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DISPERSION OF LOVE WAVES IN AN INHOMOGENEOUS LAYER DUE TO A SOURCE

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Abstract

We consider an inhomogeneous elastic layer overlying a two layer model of the earth. The Love waves excited by a source lying just above the half-space, are considered. We are using Green's function method. The resulting dispersion relation is shown to agree with earlier known result in the special case.

1. Introduction

In the study of surface waves it is a reasonable approximation to model the earth as a medium consisting of a homogeneous elastic layer overlying a homogeneous elastic half space. Various authors, for example Ghosh [3], Chattopadhya, Pal and Chakroborty [1], have considered propagation of Love waves in case of varying density or rigidity. We consider a model consisting of two layers overlying a half space. This model has been studied by Sato [5] or more recently by Kazi and Abu-Safiya [4]. To fit the case in which the upper layer has variable properties, we allow the density in this layer to be variable and regard it as 'perturbation' of the case with constant density. The disturbance is assumed to be caused by a harmonic source lying just above the interface between the half spcee and the intermediate layer. We have used Green's function method [6] to obtain the dispersion relation for the disturbance present at the surface of the earth. This dispersion relation can be reduced to the case of homogneous upper layer when the perturbation parameter ϵ is equated to zero. The case of variation in rigidity can be treaed in a similar way.

2. Formulation of the Problem

We consider the problem in which two elastic horizontal layers of uniform thickness h and H overlie a semi-infinite substratum. We take the upper layer to be inhomogeneous. In this layer the density varies as linear function of z. The origin coordinate is taken along the interface $z=0$. The z-axis is taken vertically downward,

The lower layer $(o \leq z \leq H)$ and semi-infinite medium $(z \geq H)$ are supposed to be homogeneous. A harmonic source of SH-type is assumed to be present at $S(0,H)$. Subscripts 1, 2 and 3 refer to upper layer $(-h \leq z \leq 1)$, lower layer $(0 \leq z \leq H)$ and semi-infinie substratum $(z \geq H)$ respectively.

The equations of motion are,

$$\mu_1 \nabla^2 v_1 - (\rho_1 + \epsilon z) \frac{\partial^2 v_1}{\partial t^2} = 0, \tag{1}$$

$$\mu_2 \nabla^2 v_2 - \rho_2 \frac{\partial^2 v_2}{\partial t^2} = 4\pi \, \sigma_2 (r,t), \tag{2}$$

$$\mu_3 \nabla^2 v_3 - \rho_3 \frac{\partial^2 v_3}{\partial t^2} = 0, \tag{3}$$

where the inhmogeneous term in (2) appears due to the presence of source of density $\sigma_2(r,t)$. We shall take $\sigma_2(r,t)=\delta(x)\delta(z-H)e^{i\omega t}$ which represents a time harmonic source with angular frequency ω placed at (o,y,H) in the space, μ_1 and ρ_1 are the rigidity and density of the respective medium. The density in the upper layer is taken to be $\rho_1+\epsilon z$, where ϵ measures the inhomogeneity. The boundary conditions at the interfaces and free surface are,

$$\frac{\partial v_1}{\partial z} = 0 \qquad \text{at} \quad z = -h, \tag{4a}$$

$$v_1(z) = v_2(z) \qquad \text{at} \quad z = 0, \tag{4b}$$

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z} \qquad \text{at} \quad z = H. \tag{4c}$$

$$v_2(z) = v_3(z) \qquad \text{at} \quad z = H, \tag{4d}$$

$$\mu_2 \frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z} \qquad \text{at} \quad z = H. \tag{4e}$$

3. Solution of the Problem

We assume the time dependence to be $e^{i\omega t}$ and suppress it throughout. The Fourier transform $V(\xi,z)$ and its inverse are defined as,

$$V(\xi,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(x,z)\, e^{i\xi x}\, dx,$$

$$v(x,z) = \int_{-\infty}^{\infty} v(\xi,z)\, e^{-i\xi x}\, d\xi,$$

The equations (1) - (3) thus transform into

$$\frac{dv_1}{dz^2} - \alpha^2 v_1 = - \frac{\epsilon}{\mu_1} \omega^2 z V_1$$

$$= 4\pi\sigma_1(z), \text{ (say)} \tag{5}$$

$$\frac{d^2v_2}{dz^2} - \beta^2 v_2 = - \frac{\epsilon}{\mu_2} \omega^2 z V_1$$

$$= 4\pi\sigma_2(z), \qquad \text{(say)} \tag{6}$$

$$\frac{d^2V_3}{dz^2} - \gamma^2 V_3 = 0$$

In (5) - (7), we have used,

$$\alpha^2 = \xi^2 - \frac{\rho_1\omega^2}{\mu_1} = \xi^2 - k_1^2.$$

$$\beta^2 = \xi^2 - \frac{\rho_2\omega^2}{\mu_2} = \xi^2 - k_2^2.$$

and

$$\gamma^2 = \xi^2 - \frac{\rho_3\omega^2}{\mu_3} = \xi^2 - k_3^2.$$

Let $G_1(z,z_o)$ be Green's function for upper layer $(-h \leq z \leq 0)$ satisfying the homogeneous boundary conditions $dG_1/dz = 0$ at $z = -h$ and $z=0$. $G_1(z,z_o)$ is the solution of the equation,

$$\frac{d^2G_1(z,z_o)}{dz^2} - \alpha^2 G_1(z,z_o) = \delta(z-z_o), \tag{8}$$

where z_o is a point in the upper layer $(-h \leq z \leq 0)$. Multipling equation (5) by G_1 and (8) by V_1, subtracting and integrating from $-h$ to 0, we get after some simplification,

$$V_1(z) = -G_1(0,z) \left.\frac{dV_1}{dz}\right|_{z=0} + \int_{-h}^{0} 4\pi\sigma_1(z_o)G_1(z,z_o)dz. \tag{9}$$

Again, let $G_2(z,z_o)$ be Green's function for lower layer $(0 \leq z \leq H)$ satisfying the conditions $dG_2/dz = 0$ at $z = 0$ and $z = H$. $G_2(z,z_o)$ in this case satisfies the equation,

$$\frac{d^2G_2(z,z_o)}{dz^2} - \beta^2 G_2(z,z_o) = \delta(z-z_o), \tag{10}$$

where z_o is a point in the layer $(0 \leq z \leq H)$. Now multiplying equation (6) by G_2 and (10) V_2, subtracting and integrating from 0 to H, we obtain,

$$V_2(z) = \frac{2}{\mu_2} G_1(H,z) - G_2(H,z) \frac{dV_2}{dz}\Big|_{z=0} + G_2(0,z)\frac{dV_2}{dz}\Big|_{z=0} \tag{11}$$

Taking the Fourier transform of (4b,c), using these in (9) and (11), we can write,

$$\frac{dV_1}{dz}\Big|_{z=0} = \frac{1}{B}\left[G_2(H,0)\frac{dV_2}{dz}\Big|_{z=H} - \frac{2}{\mu_2} G_2(H,0) + \int_h^0 4\pi\sigma_1(z_o)G_1(0,z_o)dz_o)\right] \tag{12}$$

where

$$B = G_1(0,0) + \frac{\mu_1}{\mu_2} G_2(0,0), \tag{12*}$$

Now suppose that $G_3(z,z_o)$ is Green's function for the half space $(z \geq H)$ satisfying the boundary conditions $dG_3/dz = 0$ at $z = H$ and as $z \to \infty$. In this case $G_3(z,z_o)$ satisfies the equation,

$$\frac{d^2G_3(z,z_o)}{dz^2} - \gamma^2 G_3(z,z_o) = \delta(z-z_o), \tag{13}$$

As before, we get,

$$V_2(z) = G_3(H,z)\frac{dV_3}{dz}\Big|_{z=H} \tag{14}$$

Applying the condition (4d,e) at the interface $z = H$, we obtain after some effort,

$$\frac{dV_2}{dz}\Big|_{z=H} = \frac{1}{C}\left[\frac{2}{\mu_2}\left\{G_2^2(H,H) - \frac{\mu_1}{\mu_2}\frac{G_2^2(0,H)}{B}\right\}\right.$$

$$-\epsilon\omega^2 \frac{G_2^2(0,H)}{\mu_2 B} \int_{-h}^{0} z_o V_1(z_o) G_1(0,z_o) dz_o \qquad (15)$$

where,

$$C = G_2(H,H) - \frac{\mu_1}{\mu_2} \frac{G_2^2(0,H)}{B} + \frac{\mu_2}{\mu_3} G_3(H,H) \qquad (15^*)$$

and the value of $\sigma_3(z)$ from (5) have been re-introduced. By putting the value of dV_2/dz at $z = H$ from (15) into equation (12) and using the resulting value of dV_1/dz at $z = 0$ in equation (9), we obtain,

$$V_1(z) = \frac{2G_1(0,z)G_2(0,H)}{\mu_2 BC} \left[C - G_2(H,H) + \frac{\mu_1}{\mu_2} \frac{G_2^2(0,H)}{B}\right]$$

$$+\epsilon\omega^2 \frac{G_1(0,z)}{\mu_1 B} \left[\delta \int_{-h}^{0} z_o V_1(z_o) G_1(z,z_o) dz_o\right.$$

$$-\epsilon\omega^2 \int_{-h}^{0} \frac{z_o}{\mu_1} V_1(z_o) G_1(z,z_o) dz_o.$$

$$+\epsilon\omega^2 \frac{G_1(0,z)G_2^2(0,H)}{\mu_2 B^2 C} \int_{-h}^{0} z_o V_1(z_o) G_1(0,z_o) dz_o.$$

$$V_1(z) = \frac{2G_1(0,z)G_2(0,H)}{\mu_2 BC} \left[\frac{\mu_2}{\mu_3} G_3(H,H) + \epsilon\omega^2 \frac{G_2(0,H)}{2B}\right.$$

$$\left. x \int_{-h}^{0} z_o V_1(z_o) G_1(0,z_o) dz_o\right] - \epsilon\omega \int_{-h}^{0} \frac{z_o}{\mu_1} V_1(z_o) G_1(z,z_o) dz_o$$

$$+\epsilon\omega^2 \frac{G_1(0,z)}{\mu_2 B} \int_{-h}^{0} z_o V_1(z_o) G_1(0,z_o) dz_o. \qquad (16)$$

In order to eliminate $V_1(z)$ from equation (16), we make use of an approximate expression obtained from (16) by neglecting terms involving ϵ on the right hand side. This gives,

$$V_1(z) = \frac{2G_1(0,z)G_2(0,H)G_3(H,H)}{\mu_2 BC} \qquad (17)$$

This value of $V_1(z_o)$ then determines $V_1(z)$ from (16) correct to first power of ϵ.

We thus conclude that the wave motion in the upper Green's function G_1, G_2 and G_3 can be found. Thus the equation (16) can be rewritten as,

$$V_1(z) = \frac{2G_1(0,z)G_2(0,H)G_3(H,H)}{\mu_2 BC} \; [1+\epsilon\omega^2 \frac{G_2^2(0,H)}{BC\,\mu_2} \int_{-h} z_o G_1^2(0,z_o)dz_o$$

$$+\epsilon\omega^2 \frac{1}{\mu_1 B} \int_{-h}^{0} z_o G_1^2(0,z_o)dz_o\,] -2\epsilon\omega^2 \frac{G_2(0,H)G_3(H,H)}{\mu_1\mu_2 BC}$$

$$\times \int_{-h}^{0} z_o G_1(-h,z_o)G_1(0,z_o)dz_o \qquad (18)$$

4. Derivation of Green's Functions

In order to find Green's functions we need to solve the homogeneous equation,

$$\frac{d^2U}{dz^2} - \alpha^2 U = 0, \qquad z \neq z_o \qquad (19)$$

Two linearly independent solution of (19) that vanish at $z = \infty$ and $z = -\infty$, are $U_1(z) = e^{-\alpha z}$ and $U_2(z) = e^{\alpha z}$ respectively. A combination of $U_1(z)$ and $U_2(z)$ is used to define Green's function for the upper layer giving,

$$G_1(z,z_o) = A\,e^{\alpha z} + B\,e^{-\alpha z} - \frac{e^{-\alpha|z-z_o|}}{2\alpha} \qquad (20)$$

where a and B can be determined using the boundary conditions $dG_1/dz = 0$ at $z = -h$ and $z = 0$. After calculating the values of A and B, we obtain from (20)

$$G_1(z,z_o) = -\frac{1}{2\alpha}\left[e^{-\alpha|z-z_o|} + e^{\alpha z}\left\{\frac{e^{\alpha(h+z_o)} + e^{-\alpha(h+z_o)}}{e^{\alpha h} - e^{-\alpha h}}\right\}\right.$$

$$+e^{-\alpha z} \left\{ \frac{e^{-\alpha(h+z_o)} + e^{-\alpha(h+z_o)}}{e^{\alpha h} - e^{-\alpha h}} \right\} \Bigg] \qquad (21)$$

Following the same procedure, we can write,

$$G_2(z,z_o) = -\frac{1}{2\beta} \left[e^{-\beta|z-z_o|} + e^{\beta z} \left\{ \frac{e^{\beta(H+z_o)} + e^{-\beta(H+z_o)}}{e^{\beta H} - e^{-\beta H}} \right\} \right.$$

$$\left. +e^{-\beta z} \left\{ \frac{e^{\beta(H+z_o)} + e^{-\beta(H+z_o)}}{e^{\beta H} - e^{-\beta H}} \right\} \right] \qquad (22)$$

$$G_3(z,z_o) = -\frac{1}{2\gamma} \left[e^{-\gamma|z-z_o|} + e^{\gamma z} e^{\gamma(2H-z_o)} \right] \qquad (23)$$

5. Dispersion Relation for Upper Layer

The integrals appearing in (16) can be evaluated by using the values of $V_1(z_o)$ from (17) and those of G_1, G_2 and G_3 from (21), (22) and (23). Doing this, we obtain,

$$\int_{-h}^{0} z_o G_1(-h,z_o) G_1(0,z_o) dz_o = -\frac{h(1+2\,\alpha h \coth(\alpha h))}{4\alpha^3 \sin h(\alpha h)} \qquad (24)$$

and

$$\int_{-h}^{0} z_o G_1^2(0,z_o) dz_o = -\frac{\cos h(2\alpha h) + 2\alpha^2 h^2 - 1}{8\alpha^4 \sin h^2(\alpha h)} \qquad (25)$$

Using (21), (22), (24) and (25) in (18), we obtain,

$$V_1(-h) = \frac{2}{BC\mu_3} \left[\frac{-1}{\alpha \sinh(\alpha h)} \right] \left[\frac{-1}{\beta \sinh(\beta H)} \right] \left[\frac{-1}{\gamma} \right]$$

$$\times \left[1 - \epsilon\omega^2 \frac{\cosh(2\alpha h) + 2\alpha^2 h^2 - 1}{8\, B^2 C\, \mu_2 \alpha^4 \beta^2 \sinh^2(\alpha h)} - \epsilon\omega^2 \frac{\cosh(2\alpha h) + 2\alpha^2 h - 1}{8\, B\, \mu_1 \alpha^4 \sinh^2(\alpha h)} \right]$$

$$+\epsilon\omega^2 \quad \frac{2h\{1+2\alpha h \coth(\alpha h)\}}{4\mu_1\mu_2 \, BC\alpha^3 \, \beta\gamma \, \sinh(\alpha h)\sinh(\beta H)} \tag{26}$$

Equation (26) can again be written as,

$$V_1(-h) = \frac{-2}{BC\mu_3\alpha\beta\gamma\sin(\alpha h)\sin(\beta H)} \left[1 - \frac{\epsilon\omega^2}{4\alpha^2}\left\{\frac{\cosh(2\alpha h)+2\alpha^2h^2-1}{2B^2C\,\mu_2\alpha^2\beta^2\sinh^2(\alpha h)}\right.\right.$$

$$\left.\left. + \frac{\cosh(2\alpha h) + 2\alpha^2h^2 - 1}{2B\,\mu_1\alpha^2\sinh^2(\alpha h)} + \frac{\mu_3 h + 2ah^2\mu_3\coth(\alpha h)}{\mu_1\mu_2}\right\}\right] \tag{27}$$

Separating the terms of ϵ, equation (27) takes the form,

$$V_1(-h) = \frac{-2}{BC\mu_3\,\alpha\beta\gamma\,\sinh(\alpha h)\sinh(\beta H)} \quad [1 - F(\epsilon)] \tag{28}$$

where,

$$F(\epsilon) = \frac{\omega^2}{\alpha^2}\left\{\frac{\cosh(2\alpha h)+2\alpha^2h^2-1}{2B^2C\,\mu_2\alpha^2\beta^2\sinh^2(\alpha h)} + \frac{\cosh(2\alpha h) + 2\alpha^2h^2-1}{2B\,\mu_1\alpha^2\sinh^2(\alpha h)}\right.$$

$$\left. + \frac{\mu_3 h+2\alpha h^2\mu_3\coth(\alpha h)}{\mu_1\mu_2}\right\} \tag{29}$$

Neglecting the terms containing higher powers of ϵ, (28) may be written as,

$$V_1(-h) = \frac{-2}{BC\,\mu_3\,\alpha\beta\gamma\,\sinh(\alpha h)\sinh(\beta H)\,(1+F(\epsilon))} \tag{30}$$

The corresponding displacement $V_1(x,-h)$ on the surface of the upper layer $(-h \leq z \leq 0)$ is obtained by taking the inverse Fourier transform. This gives,

$$V_1(-h) = \int_{-\infty}^{\infty} \frac{-2\,e^{-i\xi x}\,d\xi}{BC\,\mu_3\,\alpha\beta\gamma\,\sinh(\alpha h)\sinh(\beta H)\{(1+F(\epsilon)\}} \tag{31}$$

To carry out the integration, we have to find contributions of the poles of the integrand in (31). The poles are given by

$$BC\,\mu_3\,\alpha\beta\gamma\,\sinh(\alpha h)\sinh(\beta H)\,\{(1+F(\epsilon)\} = 0. \tag{32}$$

For $\epsilon = 0$, the relation (32) reduces to

$$BC = 0. \qquad (33)$$

Using the value of B, C from (12*), (15*) and writting $\beta_1 = (k_2^2 - \xi_2)^{1/2}$, $\alpha_1 = (k_1^2 - \xi)^{1/2}$, we arrive at,

$$\mu_1\mu_2\alpha_1\beta_1 \tan(\alpha_1 h) + \mu_1\mu_3\alpha_1\gamma \tan(\alpha_1 h) \tan(\beta_1 H) - \mu_2\mu_3\beta_1\gamma + \mu_2^2\beta_1^2 \tan(\beta_1 H) = 0. \qquad (34)$$

This agrees with the dispersion relation for Love waves propagation in the medium consisting of two homogenous layers overlying a half space (see Ewing. Jardetsky and Press or Kazi and Abu-Safiya [4]).

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TAYLOR EXPANSIONS OF THE FUNCTIONS $(1+Z)^{X/Z}$ AND $(1+Z)^{XZ}$

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Abstract

In this paper we find the Taylor expansions of the functions $(1+Z)^{X/Z}$ and $(1+Z)^{XZ}$.

1. Introduction

The stirling numbers of the first kind S(n,k) are generated by the Taylor expansions,

$$\frac{1}{k!} \ln^k(1+Z) = \sum_{n=k}^{\infty} S(n,k) \frac{Z^n}{n!}, \quad k=0,1,2,.., \ |Z|<1, \tag{1}$$

they are integral numbers and satisfy the inequalities $(-1)n+k.\ S(n+k)>0$ for $1 \le k \le n$. $n \ge 1$. Explicit formulas, recurrence relations and tables for S(n,k) can be found in Comtet [1].

According to Comtet [1], p.136 the Remark, and our paper [2], ordinary Bell polynomials $D_{nk}(X_1,\ldots,X_{n-k+1})$ are called the polynomials,

$$D_{nk}(X_1, \ldots, X_{n-k+1}) \equiv \Sigma \frac{k!(X_1)^{\nu_1} \ldots (X_{n-k+1})^{\nu_{n-k+1}}}{\nu_1! \ldots \nu_{n-k+1}!} \tag{2}$$

where the sum is taken over all nonnegative integers $\nu_1, \ldots, \nu_{n-k+1}$ satisfying,

$$\nu_1 + \nu_2 + \ldots + \nu_{n-k+1} = k_1\ \nu_1 + 2\nu_2 + \ldots + (n-k+1)\nu_{n-k+1} = n. \tag{3}$$

Relations and tables for $D_{nk}(X_1,\ldots,X_{n-k+1})$ can be found in [1] and [2],

Further, we shall use the relation,

$$\frac{1}{k!} D_{nk}\left(\frac{1}{1}\ \frac{1}{2} \cdots \frac{1}{n-k+1}\right) = (-1)^{n+k} \frac{S(n,k)}{n!} \qquad \ldots(4)$$

for $1 \leq k \leq n$. $n \geq 1$ (see [1] . p. 135. Equation [3i]).

2. Taylor expansion of the functin $(1+Z)^{X/Z}$

Let us set,

$$(1+Z)^{X/Z} = e^x \sum_{n=0}^{\infty} a_n(X)Z^n,\ a_o(X) = 1, \tag{5}$$

for $|Z| < 1$ and any complex number X.

Theorem 1 : For n = 1, 2, ... The corefficients $a_n(X)$ in (5) have the explicit representations,

$$a_n(X) = \sum_{k=1}^{n} X^k \sum_{\nu=1}^{k} \frac{(-1)^{k-\nu}}{(k-\nu)!} \cdot \frac{S(n+\nu,\nu)}{(n+\nu)!} \tag{6}$$

and

$$a_n(X) = (-1)^n \sum_{k=1}^{n} \frac{X^k}{k!} D_{nk}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right). \tag{7}$$

Proof : According to Theorem 1 in [3] (or [4]) for the nth derivative of composite functiones, applied to the composite function,

$$(1+Z)^{X/Z} = e^{t} \circ \frac{X}{Z} \ln (1+Z), \tag{8}$$

we have,

$$D^{n}_{z=0} (1+Z)^{X/Z} = e^{x} \sum_{k=1}^{n} A_{nk}(0), \ \ n \geq 1, \tag{9}$$

where,

$$A_{nk}(0) = \frac{x^{k}}{k!} \sum_{\nu=1}^{k} (-1)^{k-\nu} \binom{k}{\nu} D^{n}_{z=0} \left(\frac{\ln(1+z)}{z}\right)^{\nu}, 1 \leq k \leq n. \tag{10}$$

With the help of (1) from (10) and (9) we come to the formula

$$D^{n}_{z=0}(1+z)^{x/z} = n!\, e^{x} \sum_{k=1}^{n} X^{k} \sum_{m=1}^{k} \frac{(-1)^{k-\nu}}{(k-\nu)!} \cdot \frac{S(n+\nu,\nu)}{(n+\nu)!} \tag{11}$$

for $n=1,2,\ldots$. Thus from (11) and (5) we obtain the formula (6) $n=1,2,\ldots$.

If we apply the Faa di Brune "precise formula" for the nth derivative of composite functions, developed in our paper [2], to the composite function (8) written in the form

$$(1+z)^{x/z} \equiv e^{t} \circ \left(x \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m+1} z^{m}\right)$$

and we take into account (2) and (3), then we obtain again the formula (9) but with,

$$A_{nk}(0)=(-1)^{n} n! \frac{X^{k}}{k!} D_{nk}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right), 1 \leq k \leq n. \tag{12}$$

Therefore, from (12) and (9) we come to another formula,

$$D^n_{z=0}(1+z)^{x/z}=(-1)^n n! e^x \sum_{k=1}^{n} \frac{X^k}{k!} D_{nk}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right). \quad (13)$$

for $n=1,2,\ldots$. Thus from (13) and (5) we obtain the formula (7) for $n=1,2,\ldots$.

In particular, for $x=1$ if we set $a_n \equiv a_n(1)$, $n=0,1,2,\ldots$ then (5) is reduced to,

$$(1+Z)^{1/Z} = e \sum_{n=0}^{\infty} a_n Z_n,\ a_o=1,\ |Z|<1, \quad (14)$$

and Theorem 1 yields.

Corollary 1.1: For $n=1,2,\ldots$ the coefficients a_n in (14), have the explicit representation

$$a_n = \sum_{\nu=1}^{n} \frac{S(n+\nu,\nu)}{(k-\nu)!} \sum_{k=0}^{n-\nu} \frac{(-1)^k}{k!}$$

and

$$a_n = (-1)^n \sum_{k=1}^{n} \frac{1}{k!} D_{nk}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right).$$

The comperison of (6) and (7) gives.

Corollary 1.2: For $1 \le k \le n$. $n \ge 1$, we have the combinatorial identities,

$$D_{nk}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right) = (-1)^n k! \sum_{\nu=1}^{k} \frac{(-1)^{k-\nu}}{(k-\nu)!} \cdot \frac{S(n+\nu,\nu)}{(n+\nu)!}$$

Coroallary 1.3: For $n=1,2,\ldots$ and any complex number X, we have the summation formula,

$$\sum_{k=0}^{\infty} \frac{S(n+k,k)}{(n+k)!} X^k = e^x a_n(x) \quad (15)$$

where $a_n(x)$ are determined by (6) and (7).

Proof : From (8) and (1) we obtain the expansion,

$$(1+Z)^{x/z} = \sum_{k=0}^{\infty} \frac{X^k}{Z^k} \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} Z^n$$

$$= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} \frac{S(n+k,k)}{(n+k)!} X^k .$$

The comperison of (16) and (5) gives the formula (15), keeping in mind (6) and (7). (For $n=0$. the second sum on the right hand side of (16) is reduced to e^x, since $S(k,k)=1$ according to (1).

Theorem 2 : The coefficients $a_n(x)$ in (5) satisfy the recurrence relation,

$$na_n(x) = x \sum_{k=1}^{n} (-1)^k \frac{k}{k+1} a_{n-k}(x), \tag{17}$$

$$n = 1, 2, \ldots, a_o(x) = 1.$$

Proof : We can write (5) in the form,

$$\exp\left[x \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^k\right] = e^x \sum_{n=0}^{\infty} a_n(x) z^n \tag{18}$$

Differentiating (18) with respect to z, we obtain,

$$\sum_{n=1}^{\infty} n\, a_n(x)\, z^n = x \sum_{k=1}^{\infty} (-1)^k \frac{k}{k+1} \lambda . \sum_{n=0}^{\infty} a_n(x)\, Z^k$$

$$= x \sum_{n=1}^{n} z^n \sum_{k=1}^{n} (-1)^k \frac{k}{k+1} a_{n-k}(x),$$

whence we obtain the relation (17).

With the help of (17) we find the first few coefficients,

$$a_1(x) = -\frac{x}{2}, \quad a_2(x) = \frac{x}{24}(3x+8),$$

$$a_3(x) = -\frac{x}{48}(x+2)(x+6),$$

$$a_4(x) = \frac{x}{5760}(15x^3+240x^2+1040x+1152),$$

(The last polynomial is irreducible in the field of the rational numbers.) From Theorem 1 it follows that $(-1)^n a_n(x)>0$ for $n=1,2,\ldots$ if $x > 0$.

In particular, for $x=1$. Theorem 2 yields.

Corollary 2.1: The coefficients a_n in (14) satisfy the recurrence relation,

$$na_n = \sum_{k=1}^{n} (-1)^k \frac{\lambda}{k+1} a_{n-k}, \; n \geq 1, \; a_o = 1. \tag{20}$$

From (20) (or (19) for $x=1$) we obtain the first few coefficients,

$$a_1 = -\frac{1}{2}, \; a_2 = \frac{11}{24}, \; a_3 = -\frac{7}{16}, \; a_4 = \frac{2447}{5760}, \ldots$$

From corollary 1.1 it follows that $(-1)^k a_n>0$ for $n=1,2,\ldots$.

Theorem 3: The coefficients $a_n(x)$ in (5) satisfy the recurrence relation,

$$a'_n(x) = \sum_{k=1}^{n} \frac{(-1)^k}{k+1} a_{n-k}(x), \; n \geq 1, \; a_o(x) = 1 \tag{21}$$

Proof : First, we shall prove that the series on the right-hand side of (5) can be differentiated with respect to x. Indeed, for $|x| \leq r$, where $r>0$, from (7) we obtain the estimate,

$$|a_n(x)| \leq (-1)^n a_n(r), \; n \geq 1. \tag{22}$$

For $|x|\backslash r$. $r>0$ from (5) by aid of (22) we otain the convergent majorant series,

$$\left| \sum_{n=0}^{\infty} a_n(x) z^n \right| \leq \sum_{n=0}^{\infty} a_n(r) (-|z|)^n = e^{-z}(1-|z|)^{-z/|z|} \qquad (23)$$

From (23) it follows that the series in (5) is uniformly convergent with respect to x in the closed disc $|x| \leq r$ for any real number $r>0$. Therefore, according to the Weierstrass theorem, we can differentiate (50 with respect to x and to obtain,

$$\sum_{n=0}^{\infty} a_n(x)z^n + \sum_{n=1}^{\infty} a_n{}^1(x) z^n \qquad (24)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^k \sum_{n=0}^{\infty} a_n(x) z^n$$

$$= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \frac{(-1)^k}{k+1} a_{n-k}(x).$$

The comparison of the coefficients of z^n in (24) gives the relation (21).

From (17) and (21) we obtain.

Corollary 3.1: The coefficients $a_n(x)$ in (5) satisfy the recurrence relation,

$$xa_n'(x)+na_n(x)=x \sum_{\lambda=1}^{n} (-1)^k a_{n-k}(x),\ n\geq 1,\ a_o(x)=1. \qquad (25)$$

If we integrate (25) with respect to $a_n(x)$, then we obtain.

Corollary 3.2: The coefficients $a_n(x)$ in (5) satisfy the recurrence relation,

$$a_n(x) = \frac{1}{x^n} \int x^n \sum_{k=1}^{n} (-1)^k a_{n-k}(x)\, dx,\ n\geq 1,\ a_o(x)=1. \qquad (26)$$

With the help of (26) we can easily continue the table (19).

Theorem 4: The polynomials $w=a_n(x)$. $n=1,2,\ldots$ of degree n in x, determined by (5), satisfy the Euler differential equations,

$$\sum_{k=0}^{n}(-1)^k \frac{x^k}{k!} w^{(k)} = 0,\ w^{(0)} \equiv w, \tag{27}$$

of order $n=1,2,\ldots$ respectively, where the derivatives of w are taken with respect to x.

First Proof : The substitution $w=x^\alpha$, where α is a certain parameter, redues the equation (27) to the form,

$$\sum_{k=0}^{n}(-1)^k \binom{\alpha}{k} = 0. \tag{28}$$

The roots of the equation (28) are $\alpha=1,2,\ldots,n$. Indeed, for $\alpha=n$. $n\geq 1$, the equation (28) is reduced to $(1-1)^n\equiv 0$. For a fixed $\alpha=1,\ldots,n-1$, $n\geq 2$, we have $\binom{\alpha}{k}=0$ if $k>\alpha$, and hence, the equation (28) is redced to,

$$\sum_{k=0}^{\alpha}(-1)^k \binom{\alpha}{k} \equiv (1-1)^\alpha = 0.$$

Therefore, the general solutions of the equations (27) are,

$$w = \sum_{\lambda=1}^{n} C_k x^k,\ n=1,2,\ldots, \tag{29}$$

respectively, where C_k are arbitrary constants. The comparison of (29) and (6) (or (7)) shows that the polynomials $w=a_n(x)$, $n=1, 2, \ldots$, are corresponding particular solution of the equations (27).

Second Proof : We differentiate (21) $K-1$ times with respect to x to obtain,

$$a_n^{(k)}(X) = \sum_{\nu=1}^{n-k+1} \frac{(-1)^\nu}{\nu+1} a_{n-\nu}^{k-1}(x), \tag{30}$$

$$1\leq k\leq n,\ n\geq 1,\ a_{n-\nu}^{(0)}(x) \equiv a_{n-\nu}(x).$$

with the help of (3) we conclude that the sum,

$$\sum_{k=1}^{n} (-1)^k \frac{x^k}{k!} a_n^{(k)}(x) \tag{31}$$

$$= \sum_{\nu=1}^{n} \frac{(-1)^\nu}{\nu+1} \sum_{k=0}^{n} (-1)^{k+1} \frac{x^{k+1}}{(k+1)!} a_{n-\nu}^{(k)}(x)$$

for $n \geq 1$. Now we find that the derivative,

$$\frac{d}{dv} \sum_{k=0}^{n-\nu} (-1)^{k+1} \frac{x^{k+1}}{(k+1)} a_{n-\nu}^{(k)}(x) \tag{32}$$

$$= \sum_{k=0}^{n-\nu} (-1)^k \frac{x^k}{k!} a_{n-\nu}^{(k)}(x) +$$

$$+ \sum_{k=1}^{n-\nu} (-1)^k \frac{x^k}{k!} a_{n-\nu}^{(k)}(x)$$

$$= -a_{n-\nu}^{(k)},\ 1 \leq \nu \leq n-1,\ n \geq 2.$$

Integrating (32), we obtain the relation,

$$\sum_{k=0}^{n-\nu} (-1)^{k+1} \frac{x^{k+1}}{(k+1)!} a_{n-\nu}^{(k)}(x) = - \int_0^x a_{n-\nu}(t)\,dt \tag{33}$$

for $1 \leq \nu \leq n-1$, $(n \geq 2)$, ovidently tiue for $m = n$, $(n \geq 1)$ as welll, since $a_0^{(0)} \equiv a_0(x) = 1$. From (31) and (33) it follows that the sum,

$$\sum_{k=1}^{n} (-1)^k \frac{x^k}{k!} a_n^{(k)}(x)$$

$$= \sum_{k=1}^{n} \frac{(-1)^\nu}{\nu+1} \int_0^x a_{n-\nu}(t)\,dt$$

$$= - a_n(x),\ n \geq 1,$$

after having integrated (21), i.e.,

$$\sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} a_n^{(k)}(x) = 0, \; a_n^{(0)}(x) \equiv a_n(x), \; n \geq 1.$$

3. Taylor Expansion of the Function $(1+Z)^{ZX}$

Let us set,

$$(1+Z)^{XZ} = 1 + \sum_{n=2}^{\infty} b_n(x) Z^n \tag{34}$$

for $|Z| < 1$ and any complex number X.

Theorem 5: For n=2,3,..., the coefficients $b_n(x)$ in (34) have the explicit representations,

$$b_n(x) = \sum_{k=1}^{[n/2]} \frac{S(n-k,k)}{(n-k)!} X^k \tag{35}$$

and

$$b_n(x) = (-1)^n \sum_{k=1}^{[n/2]} \frac{X^k}{k!} D_{n-k,k}\left(\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n-2k+1} \right) \tag{36}$$

where [n/2] denotes the greatest integer less than or equal to n/2.

Proof : With the help of (1) we obtain the expansion,

$$(1+Z)^{XZ} = \exp\left[XZ \ln(1+Z)\right] \tag{37}$$

$$= \sum_{k=0}^{\infty} X^k Z^k \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} Z^n$$

$$= \sum_{n=0}^{\infty} Z^n \sum_{k=0}^{[n/2]} \frac{S(n-k,k)}{(n-k)!} X^k$$

$$= 1 + \sum_{n=2}^{\infty} Z^n \sum_{k=1}^{[n/2]} \frac{S(n-k,k)}{(n-k)!} x^k.$$

The comparison of (37) and (34) gives the formula (35). The formula (36) follows from (35) and (4).

In particular, for $x=1$ if we set $b_n \equiv b_n(1)$. $n=2,3,\ldots$, then (34) is reduced to,

$$(1+Z)^Z = 1 + \sum_{n=2}^{\infty} b_n Z^n, \ |Z| < 1, \tag{38}$$

and Theorem 5 yields.

Corollary 5.1: For $n=2,3,\ldots$, the coefficients b_n in (38) have the explicit representations,

$$b_n = \sum_{k=1}^{[n/2]} \frac{S(n-k,k)}{(n-k)!}$$

and

$$b_n = (-1)^n \sum_{k=1}^{[n/2]} \frac{1}{k!} D_{n-k,k}\left(\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n-2k+1} \right)$$

Theorem 6: The coefficients $b_n(x)$ in (34) satisfy the recurrence relation,

$$nb_n(x) = x \sum_{k=2}^{n} (-1)^k \frac{k}{k-1} b_{n-k}(x), \tag{39}$$

$$n = 2, 3, \ldots, b_o(x) = 1, b_1(x) = 0,$$

Proof : We can write (34) in the form,

$$\exp\left[x \sum_{k=2}^{\infty} \frac{(-1)^k}{k+1} z^k \right] = 1 + \sum_{n=2}^{\infty} b_n(x) z^n. \tag{40}$$

Differentiating (4) with respect to z, we obtain

$$\sum_{n=2}^{\infty} nb_n(x)Z^n = x \sum_{k=2}^{\infty} (-1)^k \frac{k}{k-1} Z^k . \sum_{n=0}^{\infty} b_n(x)Z^n$$

$$= x \sum_{n=2}^{\infty} Z^n \sum_{k=2}^{n} (-1)^k \frac{k}{k-1} b_{n-k}(x),\ b_o(x)=1,\ b_1(x)=0,$$

whence we obtain the relation (39).

With the help of (39) we find the first few coefficients,

$$b_2(x) = x,\ b_3(x) = -\frac{x}{2},\ b_4(x) = \frac{x}{6}(3x+2),$$

$$b_5(x) = -\frac{x}{4}(2x+1),\ b_6(x) = \frac{x}{120}(20x^2+55x+24),$$

$$b_7(x) = -\frac{x}{12}(2x+1)(3x+2),\ \ldots . \tag{41}$$

From Theorem 5 it follows that $(-1)^k b_n(x) > 0$ for $n=2,3,\ldots$ if $x>0$.

In particular for $x=1$ Theorem 6 yields

Corollary 6.1: The coefficients b_n in (38) satisfy the recurrence relation,

$$nb_n = \sum_{k=0}^{n} (-1)^k \frac{k}{k-1} b_{n-k},\ n\geq 1,\ b_0=1,\ b_1=0. \tag{42}$$

From (42) (or (41) for $x=1$) we obtain the first few coefficients,

$$b_2 = 1,\ b_3 = -\frac{1}{2},\ b_4 = \frac{5}{6},\ b_5 = -\frac{3}{4},$$

$$b_6 = \frac{33}{40},\ b_7 = -\frac{5}{6},\ \ldots$$

From Corollary 5.1 it follows that $(-1)^n b_n > 0$ for $n=2,3,\ldots$.

Theorem 7: The coefficients $b_n(x)$ in (34) satisfy the recurrence relation,

$$b_n'(x) = \sum_{k=2}^{n} \frac{(-1)^k}{k-1} b_{n-k}(x),\ n \geq 2,\ b_o(x)=1,\ b_1(x) = 0, \tag{43}$$

Proof : We apply the method in the proof of Theorem 3 to (36) and (34) in order to obtain (43).

From (39) and (43) we obtain.

Corollary 7.1: The coefficients $b_n(x)$ in (34) satisfy the recurrence relation,

$$xb_n'(x) - nb_n(x) = x \sum_{k=2}^{n} (-1)^{k-1} b_{n-k}'(x), \tag{44}$$

$$n \geq 2,\ b_o(x) = 1,\ b_1(x) = 0$$

If we integrate (44) with respect to $b_n(x)$, then we obtain.

Corollary 7.2: The coefficients $b_n(x)$ in (34) satisfy the recurrence relation,

$$b_n(x) = x^n \int \frac{dx}{x^n} \sum_{k=2}^{n} (-1)^{k-1} b_{n-k}(x), \tag{45}$$

$$n \geq 2,\ b_o(x) = 1,\ b_1(x) = 0.$$

With the help of (45) we can easily continue the table (41).

Theorem 8: The polynomials $W=b_n(x)$, $n=2,3,\ldots$, of degree $[n/2]$ in X, determine by (34), satisfy the Euler differential equations,

$$\sum_{k=0}^{[n/2]} (-1)^k \frac{x^k}{k!} w^{(k)} = 0,\ w^{(0)} \equiv w,$$

of order $[n/2]$, $n=2,3,\ldots$, respectively, where the derivatives of w are taken with respect to x.

Proof : This follows with the help of the methods in the proof of Theorem 4 (and (43) for the second method).

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COMPARISON OF REGULARIZATION METHODS PROPOSED FOR THE NUMERICAL SOLUTION OF ILL-POSED PROBLEMS

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Abstract

Methods are presented in this paper for estimating solutions of ill-posed problems in the form of Fredholm integral equations of the first kind, given noisy data. These are regularization methods and regularization is effected by maximum likelihood technique and regular filtering by WCV Method.

In these methods optimal amount of smoothing may be computed, based on the data and the assumed known noise variance. We shall compare our method with the method of Al-Faour [2], over the same highly ill-posed problems and for comparison purposes the results are shown in the tables and the diagrams.

Introduction

For many years ill-posed problems have been considered as mere mathematical anomalies. However it appeared in early sixties, that this was erroneous and that many ill-posed problems generally inverse problems [4], arose from practical situations. Now a days the systematic study of these problems has undoubtedly proved of great relevance in many fields of applied physics [21, 22].

Ill-posed problems are generally inverse problems. With the growing recognition of the significance of inverse problems in applications, it is important to examine the techniques available for their approximate solution.

In this paper we will consider the most prominant of these methods i.e. the method of regularization. Although this method can be applied to general inverse problems, we focus our attention on the important special case of integral equations of the first kind of convolution type.

It is well known that these equations are ill-posed in the sense that small perturbations in data g can lead to large errors in the solution f. Moreover these equations serve as a good model for general ill-posed inverse problems. Sometimes it is possible and useful to determine the degree of ill-posedness of an inverse problem by comparing it with a first kind integral equation.

The degree of ill-posedness of such type of equations depends upon the rate of decay of eigen-values, the faster decay, the more ill-posed the problem. There are the results which relate the smoothness of the kernel $K(x,y)=K(x-y)$ to the decay rate of eigenvalues. Basically the smoother the kernel, the faster the decay rate and hence the more ill-posed the problem. See [13] for a detailed discussion of these points.

For regularization to be useful in practice there must be some method for choosing a good value of (the regularization parameter) λ for any given data set. We shall consider two such methods as described below.

1. Method 1 : Rectangular filtering by WCV

Al Faour [2], discussed this method for finding an optimal rectangular filter and the choice of optimal cut-off point using Wahba's weighted cross validation (WCV) [21] method.

Descriation of the Method

The problem of finding an optimal k in the case of rectangular filtering can be set in the following manner. Consider the integral equation

$$\int_{-\infty}^{\infty} k(x-y) f(y)\, dy = g(x) \qquad -\infty < x < \infty - \tag{1.1}$$

where $g(x) = g_T(x) + E(x)$. Assuming that both functions $K(x)$ and $g(x)$ are sampled at the points $k=nh$ when $n=0, 1, \ldots N-1$ and let them be interpolated at $\{x_n\}$ by trigonometric polynomials of degree at most N with a period of Nh.

Let T_N be the space of trigonometric polynomials of degree at most N-1 with period $T=nh$.

For $g_N(x) \in T_N$, let $g_N = g(x_n) = g_N(x_n)$, $n=0, 1, 2 \ldots, N-1$

The DFT of the data set (x_n, g_n) is then defined by

$$\hat{g}_q = \sum_{n=0}^{N-1} g_n \exp(-iw_q x_n) \qquad q=0, \ldots, N-1 \tag{1.2}$$

where $$w_q = \frac{2\pi}{T}\, q = \frac{2\pi}{Nh}\, q$$

The inverse DFT is,

$$g_n = \frac{1}{N} \sum_{q=0}^{N-1} \hat{g}_q \exp(iw_q x_n) \tag{1.3}$$

which samples the finite Fourier series

$$g_N(x) = \frac{1}{N} \sum_{q=0}^{N-1} \hat{g}_q \exp(iw_q x)$$

where $$\hat{g}_q^{\bullet} = \overline{\hat{g}_{N-q}}$$

Approximating g and k by their trigonometric interpolants g_N and k_N on the interval $[O,T]$, equation (1.1) is approximated by the finite convolution equation

$$\int_0^T K_N(x - y) f_N(y)\, dy = g_N(x)$$

where K_N assumed periodically continued outside $[O,T]$ (T may be 1.0) and $f_N \in T_N$

The Fourier coefficients are then related by $\hat{k}_q \; \hat{f}_q = \hat{g}_q$ so that

$$\hat{f}_q = \frac{\hat{g}_q}{\hat{k}_q} \tag{1.4}$$

and

$$\left.\begin{aligned} f_n &= \frac{1}{N} \sum_{q=0}^{N-1} \frac{\hat{g}_q}{\hat{k}_q} \exp(iw_q x_n) \\ f_N(x) &= \frac{1}{N} \sum_{q=0}^{N-1} \frac{\hat{g}_q}{\hat{k}_q} \exp(iw_q x) \end{aligned}\right\} \tag{1.5}$$

Writing $\mathbf{f} = (f_o \ldots f_{N-1})^T$

There exists an $N \times N$ matrix K such that

$$(K\,\mathbf{f})_n = \int_0^T K_N(x_n - y) f_N(y)\, dy \tag{1.6}$$

and K is given by

$$K = \psi \; \hat{K} \psi^H$$

where $K = \text{diag}\,(\hat{K}_o \ldots \hat{K}_{N-1})$

and

$$\psi_{qr} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} qr\right) \tag{1.7}$$

where $q,r = 0, \ldots N-1$

Where ψ represents the DFT matrix in equaton (1.2) and ψ^H represents the inverse DFT matrix in equation (1.3) and we observe that ψ is unitary matrix i.e.

$$\psi \psi^H = \psi^H \psi = I$$

Now equation (1.5) should give an accurate solution of (1.1) if $\dfrac{\hat{g}_q}{\hat{k}_q}$ is known exactly.

However in practice, g_n and therefore g_q always contain some error. Thus due to ill-posedness of the problem, the solution obtained by (1.5) is unstable. To stabilize f_n we multiply the ratio (1.4) by a filter z_q. Al-Foour [2] has restricted his discussion to the rectangular filter of the form

$$Z_q = Z_{n-q} = \left.\begin{matrix} 1 & \quad 0 \le q \le Q-1 \\ 0 & \quad \text{otherwise} \end{matrix}\right\} \qquad (1.8)$$

where Q is a cut-off frequency to be determined.

In order to determine optimal Q simply from the observational data Al-Faour [2] has applied. Wahba's principle of weighted cross validation in Fourier space [21].

An optimal Q is then defined by that which minimizes the weighted mean-square prediction error,

$$V(Q) = \frac{1}{N} \sum_{q=0}^{N-1} |\hat{g}_{Q,q} - \hat{g}_q|^2 \, w_q(Q) \qquad (1.9)$$

where the weights $\psi_q[Q]$ are

$$w_q(Q) = \frac{(1-\hat{a}_{qq})^2}{[1/N \, Tr(I-A(Q))]^2} \qquad (1.10)$$

$q = 0, \ldots, N-1$

Wahba has shown in a different context that $V(Q)$ in (1.9) with weights given in (1.10) can be simplified to the form (Seee Al-Faour [2] page 50)

$$V(Q) = \frac{\frac{1}{N} \overset{N-Q}{\underset{q=Q}{R}} |\hat{g}_q|^2}{[1 - \frac{2Q+1}{N}]^2} \qquad (1.11)$$

2. Maximum likelihood method with non-negativity Constraints

We have worked out in detail the procedural steps of the method without non-negativity constraits in [7]. In order to compare this method with the method of section 1, on n same test problems we shall concentrate on maximum likelihood method with non-negativity constraints.

Description of the method :

We estimate the solution of

$$\int_{-\infty}^{\infty} k(x-y) f(y)\, dy = g(x), \quad -\infty \leq x \leq \infty$$

where we know in advance that f is non-negative and hence our estimate f_N is constrained to be non-negative.

We first describe Wahba's [22] constrained algorithm in our setting.

Let $f_N = (f(z), \ldots f(x_{N-1}))^T$ and

consider the pth order regularization functional in T_N.

$$C(f_N; \lambda) = \| \hat{K} \psi^H f_N - \psi^H g_N \|_2^2 + \lambda\, \mathbf{f}_N^H \psi\, g\, \psi^H \mathbf{f}_N \qquad (2.1)$$

where $\hat{K} = \psi^H k\, \psi$

and $$J = \mathrm{diag}\ \{(i\widetilde{w}_q)^{2p}\} \tag{2.2}$$

Let f_λ be the minimizer of (2.1) subject to $f_N \geq 0$

with components $f_{\lambda,n}$. The indices $n_1, n_2, \ldots, n_L$ for which $f_{\lambda,n} \geq 0$ are to be determined.

.Let E be the $N \times L$ indicator matrix of these indices. That is E has a unit element in the mth row and nth column if $m=n_j$, $j=1, \ldots, L$ and zeros elsewhere. In what follows we denote by 1 the set of indices $(n_1, \ldots n_L)$ of inactive constraints.

The constrained minimizer of (2.1) is

$$\mathbf{g}_\lambda = E(E^H \psi \hat{K}^H k \psi^H E + \lambda E^H \psi \hat{J} \psi^H E)^{-1} E^H \psi \hat{K}^H \psi^H \mathbf{g}_n \tag{2.3}$$

Defining

$$\mathbf{g}_\lambda = k\, \mathbf{f}_\lambda$$

There exists an $N \times N$ influence matrix $A_L(\lambda)$ satisfying

$$\mathbf{g}_\lambda = A_L(\lambda)\, \mathbf{g}_N$$

It can be shown that,

$$A_L(\lambda) = \psi \hat{K} \psi^H E(\Sigma_k + \lambda \Sigma_J)^{-1} E^H \psi \hat{K}^H \psi^H \tag{2.4}$$

where $\Sigma_k = E^H \psi K^H K \psi^H E$ and $\Sigma_J = E^H \psi J \psi^H$

with the property that

Trace $(1 - A_L(\lambda)) = N - L + \lambda$ Trace (B)

where $B = \Sigma_J (\Sigma_K + k \Sigma_J)^{-1}$

Wahba argues that the optimal λ in the constrained setting may be found by minimizing

$$V^c_{\text{approx}}(\lambda) = \frac{\| K\, w^H \mathbf{f}_\lambda - w^H \mathbf{g}_N \|^2}{[1/N\,(N-L+\lambda\ \text{Trace}\,(B)]^2} \tag{2.5}$$

Clearly $\mathbf{f}_\lambda$ depends non-linearly on E and λ and so E must be recomputed whenever $\mathbf{f}_\lambda$ is computed.

These iterations can be expensive and unhandy.

Moreover $V^c_{\text{approx}}(\lambda)$ need not be a continuous function of λ.

For a given λ Wahba [22] uses a quadratic programming algorithm to minimize (2.1) subject to $\mathbf{f}_N \geq 0$. A unique minimum always exists (See e.g. Butler [6]).

Having found E and $\mathbf{f}_N$ she then calculates B by solving L linear system defined by

$$(\Sigma_k + \lambda\, \Sigma_J)\, B = \Sigma_J \tag{2.6}$$

using Linpack (Dongarra etal [10]). She then examines the value of $V^c_{\text{approx}}(\lambda)$, adjusts λ accordingly if a minimum is not found, then repeats the process. This is an expensive procedure computationally.

Our method is simpler than Wahba's algorithm and quite less expensive computationally.

3. Maximum likelihood regularization with non-negativity

We notice that since $\psi(\widehat{K}^H K + \lambda J)\, \psi^H$

is circulant, so is $(\Sigma_k + \lambda\, \Sigma_J)^{-1}$ and consequently so is $E(\Sigma_k + \lambda\, \Sigma_J)^{-1} E^H$ in equation (2.4). Thus $A_L(\lambda)$ is clearly circulant.

In principle we can use the L - dimensional DFT to evaluate $A_L(\lambda)$. Thus avoiding the necessity of solving the L linear systems in (2.6).

Now consider,

$$\hat{K}\,\psi^{H}\,E(\Sigma_k + \lambda\,\Sigma_J)^{-1}\,E^{H}\,\psi\,\hat{K}^{H} \simeq \text{diag}\,(\Sigma_{q;\lambda}) \tag{3.1}$$

where

$$\tilde{Z}_{q;\lambda} = \begin{cases} Z_{q;\lambda}\,, & q \in I \\ 0\,, & q \notin I \end{cases} \tag{3.2}$$

For pth order regularization we recall from equation (3.1)that

$$Z_{q;\lambda} = \frac{|\hat{K}_{N,q}|^2}{|K_{N,q}|^2 + N^2\,\lambda\,w_q^{2p}}$$

From equation (3.1) it follows that

$$A_L(\lambda) = \psi\,\text{diag}\,(Z_{q;\lambda})\,\psi^{H} \tag{3.3}$$

and so from equation (3.3) we have

$$V^{c}_{\text{approx}}(\lambda) = \frac{1/N[\Sigma_{q}\in_{I}(1-Z_{q;\lambda})^2\,|g_q|^2 + \Sigma_q\notin_{I}|\hat{g}_q|^2]}{1/N\,[N-L + (R_q\in_{I}(1-Z_{q;\lambda}))^2]} \tag{3.4}$$

We minimize $V^{c}_{\text{approx}}(\lambda)$ in (3.4) by making a linear search in λ. The function is not always continuous because the index set i, changes with λ. At each step we minimize $C(f_N\ ;\ \lambda)$ in (2.1) subject to non-negativity constraints using the NAG quadratic programming subroutine E04LBF, which yields the index set I for any given λ. When a minimizing value of λ is found, the corresponding $\mathbf{f}_\lambda$ is given by NAG subroutine E04LBF.

We conclude that the indicator set I obtained through the NAG quadratic programming subroutine E04LBF, plays a key role in the algorithm. It affects the filter function and ultimately affects the expression for $V_{ML}(\lambda)$ (where *ML* means Maximum Likelihood).

If I is the indicator set underlying the matrix E, i.e. the set of inactive constraint indices, and $V_{ML}(\lambda)$ in the constrained case may be approximated by

$$V^{ML}_{approx}(\lambda) = \tfrac{1}{2} N \log [\Sigma_{q\in I} (1 - Z_{q;\lambda}) \, |g_q|^2 + \Sigma_{q\notin I} |g_q|^2] - \Sigma_{q\in I} \log (1 - Z_{q;\lambda})$$

where L is the number of inactive constraints.

To minimize $V^{ML}_{approx}(\lambda)$, we used the NAG quadratic programming subroutine E04LBF for each k evaluation in the minimization process.

Since $V^{ML}_{approx}(\lambda)$ is not necessarily a continuous function of λ, we have made a linear search in order to find λ_{opt} corresponding to the least value of $V^{ML}_{approx}(\lambda)$ and noted the corresponding solution vector f_λ.

Note :

λ is the regularization parameter and λ_{opt} is the optimal value of k to be determined.

4. Convergence of regularized solution

Assume that the function $K(x,y)$ defined in (1.1) is continuous on $[0,1] \times [0,1]$. Consider the integral operator T with kernel $K(x,y)$ defined by

$$Tf(x) = \int_0^1 K(x, y) f(y) \, dy \qquad (4.1)$$

Has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ satisfying $\lambda\ ; \rightarrow 0$ with corresponding orthonormal eigenfunctions $\phi_i \in C[0,1]$. We also need the following quadrature assumption. Assume there exist μ and ν such that $0 \geq \mu \geq 1$ and $0 < \nu < 1 - (1/4p)$ and sequence $k_n \rightarrow 0$ such that for all f; $g \in H$ (Hilbert space).

$$\left| \int_0^1 fg - \frac{1}{n} \sum_{i=1}^{n} f(xi)\, g(xi) \right| \leq k_n \, \|f\|_\nu \, \|g\|_\nu \qquad (4.2)$$

where $H_o = L^2(0,1)$. It is clear that if $\mu > \nu$ then H_μ c H_ν and $\|.\|H_\mu$ is a stronger norm than $\|.\|H_\nu$ the following theorem is proved in (Lukas, [13]).

Theorem Statement : With assumptions (4.1) and (4.2). Let $f \in H_s$ where $s \geq \max(\nu, \mu)$ and $\mu < 2 - \nu - (1/2p)$. Suppose that $\lambda \to 0$ (regularization parameter) as $n \to \infty$ in such a way that $\lambda_{n,\lambda} \to 0$.

The proof of the above theorem begins by decomposing the expected mean square error into the bias squared plus the variance. Now f is the approximate solution and f_o is the true solution.

$$E \parallel f_{n,\lambda} - f_o \parallel_\mu^2 = E \parallel Ef_{n,\lambda} - f_0 + f_{n,\lambda} - Ef_{n,\lambda} \parallel_\mu^2$$

$$= \parallel Ef_{n,\lambda} - f_o \parallel_\mu^2 + E \parallel f_{n,\lambda} - Ef_{n,\lambda} \parallel_\mu^2$$

$$+ 2E\,(Ef_{n,\lambda} - f_o, f_{n,\lambda} - Ef_{n,\lambda})_\mu$$

$$= \parallel Ef_{n,\lambda} - f_o \parallel_\mu^2 + E \parallel f_{n,\lambda} - Ef_{n,\lambda} \parallel_\mu^2 = 0$$

Remark : The optimal convergence rate with particular reference to the special case of a convolution equation

$\int_0^1 k(x-y)\, f(y)\, dy\, g(x)$ in which the function k is periodic with period 1. (See Lukas [14] page 31-32).

5. Test Problems

Problem P(I) : This example has been taken from Turchin [19].

$$\int_{-3.2}^{3.1} k(x-y) f(y)\, dy = g(x)$$

where f is the sum of two Gaussian functions

$$f(x) = 0.5 \exp\left[-\frac{(x+0.4)^2}{0.18}\right] + \exp\left[-1\,\frac{(x-0.6)^2}{0.18}\right]$$

the esential support of $g(x)$ is $-2.5 < x < 2.7$, the problem is made highly ill-posed by choosing a wider kernal.

$$K(x) = \begin{cases} (5/12)\ (-x+1.2) & 0 \leq x < 1.2 \\ (5/12)\ (x+1.2) & -1.2 \leq x < 0 \\ 0 & |x| \geq 1.2 \end{cases}$$

the function f, g and K are displayed in diag (1) with a spacing 0.1 and n the no. of grid points is 64.

Problems P(2)

This example has been taken from Medgyessy [15]. The solution functin is the sum of six Gaussians and the kernel is also Gaussian.

$$\int_{-\infty}^{\infty} K(x\text{-}y)\, f(y)\, dy = g(x).$$

where $g(x) = \sum_{k=1}^{6} A_k \exp\left[-\frac{(x-B_k)^2}{C_k}\right]$

$A1 = 10$	$B1 = 0.5$	$C1 = 0.04$
$A2 = 10$	$B2 = 0.7$	$C2 = 0.02$
$A3 = 5$	$B3 = 0.875$	$C3 = 0.02$
$A4 = 10$	$B4 = 1.125$	$C4 = 0.04$
$A5 = 5$	$B5 = 1.325$	$C5 = 0.02$
$A6 = 5$	$B6 = 1.525$	$C6 = 0.02$

The essential support of $g(x)$ is $0 < x < 2$.

The kernel is

$$K(x) = \frac{1}{\sqrt{\pi\lambda}} \exp\left(-\frac{x^2}{\lambda}\right), \quad \lambda = 0.015$$

with essential support $(-0.26, 0.26)$.

The solution is

$$f(x) = \sum_{k=1}^{6} \left\{ \left(\frac{C_k}{C_k-\lambda}\right)^{1/2} A_k \exp\left[- \frac{(x-B_k)^2}{(C_k-\lambda)} \right]\right\}$$

with essential support (0.26, 1.74)

The functions f,g and K are displayed in diag (2). With a spacing 0.1 and n the number of grid points is 64.

6. Addition of random noise to the data functions

In solving the problems P(1) and P(2) which are highly ill-posed, even then we have demonstrated their solutions by considering the data functions contaminated by varying amounts of random noise. To generate sequences of random errors of the from $\{\in_n\}$ $n = 0, 1, \ldots N\text{-}1$ We have used to NAG Algorithm G05DAA which returns pseudo-random real numbers taken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental errors we have

$$A = 0$$

$$B = \frac{(x)}{100} \quad \max_{0 \leq n \leq N-1} |g_n| \qquad (6.1)$$

Where x denotes a chosen percentage, e.g. $x = 0.7, 1.7, 3.3$ etc.

Thus the random error $\in_n$ added to g_n does not exceed $3x\%$ of the maximum value of $g(x)$.

7. Numerical Results

In this section we describe the application of the two methods discussed in section 1 and section 3 over the same test problems enlisted in section 5.

Problems P(1)

WCV method : AL-Faour [2], considered this severely ill-posed problems when $a=1.2$, the values of S_α (cut-off frequency), when gs' are subject to quadrature error $x=0.77$ only, can be found in Table (1). With the best $f_Q(x)$ and $f_\beta(x)$ [Biraud's method] Al-Faour has discussed Biraud's method also and compared it with WCV method. The results are shown in Diag (3).

It is interesting to observe that WCV predicts the minimizer of S_Q reasonally well.

Maximum Likelihood (ML) Method

This highly ill-posed problem could not be satisfactorily treated using unconstrained regularization. For constrained regularization the results are enormously superior.

With 0.7% noise and .7 % noise the ML constrained method gave the solution to a very good accuracy and better than the WCV method as shown in Diags (4,5)

Problem P(2)

WCV method : AL-Faour says let us solve this problem when g(x) is given to maximum accuracy at $x = nh$, $n = 40 \mp 1, \ldots, \mp 128$, $h = 0.025$. $V(Q)$ in this case becomes almost flat at $32 \leq Q \leq 128$ which makes it difficult to predict the optimum cut-off point.

Notice that for $47 \leq Q \leq 53$ $f(x)$ is estimated very well by both f_Q (WCV method) and f_β (Biraud's method) i.e. for the clean data as shown in Table 2 and Diag (6).

For 1.7% noise, Table 3 shows that $Q=33$ gives the best f_Q while the best $f_\beta(x)$ is for $Q=32$ as shown in Diag (7).

Maximum Likelihood (ML) Method

For clean data ML-constrained method yielded a good solution as compared with WCV method resolving all the six peaks.

With 1.7% noise again ML-constrained method gave a good solution yielding 4 clear peaks, as shown in diag (6).

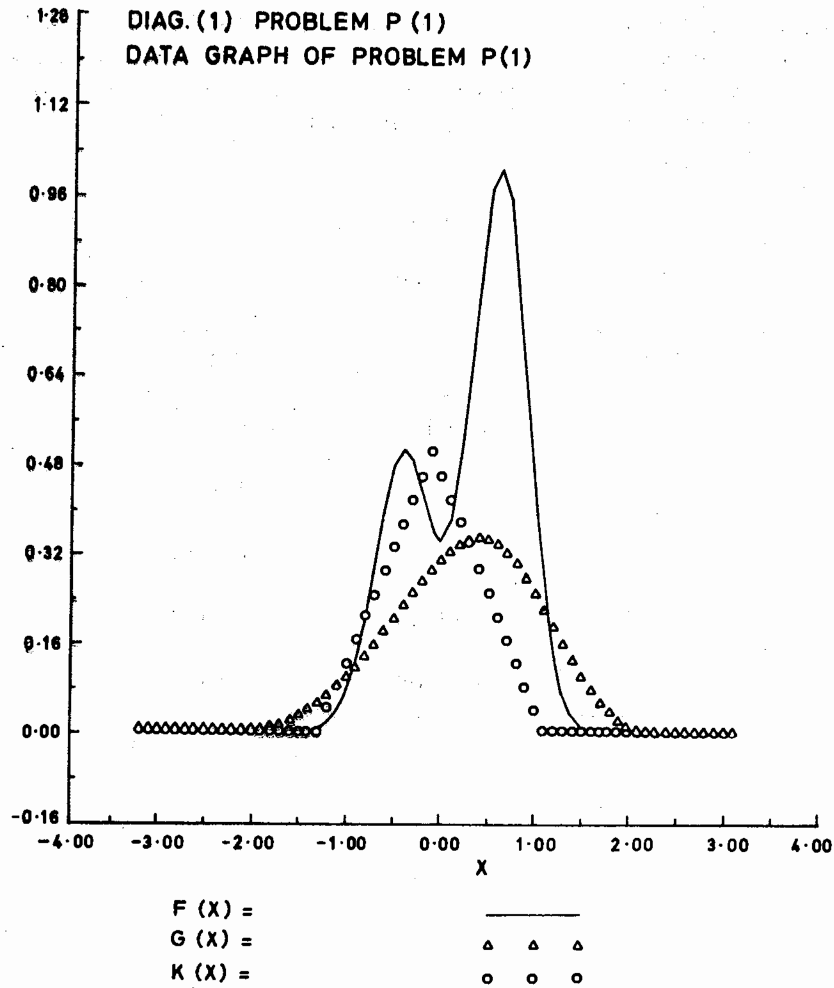
Conclusion

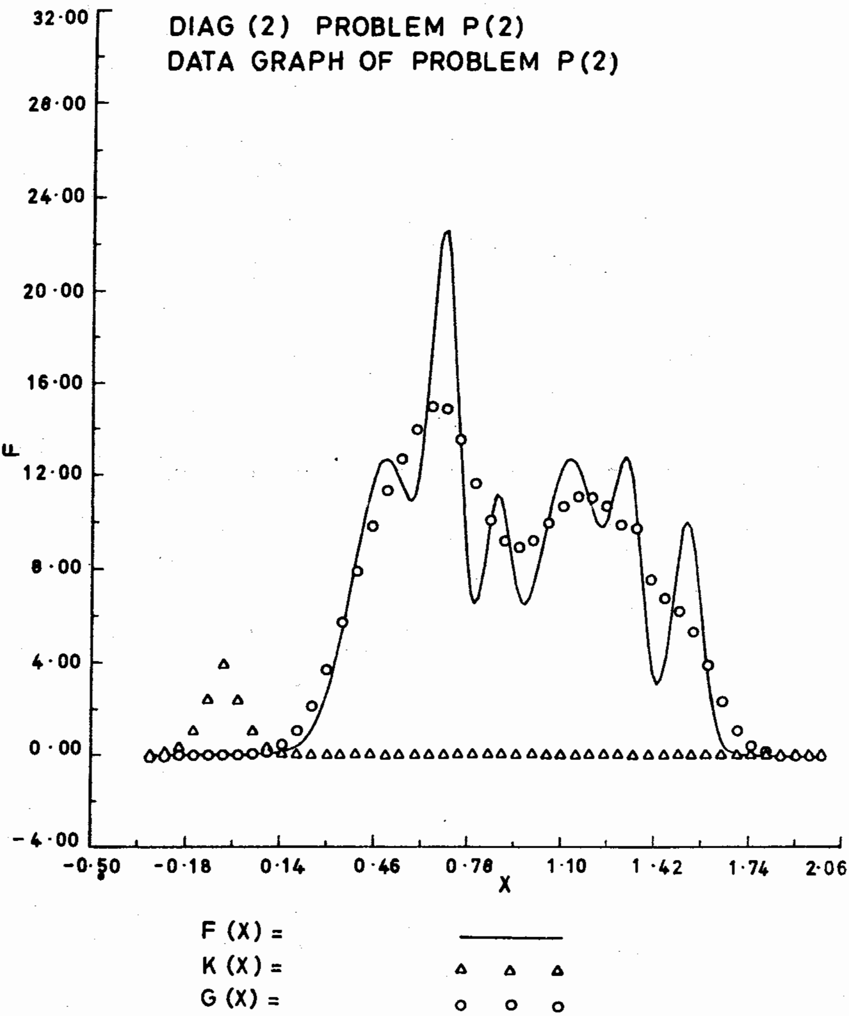
For mildly and moderately ill-posed problems and with low noise level, ML-constrained method is comparable with WCV and Biraud's methods. For highly ill-posed problems with low level noise ML-constrained compares well with WCV and Biraud's method but for higher level of noise ML-constrained is superior to WCV and Biraud's methods.

Table - 1

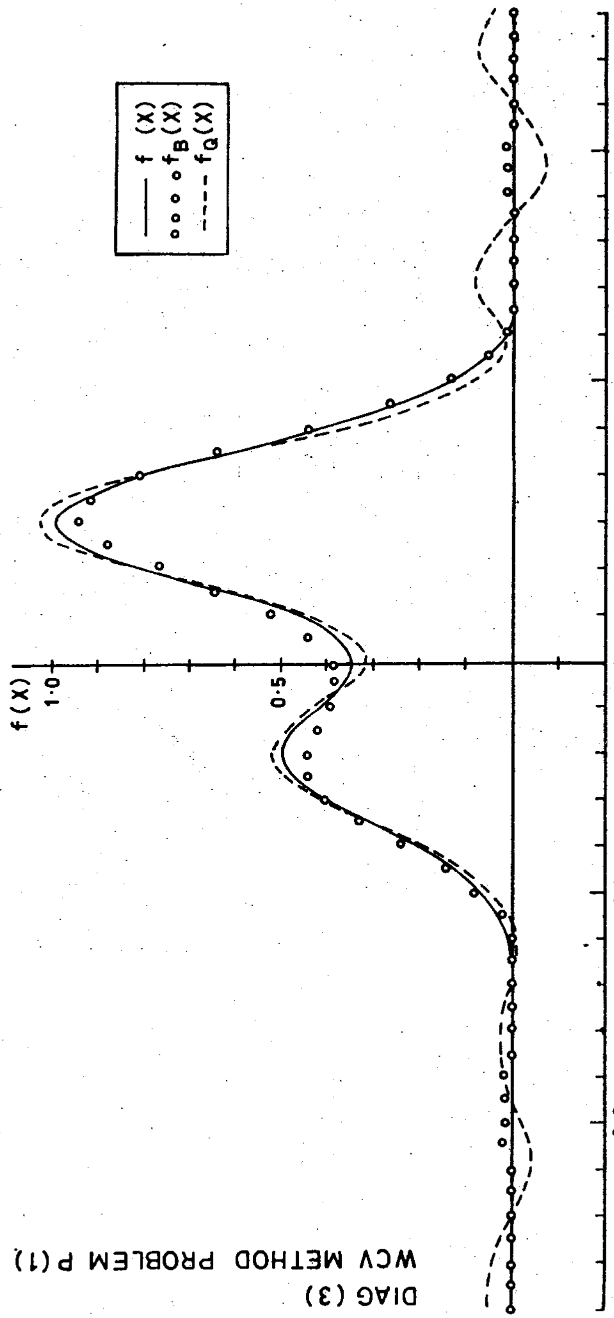
(WCV Method)

Q	SQ	SB
4	0.764732	0.553854
5	0.524304	0.795284
6	0.291213	0.142554
7	0.133506	0.052124
8	1.725080	1.431321

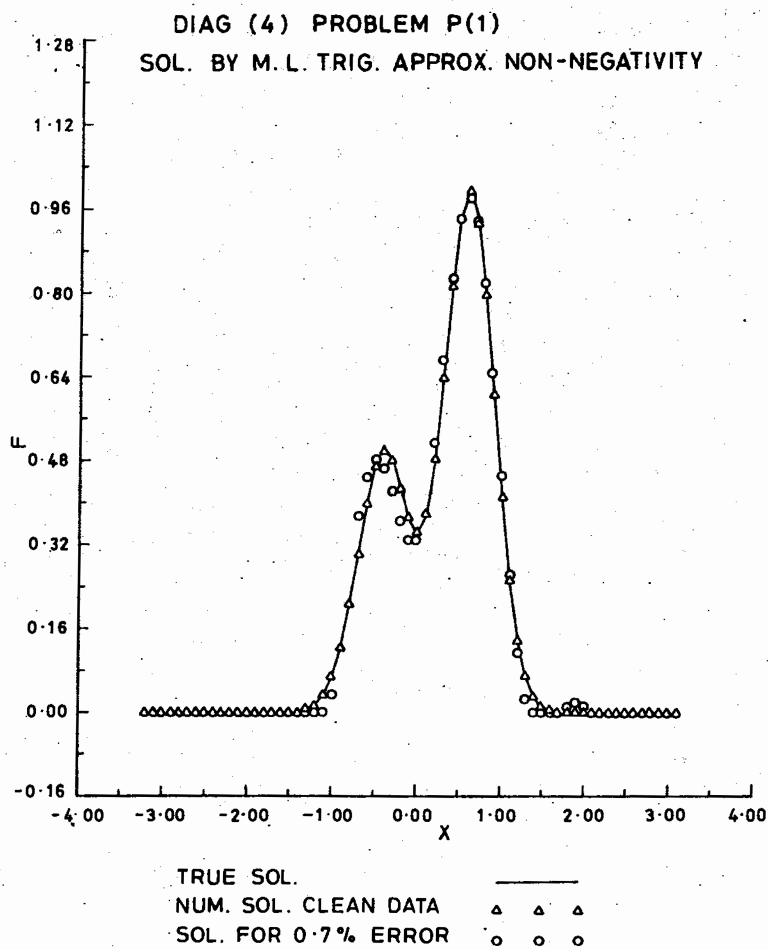

DIAG. (1) PROBLEM P (1)
DATA GRAPH OF PROBLEM P(1)
1·28
1·12
0·96
0·80
0·64
0·48
0·32
0·16
0·00
-0·16
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
F (X) =
G (X) =
K (X) =

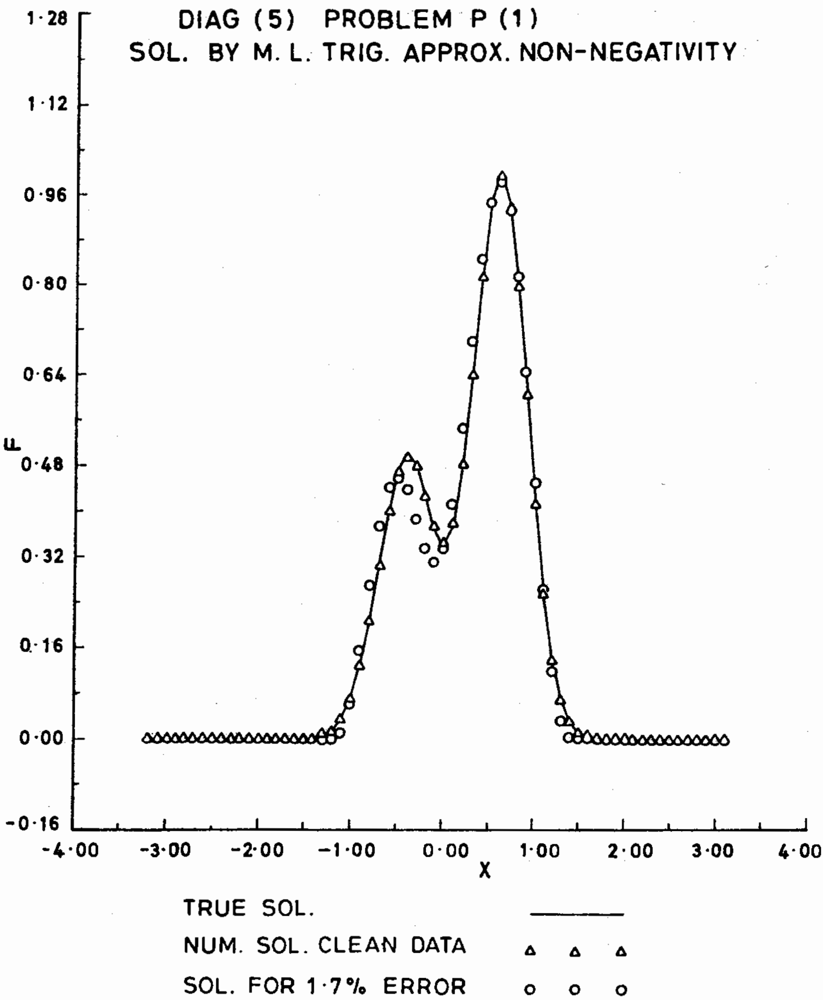


DIAG (2) PROBLEM P(2)
DATA GRAPH OF PROBLEM P(2)
32·00
28·00
24·00
20·00
16·00
12·00
8·00
4·00
0·00
-4·00
F
-0·50
-0·18
0·14
0·46
0·78
1·10
1·42
1·74
2·06
X
F (X) =
K (X) =
G (X) =

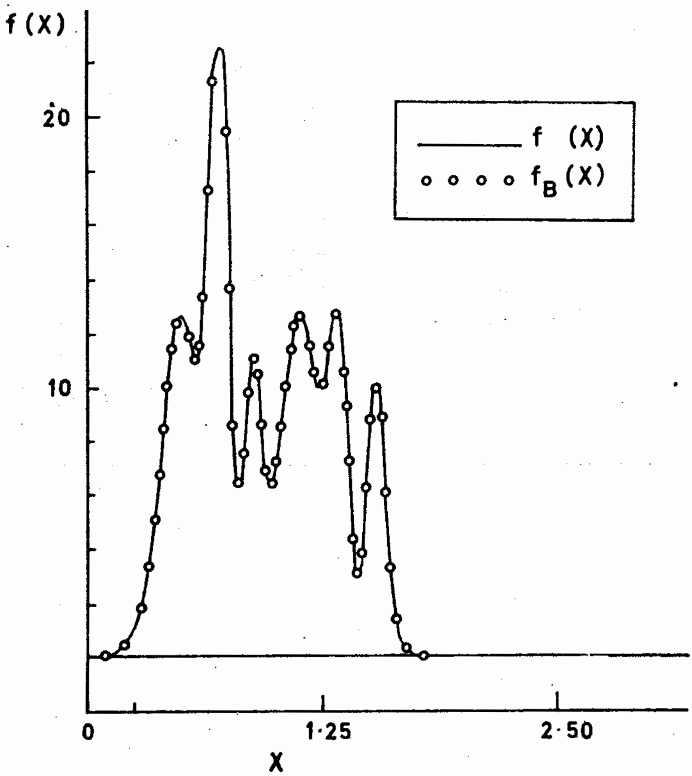


DIAG (3)
WCV METHOD PROBLEM P(1)
f(X)
1·0
0·5
0
x
-2·0
-1·0
1·0
2·0
f (X)
f_B(X)
f_Q(X)


DIAG (4) PROBLEM P(1)
SOL. BY M.L. TRIG. APPROX. NON-NEGATIVITY
1·28
1·12
0·96
0·80
0·64
0·48
0·32
0·16
0·00
-0·16
F
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 0·7% ERROR

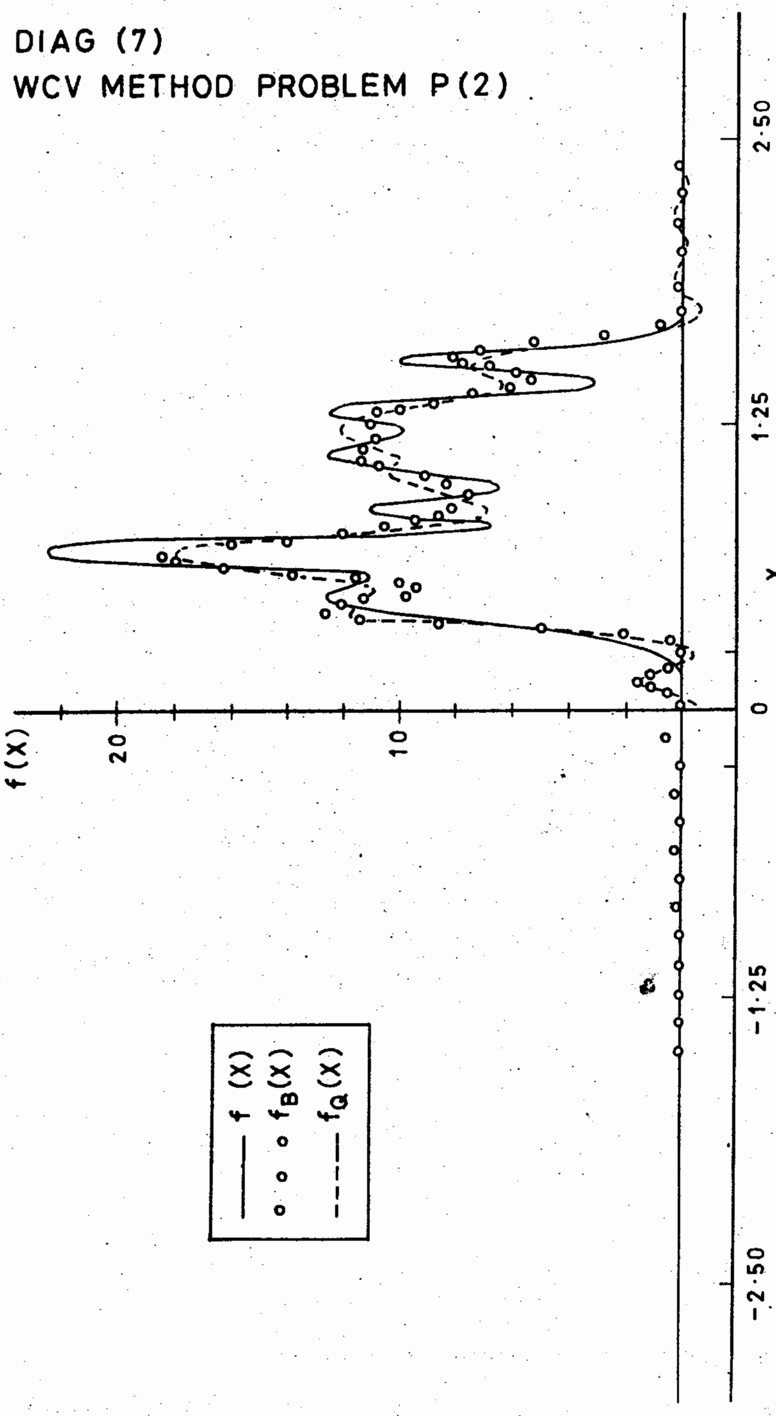

DIAG (5) PROBLEM P (1)
SOL. BY M. L. TRIG. APPROX. NON-NEGATIVITY
1·28
1·12
0·96
0·80
0·64
F
0·48
0·32
0·16
0·00
-0·16
-4·00
-3·00
-2·00
-1·00
0·00
1·00
2·00
3·00
4·00
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 1·7% ERROR

DIAG (6) PROBLEM P(2)
WCV METHOD PROBLEM P(2)



DIAG (7)

WCV METHOD PROBLEM P(2)



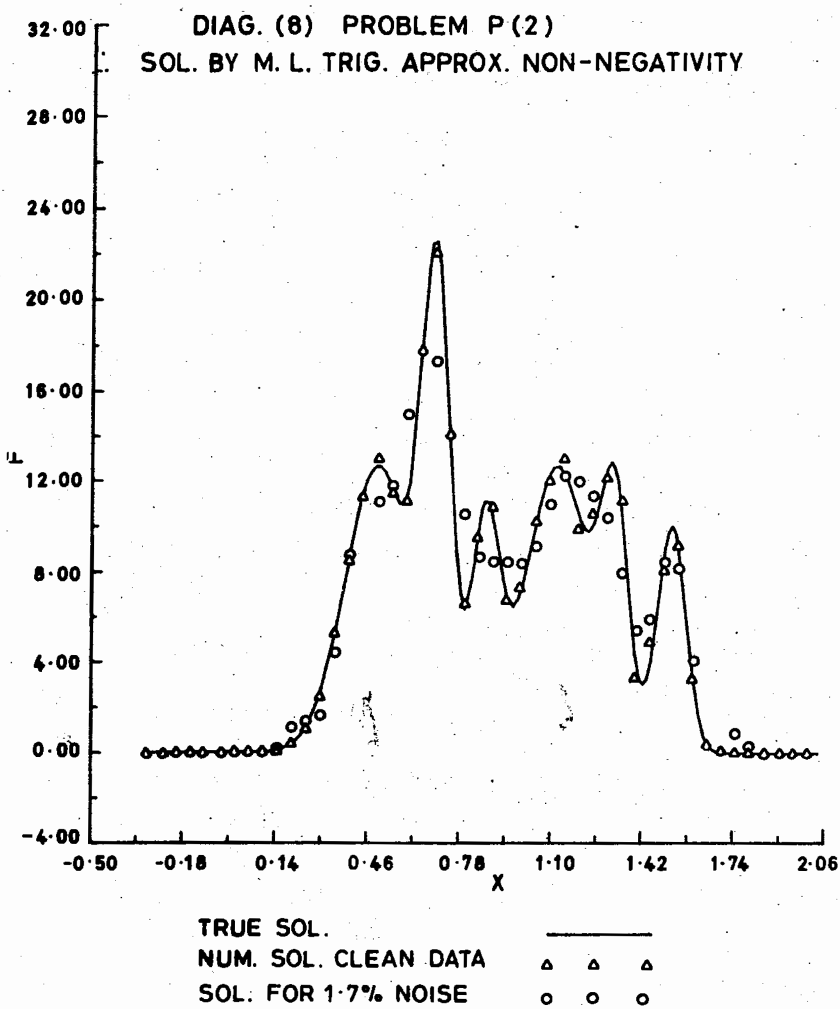

DIAG. (8) PROBLEM P(2)
SOL. BY M. L. TRIG. APPROX. NON-NEGATIVITY
32·00
28·00
24·00
20·00
16·00
12·00
8·00
4·00
0·00
-4·00
F
-0·50
-0·18
0·14
0·46
0·78
1·10
1·42
1·74
2·06
X
TRUE SOL.
NUM. SOL. CLEAN DATA
SOL. FOR 1·7% NOISE

Table - 2

(WCV Method)

Q	SQ	SB
47	1.390969	0.612892
48	0.924925	0.485427
51	0.530770	0.370364
52	0.471659	0.358874
53	0.397499	0.344118

Table - 3

(WCV Method) 1.7% error in q(x)

Q	SQ	SB
25	282.571376	241.672912
26	282.501506	226.835827
27	295.806563	222.624116
28	324.233600	268.658815
29	353.035351	349.395071

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VALUES OF CERTAIN MONOMIAL SYMMETRIC FUNCTIONS OF THE ROOTS OF UNITY

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Abstract

In this paper we find the values of the monomial symmetric functions (1).

The subject of the present paper is an old problem unsolved heretofore. Let $n(n\geq 1)$ and m be integers and let $\sigma_k(n,m)$ denote the monomial symmetric functions,

$$(1) \qquad \sigma_k(n,m) = \sum_{C} X_{j_1}^{m} X_{j_2}^{m} \ldots X_{j_k}^{m},\ k=1,\ldots,n,\ \sigma_o(n,m) = 1,$$

where C denotes that the sum is taken over all combinations $X_{j_1}, X_{j_2},\ldots,X_{j_k}$ of the roots,

$$(2) \qquad X_j = \epsilon^j, j=1,\ldots,n,\ \epsilon = exp\left(i\ \frac{2\pi}{n}\right),$$

of the equation,

$$(3) \qquad X^n - 1 = 0,$$

taken K at a time, Here our aim is the find the values $\sigma_k(n,m)$ of the symmetric functions (1). As it is well-known, the vlues $\sigma_k(n,m)=0$ if the degrees km of the functions (1) are not divided by the degree n of the equation (3). An open problem is to find the unknown values $\sigma_k(n,m)$ if km are divided by n (cf.

Comtet [1], *pp.* 158-159, No.9). The solution of this problem is contained in the following.

Theorem 1: If $n\,(n\geq 1)$ and m are integers, then we have the identity,

$$\prod_{s=1}^{n} (x - \epsilon^{ms}) = (x^{d/n} - 1)^{d},\ \epsilon = exp\left(i\ \frac{2\pi}{n}\right), \tag{4}$$

where

$$d = (n,\ |m|) \tag{5}$$

is the greatest common divisor of the numbers n and $|m|$.

Proof : Let n $(n\geq 2)$ and m $(1\leq m\leq n-1)$ be positive integers. Then for $j=m$ from (2) it follows that ϵ^{m} is a root of (3).

1. If n and m are relatively prime, *i.e.* $d=(n,m)=1$. then ϵ^{m} is a primitive root of (3). Hence, the numbers,

$$X_s^{1} = \epsilon^{ms},\ S = 1,\ \ldots,\ n, \tag{6}$$

represent the different n roots of (3), *i.e.* the numbers (6) are the same as the numbers (2) but written in another order. Therefore, we can write,

$$\prod_{s=1}^{n} (X - X_s^{1}) = \prod_{j=1}^{n} (X - X_j) = X^{n} - 1. \tag{7}$$

From (6) and (7) we obtain (4) - (5) if $d=(n,m)=1$ for the examined n and m.

2. If n and m are not relatively prime, *i.e.* $d=(n,m)>1$. than ϵ^{m} is not a primitive root of the equation,

$$X^{n/d} - 1 = 0, \tag{8}$$

Hence, the numbers

$$X_s^{11} = \epsilon^{ms},\ S = 1,\ \ldots,\ n/d, \tag{9}$$

represent the different n/d roots of (8). Now from (2)-(3) and (8)-(9) we get the equations,

(10) $$\prod_{s=1}^{n} (X - \epsilon^{ms}) = \prod_{\nu=0}^{d-1} \prod_{s=\nu n_1+1}^{\nu n_1+n_1} (X - \epsilon^{ms})$$

$$= \left\{ \prod_{s=1}^{n_1} (X - X_s^{(1)}) \right\}^d = (X^{n_1} - 1)^d,\ n_1 = \frac{n}{d}.$$

From (10) we obtain (4)-(5) if $d=(n,m)>1$ for the examined n and m.

II. Let n ($n \geq 1$) and m be arbitrary integers.

1. If $m \geq 0$, then we set,

(11) $$m = nq + r,\ q \geq 0,\ 0 \leq r \leq n-1,$$

where q is the quotient and r is the remainder from the divison of m by n, *i.e.*,

(12) $$d = (n, m) = (n, r),$$

According to (2), (11)-(12) an what has been proved in section *I*, we get the equations,

(13) $$\prod_{s=1}^{n} (X - \epsilon^{ms}) = \prod_{s=1}^{n} (X - \epsilon^{rs}) = (X^{n/d} - 1)^d.$$

which are evidently true for $r=0$ if we set $d=(n,nq)=(n,0)=n$.

From (11)-(13) we obtain (4)-(5) for the examined n and m.

2. If $m < 0$, then from (2) it follows that,

(14) $$\epsilon^{ms} = \epsilon^{|m|(n-s)},\ s=1,\ldots,n,\ \epsilon = exp\left(i\ \frac{2\pi}{n}\right)$$

According to (14) and what has been proved in section *II*. 1, we get the equations,

$$(15)\qquad \prod_{s=1}^{n} (X - \epsilon^{ms}) = \prod_{s=1}^{n} (X - \epsilon^{|m|s}) = (X^{n/d} - 1)^{d},$$

where $d=(n,|m|)$. From (15) we obtain (4)-(5) for the examined n and m.

This completes the proof of Theorem 1.

Theorem 2: If n $(n\geq 1)$, m and k $(0\leq k\leq n)$ are integers, then for the values of the functions (1) we have the formulas

$$(16)\qquad \sigma_k(n, m) = (-1)^{k-2} \binom{d}{s}$$

If K is of the form $K=ns/d$, and

$$(17)\qquad \sigma_k(n, m) = 0$$

If k is not of the form $K=ns/d$, where in both cases s $(0\leq s\leq d)$ is an integer and d is given by (5).

Remark : If $m=nq$, $q=0, \pm 1, \pm 2, \ldots$, *i.e.* $d=(n, n|q|)=n$. then the equations (17) are omitted.

Proof : From (1)-(2) and (4)-(5) we obtain the identity

$$(18)\qquad \sum_{k=0}^{n} (-1)^k \sigma_k(n,m) X^{n-k} = \sum_{s=0}^{d} (-1)^s \binom{d}{s} X^{n-n_1 s},\ n_1 = \frac{n}{d}$$

Equating the coefficients in (18) yields the equations (16) and (17).

This completes the proof of Theorem 2.

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MATRIX TRANSFORMATIONS OF X_p INTO C_s

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Abstract

The purpose of this paper is to characterize the matrices in the class (Xp, Cs).

1. Introduction

Let X and Y be any two non-empty subsets of the space of all sequences of complex numbers and let $A=(a_{nk})$, $(n, k=1,2,\ldots)$ be an infinite matrix of complex numbers. We write $Ax=(A_n(x))$ if $A_n(x) = \Sigma_k a_{nk} x_k$ converges for each n. (Throughout summation without limits runs from 1 to ∞). if $x=(x_k) \epsilon X$ implies that $Ax=(A_n(x)) \epsilon Y$, we say that A defines a matrix transofrmation from X into Y and we denote it by $A : X \to Y$. By (X,Y) we mean the class of matrices A such that $A : X \to Y$. If in X and Y there is some notion of limit or sum, then we write (X,Y,P) to denote the subset of (X,Y) which preserves the limit or sum.

Let X be the set of all real sequences $x=(x_k)$. A functional f from X into the non-negative extended real number system is called a semi-norm if for every x and y in X,

(i) $f(0) = 0$

(ii) $f(\alpha x) = |\alpha| f(x)$, for every real number α

(iii) $f(x+y) \le f(x) + f(y)$.

In stead of (i) f satisfies the condition that $f(0)=0$ if and only if $x=0$, then f is called a norm.

We denote by X_f the collection of all sequences x satisfying $f(x) < \infty$. Obviously $_fX$ is a linear space, and we called $_fX$ a normed *k*öthe sequence space of nonabsolute type with the semi-norm f. If X_f is complete with respect to the norm f, then X_f is called a Banach sequence space of nonabsolute type, since we did not assume the property $f(x) = f(|x|)$ where $|x| = (|x_k|)$.

From now on, we shall always assume that X_f is a Banach sequence space of a nonabsolute type. Given a semi-norm f, we define a new semi-norm f' as follows;

$$f'(x) = \sup \{ \, | \sum_k X_k y_k | \; ; \quad f(y) \le 1 \, \}$$

and put $f'(x) = \infty$ if the series $\Sigma_k x_k y_k$ is divergent for some y satisfying $f(y) \le 1$. The semi-norm f' is called the associate semi-norm of f. The space $X_{f'}$ consisting of all sequences x $\in X$ with $f'(x) < \infty$ is called the associate space of X_f. For any $x \in X_f$ and any y $\in X_f'$ we always have,

$$| \sum_k x_k y_k | \le \quad f(x) \cdot f'(y).$$

A semi-norm f is said to be saturated, if for every non-empty subset E of positive integers, there exists a nonempty subset F of E such tha $f(x_F) < \infty$, where the sequence $x_F = (x_k)$ is defined as $k=1$ if $k \in F$ and $x_k=0$ if $k \notin F$. It is easy to see that f is saturated if and only if X_f contains all finite sequences, *i.e.*, all sequences having only finitely many non-zero terms. The following is a consequence of the Banch-Steinhaus Theorem.

Theorem 1: Let f' be saturated and $y \in X$. Then $y \in X_{f'}$ if and only if the series $\Sigma_k x_k y_k$ is convergent for every $x \in X_f$.

In [2] Ng and Lee have introduced the Cesaro sequence spaces of nonabsolute type as follows ;

$$X_p = \{x : \| x \|_p = (\sum_n | \frac{1}{n} \sum_{k=1}^{n} x_k |^p)^{1/p} < \infty \text{ for } 1 \leq p < \infty$$

and

$$X_\infty = \{x : \| x \|_\infty = \sup \{| \frac{1}{n} \sum_{k=1}^{n} x_k |^p), n=1,2,3,...\}$$

Note that the above norms are saturated except for $p=1$. It is easy to prove that $X_p(1 \leq p \leq \infty)$ are Banach sequence spaces of a nonabsolute type.

Let Y_q be the space of all $y \in X$ such that,

(1) $|ky_k| \leq M$ for all $k=1, 2, 3, \ldots$.

(2) $\lambda_q(y) = (\Sigma_k | k (y_k - y_{k+1}) |^q)^{1/q} < \infty$, for $1 \leq q < \infty$

and $\lambda_\infty(y) = \sup \{|k(y_k - y_{k+1})| ; \quad k = 1, 2, 3, \ldots .\} < \infty$.

Theorem 2: (see, Ng and Lee [2]). The associate space $X_{p'}$ of X_p $(1 \leq p \leq \infty)$ is the space Y_q with the norm λq, where $p^{-1} + q^{-1} = 1$.

We define (see, Stieglitz and Tietz [5])

$$C_s = \{x : \{ \sum_{i=1}^{n} x_i \} \text{ is convergent}\}$$

2. Matrix Transformations

The following notation is used throughout. For all integers $n \geq 1$, we write

$$t_n(Ax) = \sum_{i=1}^{n} A_i(x) = \Sigma_k b_{nk} x_k$$

where

$$b_{nk} = \sum_{i=1}^{n} a_{ik}.$$

The following result due to Zeller ([3] p.29).

Lemma 1: A matrix A transforms a BK-space E into a BK-space F then the transformation is linear continuous.

Theorem 3: A matrix transformation $A=(a_{nk})$ maps the space X_p $(1 \leq p < \infty)$ into the space C_s if and only if

(1) $\sup_n \| \{k(b_{nk} - b_{nk+1})\}_{k\geq 1} \| < \infty,$

(2) $\sup_{k\geq 1} | k\, b_{nk} | < \infty$ for every fixed n,

(3) $\lim_n k\,(b_{nk} - b_{nk+1}) = \delta_k$ for every fixed k,

where $p^{-1} + q^{-1} = 1$.

Proof : Necessity : Let $A \in (X_p, C_s)$. Then the series

$$t_n(Ax) = \sum_k b_{nk}\, x_k$$

is convergent for every $x \in X_p$. Then by Theorem 1 and sequence $(b_{nk})_{k\geq 1}$ is an element in Y_q for every n, it follows that the condition (2) holds and

$$\| \{k(b_{nk} - b_{nk+1})\}_{k\geq 1} \| < \infty.$$

Since X_p and C_s are BK-spaces by Lemma 1 $\| t_n(Ax) \| \leq k \| x \|_p$ for some real constant K and all $x \in X_p$ or

$$\sup_{n\geq 1} |t_n(Ax)| \leq k \| s \|$$

for all $x \in X_p$ with $s=(s_k)$ where $s_k = \frac{1}{k} \sum_{i=1}^{k} x_i$. It follows that

$$\sup_{n\geq 1} \left| \frac{\sum_k k(b_{nk}-b_{nk+1})\, s_k}{\| s \|} \right| \leq K.$$

Hence we have

$$\sup_n \| \{k(b_{nk} - b_{nk+1})\}_{k=1} \| \leq K.$$

Then the condition (1) holds.

To prove that condition (3) is necessary, we take for each fixed k a sequence $x^{(k)}$ in X_p with,

$$x_j^{(k)} = \begin{cases} k & \text{if } j = k \\ -k & \text{if } j = k+1 \\ 0 & \text{if } j \neq k, k+1. \end{cases}$$

Then we see that

$$s_k = \frac{1}{k} \sum_{j=1}^{k} x_j^{(k)} = 1.$$

and

$$s_j = 0 \qquad \text{if } j \neq k$$

For this sequence $x^{(k)}$ we have,

$$t_n(Ax^{(k)})_n = \sum_j b_{nj} x_j^{(k)}$$

$$= \sum_j j(b_{nj} - b_{nj+1}) s_j$$

$$= k(b_{nk} - b_{nk+1}) \rightarrow \delta_k \quad \text{as} \quad n \rightarrow \infty.$$

Therefore (3) holds.

Sufficiency : Suppose that the conditions hold. Then by conditions (1) and (2) the series,

$$t_n(Ax) = \sum_k b_{nk} x_k$$

is convergent for every $x \in X_p$. By condition (3) we have

$$|k(b_{nk} - b_{nk+1})|^q \to |\delta_k|^q \text{ as } n \to \infty$$

and since for every positive integer m

$$\left(\sum_{k=1}^{m} |k(b_{nk} - b_{nk+1})|^q\right)^{1/q} \leq \sup_n \Sigma_k |k(b_{nk}-b_{nk+1})|^q)^{1/q} = \beta.$$

by lettng n $\to \infty$, we get

$$\left(\sum_{k=1}^{m} |\delta_k|^q\right)^{1/q} \leq \sup_n (\Sigma_k |k(b_{nk} - b_{nk+1})|^q)^{1/q}.$$

Since this is true for every positive integer m, it follows that

$$\left(\sum_{k} |\delta_k|^q\right)^{1/q} < \infty$$

Now for every sequence $x \in X_p$, we have

$$S_n = \frac{1}{n} \sum_{k=1}^{n} x_k \to 0 \quad \text{as } n \to \infty.$$

Given any $\in > 0$, there exists $N > 0$ such that

$$\left(\sum_{k=N}^{\infty} |s_k|^p\right)^{1/p} < \in/4\beta.$$

And by condition (3), there exists integer N' such that

$$\left| \sum_{k=1}^{N} \{k(b_{nk} - b_{nk+1}) - \delta_k\} s_k \right| < \varepsilon/2$$

for all $n \geq N'$. Now for all $n \geq N'$

$$\left| \sum_{k=1}^{\infty} \{k(b_{nk} - b_{nk+1}) - \delta_k\} s_k \right|$$

$$\leq \left| \sum_{k=1}^{N} \{k(b_{nk} - b_{nk+1}) - \delta_k\} s_k \right|$$

$$+ \left| \sum_{k=N+1}^{\infty} \{k(b_{nk} - b_{nk+1}) - \delta_k\} s_k \right|$$

$$< \varepsilon/2 + \Big(\sum_{k=N+1}^{\infty} | \{k(b_{nk} - b_{nk+1}) | + | \delta_k | \}^q \Big)^{1/q}$$

$$\Big(\sum_{k=N+1}^{\infty} | s_k |^p \Big)^{1/p}$$

$$< \varepsilon/2 + 2\beta \frac{\varepsilon}{4\beta} = \varepsilon.$$

So we have

$$\lim_n \sum_k k(b_{nk} - b_{nk+1}) s_k = \sum_k \delta_k s_k$$

It follows that

$$\lim_n t_n(Ax) = \lim_n \sum_k b_{nx} x_k$$

$$= \lim_n \sum_k k(b_{nk} - b_{nk+1}) s_k$$

$$= \sum_k \delta_k s_k$$

This show that $Ax \in C_s$, and $A=(a_{nk})$ maps X_p into C_s. And this completes the proof.

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PRINCIPLE OF EQUICONTINUITY FOR TOPOLOGICAL GROUPS

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Abstract

In this nite we obtain a version of the principle of equicontinuity for topological groups by making use of Baire's category theorem. Our result constitutes a generalization of the Vitali-Hahn-Saks theorem for group-valued measures due to Drewnowski. A uniform boundednes principle type theorem is also established for uniform topological smigroups.

Preliminaries

Through out this note G will denote a commutative Hausdorff topological group and R a ring of subsets of a set X.

A set function $\eta : R \rightarrow [0. \infty]$ is called a submesure if,

(i) $\eta(\phi) = 0,$

(ii) $\eta(EUF) \leq \eta(E) + \eta(F)$ with $E \cap F = \phi$

(iii) $E \subseteq F$ implies $\eta(E) \leq \eta(F)$ where $E,F \in R$.

A submeasure η on R is said to be complete if for each sequence $\{E_n\}$ in R with $\eta(E_n \Delta E_m) \rightarrow o$ as $n,m \rightarrow \infty$ (Δ stands for symmetric difference), there exists $E \in R$ such that the ring R is complete with respect to η.

A real-valued function q on G is said to be quasi-norm of G if (*a*) $q(x) \geq o$ (*b*) $q(O) = o$ (*c*) $q(x) = q(-x)$ (*d*) $q(x+y) \leq q(x) + q(y)$, $(xy \epsilon G)$. If q is a quasi-nom on G then we say that (G,q) is a quasi-normed group. Let u be a finite sumeasure on R. The triplet (R, Δ, η) is aquasi-normed topological goup. It is well known that if (G,Γ) is a topological group then Γ is determined by the family of -continuous quasi-norms on G. However in our main result we shall use meighbourhood apporach for topological groups as given in {4}.

2. PRINCIPLE OF EQUICONTINITY

Drewnowski [1] has indicated that the following result holds.

Theorem 2.1

Suppose that R is complete with respect to a submeasure η, $\{\mu_n\}$ is a sequence of η-continuous additive set functions on R to G. If, for each $E \epsilon R$, $\lim \mu_n(F) = \mu(E)$ exists then the *st* function μ is additive and η-continuous and the set functions $\mu_n(n \epsilon N)$ are equi-η-continuous.

Example 2.2

A ring R complete with respct to a submasure on it is metrically complete and so a space of second category by Barie's theorem.

Remark 3.2

There exist Baire spaces or spaces of the second category which are not metrizable and so these spaces can not be metrically complete.

We now generalize theorem 2.1 to the case in which the domain of the family of functions is a group of second category. We shall use the techique as given in ([3], theorem 2,*p* 158) to get the following "principle" of equicontinuity" for topological groups.

Theorem 2.4

Let (G,Γ) be of second category and (H,λ) a Hausdorff commutative topoloigcal group. Suppose that $\{\mu_n\}$ is a sequence of continuous homomorphisms from G to H. If for each $x \in G$, $\lim_{n\to\infty} \mu_n(x) = \mu(x)$ exists then μ is a continuous homomorphism and $\{u_n\}$ is sequence of equi-Γ-continuous homomorphisms.

Proof

Suppose that thetopology Γ is generated by a family $\{n_i:i\epsilon I\}$ of continuous quasi-norms on G. Since $\{\mu_n$ is a sequence of continuous functions therefore for each Γ-closed neighbourdood U of o in H the sets

$$G_{n,m} = \{x\epsilon\, H: \mu_n(x) - \mu_m(x) \in U\},\ n = 1,2,\ldots$$

are Γ-closd sets as the inverse image of a closed set under a continuous map is a closed set.

Put $G_p = \cap_{n,m\geq p} G_{n,m}$ $(P=1,2,3\ldots)$. Each G_p is a Γ-closed set

being the countable intersection of Γ-closed sets. It can easily be verified that $G = U^{\infty}{}_{p=1}\, G_p$. Thus G can be represented as a countable union of Γ-closed sets. As G is of second category with respect to the family $\{n_1:i\epsilon I\}$ of quasi-norms on G so at least one of the sets G_p must have non-empty interior. Thus there exists an integer q, a positive number r, a point a $\in G$ and $i_o \in I$ such that,

$$\mu_n(x) - \mu_m(x) \in U \text{ for } n,m \geq q \text{ and } x \in K,$$

where $K = \{x \in G : n_{i_o}[x-a] < r\} \subseteq Gq$. That is

$$(\mu_n - \mu_m)(x) \in U$$

provided $n,m \geq q$ and $n_{i_o}(x-a) < r$ (*)

By continuity of μ_q, given a neighbourhood U of o in H, there exists a finite set ni_q $(1\leq i\leq p)$ and $\delta i_v > o$ $(1\leq i\leq p)$ such that $\mu_q(b) \in U$ whenever $b \in \cap^{P}{}_{i=1}\, \{x \in G: n_{i_q}(b) < \delta_{i_q}\}$. Since the arguments are similar, we shall consider, in the sequel, x in any of the following two equivalent Γ-neighbourhoods in the sequel, x in any of the following two equivalent Γ-neighbourhoods of o in G given by,

$$M = \{x \in G: n_{i_q}(x) < \delta_{i_q}\} \cap \{x \in G: n_{i_o}(x) < r\}$$

$$L = \{x \in G: \tilde{n}_q(x) < \tilde{\delta}\} \text{ where } \tilde{n}_q \text{ and } \tilde{\delta} \text{ are defined by,}$$

$$\tilde{n}_q = min\,(n_{i_o}, n_{i_q}), \; \tilde{\delta} = min\,(r, \delta_{i_q}).$$

If $x \in L$ $(=M)$ then $n_{i_o}(x+a-a) = n_{i_o}(x) < r$ and so $(x+a) \in K$

Following Husain ([4], *P*.46) if V is any k-closed neighbourhood of o in H, then there exists a closed, symmetric neighbourhood U of o in H such that $U+U+U \subseteq V$.

We have by (*), $\mu_n(x) - \mu_q(x) \in U$ *fr* $n \geq q$ and $x \in K$. By hypothesis $\mu(x) = lim_{n\to\infty} \mu_n(x) \in U$ *so* $\mu(x) - \mu_q(x) \in U$ provided $x \in K$. This gives with x = a, $x + a$; $\mu(a) - \mu_q(a) \in U$ and $\mu(x+a) - \mu_q(x+a) \in U$. Now $\mu(x) = [\mu(x) + \mu(a) - \mu_q(x) - \mu_q(a) - \mu(a) + \mu_q(a)] + \mu_q(x)]$

$$=[\mu(x+a) - \mu_q(x+a)] - [\mu(a) - \mu_q(a)] + \mu_q(x) \in U - U + U \subseteq V \qquad \text{(By(**))}$$

Thus $\mu(x) \in V$ whenever X belongs to the neighbourhood M of o in G and so μ is Γ-continuous as required.

The function μ is a homomorphism because μ_n, $n = 1, 2, 3, \ldots$ are homomorphisms and addition is continuous in G.

Next, we prove that $\{\mu_n : n \in N\}$ is equi-Γ-continuous.

Clearly $\mu_n(x) = \mu_n(x) - \mu(x) + \mu(x) \in U + U \subseteq V$. Moreover each μ_i, $1 \leq i \leq q-1$ is Γ-continuous therefore there exists an Mi(a Γ-neighbourhood of o in G) such that $\mu_i(x) \in V$ whenever $x \in M_i$.

Taking $\tilde{M} = M_1 \cap M_2 \cap \ldots \cap M_{q-1} \cap M$, we have

$$\mu_n(x) \in V \text{ for } all\ n = 1,2,\ldots \text{ and } x \in \tilde{M}.$$

Hence $\{\mu_n : n \epsilon N\}$ is equi-Γ-continuous as desired.

Remark : 2.5

A vrsion of uniform boundedness principle for topoological groups has been obtained by khan and Rowlands [6] without the use of Baire's theorem.

In sequel S denotes a Hausdorff uniform semigroup (*cf.* [2]).

Definition 2.6:

An S-valued additive set function μ on R is said to be exhaustive if and only if, for every sequence $\{E_n\}$ of pairwise disjoint sets in R, $lim_{n\to\infty}\ \mu(E_n) = 0$.

For completeness sake we quote the following theorems.

Theorem 2.7 ([5], theorem 6(i)) :

Let $(\mu_i : i \epsilon I\}$ be a collection of exhaustive G-valued measures on a σ-algebra A which are pointwise bounded. Then the family $\{\mu_i : i \epsilon I\}$ is uniformly bounded.

Theorem 2.8 (Drewnowski [2]) :

Let $\mathcal{U}$ be a Hausdorff uniformity on S. Then the following are equivalent :

(i) There exists a commutativ Hausdorff topological group (G,Γ) and a map h : $S \to G$ such that h is an isomorphism of S into $h(S)$ and $h : (S,\mathcal{U}) \to (h(S),\Gamma|\ h(S))$ is a homeomorphism.

(ii) For ach $U \epsilon \mathcal{U}$ there exists a $V \epsilon \mathcal{U}$ such that if,

$$(x+z,\ y+z)\ \epsilon\ V \text{ then } (x,y)\ \epsilon\ U\ (x,y,z\ \epsilon\ S).$$

We now generalize theorem 2.7 due to Kalton for uniform semigroups.

Theorem 2.9 :

Suppose S satisfies the condition (ii) of theorem 2.8. If $\{\mu_i : i \in I\}$ is a collection of exhaustive and pointwise bounded S-valued functions on A, then $\{\mu_i, i \in I\}$ is uniformly bounded.

Proof :

By theorem 2.8, S can be embedded isomorphically and homeomorphically in a toplogical group (G,Γ); that is, there exists and isomorphism $h: S \rightarrow G$ such that $h:(S\ \Gamma_U) \rightarrow (h(S),\ \Gamma|h(S))$ is a homeomorphism. Now $\{\mu_i : i \in I\}$ is exhaustive and pointwise bounded and h is continuous so $\{ho\mu_i : i \in I\}$ is exhaustive and pointwise bounded. Now by theorem 2.7 $\{ho\mu_i : i \in I\}$ is uniformly bounded. Since h^{-1} is continuous so $\{\mu_i : i \in I\}$ is uniformly bounded as required.

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CHARACTERIZATIONS OF BCI-ALGEBRAS WITH WEAK UNIT

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Abstract

In this paper, we define weak left-self maps in *BCI*-algebras and characterize *BCI*-algebras with weak unit in terms of these self-maps.

1. Introduction

In [4], [5] and [6], the concepts of left self-map, right self-map and left regular self-maps have been introduced with some of their properties studied. In [10], *BCI*-algebras with weak unit have been investigated. In this paper, we define weak left self-maps in *BCI*-algebras and characterize *BCI*-algebras with weak unit in the language of these maps.

2. Preliminaries

A BCI-algebra X is an algebra $(X,*,0)$ of type (2,0) with following conditions for $x,y,z \in X$:

(1) $(x*y)*(x*z) \leq z*y$

(2) $x*(x*y) \leq y$

(3) $x \leq x$

(4) $x \leq y$, $y \leq x$ imply $x=y$

(5) $x \leq 0$ implies $x=0$

where $x \leq y$ *iff* $x*y=0$. If (5) is replaced by $0 \leq x$ for all $x \in X$, then X is known as *BCK*-alebra ([9]). Let X be a BCI-algebra and $M = \{x \in X: 0*x=0\} \subseteq X$; Then M is known as *BCK*-part of X.

(6) For $x \in M$, $y \in X-M$, $x*y$, $y \times x \in X-M$,

(7) $(x*y)*z = (x*z)*y$,

(8) $x*0 = x$,

(9) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$ for $x,y,z \in X$ ([9]).

Definition 1 [1] : Let X be a *BCI*-algebra. Let $x_o \in X$ be an element such that for $y \in X$ with $y * x_o = 0$ implies $y = x_o$ and define

$$A(x_o) = \{x \in X : x_o * x = 0\}$$

The point x_o is known as the initial element of $A(x_o)$; Let 1 denote the set of all initial elements in X. We call it the centre of X.

(10) Let x be a *BCI*-algebra with I as its centre. then, for $x_o \neq y_o$, x_o, $y_o \in 1$, $A(x_o) \cap A(y_o) = \phi$. and $\bigcup_{x_o \in I} A(x_o) = X$. ([1]).

(11) Let X be a *BCI*-algebra with 1 as its centre. Then I is a p-semisimple *BCI*-algebra ([2]).

(12) Let X be a *BCI*-algebra with I and M as its centre and *BCK*-part respectively. Let $x_o \in I$. The, x,y, $A(x_o)$ imply $x*y$, $y*x \in M$. ([1]).

(13) Let X be a *BCI*-algebra with I as its centre. Let $\dot{x}_o \in$I. For $x,y \in A(x_o)$, $o*x = o*y$ ([2]).

(14) Let X be a *BCI*-algebra with I as its centre. Let $o \neq x_o$, $o \neq y_o$, x_o, $y_o \in I$. If $x \in A(x_o)$, $o*x \in A(y_o)$, then $o*x = y_o \in A(y_o)$ ([12]).

(15) Let X be a *BCI*-algebra. Then following are equivalent.

(i) X is p-semismple

(ii) $x*(o*y) = y*(o*x)$

(iii) $x*y = 0$ imply $x=y$

(iv) $x*y = o*(y*x)$

for all $x,y \in X$ ([7], [10], [11]).

Definition 2 [10] : A *BCI*-algebra X is called a *BCI*-algera with weak unit if $o*x \leq x$ for all $x \in X$.

Definition 3 [1] : A BCI-algebra X is called associative if $o*x=x$ for all $x \in X$.

Lemma 1 : Let X be a *BCI*-algebra with I as its centre. X is a *BCI*-algebra with weak unit if and ony if I is associative *BCI*-algebra.

Proof : Let X be a *BCI*-algebra with weak unit. Let $x_o \in$ I. Since $o*x_o \leq x_o$, it follows that $o*x_o = x_o$. Thus I is associative.

Conversely, suppose that I is an asociative *BCI*-algebra. Let $y \in X$. Then y is contained in some $A(y_o)$ for $y_o \in I$ and $y_o \leq$ y. By (9) $o*y \leq o*y_o = y_o$ or $o*y \leq y_o \leq y$ implies $o*y \leq y$. Hence X is a *BCI*-algebra with weak unit. This completes the proof.

In [4], we defined left self-map in *BCK*-algebras, which is denoted by $l_x : X \rightarrow X$, for some fixed $x \in X$ and is given by $l_x(t) = x*t$, for all $t \in X$. We adopt the same definition for BCI-algebras. However, we generalize this concept to weak left self-map in *BCI*-algebras and shall note that left-self map and weak left self-maps coincide in *BCK*-algebras.

Definition 4: Let X be a *BCI*-algebras. For a fixed $x \in X$, the map $L_x : X \rightarrow X$ given by $L_x(t) = (x*t)*(o*x)$, for all $t \in X$, is called a weak left self-map.

Remakrs : Let X be a BCK-algebra, then for $x \in X$, $L_x : X \to X$ is a weak left self-map if and only if $L_x(t) = (x*t)*(o*x) = (x*t)*o = x*t = l_x$, thus left self-maps and weak left self-maps coincide in BCK-algebras.

Theorem 1: Let X be a BCI-algebra with weak unit and M as its BCK-part. Then followings hold :

(1) $L_x(t) \in M$, for $I \in M$.

(2) $L_x(t) \in M$ for

Proof : Let X be a BCI-algebra with weak unit and I as its centre. By lemma 1, $o*x_o = x_o$, for all $x_o \in I$. Let $t \in M$. For any fixed $x \in X$, $Lx(t) = (x*t)*(o*x)$. By (10), x is contained in some unique $A(x_o) \subseteq X$, for $x_o \in I$. Now $x_o \leq x$ implies $o*x \leq o*x_o = x_o$ or $o*x = x_o$, because $x_o \in I$ and we canl write $L_x(t) = (x*t)*x_o = (x*x_o)*t$. By (12), x, $x_o \in A(x_o)$. Imply $x*x_o \in M$. Thus $x*x_o \in M$ imply $(x*x_o)*t \in M$, because M is closed under *. Hence $L_x(t) \in M$, for $t \in M$.

(2) Let $t \in X-M$. Then $Lx(t) = (x*t)*(o*x) = (x*t)*x_o = (x*x_o)*t$. Since, $x*x_o \in M$; therefore by (6), $(x*x_o)*t \in X-M$ which given $L_x(t) \in X-M$, for L $X-M$. Hence the theorem.

Theorem 2: Let X be a BCI-algebra. Foe x,y X, if $x \leq y$, then $L_x \leq L_y$.

Proof : Let $x,y \in X$ and $x \leq y$. Then $x*y=o$ or $(x*y)*x = o*x$ or $(x*y)*y = o*x$ or $o*y = o*x$. Again $x<y$ gives $(x*t) \leq y*t$ or $(x*t)*(0*x) \leq (y*t)*(o*x)$ or $(x*t)*(0*x) \leq (y*t)*(o*y)$ or $L_x(t) \leq L_y(t)$, for $t \in X$. Hence $L_x \leq L_y$. This completes the proof.

The identity self-map $id_X : X \to X$ s given by $id_X(t) = t$ for $t \in X$.

Theorem 3: Let X be an associative BCI-algebra. For $x \in X$, $L_x = Id_x$.

Proof : Since X is associative, therefore for $x \in X$, $o*x = x$ holds. Let $t \in X$, then $L_x(t) = (x*t)*(o*x)$ or $L_x(t) = (x*t)*x = (x*x)*t = o*t = t = id_X(t)$ or $L_x(t) = id_X(t)$, for all $t \in X$, which gives $L_x = id_X$. This completes the proof.

We characterie *BCI*-algebras with weak unit in terms of weak left self-maps in the sequel.

Theorem 4: Let X be a *BCI*-algebra with I as its centre, X is a *BCI*-algebra with weak unit if and only if id_I = the restriction of L_{x_o} *to* I, $x_o \in X$.

Proof : Let X be a *BCI*-algebra with weak unit an I as its centre. By lemma I, 1 is an associative *BCI*-algebra. By theorem 3, $L_{x_o} = id_I$, for $x_o \in I$.

Conversely, $L_{x_o} = id_I$. Let x_o, $t_o \in I$, then $L_{x_o}(t_o) = (x_o*t_o)*(o*x_o)$. By (15)(ii), $L_{x_o}(t_o) = (x_o*t_o)*(o*x_o) = x_o*(o*(x_o*t_o))$. Thus $L_{x_o}(t_o)=id_I(t_o)$ implies $x_o*(o*(x_o*t_o) = t_o$ or $(x_o**o*(x_o*t_o)))*t_o = 0$ or $(x_o*t_o)*(x_o*(x_o*t_o)) = 0$ or $(x_o*t_o)*(t_o*x_o)=0$, by (15)(iv). By (15)(iii) $x_o*t_o = t_o*x_o$. Put $x_o=o$. Then $o*t_o=t_o$; for all $t_o \in I$, which implies I is associative *BCI*-algebra. By lemma 1, X is a *BCI*-algebra with weak unit. This completes the proof.

Theorem 5: Let X be a *BCI*-algebra with I as its centre. Let $x_o \in I$ and $A(x_o) \subseteq X$. Then $L_x[A(x_o)] \subseteq A(x_o)$ for $x \in A(x_o)$.

Proof : Let X be *BCI*-algebra with I as its centre and $x_o \in I$, then $A(x_o) \subseteq X$. Let $x \in A(x_o)$. We show that $L_x(t) \in A(x_o)$, for $t \in A(x_o)$. Now $L_x(t) = (x*t)*(o*x)$. By (12), x, $t \in A(x_o)$ imply $x*t \in M$. Put $x*t = m \in M$ (say). Then $L_x(t) = m*(o*x)$.

Again by(13), $x, x_o \in A(xo)$ imply $o*x=o*x_o$. By (11), $o, x_o \in I$ imply $o*x_o \in I$. Let $o*x_o=y_o \in I$, thus $L_x(t) = m*y_o$. By 15(iii), $o*x_o=y_o$ imply $o*y_o=x_o$. Now $0 \leq m$ imply $o*y_o \leq m*y_o$ or $x_o \leq m*y_o$. By definition of $A(x_o)$, $x_o \leq m*y_o = L_x(t)$ implies $L_x(t) \in A(x_o)$. Hence $L_x[A(x_o)]$ $A(x_o)$. This completes the proof.

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EQUIVALENTE OF STRONG AND REGULAR IDEALS IN BCI-ALGEBRAS

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Abstract

In this paper we show that strong ideals and regular ideals coincide in *BCI*-algebras.

1. Introduction

In [2] the concept of strong ideals was introduced. Some results related with these ideals had been established in [2] and [4].

M. Daoji [5], gave the notion of a regular ideal in *BCI*-algebras. In this paper, we show that strong ideals and regular ideals in *BCI*-algebras coincide. Some other facts about strong ideals have also been investigated.

Our notions of *BCK*/*BCI*-algebras shall be as are developed in [6], [7] and [1]. The *BCK*-Part of a given *BCI*-algebra X will be denoted by M.

Definition 2 [2]: An ideal A in a *BCI*-algebra X is called a strong ideal if for $a \in A$, $x \in X-A$, $a*x \in X-A$.

Definition 3 [5]: An ideal A in a *BCI*-algebra X is called regular if $x*y \in A$, $x \in A$ imply $y \in A$.

In theorem 3.2 of [2], it is shown that every strong ideal in a *BCI*-algera X contains its *BCK*-part M. However, the following example shows that every ideal which contains M may not be necessarily strong.

Example 1 [8] : Let Z be the set of integers and '$-$' the minus operation, then $(Z, -, 0)$ is a BCI-algebra. Since $o-x=0$ implies $x=0$ for $x \epsilon Z$, its *BCK*-part $M=\{0\}$ and hence Z is a p-semisimple algebra.

Let us consider $A = \{0, 1, 2, 3, ...\}$ and $B=\{0, -1, -2, ...\}$. Then A and B are ideals in X. Note that for $-2 \epsilon Z-A$, $3 \epsilon A$, we have $3-(-2) = 5 \in A$, but $-2 \notin A$ which implies A is not strong, but A contains $M = \{0\}$. Also note that A and B are not sub-algebras.

Theorem 1: Let X be a *BCI*-algebra and $A \subseteq X$ an ideal in X. Then followings are equivalent:

(i) A is strong.

(ii) A is regular.

Proof : (i) implies (ii). Let A be strong. We show that A is regular. Let $x*y \epsilon A$, $x \epsilon A$, then we claim that $y \epsilon A$. Suppose $y \epsilon X-A$. Then $x*y \epsilon A$ implies A is not strong, a contradiction. Hence $y \epsilon A$ and A is regular.

(ii) implies (i). Let A be regular, we show that A is strong. Let $x \epsilon A$, $y \epsilon X-A$. We claim that $x*y \epsilon X-A$. Suppose $x*y \epsilon A$. Now $x \epsilon A$, $x*y \epsilon A$ and A being a regular ideal implies $y \epsilon A$, A contradiction. Hence $x*y \epsilon X-A$ and A is strong.

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ON THE ISHIKAWA FIXED POINT ITERATIONS FOR SOME CONTRACTIVE MAPPINGS

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Abstract

In this paper, we establish some fixed point theorems for Ishikawa iterates of mappings on a normed space under various contractive conditions.

1. Introduction

Let C be a nonempty subset of a Banach space B and let T be a mapping of C into itself. The sequence $\{x_n\}$ associated with T is called an Ishikawa scheme [1] if

$$x_0 \in C,$$

$$x_{n+1} = (1-\alpha_n)x_n+\alpha_n Ty_n,\ y_n=(1-\beta_n)x_n+\beta_n Tx_n,\ n\geq 0. \qquad (1)$$

In the Ishikawa scheme (1), $\{\alpha_n\}$, $\{\beta_n\}$ satisfy $0\leq\alpha_n\leq\beta_n\leq 1$ for all n, $\lim_{n\to\infty} \beta_n = 0$ and $\Sigma\, \alpha_n \beta_n = \infty$. In this paper, we restrict them to satisfy

(i) $0\leq\alpha_n,\ \beta_n\leq 1$ (ii) $\lim_{n\to\infty} \beta_n>0$ (iii) $\lim_{n\to\infty} \alpha_n=h<1$. (2)

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In [2] and [3], it has been shown that for a mapping T and for two mappings T_1 and T_2 satisfying certain contractive conditions if the sequence of Mann iterates associated with T and with T_1 and T_2 converges, then it converges to a fixed point of T and to a common fixed point of T_1 and T_2.

In the present paper, it is proved that for the mapping T and for two mappings T_1, T_2 which satisfy conditions (I), (II) and (III) below if the sequence of Ishikawa iterates converges, then it converges to a fixed point of T and to a common fixed point of T_1 and T_2. These results extend the corresponding results of Khan [2] and Pathak [3].

The contractive conditions to be used are the following. There exists a constant q, $0 < q < 1$, such that for all $x, y \in B$,

(I) $$\|Tx-Ty\| \leq q\, max\left\{\|x-y\|,\ \frac{\|x-Tx\|\,[1-\|x-Ty\|]}{1+\|x-Tx\|},\ \frac{\|x-Ty\|\,[1-\|x-Tx\|]}{1+\|x-Ty\|},\ \frac{\|Tx-y\|\,[1-\|y-Ty\|]}{1+\|Tx-y\|},\ \frac{\|y-Ty\|\,[1-\|Tx-y\|]}{1+\|y-Ty\|}\right\}.$$

(II) $$\|T_1x-T_2y\| \leq q\, max\left\{\|x-y\|,\ \frac{\|x-T_1x\|\,[1-\|x-T_2y\|]}{1+\|x-T_1x\|},\ \frac{\|x-T_2y\|\,[1-\|x-T_1x\|]}{1+\|x-T_2y\|},\ \frac{\|T_1x-y\|\,[1-\|y-T_2y\|]}{1+\|T_1x-y\|},\ \frac{\|y-T_2y\|\,[1-\|T_1x-y\|]}{1+\|y-T_2y\|}\right\}.$$

(III) At least one of the following conditions holds :

(i) For each $x, y \in B$, $\|x-T_1x\| + \|y-T_2y\| \leq \alpha \|x-y\|$, $\alpha \geq 0$.

(ii) For each $x, y \in B$, $\|x-T_1x\| + \|y-T_2y\| \leq$

$$\beta[c\|x-y\| + \|x-T_2y\| + \|y-T_1x\|],\ 0 \le \beta < 1,\ c > 0.$$

(iii) For each $x, y \in B$, $\|T_1x-T_2y\| + \|x-T_1x\| + \|y-T_2y\|$

$$\le \gamma[\|x-T_2y\| + \|y-T_1x\|],\ 1 \le \gamma < 2.$$

(iv) $\|T_1x-T_2y\| \le \delta\, max[c\|x-y\|,\ \|x-T_2x\| + \|y-T_2y\|,$

$$\|x-T_2y\| + \|y-T_1x\|\},\ 0\le\delta<1,\ c>0.$$

2. Main Results

We establish three fixed point theorems using the technique as appeared in Rhoades [4] and Naimpally and Singh [5] for the mappings satisfying contractive conditions (I), (II) and (III) as defined before.

Theorem 2.1 : Let C be a closed, convex subset of a normed space X. Let $T : C \to C$ be a continuous self-mapping satisfying (I). Suppose $\{\alpha_n\}$, $\{\beta_n\}$ satisfy (2) with $\{\alpha_n\}$ bounded away from zero. If $\{x_n\}$ defined by (I) converges to a point u, then u is a fixed point of T.

Proof : From (I), $x_{n+1}-x_n = \alpha_n (Ty_n-x_n)$. Since $x_n \to u$, $\|x_{n+1}-x_n\| \to 0$. Since $\{\alpha_n\}$ is bounded away from zero, $\|Ty_n-x_n\| \to 0$. It also follows that $\|u-Ty_n\| \to 0$. We shall show that u is the fixed point of T. Now consider,

$$\|u-Tu\| \le \|u-x_{n+1}\| + \|x_{n+1}-Tu\|$$

$$\le \|u-x_{n+1}\| + \|(1-\alpha_n)x_n+\alpha_n Ty_n-Tu\|$$

$$\le \|u-x_{n+1}\| +(1-\alpha_n)\|x_n-Tu\| +\alpha_n\|Ty_n-Tu\|$$

$$\le \|u+x_{n+1}\| +(1-\alpha_n)\|x_n-Tu\| +\alpha_n\, max\{\|y_n-u\|,$$

$$\frac{\|y_n-Ty_n\|\,[1-\|y_n-Tu\|]}{1+\|y_n-Ty_n\|}, \qquad \frac{\|y_n-Tu\|\,[1-\|y_n-Ty_n\|]}{1+\|y_n-Tu\|},$$

$$\frac{\|Ty_n-u\|\,[1-\|u-Tu\|]}{1+\|Ty_n-u\|}, \qquad \frac{\|u-Tu\|\,[1-\|Ty_n-u\|]}{1+\|u-Tu\|}, \tag{3}$$

where

$$\begin{aligned}\|y_n-n\| &= \|(1-\beta_n)x_n+\beta_n\, Tx_n-u\| \\ &\leq (1-\beta_u)\|x_n-u\| +\beta_n \|Tx_n-u\| \\ &\leq \|x_n-u\| + \|Tx_n-x_n\|, \end{aligned} \tag{4}$$

$$\begin{aligned}\|y_n-Ty_n\| &= \|(1-\beta_n)x_n+\beta_n Tx_n-Ty_n\| \\ &\leq (1-\beta_n)\|x_n-Ty_n\| +\beta_n\|Tx_n-Ty_n\| \\ &\leq \|x_n-Ty_n\| +\beta_n \|Tx_n-x_n\| \\ &\leq \|x_n-Ty_n\| + \|Tx_n-x_n\|, \end{aligned} \tag{5}$$

$$\begin{aligned}\|y_n-Tu\| &= \|(1-\beta_n)\, x_n+\beta_n\, Tx_n-Tu\| \\ &\leq (1-\beta_n) \|x_n-Tu\| +\beta_n\|Tx_n-Tu\| \\ &\leq \|x_n-Tu\| +\beta_n \|Tx_n-x_n\| \\ &\leq \|x_n-Tu\| + \|x_n-Tx_n\|. \end{aligned} \tag{6}$$

Putting each of (4)-(6) in (3), letting $n\to\infty$, and using the continuity of T, we obtain

$$\|u-Tu\| \leq (1-h)\|u-Tu\| +hq\, max \left\{ \|u-Tu\|, \frac{\|u-Tu\|\,[1-2\|u-Tu\|]}{1+\|u-Tu\|} \right.$$

$$\left. \frac{2\|u-Tu\|\,[1-\|u-Tu\|]}{1+2\|u-Tu\|}\; 0, \quad \frac{\|u-Tu\|}{1+\|u-Tu\|} \right\}.$$

If $\|u\text{-}Tu\| \leq (1\text{-}h)\ \|u\text{-}Tu\| +hq\ \|u\text{-}Tu\|$,

then we have

$h(1-q)\|u-Tu\| \leq 0$ *i.e.*, $u=Tu$. Hence u is a fixed point of T, and if,

$$\|u-Tu\| \leq (1-h)\|u-Tu\| + \text{h}q \quad \frac{\|u-Tu\|}{1+\|u-Tu\|},$$

then $\|u-Tu\|^2 \leq -(1-q)\ \|u-Tu\|$.

Suppose that $Tu \neq u$ and let $\|u-Tu\| = \delta$. Then we have $\delta \leq -(1-q)$, a contradiction. Hence $u=Tu$, *i.e.*, u is the fixed point of T and this completes the proof.

Theorem 2.2 : Let C be a closd, convex subset of a normed space X and let T_1 and T_2 be continuous self-mappings on C satisfying (II). Then $\{x_n\}$, the sequence of Ishikawa iterates associated with T_1 and T_2, is given below:

For $x_o \in C$ set $x_{2n+1}=(1-\alpha_n)x_{2n}+\alpha_n\ T_1y_{2n}$, $y_{2n}=(1-\beta_n)x_{2n}+\beta_n\ T_1x_{2n}$ and $x_{2n+2}=(1-\alpha_n)x_{2n+1}+\alpha_n T_2y_{2n+1}$, $y_{2n+1}=(1-\beta_n)x_{2n+1}+\beta_n Tx_{2n+1}$ for $n=0,\ 1,\ 2,\ \ldots$, where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy (2) with $\{\alpha_n\}$ bounded away from zero. If $\{x_n\}$ converges to u in C and if u is a fixed point of either T_1 or T_2, then u is common fixed point of T_1 and T_2.

Proof : As in the proof of theorem 2.1, it follows that $\|T_1y_{2n}-x_{2n}\|$ and $\|u-T_1y_{2n}\|$ tend to zero as $n \rightarrow \infty$.

Now we shall show that u is the common fixed point of T_1 and T_2. Let $T_1u = u$. Then we have

$$\|u-T_2u\| \leq \|u-x_{2n+1}\| + \|x_{2n+1}-T_2u\|$$

$$\leq \|u-x_{2n+1}\| + \|(1-\alpha_n)x_{2n}+\alpha_n\ T_1y_{2n}-T_2u\|$$

$$\leq \|x_{2n+1}-u\| + (1-\alpha_n)\|x_{2n}-T_2u\| + \alpha_n\|T_1y_{2n}-T_2u\|$$

$$\leq \|x_{2n+1}-u\| + (1-\alpha_n)\|x_{2n}-T_2u\| + \alpha_n\, q\ max\ \{\|y_{2n}-u\|,$$

$$\frac{\|y_{2n}-T_1y_{2n}\|[1-\|y_{2n}-T_2u\|]}{1+\|y_{2n}-T_1y_{2n}\|},\quad \frac{\|y_{2n}-T_2u\|[1-\|y_{2n}-T_1y_{2n}\|]}{1+\|y_{2n}-T_2u\|},$$

$$\frac{\|T_1y_{2n}-u\|\ [1-\|u-T_2u\|]}{1+\|Ty_{2n}-u\|},\quad \frac{\|u-T_2u\|\ [1-\|T_1y_{2n}-u\|]}{1+\|u-T_2u\|}, \tag{7}$$

where

$$\begin{aligned}\|y_{2n}-T_1y_{2n}\| &:= \|(1-\beta_n)x_{2n}+\beta_nT_1x_{2n}-T_1y_{2n}\| \\ &\le (1-\beta_n)\|x_{2n}-T_1y_{2n}\|+\beta_n\|T_1x_{2n}-T_1y_{2n}\| \\ &\le \|x_{2n}-T_1y_{2n}\|+\|\mathrm{T}_1x_{2n}-x_{2n}\|, \end{aligned}\tag{8}$$

$$\begin{aligned}\|y_{2n}-\mathrm{u}\| &= \|(1-\beta_n)x_{2n}+\beta_nT_1x_{2n}-u\| \\ &\le (1-\beta_n)\|x_{2n}-u\|+\beta_n\|T_1x_{2n}-u\| \\ &\le \|x_{2n}-u\|+\|T_1x_{2n}-x_{2n}\|, \end{aligned}\tag{9}$$

$$\begin{aligned}\|y_{2n}-T_2u\| &= \|(1-\beta_n)x_{2n}+\beta_nT_1x_{2n}-T_2u\| \\ &\le (1-\beta_n)\|x_{2n}-T_2u\|+\beta_n\|T_1x_{2n}-T_2u\| \\ &\le \|x_{2n}-T_2u\|+\|T_1x_{2n}-x_{2n}\|, \end{aligned}\tag{10}$$

Substituting (8)-(10) in (7), letting $n\to\infty$ and using the continuity of T_1, we obtain

$$\|u-T_2u\|\le(1-h)\|u-T_2u\|+hq\ max\left(0,0,\frac{\|u-T_2u\|}{1+\|u-T_2\|},0,\frac{\|u-T_2u\|}{1+\|u-T_2u\|}\right),$$

$$i.e.,\ \|u-T_2u\|\le(1-h)\|u-T_2u\|+hq\frac{\|u-T_2u\|}{1+\|u-T_2u\|}.$$

This gives,

$$\|u-T_2u\|^2\le -(1-q)\ \|u-Tu\|$$

which implies that $T_2u=u$. Similarly we can prove that if $T_2u=u$ then $T_1u=u$ *i.e.*, u is the fixed point of T_1 and T_2.

The above results extend ([3], Theorems 1,2) to Ishikawa scheme. We next extend the result in [2].

Theorem 2.3 : Let X be a normed space and let T_1 and T_2 be continuous self-mappings on X satisfying (III). Let $\{x_n\}$, the sequence of Ishikawa iterates associated with T_1 and T_2, where $\{\alpha_n\}$, $\{\beta_n\}$, satisfy (2) with $\{\alpha_n\}$ bounded away from zero. If $\{x_n\}$ converges to u and if u is a fixed point of either T_1 or T_2, then u is a common fixed point of T_1 and T_2.

Proof : As in the proof of Theorem 2.1, it follows that $\| T_1y_{2n}-x_{2n}\|$ and $\| T_1y_{2n}-u\|$ tend to zero as $n\to\infty$. Let $T_1u=u$. Now we shall show that u is a common fixed point of T_1 and T_2. We have,

$$\begin{aligned}\| u-T_2u\| &\leq \| u-x_{2n+1}\| + \| x_{2n+1}-T_2u\| \\ &\leq \| u-x_{2n+1}\| + \| (1-\alpha_n)x_{2n}+a_nT_1y_{2n}-T_2u\| \\ &\leq \| u-x_{2n+1}\| +(1-\alpha_n)\| x_{2n}-T_2u\| +a_n\| T_1y_{2n}-T_2u\| .\end{aligned} \tag{11}$$

If T_1 and T_2 satisfy (III) (i), then,

$$\begin{aligned}\| T_1y_{2n}-T_2u\| &\leq \| T_1y_{2n}-y_{2n}\| + \| y_{2n}-u\| + \| u-T_2u\| \\ &\leq (1+\alpha)\| y_{2n}-u\| \\ &\leq (1+\alpha)[\| (1-\beta_n)x_{2n}+\beta_nT_1x_{2n}-u\|] \\ &\leq (1+\alpha)[\| x_{2n}-u\| +\beta_n\| T_1x_{2n}-u\|] \\ &\leq (1+\alpha)[\| x_{2n}-u\| + \| T_1x_{2n}-x_{2n}\|].\end{aligned} \tag{12}$$

If T_1 and T_2 satisfy (III) (ii), then,

$$\begin{aligned}\| T_1y_{2n}-T_2u\| &\leq \| T_1y_{2n}-y_{2n}\| + \| y_{2n}-u\| + \| u-T_2u\| \\ &\leq \| y_{2n}-u\| +\beta[c\| y_{2n}-u\| + \| y_{2n}-T_2u\| + \| u-T_2y_{2n}\|] \\ &\leq (1+\beta c)\| x_{2n}-u\| +(1+\beta+\beta c)\| T_1x_{2n}-x_{2n}\|\end{aligned}$$

$$+ \beta \| u - T_1 y_{2n} \| + \beta \| x_{2n} - T_2 u \| . \tag{13}$$

If T_1 and T_2 satisfy (III) (iii), then,

$$\| T_1 y_{2n} - T_2 u \| \leq \gamma [\| y_{2n} - T_2 u \| + \| u - T_1 y_{2n} \|] - \| y_{2n} - T_1 y_{2n} \| - \| u - T_2 u \|$$

$$\leq \gamma [\| x_{2n} - T_2 u \| + \| u - T_1 y_{2n} \|] + (\gamma - 1) \| T_1 x_{2n} - x_{2n} \|]$$

$$- \| x_{2n} - T_1 y_{2n} \| - \| u - T_2 u \| . \tag{14}$$

If T_1 and T_2 satisfy (III) (iv), then,

$$\| T_1 y_{2n} - T_2 u \| \leq \delta \max [c \| y_{2n} - u \| , \| y_{2n} - T_1 y_{2n} \| + \| u - T_2 u \| ,$$

$$\| y_{2n} - T_2 u \| + \| u - T_1 y_{2n} \| \}$$

$$\leq \delta [\| x_{2n} - T_2 u \| + \| T_1 x_{2n} - x_{2n} \| + \| u - T_2 u \|]. \tag{15}$$

Substituting each of (12)-(15) in (11), letting $n \to \infty$ and using the continuity of T_1, we obtain

$$\| u - T_2 u \| \leq \lambda \| u - T_2 u \| ,$$

where

$$\lambda = \max [1 - h, 1 - h(1 - \beta), 1 - h(2 - \gamma), 1 - h(1 - \delta)] < 1.$$

Hence $T_2 u = u$. Similarly we can prove that if T_2u = u, then $T_1 u = u$, *i.e.*, u is a common fixed point of T_1 and T_2.

Finally, we conclude this section by the following question. Does the conclusion of the above theorems hold if the continuity of T and T_1, T_2 is removed?

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GENERALISED HADAMARD MATRICES AND ASSOCIATED GRAPHS

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Abstract

This paper outlines graph-theoretic-techniques in the investigation of generalized Hadamard matrices. The techniques are illustrated by applying them to the matrices described by Drake and Dawson. A detailed study of the case GH(8,G), where G is a group of order 4, is reported.

In this paper, we are concerned with the construction and investigation of generalized Hadamard matrices of size 2m with entries from a group G of order m. Such matrices are associated with generalized Hadamard designs as described below, and this association introduces a natural equivalence relation on the set of generalized Hadamard matrices - an equivalence class is the set of matrices corresponding to a given design.

With each generalized Hadamard matrix we associate a set of graphs, usually directed. We ask when the corresponding class contains a matrix whose graphs are undirected. This corresponds to the matrix being symmetric

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and the design having a polarity. We examine the possibility of constructing such matrices from a plausible set of graphs.

Early constructions of such matrices have been given by Bose and Bush[1], Shrikhande[6] and Butson[2]. The case in which m is an odd prime, given in Butson[2], was generalized independently by Jungnickel[5] and Street[7]. We use the construction given by Dawson[3], and described below, which includes both Jungnickel and Street constructions. Drake[4] describes a construction when m is a power of 2. We investigate this below also.

A **generalized Hadamard matrix** GH(2m,G) is a $2m \times 2m$ matrix $H=[h_{ij}]$ with entries from a group G of order m, written additively, such that for each $i,i'=1,...,2m$ with $i \neq i'$ the sequence $(h_{ij}-h_{i'j}:j=1,..,2m)$ contains each element from G exactly twice. It follows that for each $j,j'=1,..,2m$ with $j \neq j'$ the sequence $(h_{ij}-h_{ij'}:i=1,..,2m)$ has the same property - see, for example, Jungnickel[5]. If the entries in H are replaced by the corresponding matrices in the regular representation, the resulting matrix is the incidence matrix of a generalized Hadamard design Γ, i.e., a 1-$(2m^2,2m,2m)$ design. Other incidence matrices of Γ may be obtained by applying the same process to any matrix obtained from H by any combination of the following operations:

(i) permuting the rows,
(ii) permuting the columns,
(iii) adding to the elements of a row a fixed element of G,
(iv) adding to the elements of a column a fixed element of G,
(v) applying an automorphism of G to the entries of H.

We now show how to associate graphs with a GH(2m,G) H. First we normalize H - i.e. we subtract suitable elements from the rows and columns to ensure that the first row and column, of the resulting matrix consists entirely of 0's. Now let $\tilde{H}$ be the matrix obtained by deleting the first row and column, i.e., the core of H. For each $g \epsilon G$, let $A_g=[a_{ij}^{(g)}]$ be the $(2m-1)\times(2m-1)$ matrix defined by

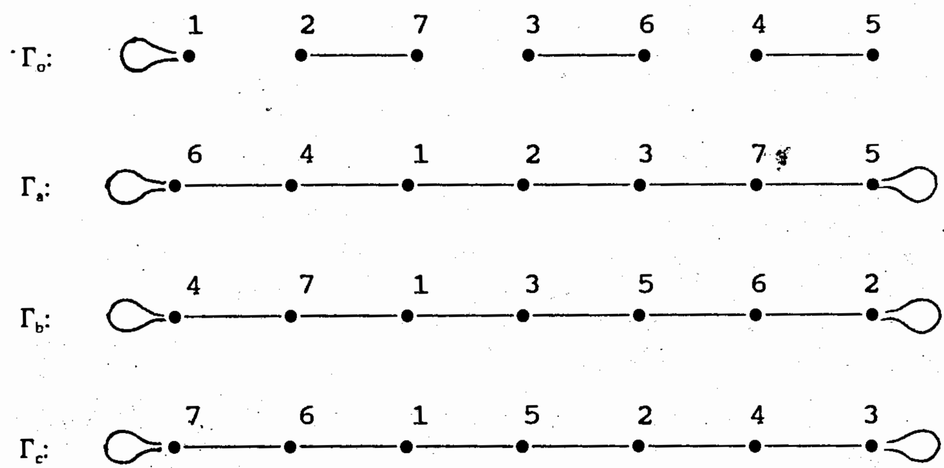
$$a_{ij}^{(g)} = \begin{cases} 1 \text{ if } (\tilde{H})_{ij}=g \\ 0 \text{ if } (\tilde{H})_{ij}\neq g \end{cases}$$

A_g is the adjacency matrix of a graph Γ_g with vertex set $V=\{1,..,2m-1\}$ and such that i is joined to j (by a directed edge) if and only if $a_{ij}^{(g)}=1$. If, for some pair i, j with $i \neq j$, i is joined to j and j is joined to i, we may replace the pair of directed edges by an undirected edge. $a_{ii}^{(g)} = 1$ means that Γ_g has a loop at i. We note that formally $\widetilde{H} = \sum_{g \in G} g A_g$.

In the case of $GH(8, Z_2^2)$ with

$$\widetilde{H} = \begin{vmatrix} 0 & a & b & a & c & c & b \\ a & b & a & c & c & b & 0 \\ b & a & c & c & b & 0 & a \\ a & c & c & b & 0 & a & b \\ c & c & b & 0 & a & b & a \\ c & b & 0 & a & b & a & c \\ b & 0 & a & b & a & c & c \end{vmatrix}$$

we have,



In the general case, each vertex in Γ_0 has in-degree and out-degree 1, and each vertex in $\Gamma_g (g \neq 0)$ has in-degree and out-degree 2.

Given $\widetilde{H}$, we ask which permutation of its rows and columns will produce a symmetric matrix. For a permutation π of $1, \ldots, 2m-1$, let P_π be the corresponding permutation matrix, i.e., $P_\pi = [p_{i,j}]$ where $p_{i,j} = \delta_{i,\pi(j)} = 1$ or 0 according as $i = \pi(j)$ or $i \neq \pi(j)$. Then $P_{\pi_1} P_{\pi_2} = P_{\pi_1 \pi_2}$ and $P_{\pi^{-1}} = P_\pi^{-1} = P_\pi^t$.

The matrix $\tilde{K}$ obtained from $\tilde{H}$ by permuting the rows according to π_1 and the columns according to π_2 is $P_{\pi_1}^t \tilde{H} P_{\pi_2}$. Thus $\tilde{K}$ is symmetric if and only if $P_{\pi_1}^t \tilde{H} P_{\pi_2} = P_{\pi_2}^t \tilde{H} P_{\pi_1}$, i.e., $P_\delta \tilde{H} P_\delta = \tilde{H}$ where $\delta = \pi_2 \pi_1^{-1}$, or equivalently $\tilde{H} P_\delta$ is symmetric. The latter is true precisely when $A_g P_\delta$ is symmetric for all $g \epsilon G$.

If $\tilde{H}$ is already symmetric, then each A_g is symmetric. So $A_g A_h P_\delta = A_g P_\delta^t P_h = P_\delta A_g A_h$ for all $g,h \epsilon G$. Thus, δ must induce an automorphism of the graphs with adjacency matrices $A_g A_h$ $(g,h \epsilon G)$. Moreover, every δ corresponding to such an automorphism and for which P_δ is inverted by the permutation matrix A_o has the property that $\tilde{H} P_\delta$ is symmetric. By considering the obviously linearly independent set of matrices $\{A_g A_o : g \epsilon G\}$, we see that the centralizer algebra of P_δ in the full matrix algebra has dimension at least m. In the above example, if we write C for the cycle matrix of size 7, we have $A_o^2 = 1$, $A_a A_o = C^4 + C^6$, $A_b A_o = C + C^5$, $A_c A_o = C^2 + C^3$, $A_o A_a = C + C^3$, $A_o A_b = C^2 + C^6$, $A_o A_c = C^4 + C^5$. An easy calculation shows that the centralizer algebra contains $\{C_i : i = 0,\ldots,6\}$. Hence $P_\delta = C^i$ for some i; indeed, for all such i, $\tilde{H} P_\delta$ is symmetric.

If $\tilde{H}$ is not necessarily symmetric, we wish to find δ such that $A_g P_\delta$ is symmetric for all $g \epsilon G$. Recall that A_o is a permutation matrix, P_ζ say. Let $\pi = \zeta\delta$. Then $A_o P_\delta$ is symmetric if and only if $\pi^2 = 1$. Also $P_\delta A_g$ is symmetric if $A_g P_\delta$ is symmetric. So we necessarily have $P_\delta A_o A_g^t A_o = A_o^t A_g P_\delta A_o$ for all $g \epsilon G$, i.e. $P_\pi A_g^t A_o = A_o^t A_g P_\pi$. If $\Delta_{g,h}$ denotes the graph with adjacency matrix $A_g^t A_h$, we require a permutation π of order 2 which interchanges $\Delta_{g,o}$ with $\Delta_{o,g}$ for each $g \epsilon G \backslash \{0\}$. And from each such π, we can reconstruct an appropriate δ.

We now describe the graphs associated with a GH(2q,G), where q is a power of 2 and G is elementary abelian. We use the construction of Drake[4]. Let H' be the $2q \times 2q$ matrix whose rows and columns are indexed by F_{2q} (the field of 2q elements) and such that $(H')_{x,z} = xz$. Then H' is a $GH(2q, F_{2q})$. Now take any additive group homomorphism $F_{2q} \rightarrow F_q$, and let H be the matrix obtained from H' by applying this homomorphism to the entries. Then H is a $GH(2q, F_g)$. We may suppose that the rows and columns are listed in the same order, with the row and column indexed by 0 coming first. Then H', and consequently H, are symmetric and both are already normalized.

The graphs associated with H are now easily described. First we take the group homomorphism above to be the one with kernel $\{0,1\}$ and denote the image of an element x by $\overline{x}$ - so $\overline{x+1}=\overline{x}$. The vertex set of each of the graphs is $V=F_{2q}\backslash\{0\}$.

Γ_0 has one loop at 1, and the remaining edges join x and x^{-1} for $x\neq 1$. If $a\epsilon F_{2q}\backslash\{0,1\}$, the edges of $\Gamma_{\overline{a}}$ join x to ax^{-1} and to $(a+1)x^{-1}$, for each $x\epsilon V$. Since each element of $F_{2q}\backslash\{0\}$ has a unique square root, each $\Gamma_{\overline{a}}$ has exactly two loops, at $\sqrt{a}$ and $\sqrt{(a+1)}$. These vertices are joined by a path of length 2r, where 2r+1 is the order of the element a/(a+1). The remaining components of $\Gamma_{\overline{a}}$ are all circuits. Since the products of the vertices on consecutive edges are a and a+1, these circuits have even length. Going in one direction around such a circuit, the vertices are x, $(a+1)x^{-1}$, $(a+1)^{-1}ax$, ..., $(a+1)^s a^{-s+1}x^{-1}$, $(a+1)^{-s}a^s x$, ... The length of this circuit is 2(2r+1), where 2r+1 is the order of a/(a+1) as above.

Next, we consider the construction of Dawson[3] of a GH(2q,G), where q is an odd prime power and G is an elementary abelian group of order q. Let

$$H = \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix}$$

where H_{ik} (i,k = 1,2) is a $q\times q$ matrix over F_q whose rows and columns are indexed by the elements of F_q and such that $(H_{ik})_{xz} = \alpha_{ik}z^2+\beta_{ik}zx+\gamma_{ik}x^2$, where α_{ik}, β_{ik} and γ_{ik} are elements of F_q satisfying

$$\frac{\alpha_{11}-\alpha_{21}}{\beta_{11}\beta_{21}} = \frac{\alpha_{12}-\alpha_{22}}{\beta_{12}\beta_{22}} \qquad (1)$$

$$\frac{\gamma_{11}-\gamma_{12}}{\beta_{11}\beta_{12}} = \frac{\gamma_{21}-\gamma_{22}}{\beta_{21}\beta_{22}} \qquad (2)$$

$$4\times\frac{(\alpha_{11}-\alpha_{21})(\gamma_{11}-\gamma_{12})}{\beta_{11}\beta_{21}\,\beta_{11}\beta_{12}} = \frac{\beta_{11}\beta_{22}-\beta_{12}\beta_{21}}{\beta_{11}\beta_{12}\,\beta_{21}\beta_{22}} \qquad (3)$$

$\beta_{11}\beta_{12}\beta_{21}\beta_{22}$ is a non-square in F_q. (4)

We can find a matrix equivalent to H with $\alpha_{11}=\gamma_{11}=\alpha_{12}=\gamma_{21}=0$, $\beta_{11}=\beta_{12}=\beta_{21}=1$. We call this latter matrix H and set $\alpha_{22}=\alpha$, $\beta_{22}=\beta$, $\gamma_{22}=\gamma$. The associated quadratic forms are then $(H_{11})_{xz}=xz$, $(H_{12})_{xz}=xz+\frac{\gamma}{\beta}x^2$, $(H_{21})_{xz}=xz+\frac{\alpha}{\beta}z^2$ and $(H_{22})_{xz}=\alpha z^2+\beta xz+\gamma x^2$. The conditions satisfied by α, β and γ are

$$4\alpha\gamma=\beta(\beta-1) \qquad (3')$$

β is a non-square in F_q. (4′)

Clearly H is normalized and H_{11} is symmetric. We examine when H may be "symmetrized" by permuting the last q rows and the last q columns among themselves. Let v denote a label for one of the last q columns and let v′ denote the label for the corresponding row. Let x denote a label for one of the first q rows.

If H is symmetric, then $(H_{12})_{xv}=(H_{21})_{v'x}$ for all $x\epsilon F_q$, i.e., $xv+\frac{\gamma}{\beta}x^2=v'x+\frac{\alpha}{\beta}x^2$. Thus $\alpha=\gamma$ and $v'=v$.

If $\gamma=\alpha$ and $v'=v$ for all $v\epsilon F_q$, it is immediate that H is symmetric. In this case, we may replace (3′) and (4′) above by

β and $\beta-1$ are non-squares in F_q (5)

$$4\alpha^2=\beta(\beta-1). \qquad (6)$$

If $q\equiv 1 \pmod 4$, there are $(q-1)/4$ non-squares β in F_q satisfying (5); if $q\equiv 3 \pmod 4$ there are $(q+1)/4$ such non-squares.

To describe the associated graphs, let $V=\{(x,i): x\epsilon F_q, i\epsilon\{1,2\}\}\backslash\{(0,1)\}$. (x,1) is the vertex corresponding to the row/column labelled x among the first q of H, and (x,2) is the vertex corresponding to the row/column labelled x among the last q. We use ~ to denote adjacency.

In Γ_0, clearly $(x,1)\not\sim(z,1)$ for all $x,z\epsilon F_q\backslash\{0\}$. Also $(x,2)\sim(z,2)$ if and only if $x=z=0$, since $\alpha z^2+\beta zx+\alpha x^2=\alpha((z+\frac{\beta x}{2\alpha})^2-\beta(\frac{x}{2\alpha})^2)$. Finally, $(x,1)\sim(z,2)$ if and only if $\alpha x+\beta z=0$.

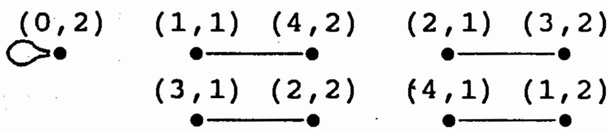
Now let $a\epsilon F_q\backslash\{0\}$. In Γ_a, $(x,1)\sim(z,1)$ if and only if $z=\frac{a}{x}$; $(x,1)\sim(z,2)$ if and only if $z=-\frac{\alpha}{\beta}x+\frac{a}{x}$; $(x,2)\sim(z,2)$ if and only if $\alpha z^2+\beta zx+\alpha x^2=a$.

We give, as an example, the graphs associated with Butson matrix below. The core of the matrix is

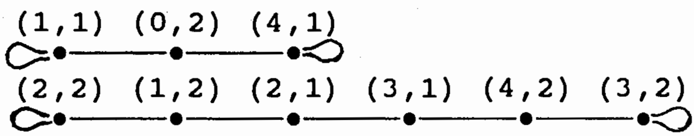
1	2	3	4	1	2	3	4	0
2	4	1	3	4	1	3	0	2
3	1	4	2	4	2	0	3	1
4	3	2	1	1	0	4	3	2
1	4	4	1	0	3	2	2	3
2	1	2	0	3	4	1	4	3
3	3	0	4	2	1	1	2	4
4	0	3	3	2	4	2	1	1
0	2	1	2	3	3	4	1	4

The associated graphs are

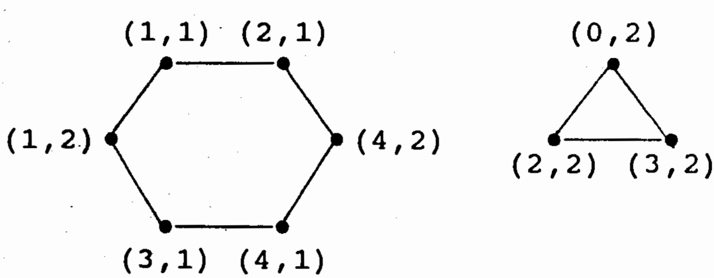
Γ_0:



Γ_1:



Γ_2:



Γ_3 and Γ_4 are isomorphic to Γ_2 and Γ_1, respectively. The corresponding vertices may be obtained by multiplying the first components of the vertices by 2.

In the special case of GH(8,G), where G is a graph of order 4, we began by enumerating all 7x7 matrices in the 4 symbols 0,1,2 and 3 with the following properties.

(i) Every row has one 0 and two each of 1,2 and 3;
(ii) Every pair of distinct rows/columns has just one position with the same symbol;
(iii) The first row is 0,1,1,2,2,3,3;
(iv) The rows are ordered lexicographically.

Clearly the core of any GH(8,G) must satisfy the above properties. We found with the aid of a computer that there were only two such matrices

$$A = \begin{vmatrix} 0&1&1&2&2&3&3 \\ 1&0&2&1&3&2&3 \\ 1&2&0&3&1&3&2 \\ 2&1&3&0&3&1&2 \\ 2&3&1&3&0&2&1 \\ 3&2&3&1&2&0&1 \\ 3&3&2&2&1&1&0 \end{vmatrix} \qquad B = \begin{vmatrix} 0&1&1&2&2&3&3 \\ 1&2&3&0&1&2&3 \\ 1&3&2&2&3&1&0 \\ 2&0&2&3&1&3&1 \\ 2&1&3&1&3&0&2 \\ 3&2&1&3&0&1&2 \\ 3&3&0&1&2&2&1 \end{vmatrix}$$

We then tried all possible substitutions of group elements using Z_4 and Z_2xZ_2 for the four symbols, replacing 0 by the identity element in all cases. The only situation which gave rise to a core of a Hadamard matrix was the matrix B, where 1,2 and 3 are replaced by distinct non-identity elements of the group Z_2xZ_2.

We conclude by listing some of the questions raised by the above investigation.

a. Is there a straightforward graph-theoretic characterization of the sets of graphs associated with a generalized Hadamard matrix?

b. Given such a set of graphs, is there an efficient algorithm for constructing the matrix?

c. Given two such sets of graphs, how can one determine whether the associated matrices are equivalent? In particular, does one such set of graphs determine the graph associated with the generalized Hadamard matrix uniquely?

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