bmatrix} 4.01524 \text{ E} + 01 \\ 7.63785 \text{ E} + 01 \\ 1.05135 \text{ E} + 02 \\ 1.23611 \text{ E} + 02 \\ 1.29999 \text{ E} + 02 \\ 1.23675 \text{ E} + 02 \\ 1.05257 \text{ E} + 02 \\ 7.65462 \text{ E} + 01 \\ 4.03495 \text{ E} + 01 \end{bmatrix}.$$

Using the iterative algorithm (2), after four iterations we get

$$z_4 = \begin{bmatrix} 3.35740 \text{ E} + 01 \\ 6.52027 \text{ E} + 01 \\ 9.15664 \text{ E} + 01 \\ 1.09168 \text{ E} + 02 \\ 1.15363 \text{ E} + 02 \\ 1.09168 \text{ E} + 02 \\ 9.15664 \text{ E} + 01 \\ 6.52027 \text{ E} + 01 \\ 3.35740 \text{ E} + 01 \end{bmatrix}.$$

We can easily see that $\| F(x^*) \| \leqslant 3.577082405 \text{ E}-06$. Therefore, we may choose $z_4 = x^*$ and $z_0 = x_0$ for our Theorem 2. We get the following results

$$\| F'(x^*)^{-1} \| \leqslant \bar{b} = 2.55882 \text{ E} + 01,$$

$$k = 2.9100265 \text{ E} - 02,$$

$$c = \frac{3}{2} h^2 = .015$$

and

$$p = \tfrac{1}{2}.$$

Using the above values and (17) we obtain

$$q = .425911478 < 1.$$

That is, the hypothses of Theorem 2 for equation (24) are satisfied in U^*.

Therefore the iteration given by (18) remains in U^* and converges to the solution x^* of (24) as $n \to \infty$.

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ON UNIQUENESS OF GENERALIZED DIRECT PRODUCTS OF RINGS

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One of the standard methods of investigating algebraic structures like Groups, Rings, Modules and Lie Algebras is by constructing them with the help of given algebraic structure of smaller order/dimension and, by using the properties of latter's, one can obtain the necessary facts about the former's. In fact such constructions like direct products and semidirect products can be found in the fundamentals of Theory of Abstract Algebra. Other constructions proving their own singificance include generalized direct products which were introduced by B. H. Neumann and H. Neumann [1]. If on one hand generalized direct products behave almost like direct products, then on the other hand situation is somewhat complicated.

The present article is devoted to demonstrate one of the reasons of such complicated behaviour of generalized direct products. As all these results are in extension of [2], all notations and terminology will be the same as in [2]. Further, if A is a ring, then by annihilator Ann(A) of A we mean the set of those elements a in A such that $xa = ax$ for all $x \in A$.

Note that in case of associative rings Ann(A) is ideal in A.

Unless otherwise specified all rings under consideration are associative, and, therefore, their annihilators are ideals.

Definition 1. A finite set $\{A_1, A_2, \ldots, A_m\}$ of the proper subrings of A describes generalized direct decomposition of A if

(1) A is generated by $A_1, \ldots A_m$ i.e. $A = \langle A_1, \ldots A_m \rangle$

(2) for $i \neq j$, $A_i \cdot A_j = A_j \cdot A_i = \{0\}$.

If A possesses such decomposition, then A is said to be generalized directly decomposable and we write $A =$ g.d d. $\{A_1, \dots A_m\}$. Otherwise A is said to be generalized directly indecomposable.

Definition 2. Let A_1 and A_2 be two rings, $H_1 \leqslant Ann(A_1)$, $H_2 \leqslant Ann(A_2)$. Let $\theta : H_1 \rightarrow H_2$ be an isomorphism. Then $R = < (x_1, \theta(x_1)) ; x_1 \in H_1 >$ is an ideal in $A_1 \oplus A_2$. The factor ring $A = A_1 \oplus A_2 / R$ is said to be generalized direct product of A_1 and A_2 amalgamating a subring H_1 with respect to θ; and we write $A = A(\theta) =$ g.d.p. $(A_1, A_2; H_1, H_2; \theta)$.

O. Schreier [3] proved that the generalized direct products of groups amalgamating a single central subgroup always exist. An analogue of Schreier's result for rings is as follows :

Proposition. Generalized direct product of two rings A_1 and A_2 amalgamating a single subring of the annihilator always exists and is unique up to isomorphism.

Proof. Let $A_1 = < X_1 ; R_1 >$, $A_2 = < X_2 ; R_2 >$ (where X_i is the number of generators and R_i is the number of relations of A_i, respectively) and $H_1 \leqslant Ann(A_1)$, $H_2 \leqq Ann(A_2)$. Let $\theta : H_1 \rightarrow H_2$ be an isomorphism. Then $R = < (x_1, \theta(x_1)) ; x_1 \in H_1 >$ is an ideal in $A_1 \oplus A_2$, and $A = A_1 \oplus A_2 / R = < X_1, X_2 : R_1, R_2, X_1 . X_2 = X_2 . X_1 = 0, h = \theta(h)$ for all $h \in H_1 >$, by definition, is a generalized direct product of A_1 and A_2 amalgamating the subring H_1 which always exists. To prove the uniqueness let $A =$ g. d. d. $\{A_1, A_2\}$. Take

$H_1 = H = H_2$ and $\theta = Id_H$. The mapping $A \rightarrow A_1 \oplus A_2 / R$ defined as $a \rightarrow a + R$ is onto homomorphism with kernal $\{0\}$ and, therefore, $A_1 \oplus A_2 / R \cong A$ which proves the uniqueness.

It is well known that the direct sum of the direct factors of a direct decomposition of a ring A is always isomorphic to A. However if $A(\theta) = g.d.p.(A_1, A_2; H_1, H_2; \theta)$ and $A(\phi) = g.d.p.(A_1, A_2; H_1, H_2; \phi)$ are two generalized direct products of A_1 and A_2 amalgamating single subring H_1, then $A(\theta)$ may not be isomorphic to $A(\phi)$.

Example. Let $A_1 = < a, b; 0 = 4a = 2b = ab > \cong Z_4 \oplus Z_2$, $H_1 = < 2a, b >$ $A_2 = < c, d; 4c = 0 = 2d = cd > \cong Z_4 \oplus Z_2$, $H_2 = < 2c, d >$ and θ_1, θ_2 are isomorphisms from H_1 into H_2 defined as follows :

$$\theta_1 : \begin{cases} 2a \rightarrow 2c \\ b \rightarrow d \end{cases} \qquad \theta_2 : \begin{cases} 2a \rightarrow d \\ b \rightarrow 2c. \end{cases}$$

Then $R_1 = < (2a, 2c), (b, d) >$, $R_2 = < (2a, d), (b, 2c) >$ are ideals in $A_1 \oplus A_2$, and therefore, $A_1 \oplus A_2 / R_1 = A(\theta_1) = < a+R_1, b+R_1, c+R_1, d+R_1 >$ which is isomorphic to $Z_4 \oplus Z_2$. Whereas

$$A_1 \oplus A_2 / R_2 = A(\theta_2) = < a+R_2, b+R_2, c+R_2, d+R_2 > \text{ is isomorphic to } Z_4 \oplus Z_4.$$

In general if $\{A_1, A_2\}$ is a generalized direct decomposition of A, then a generalized direct product of A_1 and A_2 amalgamating a single subring of the annihilator is not necessarily isomorphic to A. In fact to define a generalized direct product of A_1 and A_2 (where

$\{A_1, A_2\}$ is a g.d.d. of A) one needs more information, perhaps about the subrings of Ann(A_1), Ann(A_2) and the isomorphism between them. It may happen that not all of the generalized direct products of A_1 and A_2 are isomorphic to A. However, A is always isomorphic to at least one of the generalized direct products of A_1 and A_2 amalgamating a single subring of the annihilator.

Now we describe the conditions under which two generalized direct products of rings are isomorphic.

Theorem. Let $A(\theta) = \text{g.d.p.}(A_1, A_2; H_1, H_2; \theta)$ and $A(\phi) = \text{g.d.p.}(A_1, A_2; H_1, H_2; \phi)$ be two generalized direct products of two rings A_1 and A_2 amalgamating a single subring H_1. If $\phi^{-1}\theta$ can be extended to an automorphism of the ring A_1, then $A(\theta) \cong A(\phi)$.

Proof. Let $R_1 = \langle (x_1, \theta_1(x_1)); x_1 \in H_1 \rangle$ and $R_2 = \langle (x_1, \phi(x_1)); x_1 \in H_1 \rangle$. Suppose that $\phi^{-1}\theta$ can be extended to an automorphism (say α) of A_1; i.e. $\alpha \mid H_1 = \phi^{-1}\theta$. Consider the mapping $\gamma : A_1 \oplus A_2 \rightarrow A_1 \oplus A_2 / R_1$ defined by the following formula ;

$$\gamma(a_1, a_2) = (\alpha^{-1}(a_1), a_2), \text{ where } a_1 \in A_1, \ a_2 \in A_2.$$

Then one can easily check that γ is onto homomorphism with ker $\gamma = R_2$. Hence $A(\theta)$ is isomorphic to $A(\phi)$.

Definition 3. Let $A(\theta) = \text{g.d.p.}(A_1, A_2; H_1, H_2; \theta)$. If $H_1 = \text{Ann}(A_1)$ and $H_2 = \text{Ann}(A_2)$, then $A(\theta)$ is called the central product of A_1 and A_2 with respect to θ and we write $A(\theta) = \text{c.p.}(A_1, A_2; \theta)$.

Note that central product of A_1 , A_2 is a special case of g d.p. $(A_1, A_2 ; H_1, H_2 ; \theta)$; therefore, as was mentioned above, two central products of A_1 and A_2 are not necessarily isomorphic. However, the following result describes the conditions under which two central products of A_1 and A_2 are isomorphic.

Corollary. Let A_1 and A_2 be two rings. If every automorphism of Ann (A_1) can be extended to an automorphism of A_1 , then all central products of A_1 and A_2 are isomorphic.

Proof. Putting H_1 = Ann (A_1) in the above theorem we have

$\alpha \mid_{\mathrm{Ann}(A_1)} = \phi^{-1}\theta$, which of course can be extended to an automorphism of A_1

These results are contained in the authors' Ph D. dissertation. The author is thankful to Dr. Yu. A. Bahturin for suggesting the problem, assistance and encouragement in carrying out the research.

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ON SOME IDEALS IN BCI-ALGEBRAS

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Abstract

In this paper we study relationship between the ideals in the BCI-algebra X and ideals in the centre I of X.

Introduction

In 1980 K. iseki [7]. introduced the concept of BCI-algebras and since then so many researchers have contributed a lot to the development of the d scipline. In [1], we classified BCI-algebras and defined the centre I of BCI-algebra X. In [2], it is shown that I is a p semisimple algebra. In this paper we study the relationship between ideals in I and ideals in X.

Preliminaries

A BCI-algebr X is an algebra $(X,*,o)$ with the following conditions for all $x,y,z \in X$:

(1) $((x*y) * (x*z)) * (z*y) = o$,
(2) $(x* (x*y)) * y = o$,
(3) $x*x = o$,
(4) $x*y = o = y*x$ implies $x = y$,
(5) $x*o = o$ implies $x = o$, where
$x \leq y$ iff $x*y = o$. ([7]).

Let X be a BCI-algebra and $M = \{x \in X : o*x = o\}$ its BCK-part. Then, X is called proper if $X - M \neq \phi$. We note that a BCK-algebra is trivially a BCI-algebra.

(6) For $x \in M$, $y \in X-M$, $x*y$, $y*x \in X-M$ ([7]).
(7) $(x*y) * z = (x*z) * y$, for $x,y,z \in X$ ([7]).

(8) $x*o = x$, for $x \in X$ ([7]).

(9) $x \leqslant y$ implies $x*z \leqslant y*z$ and $z*y \leqslant z*x$, for all $x*y, z \in X$ ([7]).

Definition 1 [7]. Let X be a BCI-algebra. Then $x, y \in X$ are comparable iff $x*y = o$ or $y*x = o$.

Definition 2 [1]. Let X be a BCI-algebra. We choose an element $x_o \in X$ such that there does not exist $y \neq x_o$ with $y*x_o = o$ and define

$$A(x_o) = \{x \in X : x_o * x = o\}.$$

Obviously, $A(x_o) \subseteq X$ and $x_o \in A(x_o)$: that is $A(x_o)$ is non-empty. The point x_o is known as the initial element of $A(x_o)$; that is, if for some $y \in X$, $y*x_o = o$, then $y = x_o$.

Let I denote the set of all initial elements in X, we call it centre of X. We note that $M = A(o)$, and if $o \neq x_o \in I$, then $A(x_o) \subseteq X - M$.

(10) Let X be a BCI-algebra with I as its centre, then for $x_o, y_o \in I$ $A(x_o) \cap A(y_o) = \phi$.

Further, it is obvious that if $x\ y \in X$ are comparable, than both are contained in the same $A(x_o)$, for $x_o \in I$ ([1]).

(11) Let X be a BCI-algebra with I as its centre; then I is a P-semisiple algebra ([2]).

(12) Let X be a BCI-algebra with M as its BCK-Part. Let $A(x_o) \subseteq X$ for $x_o \in I$, then for $x, y \in A(x_o)$, $x*y$, $y*x \in M$ ([1]).

Definition 3 [7]. Let X be a BCI-algebra and $A \subseteq X$. A is called an ideal in X if,

(i) $o \in A$.

(ii) $x \in A$, $y*x \in A$ imply $y \in A$.

Definition 4 [4]. Let X be a BCI-algebra and A an ideal in X, A is a closed ideal in X if, $o*a \in A$, for all $a \in A$.

Definition 5 [3]. An ideal A in X is strong, if for $x \in A$, $y \in X-A$, $x*y \in X-A$.

Definition 5 [4]. Let A be an ideal in X. Let a be any fixed element of A. If for some $x \in X-A$, $a*x \in A$, then A is called a weak ideal in X.

Note that in BCK-algebra, every ideal is a weak ideal, because $o*x = o \in A$, for all $x \in X-A$.

(13) Let X be a BCI-algebra with I as its centre. Let $o \in N \subset I$ and $H = \bigcup_{x_o \in N} A(x_o)$. H is a closed ideal in X iff N is a closed ideal in I ([2]).

(14) Let X be a p-semisimple algebra and $A \subseteq X$ an ideal in X. A is closed iff A is strong ([3]).

(15) Let X be a BCI-algebra and H a strong ideal in X, then H is closed ([3)].

(16) Let X be a BCI-algebra with I as its centre. Let H be a strong ideal in X. Then, $H = \bigcup_{x_o \in N} A(x_o)$, where $N = I \cap H$ ([3]).

(17) Every sub-algebra in a p-semisimple algebra is an ideal in X ([7]).

(18) Let X be a BCI-algebra, then following are equivalent:

(i) X is p-semisimple

(ii) $x*y = o$ implies $x = y$

(iii) $a*x = b*x$ implies $a = b$

(iv) $x*a = x*b$ implies $a = b$ for $a,b,x,y \in X$ ([6,9.10]).

Definition 8 [4]. An ideal A in a BCI-algebra X is called an obstinate ideal in X if, for $x,y \in X-A$, $x*y$, $y*x \in A$.

We note that all the obstinate ideals which appear in [4] are strong ideals. It is interesting to know an obstinate ideal in proper BCI-algebra which partly contains M and partly contains X – M ; that is, a weak obstinate BCI-ideal. The following example explains that such weak obstinate ideals do exist in proper BCI-algebras.

Example 1. Let x = (o,a,b,x,y) be a BCI-algebra in which * is defined by the following table.

*	o	a	b	x	y
o	o	o	o	x	x
a	a	o	o	x	x
b	b	b	o	y	x
x	x	x	x	o	o
y	y	y	x	b	o

Note that A = (o,a,x) is a weak-ideal which is obstinate.

Lemma 1. Let X be a BCI-algebra with I as its centre. Let $N \subseteq I$ and $H = \bigcup_{x_o \in N} A(x_o)$. H is an ideal in X iff N is an ideal in I.

Proof. Let $H = \bigcup_{x_o \in N} A(x_o)$ be an ideal in X, Obviously, $N \subseteq H$.

Now $N \subseteq I$ and $N \subseteq H$ implies $N = H \cap I$ and $I \cap (X - H) = I - N$. We show that N is an ideal in I, simply by showing that,

(i) $o \in N$,

(ii) $x_o \in N$, $y_o \in I - N$ imple $y_o * x_o \in I - N$.

Since H is an ideal, therefore $o \in H$. But by (11), $o \in I$. Thus $o \in H \cap I = N$ implies $o \in N$. Let $x_o \in N$, $y_o \in I - N$. We show that

$y_o * x_o \in I-N$. Since $I-N \subseteq X-H$, therefore $y_o \in I-N \subseteq X-H$ imples $y_o \in X-H$. Also $x_o \in N \subseteq H$ implies $x_o \in H$. Since H is an ideal, therefore $y_o * x_o \in X-H$. Now by (11) I is closed under $*$, therefore for $x_o \in N \subseteq I$, $y_o \in I-N \subseteq I$ implies $y_o * x_o \in I$. $y_o * x_o \in I$ and $y_o * x_o \in X-H$ implies $y_o * x_o \in (X-H) \cap I = I-N$. $y_o * x_o \in I-N$ for all $y_o \in I-N$. Thus, N is an ideal in I.

Conversely. Let N be an ideal in I. We show that $H = \bigcup_{x_o \in N} A(x_o)$ is an ideal in X : simply by showing.

(i) $o \in H$.

(ii) $x \in H$, $y \in X-H$ implies $y*x \in X-H$.

$o \in N \subseteq H$ implies $o \in H$. Let $x_o \in N$ then $A(x_o) \subseteq H$.

Let $y_o \in I-N$ then by construction of H. $A(y_o) \subseteq X-H$. Let $x_o \neq y_o$ and $x \in A(x_o)$, $y \in A(y_o)$, we show that $y*x \in X-H$. Since N is an ideal in I, therefore for $x_o \in N$, $y_o \in I-N$, $y_o * x_o \in I-N$. Let $x_o * x_o = z_o \in I-N$. Then $A(z_o) \subseteq X-H$. Since $x \in A(x_o)$, therefore $x_o \leqslant x$. By (9), $y*x \leq y*x_o$. By defination of $A(y_o)$, $y_o \leqslant y$. By (9) $y_o * x_o \leqslant y*x_o$ or $z_o \in y*x_o$. Thus $y*x_o \in A(z_o) \subseteq X-H$. By (10) $y*x \leqslant y*x_o$ implies $y*x \in A(z_o) \subseteq X-H$. Hence H is an ideal in X. This completes the proof.

Theorem 1. Let X be a BCI-algebra with I as its centre. Let $N \subseteq I$ and $H = \bigcup_{x_o \in N} A(x_o)$. H is an obstenate ideal in X iff N is an obstinate ideal in I.

Proof. Let $H = \bigcup_{x_o \subseteq N} A(x_o)$ be an obstinate ideal in X; By

Lemma 1. N is an ideal in I. We only establish the obstinacy of N. Obviously, $N \cap H = I$. Let $x_o, y_o \in I - N$. Then $A(x_o)$, $A(y_o) \subseteq X - H$ imply $x_o, y_o \in X - H$. Since H is obstinate, therefore $x_o, y_o \in X - H$ imply $x_o * y_o \in H$. Further, by (11), I is closed, therefore, $x_o * y_o \in I$, $x_o * y_o \in H$ imply $x_o * y_o \; I \cap H = N$; that is, $x_o * y_o \in N$, which gives that N is obstinate.

Conversely. Let N be an obstinate ideal in I, we show that $H = \bigcup_{x_o \in N} A(x_o)$ is an obstinate ideal in X. By Lemma 1. H is an id al i X. We only establish that H is obstinate. Obviously, $N = H \cap I$. Let $x_o, y_o \in I - N$. Then $A(x_o)$, $A(y_o) \subseteq X - H$.

Case (i) let $x \in A(x_o)$, $y \in A(y_o)$. N is obstinate, $x_o, y_o \in I - N$ give $x_o * y_o$, $y_o * x_o \in N$. Let $x_o * y_o = n_o \in N$, then $A(n_o) \subseteq H$. By definition $y \in A(y_o)$ gives $y_o \leqslant y$. Now for $x \in A(x_o)$, by (9), $x * y < x * y_o$. Since $x_o \leqslant x$, therefore $x_o * y_o \leqslant x * y_o$ or $n_o \leqslant x * y_o$ or $x * y_o \in A(n_o) \subseteq H$. Further by (10), $x * y \leqslant x * y_o$ implies $x * y \in A(n_o) \subseteq H$ and H is obstionate.

Case (ii) Let $x, y \in A(x_o) \subseteq X - H$, for $x_o \in I - N$. By (12), $x * y$, $y * x \in M = A(o) \subseteq H$.

From case (i) and (ii), it follows that H is obstinate. This completes the proof.

Theorem 2. Let X be a BCI–algebra with I as its centre. Let $N \subseteq I$ and $H = \bigcup_{x_o \in N} A(x_o)$. H is a strong ideal in X iff N is a strong ideal in I.

Proof. Let $H = \bigcup_{x_o \in N} A(x_o)$ be a strong ideal in X. Obviously, $N = H \cap I$ and $(X - H) \cap I = I - N$. By Lemma 1. N is an ideal in I.

We only show that N is strong. Let $A(x_o) \subseteq H$ and $A(y_o) \subseteq X\text{-}H$ then $x_o \in N, y_o \in I—N$. Since H is strong, therefore $x_o * y_o \in X - H$. Further by(11), $x_o, y_o \in I$ imply $x_o * y_o \in I$. Now $x_o * y_o \in X—H$ and $x_o * y_o \in I$ imply $x_o * y_o \in (X—H) \cap I = I—N$; that is $x_o * y_o \in I - N$, which gives N is strong.

Conversely, N is strong ideal in I. By Lemma 1, $H = \bigcup_{x_o \in N} A(x_o)$ is an ideal in X. We only show that H is strong. Let $x \in A(x_o) \subseteq H$, $y \in A(y_o) \subseteq X-H$, for $x_o \in N, y_o \in I—N$. We prove that $x*y \in X—H$. Since N is strong ideal in I, therefore $x_o * y_o \in I—N$. Let $x_o * y_o = z_o \in I-N$, then $A(z_o) \subseteq X-H$. By defination $x \in A(x_o)$ gives $x_o \leqslant x$. For $y \in A(y_o)$, by (9), we can write $x_o * y \leqslant x*y$. Similarly, $y_o \leqslant y$ implies $x_o * y \leqslant x_o * y_o = z_o$ or $x_o * y = z_o$. because $z_o \in I$. Thus $z_o \leqslant x*y$ implies $x*y \in A(z_o) \subseteq X—H$ and hence H is strong. This completes the proof.

Lemma 2. Let X be a finite p-semisimple algebra and $A \subseteq X$ be a proper ideal in X. Then $O(A) \leqslant O(X—A)$.

Proof. Suppose O X—A) < O (A) Let $a \in A$. $x \in X—A$, then $x*a \in X—A$, because otherwise $x*a \in A$, $a \in A$ and A being an ideal implies $x \in A$, a contradiction Now $x*a \in X—A$ for all $a \in A$, $x \in A—A$ Since $O(X-A) < O(A)$, therefore for some distinct $a_1 . a_2 \in A$, $x*a_1 = x*a_2$ holds. By (18) (iv) $x*a_1 = x*a_2$ implies $a_1 = a_2$, a contradiction. This complectes the proof.

Lemma 3. Let X be a finite p-semisimple algebra and $A \subseteq X$ be a proper ideal in X, then A is closed.

Proof. It is sufficient to show that $o*a \in A$, for all $a \in A$. Suppose $o*b \in X—A$ for some $b \in A$. Let $o*b = c \in X—A$.

Now for $x \in X-A$, $b \in A$, we have $x*b \in X-A$, because otherwise $x \in A$, a contradiction Let $O(A)=m$, $O(X)=N$ and $O(X-A)=n$, y Lemma 2 $m<n$. Let $A=(o=x_1, x_2 \dots x_m)$ a and $X-A=$ $y_1, y_2 \dots y_n$). Now $y_1 * b \dots y_n * b$ are n distinct elements of $X-A$, because other wise (18) qives that atleast any two elements of $X-A$ are equal, which is false. Further $o*b \in X-A$. Thus $o*b=y*b$ for some $y \in X-A$. Again (18) gives $y=o$, a contradiction, because $o \subseteq A$. Thus our supposition is incorrect. Hence A is closed.

Theorem 3. Let X be a finite p-semisimple algebra and $A \subseteq X$ be proper ideal in X. Then A is strong.

Proof. It follows from lemma 3 and (14).

Lemma 4. Let X be a p-semisimple a algebra. Let A.B. be two sub-algebras of X such that A and B are not properly contained in each other. Then $A \cup B$ in not sub-algebra of X.

Proof. By (17) every sub-algebra in X is an ideal in X, therefore A, B are ideals in X. Suppose $A \cup B$ is a sub-algebra, than $A \cup B$ is an ideal in X. For $a \in A$, $b \in B$ imply $a*b \in A \cup B$.

There are three possibilities namely,

(i) $a*b \in A$.

(ii) $a*b \in B$.

(iii) $a*b \in A \cap B$.

Cast (i) Let $a*b \in A$. Then $a*b=c \in A$ (say) Now $a*b=c$ implies $(a*b)*c=o$ or $(a*c)*b=o$. By (18) $a*c=b \in B$, which implies A is not closed, a contradiction Thus $a*b \notin A$.

Case (ii) Let $a*b \in B$. Since B in an ideal in X, therefore, $a*b \in B$, $b \in B$ implies $a \in B$, a contradiction. Thus $a*b \notin B$.

Case (iii). From case (I) and (ii) $a*b \notin A, B$, that is $a*b \notin A \cap B$. Hence $A \cup B$ is not a sub-algebra. This completes the proof.

Corollary 1. Union of two ideals A, B (such that A and B are not properly contained in each other) in a p-semisimple algebra X is not an ideal in X.

Theorem 4. Let X be a BCI-algebra with I as its centre Let A, B be proper BCI-sub-algebras such that $A \cap I = N_1$, $B \cap I = N_2$. If N_1, and N_2 are not properly contained in each other, then $A \cup B$ in not closed.

Proof. Let $A \cap I = N_1$, $B \cap I = N_2$. Let $x_0, y_0 \in N_1$. Since $N_1 \subseteq A$ and A is closed, therefore $x_0 * y_0 \in A$. Since $N_1 \subseteq I$ and I is closed, therefore $x_0 * y_0 \in I$. Now $x_0 * y_0 \in A$ and $x_0 * y_0 \in I$ both in imply $x_0 * y_0 \in A \cap I = N_1$, which gives N_1, is closed. Similarly N_2 is closed. Let $(A \cup B) \cap I = N$. Then by Lemma 4, $N_1 \cup N_2 = N \subseteq I$ is not closed, which implies $A \cup B$ is not closed. This completes the proof.

Corollary 2. Union of arbitrary distinct (which are not properly contained in each other) closed ideals in a BCI-algebra X is not an ideal in X.

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EXTENSIONS OF SOME FIXED POINT THEOREMS OF KANNAN AND WONG TO PARANORMED SPACES

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Abstract

Let X be a convex subset of a paranormed space E and T a self mapping on X. We obtain some results on the convergence of certain sequences to fixed points of T under various contractive conditions.

Let X be a convex subset of a linear space E and T a self mapping on X, R. Kannan [2] and C S. Wong [6] obtained fixed point theorems on the convergence of certain sequences involving T in the case of E a normed or Banach space. Later J. Achari [1] and S. L. Singh ([3], [4]) proved some of these results for more general contractive conditions. The purpose of this paper is to extend them to the case of E a paranormed space.

In the sequel we shall assume that the topology of E is generated by a total paranorm q having the following properties (see [5], p. 52):

(a) $q(x) \geq 0$, and $q(x) = 0$ iff $x = 0$.

(b) $q(-x) = q(x)$.

(c) $q(x+y) \leq q(x) + q(y)$.

(d) If $\{a_n\}$ is a sequence of real or complex scalars with $a_n \to a$ and $\{x_n\}$ is a sequence in E with $x_n \to x$, then $q(a_n x_n - ax) \to 0$.

Every metric linear space is a paranormed space. Note that a total paranorm q need not satisfy $q(ax) = |a| q(x)$ (property of a norm) or $q(ax) \leq q(x)$ for $|a| \leq 1$ (property of an F-norm).

Throughout this paper, X denotes a conxex subset of E.

We first obtain a generalization of ([1]), Theorem 3) and ([2], Theorem 6).

Theorem 1. Suppose E is complete and X a closed convex subset of E. Let $T : X \rightarrow X$ be a mapping satisfying

$$q(Tx-Ty) \leq r \max\{q(x-y), q(x-Tx), q(y-Ty), q(x-Ty), q(y-Tx)\}, \quad (1)$$

for all $x, y \in X$, where $0 \leq r < 1$. For each $n \geq 1$, let a_n be a solution of the equation $Tx - r = A_n$, wnere $A_n \in X$. If $\lim_{n \to \infty} A_n = 0$, then $\{a_n\}$ converges and its limit point is a unique solution of the equation $Tx = x$.

Proof. For n, m $\geq$ 1, we have by using (1) that

$$\begin{aligned} q(a_n - a_m) &\leq q(a_n - Ta_n) + q(Ta_n - Ta_m) + q(Ta_m - a_m) \\ &\leq q(A_n) + r \max\{q(a_n - a_m), q(A_m), q(A_m), q(a_n - a_m) + q(A_m), \\ &\qquad q(a_n - a_m) + q(A_n)\} + q(A_m) \end{aligned}$$

or

$$q(a_n - a_m) \leq \frac{1+r}{1-r}(q(A_n) + q(A_m)).$$

Since $\lim_{n \to \infty} A_n = 0$, we conclude that $\{a_n\}$ is a Cauchy sequence in X. Thus there exists some $u \in X$ such that $\lim_{n \to \infty} a_n = u$.

We now show that $Tu = u$. Using (1) again, we obtain

$$q(Tu-u) \leq q(Tu-Ta_n) + q(Ta_n - a_n) + q(a_n - u)$$
$$\leq r \max\{q(u-a_n), q(u-Tu), q(A_n),$$
$$q(u-a_n) + q(A_n), q(a_n - u) + q(u-Tu)\}$$
$$+ q(A_n) + q(a_n - u).$$

Letting $n \to \infty$, we obtain $q(Tu-u) \leq r\, q(Tu-u)$, and thus $Tu = u$.

For uniqueness, suppose that also $Tv = v$ for some $v \in X$. Then

$$q(u-v) = q(Tu-Tv)$$
$$\leq r \max\{q(u-v), q(u-Tu), q(v-Tv),$$
$$q(u-Tv), q(v-Tu)\}$$
$$= r\, q(u-v).$$

Before stating the next result, we need the following.

Definition. For any $x_0 \in E$ and $0 < t < 1$, let $x_{n+1} = tTx_n$ for $n \geq 0$. Then $\{x_n\}_{n=0}^{\infty}$ is called a sequence of *Picard iterates* of T.

The following result extends ([1], Theorem 1) as well as the theorem of [3].

Theorem 2. Let $T : X \to X$ be a mapping satisfying

$$q(Tx-Ty) \leq r \max\{kq(x-y), q(x-Tx), q(y-Ty), q(x-Ty),$$
$$q(y-Tx)\} + s \max\{q(x-T^2x), q(Tx-T^2x),$$
$$q(y-T^2x), q(Ty-T^2x)\}$$

for all $x, y \in X$, where $k, r, s \geq 0$ with $r + s < 1$. If, for some $x_0 \in X$ and $0 < t < 1$, the sequence $\{x_n\}$ of Picard iterates converges to a point u X, then u is a fixed point of T.

Proof. Since $\lim_{n\to\infty} x_n = u$ and $q(Tx_n - x_n) = q(t^{-1}(x_{n+1} - x_n))$, is follows that $\lim_{n\to\infty} Tx_n = u$. We now show that also $\lim_{n\to\infty} T^2 x_n = u$, as follows. Taking $x = x_n$ and $y = Tx_n$ in (?), we obtain

$$\begin{aligned} q(Tx_n - T^2 x_n) &= q(Tx_n - T(Tx_n)) \\ &\leqslant r \max\{kq(x_n - Tx_n), q(x_n - Tx_n), \\ &\quad q(Tx_n - T^2 x_n), q(x_n - T^2 x_n), \\ &\quad q(Tx_n - Tx_n)\} + s\max\{q(x_n - T^2 x_n), \\ &\quad q(Tx_n - T^2 x_n), q(Tx_n - T^2 x_n), \\ &\quad q(T^2 x_n - T^2 x_n)\}. \end{aligned}$$

Letting $n\to\infty$, we have

$$q(u - \lim_{n\to\infty} T^2 x_n) \leqslant (r+s)\, q(u - \lim_{n\to\infty} T^2 x_n).$$

Since $r+s < 1$, we have $\lim_{n\to\infty} T^2 x_n = u$.

Now, for any $n \geqslant 1$.

$$\begin{aligned} q(Tu-u) &\leqslant q(Tu - Tx_n) + q(Tx_n - x_n) + q(x_n - u) \\ &\leqslant r \max\{kq(u - x_n), q(u - Tu), q(x_n - Tx_n), \\ &\quad q(u - Tx_n), q(x_n - Tu)\} + s\max\{q(u - T^2 x_n), \\ &\quad q(Tu - T^2 x_n), q(x_n - T^2 x_n), q(Tx_n - T^2 x_n)\} \\ &\quad + q(Tx_n - x_n) + q(x_n - u). \end{aligned}$$

Since n is arbitrary, we let $n \to \infty$ and obtain

$$q(Tu-u) \leqslant (r+s)\, q(Tu-u).$$

Consequently $Tu = u$, as required.

The following theorem extends ([4], Theorems 3 and 4).

Theorem 3. Let $T : X \to X$ be a mapping satisfying at least one of the following conditions :

$$q(Tx-Ty) \leqslant a \max\{kq(x-y), \tfrac{1}{2}[q(x-Tx)+q(y-Ty)]\} + b[q(x-Ty)+q(y-Tx)], \quad (3)$$

$$q(Tx-Ty) \leqslant a \max\{kq(x-y)\, \tfrac{1}{2}[q(x-Ty)+q(y-Tx)]\} + b[q(x-Tx)+q(y-Ty)] \quad (4)$$

for all $x, y \in X$, where $k, a, b \geq 0$ with $a + 2b < 2$. If, for some $x_0 \in X$ and $0 < t < 1$, the sequence $\{x_n\}$ of Picard iterates converges to $u \in X$, then $Tu = u$.

Proof. The proof is similar to that of Theorem 2, and is therefore omitted.

Remark. The above result was obtained in [4] under the restrictions that $k = 1$, $b \leqslant 1$, and $a + 2b = 1$ with q being a norm on E.

Finally, we give an example of a paranormed space and a mapping which satisfies the contractive conditions of Theorems 1 and 2.

Example. Let $E = R$, the set of real numbers, and q be the total paranorm defined by $q(x) = |x|/(1+|x|)$ for $x \in R$. Let $X = [0, 1]$, and define $T : X \to X$ by

$$T(x) = \begin{cases} 1/8 & \text{if } 0 \leqslant x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Then T satisfies the condition (2) for $r = \frac{2}{3}$, $s = \frac{1}{4}$, and any $k \geq 0$ as follows. If $x = y = 1$ or $x, y \in [0, 1)$, then (2) is trivially satisfied.

If $x=1$ and $0 \leq y < 1$, then

$$q(Tx-Ty) \leq q\left(\frac{1}{8}\right) = \frac{1}{9},$$

$$r\, q(x-Ty) = \frac{2}{3}\, q(1) = \frac{1}{3} > \frac{1}{9},$$

and also

$$s\, q(x-T^2x) = \frac{1}{4}\, q(1) = \frac{1}{8} > \frac{1}{9}.$$

Taking $r = \frac{2}{3}$, $s=0$, and $k=1$, the condition (1) is clearly satisfied. Note that $x = \frac{1}{8}$ is the fixed point of T.

Remark. The results of this paper need not hold for an arbitrary semi-normed or non Hausdoeff locally convex space E ; for, if q is a semi-norm on E, then $q(Tu-u)=0$ does not necessarily imply that $Tu=u$. However, one may possibly try them for strictly convex locally convex spaces.

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A DISPROOF OF A CONJECTURE OF ROBERTSON AND GENERALIZATIONS

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Abstract

In this paper we disprove our general conjecture that, for $y \geqq y_0$, where y_0 is a certain number of the interval $0 < y_0 \leqq 1/2$, all coefficients of the powers of X in the expansion (1) are nonnegative. In particular, for $y = 1/2$, the special conjecture of Robertson is disproved.

In [1] pp. 264, 274–280, Section 4, we established the Taylor expansion

$$d(z) \overset{!}{=} \left\{ \frac{\left(\frac{1+z}{1-z}\right)^x - 1}{2xz} \right\}^y = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} d_{nj}(y)\, x^j \tag{1}$$

for $|z| < 1$, where x and y are arbitrary complex numbers, $1^x = 1^y = 1$, and the coefficients $d_{nj}(y)$ are found explicitly. In particular, the coefficients $d_{nj}(y)$ are equal to zero if n and j are of different parity. We proved that the coefficients $d_{nj}(y)$, $0 \leqq j \leqq n$, in (1) are nonnegative for $n = 1, \ldots, [y]+1$ if $y > 0$ is not a positive integer ([y] denotes the greatest integer less than y), and for all $n = 1, 2, \ldots$ if y is a positive integer ($y = 1, 2, \ldots$). Therefore, we conjectured that the coefficients $d_{nj}(y)$, $0 \leqq j \leqq n$, in (1) are nonnegative for the "tail-end" as well, i.e. far $n > [y]+1$ if $y \geqq y_o$ is not a positive integer where y_o is a certain number of the interval $0 < y_o \leqq 1/2$. In particular, for $y=1/2$ this conjecture is due to Robertson [2], (pp 8, 20-21, 176). In [3] the Robertson conjecture has been disproved. At the author's request, (1) Staffan Wrigge and Arne Fransen of National Defence Research Institute, Systems Analysis Department (F0A 1), P O Box 27322, S–10254, Stockholm, Sweden, (2) Earl Dilcher of Dalhousie University, Department of Mathematics, Statistics and Computing Science, Halifax, Nova Scotia, Canada B 3H 3J5 and (3) Pierre Barrucand of University Pierre et Marie-Curie (Paris VI), Institut de mathematiques pures et appliquees, 4, place Jussieu, 75252 Paris Cedex 05, France, computed independently the coefficients d_{nj} $(1/2)$, $0 \leqq j \leqq n$, for (1) $9 \leqq n \leqq 15$, (2) $8 \leqq n \leqq 20$ and (3) $0 \leqq n \leqq 29$, respectively, and found that the first negative coefficient occurs for $n = 13$ and $j = 13$. In addition, the author had suggested the Robertson conjecture to the attention of A A. Jagers of University of Twente, Department of Mathematics, Enschede, The Netherlands, who in his turn suggested it to F.W. Steutel of Eindhoven University of Technology. Department of Mathematics and Computing Science, P.O. Box 513, 5600 MB Eindhoven, the Netherlands. In a private communication to the author, Steutel [4] also disproved the Robertson conjecture. Now in this paper we shall show that our general conjecture is false for any rational (but not integer) $y > 0$ as well. For this we need two lemmas.

Lemma 1. (Steutel [4] and [5], p. 137).

Let $$p(z) = \sum_{n=0}^{\infty} p_n z^n \qquad (2)$$

be a (possibly formal) power series generating a strictly logarithmically convex sequence of positive numbers p_n, $n=0, 1, 2, \ldots$ i.e.

$$p_n^2 < p_{n+1}\, p_{n-1}, \; n = 1, 2, \ldots, \qquad (3)$$

and let

$$(p(z))^y := \sum_{n=0}^{\infty} p_n(y)\, z^n,\ y > 0,\ p_o(y) > 0, \tag{4}$$

Then

$$p_n(y) > 0,\ n = o, 1, 2, \ldots \tag{5}$$

Corollary. Under the conditions of Lemma 1, if $y < 0$ in (4), then there exists at least one subscript $n > 0$ such that $p_n(y) < 0$.

Proof. The Corollary follows from (5) and the identities

$$\sum_{k=0}^{n} p_k(y)\, p_{n-k}(-y) \equiv 0.\ n=1, 2, \ldots,\ y < 0,\ p_o(y) > 0,$$

resulting from the multiplication of the series (4) and the series obtained from (4) after substituting y for $-y$.

Lemma 2. Let

$$B(t) := \left(\frac{e^t - 1}{t}\right)^y := \sum_{n=0}^{\infty} B_n(y)\, t^n,\ |t| < 2\pi,\ 1^y = 1. \tag{6}$$

Then

(i) for any rational (but not integer) $y > 0$, and (ii) for any rational number $y < 0$, there exists at least one subscript $n > 0$ such that $B_n(y) < 0$.

(iii) for any irrational number $y > 0$, there exists at least one subscript $n > 0$ such that eihter $B_n(y) < 0$ or $B_n(-y) < 0$.

Proof. (i) Let $p \geqq 1$ and $q \geqq 2$ be integers such that p is not a multiple of q. Then for $y = p/q$ from (6) we obtain

$$\left(\frac{e^t - 1}{t}\right)^p = \left(\sum_{n=0}^{\infty} B_n t^n\right)^q;\ B_n := B_n\left(\frac{p}{q}\right),\ |t| < 2\pi, \tag{7}$$

where $B_o = 1$ and all B_n . $n=1, 2, \ldots$, are real numbers. We have

$$\left(\frac{e^t - 1}{t}\right)^p = p!\sum_{n=0}^{\infty} \frac{S(n+p, p)}{(n+p)!}\, t^n \tag{8}$$

where

$$S(n+p, p) = \frac{1}{p!} \sum_{\nu=1}^{p} (-1)^{p-\nu} \binom{p}{\nu} \nu^{n+p}, \quad n \geqq 0, \tag{9}$$

are the Stirling numbers of the second kind which are positive integers (see, for example, in [6], pp. 313 and 310, Formulas (21) and (5)—(6), respectively, or in [7], Chapter V), and

$$\left(\sum_{n=0}^{\infty} B_n t^n \right)^q = \sum_{n=0}^{\infty} t^n \sum B_{k_1} B_{k_2} \dots B_{k_q} \tag{10}$$

where the inner sum is taken over all nonnegative integers k_1, k_2, ..., k_q satisfying

$$k_1 + k_2 + \dots + k_q = n, \ n \geqq 0. \tag{11}$$

From (7), (8), (10) and (11) it follows that

$$\frac{p!\, S(n+p, p)}{(n+p)!} = \sum B_{k_1} B_{k_2} \dots B_{k_q} \quad n \geqq 0. \tag{12}$$

(From (12) and (11) for $n = 0$ we obtain again $B_o = 1$, and for $n=1$ we obtain $B_1 = p/2q$ since $S(p+1, p) = p(p+1)/2$ (see, for example, in [7], p. 227) of course, these values of B_o and B_1 follow directly from (7).) From (12) and (11) for $n \geqq 2$ we obtain

$$B_n = \left(\frac{1}{q} \frac{p!\, S(n+p, p)}{(n+p)!} - \sum B_{k_1} B_{k_2} \dots B_{k_q} \right) \tag{13}$$

where the sum is taken over all nonnegative integers k_1, k_2, ... k_q satisfying simultaneously the inequalities $0 \leqq k_j \leqq n-1$, $j=1,2,\dots,q$, and the equation (11). Now if we assume that all B_n, $n=0, 1, 2, \dots$ are nonnegative, then from (13) it follows that

$$0 \leqq B_n \leqq \frac{p!\, S(n+p, p)}{q\,(n+p)!}, \quad n \geqq 2. \tag{14}$$

Further, with the help of (9) we obtain that

$$\lim_{n\to\infty}\left(\frac{p\,!\,S(n+p,\,p)}{q\,(n+p)\,!}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{S(n+p+1,\,p)}{(n+p+1)\,S(n+p,\,p)}$$

$$= \lim_{n\to\infty}\frac{p}{n+p+1}\cdot\frac{\sum_{\nu=1}^{p}(-1)^{p-\nu}\binom{p}{\nu}\left(\frac{\nu}{p}\right)^{n+p+1}}{\sum_{\nu=1}^{p}(-1)^{p-\nu}\binom{p}{\nu}\left(\frac{\nu}{p}\right)^{n+p}} = 0. \tag{15}$$

Therefore, from (15) and (14) we conclude that the series (6) for $y=p/q$ has an infinite radius of convergence; but this is a contradiction as $t=2m\pi i$, $m=\pm 1, \pm 2, \ldots$, are branch points of the function (6) for $y=p/q$. Hence, for $y=p/q$, not all coefficients $B_n(p/q)$, $n=0, 1, 2, \ldots$, in the series (6) ara nonnegative.

(ii) Let $p \geqq 1$ and $q \geqq 1$ be integers. Then for $y=-p/q$ from (6) we obtain the identity

$$\left(\sum_{n=0}^{\infty} B_n t^n\right)^q \left(\frac{e^t-1}{t}\right)^p \equiv 1.\ B_n := B_n\left(-\frac{p}{q}\right), \tag{16}$$

having in mind (10) and (8). If all $B_n \geqq 0$ for $n=0, 1, 2, \ldots$, then the coefficients of t^n for $n=1, 2, \ldots$ in the expansion of the left-hand side of (16) will be positive but not equal to zero according to the right-hand side of (16). Hence, for $y=-p/q$, not all coefficients $B_n(-p/q)$, $n=0, 1, 2, \ldots$, in the series (6) are nonnegative.

(iii) Let $y>0$ be an irrational number. Then from (6) we deduce the identities

$$\sum_{k=0}^{n} B_k(y)\,B_{n-k}(-y) \equiv 0,\ n=1, 2, \ldots \tag{17}$$

From (17) it follows that not all coefficients $B_n(y)$ and $B_n(-y)$ for $n=1, 2, \ldots$ are nonnegative.

This completes the proof of Lemma 2.

In particular, for $y = 1/q$, where $q \geqq 2$ is an integer, Lemma 2 is due to Steutel [1].

An open problem is whether for any irrational number $y > 0$ there always exists at least one subscript $n > 0$ such that $B_n(y) < 0$; the same question holds for $B_n(-y)$.

Theorem. For any rational (butnot integer) $y > 0$, not all coefficients $d_{nj}(y)$ in (1) are nonnegative.

Remark. Evidently, for any integer $y < 0$, all coefficients $d_{nj}(y)$ in (1) are positive.

Proof. Let

$$t := x \log \frac{1+z}{1-z}, \tag{18}$$

$$A(z) := \frac{1}{2z} \log \frac{1+z}{1-z} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n+1}, \quad |z| < 1, \tag{19}$$

$$(A(z))^y := \sum_{n=0}^{\infty} A_n(y) z^{2n}, \quad |z| < 1, y > 0, A_0(y) = 1, \tag{21}$$

where according to (2)–(5) in Steutal's lemma 1 we shall have

$$A_n(y) > 0, \; n=0, 1, 2, \dots \; (y > 0). \tag{21}$$

With the help of (18)–(20) and (6) the equation (1) takes the form

$$d(z) = (A(z))^y \, B(t) \tag{22}$$

$$= \sum_{j=0}^{\infty} B_j(y) (2xz)^j (A(z))^{y+j}$$

$$= \sum_{n=0}^{\infty} z^{2n} \sum_{j=0}^{n} A_{n-j}(y+2j) \, B_{2j}(y) (2x)^{2j}$$

$$+ \sum_{n=0}^{\infty} z^{2n+1} \sum_{j=0}^{n} A_{n-j}(y+2j+1) B_{2j+1}(y)(2x)^{2j+1}$$

for $|z| < 1$. From (1) and (22) we obtain the following formulas for the nonvanishing coefficients

$$d_{2n,2j}(y) = 2^{2j} A_{n-j}(y+2j) B_{2j}(y) \tag{23}$$

and

$$d_{2n+1,2j+1}(y) = 2^{2j+1} A_{n-j}(y+2j+1) B_{2j+1}(y) \tag{24}$$

for $0 \leqq j \leqq n$, $n = 0, 1, 2, \ldots$. It is clear from (21), (23) and (24) that the signs of $d_{2n,2j}(y)$ and $d_{2n+1,2j+1}(y)$ are determined by those of $B_{2j}(y)$ and $B_{2j+1}(y)$, respectively. Therefore, for any rational but not integer) $y > 0$, according to our lemma 2, there exists at least one integer $j > 0$ such that either $B_{2j}(y) < 0$ or $B_{2j+1}(y) < 0$, i.e. either $d_{2n,2j}(y) < 0$ or $d_{2n+1,2j+1}(y) < 0$ for all integers $n \geqq j$, respectively. (For example, in [3] we have shown that $d_{13,13}(\frac{1}{2}) < 0$, i.e. $B_{13}(\frac{1}{2}) < 0$. Hence, $d_{2n+1,13}(\frac{1}{2}) \leqq 0$ for all integers $n \geqq 6$).

This completes the proof of the Theorem.

In particular, for $y = 1/q$, where $q \geqq 2$ is an integer, the Theorem is due to Steutel [4].

From the Theorem proved it follows that our general conjecture for any rational (but not integer) $y > 0$ as well as the special conjecture of Robertson for $y = \frac{1}{2}$ for the coefficients in (1) are false. But our general conjecture is open for the irrational numbers $y > 0$.

Application. For $j = n$ and $y = \frac{1}{2}$ from (23) and (24) we obtain the formula

$$d_{nn}(\tfrac{1}{2}) = 2^n B_n, \; n = 0, 1, 2, \ldots, \tag{25}$$

where the numbers B_n are generated by the expansion (6) for $y = \frac{1}{2}$ and $B_n := B_n(\frac{1}{2})$. From (13) for $p = 1$ and $q = 2$ we obtain the recurrence relation

$$B_n = \tfrac{1}{2}\left(\frac{1}{(n+1)!} - \sum_{k=1}^{n-1} B_k B_{n-k}\right), \; n \geq 2, \; B_0 = 1, \; B_1 = \tfrac{1}{4}, \tag{26}$$

for the calculation of the numbers B_n. Thus from (26) we obtain successively

$$B_2 = \frac{5}{2^5 . 3}, \; B_3 = \frac{1}{2^7}, \; B_4 = \frac{79}{2^{11}.3^2.5}, \tag{27}$$

$$B_5 = \frac{3}{2^{13}.5}, \; B_6 = \frac{71}{2^{16}.3^3.7}, \; B_7 = \frac{113}{2^{18}.3^3.5.7},$$

$$B_8 = \frac{3053}{2^{23}.3^4.5^2.7}, \; B_9 = \frac{1}{2^{25}.3^3.5^2}$$

$$B_{10} = \frac{17}{2^{28}.3^2.5.7.11}, \; B_{11} = \frac{19}{2^{30}.3^2.7.11},$$

$$B_{12} = \frac{935917}{2^{34}.3^4.5^3.7^2.11.13},$$

$$B_{13} = -\frac{20287103}{2^{36}.3^6.5^3.7^2.11.13} < 0,$$

$$B_{14} = -\frac{2452337}{2^{39}.3^7.5^2.7^2.11.13} < 0 \,!$$

The last two equations in (27), having in mind (25), show that the Robertson conjecture is false. More general, with the help of (27) from (23) and (24) we conclude that

$$d_{2n,2j}(\tfrac{1}{2}) > 0, \; 0 \leqq j \leqq n, \; n = 0, 1, 2, 3, 4, 5, 6,$$

$$d_{2n+1,2j+1}(\tfrac{1}{2}) > 0, \; 0 \leqq j \leqq n, \; n = 0, 1, 2, 3, 4, 5,$$

and

$$d_{2n,\,2j}\left(\frac{1}{2}\right) > 0,\ 0 \leq j \leq 6,\ n \geqslant 7,$$

$$d_{2n+1,\,2j+1}\left(\frac{1}{2}\right) > 0,\ 0 \leqslant j \leqslant 5,\ n \geqslant 6,$$

but

$$d_{2n+1,\,13}\left(\frac{1}{2}\right) < 0,\ n \geqslant 6,$$

$$d_{2n,\,14}\left(\frac{1}{2}\right) < 0,\ n \geqq 7,$$

respectively. Again from the last two inequalities it follows that the Robertson conjecture is false.

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ON THE COEFFICIENTS OF THE POWERS OF THE UNIVALENT FUNCTIONS OF THE CLASS S

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Abstract

In this paper we give a simple proof of the inequalities (7) for the cases $n = 1$, $[\lambda] + 1$ if $\lambda > 1$ is not an integer ($[\lambda]$ denotes the greatest integer less than λ), and for all $n = 1, 2, \ldots$ if $\lambda < 1$ is an integer, respectively.

Let S be the class of functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \tag{1}$$

that are analytic and univalent in the disc $|z| < 1$, and let

$$\left[\frac{f(z)}{z}\right]^{\lambda} \equiv 1 + \sum_{n=1}^{\infty} f_n(\lambda) z^n \tag{2}$$

for $f(z) \in S$ and any complex number λ.

Let $k(z, \epsilon) \in S$ be the Koebe function

$$k(z, \epsilon) = \frac{z}{(1-\epsilon z)^2} = \sum_{n=1}^{\infty} n \epsilon^{n-1} z^n, \quad |\epsilon| = 1, \tag{3}$$

and for any complex number λ, let

$$\left[\frac{k(z, \epsilon)}{z}\right]^{\lambda} \equiv 1 + \sum_{n=1}^{\infty} k_n(\lambda, \epsilon) z^n, \tag{4}$$

where

$$k_n(\lambda, \epsilon) = \binom{2\lambda+n-1}{n} \epsilon^n, \quad n = 1, 2, \ldots \tag{5}$$

For $\lambda=1$, Louis de Branges [1] proved the Bieberbach conjecture for the class S that

$$|f_n(1)| \leqq n+1, \; f_n(1) \equiv a_{n+1}, \tag{6}$$

for $n=1, 2, \ldots$ where for some n the equality holds only for the Koebe function (3) with (4)–(5).

For a positive integer $\lambda > 1$ in (2) and (4)–(5), from the results due to Milin [2], p. 101, Theorem 3.9 and Grinshpan [3], p. 88, and from the inequalities (6) it follows that the inequalities

$$|f_n(\lambda)| \leqq \binom{2\lambda+n-1}{n}, \; n=1, 2, \ldots, \tag{7}$$

hold, where for some n the equality holds only for the Koeba function (3) with (4)–(5). A direct and simpler proof of the inequalities (7) is given by us in [4]–[6]. The problem for the correctness of the inequalities (7) if $\lambda > 1$ is not an integer has been solved affirmatively by Louis de Branges [7], Hayman and Hummel [8], and Milin and Grinshpan [9].

In this paper we give a direct and simpler proof of the inequalities (7) for the cases $n=1, \ldots, [\lambda]+1$ if $\lambda > 1$ is not an integer ($[\lambda]$ denotes the greatest integer less than λ), and, again, for all $n=1, 2, \ldots$ if $\lambda > 1$ is an integer, respectively, where, for some n, the equality holds only for the Koebe fuoction (3).

Proof. From (1), (2) and our paper [10], p. 84, formulas (25)–(26), we obtain the formula

$$f_n(\lambda) = \sum_{r=1}^{n} (\lambda)_r \, C_{nr}(a_2, \ldots, a_{n-r+2}) \tag{8}$$

for $n=1, 2, \ldots$ and any λ, where

$$(\lambda)_r = \lambda(\lambda-1)\ldots(\lambda-r+1), \; r=1, 2, \ldots, \tag{9}$$

and

$$C_{nr}(a_2, \ldots, a_{n-r+2}) = \Sigma \frac{(a_2)^{\nu_1} \ldots (a_{n-r+2})^{\nu_{n-r+1}}}{\nu_1! \ldots \nu_{n-r+1}!} \tag{10}$$

where the sum is taken over all nonnegative integera $\nu_1, \nu_2, \ldots,$ ν_{n-r+1} satisfying

$$\nu_1 + \nu_2 + \ldots + \nu_{n-r+1} = r,$$

$$\nu_1 + 2\nu_2 + \ldots + (n-r+1)\nu_{n-r+1} = n \tag{11}$$

In particular, for the Koebe function (3) with (4), from (8)–(11) we obtain the farmula

$$k_n(\lambda, \epsilon) = \epsilon^n \sum_{r=1}^{n} (\lambda)_r C_{nr}(2, \ldots, n-r+2) \tag{12}$$

for $n=1, 2, \ldots$ and any λ. Now the comparison of (12) and (5) yields the identities

$$\sum_{r=1}^{n} (\lambda)_r C_{nr}(2, \ldots, n-r+2) = \binom{2\lambda+n-1}{n} \tag{13}$$

for $n=1, 2, \ldots$ and any λ.

Therefore, from (8)—(13). with (6) in mind we obtain the sharp estimates

$$|f_n(\lambda)| \leqq \sum_{r=1}^{n} (\lambda)_r \; |C_{nr}(a_2, \ldots, a_{n-r+2})| \tag{14}$$

$$\leqq \sum_{r=1}^{n} (\lambda)_r \; C_{nr}(2, \ldots, n-r+2) = \binom{2\lambda+n-1}{n}$$

for $n=1, \ldots, [\lambda]+1$ if $\lambda > 1$ is not an integer, and for all $n=1, 2, \ldots$ if $\lambda > 1$ is an integer, respectively, where for some n the equality holds only for the Koebe function (3).

Remark. The identities (13) can be written in the following form. We have the identities (see [4], p. 971, Identities (25))

$$r!\, C_{nr}(2, \ldots, n-r+2) = \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \binom{2j+n-1}{n} \tag{15}$$

for $n=r, r+1, \dots$ and $r=1, 2, \dots$ From (15) it follows that

$$\sum_{r=1}^{n} (\lambda)_r \, C_{nr}(2, \dots, n-r+2)$$

$$= \sum_{r=1}^{n} \binom{\lambda}{r} \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \binom{2j+n-1}{2}$$

$$= \sum_{j=1}^{n} \binom{2j+n-1}{2} \sum_{r=j}^{n} (-1)^{r-j} \binom{r}{j} \binom{\lambda}{r}$$

$$= \sum_{j=1}^{n} \binom{2j+n-1}{n} \binom{\lambda}{j} \sum_{r=0}^{n} (-1)^{r} \binom{\lambda-j}{r}$$

$$= \sum_{j=1}^{n} (-1)^{n-j} \binom{2j+n-1}{n} \binom{\lambda}{j} \binom{\lambda-j-1}{n-j} \qquad (16)$$

for $n=1, 2, \dots$ and any λ Now the comparison of (16) and (13) yields the combinatorial identities

$$\sum_{j=1}^{n} (-1)^{n-j} \binom{2j+n-1}{n} \binom{\lambda}{j} \binom{\lambda-j-1}{n-j} = \binom{2\lambda+n-1}{n}$$

valid for $n=1, 2, \dots$ and arbitrary λ.

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SOME FIXED POINT THEOREMS FOR ITERATES OF QUASI NONEXPANSIVE MAPPINGS IN LOCALLY CONVEX SPACES

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Abstract

Under certain conditions, we establish some fixed point theorems for iterates of quasi-nonexpansive self-mappings in a locally convex space. An example is given to justify our results.

1. Introduction

Let T be a self mapping on a linear topological space X. In recent years several authors have obtained fixed point theorems for iterates assuming T is quasi-nonexpansive mapping and X is a Banach space under some conditions, see [1], [2], [3], [4].

In this paper we use this approach to study the convergence of iterates of quasi-nonexpansive mapping in a locally convex space. We obtain the locally convex versions of the two theorems of W.V. Petryshyn and T.E. Williamson JR. [4] As consequences we proved two fixed point theorems for iterates under certain conditions in a locally convex space.

In the sequel, we assume that X is a locally convex space whose topology is generated by a family $\{ p_\gamma : \gamma \in I \}$ of continuous semi-norms and satisfying the axiom of seperation, see [5], pp. 24 –26].

We adapt here a definition of quasi-nonexpansive mapping in a Banach space as stated in [6] to be held in a locally convex space

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as follows :

Definition 1 1. Let X be a locally convex space topologized by a family of continuous semi-norms $\{ p_\gamma : \gamma \in I \}$ and satisfying the axiom of seperation. Suppose C is a closed convex subset of X. A self-mapping T on C is said to be quasi-nonexpansive if Γ has a fixed point u ε C such that $p_\gamma (x-u) \neq 0$ then

$$p_\gamma (Tx-u) \leqslant p_\gamma (x-u) \tag{1}$$

is true for all x ε C.

In what follows, we suppose that the mapping T on C is quasi-nonexpansive and the set of all fixed points of T is denoted by Fix (T). Also we define :

$$p_\gamma (x, \text{Fix}(T)) = \text{Inf} \{p_\gamma (x-u) ; u \in \text{Fix}(T), \gamma \in I\}. \tag{2}$$

Our investigation of the convergence of iterates of quasi-nonexpansive mapping T on C is carried out under the conditions :

(i) Fix (T) $\neq \phi$.

(ii) $p_\gamma (x-Tx) \neq 0$ for all x ε C.

The following definition is used later.

Definition 1.2. [7]. A sequence $\{ x_n \}$ in a locally convex space X is said to be Cauchy sequence iff $p_\gamma (x_n - x_m) \to 0$ as $n, m \to \infty$ for all $\gamma \in I$. X is quasi-complete if every bounded closed subset of X is complete.

Remark 1.1. Clearly every complete space is quasi-complete space aud every quasi-complete space is sequentially complete [8, pp. 210], but not conversly.

2. Main Results

Throughout X denotes a quasi-complete locally convex space whose topology is generated by a family $\{p_\gamma : \gamma \in I\}$ of continuous semi-norms.

Theorem 2.1. Let C be a closed bounded subset of X. Suppose T is a continuous self-mapping on C into itself such that ;

(i) Fix (T) $\neq \phi$.

(ii) T is quasi-nonexpansive.

(iii) $p_\gamma (x - Tx) \neq 0$ for all $x \in C$.

(iv) There exists $x \in C$ such that $x_n = T^n x \in C$ for each $n \geq 1$.

Then, $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T in C.

Proof. We show that $\{x_n\}$ is a Cauchy sequence in C. Given $\varepsilon > 0$, then there exists $N > 0$ such for all r, s, N, $p_\gamma (x_N$, Fix (T)) $< \frac{\varepsilon}{2}$, Hence for all $r, s > N$ we obtain.

$$p_\gamma (x_r - x_s) \leq p_\gamma (x_r - u) + p_\gamma (x_s - u),\ u \in \text{Fix}\ (T).$$

Since T is quasi-nonexpansive, one gets :

$$p_\gamma (x_r - u) = p_\gamma (T^r x - u) \leq p_\gamma (T^N x - u).$$

and

$$p_\gamma (x_s - u) = p_\gamma (T^s x - u) \leq p_\gamma (T^N x - u).$$

Taking the infimum over $u \in$ Fix (T), we get

$$p_\gamma (x_r - x_s) \leq 2\, p_\gamma (x_N\ .\ \text{Fix}\ (T)) < \varepsilon\ ,$$

so $\{x_n\}$ is a Cauchy sequence and hence converges to $y \in C$. Furthermore since T is cotinuous, Fix (T) is closed in C and therefore $y \in$ Fix (T).

Theorem 2.2. Let C be a closed convex subset of X. Suppose T is a continuous self-mapping on C into itself which satisfies ;

(i) Fix (T) $\neq \phi$.

(ii) T is quasi-nonexpansive.

(iii) $p_\gamma (T_\lambda (x) - x) \neq 0$ for all $x \in C$.

Then, for each $x \in C$ and $0 < \lambda < 1$, the sequence $\{T_\lambda^n(x)\}_{n=1}^{\infty}$ of iterates, where $T_\lambda : C \to C$ is defined by $T_\lambda(x) = \lambda Tx + (1-\lambda)x$ converges to a fixed point of T in C.

Proof. To prove this theorem. It sufficies to show that T_λ satisfies the conditions of theorem 2.1. Now since C is a closed and convex in X, T_λ is well defined on C and Fix (T) = Fix (T_λ) $\neq \phi$.

Since for each $\lambda \in (0, 1)$, $x \in C$ and $n \in$ Fix (T), we have

$$p_\gamma(T_\lambda x - u) = p_\gamma(\lambda Tx + (1-\lambda)x - \lambda u - (1-\lambda)u)$$

$$\leqslant \lambda p_\gamma(Tx - u) + (1-\lambda) p_\gamma(x-u)$$

$$\leqslant p_\gamma(x-u),$$

this implies that T_λ is quasi-nonexpansive mapping

Now

$$p_\gamma(T_\lambda x - x) = p_\gamma(\lambda Tx + (1-\lambda)x - \lambda x - (1-\lambda)x)$$

$$= \lambda p_\gamma(Tx - x) \neq 0, \lambda \in (0, 1).$$

Then by hypothesis, there exi t $x \in C$ such that $\{T_\lambda^n(x)\}_{n=1}^{\infty} \in C$. Hence Theorem 2.2. follows from Theorem 2.1, and the theorem is proved.

Remark 2.1. The above Theorems are the extension of Theorems (1.1) and (1.1)' (see [3]) in a locally convex space setting.

3. Application

In this section we present some applications of Theorems 2.1 and 2.2. The first is :

Theorem 3.1. Let C be a nonempty closed convex subset of X, and T be a continuous self-mapping on C into itself. Suppose for each $\lambda \in I$, there exist nonnegative functions $d_i\ (\gamma, ..)$, $i = 1, 2, 3$ of $C \times C$ into $[0, \infty)$ such that the following are satisfied for $x, y \in C$:

I. (a) $3d_1(\gamma, x, y) + 2d_2(\gamma, x, y) + 4d_3(\gamma, x, y) \leqslant 1$

(b) $d_1(\gamma, x, y) + 2\,d_3(\gamma, x, y) < 1$

II. $$p_\gamma(Tx - Ty) \leqslant a_1(\gamma)\, p_\gamma(x - y) + a_2(\gamma)\,[p_\gamma(x - Tx) + p_\gamma(y - Ty)] + a_3(\gamma)\,[p_\gamma(x - Ty) + p_\gamma(y - Tx)],$$

where $a_i(\gamma) = d_i(\gamma, x, y)$.

Then for each $x \in C$ and $0 < \lambda < 1$, the sequence $\{T_\lambda^n(x)\}_{n=1}^{\infty}$ of iterates, where $T_\lambda : C \to C$ is defined by $T_\lambda(x) = \lambda Tx + (1-\lambda)x$, $x \in C$, converges to a member of Fix (T).

Proof. By Schauder—Tychonoff theorem [9, pp. 456] T has at least one fixed point and it is easily seen that Fix (T) = Fix $(T_\lambda) \neq \phi$. Also T satisfies the condition $p_\gamma(Tx - u) \leqslant p_\gamma(x - u)$, $p_\gamma(x - u) \neq 0$ for all $x \in C$, where u is a fixed point of T.

For,

$$\begin{aligned} p_\gamma(Tx - u) = p_\gamma(Tx - Tu) &\leqslant a_1(\gamma)\, p_\gamma(x - u) + a_2(\gamma)\,[p_\gamma(x - Tx)] \\ &\quad + a_3(\gamma)\,[p_\gamma(x - Tu) + p_\gamma(u - Tx)] \\ &\leqslant [a_1(\gamma) + a_3(\gamma)]\, p_\gamma(x - u) \\ &\quad + a_2(\gamma)\,[p_\gamma(x - Tx)] \\ &\quad + a_3(\gamma)\, p_\gamma(u - Tx). \qquad (1) \end{aligned}$$

This implies that :

$$p_\gamma(Tx - u) \leqslant \left\{\frac{a_1(\gamma) + a_2(\gamma) + a_3(\gamma)}{1 - a_2(\gamma) - a_3(\gamma)}\right\} p_\gamma(x - u)$$

$$= \{1 - \frac{2a_1(\gamma) + 2a_3(\gamma)}{1 - a_2(\gamma) - a_3(\gamma)}\} p_\gamma(x-u),$$

i.e.,

$$p_\gamma(Tx - u) \leqslant \{\frac{1 - (2a_1(\gamma) + 2a_3(\gamma))}{1 - a_2(\gamma) - a_3(\gamma)}\} p_\gamma(x-u).$$

From (I – a) we obtain :

$$P_\gamma(Tx - u) \leqslant p_\gamma(x-u). \tag{2}$$

Hence T is quasi-nonexpansive.

Also, we have

$$p_\gamma(T_\lambda x - u) = p_\gamma(T_\lambda(x) - T_\lambda(u))$$

$$= p_\gamma(\lambda(Tx - u) + (1-\lambda)(x-u)).$$

From (2), we see that

$$p_\gamma(T_\lambda x - u) \leqslant p_\gamma(x-u).$$

Hence T_λ is a quasi-nonexpansive mapping.

Suppose $p_\gamma(Tx - u) \leqslant p_\gamma(x-u)$, $p_\gamma(x-u) \neq 0$. Then using) (1) we have

$$[1 - a_1(\gamma) - 2a_3(\gamma)] p_\gamma(x-u) \leqslant a_2(\gamma) p_\gamma(x - Tx).$$

Since by 1–(b) the left hand side is nonzero, it follows that $p_\gamma(x - Tx) \neq 0$. Also one can show that $p_\gamma(T_\lambda x - x) \neq 0$.

Applying Theorem 2.2., the sequence $\{T_\lambda^n(x)\}_{n=1}^{\infty}$ of iterates converges to the fixed point of T in C. This completes the proof of the theorem.

Remark 3.1. The above theorem extends Theorem 4 in [1] in a locally convex space setting.

Another consequence of Theorem 2.2. is the following Theorem.

Theorem 3.2. Let C be a nonempty closed convex sbset of X. Suppose T is a self-mapping on C into itself such that:

$$p_\gamma (Tx-Ty) \leqslant \delta \max \{ p_\gamma (x-y), \tfrac{1}{2} p_\gamma (x-Tx) + p_\gamma (y-Ty),$$
$$\tfrac{1}{2} [p_\gamma (x-Ty) + p_\gamma (y-Tx)] \},$$

for all x, y ε C and $0 < \delta \leqslant 1$.

Then, the sequence $\{ T_\lambda^n x \}_{n=1}^{\infty}$ of iterates, where $T_\lambda : C \to C$ is defined by $T_\lambda x = \lambda Tx + (1-\lambda) x$, $0 < \delta < 1$, converges to a member of Fix (T).

Proof. By Schauder Tychonoff Theorem T has at least one fixed point and it is easily seen that Fix (T) = Fix (T_λ) $\neq \phi$. Also T satisfies the condition $p_\gamma (Tx-u) \leqslant p_\gamma (x-u)$, $p_\gamma (x-u) \neq 0$, x ε C, where u ε Fix (T). For

$$p_y (T_x - u) = p_y (Tx-Tu)$$
$$\leqslant \delta \max \{ p_\gamma (x-u), \tfrac{1}{2} p_\gamma (x-Tx) + p_\gamma (u-Tu),$$
$$\tfrac{1}{2} p_\gamma (x-Tu) + \tfrac{1}{2} p_\gamma (u-Tx) \}.$$

Hence, $p_\gamma (Tx-u) \leqslant \delta \max \{ p_\gamma (x-u), \tfrac{1}{2} p_\gamma (x-u) + \tfrac{1}{2} p_\gamma (u-Tx) \}$.

If, $p_\gamma (Tx-u) \leqslant \delta p_\gamma (x-u) \leqslant p_\gamma (x-u) \leqslant p_\gamma (x-u)$ (as $0 < \delta \leqslant 1$), then T is quasi-nonexpansive mapping.

If, $p_\gamma (Tx-u) \leqslant \delta [\tfrac{1}{2} p_\gamma (x-u) + \tfrac{1}{2} p_\gamma (Tx-u)]$. We obtain

$$(2-\delta) p_\gamma (Tx-u) \leqslant \delta p_\gamma (x-u), \; 0 < \delta \leqslant 1,$$

this implies that $p_\gamma (Tx-u) \leqslant p_\gamma (x-u)$.

Then T is quasi-nonexpansive.

Also as in the proof of Theorem 3.1, one easily show that T_λ is quasi-nonexpansive and $p_\gamma\ (T_\lambda\ x - x) \neq 0$. Applying Theorem 2.2 then the sequence of iterates $\{T_\lambda^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

Now, we give an example of a non-normable locally convex space and a quasi-nonexpansive mapping that has a fixed point.

4. *Example.* Let Ω be an open subset of R^n and $X = C(\Omega)$ be the space of continuous real valued functions on Ω. Let $\triangle$ be the family of closed subsets of Ω. For $\gamma \in \triangle$, define :

$$p_\gamma(f) = \max_{x \in \gamma} |f(x)|, \quad f \in X.$$

Then p_γ is a semi-norm, and the family $\{p_\gamma : \gamma \in \triangle\}$ generates a topology under which X is a locally convex space. For a special case, let $(\Omega = (1, -1)$ and $X = C(-1, 1)$. Let $C = \{f \in X : [0, \frac{3}{4}] \to [0, \frac{3}{4}]$ Then C is a closed convex subset of X. Define :

$$T : C \to C \text{ by } (Tf)(x) = (\sin x)\ f(x).$$

Clearly T has a fixed point f = 0 in C.

Also

$$p_\gamma(Tf - 0) = \max_{x \in \gamma} |(\sin x)\ f(x) - (\sin x)\ 0|$$

$$= \max_{x \in \gamma} |\sin x|\ |f(x)|$$

$$\leqslant \max_{x \in \gamma} |f(x)|$$

$$= p_\gamma(f - 0).$$

Hence T is quasi-nonexpansive.

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ON A SUBCLASS OF P VALENT ANALYTIC FUNCTIONS

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Abstract

A sharp coefficient estimates, distortion theorems are determined for the class $R_p^{\lambda}(\alpha, \beta, A, B)$ of functions $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$ and satisfying the condition

$$\left| \frac{\dfrac{f'(z)}{pz^{p-1}} - 1}{(B-A)\left(\dfrac{f'(z)}{pz^{p-1}} - 1 + (1-\alpha)\cos\lambda\, e^{-i\lambda}\right) + A\left(\dfrac{f'(z)}{pz^{p-1}} - 1\right)} \right| < 1$$

for some $\alpha, \beta, \lambda, A, B$ ($0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$, $|\lambda| < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1$, $0 < B \leqslant 1$) with p a positive integer. A sufficient condition for a function to beiong to $R_p^{\lambda}(\alpha, \beta, A, B)$ has also been determined. We shall also prove that a subclass of p-valent analytic functions is closed under convolution.

1. Introduction

Let E be the class of functions $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$

which are regular and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in E$ is said to be in $R_p^{\lambda}(\alpha, \beta, A, B)$ if it satisfies the condition

$$\left| \frac{\dfrac{f'(z)}{pz^{p-1}} - 1}{(B-A)\left(\dfrac{f'(z)}{pz^{p-1}} - 1 + (1-\alpha)\cos\lambda\, e^{-i\lambda}\right) + A\left(\dfrac{f'(z)}{pz^{p-1}} - 1\right)} \right| < 1 \tag{1.1}$$

for some $\alpha, \beta, \lambda, A, B$ ($o \leqslant \alpha < p$, $o < \beta \leqslant 1$, $|\lambda| < \frac{\pi}{2}$, $\leqslant A < B \leqslant 1$, $o < B \leqslant 1$) with p a positive integer and for all $z \in U$. It is easily seen that for $f(z) \in R_p^{\lambda}(\alpha, \beta, A, B)$, the values $\dfrac{f'(z)}{pz^{p-1}}$ lie inside the circle in the right half-plane with center at

$$\frac{1 - [(B-A)\beta + A]\{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda}\}}{1 - [(B-A)\beta + A]^2}$$

and radius

$$\frac{(B-A)\beta(1-\alpha)\cos\lambda}{1 - [(B-A)\beta + A]^2}$$

Further, it follows from Schwarz's Lemma [4] that if $f(z) \in R_p^{\lambda}(\alpha, \beta, A, B)$, then

$$\frac{f'(z)}{pz^{p-1}} = \frac{1 + \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda}\}\, w(z)}{1 + [(B-A)\beta + A]\, w(z)}$$

where $w(z)$ is regular in U and satisfies the conditions $w(o) = o$, and $|w(z)| < 1$ for $z \in U$.

We note that :

1. For $A = -1$ and $B = 1$, we get the class introduced and studied by Mogra [3].

2. For p=1, we get the class introduced and studied by Aouf and Owa [2].

3. For p=1. A= −1 and B=1, we get the class introduced and studied by Ahuja [1].

4. For $\lambda=o$, $\alpha=o$, $A=-1$ and $B=1$ and replacement β by $\frac{2\delta-1}{2\delta}$, $\delta>\frac{1}{2}$, we get the class introduced and studied by Sohi [5].

Also by taking different values of the parameters α, β, λ, A and B, the class R_p^λ (α, β, A, B) reduces to the following subclasses of p-valent analytic functions introduced by Mogra [3] :

$$R_p^\lambda(\alpha)=R_p^\lambda(\alpha,1,-1,1)$$

$$=\left\{f\in E:\operatorname{Re}\left(e^{i\lambda}\frac{f'(z)}{pz^{p-1}}\right)>\alpha\cos\lambda, o\leqslant\alpha<p, |\lambda|<\frac{\pi}{2}, \in U\right\},$$

$$R_{p,\delta}^\lambda=R_p^\lambda\left(o,\frac{2\delta-1}{2\delta},-1,1\right)$$

$$=\left\{f\in E:\left|\frac{e^{i\lambda}\frac{f'(z)}{pz^{p-1}}-i\sin\lambda}{\cos\lambda}-\delta\right|<\delta, \delta>\tfrac{1}{2}, |\lambda|<\frac{\pi}{2}, z\in U\right\},$$

$$(R_p^\lambda)^\sigma=R_p^\lambda\left(1-\sigma,\tfrac{1}{2},-1,1\right)$$

$$=\left\{f\in E:\left|\frac{e^{i\lambda}\frac{f'(z)}{pz^{p-1}}-i\sin\lambda}{\cos\lambda}-1\right|<\sigma, o<\sigma\leqslant 1,\right.$$

$$\left.|\lambda|<\frac{\pi}{2}, z\in U\right\},$$

$$(R_p^\lambda)\gamma=R_p^\lambda\left(\frac{1-\gamma}{1+\gamma},\frac{1+\gamma}{2},-1,1\right)$$

$$= \left\{ f \in E : \left| \frac{\dfrac{e^{i\lambda} \dfrac{f'(z)}{pz^{p-1}} - i \sin\lambda}{\cos\lambda} - 1}{\dfrac{e^{i\lambda} \dfrac{f'(z)}{pz^{p-1}} - i \sin\lambda}{\cos\lambda} + 1} \right| < \gamma, o < \gamma \leq 1, \; |\lambda| < \frac{\pi}{2}, z \in U \right\}.$$

We, further, observe that for special choice of the parameters α, β, λ, A and B our class rise to the following new subclasses of p-valent analytic functions :

$$1- R^{\lambda}_{p,\delta,\alpha} = R^{\lambda}_{p} \left(\alpha, \frac{2\delta - 1}{2\delta}, -1, 1 \right)$$

$$= \left\{ f \in E : \left| \frac{e^{i\lambda} \dfrac{f'(z)}{pz^{p-1}} - \alpha \cos\lambda - i \sin\lambda}{(1-\alpha)\cos\lambda} - \delta \right| < \delta, \; \delta > \tfrac{1}{2}, o \leq \alpha < p, |\lambda| < \frac{\pi}{2}, z \in U \right\}.$$

$$2- R_{p}(\gamma, A, B) = R^{o}_{p} \left(\frac{-A + A\gamma}{B\gamma - A}, \frac{B\gamma \quad A}{B - A}, A, B \right)$$

$$= \left\{ f \in E : \left| \frac{\dfrac{f'(z)}{pz^{p-1}} - 1}{B \dfrac{f'(z)}{pz^{p-1}} - A} \right| < \gamma, o < \gamma \leq 1, -1 \leq A < B \leq 1, \; o < B \leq 1, z \in U \right\},$$

$$3- R^{\lambda}_{p,\alpha}(A, B) = R^{\lambda}_{p}(\alpha, 1, A, B)$$

$$= \left\{ f \in E : \left| \frac{\dfrac{f'(z)}{pz^{p-1}} - 1}{B \dfrac{f'(z)}{pz^{p-1}} - [B + (A - B)(1-\alpha)\cos\lambda \, e^{-i\lambda}]} \right| < 1, z \in U \right\}$$

$$4- R^{\lambda}_{p,\alpha,\beta}(A, B) = R^{\lambda}_{p} \left(\frac{-A + A\beta - (A-B)\alpha\beta}{B\beta - A}, \frac{B\beta - A}{B - A}, A, B \right)$$

$$= \left\{ f \in E \left| \frac{\frac{f'(z)}{pz^{p-1}} - 1}{B \frac{f'(z)}{pz^{p-1}} - [B + (A-B)(1-\alpha)\cos\lambda\, e^{-i\lambda}]} \right| \right.$$

$$\left. < \beta,\ 0 \leqslant \alpha < p,\ 0 < \beta \leqslant 1,\ -1 \leqslant A < B \leqslant 1,\ 0 < B \leqslant 1,\ z \in U \right\}$$

As noticed above, the class $R_p^\lambda(\alpha, \beta, A, B)$ includ s the various subclasses of p-valent analytic functions, a study of its properties will lead to a unified study of these classes. In the present paper, we determine a sufficient condition, coefficient estimates, distortion theorems for $f(z) \in R_p^\lambda(\alpha, \beta, A, B)$. We shall further proves that the subclass $R^\lambda_{p,\delta,\alpha}$ of E, is closed under convolution.

2. A sufficient condition

Theorem 1. Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ be analytic and p-valent in U. If for some α, λ, A and B ($0 \leqslant \alpha < p$, $|\lambda| < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1$),

$$\sum_{k=1}^{\infty} (p+k)\,|a_{p+k}| \leqslant \frac{(B-A)\,\beta P\,(1-\alpha)\cos\lambda}{1-A-(B-A)\beta},$$

$$\text{whenever } 0 < \beta \leqslant \frac{-A}{(B-A)}, \quad (2.1)$$

and

$$\sum_{k=1}^{\infty} (p+k)\,|a_{p+k}| \leqslant \frac{(B-A)\,\beta p\,(1-\alpha)\cos\lambda}{1+A+(B-A)\beta},$$

$$\text{whenever } \frac{-A}{(B-A)} \leqslant \beta \leqslant 1, \quad (2.2)$$

then $f(z) \in R_p^\lambda(\alpha, \beta, A, B)$.

Proof. Suppose that (2.1) holds for $0 < \beta \leq \frac{-A}{(B-A)}$ and that

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} \, z^{p+k}$$

then for $z \in U$,

$$| f'(z) - pz^{p-1} | - | (B-A)\beta (f'(z) - pz^{p-1} + p(1-\alpha)\cos\lambda\, e^{-i\lambda} z^{p-1}) + A(f'(z) - pz^{p-1}) |$$

$$= \left| \sum_{k=1}^{\infty} (p+k)\, a_{p+k}\, z^{p+k-1} \right| - \left| (B-A)\beta p(1-\alpha) \cos\lambda\, e^{-i\lambda} z^{p-1} - \sum_{k=1}^{\infty} (p+k)(-A-(B-A)\beta)\, a_{p+k}\, z^{p+k-1} \right|$$

$$\leq \sum_{k=1}^{\infty} (p+k) |a_{p+k}| r^{p+k-1} - \{(B-A)\beta p(1-\alpha)\cos\lambda\, r^{p-1} - \sum_{k=1}^{\infty} (p+k)(-A-(B-A)\beta) |a_{p+k}| r^{p+k-1}\}$$

$$< \{ \sum_{k=1}^{\infty} [(p+k) + (-A-(B-A)\beta)(p+k)] |a_{p+k}| - (B-A)\beta p(1-\alpha)\cos\lambda \} r^{p-1}$$

$$= \{ \sum_{k=1}^{\infty} (1-A-(B-A)\beta)(p+k) |a_{p+k}| - (B-A)\beta p(1-\alpha)\cos\lambda \} r^{p-1}$$

The last quantity is nonpositive by (2.1), so that $f(z) \in R_p^\lambda(\alpha, \beta, A, B)$. Next, we assume that (2.2) holds for $\frac{-A}{(B-A)} \leq \beta \leq 1$. Then

$$| f'(z) - pz^{p-1} | - | (B-A)\beta (f'(z) - pz^{p-1} + p(1-\alpha)\cos\lambda\, e^{-\lambda i} z^{p-1}) + A(f'(z) - pz^{p-1}) |$$

$$= \left| \sum_{k=1}^{\infty} (p+k)\, a_{p+k}\, z^{p+k-1} \right| - \left| (B-A)\,\beta p\,(1-\alpha) \cos\lambda\, e^{-i\lambda} z^{p-1} + \sum_{k=1}^{\infty} (A+(B-A)\,\beta)\,(p+k)\, a_{p+k}\, z^{p+k-1} \right|$$

$$< \left\{ \sum_{k=1}^{\infty} (1+A+(B-A)\,\beta)\,(p+k)\, |a_{p+k}| - (B-A)\,\beta p\,(1-\alpha)\cos\lambda \right\} r^{p-1}$$

$\leqslant o$, by (2.2).

This proves that $f(z) \in R_p^{\lambda}(\alpha, \beta, A, B)$. Hence the theorem.

We note that

$$f(z) = z^p + \frac{(B-A)\,\beta p\,(1-\alpha)\cos\lambda\, e^{-i\lambda}}{(p+k)\,(1-A-(B-A)\,\beta)}\, z^{p+k}$$

is an extremal function with respect to Ist part the theorem and

$$f(z) = z^p + \frac{(B-A)\,\beta p\,(1-\alpha)\cos\lambda\, e^{-i\lambda}}{(p+k)\,(1+A+(B-A)\,\beta)}\, z^{p+k}$$

is an extremal function with respect to IInd part of the theorem since

$$\left| \frac{\dfrac{f'(z)}{pz^{p-1}} - 1}{(B-A)\,\beta\left(\dfrac{f'(z)}{pz^{p-1}} - 1 + (1-\alpha)\cos\lambda\, e^{-i\lambda}\right) + A\left(\dfrac{f'(z)}{pz^{p-1}} - 1\right)} \right| = 1$$

for $z = 1$, $o \leqslant \alpha < p$, $o < \beta \leqslant 1$, $|\lambda| < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$ and $k = 1, 2, 3, \ldots$.

We also observe that the converse of the above theorem may not be true. For example, consider of the function f (z) given by

$$\frac{f'(z)}{pz^{p-1}} = \frac{1 \quad \{[(B-A)\,\beta + A] - (B-A)\,\beta\,(1-\alpha)\cos\lambda\, e^{-i\lambda}\}\, z}{1 - [(B-A)\,\beta + A]\, z}$$

It is easily seen that $f(z) \in R_p^\lambda(\alpha, \beta, A, B)$ but

$$\sum_{k=1}^{\infty} \frac{(p+k)(1-A-(B-A)\beta}{(B-A)\beta p(1-\alpha)\cos\lambda} |a_{p+k}|$$

$$= \sum_{k=1}^{\infty} \frac{(p+k)(1-A-(B-A)\beta)}{(B-A)\beta p(1-\alpha)\cos\lambda} \cdot \frac{(B-A)\beta p(1-\alpha)\cos\lambda}{(p+k)}$$

$$[(B-A)\beta+A]^{k-1}$$

$$= \sum_{k=1}^{\infty} (1-A-(B-A)\beta)[(B-A)\beta+A]^{k-1} > 1$$

for α, β, λ, A and B satisfying $o \leqslant \alpha < p$, $o < \beta \leqslant \frac{-A}{(B-A)}$, $|\lambda| < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$, and also

$$\sum_{k=1}^{\infty} \frac{(p+k)(1+A+(B-A)\beta)}{(B-A)\beta p(1-\alpha)\cos\lambda} |a_{p+k}|$$

$$= \sum_{k=1}^{\infty} \frac{(p+k)(1+A+(B-A)\beta)}{(B-A)\beta p(1-\alpha)\cos\lambda} \cdot \frac{(B-A)\beta p(1-\alpha)\cos\lambda}{(p+k)}$$

$$[(B-A)\beta+A]^{k-1}$$

$$= \sum_{k=1}^{\infty} (1+A+(B-A)\beta)[(B-A)\beta+A]^{k-1} > 1$$

for α, β, λ, A and B satisfying $o \leqslant \alpha < p$, $\frac{-A}{(B-A)} \leqslant \beta \leqslant 1$, $|\lambda - < \frac{\pi}{2}$, $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$ and $z \in U$.

Corollary 1. Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ be analytic and p-valent in U. If for some α, λ, $o \leqslant \alpha < p$, $|\lambda| < \frac{\pi}{2}$,

$$\sum_{k=1}^{\infty} (p+k)|a_{p+k}| \leqslant (2\delta-1)p(1-\alpha)\cos\lambda, \text{ whenever } \tfrac{1}{2} < \delta \leqslant 1,$$

$$\sum_{k=1}^{\infty} (p+k)|a_{p+k}| \leqslant p(1-\alpha)\cos\lambda, \text{ whenever } \delta \geqslant 1,$$

then $f(z)$ belongs to $R^{\lambda}_{p,\delta,\alpha}$.

Corollary 2. Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ be analytic and p-valent in U. If for some γ, A, B ($0 < \gamma \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$),

$$\sum_{k=1}^{\infty} (p+k) \, | a_{p+k} | \leq \frac{(B-A)\gamma}{(1+B\gamma)},$$

then $f(z) \in R_p(\gamma, A, B)$.

Corollary 3. Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ be analytic and p-valent in U. If for some α, λ, A, B ($0 \leq \alpha < p$, $|\lambda| < \frac{\pi}{2}$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$),

$$\sum_{k=1}^{\infty} (p+k) \, | a_{p+k} | \leq \frac{(B-A)\, p\, (1-\alpha) \cos\lambda}{(1+B)},$$

then $f(z) \in R^{\lambda}_{p,\alpha}(A, B)$.

Corollary 4. Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ be analytic and p-valent in U. If for some α, β, λ, A, B ($0 \leq \alpha < p$, $0 < \beta \leq 1$, $|\lambda| < \frac{\pi}{2}$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$),

$$\sum_{k=1}^{\infty} (p+k) \, | a_{p+k} | \leq \frac{(B-A)\, \beta p\, (1-\alpha) \cos\lambda}{(1+B\beta)},$$

then $f(z)$ belongs to $R^{\lambda}_{p,\alpha,\beta}(A, B)$.

Remark 1.

1. Putting $A = -1$ and $B = 1$ in Theorem 1, we get the corresponding sufficient condition obtained by Mogra [3].

2. Putting $p = 1$ in Theorem 1, we get the corresponding sufficient condition by Aouf and Owa [2].

3. Putting $p=1$, $A=-1$ and $B=1$ in Theorem 1, we get the corresponding sufficient condition obtained by Ahuja [1].

Motivated by Theorem 1, we introduce a new subclass of p-valent analytic functions in the unit disc U. We say that a function $f(z) \in E$ is in the class $\bar{R}_p^{\lambda}(\alpha, \beta, A, B)$ if and only if the condition (2.1) holds for $0 < \beta \leq \frac{-A}{(B-A)}$ and the condition (2.2) holds for $\frac{-A}{(B-A)} \leq \beta \leq 1$. Clearly $\bar{R}_p^{\lambda}(\alpha, \beta, A, B) \subset R_p^{\lambda}(\alpha, \beta, A, B)$. Then the following theorem is in order.

Theorem 2. If

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$$

and

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

belong to $\bar{R}_p^{\lambda}(\alpha, \beta, A, B)$, then so does $F(z)$, where $F(z)$ is defined by

$$F(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

Proof. Since $f(z) \in R_p^{\lambda}(\alpha, \beta, A, B)$, we have

$$\sum_{k=1}^{\infty} (p+k) |a_{p+k}| \leq \begin{cases} \dfrac{(B-A)\beta p(1-\alpha)\cos\lambda}{1-A-(B-A)\beta} & \text{if } 0 < \beta \leq \dfrac{-A}{(B-A)} \\[2ex] \dfrac{(B-A)\beta p(1-\alpha)\cos\lambda}{1+A+(B-A)\beta} & \text{if } \dfrac{-A}{(B-A)} \leq \beta \leq 1. \end{cases} \tag{2.3}$$

This yields

$$|a_{p+k}| \leq \begin{cases} \dfrac{(B-A)\beta p(1-\alpha)\cos\lambda}{(1-A-(B-A)\beta)(p+k)} & \text{if } o<\beta\leq \dfrac{-A}{(B-A)} \\ \dfrac{(B-A)\beta p(1-\alpha)\cos\lambda}{(1+A+(B-A)\beta)(p+k)} & \text{if } \dfrac{-A}{(B-A)}\leq\beta\leq 1 \end{cases}$$

for all $k \geq 1$. Therefore, it follows that

$$|a_{p+k}| < 1 \ (k \geq 1). \tag{2.4}$$

Using (2.4) we obtain

$$\sum_{k=1}^{\infty} (p+k)|a_{p+k}|^2 \leq \sum_{k=1}^{\infty} (p+k)|a_{p+k}| \tag{2.5}$$

Similarly, since $g(z) \in R_p^{-\lambda}(\alpha, \beta, A, B)$ we have

$$\sum_{k=1}^{\infty} (p+k)|b_{p+k}| \leq \begin{cases} \dfrac{(B-A)\beta p(1-\alpha)\cos\lambda}{1-A-(B-A)\beta} & \text{if } o<\beta\leq \dfrac{-A}{(B-A)} \\ \dfrac{(B-A)\beta p(1-\alpha)\cos\lambda}{1+A+(B-A)\beta} & \text{if } \dfrac{-A}{(B-A)}\leq\beta\leq 1 \end{cases} \tag{2.6}$$

and

$$\sum_{k=1}^{\infty} (p+k)|b_{p+k}|^2 \leq \sum_{k=1}^{\infty} (p+k)|b_{p+k}|. \tag{2.7}$$

Now we have

$$\sum_{k=1}^{\infty} (p+k)|a_{p+k} b_{p+k}|$$

$$\leq \left(\sum_{k=1}^{\infty} (p+k)|a_{p+k}|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (p+k)|b_{p+k}|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=1}^{\infty} (p+k)|a_{p+k}|\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (p+k)|b_{p+k}|\right)^{\frac{1}{2}} \tag{2.8}$$

where we have applied Schwarz's inequality [4] and the relations (2.5) and (2.7). Applying (2.3) and (2.6) to the relation (2.8) we get

$$\sum_{k=1}^{\infty} (p+k)\,|a_{p+k}\, b_{p+k}|$$

$$\leqslant \begin{cases} \dfrac{(B-A)\,\beta\, p\,(1-\alpha)\cos\lambda}{1-A-(B-A)\,\beta} & \text{if } o<\beta\leqslant \dfrac{-A}{(B-A)} \\[2ex] \dfrac{(B-A)\,\beta\, p\,(1-\alpha)\cos\lambda}{1+A+(B-A)\,\beta} & \text{if } \dfrac{-A}{(B-A)}\leqslant\beta\leqslant 1. \end{cases}$$

This proves that $F(z) \in R_p^{-\lambda}(\alpha, \beta, A, B)$.

3. Coefficient estimates

Theorem 3. If $f(z)=z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ is in $R_p^{\lambda}(\alpha,\beta,A,B)$ for some α,β,λ, A, B satisfying $(o\leqslant\alpha<p,\ o<\beta\leqslant(\frac{1-A}{B-A}),\ |\lambda|<\frac{\pi}{2}$, $-1\leqslant A<B\leqslant 1,\ o<B\leqslant 1)$, then

$$|a_{p+k}| \leqslant \frac{(B-A)\,\beta\, p\,(1-\alpha)\cos\lambda}{p+k},\ k=1, 2, \ldots.$$

The inequality is sharp.

Proof. Since $f(z) \in R_p^{\lambda}(\alpha, \beta, A, B)$, we have

$$\frac{f'(z)}{pz^{p-1}} = \frac{1+\{[(B-A)\beta+A]-(B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda}\}w(z)}{1+[(B-A)\,\beta+A]\,w(z)} \tag{3.1}$$

where $w(z) = \sum_{m=1}^{\infty} t_m z^m$ is regular in U and satisfies the conditions $w(o)=o$ and $|w(z)| < 1$ for $z \in U$. From (3.1), we have

$$\{(B-A)\,\beta\, p\,(1-\alpha)\cos\lambda\, e^{-i\lambda} z^{p-1} + \sum_{m=1}^{\infty} [(B-A)\beta+A]\,(p+m)$$

$$a_{p+m}\ z^{p+m-1}\}\ [\sum_{m=1}^{\infty} t_m\ z^m] = -\sum_{m=1}^{\infty}(p+m)\ a_{p+m}$$

$$z^{p+m-1} \tag{3.2}$$

Equating corresponding coefficients on both sides of (3.2) we observe that the coefficient a_{p+k} on the right of (3.2) depends only on $a_{p+1}, a_{p+2}, \ldots, a_{p+k-1}$ on the left of (3.2) for $k \geq 1$. Hence for $k \geq 1$, it follows from (3.2) that

$$\{(B-A)\beta p(1-\alpha)\cos\lambda\ e^{-i\lambda}\ z^{p-1} + \sum_{m=1}^{k-1}[(B-A)\beta + A](p+m)$$

$$a_{p+m}\ z^{p+m-1}\}\ w(z) = -\sum_{m=1}^{k}(p+m)\ a_{p+m}\ z^{p+m-1}$$

$$-\sum_{m=k+1}^{\infty} c_m\ z^{p+m-1}$$

where c_m being complex numbers. Then, since $|w(z)| < 1$, we get

$$|(B-A)\beta p(1-\alpha)\cos\lambda\ e^{-i\lambda}\ z^{p-1} + \sum_{m=1}^{k-1}[(B-A)\beta+A](p+m)$$

$$a_{p+m}\ z^{p+m-1}| \geq |\sum_{m=1}^{k}(p+m)\ a_{p+m}\ z^{p+m-1}$$

$$+\sum_{m=k+1}^{\infty} c_m\ z^{p+m-1}|. \tag{3.3}$$

Squaring both sides of (3.3) and integrating round $|z| = r$, $o < r < 1$, we obtain

$$\sum_{m=1}^{k}(p+m)^2\ |a_{p+m}|^2\ r^{2(p+m-1)} + \sum_{m=k+1}^{\infty}|c_m|^2$$

$$r^{2(p+m-1)} \leq (B-A)^2\ \beta^2\ p^2\ (1-\alpha)^2\cos^2\lambda\ r^{2(p-1)}$$

$$+[(B-A)\beta+A]^2\sum_{m=1}^{k-1}(p+m)^2\ |a_{p+m}|^2\ r^{2(p+m-1)}.$$

If we take limit as r approaches 1, then

$$\sum_{m=1}^{k} (p+m)^2 \, | a_{p+m} |^2 \leqslant (B-A)^2 \beta^2 p^2 (1-\alpha)^2 \cos^2 \lambda$$

$$+ [(B-A)\beta + A]^2 \sum_{m=1}^{k-1} (p+m)^2 \, | a_{p+m} |^2$$

or

$$(p+k)^2 \, | a_{p+k} |^2 \leqslant (B-A)^2 \beta^2 p^2 (1-\alpha)^2 \cos^2 \lambda$$

$$- \{1 - [(B-A)\beta + A]^2\} \sum_{m=1}^{k-1} (p+m)^2 \, | a_{p+m} |^2$$

Since $\quad o < \beta \leqslant \left(\frac{1-A}{B-A}\right),$

$$(p+k)^2 \, | a_{p+k} |^2 \leqslant (B-A)^2 \beta^2 p^2 (1-\alpha)^2 \cos^2 \lambda$$

whence follows that

$$| a_{p+k} | \leqslant \frac{(B-A)\beta p (1-\alpha) \cos \lambda}{p+k}, \; k \geqslant 1 .$$

Consider the function

$$f(z) = \int_0^z pt^{p-1} \frac{1 - \{[(B-A)\beta + A] - (B-A)\beta (1-\alpha) \cos \lambda \, e^{-i\lambda}\} t^k}{1 - [(B-A)\beta + A] t^k} \, dt, \; z \in U,$$

where $o \leqslant \alpha < p$, $o < \beta \leqslant \left(\frac{1-A}{B-A}\right)$, $|\lambda| < \frac{\pi}{2}$, and $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$.

Then it is easy to check that $f(z) \in R_p^{\lambda}(\alpha, \beta, A, B)$ and the function

$$f(z) = z^p + \frac{(B-A)\beta p (1-\alpha) \cos \lambda \, e^{-i\lambda}}{p+k} z^{p+k} + \cdots .$$

for all $k \geqslant 1$ and $z \in U$ showing that the estimates are sharp.

Remark 2.

Taking appropriate values of the parameters α, β, λ, A, B in Theorem 3 we may get the corresponding coefficient estimates for functions in the classes

$$R^{\lambda}_{p,\delta,\alpha},\ R_p(\gamma, A, B),\ R^{\lambda}_{p,\alpha}(A, B) \text{ and } R^{\lambda}_{p,\alpha,\beta}(A, B).$$

Remark 3.

By taking appropriate values of the parameters α, β, λ, A, B and p in Theorem 3 we obtain the corresponding results established by Mogra [3], Aouf and Owa [2], Ahuja [1] and Sohi [5].

4. Distortion theorems.

Theorem 4. If $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ belongs to the class $R^{\lambda}_p(\alpha, \beta, A, B)$, then for $z \in U$,

$$|f(z)| \leqslant \int_0^{|z|} pt^{p-1} \frac{1+(B-A)\beta(1-\alpha)\cos\lambda . t+[(B-A)\beta+A]\{[B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}t^2}{1-[(B-A)\beta+A]^2 t^2}\, dt, \tag{4.1}$$

and

$$|f(z)| \geqslant \int_0^{|z|} pt^{p-1} \frac{1-(B-A)\beta(1-\alpha)\cos\lambda . t+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}t^2}{1-[(B-A)\beta+A]^2 t^2}\, dt. \tag{4.2}$$

For $\beta = \left(\frac{-A}{B-A}\right)$, the above estimates reduce to

$$|f(z)| \leqslant r^p - \frac{Ap(1-\alpha)\cos\lambda . r^{p+1}}{p+1}$$

$$|f(z)| \geqslant r^p + \frac{Ap(1-\alpha)\cos\lambda . r^{p+1}}{p+1}, \quad (|z| = r).$$

The bounds are sharp.

Proof Since $f(z) \in R_p^{\lambda}(\alpha, \beta, A, B)$, we observe that the condition (1.1) coupled with an application of Schwarz's Lemma [4] implies

$$\left| \frac{f'(z)}{pz^{p-1}} - a \right| < b \tag{4.3}$$

where

$$a = \frac{1-[(B-A)\beta+A]\{[(B-A)\beta+A]-(B-A)\beta(1-\alpha)\cos\lambda\, e^{-i\lambda}\}r^2}{1-[(B-A)\beta+A]^2\, r^2}, \tag{4.4}$$

$$b = \frac{(B-A)\beta(1-\alpha)\cos\lambda\,.\,r}{1-[(B-A)\beta+A]^2\, r^2}, \quad |z| = r. \tag{4.5}$$

Hence, we have

$$\frac{1-(B-A)\beta(1-\alpha)\cos\lambda\,.\,r+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}r^2}{1-[(B-A)\beta+A]^2\, r^2}$$

$$\leqslant \operatorname{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) \leqslant$$

$$\frac{1+(B-A)\beta(1-\alpha)\cos\lambda\,.\,r+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}r^2}{1-[(B-A)\beta+A]^2\, r^2}, \tag{4.6}$$

Let

$$g(z) = \frac{1+(B-A)\beta(1-\alpha)\cos\lambda\,.\,z+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}r^2}{1-[(B-A)\beta+A]^2\, z^2}$$

Since $g(o) = 1 = f'(o)$ and $g(z)$ is univalent in U, it follows that f' is subordinate to g. Hence

$$|f'(z)| \leqslant$$

$$pr^{p-1}\ \frac{1+(B-A)\beta(1-\alpha)\cos\lambda\,.\,r+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}r^2}{1-[(B-A)\beta+A]^2\, r^2}, \tag{4.7}$$

Now, in view of

$$| f(z) | = | \int_{o}^{z} f'(s)\, ds | \leqslant \int_{o}^{|z|} | f'(t\, e^{i\theta}) |\, dt$$

and with the aid of (4.7) we may write

$$| f(z) | \leqslant \int_{o}^{|z|} p t^{p-1} \frac{1+(B-A)\beta(1-\alpha)\cos\lambda \,.\, t+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}\, t^2}{1-[(B-A)\beta+A]^2\, r^2}\, dt$$

which gives (4.1). In order to obtain the lower bound for $f(z)$ we integrate along the path L whose image is the line segment $[o, f(z)]$. Thus

$$| f(z) | = | \int_{L} f'(s)\, ds | \geqslant \int_{L} | f'(s) |\, ds$$

$$\geqslant \int_{o}^{|z|} p t^{p-1} \frac{1-(B-A)\beta(1-\alpha)\cos\lambda \,.\, t+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]\}\, t^2}{1-[(B-A)\beta+A]^2\, t^2}\, dt.$$

This proves the theorem.

By taking the function

$$f(z) = \int_{o}^{z} p t^{p-1} \frac{1+(B-A)\beta(1-\alpha)\cos\lambda . t+[(B-A)\beta+A]\{(B-A)\beta(1-\alpha)\cos^2\lambda-[(B-A)\beta+A]^2\}\, t^2}{1-[(B-A)\beta+A]^2\, t^2}\, dt$$

one can show that the estimates are sharp.

Remark 4. The corresponding distortion theorems for functions belonging to the classes $R^{\lambda}_{p, \delta, \alpha}$, $R_p(\gamma, A, B)$, $R^{\lambda}_{p, \alpha}(A, B)$ and $R^{\lambda}_{p, \alpha, \beta}(A, B)$ can be obtained from Theorem 4 by taking appropriate values of the parameters.

Remark 5.

1. Putting $A = -1$ and $B = 1$ in Theorem 4, we get the distortion theorem otained by Mogra [3].

2. Putting p=1 in Theorem 4, we get the distortion theorem obtained by Aouf and Owa [2].

3. Putting p=1, A=−1 and B=1 in Theorem 4, we get the distortion theorem obtained by Ahuja [1].

4. Putting $p=1$, $\alpha=0$, $\beta=\frac{2\delta-1}{2\delta}$ $(\delta>\frac{1}{2})$, $A=-1$ and $B=1$ in Theorem 4, we get the distortion theorem obtained by Sohi [5].

5. Convex set of functions

Theorem 5. If f (z) and g (z) belong to the class $R^{\lambda}_{p,\delta,\alpha}$, then $\nu f(z)+(1-\nu)g(z)$, $0\leqslant\nu\leqslant1$, belongs to the class $R^{\lambda}_{p,\delta,\alpha}$.

Proof. Since f (z) and g (z) belong to the class $R^{\lambda}_{p,\delta,\alpha}$, we have

$$\left|\frac{e^{i\lambda}\frac{f'(z)}{pz^{p-1}}-\alpha\cos\lambda-i\sin\lambda}{\delta(1-\alpha)\cos\lambda}-1\right|<1 \tag{5.1}$$

and

$$\left|\frac{e^{i\lambda}\frac{g'(z)}{pz^{p-1}}-\alpha\cos\lambda-i\sin\lambda}{\delta(1-\alpha)\cos\lambda}-1\right|<1 \tag{5.2}$$

for some δ,λ,α satisfying $\delta>\frac{1}{2}$, $|\lambda|<\frac{\pi}{2}$ and $0\leqslant\alpha<p$. Using (5.1) and (5.2), it follows that

$$\left|\frac{e^{i\lambda}\frac{(\nu f'(z)+(1-\nu)g'(z))}{pz^{p-1}}-\alpha\cos\lambda-i\sin\lambda}{\delta(1-\alpha)\cos\lambda}-1\right|$$

$$\leqslant\nu\left|\frac{e^{i\lambda}\frac{f'(z)}{pz^{p-1}}-\alpha\cos\lambda-i\sin\lambda}{\delta(1-\alpha)\cos\lambda}-1\right|$$

$$+(1-\nu)\left|\frac{e^{i\lambda}\,\dfrac{f'(z)}{pz^{p-1}}-\alpha\cos\lambda-i\sin\lambda}{\delta(1-\alpha)\cos\lambda}-1\right|$$

$$<\nu+(1-\nu)=1,$$

for all $z \in U$. This proves that $\nu f(z)+(1-\nu)g(z)$ belongs to $R^{\lambda}_{p,\delta,\alpha}$.

6. Convolution of functions.

Theorem 6. If

$$f(z)=z^p+\sum_{k=1}^{\infty} a_{p+k}\, z^{p+k}$$

and

$$g(z)=z^p+\sum_{k=1}^{\infty} b_{p+k}\, z^{p+k}$$

belong to $R^{\lambda}_{p,\delta,\alpha}$, then

$$F(z)=z^p+\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right) a_{p+k}\, b_{p+k}\, z^{p+k}$$

is also a member of $R^{\lambda}_{p,\delta,\alpha}$

Proof. Since $f(z)$ and $g(z)$ belong to $R^{\lambda}_{p,\delta,\alpha}$, we have

$$\left|\frac{e^{i\lambda}\,\dfrac{f'(z)}{pz^{p-1}}-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}-\delta\right|<\delta,\ \delta>\tfrac{1}{2},\ z\in U,$$

and

$$\left|\frac{e^{i\lambda}\,\dfrac{g'(z)}{pz^{p-1}}-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}-\delta\right|<\delta,\ \delta>\tfrac{1}{2},\ z\in U.$$

It is well known [4] that if $h(z) = \sum_{n=o}^{\infty} c_n z^n$ is regular in U and $|h(z)| \leqslant D$, then

$$\sum_{n=o}^{\infty} |c_n|^2 \leqslant D^2 . \tag{6.1}$$

Applying the estimate (6.1) to the function

$$\left\{ \frac{(1+i\tan\lambda)\dfrac{f'(z)}{pz^{p-1}} - i\tan\lambda - \alpha}{(1-\alpha)} - \delta \right\},$$

we get

$$(1-\delta)^2 + \left|\frac{1+i\tan\lambda}{1-\alpha}\right|^2 \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right|^2 \leqslant \delta^2$$

which yields

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right|^2 \leqslant (2\delta-1)(1-\alpha)^2 \cos^2\lambda, \delta > \tfrac{1}{2}.$$

Similarly, applying the estimate (6.1) to the function

$$\left\{ \frac{(1+i\tan\lambda)\dfrac{g'(z)}{pz^{p-1}} - i\tan\lambda - \alpha}{(1-\alpha)} - \delta \right\},$$

we obtain

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| b_{p+k} \right|^2 \leqslant (2\delta-1)(1-\alpha)^2 \cos^2\lambda, \delta > \tfrac{1}{2}.$$

Since

$$F(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) a_{p+k}\, b_{p+k}\, z^{p+k},$$

we have

$$\left| \frac{(1+i\tan\lambda)\dfrac{F'(z)}{pz^{p-1}} - i\tan\lambda - \alpha}{1-\alpha} - \delta \right|^2$$

$$= \left| (1-\delta) + \left(\frac{1+i\tan\lambda}{1-\alpha}\right) \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 a_{p+k} b_{p+k} z^k \right|^2$$

$$\leq (1-\delta)^2 + 2\left(\frac{1-\delta}{1-\alpha}\right) \sec\lambda \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right| \left| b_{p+k} \right| r^k$$

$$+ \frac{\sec^2\lambda}{(1-\alpha)^2} \left(\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right| \left| b_{p+k} \right| r^k \right)^2, \quad (|z| = r).$$

$$\leq (1-\delta)^2 + 2\left(\frac{1-\delta}{1-\alpha}\right) \sec\lambda \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right| \left| b_{p+k} \right|$$

$$+ \frac{\sec^2\lambda}{(1-\alpha)^2} \left(\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right| \left| b_{p+k} \right| \right)^2$$

$$\leq (1-\delta)^2 + 2\left(\frac{1-\delta}{1-\alpha}\right) \sec\lambda \left(\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right|^2 \right)^{\frac{1}{2}} .$$

$$\left(\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| b_{p+k} \right|^2 \right)^{\frac{1}{2}}$$

$$+ \frac{\sec^2\lambda}{(1-\alpha)^2} \left(\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| a_{p+k} \right|^2 \right) . \left(\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 \left| b_{p+k} \right|^2 \right)$$

$$\leq (1-\delta)^2 + 2(1-\delta)(2\delta-1)(1-\alpha)\cos\lambda$$
$$+ (2\delta-1)^2 (1-\alpha)^2 \cos^2\lambda.$$

Consequently

$$\left| \frac{e^{i\lambda} \dfrac{F'(z)}{pz^{p-1}} - \alpha\cos\lambda - i\sin\lambda}{(1-\alpha)\cos\lambda} - \delta \right|^2 < \delta^2$$

if

$$(1-\delta)^2 + 2(1-\delta)(2\delta-1)(1-\alpha)\cos\lambda$$
$$+ (2\delta-1)^2 (1-\alpha)^2 \cos^2\lambda < \delta^2$$

that is, if

$$(2\delta-1)\,[(1-\alpha)\cos\lambda - 1]\,[(2\delta-1)(1-\alpha)\cos\lambda + 1] < 0$$

which is true for δ, λ, α satisfying $\delta > \frac{1}{2}$, $|\lambda| < \frac{\pi}{2}$ and $0 \leqslant \alpha < p$.

Hence $F(z) \in R^{\lambda}_{p,\delta,\alpha}$

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