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SOLUTION OF THE $n \times n$ MATRIX SPECTRAL PROBLEM

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ABSTRACT

The inverse spectral transform introduced by Gardner *et al.* [1] and used by Ablowitz *et al.* [2] and others has proved to be a useful method for solving non-linear evolution equations. This paper is an attempt to solve the spectral problem which is the generalization of the spectral problem solved by Kalim [3]. The method which is given in this paper helps in solving the spectral problem when the potentials are $n \times n$ matrices.

INTRODUCTION

In this paper the spectral problem

$$\frac{\partial}{\partial x} u = \{\mathbf{A}(\zeta) + \mathbf{B}(x, \zeta)\} \cdot u, \tag{1}$$

where u is an n-element column vector and $\mathbf{A}(\zeta)$ and $\mathbf{B}(x, \zeta)$ are $n \times n$ matrices. The problem is solved by using the inverse spectral transform and with the help of Fredholm theory. A set of spectral data which is sufficient for the reconstruction of $\mathbf{B}(x, \zeta)$ is found, and then the algorithm (the inverse spectra transform) for the reconstruction of $\mathbf{B}(x, \zeta)$ is described.

THE DIRECT SPECTRAL PROBLEM

Eigenvalues and left and right eigenvectors of $\mathbf{A}(\zeta)$ in (1) are $\lambda_i(\zeta)$, $\tilde{v}_i(\zeta)$ and $v_i(\zeta)$, $i = 1, 2, \ldots, n$ respectively and are written by

$$\tilde{v}_i(\zeta) \cdot \mathbf{A}(\zeta) = \lambda_i(\zeta)\, \tilde{v}_i(\zeta)$$

$$\mathbf{A}(\zeta) \cdot v_i(\zeta) = \lambda_i(\zeta)\, v_i(\zeta). \tag{2}$$

We take up the simplest case by assuming that, apart from the isolated points, the $\lambda_i(\zeta)$ are distinct and that they and $\tilde{v}_i(\zeta)$ and $v_i(\zeta)$ are regular throughout the complex ζ-plane. This assumption is sufficient but not necessary. It appears, though it is not shown here that the more general condition that, apart from isolated points, there are no eigenvectors which are distinct and regular with respect to ζ on some Riemann surface is also sufficient.

We define the function $\Phi_i(x, \zeta)$, $i = 1, 2, \ldots, n$ throughout the complex ζ-plane such that

(*i*) $u_i = \Phi_i(x, \zeta)\, e^{\lambda_i(\zeta)x}$ satisfies (1)

(*ii*) $\Phi_i(x, \zeta) \rightarrow v_i(\zeta)$ as $x \rightarrow -\infty$

(*iii*) For any given ζ, $\Phi(x, \zeta)$ is bounded for $-\infty < x < \infty$. Obviously these conditions require that $\mathbf{B}(x, \zeta)$ should tend to zero in some sense as $x \rightarrow \pm\infty$.

We divide the complex ζ-plane into regions each of which is labelled by the permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ where

$$\text{Re}\,\{\lambda_{i_1}\} > \text{Re}\,\{\lambda_{i_2}\} \ldots > \text{Re}\,\{\lambda_{i_n}\}.$$

On a boundary between two regions

$$\text{Re}\,\{\lambda_i\} \doteq \text{Re}\,\{\lambda_j\}$$

for at least one pair $i \neq j$, $i, j = 1, 2, \ldots, n$.

We define a projection operator $\mathbf{P}_i(\zeta)$ which will project out the $v_i(\zeta)$ component of an arbitrary vector:

$$\mathbf{P}_i(\zeta) = \frac{1}{\tilde{v}_i(\zeta) \cdot v_i(\zeta)}\, v_i(\zeta)\, \tilde{v}_i(\zeta). \tag{3}$$

Also $$\mathbf{P}_i^{(u)}(\zeta) = \sum_{\text{Re}\,\{\lambda_j(\zeta) - \lambda_i(\zeta)\} > 0} \mathbf{P}_j(\zeta) \tag{4a}$$

and $$\mathbf{P}_i^{(l)}(\zeta) = \sum_{\text{Re}\,\{\lambda_n(\zeta) - \lambda_i(\zeta)\} < 0} \mathbf{P}_j(\zeta) \tag{4b}$$

These projection matrices are analytic in the interior of any region and

$$\mathbf{P}_i(\zeta) + \mathbf{P}_i^{(u)}(\zeta) + \mathbf{P}_i^{(l)}(\zeta) = \mathbf{I}. \tag{5}$$

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(2)

We define

$$\mathbf{K}_i(x, y, \zeta) = \exp \{ (x - y) \ (\mathbf{A}(\zeta) - \lambda_i(\zeta)\,\mathbf{I}) \} \,.$$
$$\{ \theta\,(x - y)\ \mathbf{I} - \mathbf{P}_i^{(u)}(\zeta) \} \,.\, \mathbf{B}(y, \zeta) . \quad (6)$$

where $\theta\,(x - y)$ is the step function

$$\theta\,(x - y) = \left. \begin{matrix} 0 \ \text{if} \ x < y \\ 1 \ \text{if} \ x < y \end{matrix} \right\} \quad (7)$$

Now we consider the Fredholm equation

$$\Phi_i(x, \zeta) = v_i(\zeta) + \int_{-\infty}^{\infty} \mathbf{K}_i\,(x, y, \zeta) \,.\, \Phi_i(y, \zeta)\, dy. \quad (8)$$

The Fredholm determinant $f_i(\zeta)$ and minor $\mathbf{F}_i(x, y, \zeta)$ are given by

$$f_i(\zeta) \doteq \sum_{m=0}^{\infty} f_i^{(m)} \,.\, \zeta^m , \quad (9)$$

where

$$f_i^{(m)} = \frac{(-1)^m}{m1} \sum_{j_1 j_2 \cdots j_m = 1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}$$

$$K^{(m)} \begin{bmatrix} x_1, j_1, x_2, j_2, \ldots, x_m, j_m \\ x_1, j_1, x_2, j_2, \ldots, x_m, j_m \end{bmatrix} \quad (10)$$

where
$$K^{(m)} \begin{bmatrix} x_1, d_1, x_2, d_2, \ldots, x_m, d_m \\ y_1, k_1, y_2, k_2, \ldots, y_m, k_m \end{bmatrix}$$

$$= \det \begin{bmatrix} K_{j_1 k_1}\,(x_1, y_1) & K_{j_1 k_2}\,(x_1, y_2) \ldots \\ K_{j_2 k_1}\,(x_2, y_1) & K_{j_2 k_2}\,(x_2, y_2) \ldots \end{bmatrix}_{m \times m} \quad (11)$$

$$\mathbf{F}_i(x, y, \zeta) = \sum_{m=0}^{\infty} \mathbf{F}_i^{(m)}\,(x, y)\, \zeta^m \quad (12).$$

where the jk the element of $\mathbf{F}_i^{(n)}\,(x\,,\,y)$ is

$$F_{jk}^{(m)}(x, y) = \frac{(-1)^m}{m!} \sum_{j_1 j_2 \cdots j_m = 1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots\cdots\cdots \int_{-\infty}^{\infty} \cdot$$

$$K^{(m+1)} \begin{bmatrix} x, j, x_1, j_1, x_2, j_2, \ldots, x_m, j_m \\ \cdot \\ y, k, x_1, j_1, x_2, j_2, \ldots, x_m, j_m \end{bmatrix}$$

$$dx_1, dx_2, \ldots\ldots\ldots\ldots dx_m \qquad (13)$$

Standard Fredholm theory requires that region of integration in the Fredholm equation should be finite but in our case it is infinite as can be seen from (8). To overcome this difficulty we now impose conditions on $\mathbf{B}(y, \zeta)$. It must be such that there exists a real function $K(y, \zeta)$ satisfying

$$\left| [\, \mathbf{K}_i(x, y, \zeta)\,]_{jk} \right| \leq K(y, \zeta) \qquad (14)$$

for $\quad i, j, k = 1, 2, \ldots, n, \qquad -\infty < x < \infty$ and

$$\int_{-\infty}^{\infty} K(y, \zeta)\, dy = M(\zeta), \qquad (15)$$

where $M(\zeta) \to 0$ as $\zeta \to \infty$ and is bounded throughout the complex ζ-plane. Under these circumstances the series for the Fredholm determinant and minor converge and it follows that $f_i(\zeta)$ and $\mathbf{F}_i(x, y, \zeta)$ are analytic with respect to ζ in the interior of any of these regions and that their limits as ζ approaches a boundary exist and are finite. Thus provided $f_i(\zeta) \neq 0$ the resolvent is

$$\mathbf{R}_i(x, y, \zeta) = \frac{1}{f_i(\zeta)} \mathbf{F}_i(x, y, \zeta) \qquad (16)$$

and the solution of (8) is

$$\Phi_i(x, \zeta) = v_i(\zeta) + \int_{-\infty}^{\infty} \mathbf{R}_i(x, y, \zeta)\, v_i(\zeta)\, dy \qquad (17)$$

This solution is bounded for $-\infty < x < \infty$ and since it is easily shown that if (14) is satisfied then

$$\left| [\mathbf{K}_i\,(x, y, \zeta)\,]_{jk} \right| \leq K(y, \zeta) \exp\{P(x-y)\} \qquad (18)$$

for $y > x$ where P is the smallest non-negative number in the set $\{\text{Re}\ \lambda_j(\zeta) - \lambda_i(\zeta),\ j \neq 1\}_k$ it follows from (8) that in the interior of any of the regions in the ζ-plane,

$$\Phi_i(x, \zeta) \rightarrow v_i(\zeta) \text{ as } x \rightarrow -\infty$$

Thus we have that

$$\Phi_i(x, \zeta) = \exp\{\lambda_i(\zeta)\, x\}\ \Phi_i(x, \zeta),$$

where $\Phi_i(x, \zeta)$, $i = 1, 2, \ldots, n$ form a *et* of Jost functions.

$$\Phi_i(x, \zeta) = e^{-\lambda_i(\zeta)x}\ \Phi_i(x, \zeta) \tag{19}$$

satisfy conditions (ii) and (iii) at any point in the interior of any region of the complex ζ-plane where $f_i(\zeta) \neq 0$ it is easy to check that condition (i) is also satisfied. We take (8) and (19) as the definitions of the Jost function $\Phi_i(x, \zeta)$. It is regular throughout the complex ζ-plane apart from poles where $f_i(\zeta) = 0$ and finite singularities on the boundaries between the regions. Conditions (14) and (15) ensure that

$$\lim_{\zeta\to\infty} \{\exp(-\lambda_i(\zeta)x)\ \phi_i(x, \zeta) - v_i(\zeta)\}$$

$$= \lim_{\zeta\to\infty} \{\phi_i(x, \zeta) - v_i(\zeta)\} = 0 \tag{20}$$

THE SPECTRAL DATA

The spectral data consists of information about the singularities to the Jost functions $\phi_i(x, \zeta)$. First we consider the poles. These occur where the Fredholm determinant $f_i(\zeta)$ vanishes. We consider only the simplest case by assuming that the zeros $\zeta_i^{(k)}$, $k = 1, 2, \ldots, m_i$ of $f_i(\zeta)$ are simple, do not coincide with a zero of $f_j(\zeta), j \neq i$ and do not lie on a boundary between two regions. In this case, the pole in $\phi_i(x, \zeta)$ is simple and the residue is:

$$\text{Res}\ \phi_i(x, \zeta_i^{(k)}) = \lim_{\zeta\to\zeta_i^{(k)}} \{(\zeta - \zeta_i^{(k)})\ \phi_i(x, \zeta)\}$$

$$= \lim_{\zeta\to\zeta_i^{(k)}} \left\{\frac{(\zeta - \zeta_i^{(k)})}{f_i(\zeta) - f_i(\zeta_i^{(k)})}\ f_i(\zeta)\ \phi_i(x, \zeta)\right\},$$

where $f_i(\zeta_i^{(k)}) = 0$

$$= \lim_{\zeta \to \zeta_i^{(k)}} \left\{ \frac{f_i(\zeta)}{f_i'(\zeta)} \phi_i(x, \zeta) \right\}, \tag{21}$$

where $f_i'(\zeta) = \dfrac{d}{d\zeta} f_i(\zeta)$ (22)

If $u_i(x, \zeta)$, $i = 1, 2, \ldots, n$ are solutions of (1) then the wronskian is given by

$$W(u_1, u_2, \ldots, u_n) = \det [u_1, u_2, \ldots, u_n] \tag{23}$$

and $\dfrac{\partial}{\partial x} W = T_r \{ \mathbf{A}(\zeta) + \mathbf{B}(x, \zeta) \} W$ (24)

By choosing $u_i(x, \zeta) = \phi_i(x, \zeta)$ $(i = 1, 2, \ldots, n)$ (24) becomes

$$W(\phi_1(x, \zeta), \phi_2(x, \zeta), \ldots, \phi_n(x, \zeta))$$

$$= \alpha \exp \left\{ T_r (\mathbf{A}(\zeta))\, x + \int_{\infty}^{x} T_r (\mathbf{B}(y, \zeta))\, dy \right\} \tag{25}$$

and $\operatorname{Res} \phi_i(x, \zeta_i^{(k)}) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \gamma_{ij}^{(k)} \phi_j(x, \zeta_i^{(k)})$ (26)

The quantities $\zeta_i^{(k)}$ and $\gamma_{ij}^{(k)}$ constitute the discrete part of the spectral data.

Next we consider the finite singularities on the boundaries between regions. If on a given boundary there is only one pair i, j such that $\operatorname{Re}(\lambda_i - \lambda_j) = 0$ we shall say the boundary is simple. The two sides of a simple boundary can be labelled $\pm$ve according to the sign of $\operatorname{Re}(\lambda_j - \lambda_i)$ and we shall use the superfix $\pm$ to denote the limit of a quantity as the boundary is approached on the $\pm$ve side.

From the definition of $\mathbf{K}_i(n, y, \zeta)$ it is clear that since

$$\mathbf{P}_i^{(u)+}(\zeta) - \mathbf{P}_i^{(u)-}(\zeta) = \mathbf{P}_j(\zeta) \tag{27}$$

$$\therefore \quad \mathbf{K}_i^{+}(x, y, \zeta) - \mathbf{K}_i^{-}(x, y, \zeta) = \exp \{ (x - y) (\mathbf{A}(\zeta) - \lambda_i(\zeta) \mathbf{I}) \} - \mathbf{P}_j(\zeta)\, \mathbf{B}(y, \zeta)$$

$$= \exp \{ (x - y) (\lambda_j(\zeta) - \lambda_i(\zeta) - \lambda_i(\zeta) \} \mathbf{P}_j(\zeta)\, \mathbf{B}(y, \zeta) \tag{28}$$

and $\mathbf{K}_i(x, y, \zeta) = \exp \{ (x - y) (\lambda_j(\zeta) - \lambda(\zeta) \} \mathbf{K}_j^{+}(x, y, \zeta)$ (29)

The resolvent equation

$$\mathbf{R}_i\,(x, y, \zeta) - \mathbf{K}_i\,(x, y, \zeta)$$

$$= \int_{-\infty}^{\infty} \mathbf{R}_i(x, z, \zeta)\,.\,\mathbf{K}_i(z, y, \zeta)\,dz$$

$$= \int_{-\infty}^{\infty} \mathbf{K}_i(x, z, \zeta) - \mathbf{R}_i(z, y, \zeta)\,dz \tag{30}$$

gives $\bar{\mathbf{R}}_i\,(x, y, \zeta) = \exp\{(x - y)\,(\lambda_j\,(\zeta) - \lambda_i(\zeta)\}\,\mathbf{R}_j^{\,+}\,(x, y, \zeta)$ (31)

And finally we know

$$\phi_i(x, \zeta)\,e^{-\lambda_i(\zeta)x} = v_i(\zeta) + \int_{-\infty}^{\infty} \mathbf{R}_i(x, y, \zeta)\,.\,v_i(\zeta)\,dy$$

$$\phi_i^{\,+}(x, \zeta)\,e^{-\lambda_i(\zeta)x} = v_i(\zeta) + \int_{-\infty}^{\infty} \mathbf{R}_i^{\,+}(x, y, \zeta)\,.\,v_i(\zeta)\,dy$$

$$\phi_i^{\,-}(x, \zeta)\,e^{-\lambda_i(\zeta)x} = v_i(\zeta) + \int_{-\infty}^{\infty} \bar{\mathbf{R}}_i(x, y, \zeta)\,.\,v_i(\zeta)\,dy$$

$$\therefore \quad \phi_i^{\,+}(x, \zeta) - \phi_i^{\,-}(x, \zeta) = \exp\{\lambda_i(\zeta)x\}$$

$$\int_{-\infty}^{\infty} \{\,\mathbf{R}_i^{\,+}(x, y, \zeta) - \bar{\mathbf{R}}_i(x, y, \zeta)\,\}\,.\,v_i(\zeta)\,dy$$

$$= \exp\,\{\lambda_i(\zeta)x\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\,\delta\,(x - w)\,\mathbf{I} + \mathbf{R}_i^{\,+}\,(x, w, \zeta)\,]$$

$$- [\,\mathbf{K}_i^{\,+}\,(w, z, \zeta) - \bar{\mathbf{K}}_i\,(w, z, \zeta)\,]$$

$$- [\,\delta\,(z - y)\,\mathbf{I} + \bar{\mathbf{R}}_i\,(z, y, \zeta)\,]\,v_i(\zeta)\,dw\,dz\,dy$$

$$= \exp\,\{\lambda_i(\zeta)x\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\,\{\,\delta\,(x - w)\,\mathbf{I} + \mathbf{R}_j^{\,+}\,(x, w, \zeta)\,]$$

$$= \exp\,\{\,(\lambda_i(\zeta) - \lambda_j(\zeta)\,z\,\}\,.\,\mathbf{P}_j(\zeta)\,.\,\mathbf{B}(z, \zeta)\,.\,\phi_{-i}(z, \zeta)\,dw\,dz$$

$$= \phi_j(x, \zeta)\,\mathrm{Q}_{ji}(\zeta)\,, \tag{32}$$

$$\text{where } Q_{ji}(\zeta) = \frac{1}{\tilde{v}(\zeta) \cdot v_j(\zeta)} \tilde{v}_j(\zeta) - \int_{-\infty}^{\infty} \exp\{(\lambda_i(\zeta) - \lambda_j(\zeta)) z\}$$

$$\mathbf{B}(z, \zeta) \cdot \phi_{-i}(z . \zeta)\, dz \qquad (33)$$

Now we will discuss the case for the compound boundaries. On the compound boundary

$$\text{Re}\, \lambda_i^{(\zeta)} < \text{Re}\, \lambda_j^{(\zeta)} < \text{Re}\, \lambda_k^{(\zeta)}$$

is on the +ve side and $\text{Re}\, \lambda_i^{(\zeta)} > \text{Re}\, \lambda_j^{(\zeta)} > \text{Re}\, \lambda_k^{(\zeta)}$ on the −ve side. Thus

$$\mathbf{P}_i^{(u)+}(\zeta) - \mathbf{P}_i^{(u)-}(\zeta) = \mathbf{P}_j(\zeta) + \mathbf{P}_k(\zeta) \qquad (34)$$

$$\mathbf{K}_i^{+}(x, y, \zeta) - \mathbf{K}_i(x, y, \zeta) = \exp\{(x-y)(\mathbf{A}(\zeta) - \lambda_i(\zeta)\mathbf{I}\} - \{\mathbf{P}_j(\zeta) + \mathbf{P}_k(\zeta)\}\, \mathbf{P}(y, \zeta) \qquad (35)$$

and $$\mathbf{K}_i(x, y, \zeta) = \exp\{(x-y)(\lambda_k(\zeta) - \lambda_i(\zeta))\, \mathbf{K}_k^{+}(x, y, \zeta) \qquad (36)$$

Arguing in the same way as for the simple boundary we get

$$\phi_i^{+}(x, \zeta) - \phi_i^{-}(x, \zeta) = \phi_j^{+}(x, \zeta)\, Q_{ji} + \phi_k^{+}(x, \zeta)\, Q_{ki}, \qquad (37)$$

where $Q_{ji}(\zeta)$ is give by (32) and $Q_{ki}(\zeta)$ is given by

$$Q_{ki}(\zeta) = \frac{1}{v_k(\zeta) \cdot v_k(\zeta)} \tilde{v}_k$$

$$\int_{-\infty}^{\infty} \exp\{(\lambda_i(\zeta) - \lambda_k(\zeta) z\}\, \mathbf{B}(z, \zeta) \cdot \phi_i^{-}(z, \zeta)\, dz \qquad (38)$$

The quantities $Q_{ij}(\zeta)$ along all the boundaries constitute the continuum part of the spectral data. The spectral data is

$$S = \{\zeta_i^{(k)}, \gamma_{ij}^{(k)}, Q_{ij}(\zeta), i, j = 1, 2, \ldots, n, i = j$$

K = 1, 2,..., m_i and ζ runs over all the boundaries between the regions. (39)

THE INVERSE SPECTRAL PROBLEM

The inverse spectral problem that of the reconstruction of the matrix $\mathbf{B}(x, \zeta)$ from the spectral data S. We notice that the quantities

$$\phi_i(x, \zeta) = \exp\{-\lambda_i(\zeta)\, x\}\, \phi_i(x, \zeta) \qquad (40)$$

(from (19)) have the following properties:

(i) $\phi_i(x, \zeta) - v_i(\zeta) \to 0$ as $\zeta \to \infty$ from (20) (41)

(ii) $\phi_i(x, \zeta)$ has simple poles at $\zeta = \zeta_i^{(k)}$

$k = 1, 2, \ldots, m_i$ with residues

$$\text{Res } \Phi_i (x, \zeta_i^{(k)}) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \gamma_{ij}^{(k)} \exp \{(\lambda_j(\zeta_i^{(k)} - \lambda_i(\zeta_i^{(k)})x\} \Phi_j(x, \zeta_k^{(k)})$$

from (26)

(iii) On a boundary

$$\Phi_i^+(x,\zeta) - \Phi_i^-(x,\zeta) = \Sigma_j Q_{ij}(\zeta) \exp \{(\lambda_j(\zeta) - \lambda_i(\zeta))x\} \Phi_j^+(x,\zeta) \quad (43)$$

These properties are sufficient to define $\Phi_i(x, \zeta)$

$$\underline{\Phi}_i(x, \zeta) = v_i(\zeta) - \sum_{k=1}^{m_i} \text{Res } \Phi_i(x, \zeta_i^{(k)})$$

$$+ \frac{1}{2\pi i} \int \frac{\{\Phi_i^+(x, \zeta') - \Phi_i^-(x, \zeta')}{\zeta' - \zeta} d\zeta' \quad (44)$$

$$\therefore \quad \Phi_i(x, \zeta) = v_i(\zeta) -$$

$$\sum_{k=1}^{m_i} \sum_{\substack{j=1 \\ j \neq i}}^{m_i} \gamma^k_{ij} \exp \frac{\{(\lambda_j(\zeta_i^{(k)}) - \lambda_i(\zeta_i^{(k)}))\, x\}}{\zeta_i^{(k)} - \zeta} \Phi_j(x, \zeta_i^{(k)})$$

$$+ \frac{1}{2\pi i} \int \Sigma_j \frac{Q_{ij}(\zeta) \exp \{(\lambda_j(\zeta') - \lambda_i(\zeta'))x\}}{\zeta' - \zeta} \Phi_j^{\pm}(x, \zeta')\, d\zeta' \quad (45)$$

where the integral is along all the boundaries the direction of integration being so that the +ve side is on the left. By choosing appropriate values for ζ, the left hand side of the equation (45) can be written as

$\Phi_j(x, \zeta_i^{(k)}), i, j = 1, 2, \ldots, n, i \neq j, k = 1, 2, \ldots, m_i$, *i.e.*,

$$\Phi_j(x, \zeta_i^{(k)}) = v_j(\zeta) - \sum_{p\, k=1}^{m_i} \sum_{q\, i=1}^{m_i}$$

$$\frac{\gamma_{jq}^{(p)} \exp\{(\lambda_q(\zeta_j^{p}) - \lambda_j(\zeta_j^{p}))x}{\zeta_j^{(p)} - \zeta_i^{(k)}} \Phi_q(x, \zeta_j^{(p)})$$

$$+ \frac{1}{2\pi i} \int \sum_{l=1}^{m_i} \sum_{i\,q=1}^{n} \frac{Q_{jq}(\zeta') \exp\{(\lambda_q(\zeta') - \lambda_j(\zeta'))x\}}{\zeta' - \zeta_i^{(k)}}$$

$$\Phi_q^{\pm}(x, \zeta')\, d\zeta' \qquad (46)$$

or when ζ approaches the boundary from the appropriate sides we have the left hand side of equation (45) $\Phi^{\pm}(x, \zeta')$. There is singularity in the integrand. To overcome this difficulty we draw a semi-circle around the singularity, keeping the singularity on the left side of the contour. Thus we have a set of linear matrix/Fredholm equations in the unknowns $\Phi_j(x, \zeta_i^{(k)})$ and $\Phi_j^{\pm}(x, \zeta')$. The question of the existence and uniqueness of the solution to these equations has yet to be investigated, but in many cases of practical interest there appears to be no difficulty. Equations (40) and (45) give $\Phi_i(x, \zeta)$, $i = 1, 2, \ldots, n$ throughout the complex ζ-plane and hence $\mathbf{B}(x, \zeta)$ can be found from (1).

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4 CYCLIC 4-STEP A-STABLE METHOD

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ABSTRACT

The solution of stiff system of differential equations contains slowly as well as fast decaying component. The numerical method with finite stability regions are therefore not suitable for solving such systems. One is therefore interested in methods which are absolute stable in the left-half complex plane or in at least an infinite portion of it.

In this work 4 cyclic 4-step A-stable methods of order two are constructed. It seems that the methods of order higher than two may not be possible. The methods are applied to stiff and non-stiff problems. The numerical results are then compared with some published results.

1. INTRODUCTION

Let us consider the initial value problem

$$y' = f(t, y) \qquad (1.1)$$

$$y(0) = y_0, 0 \leq t \leq T$$

For the numerical approximate solution to the initial value problem (1.1) m cyclic k-step method, namely

$$Z_{m(n+1)} = AZ_{mn} + mhB_0 f(Z_{m(n+1)}) + mhB_1 f(Z_{mn})$$

$$(n = 0, 1, \ldots) \qquad (1.2)$$

(P. Albrecht [6]) are to be considered, where A, B_0, B_1 are the $k \times k$ matrices,

$$f(Z_{mn}) = \begin{bmatrix} f(t_{mn+k-1}, y_{mn+k-1}) \\ f(t_{mn+k-2}, y_{mn+k-2}) \\ \cdot \\ \cdot \\ \cdot \\ f(t_{mn}, y_{mn}) \end{bmatrix}$$

$$Z_{mn} = \begin{bmatrix} y_{mn+k-1} \\ y_{mn+k-2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{mn} \end{bmatrix}$$

The linear k-step method of the type

$$\sum_{\nu=0}^{k} a_\nu y_{n+\nu} + h \sum_{\nu=0}^{k} b_\nu f(t_{n+\nu}, y_{n+\nu}) = 0$$

is special case of (1.2) when $m = 1$ (Urabe [7]).

DEFINITION 1.1

Putting $y' = \lambda y$ in (1.2), we get a polynomial

$$\det [(I-HB_0)\mu - (A+HB_1)] = Q(\mu, H) \tag{1.3}$$

where $H = h\lambda$

We call it a stability polynomial.

THEOREM 1.1 (MIR [4])

Let there exist a unique solution of the initial value problem (1.1) and $y(t)$ the exact solution to (1.1). Let $f(t, y(t))$ be sufficiently smooth. Then m cyclic k-step method (1.2) is consistent with the initial value problem (1.1) if the following conditions hold:

$$Ae = e$$

and $(A - I)C = m(I - \phi)e,$

where $\phi = B_0 + B_1$,

$$C = \begin{pmatrix} k-1 \\ k-2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}, \text{ I is } k \times k \text{ identity matrix}$$

The method (1.2) is of order q, when simultaneously the following conditions hold:

$$(A-I)\, C^p = pm(I-\phi)\, C^{p-1} + \sum_{\nu=0}^{p-2} \binom{p}{\nu} m^{p-\nu}(I-(p-\nu)B_0)C^{\nu}$$

$$(p = 0, 1, 2, \ldots\ldots\ldots\ldots, q) \qquad (1.4)$$

where the sum on the right-hand side for $p \leq 1$ is considered to be empty and

$$C = \begin{pmatrix} (k-1)^p \\ (k-2)^p \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}, \quad C^0 = e$$

DEFINITION 1.2 ([2])

Let $\mu_i(H)$, $i = 1\,(1)k$ be the eigenvalues of the stability polynomial $Q(\mu, H)$ (1.3). The method (1.2) is said to be absolute stable if $|\mu_i(H)| < 1$, $i = 1(1)k$, where $H = \{H \varepsilon C: |\mu_i.(H)| < 1, i = 1(1)k\}$ is called its regions of absolute stability.

DEFINITION 1.3 ([5])

A method which is absolute stable in the left-half complex plane C is called A-stable, that is, a method is A-stable if

$$1H \supset \{ H : \text{Re}(H) < 0 \}$$

THEOREM (1.2) (Dahlquist [1])

An explicit linear multistep method cannot be A-stable. It follows that explicit m cyclic k-step method cannot be A-stable.

2. CONSTRUCTION OF 4 CYCLIC 4 STEP A-STABLE METHODS

To construct 4 cyclic 4-step A-stable method of order two we determine the 4 × 4 matrices A, B_0, B_1 so that they satisfy the first three linear equations in (1.4).

Each of these equations represents 4 single equations. Therefore, we have to solve 12 equations for 48 unknowns, *i.e.*, 4-systems having each three equations with 12 unknowns and 9 free parameters.

In order to make the method A-stable, we shall choose the 9 surplus unknowns in such a way that

$$| \mu_i (H) | < 1 \ (i = 1, 2, 3, 4) , \text{Re}(H) < 0$$

Adopting the above mentioned procedure, we have, therefore the following equations.

$$Ae = e$$

$$(A - I) C = 4 (I - B_0 - B_1) e$$

$$(A - I) C^2 = 8 (I - B_0 - B_1) C + 16 (I - 2B_0) e \qquad (2.1)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$B_1 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$B_0 = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

and $$C = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

The four systems from (2.1) are given by

First System

$$a_{11} + a_{12} + a_{13} + a_{14} = 1$$
$$3a_{11}+2a_{12}+a_{13} = 7-4b_{11}-4b_{12}-4b_{13}-4b_{14}-4c_{11}-4c_{12}-4c_{13}-4c_{14}$$
$$9a_{11}+4a_{12}+a_{13}=49-56b_{11}-48b_{12}-40b_{13}-32b_{14}-24c_{11}-16c_{12}-8c_{13} \quad (2.2)$$

Second System

$$a_{21} + a_{22} + a_{23} + a_{24} = 1$$
$$3a_{21}+2a_{22}+a_{23} = 6-4b_{21}-4b_{22}-4b_{23}-4b_{24}-4c_{21}-4c_{22}-4c_{23}-4c_{24}$$
$$9a_{21}+4a_{22}+a_{23}=36-56b_{21}-48b_{22}-40b_{23}-32b_{24}-24c_{21}-16c_{22}-8c_{23} \quad (2.3)$$

Third System

$$a_{31} + a_{32} + a_{33} + a_{34} = 1$$
$$3a_{31}+2a_{32}+a_{23} = 5-4b_{31}-4b_{32}-4b_{33}-4b_{34}-4c_{31}-4c_{32}-4c_{33}-4c_{34}$$
$$9a_{31}+4a_{32}+a_{33}=25-56b_{31}-48b_{32}-40b_{33}-32b_{34}-24c_{31}-16c_{32}-8c_{33} \quad (2.4)$$

Fourth System

$$a_{41} = -22b_{41} - 18b_{42} - 14b_{43} - 10b_{44} - 6c_{41} - 2c_{42} + 2c_{43} + 6c_{44} - a_{44}$$

$$a_{42} = -3 + 40b_{41} + 32b_{41} + 32b_{42} + 24b_{43} + 16b_{44} + 8c_{41} + 0c_{42} - 8c_{43} - 16c_{44} + 3a_{44}$$

$$a_{43} = 1 - 18b_{41} - 14b_{42} - 10b_{43} - 6b_{44} - 2c_{41} + 2c_{42} + 6c_{43} + 10c_{44} - 3a_{44} \qquad (2.9)$$

METHOD 2.1

We shall now choose the free parameters for the four systems (2.6) – (2.9) in such a way that the conditions of A-stability, namely

$$|\mu_i(H)| < 1 \;,\; (i = 1, 2, 3, 4)\;,\; \mathrm{Re}(H) < 0,$$

are satisfied. The characteristic equation (1.3) gives

$$\left| \begin{pmatrix} 1-Hb_{11} & -Hb_{12} & -Hb_{13} & -Hb_{14} \\ -Hb_{21} & 1-Hb_{22} & -Hb_{23} & -Hb_{24} \\ -Hb_{31} & -Hb_{32} & 1-Hb_{33} & -Hb_{34} \\ -Hb_{41} & -Hb_{42} & -Hb_{43} & 1-Hb_{44} \end{pmatrix} \mu - \begin{pmatrix} a_{11}+Hc_{11} & a_{12}+Hc_{12} & a_{13}+Hc_{13} & a_{14}+Hc_{14} \\ a_{21}+Hc_{21} & a_{22}+Hc_{22} & a_{23}+Hc_{23} & a_{24}+Hc_{24} \\ a_{31}+Hc_{31} & a_{32}+Hc_{32} & a_{33}+Hc_{33} & a_{34}+Hc_{34} \\ a_{41}+Hc_{41} & a_{42}+Hc_{42} & a_{43}+Hc_{43} & a_{44}+Hc_{44} \end{pmatrix} \right| = 0$$

This gives

$$\begin{vmatrix} \mu(1-Hb_{11})-a_{11}-Hc_{11} & -\mu Hb_{12}-a_{12}-Hc_{12} & -\mu Hb_{13}-a_{13}-Hc_{13} & -\mu Hb_{14}-a_{14}-Hc_{14} \\ -\mu Hb_{21}-a_{21}-Hc_{21} & -\mu(1-Hb_{22})-a_{22}-Hc_{22} & -\mu Hb_{23}-a_{23}-Hc_{23} & -\mu Hb_{24}-a_{24}-Hc_{24} \\ -\mu Hb_{31}-a_{31}-Hc_{31} & -\mu Hb_{32}-a_{32}-Hc_{32} & \mu(1-Hb_{33})-a_{33}-Hc_{33} & -\mu Hb_{34}-a_{34}-Hc_{34} \\ -\mu Hb_{41}-a_{41}-Hc_{41} & -\mu Hb_{42}-a_{42}-Hc_{42} & -\mu Hb_{43}-a_{43}-Hc_{43} & \mu(1-Hb_{44})-a_{44}-Hc_{44} \end{vmatrix} = 0 \qquad (2.10)$$

Let us choose,

$$b_{12} = b_{13} = b_{14} = c_{12} = c_{13} = c_{14} = a_{12} = a_{13} = a_{14} = 0$$

This reduces equations (2.6) to

$$a_{11} = 7 - \frac{26}{3} b_{11} - \frac{10}{3} c_{11} \tag{2.11}$$

$$7 = 11b_{11} + 3c_{11} \tag{2.12}$$

$$-3 = -5b_{11} - c_{11}$$

Solving equations (2.12) simultaneously, we get

$$b_{11} = c_{11} = \frac{1}{2}$$

from (2.11), we therefore have $a_{11} = 1$.

Using these values of free parameters alongwith $a_{12} = 0$ (2.10) gives

$$(\mu(1-H/2)-1-H/2)\begin{vmatrix} \mu(1-Hb_{22})-a_{22}-Hb_{22} & -\mu Hb_{23}-a_{23}-Hc_{23} & -\mu Hb_{24}-a_{24}-Hc_{24} \\ -\mu Hb_{32} \quad a_{32}-Hc_{32} & \mu(1-Hb_{33})-a_{33}-Hc_{33} & -\mu Hb_{34}-a_{34}-Hc_{34} \\ -\mu Hb_{42}-a_{42}-Hc_{42} & -\mu Hb_{43}-a_{43}-Hc_{43} & \mu(1-Hb_{44})-a_{44}-Hc_{44} \end{vmatrix}$$

$$= 0$$

This implies $\mu_1 = \dfrac{1 + H/2}{1 - H/2}$ (2.13)

$$\begin{vmatrix} \mu(1-Hb_{22})-a_{22}-Hb_{22} & -\mu Hb_{23}-a_{23}-Hc_{23} & -\mu Hb_{24}-a_{24}-Hc_{24} \\ -\mu Hb_{32}-a_{32}-Hc_{32} & \mu(1-Hb_{33})-a_{33}-Hc_{33} & -\mu Hb_{34}-a_{34}-Hc_{34} \\ -\mu Hb_{42}-a_{42}-Hc_{42} & -\mu Hb_{43}-a_{43}-Hc_{43} & \mu(1-Hb_{44})-a_{44}-Hc_{44} \end{vmatrix} = 0 \tag{2.14}$$

Choosing

$$b_{22} = a_{22} = c_{22} = 1\,,\ b_{23} = c_{23} = b_{24} = c_{24} = a_{23} = a_{24} = 0$$

reduces equations (2.7) to

$$a_{21} = -\frac{14}{3} - \frac{26}{3} b_{21} - \frac{10}{3} c_{21} \tag{2.15}$$

$$-5 = 11b_{21} + 3c_{21}$$

$$4 = 10b_{21} - 2c_{21} \tag{2.16}$$

Solving equations (2.16) simultaneously, we get

$$b_{21} = -\frac{1}{4}\,,\ c_{21} = -\frac{3}{4}$$

From (2.15), we therefore have $a_{21} = 0$

Let us now choose

$$b_{31} = b_{33} = b_{34} = c_{32} = c_{33} = c_{34} = a_{33} = a_{32} = a_{34} = 0$$

This reduces equations (2.11) to

$$a_{31} = 5 - \frac{40}{3} b_{32} - \frac{16}{3} c_{31} \qquad (2.17)$$

$$3 = 14b_{32} + 2c_{32} \qquad (2.18)$$

$$5 = 18b_{32} + 6c_{31}$$

Solving equations (2.18) simultaneously, we have

$$b_{32} = \frac{1}{6}, \quad c_{31} = \frac{1}{3}$$

From equations (2.17), we therefore have

$$a_{31} = 1$$

Taking $b_{42} = b_{43} = c_{42} = c_{43} = a_{43} = a_{42} = 0$, $b_{44} = a_{44} = c_{44} = 1$ reduces equations (2.9) to

$$a_{41} = -2 - 22b_{41} - 6c_{41} \qquad (2.19)$$

$$1 = 9b_{41} + c_{41} \qquad (2.20)$$

$$0 = 5b_{41} + c_{41}$$

Solving equations (2.20) simultaneously, we get

$$b_{41} = \frac{1}{4}, \quad c_{41} = \frac{-5}{4}$$

From equations (2.19) , we therefore have

$$a_{41} = 0$$

Using these values of free parameters alongwith $a_{22} = a_{33} = a_{44} = 0$, equation (2.14) reduces to

$$\begin{vmatrix} \mu(1-H)-1-H & 0 & 0 \\ -\mu H/6 & \mu & 0 \\ 0 & 0 & \mu(1-H)-1-H \end{vmatrix} = 0 \qquad (2.21)$$

Solving (2.21), we get

$$\mu_2 = 1+H/1-H$$

$$\mu_3 = 1+H/1-H\,,\ \mu_4 = 0 \qquad (2.22)$$

Obviously the eigenvalues in (2.13) and (2.22) satisfy the condition of A-stability and thus makes the method (1.2) A-stable. The elements of A, B_0 and B_1 are compiled in the following table.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ 0 & 1/6 & 0 & 1 \\ 1/4 & 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ -5/4 & 0 & 0 & 1 \end{bmatrix} \qquad (2.23)$$

METHOD 2.2

Adopting the same procedure and choosing again the same free parameters as discussed previously for the method (2.1) we have with $a_{11} = 1$, $a_{13} = a_{14} = a_{12} = b_{12} = b_{13} = b_{14} = c_{12} = c_{13} = c_{14} = 0$, $b_{11} = 1/2$, $c_{11} = 1/2$,

$$\mu_1 = \frac{1 + H/2}{1 - H/2} \qquad (2.24)$$

and with $b_{22} = a_{22} = c_{22} = b_{23} = c_{23} = b_{24} = c_{24} = 0$, $a_{22} = a_{23} = a_{24} = 0$, $b_{21} = 9/32$, $c_{21} = 15/32$, $a_{21} = 1$, $a_{32} = a_{33} = a_{34} = 0$, $b_{32} = 1/6$, $c_{31} = 1/3$, $a_{31} = 1$, $a_{44} = a_{43} = a_{42} = 0$, $b_{41} = 1/32$, $c_{11} = 7/32$, $c_{41} = 7/32$, $a_{41} = 1$

$$\mu_2 = \mu_3 = \mu_4 = 0 \qquad (2.25)$$

The eigenvalues in (2.24) and (2.25) satisfy the conditions of A-stability and thus makes the method (1.2) A-stable.

Now the elements of A, B_0 and B_1 are compiled in the following table.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 9/32 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 1/32 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 15/32 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 7/32 & 0 & 0 & 0 \end{bmatrix} \qquad (2.26)$$

3. APPLICATIONS TO NON-STIFF PROBLEM

In this section, the method 2.2 is applied to a non-stiff initial value problem

$$y' = x + y \, , y(0) = 1 \qquad (3.1)$$

having the exact solution

$$y(x) = 2e^{x} - x - 1$$

The initial value problem (3.1) is computed on a VAX–11/730 with step sizes h = 0.01 and h = 0.001. The explicit Runge-Kutta method of order 3 and three stages ([3]) is

$$y_{n+1} = y_n + h/4 \, (K_1 + 3K_3),$$

where $K_1 = f(x_n, y_n)$,

$$K_2 = f(x_n + h/3, y_n + h/3 \, K_1) \qquad (3.2)$$

$$K_3 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hK_2\right)$$

We use the method (3.2) as a predictor and the method (2.2) as a corrector. The numerical results are given in the tables (3.1) – (3.2).

Table 3.1

	4 cyclic 4-step A-stable Method (2.2) of order two	Exact	Relative Error
X_n	Y_n	Y_n	R_n
0.4	0.1583781242D+01	0.158364937D+01	8.32D−05
0.5	0.1797626853D+01	0.1797442436D+01	1.02D−04
0.6	0.2153223515D+01	0.2152961731D+01	1.21D−04
0.7	0.2452204227D+01	0.2451871157D+01	1.35D−04
0:8	0.2793139458D+01	0.2792733908D+01	1.45D−04
0.9	0.3180459976D+01	0.3179962635D+01	1.56D−04
1.0	0.3619023800D+01	0.3618433952D+01	1.63D−04
1.1	0.4114243507D+01	0.4113536835D+01	1.71D−04

Table 3.2

	4 cyclic 4-step A-stable Method (2.2) of order two	Exact	Relative Error
X_n	Y_n	Y_n	R_n
0.4	0.1583651066D+01	0.1583649397D+01	1.05D−06
0.5	0.1797444582D+01	0.1797442436D+01	1.19D−06
0.6	0.2044240236D+01	0.2044237852D+01	1.16D−06
0.7	0.2327508926D+01	0.2327505350D+01	1.53D−06
0.8	0.2651085854D+01	0.2651081800D+01	1.52D−06
0.9	0.3019210815D+01	0.3019206047D+01	1.57D−06
1.0	0.3436569214D+01	0.3436563492D+01	1.66D−06
1.1	0.3908338308D+01	0.3908332348D+01	1.52D−06

The R_n column shows $| (y_n - y(x_n)/y(x_n) |$ at eight values of x, 0.4 to 1.1. Comparing the results of 4 cyclic 4-step A-stable method (2.1) with exact results for step-lengths 0.01 and 0.001, we note that the results are quite accurate although the method is of consistent order two only. The method is also very fast, since the accuracy is reached at the maximum at second iteration. The method (2.1) gives similar results.

4. APPLICATIONS TO STIFF PROBLEM

In this section, the methods (2.1) and (2.2) are applied to the stiff initial value problem ([3])

$$\begin{aligned} y' &= -200\,(y - F(x)) + F'(x)\,, \\ y(0) &= 10, \end{aligned} \qquad (4.1)$$

where $F(x) = 10 - (10 + x)\,e^{-x}$

Exact solution of (4.1) is

$$Y(X) = F(X) + 10e^{-200X}$$

The stiff problem (4.1) is computed numerically on a VAX−11/730 with step sizes 0.01, 0.005 and compared with the numerical results of fourth-order Runge-Kutta Method, Adman Fourth-order predictor corrector method and Calahan's Method ([3]).

We used the method (3.1) as a predictor and the 4 cyclic 4-step A-stable methods (2.1) and (2.2) as correctors. The numerical results are presented in table 4.1. The R_n column shows $| (y_n - y(x_n))/y(x_n) |$ at two values of x; 0.4 and 1.0. The point $x = 0.4$ and $x = 1.0$ are chosen as representation of error.

Table 4.1

Method	h	R_n	
		$x = 0.4$	$x = 1.0$
* RK_4	0.01	1.0D−05	2.0D−09
** DEQ	0.005	3.0D−09	2.0D−09
*** CAL	0.01	1.7D−02	4.0D−08
Method 2.1	0.01	1.07D−05	1.56D−06
" " " "	0.005	1.76D−07	2.9D−07
Method 2.2	0.01	1.7D−05	4.4D−06
" " " "	0.005	1.6D−07	0.0

The R_n column shows $\left| (y_n - y(x_n)/y(x_n) \right|$ at eight values of x, 0.4 to 1.1. Comparing the results of 4 cyclic 4-step A-stable method (2.1) with exact results for step-lengths 0.01 and 0.001, we note that the results are quite accurate although the method is of consistent order two only. The method is also very fast, since the accuracy is reached at the maximum at second iteration. The method (2.1) gives similar results.

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$$y(0) = 10, \qquad (4.1)$$

where $F(x) = 10 - (10 + x)\,e^{-x}$

Exact solution of (4.1) is

$$Y(X) = F(X) + 10e^{-200X}$$

The stiff problem (4.1) is computed numerically on a VAX−11/730 with step sizes 0.01, 0.005 and compared with the numerical results of fourth-order Runge-Kutta Method, Adman Fourth-order predictor corrector method and Calahan's Method ([3]).

We used the method (3.1) as a predictor and the 4 cyclic 4-step A-stable methods (2.1) and (2.2) as correctors. The numerical results are presented in table 4.1. The R_n column shows $\left| (y_n - y(x_n))/y(x_n) \right|$ at two values of x; 0.4 and 1.0. The point $x = 0.4$ and $x = 1.0$ are chosen as representation of error.

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Method 2.1	0.01	1.07D−05	1.56D−06
" " " "	0.005	1.76D−07	2.9D−07
Method 2.2	0.01	1.7D−05	4.4D−06
" " " "	0.005	1.6D−07	0.0

The above results show that the method CAL has less accuracy. The 4 cyclic 4-step A-stable methods (2.1) and (2.2) of order two give the same accuracy as the methods RK_4 and DEQ. However, the method (2.2) gives higher accuracy at the point $x = 1.0$ as compared to the method DEQ.

* Fourth-order Runge-Kutta Method

** Adams Fourth-order predictor-corrector Method

*** Calahan's Method

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CONSTRUCTION OF THE EVOLUTION EQUATIONS

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ABSTRACT

This paper is an attempt to convert the spectral problem

$$\frac{\partial}{\partial x} u = \{ A(\zeta) + B(x, \zeta) \} . u \tag{1}$$

where u is an n-element column vector and $A(\zeta)$ and $B(x, \zeta)$ are $n \times n$ matrices into non-linear evolution equations. The spectral problem (1) is already solved by [1].

INTRODUCTION

Linear scattering problem

$$dv = \Omega v \tag{2}$$

in which Ω is an $N \times N$ matrix consisting of a one-parameter (ζ, the spectral parameter) is family of one-forms. For appropriate parameterization of Ω the integrability condition of equation (2), *i.e.*,

$$\theta = 0 \quad , \quad \theta = d\Omega - \Omega \wedge \Omega \tag{3}$$

gives integrable evolution equations.

The above expression (3) of a non-linear evolution equation is sometimes called a Lax pair representation. The simplest structure of Ω is

$$\Omega = (\zeta R_0 + P)\, dx + Q(\zeta)\, dt \,, \; \mathrm{Tr}\, \Omega = 0 \tag{4}$$

in which R_0 and P are $N \times N$ matrices independent of the spectral parameter ζ. The equation (1) is equivalent to (4) and

$$\frac{\partial}{\partial t} u = Qu$$

where $A(\zeta) = \zeta R_0$ and $B(x, \zeta) = P$ (5)

we construct non-liner evolutes by determining Q in equation (4) from the zero-curvature equation (3). We follow the original idea due to Kotera and Swada [2], and Newell [3] and then developed by Alberty *et al.* [4].

R_0 is a diagonal N×N matrix of constant entries.

$$R_0 = \text{diag.} \ (\beta_1, \beta_2, \ldots\ldots\ldots\ldots\ldots \beta_N) \tag{6}$$

in which β_1 are in general complex and distinct

$$\beta_i \neq \beta_j \quad \text{if} \quad i \neq j \tag{7}$$

The "potential" matrix P is independent of ζ and is off diagonal,

$$P_{ii} = 0 \ , \ i = 1, 2, \ldots, N \tag{8}$$

and its entries are dependent variables (fields) assumed to obey the boundary conditions.

$$P_{ij}(x, t) \ , \frac{\partial}{\partial x} P_{ij} \ , \ldots, \left(\frac{\partial}{\partial x}\right)^m P_{ij}, \ldots, 0 \ \text{as} \ |x| \rightarrow \infty \tag{9}$$

The linear problem (2) is explicitly

$$V_x = (\zeta R_0 + P) V \ \text{and} \ V_t = QV \tag{10}$$

where $$V_x = \frac{\partial}{\partial x} V \ \text{and} \ V = (V_1, \ldots\ldots V_N)^T \tag{11}$$

and $$V_t = \frac{\partial}{\partial t} V$$

Then the zero-curvature equation (3) expressed explicitly as

$$P_t - Q_x + [\, \zeta R_0 + P \, , Q \,] = 0 \tag{12}$$

guarantees the integrability (compatibility) of equations (10). Now we find the general expression for Q in terms of P and x-derivatives so that it gives a ζ-independent non-linear evolution equation for P through equation (12).

Let us start by assuming that Q is a polynomial in ζ,

$$Q = \sum_{r=0}^{n} \zeta^r Q_{n-r} \ , \ T_r Q_r = 0 \qquad (13)$$

by substituting this into equation (12) and equating power of ζ we get the following recursion formula for Q_r, $r = 0, 1, \ldots\ldots\ldots n$.

$$P_t = (Q_n)_x + [Q_n, P] \qquad (14a)$$

$$[R_0, Q_r] = (Q_{r-1})_x + [Q_{r-1}, P] \ , \ 1 \leq r \leq n \qquad (14b)$$

$$[R_0, Q_0] = 0 \qquad (14c)$$

Since R_0 is a diagonal matrix with distinct entries, equation (14c) requires that Q_0 be diagnonal matrix either

$$Q_0 = C = \text{diag.}\ (c_1, c_2, \ldots, c_N), \ \sum_{i=1}^{N} c_i = 0 \qquad (15)$$

We also require that c_i is x-independent but can possibly depend on t. From the diagonal part of the equations (14a) and (14b) we obtain

$$Q_{rD} = -I\,[Q_{rF}, P]_D \ , \ 1 \leq r \leq n \qquad (16)$$

where I is an integral operator

$$If(x) = \int_{-\infty}^{x} f(y)\,dy \qquad (17)$$

and the suffices D and F denote the diagonal and off diagonal part of a matrix, respectively. The off-diagonal part of equation (14b) gives the relation between Q_{rF} and Q_{r-1}

$$[R_0, Q_{rF}] = (Q_{r-1}, F) + [Q_{r-1}, P]_F \ , \ 1 < r < n \qquad (18)$$

Thus Q_{rF} is determined from Q_{r-1} whilst Q_{rD} is expressed in terms of Q_{rF}. In this way every Q_r is expressed in terms of P, its x derivatives and its x-integrals. By substituting these results into equation (14a) we obtain a non-linear evolution equation

$$P_t = D_R{}^n\ [C\,.\,P] \qquad (19)$$

An integro-differential operator D operating on an off-diagonal matrix $H = (H_{lm})$ is defined as

$$DH = \frac{\partial}{\partial x} H + [H , P]_F - [I (H , P)_D , P]_F \tag{20}$$

and the operator D_R in equation (19) means

$$D_R H = DH_R \tag{21}$$

in which H_R is an off-diagonal matrix

$$(H_R)_{lm} = \frac{H_{lm}}{\beta_l - \beta_m} , \quad l \neq m$$

Example

$$P = \begin{pmatrix} 0 & P_{12} & P_{13} & P_{14} \\ P_{21} & 0 & P_{23} & P_{24} \\ P_{31} & P_{32} & 0 & P_{34} \\ P_{41} & P_{42} & P_{43} & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \beta_4 \end{pmatrix}$$

$$C = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix}$$

$$[C,P] = \begin{pmatrix} 0 & (c_1 - c_2)P_{12} & (c_1 - c_3)P_{13} & (c_1 - c_4)P_{14} \\ (c_2 - c_1)P_{21} & 0 & (c_2 - c_3)P_{23} & (c_3 - c_4)P_{24} \\ (c_3 - c_1)P_{31} & (c_3 - c_2)P_{32} & 0 & (c_3 - c_4)P_{34} \\ (c_4 - c_1)P_{41} & (c_4 - c_2)P_{42} & (c_4 - c_3)P_{43} & 0 \end{pmatrix}$$

$$[C,P]_R = \begin{pmatrix} 0 & \frac{(c_1 - c_2)P_{12}}{\beta_1 - \beta_2} & \frac{(c_1 - c_3)P_{13}}{\beta_1 - \beta_3} & \frac{(c_1 - c_4)P_{14}}{\beta_1 - \beta_4} \\ \frac{(c_2 - c_1)P_{21}}{\beta_2 - \beta_1} & 0 & \frac{(c_2 - c_3)P_{23}}{\beta_2 - \beta_3} & \frac{(c_2 - c_4)P_{24}}{\beta_2 - \beta_4} \\ \frac{(c_3 - c_1)P_{31}}{\beta_3 - \beta_1} & \frac{(c_3 - c_2)P_{32}}{\beta_3 - \beta_2} & 0 & \frac{(c_3 - c_4)P_{34}}{\beta_3 - \beta_4} \\ \frac{(c_4 - c_1)P_{41}}{\beta_4 - \beta_1} & \frac{(c_4 - c_2)P_{42}}{\beta_4 - \beta_2} & \frac{(c_4 - c_3)P_{43}}{\beta_4 - \beta_3} & 0 \end{pmatrix}$$

First term in (20) is

$$\frac{c_l\ c_m}{\beta_l\ \beta_m}\ (P_{lm})_x$$

Second term in (20) is

$$[\ [C, P]_R\ ,\ P]_{lm}$$

$$= \{ [C, P]_R \}_{lk} P_{km} - P_{lk}\ \{ [C, P] \}_{km}$$

$$= P_{lk} P_{km}\ \{ \frac{c_l - c_k}{\beta_l - \beta_k} - \frac{c_m - c_k}{\beta_m - \beta_k}\ l \neq m \}$$

Third term is (20) is

$$[\ [C, P]_R, P\ ]_{lm}$$

But in this $l = m$ we get

$$P_{lk} P_{kl}\ [\frac{c_l - c_k}{\beta_l - \beta_k} - \frac{c_k - c_l}{\beta_k - \beta_l}] = 0$$

So substituting these values in (19) we get

$$\frac{\partial}{\partial x} P_{lm} = \frac{c_l - c_m}{\beta_l - \beta_m} [P_{lm}]_x +$$

$$\sum_{k=1}^{N} P_{lk} P_{kl}\ [\frac{c_l - c_k}{\beta_l - \beta_k} - \frac{c_k - c_l}{\beta_k - \beta_l}]$$

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ON THE INFINITESIMAL ISOMETRIC DEFORMATION OF CARTESIAN PRODUCT OF RIEMANNIAN MANIFOLDS

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ABSTRACT

In this paper, the isometric and the infinitesimal isometric deformation of Cartesian product of Riemannian manifolds are defined and studied. Some geometric properties of such deformations are established throughout giving some theorems. Also, the Cartesian product of the normal infinitesimal isometric deformations is studied.

1. INTRODUCTION

In this paper, all manifolds and maps are assumed sufficiently differentiable for all computations to make sense. All manifolds are assumed connected. A submanifold S of a Riemannian manifold $\overline{M}$ consists of a manifold M and an immersion r of M into $\overline{M}$

Let S=(M, r) be a submanifold of a Riemannian manifold $\overline{M}$. Let I = $[-\delta, \delta]$ for some $\delta > 0$. A map

$$\gamma : I \times M \rightarrow \overline{M}$$

is said to be a deformation of S if $\gamma_0 = r$ and γ_t is an immersion for each $t \varepsilon$ I. We have written $\gamma_t(x)$ for $\gamma(t, x)$. Each immersion γ_t induces a Riemannian metric g_t on M. Each closed curve on M has a length L(t) measured by the metric g_t[1].

Let γ be a deformation of S. We say that γ is an isometric deformation (ID) of S if $g_t = g_0$ for each $t \varepsilon$ I. We say that γ is an infinitesimal isometric deformation (IID) of S if $g'(0) = 0$. When we write $g'(0)$, we regard $g_t = g(t)$ as a curve in the finite dimensional vector space of tensors of type (0, 2) at a point of M. It is easy to check that γ is an (ID) if and only if $L(t)$ is independent of t for each closed curve on M. Furthermore, γ is an IID if and only if $L'(0) = 0$ for each curve.

For each point x in M, let Z_x be the tangent vector to the curve $t \to \gamma(t, x)$ at $t = 0$. Thus Z is a section of E, where E is the restriction of the tangent bundle $T(\overline{M})$ to M, whose value at x is the initial velocity of the motion of x under the deformation. We call Z the deformation field of γ. Actually, the vector field Z determines the infinitesimal properties of γ. In particular, we have the following characterization of infinitesimal isometric deformations [1].

THEOREM 1-1

A deformation $\gamma : I \times M \to \overline{M}$ is IID if and only if for X, $Y \in \mathfrak{X}(M)$, we have

$$g(\overline{D}_X Z, Y) + g(X, \overline{D}_Y Z) = 0 \tag{1.1}$$

where $\mathfrak{X}(M)$ is the set of all vector fields on $\overline{M}$ and $\overline{D}$ is the covarient differentiation operator of the Riemannian manifold $(\overline{M}, g)$

DEFINITION 1-1

A deformation Z of S is normal if the tangential component of Z is everywhere zero.

THEOREM 1-2

If S is a hypersurface, and Z is a normal IID of S, then S is totally geodesic whenever Z is nonzero.

For the main aim of this paper, we shall need the following section.

2. PRELIMINARIES

Let M_1 and M_2 be two C^∞ complete Riemannian manifolds with Riemannian connexions $D^{(1)}$ and $D^{(2)}$ and Riemannian metrices $g^{(1)}$ and $g^{(2)}$, respectively.

A Riemannian metric $\overline{g}$ on $M_1 \times M_2$ which we shall consider throughout this paper can be defined as follows [2]:

$$\overline{g}(X, Y) = \overline{g}((X_1, X_2), (Y_1, Y_2))$$
$$= g^{(1)}(X_1, Y_1) + g^{(2)}(X_2, Y_2) \qquad (2.1)$$

where X_i, $Y_i \; \varepsilon \; \mathfrak{X}(M_i)$, $\mathfrak{X}(M_i)$ is the set of all vector fields on M_i, $i = 1,2$. Similarly, a connexion D on $M_1 \times M_2$ can be given as

$$D_X Y = D_{(X_1, X_2)}(Y_1, Y_2) = (D^{(1)}_{X_1} Y_1, D^{(1)}_{X_2} Y_2) \qquad (2.2)$$

It is clear that D is an infinitesimal connexion [2]. Moreover, D is a metric and Riemannian connexion [3].

In (1981), H.B. Pandey [2] studied some geometric properties of the Cartesian product of two C^∞ manifolds M_1 and M_2 and established that if both M_1 and M_2 have the property under consideration (such as almost complex, Kahelerian, almost Tachibana), then $M_1 \times M_2$ has the same property and vice versa.

In (1988), A. E. Rakia [3] established more interesting results concerning the geometry of manifolds product. For example, the manifold $M_1 \times M_2$ is free from conjugate (resp. focal) points if and only if both M_1 and M_2 are free from conjugate (resp. focal) points. Also, the Cartesian product of Jacobi vector fields as well as asymptotic geodesics in $M_1 \times M_2$ have been studied. In addition to these results, we have the following [3].

Proposition 2-1

Let M and N be hypersurfaces of the Riemannian manifolds $\overline{M}$ and $\overline{N}$ such that u and v are normal vectors to M and N at $p \; \varepsilon$ M, $q \; \varepsilon$ N, respectively. Then, $(u, v) \; \varepsilon \; T_{(p,q)}(\overline{M} \times \overline{N})$ is a normal vector to $M \times N$ at $(p, q) \; \varepsilon \; M \times N$ under the metric $\overline{g}$ defined by equation (2.1), where $T_{(p,q)}(\overline{M} \times \overline{N})$ is a tangent space to $\overline{M} \times \overline{N}$ at (p, q).

Proposition 2-2

$M \times N \subset \overline{M} \times \overline{N}$ is totally geodesic if and only if both $M \subset \overline{M}$ and $N \subset \overline{N}$ are totally geodesics.

3. MAIN RESULTS

Let $S_i = (N_i, r_i)$ be a Riemannian submanifold of a Riemannian manifold $\overline{N}_i$ such that r_i is an immersion of N_i into $\overline{N}_i$, $i = 1, 2$. Let $S^* = S_1 \times S_2 = (N_1 \times N_2, r_1 \times r_2)$ be a submanifold of $\overline{N}_1 \times \overline{N}_2$ due to the Cartesian product of S_1 and S_2, where $r_1 \times r_2$ is also an immersion [3] of $N_1 \times N_2$ into $\overline{N}_1 \times \overline{N}_2$. Consider $I = [-\delta, \delta]$ for some $\delta > 0$. A map

$$\gamma^* : I \times (N_1 \times N_2) \rightarrow \overline{N}_1 \times \overline{N}_2$$

is said to be a deformation of S^* if $\gamma_0^* = (r_1 \times r_2)$ and γ_t^* is an immersion for each $t \varepsilon I$. We write $\gamma_t^*(x) = \gamma^*(t, x)$ where $x = (x_1, x_2) \varepsilon N_1 \times N_2$. Each immersion γ_t^* induces a Riemannian metric $\overline{g}_t^0$ on $N_1 \times N_2$. Each closed curve on $N_1 \times N_2$ has a length $L(t)$ measured by the metric $\overline{g}_t$. If $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, $t \varepsilon [a, b]$ is a curve in $N_1 \times N_2$ such that α_i is a regular curve in N_i, $i = 1, 2$ then,

$$L_a^b(\alpha) < L_a^b(\alpha_1) + L_a^b(\alpha_2),$$

where $L_a^b(\alpha)$ denotes the length of α from $t = a$ to $t = b$. In general, we have

$$L_a^b(\alpha) \leq L_a^b(\alpha_1) + L_a^b(\alpha_2)$$

DEFINITION 3-1

Let γ^* be a deformation of S^*. We say that γ^* is an ID of S^* if $\overline{g}_t = \overline{g}_0$ for each $t \varepsilon I$. We say that γ^* is an IID of S^* if $\overline{g}'(0) = 0$.

As a direct result from theorem (1-1), we obtain

THEOREM 3-1

A deformation γ^* is an IID if and only if

$$\overline{g}\ (D_X Z, Y) + \overline{g}\ (X, D_Y Z) = 0$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \ \varepsilon\ \mathfrak{X}(N_1 \times N_2)$ and $Z = (Z_1, Z_2)$ is the deformation field of γ^*.

Corollary 3-1

A deformation γ^* is an IID if and only if for $X, Y \ \varepsilon\ \mathfrak{X}(N_1 \times N_2)$

$$\sum_1^2 \{ g^{(i)} (D_{X_i}^{(i)} Z_i , Y_i) + g^{(i)} (X_i , D_{Y_i}^{(i)} Z_i) \} = 0$$

Now, if each induced deformation is IID, then the whole deformation of the product is IID as the following theorem says.

THEOREM 3-2

If γ_i is IID of N_i, $i = 1, 2$. Then, the deformation vector field Z on $N_1 \times N_2$ is an IID.

Proof

To prove this theorem, it suffices to show that

$$\overline{g}\ (D_X Z, Y) + \overline{g}\ (X, D_Y Z) = 0 \qquad (3.1)$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \ \varepsilon\ \mathfrak{X}(N_1 \times N_2)$ and $Z = (Z_1, Z_2)$ is the deformation field of γ^*.

By using equations (2.1) and (2.2), equation (3.1) becomes

$$\begin{aligned} &\overline{g}\ ((D_{X_1}^{(1)} Z_1, D_{X_2}^{(2)} Z_2), (Y_1, Y_2)) + \overline{g}\ ((X_1, X_2), \\ &(D_{Y_1}^{(1)} Z_1, D_{Y_2}^{(2)} Z_2)) \\ &= g^{(1)} (D_{X_1}^{(1)} Z_1, Y_1) + g^{(2)} (D_{X_2}^{(2)} Z_2, Y_2) \\ &+ g^{(1)} (X_1, D_{Y_1}^{(1)} Z_1) + g^{(2)} (X_2, D_{Y_2}^{(2)} Z_2) \qquad (3.2) \end{aligned}$$

Since γ_i, $i = 1, 2$, is IID, then equation (3.2) reduce to

$$\sum_1^2 \{ g^{(i)} (D_{X_i}^{(i)} Z_i, Y_i) + g^{(i)} (X_i, D_{Y_i}^{(i)} Z_i) \} = 0$$

which complete the proof of the theorem.

In general cases, this theorem is valid as follows:

If γ_i, $i = 1, 2, \ldots, n$ is IID, then the deformation vector field Z on $N_1 \times N_2 \times \ldots \times N_n$ is also IID.

Corollary 3-2

If each induced deformation is IID, then the whole deformation of the product is IID. But if the deformation of the Riemannian product is IID, then not necessarily each induced deformation is IID. The reason could be given as follows:

For $X, Y \in \mathfrak{X}(N_1 \times N_2)$, we have

$$\overline{g}_t(X, Y) = g_t^{(1)}(X_1, Y_1) + g_t^{(2)}(X_2, Y_2)$$

or, simply

$$\overline{g}_t = g_t^{(1)} + g_t^{(2)}$$

Consequently,

$$\overline{g}'(0) = g^{(1)'}(0) + g^{(2)'}(0)$$

which means that, if $\overline{g}'(0) = 0$, this would not mean necessarily that $g^{(1)'}(0) = 0$ and $g^{(2)'}(0) = 0$.

THEOREM 3-3

If Z_i is a normal deformation of a hypersurface $S_i \subset \overline{N}_i$, $i = 1, 2$, then the Cartesian product $Z_1 \times Z_2$ is also a normal deformation of the submanifold $S_1 \times S_2 \subset \overline{N}_1 \times \overline{N}_2$.

Proof:

Since Z_i is normal deformation of S_i, $i = 1, 2$, then the tangential component of Z_i is everywhere zero. By proposition (2-1), we see that the Cartesian product of the normal components of Z_i is also a normal component to $Z_1 \times Z_2$, *i.e.*, the tangential component of $Z_1 \times Z_2$ is everywhere zero. This means that $Z_1 \times Z_2$ is normal deformation of the submanifold $S_1 \times S_2$.

Finally, under the above mentioned theorems (3-2), (3-3) and proposition (2-2), we obtain the following theorem:

THEOREM (3-4)

If Z_i is a normal IID of the hypersurface $S_i \subset \overline{N}_i$, $i = 1, 2$. Then, the Cartesian product $Z_1 \times Z_2$ is normal IID of the submanifold $S_1 \times S_2 \subset \overline{N}_1 \times \overline{N}_2$ and $S_1 \times S_2$ is totally geodesic submanifold.

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"ON THE INTRINSIC SELF ADJOINTNESS OF THE SCHRODINGER DIFFERENTIAL OPERATOR"

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ABSTRACT

The intrinsic self adjointness of the Schrodinger kinetic energy operator of a system of N different particles is restored.

1. INTRODUCTION

The kinetic energy differential operator which belongs to Schrodinger wave equation is not intrinsically self adjoint [2, 3]. This happens upon extracting the adjoint of the operator due to the presence of non vanishing boundary terms except under some appropriate boundary conditions. Hermitization methods have been introduced and employed for restoring the intrinsic self adjointness [1, 2, 3, 5]. In this paper we use these methods to extract all possible hermitized versions which correspond to arbitrary linear differential operator, this was shown in sec. 2. In sec. 3, the Schrodinger kinetic energy operator of a system of n different particles is demonstrated not to be intrinsically self adjoint. In sec. 4, we discuss the effect of non-self adjointness of the Schrodinger operator on the completeness of its representation. In sec. 5, we establish the hermitization procedure and generate an infinite set of hermitized versions of any arbitrary linear differential operator. Sec. 6, is devoted to the applications of the procedure for hermitizing the momentum and kinetic energy differential operators of a system of n particles.

2. THE ADJOINT OF A LINEAR OPERATOR

Let $$A = \sum_{\alpha=0}^{m} \sum_{j=1}^{n} P_{\alpha j}(x_j)\, D^{\alpha}_{x_j} \qquad (2.1)$$

be a linear differential operator in which $P_{\alpha j}(x_j)$ are continuous and have continuous derivatives up to the m^{th} order for all $x_j \varepsilon [a_j, b_j]$; $j = 1, 2, \ldots, n$, and $D^{\alpha}_{x_j}$ denote the ordinary differential operator $\frac{d^{\alpha}}{dx_j^{\alpha}}$.

We will evaluate first the adjoint of the operator $D^{\alpha}_{x_j}$. Consider the matrix element

$$< f_j \,|D^{\alpha}_{x_j}|\, g_j > \;=\; (-1)^{\alpha} \int_{a_j}^{b_j} dx f_j^{*}(x)\; g_j^{[\alpha]}(x), \qquad (2.2)$$

where f_j , g_j are arbitrary functions which possess continuous derivatives up to the n^{th} order for all $x \,\varepsilon\, [a_j , b_j]$. On integrating rating the right hand side of (2.2) by parts α-times, we end up with

$$< f_j \,|D^{\alpha}_{x_j}|\, g_j > \;=\; (-1)^{\alpha} \int_{a_j}^{b_j} dx\, f_j^{*[\alpha]}(x)\; g_j(x)$$

$$+ \sum_{k=1}^{\alpha} (-1)^{k-1} f_j^{*[k-1]}(x) \quad g_j^{[\alpha-k]}(x) \Big|_{a_j}^{b_j} \qquad (2.3)$$

By using the extended Dirac delta function [2] which is

$$\tilde{\delta}\,(x\,;a_j\,,b_j) \;=\; \tilde{\delta}\,(x-b_j) - \tilde{\delta}\,(x-a_j)$$

The second term on the right hand side of equation (2.3) can be written in the form of an integral as

$$\sum_{k=1}^{\alpha} (-1)^{k-1} f_j^{*[k-1]}(x) \quad g_j^{[\alpha-k]}(x) \Big|_{a_j}^{b_j}$$

$$= \sum_{k=1}^{\alpha} (-1)^{k-1} \int_{a_j}^{b_j} dx\, f_j^{*[k-1]}(x) \quad \tilde{\delta}\,(x, a_j, b_j)\, g_j^{[\alpha-k]}(x) \qquad (2.4)$$

Hence $\sum_{k=1}^{\alpha} (-1)^{k-1} f_j^{*[k-1]}(x) \quad g_j^{[\alpha-k]}(x) \Big|_{a_j}^{b_j}$

$$= \langle f_j \mid \sum_{k=1}^{\alpha} (-1)^{k-1} \overleftarrow{D}_{x_j}^{k-1} \tilde{\delta}_{x_j} D_{x_j}^{\alpha-k} \mid g_j \rangle, \qquad (2.5)$$

where $\overleftarrow{D}x_j$ is the left handed differential operator. Therefore equation (2.3) can be put in the form

$$\langle f_j \mid D_{x_j}^{\alpha} \mid g_j \rangle = \langle g_j \mid (-1)^{\alpha} D_{x_j}^{\alpha} + \sum_{k=1}^{\alpha} (-1)^{k-1} \overleftarrow{D}_{x_j}^{\alpha-k} \tilde{\delta}_{x_j} D_{x_j}^{k-1} \mid f_j \rangle \qquad (2.6)$$

This equation gives us the adjoint of the differential operator $D_{x_j}^{\alpha}$. Explicitly, it is

$$D_{x_j}^{+\alpha} \Rightarrow (-1)^{\alpha} D_{x_j}^{\alpha} + \sum_{k=1}^{\alpha} (-1)^{k-1} \overleftarrow{D}_{x_j}^{\alpha-k} \tilde{\delta}_{x_j} D_{x_j}^{k-1} \qquad (2.7)$$

3. THE ADJOINT OF N PARTICLES SCHRODINGER DIFFERENTIAL OPERATOR

The Schrodinger differential operator of a system of n different particles is

$$H = -\sum_{j=1}^{n} \frac{\hbar^2}{2m_j} D_{x_j}^2 + v(x_{ij}) \qquad (3.1)$$

where m_j is the mass of the particles j and $v(x_{ij})$ is a function only of the distances $x_{ij} = | x_i - x_j |$ between the particles.

Putting $\alpha = 2$ in equation (2.7), we obtain the adjoint of the Hamiltonian operator

$$H^{+} = H - \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} [\overleftarrow{D}_{x_j} \tilde{\delta}_{x_j} - \tilde{\delta}_{x_j} D_{x_j}] \qquad (3.2)$$

which shows that the Hamiltonian operator is not intrinsically self adjoint even if we restricted the potential V to be self-adjoint operator. However, self adjointness can be restored if the boundary terms are forced to vanish upon imposing appropriate boundary conditions.

Using the same process we can express the adjoint of the monentum operator and the kinetic energy operator respectively.

$$P^{+} = P + i\hbar \sum_{j=1}^{n} \tilde{\delta}_{x_j} \qquad (3.3)$$

and
$$T^{+} = T - \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} \left[\overleftarrow{D}_{x_j} \tilde{\delta}_{x_j} + \tilde{\delta}_{x_j} D_{x_j} \right] \qquad (3.4)$$

We can express equation (3.2) in the following matrix representation

$$< \sum_{j=1}^{n} \phi_j |H| \sum_{j=1}^{n} \Psi_j >$$

$$= < \sum_{j=1}^{n} \Psi_j |H| \sum_{j=1}^{n} \phi_j >^{*} + \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} W[\phi_j , \Psi_j] \Big|_{a_j}^{b_j} \qquad (3.5)$$

where $W[\phi_j , \Psi_j]$ denote the Wronskian determinant

$$W[\phi_j , \Psi_j] = \phi_j \Psi_j' - \phi_j' \Psi_j , \qquad (3.6)$$

and ϕ_j , Ψ_j , $j = 1, 2, \ldots\ldots\ldots , n$ are state functions that possess continuous derivatives up to the second order for all $x_j \varepsilon [a_j , b_j]$.

In the following section we will study the effect of the non hermiticity of the Hamiltonian operator on the completeness of its representation.

4. COMPLETENESS

Consider an orthonormal set of discrete functions $\{ u_l (x_j) \}$. We expand an arbitrary function v_j as a uniformally convergent series in the region $x_j . \varepsilon [a_j , b_j]$, $j = 1, 2, \ldots\ldots , n$

$$v_j = \sum_{l=0}^{\infty} a_{lj} . u_l (x_j) , \qquad (4.1)$$

where $\alpha_{lj} = < u_l (x_j) \mid v_j >$ (4.2)

In Substituting for the functions Ψ_j in equation (3.5) the difference $v_j - \sum_{l=0}^{\infty} \alpha_{lj} u_l (x_j)$ which is zero by hypothesis, we get

$$< \sum_{j=1}^{n} \phi_j \, |H| \, \sum_{j=1}^{n} (v_j - \sum_{l=0}^{\infty} \alpha_{lj} \, u_l(x_j) >$$

$$= < \sum_{j=1}^{n} (v_j - \sum_{l=0}^{\infty} \alpha_{lj} \, u_l \, (x_j)) \; |H| \; \sum_{j=1}^{n} \phi_j >^*$$

$$+ \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} \, W \, [\, \phi_j \, , (v_j - \sum_{l=0}^{\infty} \alpha_{lj} \, . \, u_l \, (x_j)) \,] \, \Big|_{a_j}^{b_j} \qquad (4.3)$$

Now, the first term of the right-hand side vanishes by hypothesis, thus we can express equation (4.3) as

$$< \sum_{j=1}^{n} \phi_j \, |H| \, \sum_{j=1}^{n} v_j > \; = \; < \sum_{j=1}^{n} \phi_j \, |H| \, \sum_{j=1}^{n} \; \sum_{l=0}^{\infty} \alpha_{lj} \, u_l \, (x_j) >$$

$$+ \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} \, W \, [\, \phi_j \, , (v_j - \sum_{l=0}^{\infty} \alpha_{lj} \, . \, u_l \, (x_j)) \,] \, \Big|_{a_j}^{b_j} \qquad (4.4)$$

Recalling equations (3.2), (3.5) & (4.2), equation (4.4) can be written in the form

$$< \sum_{j=1}^{n} \phi_j \, |H| \, \sum_{j=1}^{n} v_j > \; = \; \sum_{l=0}^{\infty} \; < \sum_{j=1}^{n} \phi_j \, |H| \, \sum_{j=1}^{n} \alpha_{lj} \, u_l \, (x_j) >$$

$$+ \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} < \phi_j \, | \, \overleftarrow{D}_{x_j} \, \delta_{x_j} - \delta_{x_j} \, D_{x_j} \, | \, v_j - \sum_{l=0}^{\infty} \alpha_{lj} \, u_l \, (x_j) > \qquad (4.6)$$

This shows that the operator H commutes with the summation $\sum_{l=0}^{\infty}$ everywhere for $x_j \; \varepsilon \; [a_j \, , \, b_j]$ except at the boundary points a_j, b_j, $j = 1, 2, \ldots, n$ because of the existence of the boundary terms of the right hand side of equation (4.6). These boundary terms vanish if the functions v_j obey the same boundary conditions as the discrete sets $\{ \, u_l \, (x_j) \, \}$ at the boundary points a_j , b_j , *i.e.*

$$v'_j \, \Big|_{x_j = a_j . b_j} = \sum_{l=0}^{\infty} a_{lj} \, u'_l \, (x_j) \, \Big|_{x_j = a_j . b_j} \; , \; j = 1, 2, \ldots, n (4.7)$$

Such boundary conditions requirements can only be satisfied for bound state problems. However, in scattering problems the derivative of the basis $\{ u_l (x_j) \}$ and the expanded functions v_j do not match. On the other hand, if the kinetic energy differential operator is to be replaced by any of its hermitized versions, the boundary conditions (4.7) can be avoided. This can be done by using the hermitization technique [3].

5. HERMITIZATION

In this section we present a formal procedure for extracting all possible hermitized versions that correspond to any arbitrary differential operator. It can be done by introducing an associate operator which is simply a linear combination of all the individual ordinary differential operators that belong to the adjoint of the composite linear differential operator under consideration. We now hermitize the ordinary differential operator $D_{x_j}^{\alpha}$ in which α is arbitrary and we denote by $D_{x_j}^{-\alpha}$ the corresponding associate differential operator. We expand $D_{x_j}^{-\alpha}$ in terms of all individual differential operators.

$$D_{x_j}^{-\alpha} = C_{j,\alpha} D_{x_j}^{\alpha} + \sum_{\mu=1}^{\alpha} D_{x_j}^{\leftarrow\alpha-\mu} \tilde{\delta}_{x_j} C_{j,\mu-1} D_{x_j}^{\mu-1} \quad (5.1)$$

where $\{C_r\}$; $r = 1, 2, \ldots\ldots, n$ are complex expansion coefficients. The adjoint of the expression (5.1) is

$$D_{x_j}^{-\alpha+} = C_{j,\alpha}^{*} D_{x_j}^{\alpha+} + \sum_{\mu=1}^{\alpha} D_{x_j}^{\leftarrow\mu-1} C_{j,\mu-1}^{*} \tilde{\delta}_{x_j} D_{x_j}^{\alpha-\mu} \quad (5.2)$$

This can be written, by using (2.7), in the form

$$D_{x_j}^{-\alpha+} = C_{j,\alpha}^{*}, \left[(-1)^{\alpha} D_{x_j}^{\alpha} + \sum_{k=1}^{\alpha} (-1)^{k-1} D_{x_j}^{\leftarrow\alpha-k} \tilde{\delta}_{x_j} D_{x_j}^{k-1} \right]$$

$$+ \sum_{\mu=1}^{\alpha} D_{x_j}^{\leftarrow\mu-1} C_{j,\mu-1}^{*} \tilde{\delta}_{x_j} D_{x_j}^{\alpha-\mu} \quad (5.3)$$

The above expression can be simplified by performing the sum of the two series on the right hand side backwardly, this gives

$$D_{x_j}^{-\alpha+} = (-1)^{\alpha} C_{j,\alpha}^{*} D_{x_j}^{\alpha} + \sum_{k=1}^{\alpha} D_{x_j}^{\leftarrow\alpha-k} \tilde{\delta}_{x_j} \left[(-1)^{k-1} C_{j,\alpha}^{*} + C_{j,\alpha-k}^{*} \right] D_{x_j}^{k-1} \quad (5.4)$$

Now, the operator $D_{x_j}^{-\alpha}$ could be rendered to be intrinisically self adjoint by equating the relevant coefficients on the right hand sides of equations (5.1) and (5.4), doing that we get

$$C_{j,\alpha} = (-1)^{\alpha} C^{*}_{j,\alpha} \tag{5.5}$$

and $C_{j,k-1} = (-1)^{k-1} C^{*}_{j,\alpha} + C^{*}_{j,\alpha-k}$, $k = 1, 2, \ldots \alpha$, $j = 1, 2, \ldots, n$ (5.6)

in addition to the restriction

$$| C_{j,\alpha} | = 1 \tag{5.7}$$

The set of algebraic equations (5.5 − 5.7) gives the necessary relationships between the expansion coefficients { C_r } which upon substituting in equation (5.1) gives infinite set of hermitized versions for the differential operators $D_{x_j}^{-\alpha}$. Hence the associate differential operator for any composite linear differential operator as (2.1) can be constructed.

6. APPLICATIONS

For the case $\alpha = 1$, the corresponding associate differential operator reads

$$\overline{D}_{x_j} = C_{j,1} D_{x_j} + \tilde{\delta}_{x_j} C_{j,0}, \; j = 1, 2, \ldots\ldots, n \tag{6.1}$$

where $C_{j,1} = -C^{*}_{j,1}$, $| C_{j,1} | = 1$

and $C_{j,0} = C^{*}_{j,1} + C^{*}_{j,0}$ (6.2)

These give

$$C_{j,1} = i$$

and $C_{j,0} = \alpha_j - i/2$ (6.3)

where α_j , $j = 1, 2, \ldots\ldots\ldots, n$ are free real parameters. Therefore (6.1) becomes

$$\overline{D}_{x_j} = iD_{x_j} + (\alpha_j - i/2)\, \tilde{\delta}_{x_j}$$

Hence the associate differential momentum operator can be expressed in the form

$$\overline{P} = P - \sum_{j=1}^{n} \hbar\,(\alpha_j - i/2)\,\delta_{x_j} \tag{6.5}$$

which represents an infinite set of hermitized versions of the momentum operator regarding the arbitrariness of the parameters α_j.

For the case $\alpha = 2$, the corresponding associate differential operator is

$$\overline{D}^2_{x_j} = C_{j,2}\, D^2_{x_j} + \overleftarrow{D}_{x_j}\, C_{j,0}\, \tilde{\delta}_{x_j} + C_{j,1}\, \tilde{\delta}_{x_j}\, D_{x_j} \tag{6.6}$$

where $C_{j,2} = C^*_{j,2}$; $| C_{j,2} | = 1$,

and $C_{j,0} = C^*_{j,2} + C^*_{j,1}$

Choosing $C_{j,0} = \sigma_j$ where σ_j , $j = 1, 2, \ldots\ldots\ldots, n$ are free complex parameters. Then we get

$$C_{j,1} = \sigma^*_j - 1 \text{ for all } j = 1, 2, \ldots\ldots\ldots, n \tag{6.7}$$

Hence equation (6.6) could be rewritten as

$$\overline{D}^2_{x_j} = D^2_{x_j} + \overleftarrow{D}_{x_j}\, \sigma_j\, \tilde{\delta}_{x_j} + (\sigma^*_j - 1)\, \tilde{\delta}_{x_j} . D_{x_j} \tag{6.8}$$

This enables us to express the associate differential operator of the kinetic energy $T = -\sum_{j=1}^{n} \frac{\hbar^2}{2m_j} D^2_{x_j}$ as

$$\overline{T} = -\sum_{j=1}^{n} \frac{\hbar^2}{2m_j}\, \overline{D}^2_{x_j}$$

$$= T - \sum_{j=1}^{n} \overleftarrow{D}_{x_j} \frac{\hbar^2}{2m_j} \sigma_j\, \tilde{\delta}_{x_j} + \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} (\sigma^*_j - 1)\, \tilde{\delta}_{x_j} . D_{x_j}$$

(6.9)

If we put $n = 1$ in equations (6.5) and 6.9 we get the results of Morsy [5].

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APPROXIMATING NEWTON-LIKE ITERATIONS IN BANACH SPACE

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ABSTRACT

We approximate Newton-like iterations in a Banach space setting by solving a linear algebraic system of finite order. The approximate inverses of the Frechet-derivatives involved are obtained recursively. Some special cases are studied in detail.

Key words and phrases: Banach space, Nonlinear operator equation, approximate inverse.

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1. INTRODUCTION

Consider the nonlinear operator equation

$$F(x) = 0 \tag{1}$$

where F is a nonlinear operator mapping a subset E_3 of a Banach space E_1 into another Banach space E_2. Newton-like methods of the form

$$y_{n+1} = y_n - A(y_n)^{-1} F(y_n), \quad n \geq 0 \tag{2}$$

have been used extensively to approximate a solution x^* of equation (1) in E_3. (see e.g. [1] — [4], [7] — [14]). Here the linear operator $A(y_n)$ is a consistent approximation to the Frechet-derivative $F'(y_n)$ of $F(y_n)$. For $A(y_n) = F'(y_n)$ one obtains the famous Newton-Kantorovich method, whereas for $A(y_n) = \delta F(y_n, y_{n-1})$, $n \geq 0$ (secant mappings) the secant method is obtained. Several authors have provided sufficient conditions for the convergence of iteration (2) to a locally unique solution x^* of equation (1) under various assumptions (see, e.g. [1], [3], [4], [7] — [14] and the references there). The iterates x_n in (2) can only be computed if the inverses

$A(y_n)^{-1}$ exist for all $n \geq 0$. However, there are many problems of interest where for various reasons this cannot happen. The linear operator $A(y_n)$ may not be continuously invertible, as for instance when dealing with small divisors (see, e.g. [8] and the references there). Moreover the iterates given by (2) cannot be easily computed in infinite dimensional spaces since the inverses of the linear operators $A(y_n)$, $n \geq 0$ may be too difficult or impossible to compute. That is why we consider instead the iterates

$$x_{n+1} = x_n - (PA(x_n))^{-1}(x_n), \quad n \geq 0 \qquad (3)$$

for approximating a locally unique solution x^* of equation (1). Here, P is a projection operator ($P^2 = P$) mapping E_2 into a space E_{2P} of finite dimension N. It is easy to see that the solution of (3) reduces to solving a system of linear algebraic equations of, at most order N. Note that it can easily be shown by induction on n that $F(x_n)$ belongs to the domain of $(PA(x_n))^{-1}$ for all $n \geq 0$. Therefore, if the inverses exist (as it will be shown later in Theorem 1), then the iterates x_n can be computed for all $n \geq 0$.

In this paper we provide sufficient conditions for the convergence of iteration (3) to a solution x^* of equation (1), where at each step the inverse of the derivative, is replaced by a linear operator, which is obtained recursively from the previous one. Some special cases are examined in detail.

Finally at the end of this paper we show that our results contain previous one's as special cases.

2. CONVERGENCE RESULTS

Denote by $U(x_0, R)$ a closed ball centered at $x_0 \in E_3$ and of radius $R > 0$ such that $U(x_0, R) \supset E_3$. Let us also denote by $L(E_1, E_2)$ the space of all bounded linear operators from E_1 into E_2. Let E_4 be a Banach space, $A: E_3 \to L(E_1, E_2)$, $P: E_2 \to E_{2P}$, $M_{-1} \in L(E_4, E_{2P})$, $L_{-1} \in L(E_4, E_1)$. For $n \geq 0$, choose $N_n \in L(E_4, E_4)$, and as in [8, p. 117] define

$$M_n = M_{n-1} N_n + PA(x_n)L_{n-1} \qquad (4)$$

$$L_n = L_{n-1} + I_{n-1} N_n \qquad (5)$$

$$x_{n+1} = x_n + L_n y_n \qquad (6)$$

y_n being a solution of

$$M_n y_n = -(F(x_n) + Z_n) \tag{7}$$

for a suitable $Z_n \in E_2$. Moreover set $Q = I - P$

We will provide sufficient conditions on the operator PA, the starting guesses x_0, N_{-1}, L_{-1}, the operators N_n's and on the Z_n's to guarantee the convergence of iteration $\{x_n\}$, $n \geq 0$ given by (6) to a solution x^* of equation (1). Moreover we will provide upper bounds on the distances $||x_{n+1} - x_n||$ and $||x_n - x^*||$ for all $n \geq 0$.

By definition the operators M_n depend on $x_n \in E_3$. That is $M_n = M(x_n)$. We will assume that the linear operator $M(x_n)$ satisfy the estimates

$$||M(x) - M(x_0)|| \leq v(||x - x_0||) \tag{8}$$

for all $x \in U(x_0, R)$, where v is a non-decreasing non-negative function with $v(0) = 0$.

Moreover we assume that the following estimates are true

$$||PA(x) - PA(x_0)|| \leq w_0(||x - x_0||) \tag{9}$$

$$||QF(x) - QF(y)|| \leq w_1(||x - y||) \tag{10}$$

and $$||PF'(x+t(y-x)) - PA(x)|| \leq w(||x-x_0|| + t||y-x||) \tag{11}$$

for all $t \in [0, 1]$ and all $x, y \in U(x_0, r) \subset U(x, R)$, where w_0, w_1 and w are non-decreasing non-negative functions with $w_0(0) = w_1(0) = w(0) = 0$.

Let us assume that there exist numbers α_0, α, β, γ, δ such that

$$||PF(x_0)|| \leq \alpha_0, \; ||L_0 y_0|| \leq \alpha \tag{12}$$

$$||M_0^{-1}||, \; ||M_{-1}^{-1}|| \leq \beta \tag{13}$$

$$||L_{-1}|| \leq \gamma \tag{14}$$

and $$||M_{-1} - PA(x_0) L_{-1}|| \leq \delta \tag{15}$$

Moreover define the function $d(r)$ by

$$d(r) = \frac{1}{1 - v(r)} \tag{16}$$

and assume that

$$v(R) \leq 1 \tag{17}$$

We can now state and prove the main result.

THEOREM

Let $F : E_3 \subset E_1 \rightarrow E_2$. Assume

(*i*) F is Frechet-differentiable on E_3;

(*ii*) The hypotheses (8)-(17) are true for all x , $y \in U(x_0 , R)$, $t \in [0 , 1]$ for some $R > 0$, with $U(x_0 , R) \subset E_3$.

(*iii*) There exist non-negative sequences $\{a_n\}$, $\{\overline{a}_n\}$, $\{b_n\}$ and $\{c_n\}$ such that for all $n \geq 0$.

$$||N_n|| \leq a_n , ||1 + N_n|| \leq \overline{a}_n \quad (18)$$

and $$||Z_n|| \leq c_n ||PF(x_n)|| \quad (19)$$

(*iv*) The scalar sequence $\{t_n\}$, $n \geq 0$ given by

$$t_{n+2}=t_{n+1}+e_{n+1}d_{n+1}(1+c_{n+1}) [I_n + \sum_{i=1}^{n} h_i w(t_i)$$

$$(t_i - t_{i-1})+w(t_{n+1})(t_{n+1}-t_n)+w_1(t_{n+1}-t_n)] , n \geq 0 \quad (20)$$

$$t_0 = 0 , t_1 = \alpha$$

is bounded above by a t_0^* with $0 < t_0^* \leq R$, where

$$e_0 = \gamma . \overline{a}_0, e_n = I_{n-1} . \overline{a}_n, n \geq 1, d_n=d(t_n)=\frac{\beta}{1-b_n}, n \geq 0$$

$$I_n = \varepsilon_n\varepsilon_{n-1} \ldots\ldots \varepsilon_0\alpha_0 , n \geq 0, \varepsilon_n = p_nd_n(1+c_n) + c_n$$

$$p_n = q_{n-1}a_n,n \geq 1,p_0=\delta a_0,q_n=p_n+w_0(t_{n+1})e_n,n \geq 1 \text{ and}$$

$$h_i = \prod_{m=i}^{n} \varepsilon_m , i \leq n.$$

(*v*) The following estimate is true $\varepsilon_m \leq \varepsilon < 1$ for all $m \geq 0$. Then

(a) the scalar sequence $\{t_n\}$ is non-decreasing and converges to a t^* with $0 < t^* \leq t_0^*$ as $n \rightarrow \infty$.

(b) the sequence (6) is well defined, remains in $U(x_0 , t^*)$ and converges to a solution x^* of equation (1) with

$$||x_{n+1} - x_n|| \leq t_{n+1} - t_n \tag{21}$$

and $$||x_n - x^*|| \leq t^* - t_n \text{ for all } n \geq 0 \tag{22}$$

Proof

(a) By definition $t_1 \geq t_0$ and $t_2 \geq t_1$. Assume $t_{m+1} \geq t_m$ for $m = 0, 1, 2, \ldots, n$, then by (2) $t_{m+2} \geq t_{m+1}$. That is we showed by induction that the scalar sequence $\{t_n\}$, $n \geq 0$ is increasing. By hypothesis $\{t_n\}$ is bounded above by a t_0^* with $0 < t_0^* \leq R$. Hence it converges to some t^* with $0 < t^* \leq t_0^*$.

(b) We use induction to show that for all $m \geq 0$ the following estimates are true respectively

$$||L_m|| \leq e_m \tag{23}$$

$$||x_{m+1} - x_m|| \leq t_{m+1} - t_m \tag{24}$$

$$x_m \in U(x_0, t^*) \tag{25}$$

$$||M_m^{-1}|| \leq d_m \tag{26}$$

$$||M_n - A(x_{m+1}) L_m|| \leq q_m \tag{27}$$

$$||M_m - A(x_m) L_m|| \leq p_m \tag{28}$$

$$||F(x_{m+1})|| \leq \varepsilon_m ||F(x_m)|| + w(t_{m+1})(t_{m+1} - t_m) \tag{29}$$

We use (5) for $n = 0$ to obtain $L_0 = L_{-1}(1 + N_0)$. By taking norms and using (14) and (18) we get

$$||L_0|| \leq ||L_{-1}|| \, ||1 + N_0|| \leq \gamma . \bar{a}_0 = e_0$$

That is (23) is true for $n = 0$. Let us assume that (23) is true for $n = 0, 1, 2, \ldots, m - 1$, then by (5)

$$||L_m|| \leq ||L_{m-1}|| \, ||1 + N_m|| \leq e_{m-1} . \bar{a}_m = e_m$$

which completes the induction for (23)

By (6) and (12)

$$||x_1 - x_0|| = ||R_0 y_0|| \leq t_1 - t_0 = \alpha \leq t^*$$

which implies $x_1 \in U(x_0, t^*)$ and that (24) is true for $m = 0$. Assume (24) and (25) are true for $k = 0, 1, 2, \ldots, m$, then

$$||x_{k+1} - x_0|| \leq \sum_{i=0}^{k} ||x_{i+1} - x_i|| \leq t_{k+1} - t_0 \leq t^*$$

that is $x_{m+1} \in U(x_0 , t^*)$ also. By (17) and the Banach Lemma on invertible operators it can easily be seen that (26) is true for all $m \leq 0$. We will complete the induction for (24) later. First we need to show (27), (28) and (29). By (4) observe that

$$||M_0 - PA(x_0) L_0|| = ||(M_{-1} - PA(x_0) L_{-1})N_0||$$
$$\leq ||M_{-1} - PA(x_0)L_{-1}|| \cdot ||N_0|| \leq \delta a_0 = p_0$$

$$||M_0 - PA(x_1)L_0|| = ||(M_0 - PA(x_0)L_0) + (PA(x_0) - PA(x_1)L_0||$$
$$\leq p_0 + w_0 (||x - x_0||) e_0 \leq p_0 + w_0 (t_1) e_0 = q_0$$

That is (27) and (28) are true for m = 0. The rest of the induction for (27) and (28) can follow using the same approach as the one in the above two inequalities.

Using the identity

$$F(x_{k+1}) = P[F(x_{k+1}) - F(x_k) - A(x_k)(x_{k+1} - x_k)] + P[F(x_k) + A(x_k)(x_{k+1} - x_k)] + Q(F(x_{k+1}) - F(x_k)) \qquad (30)$$

(7), (19), (26) and (28) we obtain

$$||(P(F(x_k) + A(x_k)(x_{k+1} - x_k)))||$$
$$= ||(M_k - PA(x_k)L_k) M_k^{-1} (F(x_k) + Z_k) - Z_k||$$
$$< p_k d_k (||PF(x_k)|| + c_k ||PF(x_k)||) + c_k ||PF(x_k)||$$
$$= [p_k d_k (1 + c_k) + c_k] ||PF(x_k)|| = \varepsilon_k ||PF(x_k)|| \qquad (31)$$

Also, by (11) and (21)

$$||(P(F(x_{k+1}) - F(x_k) - A(x_k)(x_{k+1} - x_k)))||$$
$$= \left|\left| \int_0^1 P[F(x_k + t(x_k - x_0)) - A(x_k)] (x_{k+1} - x_k)\, dt \right|\right|$$
$$\leq \int_0^1 w (||x_k - x_0|| + t ||x_k - x_{k+1}||) ||x_{k+1} - x_k|| \, dt$$
$$\leq w (t_{k+1}) (t_{k+1} - t_k) \qquad (32)$$

It now follows from (30), (31) and (32) that (29) is true or all $m \geq 0$. Moreover by (10) and (21)

$$||QF(x_{k+1}) - F(x_k))|| \leq w_1 (t_{k+1} - t_k) \tag{33}$$

We must now complete the induction for (21). Indeed from (6), (26), (23) and (29)

$$||x_{k+2} - x_{k+1}|| = ||L_{k+1} y_{k+1}|| = ||L_{k+1} M_{k+1}^{-1} (PF(x_{k+1}) + Z_{k+1})||$$

$$\leq ||L_{k+1}|| \cdot ||M_{k+1}^{-1}|| (1 + c_{k+1} ||PF(x_{k+1})||$$

$$\leq e_{k+1} d_{k+1} (1 + c_{k+1}) \Big\{ I_k + \sum_{i=1}^{k} h_i w(t_i)(t_i - t_{i-1}) + w(t_{k+1})(t_{k+1} - t_k)$$

$$+ w_1 (t_{k+1} - t_k) \Big\}$$

$$= t_{k+2} - t_{k+1}, \text{ by (20)},$$

The estimates (23) — (29) have now been verified for all $m \geq 0$. It now follows from (a) and (21) that $\{x_n\}$, $n \geq 0$ is a Cauchy sequence in a Banach space and as such it converges to some $x^* \in U(x_0, t^*)$. Letting $m \to \infty$ in (29) and using the continuity of F and (v) we obtain $F(x^*) = 0$. Finally (22) follows easily from (21). That completes the proof of the theorem.

The conditions of the above theorem are made as general as possible. However some special cases must be examined now.

Remark 1: Several sufficient conditions can be given that will guarantee $t_n \leq t_0^*$ for all $n \geq 0$. Let us examine a very natural one. Assume that there exist numbers $c, d, a, \bar{a}$ such that

$a = a_0$, $c \geq 0, d, a, \bar{a} > 0$, $\bar{a} < a < 1$, $c_n \leq c, d_n \leq d, a_n \leq a$, $\bar{a}_n \leq \bar{a}$ for all $n \geq 0$.

Then for all $n \geq 1$

$p_n = q_{n-1} a_n = (p_{n-1} + w_0(t_n) e_n)\ a_n = a_{n-1} a_n (p_{n-2} + w_0(t_{n-1}) e_{n-1}) + w_0 (t_n) e_n a_n$

$$= \ldots = \delta s_0 + \sum_{i=1}^{n-1} w_0(t_i)\, e_i + w_0(t_n)\, e_n a_n \; \left(s_i = \prod_{k=i}^{n} a_k\right)$$

$$\leq \; \delta a^{n+1} + \gamma \bar{a} w_0(t^*)\, a^n \sum_{i=1}^{n-1} \alpha_1^i + a\gamma w_0(t^*)\, (\bar{a})^{n+1} \left(\alpha_1 = \frac{\bar{a}}{a}\right)$$

$$\leq \; \bar{p}_n ,$$

where $\bar{p}(t^*) = \bar{p}_n = \delta a^{n+1} + \dfrac{\gamma \bar{a} w_0(t^*) a^n}{1 - \alpha_1} + a\gamma w_0(t^*)\, a^{-n+1}, \; n \geq 1$

Note that $\bar{p}_n \leq \bar{p}_1 = \bar{p}_1(t^*)$

Moreover we have

$$e_{n+1}\, d_{n+1}\, (1 + c_{n+1}) \leq \gamma a^2 d\, (1 + c) \text{ for all } n \geq 0,$$

$$l_n \leq f(t^*)\, \alpha_0 , \text{ provided that } f(t^*) \leq 1$$

where $f(r) = d\,(1 + c)\, p_1(r) + c$. Further more we have

$$\sum_{i=1}^{n} h_i w(t_i)\, (t_i - t_{i-1}) \leq t^* w(t^*) \sum_{i=1}^{n} h_i \leq t^* w(t^*) \sum_{i=1}^{n} f(t^*)^{n-i}$$

$$\leq t^* w(t^*)\, f(t^*)^n \sum_{i=1}^{n} f(t^*)^{-i}$$

$$\leq \frac{t^* w(t^*)}{1 - f(t^*)}$$

Finally define the function T on (0, R) by

$$T(r) = \alpha + \gamma d\, (1 + c)\, a^{-2} \left[f(r)\, \alpha_0 + \frac{r\, w(r)}{1 - f(r)} + r w(r) + w_1(r) \right].$$

Assume that there exists a minimum positive number $r^* \in$ (0, R] such that

$$f(r^*) < 1 \text{ and } T(r^*) \leq r^*$$

Then it can easily be seen $t_n \leq r^* = t^*$, $\varepsilon_n \leq \varepsilon < 1$ for all $n \geq 0$ and that the sequence $\{x_n\}$, $n \geq 0$ converges to a unique solution x^* of equation (1) in $U(x_0 , r^*)$.

Remarks 2: Condition (17) can be replaced by

$$||M_{n-1}^{-1}|| \cdot ||M_{n-1} - M_n|| \leq \bar{b}_n < 1 \text{ for all } n \geq 0$$

Then a theorem similar to the one above can easily be proved. Just replace $\{d_n\}$, by the sequence $\{\bar{d}_n\}$ defined by

$$\bar{d}_0 = \beta, \bar{d}_n = \frac{\bar{d}_{n-1}}{1 - b_n}, n \geq 1.$$

Remark 3: Hypothesis (v) above is used to show that $F(x^*) = 0$. It can be replaced by the conditions $\mathrm{Kern}(L_n) = 0$ for all $n \geq 0$ and $\lim_{n \to \infty} Zn = 0$, since by (6) and (7) we can then show $F(x^*) = 0$ also.

Remarks 4: We can extend the above results further to cover the case when F is not Frechet differentiable. Consider the equation

$$F_1(x) = 0 \tag{34}$$

with $F_1(x) = F(x) + S(x)$

where F is as before and S is a non-linear operator defined on E_3 with values on E_2 such that

$$||S(x) - S(y)|| \leq w_2 (||x - y||) \text{ for all } x, y \in E_3$$

for some nondecreasing function w_2 defined on $\mathbb{R}+$ with $w_2(0) = 0$. Note that the differentiability of S (i.e. of F_1) is not assumed here. Then a theorem similar to the main theorem can be proved immediately for equation (34) if we just replace F by F_1 in (7), (12) and (19). The scalar sequence $\{\bar{t}_n\}$, $n \geq 0$ corresponding to $\{t_n\}$ will be defined the same way, but inside the bracket in (20) there will be an extra term of the form $w_2(\bar{t}_{n+1})$, $n \geq 0$.

Remark 5: Special cases of the main theorem compare favorably with

(a) The results obtained in [8]. Choose $t = 1$ in (9), $P = 1$ and the functions w_0 and w to be defined by the right hand sides of the inequalities (36) and (37) respectively [8, p. 1126–1125]. Even then our conditions are more general.

(b) The results obtained in [3]. Just choose $P = 1$, $N_n = 0$, $Z_n = 0$, $L_n = 1$ for all $n \geq 0$.

(c) The results in [8]. Take $P = 1$, $A(x) = F'(x)$ in [[7], p. 251].

(d) The results in [1], [2], [4] – [14] by choosing the operators accordingly as in (a) – (c).

The verification of (a), (b), (c) and (d) above as almost immediate is left to the motivated reader.

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RATIONAL CONTINUITY

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ABSTRACT

If the mathematics of rational cubic curves is visualized via its homogeneous counterpart, it is required to establish some equivalent constraints on the homogeneous curve regarding continuity of the kind which is desired for the rational curve. The constraints on the homogenous curves are derived that are exactly equivalent to the parametric and σ-continuity of the rational cubic curves. For every degree of continuity, the rational continuity constraints contain a degree of freedom not present in the corresponding continuity constraints for projected curves.

Keywords: Spline, B-spline, rational spline, NURBS, interpolation, Bézier curve.

1. INTRODUCTION

DEFINITION 1

We will call a function $P(t)$ σ-continuous at $t = t_i$ if it satisfies the following constraints

$$\begin{bmatrix} P(t_{i+}) \\ P^{(1)}(t_{i+}) \\ P^{(2)}(t_{i+}) \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & \sigma_{1.i} & \\ 0 & \sigma_{2.i} & \sigma_{3.i} \end{bmatrix} \begin{bmatrix} P(t_{i-}) \\ P^{(1)}(t_{i-}) \\ P^{(2)}(t_{i-}) \end{bmatrix}, \quad (0)$$

where the *connection matrix* will be denoted by S_i.

DEFINITION 2

$P \in C^k[t_0,t_n]$ will mean that each component function of $P : [t_0,t_n] \rightarrow R^N$ is k-times continuously differentiable (this continuity is known as *parametric continuity* for parametric curves) on $[t_0 , t_n]$. Similarly the notations G^k and F^k will be fixed for *geometric* (reparametrization) and *Frenet frame* continuity.

The rational curve has manifested itself in various forms including NURBS (Non-uniform rational B-splines) [Farin '88] the rational Bézier curve [Farin'83, '88], rational Beta-spline [Boehm'87] and rational σ-splines [Sarfraz'90]. As a single rational function usually does not have enough freedom to represent a given curve, several rational segments are used instead. To generate a curve of satisfactory smoothness, the segments must connect with some amount of continuity. Thus, the use of rational curves, independent of the particular variety, creates a common problem, that of connecting rational segments to form piecewise rational curves that are smooth.

Let $P(t)$ be a parametric rational curve. There are at least two ways to visualize the mathematics of a rational curve.

1. The curve P can be thought of as a vector-valued function, each component of which is a rational function i.e.

$$P(t) = (P_1(t), P_2(t), \ldots, P_N(t)) \tag{1}$$

where each component function P_i, is the rational function

$$P_i(t) = \frac{F_i(t)}{F_{N+1}(t)} \tag{2}$$

and $F_i(t)$ are all polynomial functions.

2. P can be thought of as the composition of a vector valued polynomial function F with a projection function that takes $(x_1, \ldots, x_N, x_{N+1})$ to $\left(\frac{x_1}{x_{N+1}}, \ldots, \frac{x_N}{x_{N+1}}\right)$ i.e.

$$F(t) = (F_1(t), \ldots, F_N(t), F_{N+1}(t)) \tag{3}$$

F is refered to as the *homogenous curve* associated with P and P as the *projection* of F.

To illustrate more concretely, consider a curve formation such as the rational Bézier curve, the rational B-spline curve or the rational Beta-spline curve. Each is a function $P : R \rightarrow R^N$ that can be expressed as

$$P(t) = \frac{\sum_{i=0}^{m} w_i V B_i(t)}{\sum_{i=0}^{m} w_j B_j(t)} \qquad (4)$$

where V_i are the control vertices, $B_i(t)$ are the basis functions and w_i are the weights. Such curves have been considered in this form in [Sarfraz'90] but one can also consider such a curve as polynomial Bézier curve, B-spline or Beta-spline curve in higher dimensional space as

$$F(t) = \sum_{i=0}^{m} W_i B_i(t) \qquad (5a)$$

where W_i are control vertices in R^{N+1} whose coordinates are $W_i = (w_i V_i , w_i)$ for $i = 0 , \ldots , m$. Of course, when the R^N coordinates of P are required, the division must be performed. In Figure 1 it is shown how one might view an R^2 rational curve as an R^3 polynomial curve. Each component of the R^3 curve is strictly polynomial and the points on the R^2 curve are obtained by projecting the R^3 curve onto the $w = 1$ plane.

This strategy of manipulating rational curves has the advantage that algorithms like evaluation, subdivision, degree elevation, etc; to manipulate rational curves, can often be obtained by using the corresponding algorithm for polynomial curves. i.e. in this way one has a large body of information on polynomial curves which is almost always applicable to rational curves. For example, to increase the degree of a rational curve P, one can increase the degree of the polynomial curve F via a well-known algorithm for a polynomial curve: to subdivide P, one can subdivide F, and so on. If we project F to obtain P it will be as if the manipulation has been performed on P.

Barsky, Goldman and Micchelli [Goldman and Micchelli'89 Goldman and Barsky'89] used this method of reducing a problem associated with a rational curve to the analogous problem for its homogenous counterpart for the problem of continuity between rational segements. Specifically, they proved that parametric, geometric or Frenet frame continuity for a rational curve can be obtained by requiring the associated homogeneous curve to be parametrically, geometrically or Frenet frame continuous: methods developed by these authors are more general. In this paper we will concentrate, using elementary techniques, on two kinds of continuities for parametric curves:

(a) parametric continuity, for rational curves of arbitrary degree, in Section 2 and

(b) σ-continunity (more general form of geometric continuity. It should also be noted that $G^2 = F^2$ i.e. the geometric continuity of order 2 coincides with Frenet frame continuity of order 2) for rational cubics, in Section 3.

2. PARAMETRIC CONTINUITY

If we have a homogenous curve F that is C^k for an arbitrary positive integer k (with $F_{N+1}(t) \neq 0, \forall t$), then the projection P of F will also be C^k. The converse is not true: there are homogeneous curves F that are not C^k even though their projections P are C^k : e.g. consider the rational cubic:

$$P(t) = P_i(t_i; v_i, w_i) = \qquad (5b)$$

$$\frac{(1-\theta)^3 X_i + \theta(1-\theta)^2(v_i X_i + h_i D_i) + \theta^2(1-\theta)(w_i X_{i+1} - h_i D_{i+1}) + \theta^3 X_{i+1}}{(1-\theta)^3 + v_i\theta(1-\theta)^2 + w_i\theta^2(1-\theta) + \theta^3}$$

where the notations X_i and $D_i \in R^N$ are, respectively, the data values and the first derivative values at the knots t_i , $i = 0, \ldots\ldots, n$ with $t_0 < t_1 <, \ldots\ldots\ldots, < t_n$, $h_i = t_{i+1} - t_i$, $\theta = (t - t_i)/h_i$ and v_i , $w_i \geq 0$. This is C^1 whereas the numerator and denominator are only C^0. Thus, in general, it is not necessary for each of two scalar functions F_i nor F_{N+1} to be C^k in order that $\frac{F_i}{F_{N+1}}$ be C^k.

What constraints must we impose on two scalar functions F_i and F_{N+1}, in order that, the quotient $\frac{F_i}{F_{N+1}}$ is C^k ? For this, consider a scalar function $f(t)$ that is composed of two C^k functions, $P(t)$ for $t \leq t_j$ and $Q(t)$ for $t > t_j$, where $P(t) = \frac{F_i(t)}{F_{N+1}(t)}$ and $Q(t) = \frac{Q_i(t)}{Q_{N+1}(t)}$, as in the case with piecewise rational functions. Clearly $F(t)$ will be C^k if and only if

$$\left(\frac{F_i(t)}{F_{N+1}(t)} \right)^{(l)} \Bigg|_{t=t_j} = \left(\frac{Q_i(t)}{Q_{N+1}(t)} \right)^{(l)} \Bigg|_{t=t_j} , \; l = 0, \ldots, k, \quad (6)$$

which is equivalent to

$$[F_i(t)]^{(l)} \Big|_{t=t_j} = \left(\frac{F_{N+1}(t) Q_i(t)}{Q_{N+1}(t)} \right)^{(l)} \Bigg|_{t=t_j} , \; l = 0, \ldots, k, \quad (7)$$

Conditions (6) and (7) are equivalent since if we assume (6) holds, then

$$\left(F_{N+1}(t) \frac{Q_i(t)}{Q_{N+1}(t)} \right)^{(l)} = \sum_{m=0}^{l} \binom{l}{m} F_{N+1}^{(l-m)}(t) \left(\frac{Q_i(t)}{Q_{N+1}(t)} \right)^{(m)}$$

(by Leibnitz theorem)

$$= \sum_{m=0}^{l} \binom{l}{m} F_{N+1}^{(l-m)}(t) \left(\frac{F_i(t)}{F_{N+1}(t)} \right)^{(m)}$$

(by substitution of (6))

$$= \left(F_{N+1}(t) \frac{F_i(t)}{F_{N+1}(t)} \right)^{(l)}$$

(by Leibnitz theorem)

$$= [\, F_i(t) \,]^{(l)}$$

Using similar arguments, (6) follows with the assumption of (7).

Conversely, let

$$\alpha_{l-mj} = \left(\frac{F_{N+1}(t_j)}{Q_{N+1}(t_j)} \right)^{(l-m)}$$

then using Leibnitz theorem, (7) can be rewritten as

$$F_i^{(l)}(t_j) = \sum_{m=0}^{l} \binom{l}{m} \alpha_{l-mj} \, Q_i^{(m)}(t_j), \; l = 0, \ldots, k \quad (8)$$

Thus we have proved the following:

Theorem 3: The projection of $F(t)$ will be C^k at $t = t_j$ if and only if there exist α's so that

$$F^{(l)}(t_{i+}) = \sum_{m=0}^{l} \binom{l}{m} \alpha_{l-m,i} F^{(m)}(t_{i-}), \quad l = 0, \ldots, k \quad (9)$$

Example 4: Let us consider the rational cubic in (5b) with $v_i = w_i = r_i$, we will be interested in continuity of order two which, in matrix notation, can be expressed as

$$[F_{i+}] = A_i \, [F_{i-}] \qquad (10)$$

where

$$[F_{i+}] = \begin{bmatrix} F(t_{i+}) \\ F^{(1)}(t_{i+}) \\ F^{(2)}(t_{i+}) \end{bmatrix}, \quad [F_{i-}] = \begin{bmatrix} F(t_{i-}) \\ F^{(1)}(t_{i-}) \\ F^{(2)}(t_{i-}) \end{bmatrix} \qquad (11)$$

and

$$A_i = \begin{bmatrix} \alpha_{0.i} & & \\ \alpha_{1.i} & \alpha_{0.i} & \\ \alpha_{2.i} & 2\alpha_{1.i} & \alpha_{0.i} \end{bmatrix}$$

Now we determine the parameters α for the rational spline curve so obtained. Let $F_1(t)$ and $F_2(t)$ denote the numerator and denominator, then the homogeneous counterpart of $P(t)$ is

$$F(t) = (F_1(t) \, , F_2(t))$$

The parameters $\alpha_{j,i}$, $i = 1$, ... , $n - 1$, $j = 0, 1, 2$ can be determined in terms of tension parameters r_i's from the constraints (10).

The first constraint

$$F_2(t_{i+}) = \alpha_{0,i} \, F_2(t_{i-})$$

gives

$$\alpha_{0,i} = 1 \qquad (12)$$

The second constraint

$$F_2^{(1)}(t_{i+}) = \alpha_{0,i} F_2^{(1)}(t_{i-}) + \alpha_{1,i} F_2(t_{i-})$$

gives

$$\frac{(r_i - 3)}{h_i} = \alpha_{1,i} + \frac{\alpha_{0,i}(3 - r_{i-1})}{h_{i-1}}$$

Substituting (12) and then rearranging yields

$$\alpha_{1,i} = \frac{(r_i - 3)}{h_i} + \frac{(r_{i-1} - 3)}{h_{i-1}} \qquad (13)$$

The third constraint

$$F_2^{(2)}(t_{i+}) = \alpha_{0,i}\, F_2^{(2)}(t_{i-}) + 2\alpha_{1,i}\, F_2^{(1)}(t_{i-}) + \alpha_{2,i} F_2(t_{i-})$$

gives

$$\frac{-2(r_i - 3)}{h^2{}_i} = \alpha_{2,i} + \frac{2\alpha_{1,i}(3 - r_{i-1})}{h_{i-1}} + \frac{2\alpha_{0,i}(3 - r_{i-1})}{h^2{}_{i-1}}$$

Substituting (12), (13) and then rearranging yields

$$\alpha_{2,i} = 2\left\{\frac{r_{i-1} - 3}{h_{i-1}}\left(\frac{r_{i-1} - 3}{h_{i-1}} + \frac{r_i - 3}{h_i}\right) - \frac{r_i - 3}{h_i} + \frac{r_i - 3}{h_i}\right\} \qquad (14)$$

3. σ-CONTINUITY

Using the same notations as in Section 2, we derive in this section the constraints which are necessary and sufficient for the projection $P(t)$ of the homogeneous curves to be σ-continuous of order 2 as defined in Section 1.

(Positional Continuity): This follows straightaway from (8) for $l = 0$, i.e.

$$F_i(t_j) = \alpha_{0,j}\, Q_i(t_j)\,,\ i = 0, \ldots, N + 1 \qquad (15)$$

(σ-Continuity of order one): In addition to equations (15), we have

$$\left(\frac{F_i(t)}{F_{N+1}(t)}\right)^{(l)}\Bigg|_{t=t_j} = \sigma_{1,j}\left(\frac{Q_i(t)}{Q_{N+1}(t)}\right)^{(l)}\Bigg|_{t=t_j}$$

Differentiation gives

$$\frac{F_i^{(1)}(t_j)F_{N+1}(t_j) - F_i(t_j)F_{N+1}^{(1)}(t_j)}{F^2_{N+1}(t_j)}$$

$$= \sigma_{1,j} \frac{Q_i^{(1)}(t_j)Q_{N+1}(t_j) - Q_i(t_j)Q_{N+1}^{(1)}(t_j)}{Q_{N+1}^2(t_j)}$$

Substitution of (15) yields:

$$\frac{F_i^{(1)}(t_j)\alpha_{0,j}Q_{N+1}(t_j) - \alpha_{0,j}Q_i(t_j)\, F_{N+1}^{(1)}(t_j)}{\alpha^2{}_{0,j}Q_{N+1}^2(t_j)}$$

$$= \sigma_{1,j} \frac{Q_i^{(1)}(t_j)Q_{N+1}(t_j) - Q_i(t_j)Q_{N+1}^{(1)}(t_j)}{Q_{N+1}^2(t_j)}$$

Cross multiplying and reorganizing terms yields

$$\left(\alpha_{0,j}\, \sigma_{1,j}\, Q_{N+1}^{(1)}\,(t_j) - F_{N+1}^{(1)}\,(t_j) \right) Q_i\,(t_j)$$

$$= \left(\alpha_{0,j}\, \sigma_{1,j}\, Q_i^{(1)}\,(t_j) - F_i^{(1)}\,(t_j) \right) Q_{N+1}\,(t_j)$$

If we divide and introduce a parameter $\alpha_{1,j}$, then

$$\frac{F_{N+1}^{(1)}\,(t_j) - \alpha_{0,j}\, \sigma_{1,j}Q_{N+1}^{(1)}\,(t_j)}{Q_{N+1}(t_j)}$$

$$= \frac{F_i^{(1)}\,(t_j) - \alpha_{0,j}\, \sigma_{1,j}Q_i^{(1)}\,(t_j)}{Q_i(t_j)}$$

$$= \alpha_{1,j}$$

We can separate the constraints as

$$F_i^{(1)}\,(t_j) = \alpha_{0,j}\sigma_{1,j}\, Q_i^{(1)}\,(t_j) + \alpha_{1,j}\, Q_i(t_j)\,,\ i = 1\,, \dots\,, N \text{ and}$$

$$F_{N+1}^{(1)}(t_j) = \alpha_{0,j}\sigma_{1,j}\, Q_{N+1}^{(1)}(t_j) + \alpha_{1,j}\, Q_{N+1}(t_j) \tag{16}$$

which is equivalent to

$$F^{(1)}\,(t_j) = \alpha_{0,j}\sigma_{1,j}\, Q^{(1)}\,(t_j) + \alpha_{1,j}\, Q(t_j)\,, \tag{17}$$

where $F = (F_1\,, \dots\,, F_N\,, F_{N+1})\,,\ Q = (Q_1\,, \dots\,, Q_N\,, Q_{N+1})$

(σ-continunity of order two): In addition to equations (16) and (17),

$$\left(\frac{F_i(t)}{F_{N+1}(t)} \right)^{(2)} \Bigg|_{t=t_j} = \sigma_{3,j} \left(\frac{Q_i(t)}{Q_{N+1}(t)} \right)^{(2)} + \sigma_{2,j} \left(\frac{Q_i(t)}{Q_{N+1}(t)} \right)^{(1)} \Bigg|_{t=t_j}$$

Differentiation gives

$$\Big\{F_i^{(2)}(t_j)F_{N+1}^3(t_j) - F_i(t_j)\, F_{N+1}^{(2)}(t_j)\, F_{N+1}^2(t_j) - 2\, F_i^{(1)}(t_j)\, F_{N+1}^{(1)}(t_j)$$

$$F_{N+1}^2(t_j) + 2\, F_i(t_j)\, (F_{N+1}^{(1)}(t_j))^2\, F_{N+1}(t_j)\Big\} \Big/ F_{N+1}^4(t_j)$$

$$= \sigma_{3,j} \Big\{Q_i^{(2)}(t_j)\, Q_{N+1}^3(t_j) - Q_i(t_j)\, Q_{N+1}^{(2)}(t_j)\, Q_{N+1}^2(t_j) - 2Q_i^{(1)}(t_j)\, Q_{N+1}^{(1)}$$

$$(t_j)\, Q_{N+1}^2(t_j) + 2Q_i(t_j)\, (Q_{N+1}^{(1)}(t_j))^2\, Q_{N+1}(t_j)\Big\} \Big/ Q_{N+1}^4(t_j) + \sigma_{2,j}$$

$$\Big\{Q_i^{(1)}(t_j)\, Q_{N+1}(t_j) - Q_i(t_j)\, Q_{N+1}^{(1)}(t_j)\Big\} \Big/ Q_{N+1}^2(t_j)$$

Substitution of (15) and (16) yields

$$\Big\{ F_i^{(2)}\, \alpha_{0,j}^3\, Q_{N+1}^3 - \alpha_{0,j} Q_i \alpha_{0,j}^2 Q_{N+1}^2\, F_{N+1}^{(2)} - 2\alpha_{0,j}^2 Q^2{}_{+1}$$

$$\left(\alpha_{0,j}\sigma_{1,j} Q_i^{(1)} + \alpha_{1,j} Q_i \right) \left(\alpha_{0,j}\, \sigma_{1,j}\, Q_{I+1}^{(1)} + \alpha_{1,j}\, Q_{N+1} \right) + 2\alpha_{0,j}\, Q_i\, \alpha_{0,j}$$

$$Q_{N+1} \left(\alpha_{0,j}\, \sigma_{1,j}\, Q_{N+1}^{(1)} + \alpha_{1,j}\, Q_{N+1} \right)^2 \Big\} \Big/ \alpha_{0,j}^4\, Q_{N+1}^4$$

$$= \sigma_{3,j} \Big\{ \{Q_i^{(2)}\, Q_{N+1}^3 - Q_i\, Q_{N+1}^{(2)}\, Q_{N+1}^2 - 2\, Q_i^{(1)}\, Q_{N+1}^{(1)}\, Q_{N+1}^2 +$$

$$2Q_i\, (Q_{N+1}^{(1)})^2\, Q_{N+1}\} + \sigma_{2,j} \left(Q_{N+1}^3 Q_i^{(1)} - Q_{N+1}^2 Q_i Q_{N+1}^{(1)}\right)\Big\} \Big/ Q_{N+1}^4(t_j)$$

Cross multiplying and reorganizing terms yields

$$\Big\{F_i^{(2)} - \alpha_{0,j}\, \sigma_{3,j}\, Q_i^{(2)} - \left(\alpha_{0,j} \left(\sigma_{2,j} + 2\, (\sigma_{1,j}^2 - \sigma_{3,j}) \frac{Q_{N+1}^{(1)}}{Q_{N+1}} \right) \right.$$

$$\left. + 2\alpha_{1,j}\sigma_{1,j} \right) Q_i^{(1)} \Big\}\, Q_{N+1}$$

$$\Big\{F_{N+1}^{(1)} - \alpha_{0,j}\, \sigma_{3,j}\, Q_{N+1}^{(2)} - \left(\alpha_{0,j} \left(\sigma_{2,j} + 2\, (\sigma_{1,j}^2 - \sigma_{3,j}) \frac{Q_{N+1}^{(1)}}{Q_{N+1}} \right) \right.$$

$$\left. + 2\alpha_{1,j}\sigma_{1,j} \right) Q_{N+1}^{(1)} \Big\}\, Q_i$$

If we divide and introduce a parameter $\alpha_{2,j}$ then

$$\left\{ F_i^{(2)} \alpha_{0,j}\, \sigma_{3,j}\, Q_i^{(2)} - \left(\alpha_{0,j}\, \sigma_{2,j} + 2\alpha_{1,j}\, \sigma_{1,}\right) Q_i^{(1)} \right\} / Q_i^{(1)} \qquad (18)$$

$$= \left\{ F_{N+1}^{(2)} - \alpha_{0,j}\, \sigma_{3,j}\, Q_{N+1}^{(2)} - \left(\alpha_{0,j}\, \sigma_{2,j} + 2\alpha_{1,j}\, \sigma_{1,j}\right) Q_{N+1}^{(1)} \right\} / Q_{N+1}$$

$$= \alpha_{2,j},$$

where $$\hat{\sigma}_{2,j} = \sigma_{2,j} + 2\,(\sigma_{1,j}^{2} - \sigma_{3,j})\, \frac{Q_{N+1}^{(1)}}{Q_{N+1}} \qquad (19)$$

Equivalently, separation in (18) can be done as:

$F_i^{(2)}(t_j) = \alpha_{0,j}\, \sigma_{3,j}\, Q_i^{(2)}(t_j) + \left[\alpha_{0,j}\, \sigma_{2,j} + 2\alpha_{1,j}\, \sigma_{1,j}\right] Q_i^{(2)}(t_j) + \alpha_{2,j}$
$Q_i(t_j)$. $i = 0, 1, \ldots, N$, and

$F_{N+1}^{(2)}(t_j) = \alpha_{0,j}\, \sigma_{3,j}\, Q_{N+1}^{(2)}(t_j) + \left[\alpha_{0,j}\, \hat{\sigma}_{2,j} + 2\alpha_{1,j}\, \sigma_{1,j}\right] Q_{N+1}^{(1)}(t_j)\, \alpha_{2,j}$
$Q_{N+1}(t_j)$ (20)

which is equivalent to

$$F^{(2)}(t_j) = \alpha_{0,j}\, \sigma_{3,j}\, Q^{(2)}(t_j) + \left[\alpha_{0,j}\, \hat{\sigma}_{2,j} + 2\alpha_{1,j}\, \sigma_{1,j}\right] Q^{(1)} + \alpha_{2,j}\, Q(t_j) \qquad (21)$$

The constraints (15), (17) and (21) can be written in matrix notation as

$$[F_j] = A_j\, \hat{S}_j\, [Q_j],$$

where

$$A_j = \begin{bmatrix} \alpha_{0,j} & & \\ \alpha_{1,j} & \alpha_{0,j} & \\ \alpha_{2,j} & 2\alpha_{1,j} & \alpha_{0,j} \end{bmatrix}, \quad \hat{S}_j = \begin{bmatrix} 1 & & \\ 0 & \sigma_{1,j} & \\ 0 & \sigma_{2,j} & \sigma_{3,j} \end{bmatrix} \qquad (22)$$

Thus we have the following:

Proposition 5: The projection $P(t)$ of the homogenous curve $F(t)$ will be σ-continuous if and only if there exist connection matrices A_i and $\hat{S}_i$ as in (22) such that

$$[F_{i+}] = A_i \hat{S}_i\, [F_{i-}], \quad i = 1, \ldots, n-1 \qquad (23)$$

Remark 6: For the particular case of σ-continuity when $\sigma_{3,i} = \sigma_{1,i}^2$, it follows from (19) that $\hat{\sigma}_{2,i} = \sigma_{2,i}$ and thus $\hat{S}_i = S_i$. In this case the σ-continuity constraints coincide with Beta constraints [Barsky'81].

Example 7: Let $F_1(t)$ and $F_2(t)$ denote numerator and denominator of the rational cubic in (5b), then the homogeneous counterpart is

$$F(t) = (F_1(t), F_2(t))$$

As can be easily evaluated, at $t = t_i$,

$$F_2(t_{i+}) = 1 = F_2(t_{i-}),$$

$$F_2^{(1)}(t_{i+}) = \frac{v_i - 3}{h_i}, \quad F_2^{(1)}(t_{i-}) = \frac{3 - w_{i-1}}{h_{i-1}} \tag{24}$$

$$F_2^2(t_{i+}) = \frac{6 - 4v_i}{h_i^2}, \quad F_2^{(1)}(t_{i-}) = \frac{6 - 4w_{i-1}}{h_{i-1}^2}$$

Now consider the σ-continuity constraints (23) one by one. The first contraint simply gives

$$\alpha_{0,i} = 1 \tag{25}$$

The second constraint, after substitution of (24) and (25), gives

$$\alpha_{1,j} = \frac{v_i - 3}{h_i} + \sigma_{1,i}\frac{w_{i-1} - 3}{h_{i-1}} \tag{26}$$

The third constraint, after substitution of (24) and (25), gives

$$\sigma_{2,i} = \frac{6 - 4v_i}{h_i^2} + (2\alpha_{1,i}\sigma_{1,i} + \hat{\sigma}_{2,i})\frac{w_{i-1} - 3}{h_{i-1}} + \sigma_{3,i}\frac{4w_{i-1} - 6}{h_{i-1}^2} \tag{27}$$

4. SOME SPECIAL CASES

(a) If $\alpha_{0,i} = 1$ and $\alpha_{j,i} = 0$ for $i \geq 1$, the rational parametric continuity (9) reduces to simple parametric continuity.

(b) If $\sigma_{1,i} = \sigma_{3,i} = 1$ and $\sigma_{3,i} = 0$, the rational σ-continuity (24) reduces to rational parametric continuity.

(c) For the choice of α's as in case A, the rational σ-continuity reduces to simple σ-continuity.

(d) If the α's are as in case A and $\sigma_{3,i} = \sigma_{1.i}^{2}$, then the rational σ-continuity reduces to simple β-continuity.

(e) If the α's are as in case A and the σ's are as in case B, the rational σ-continuity reduces to simple parametric continuity.

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STRONG TOPOLOGY ON GENERAL 2-SEMI-INNER PRODUCT SPACES OF TYPE (P)

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ABSTRACT

In this paper we introduce a straight forward algebraic generalization of 2-semi-inner product spaces and 2-semi-inner product spaces of type (p). We introduce and study strong topology on such spaces.

1. INTRODUCTION

Prugovecki [5] introduced a straight forward algebraic generalization of inner product spaces, which he called generalized inner product spaces, and then he enumerated and derived some fundamental properties of different topologies in these spaces. In a different direction, Lumer [3] introduced the concept of semi-inner product space as a generalization of inner product space. Using the concept of Prugovecki [5] and Lumer [3], Nath [4] introduced what he called generalized semi-inner product spaces, and then he studied strong topologies on these spaces.

Later, Siddiqui and Rizvi [6] defined the concept of 2-semi-inner product space and obtained certain results. This concept led Abo-Hadi [1] to define the concept of 2-semi-inner product space of type (p) and he obtained certain results on such spaces. In the present paper we introduce the concept of generalized 2-semi-inner product space and that of generalized 2-semi-inner product space of type (p) and study the strong topology in such spaces.

2. PRELIMINARIES

2.1 Definition [2]

Let E be a vector space with dim (E) > 1 and $||.,.||$ non-negative real function on E × E which satisfies the following axioms:

(N_1) $||x, y|| = 0$ if x and y are linearly dependent.

(N_2) $||x, y|| = ||y, x||$

(N_3) $||\lambda x, y|| = |\lambda|\ ||x, y||$, where $\lambda \varepsilon R$

(N_4) $||x + y, z|| \leq ||x, y|| + ||y, z||$, where $x, y, z \varepsilon E$

Then we say that $||.,.||$ is a 2-norm on E. E, equipped with a 2-norm, is called a 2-normed space.

2.2 Definition [1]

Let E be a vector space with dim (E) > 1 and $[.,./.]$ a real function on E × E × E which satisfies the following axioms:

(S_1) $[x_1 + x_2, y/z] = [x_1, y/z] + [x_2, y/z]$

(S_2) $[\lambda x, y/z] = \lambda [x, y/z]$

for every $x_1, x_2, x, y, z \varepsilon E$ and for every $\lambda \varepsilon R$

(S_3) $[x, x/y] > 0$ if x and y are linearly independent

(S_4) $|[x, y/z]| \leq [x, x/z]^{1/p} [y, y/z]^{p-1/p}$, $1 < p < \infty$

Then we say that $[.,./.]$ is a 2-semi-inner product of type (p) on E.

E, equipped with a 2-semi-inner product of type(p), is called a 2-semi-inner product space of type(p).

2.3 Remark

If $p = 2$, this concept is called a 2-semi-inner product space which is due to Siddiqui and Rizvi [6].

2.4 Theorem [1]

Every 2-normed space can be made into a 2 semi-inner product space o type(p).

2.5 Theorem [1]

A 2-semi-inner product space of type(p) is 2-normed space with the 2-norm $||x , y|| = [x , x/y]^{1/p}$ provided $[x , x/y] = [y , y/x]$.

2.6 Remark

In particular, for $p = 2$, these results hold good for 2-semi-inner product spaces introduced in [6].

3. GENERALIZED 2-SEMI-INNER PRODUCT SPACES OF TYPE(p)

3.1 Definition

A vector space E is called a generalized 2-semi-inner product space of type(p) if

(G_1) there is a subset M o E which is a 2-semi-inner product space of type (p).

(G_2) there is a non-empty set L of linear operators on E which has the following properties:

(i) each element of L maps E into M.

(ii) if $Tx=0$ for all $T \varepsilon L$, then $x = 0$. We denote a generalized 2-semi-inner product space of type (p) by the triple (E,L,M).

3.2 Remark

If $p = 2$, we call this concept a generalized 2-semi-inner product space.

3.3 Remark

Every 2-semi-inner product space of type (p) [in particular, 2-semi-inner produce space] is a generalized 2-semi-inner produce space of type (p) [a generalized 2-semi-inner product space], with M=E and L=[1], I the indentity operator on E.

3.4 Remark

It would be interesting to find a non-trivial example of a generalized 2-semi-inner product space of type(p) which is not a 2-semi-inner product space of type (p).

4. STRONG TOPOLOGY

4.1 Definition

Let (E, L, M) be a generalized 2-semi-inner product space of type (p).

To each $x \varepsilon$ E, the family of sets defined by $V(x;T_1, \ldots, T_n;\varepsilon)$

$= \{y \varepsilon E : [T_k(y-x), T_k(y-x) / z]^{1/p} < \varepsilon, k = 1, 2, \ldots, n\}$,

$\varepsilon > 0$, $z \varepsilon M$, $T_1, \ldots, T_n \varepsilon L$ and $n = 1, 2, \ldots$, forms a neighbourhood basis at x for a topology on E which we call the strong topology.

4.2 Remark

In particular, for $p = 2$, we have the strong topology for generalized 2-semi-inner product spaces.

4.3 Lemma

Each $V(0; T_1, \ldots, T_n; \varepsilon)$ is circled and convex.

Proof

Let $V = V(0; T_1, \ldots, T_n; \varepsilon)$

To show V is circled:

Let $\lambda \varepsilon R$ with $|\lambda| \leq 1$ and $x \varepsilon V$.

$[T_k(\lambda x), T_k(\lambda x)/z]^{1/p} = ||T_k(\lambda x), z|| = |\lambda| \, ||T_k(x), z||$

$\Rightarrow$ $||T_k(\lambda x), z|| < \varepsilon, k = 1, 2, \ldots, n$

$\Rightarrow$ $\lambda x \varepsilon V$.

Thus V is circled. To show V is convex:

Let $\lambda \varepsilon R$, $0 < \lambda < 1$ and $x, y \varepsilon V$.

$$||T_k[\lambda x + (1-\lambda)y], z|| = ||T_k(\lambda x) + T_k[(1-\lambda)y], z||$$
$$= ||\lambda T_k(x) + (1-\lambda) T_k(y), z||$$
$$\leq \lambda ||T_k(x), z|| + (1-\lambda) ||T_k(y), z||$$
$$< \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon$$

$\Rightarrow$ $||T_k[\lambda x + (1-\lambda)y], z|| < \varepsilon, k = 1, 2, \ldots, n$

$\Rightarrow$ $\lambda x + (1-\lambda) y \varepsilon V$. Thus V is convex.

4.4 Remark

In particular, this result holds for generalized 2-semi-inner product spaces.

4.5 Lemma

Let (E, L, M) be a generalized 2-semi-inner product space of type (p). If a topology on it is introduced in which the sets $V(x ; T ; \varepsilon)$ are neighbourhoods of x $\forall \varepsilon > 0$, $T \varepsilon L$, then the resulting topological space is Hausdorff.

Proof

Here $[Tx , Tx/z] = ||Tx , z||^p$

Suppose E is not a Hausdorff space.

Then, there exist at least two points x_1, $x_2 \varepsilon E$ and $x_1 \neq x_2$ for which any two neighbourhoods have common points.

Thus for any two neighbourhoods $V(x_1 ; T ; \frac{1}{n})$ and $V(x_2;T;\frac{1}{n})$

there exists at least one $y_n \varepsilon E$ such that $y_n \varepsilon V(x_1;T;\frac{1}{n}) \cap V(x_2;T;\frac{1}{n})$

So $\forall z \varepsilon E$ we have

$$||T(y_n - x_1) , z|| < \frac{1}{n} \text{ and } ||T(y_n - x_2) , z|| < \frac{1}{n}$$

Now $$||T(x_1 - x_2) , z|| = ||T(x_1 - y_n + y_n - x_2) , z||$$
$$= ||T(x_1 - y_n) + T(y_n - x_2) , z||$$

$$||T(x_1 - y_n) + T(y_n - x_2) , z|| \leq ||T(x_1 - y_n) , z|| + ||T(y_n - x_2) , z|| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \forall n > 0, \forall z \varepsilon E$$

So $T(x_1 - x_2) = \lambda z \quad T \varepsilon L , z \varepsilon E$

In particular, for $z = 0$, $T(x_1 - x_2) = 0 , \forall T \varepsilon L$

Hence $x_1 - x_2 = 0$. Thus $x_1 = x_2$ which is a contradiction

4.6 Remark

In particular, this result holds for generalized 2-semi-inner product spaces.

4.7 Theorem

Let (E, L, M) be a generalized 2-semi-inner product space of type (p). Then E, equipped with the strong topology, is aHausdorff locally convex space.

Proof:

First we show that the strong topology is compatible with the vector space operations.

(i) To prove the addition is continuos:

For any $V(O\,;T_1, ..., T_n\,;\varepsilon) = V$, we show that

$V(O\,;T_1, ..., T_n\,;\varepsilon/2) + V(O\,;T_1, ..., T_n\,;\varepsilon/2)$
$\subset V(O\,;T_1, ..., T_n\,;\varepsilon)$

Let $x, y \,\varepsilon\, V(O\,;T_1, ..., T_n\,;\varepsilon/2)$

Then $||T_k(x), z|| < \varepsilon/2$ and $||T_k(y), z|| < \varepsilon/2$

Now $||T_k(x+y), z|| = ||T_k(x) + T_k(y), z|| < ||T_k(x), z|| + ||T_k(y), z|| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Thus $||T_k(x+y), z|| < \varepsilon$. So $x + y \,\varepsilon\, V$.

(ii) To prove the scalar multiplication is continuous:

For any $V(O\,;T_1, ..., T_n\,;\varepsilon) = V$

we have $\lambda V(O\,;T_1, ..., T_n\,;\varepsilon/\lambda) \subset V(O\,;T_1, ..., T_n\,;\varepsilon/\lambda)$

because if $x \,\varepsilon\, \lambda V(O\,;T_1, ..., T_n\,;\varepsilon/\lambda)$

Then $||T_k(x), z|| < \varepsilon/\lambda$

So $||T_k(\lambda x), z|| < \varepsilon$

Thus $\lambda x \,\varepsilon\, V(O\,;T_1, ..., T_n\,;\varepsilon)$

It follows from lemma (4.3) that E is a locally convex space and from lemma (4.5) that E is a Hausdorff space.

4.8 Remark

In particular, this result holds for generalized 2-semi-inner product spaces.

4.9 Theorem

Let (E, L, M) be a generalized 2-semi-inner product space of type(p) with the strong topology. E is metrizable if there exists a countable subset β of L with the following property;

For any T ε L, there exists an S ε α such that $[Tx , Tx/z]^{1/p} < [Sx,Sx/z]^{1/p}$, x ε E, where α is the linear manifold generated by β.

Proof:

It is sufficient to show that the family of sets $\{V(O ; S_1 , \ldots , S_n ; \frac{1}{n}) : S_1 , \ldots , S_k \, \varepsilon \, \beta \; k , n = 1, 2, \ldots \}$ is a neighbourhood basis at O for the strong topology. For every T ε L, we can find an S ε α for which

$$V(O ; S ; \varepsilon) \subset V(O ; T ; \varepsilon) \qquad \ldots\ldots (1)$$

because $||T(x) , z|| \leq ||S(x) , z||$

Clearly we have $S = \lambda_1 S_1 + \ldots + \lambda_k S_k$,

where $S_1 , \ldots , S_k \, \varepsilon \, \beta$

So $\forall x \, \varepsilon$ E , $[S(x) , S(x)/z]^{1/p} = ||S(x) , z||$

$$= ||(\lambda_1 S_1 + \ldots + \lambda_k S_k)(x) , z|| \leq |\lambda_1| \, ||S_1(x) , z|| + \ldots + |\lambda_k| \, ||S_k(x) , z|| \qquad \ldots\ldots (2)$$

Thus, if we choose an integer n such that

$$\frac{1}{n} < \frac{\varepsilon}{k|\lambda_1|}, \ldots , \frac{1}{n} < \frac{\varepsilon}{k|\lambda_k|}, \text{ then } x \, \varepsilon \, V(O ; S_r ; 1/n)$$

$$\text{implies } ||S_r(x) , z|| < \frac{1}{n} < \frac{\varepsilon}{k|\lambda_r|}, \; r = 1, 2, \ldots , k$$

So (2) becomes

$$||S(x) , z|| < \frac{\varepsilon}{k} + \ldots + \frac{\varepsilon}{k} \, (k \text{ times}) = k \frac{\varepsilon}{k} = \varepsilon$$

i.e. $\quad x \, \varepsilon \, V(O ; S_1 ; \frac{1}{n}) \cap \ldots \cap V(O ; S_k ; \frac{1}{n})$ implies $x \, \varepsilon \, V(O ; S ; \varepsilon)$

i.e. $\quad x \, \varepsilon \, V(O ; S_1 , \ldots , S_k ; \frac{1}{n}) \Rightarrow x \, \varepsilon \, V(O ; S ; \varepsilon)$

So $V(O ; S_1, \ldots, S_k, \frac{1}{n}) \subset V(O ; S ; \varepsilon)$

Using (1), $V(O ; T ; \varepsilon) \supset V(O ; S ; \varepsilon) \supset V(O;S_1,\frac{1}{n}) \cap \ldots \cap$

$V(O ; S_k ; \frac{1}{n}) \supset V(O ; S_1, \ldots S_k ; \frac{1}{n})$

Thus the family of sets

$$\{ V(O ; S_1, \ldots, S_k ; \frac{1}{n}) : S_1, \ldots, S_k \varepsilon \beta, k, n = 1, 2, \ldots \}$$

is a neighbourhood basis at O which is countable, since β is countable. Hence E is metrizable in the strong topology.

4.10 Remark

In particular, this result holds for generalized 2-semi-inner product spaces.

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ON A STIRLING-LIKE METHOD

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ABSTRACT

In this paper, new sufficient conditions for the convergence of a Stirling-like method to a locally unique solution of a non-linear operator equation are given. The Stirling-like method is also compared favorably to Newton's method under very natural assumptions.

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Key Words and Phrases: Stirling's method, Newton's method, Banach space.

1. INTRODUCTION

A fixed point x^* of an operator F defined on a subset E_2 of a Banach space E_1 and taking values into itself satisfies the equation

$$x = F(x) \tag{1}$$

We want to construct a sequence $\{x_n\}$ $n > 0 \subset E_2$ converging to x^* for a suitable starting value x_0. To achieve this construction we attach to equation (1) the iteration

$$x_{n+1} = x_n - (P(I - F'(F(x_n))))^{-1} (x_n - F(x_n)), n > 0, \tag{2}$$

where P is a linear projection ($P = P^2$) which projects E_1 on its subspace E_P and set $Q = I - P$.

The above method is called Stirling-like method. (Note that for P = I the usual Stirling's method is obtained [13], [14]). Stirling's method can be viewed as a combination of the method of successive substitutions and Newton's method. It is consequently reasonable to examine the convergence of the method of successive substitutions. In terms of computational effort, Stirling's and Newton's methods require essentially the same labor per step, as each requires the evaluation of F, F′ and the solution of a linear equation, assuming that F and its derivative are evaluated independently.

It can easily be shown by induction on n that under the hypothesis of Theorem 1 that follows $x_n - F(x_n)$ belong to the domain of $(P(I - F'(F(x_n))))^{-1}$ for all $n \geq 0$. Therefore if the inverses exist (as it will be shown later in Theorem 1), then the iterates x_n can be computed for all $n > 0$. For P = I the iterates generated by (2) cannot be easily computed in infinite dimensional spaces since the inverses of the linear operators involved may be too difficult or impossible to find. It is easy to see, however, that the solution of equation (2) reduces to solving certain operator equations in the space E_p. If, moreover, E_p is a finite dimensional space of dimension N, we obtain a system of linear algebraic equations of at most order N (see, e.g.[6] and [11] also).

Sufficient conditions for the convergence of iteration (2) to a fixed point x^* of equation (1) have already been given in [13] and [14] (for P = I). In this paper we provide new more general sufficient conditions which contain all previous ones as special cases (when P = I or not). Moreover under very natural conditions we show that the Stirling-like method converges to x^* faster than Newton's method.

2. CONVERGENCE RESULTS

Let $x_0 \in E_2$ and denote by $U(x_0, R)$ the closed ball centered at x_0 and of radius $R > 0$. We assume:

(a) the operator F is Frechet-differentiable for all $x \in U(x_0, R)$;

(b) the inverse of the linear $I - F'(F(x_0))$ exists;

(c) the ball $U(x_0, R) \subset E_2$;

(d) the following estimates are true for all $x, y, z \in U(x_0, R)$ and all $t \in [0, 1]$

$$||(P(I-F'(F(x_0))))^{-1}(PF'(x+t(x-y))-PF'(z))|| \leq v(||x-z|| + t||x-y||), \tag{3}$$

$$||(P(I-F'(F(x_0))))^{-1}Q(F'(x+t(y-x)-I)|| \leq v_1(||x-y||) \tag{4}$$

and $||F(x) - F(y)|| \leq w(||w - y||)$, (5) where v, v_1, w are non-negative and non-decreasing functions with $v(0) = v_1(0) = w(0) = 0$

(e) there exist numbers α and β such that

$$||(P(I - F'(F(x_0))))^{-1}(x_0 - F(x_0))|| \leq \alpha \tag{6}$$

and

$$||I - F'(F(x_0))|| \leq \beta \tag{7}$$

(f) define the function T on [0 , R) by

$$T(r) = \alpha + \frac{v[(\beta(1 + vow(r)) + 1)r]r + v_1(r)r}{1 - vow(r)} \tag{8}$$

where vow denotes the usual composition of two real functions. Assume that there exists a minimum number $r^* \in (0, R]$ such that

$$T(r^*) \leq r^* \text{ and } ||x_0 - F(x_0)|| + w(r^*) \leq r^* \tag{9}$$

(g) the following estimates are true

$$vow(r^*) < 1, \tag{10}$$

$$\alpha + v[(\beta + 1)\alpha]\alpha + v_1(\alpha)\alpha \leq r^* \tag{11}$$

and

$$w(||x - y||) < ||x - y|| \text{ for all } x, y \in U(x_0, r^*) \tag{12}$$

Finally, let us define the scalar sequence $\{t_n\}$, $n > 0$ by

$$t_0 = 0 \ , \ t_1 = \alpha \tag{13}$$

and

$$t_{n+2} = t_{n+1} + \frac{v[(\beta(1+vow(t_n))+1)(t_{n+1}-t_n)](t_{n+1}-t_n)+v_1(t_{n+1}-t_n)(t_{n+1}-t_n)}{1-vow(t_n)},$$

$$n > 0. \tag{14}$$

We can now state and prove the main result.

Theorem 1

Let $F : E_2 \to E_1$ be a nonlinear operator. Assume hypotheses (a) – (g) are true. Then

(i) the scalor sequence $\{t_n\}$, $n \geq 0$ defined by (13) – (14) is non-negative, nondecreasing and converges to some $t^* \in [0,r^*]$;

(ii) the iteration $\{x_n\}$, $n \geq 0$ generated by (2) is well defined, remains in $U(x_0, r^*)$ and converges to a unique fixed point x^* of equation (1) in $U(x_0, r^*)$.

Moreover the following estimates are true

$$||x_{n+1} - x_n|| \leq t_{n+1} - t_n \tag{15}$$

and $$||x_n - x^*|| \leq t^* - t_n \text{ for all } n > 0 \tag{16}$$

Proof:

(i) By (12)-(13) and the definition of v and w we get $t_2 \geq t_1$. Let us assume $t_{k+1} \geq t_k$ for $k = 0, 1, 2, \dots, n$. Then by (13) we get $t_{k+2} \geq t_{k+1}$, which shows that the sequence $\{t_n\}$, $n \geq 0$ is nondecreasing. We will show that it is also bounded above by r^*. By (13), (14) (for $n = 0$) and (11) we get $t_k \leq r^*$, $k = 0, 1, 2$. Let us assume $t_{k+1} \leq r^*$, $k = 0, 1, 2, \dots\dots, n$. Then

$$t_{k+2}=t_k+\frac{v[(\beta(1+vow(t_{k-1}))+1)(t_k-t_{k-1})](t_k-t_{k-1})+v_1(t_k-t_{k-1})(t_k-t_{k-1})}{1-vow(t_{k-1})}$$

$$+\frac{v[(\beta(1+vow(t_k))+1)(t_{k+1}-t_k)](t_{k+1}-t_k)+v_1(t_{k+1}-t_k)(t_{k+1}-t_k)}{1-vow(t_k)}$$

$$\geq t_k+\frac{v[(\beta(1+vow(t^*))+1)r^*](t_{k+1}-t_{k-1})+v_1(r^*)(t_{k+1}-t_{k-1})}{1-vow(r^*)}$$

$$\leq \dots \leq T(r^*) \leq r^* \text{ (by (8))}$$

We have now showed that the scalar sequence $\{t_n\}$, $n \geq 0$ is nondecreasing and bounded above by r^* and as such it converges to some $t^* \in [0, r^*]$ as $n \to \infty$.

(ii) We first show $x_n \in U(x_0)r^*$ and that (14) is true for all $n \geq 0$. By (2), (6) and (11) $x_1 \in U(x_0, r^*)$ and (15) is true for $n = 0$. Let us assume $x_k \in U(x_0, r^*)$ and that (15) is true for $k = 0, 1, 2, \dots\dots, n+1$. Then

$$||x_{k+2} - x_0|| \leq \sum_{j=0}^{k} ||x_{j+2}-x_{j+1}|| \leq \sum_{j=0}^{k} (t_{j+2}-t_{j+1}) \leq t_{k+2} - t_0$$

$$= t_{k+2} \leq r^* \text{ (by (i))},$$

which shows $x_n \in U(x_0, r^*)$ for all $n \geq 0$.

Using the identity

$$P(I - F'(F(x_k))) = P(I-F'(F(x_0)))$$

$$[1 + (P(I - F'(F(x_0))))^{-1} (PF'(F(x_k)))], \qquad (3)$$

(for $t = 0$, $z = x_0$, $x = x_k$) , (5), (10) and the Banach lemma on invertible operators we deduce that $P(I - F'(F(x_0)))$ is invertible and

$$||(P(I-F'(F(x_k))))^{-1} P(I-F'(F(x_0)))|| \leq \frac{1}{1-vow(||x_0-x_k||)} \leq \frac{1}{1 - vow(r^*)} \qquad (17)$$

(since $vow(||x_0 - x_k||) \leq vow(r^*) < 1$)

Note also that from (9) and the estimate

$$||x_0-F(x_k)|| \leq ||x_0-F(x_0)||+||F(x_0)-F(x_k)|| \leq ||x_0-F(x_0)|| + w(||x_0 - x_k||) \leq ||x_0 - F(x_0)|| + w(t_k - t_0) \leq x_0 - F(x_0)|| + w(r^*)) \leq r^*$$

we obtain $F(x_k) \in U(x_0, r^*)$

Using (2) for $n = k + 1$, (15) , (17) we get

$$||x_{k+2} - x_{k+1}|| = ||(P(I - F'(x_{k+1}))))^{-1} (x_{k+1} - F(x_{k+1}))||$$
$$\leq ||(P(I - F'(F(x_{k+1}))))^{-1} P(I - F'(F(x_0)))||$$

$$\{||(P(I-F'(F(x_0))))^{-1} \int_0^1 [PF'(x_k+t(x_{k+1} - x_k)) - PF'(F(x_k))]$$

$$(x_k-x_{k+1})||+||(P(I-F'(F(x_0))))^{-1} Q \int_0^1 [(F'(x_{k+1}+t(x_k-x_{k+1}))-I]$$

$$(x_{k+1}-x_k)dt||\} \leq \frac{1}{1-vow(||x_k-x_0||)} [\int_0^1 v(||F(x_k)-x_k|| + t$$

$$||x_{k+1}-x_k||) ||x_{k+1}-x_k||dt + v_1(||x_{k+1}-x_k|| ||x_{k+1}-x_k||)]$$

$$\leq \frac{1}{1-vow(||x_k-x_0||)} [\int_0^1 v [(||I-F'(F(x_0))||+$$

$||F'(F(x_0)-F'F(x_k))||)\ ||(x_{k+1}-x_k||+t||x_{k+1}-x_k||]\,dt$
$+v_1(||x_{k+1}-x_k||)||x_{k+1}-x_k||]$

$$\leq \frac{1}{1-vow\,(k_k-k_0)}\left[\int_0^1 v\,[(\beta(1+vow(||x_k-x_0||)+t)||\,x_{k+1}-x_k||]\right.$$

$$||x_{k+1}-x_k||\,dt+v_1(||x_{k+1}-x_k||)\,||x_{k+1}-x_k|| \qquad (18)$$

$$\leq \frac{1}{1-vow\,(t_k-t_0)}\,v\,[(\beta(1+vow(t_k-t_0)+1)(t_{k+1}-t_k)]$$

$$(t_{k+1}-t_k)+v_1(t_{k+1}-t_k)(t_{k+1}-t_k) = t_{k+2}-t_{k+1}$$

which completes the induction for (15). By (15) and (i) it now follows that iteration (2) is a Cauchy sequence in a Banach space and as such it converges to some $x^* \in U(x_0\,,\,r^*)$. Using the continuity of F and letting $n \to \infty$ in (2) we deduce $x^* = F(x^*)$. That is x^* is a fixed point of F.

Finally to show uniqueness, let us assume that $y^* = F(y^*)$ with $y^* \in U(x_0\,,\,r^*)$. Then by (12) and (1)

$$||x^*-y^*|| = ||F(x^*)-F(y^*)|| \leq w(||x^*-y^*||) < ||x^*-y^*||$$

which implies $x^* = y^*$.

The proof of the theorem is now complete.

Sometimes it may be very difficult or almost impossible to invert the linear operator $P(I - F'(F(x_n)))$ for all $n \geq 0$. We can suggest the use of the modified Stirling's method

$$y_{n+1}=y_n-(P(I-F'(F(y_0)))^{-1}\,(y_n-F(y_n)),y_0=x_0,n\geq 0 \qquad (19)$$

Let us define the scalar sequence

$$s_{n+2}=s_{n+1}+v[w(s_n)+(\beta+1)(s_{n+1}-s_n)](s_{n+1}-s_n) + v_1\ (s_{n+1}-s_n)\,(s_{n+1}-s_n),$$

$$s_0 = 0\,,\, s_1 = \alpha\,,\, n \geq 0 \qquad (20)$$

and the function T_1 on [0,R] by

$$T_1(r) = \alpha + v(w(r) + (\beta + 1)r)r + v_1\,(r)r \qquad (21)$$

Then by replacing the role of the sequence $\{t_n\}$, $n \geq 0$ by $\{s_n\}$, $n \geq 0$ and the function T by T_1 in the previous hypotheses and following the proof of Theorem 1 we can show

Theorem 2

Let $F : E_2 \to E_1$ be a nonlinear operator. Assume hypotheses (a) − (h) excluding (10) from hypothesis (g) are true. Then

(i) the scalar sequence $\{s_n\}$, $n \geq 0$ defined by (20) is nonnegative nondecreasing and converges to some $t_1 \in [0,r^*]$.

(ii) the iteration $\{y_n\}$ $n \geq 0$ generated by (19) is well defined, remains in $U(x_0, r^*)$ and converges to a unique fixed point y^* of equation (1) in $U(x_0, r^*)$.

Moreover the following estimates are true

$$||y_{n+1} - y_n|| \leq s_{n+1} - s_n \tag{22}$$

and $$||y_n - y^*|| \leq t_1 - t_n \text{ for all } n \geq 0 \tag{23}$$

Remark 1

The error estimates (15) and (16) (similarly for estimates (22) and (23)) can be improved if there exists a function u such that $u' = v$. Using (18) we can show that estimates (15) and (16) are still valid if we replace the sequence $\{t_n\}$ $n \geq 0$ by the sequence $\{t^1{}_n\}$ $n \geq 0$ given by

$$t_0^1 = 0 \text{ , } t_1^1 = \alpha$$

and $$t_{n+2}^1 = t_{n+1}^1 +$$

$$[u[(\beta(1+vow(t_n^1))+(t_{n+1}^1-t_n^1)]-u[(\beta(1+vow(t_n^1))$$
$$(t_{n+1}^1-t_n^1)))]+v_1(t_{n+1}^1-t_n^1)\,(t_{n+1}^1-t_n^1)]/[1 - vow\,(t_n^1)]$$

for all $n \geq 0$. (24)

We will now introduce Newton's method

$$z_{n+1} = z_n - (P(I - F'(z_n)))^{-1}(z_n - F(z_n)),\ z_0 = x_0,\ n \geq 0 \tag{25}$$

Let us assume that the inverse of $I - F'(z_0)$ exists and

$$||(P(I - F'(x_0)))^{-1} Q(F(x) - F(y))|| \leq v_2(||x - y||) \tag{26}$$

where v_2 is a nonnegative, nondecreasing function with $v_2(0) = 0$. Define the number $\bar{\alpha}$ by

$$||(P(I - F'(z_0)))^{-1}(z_0 - F(z_0))|| \leq \bar{\alpha} \tag{27}$$

the function P on [0 , R) by

$$P(r) = \bar{a} + \frac{v(r)\, r + v_2(r)}{1 - v(r)} \tag{28}$$

and the scalar iteration $\{q_n\}$, $n \geq 0$ by

$$q_0 = 0 \quad , \quad q_1 = \bar{\alpha} \tag{29}$$

and $$q_{n+2} = q_{n+1} + \frac{v(q_{k+1} - q_n)(q_{n+1} - q_n) + v_2(q_{n+1} - q_n)}{1 - v(q_n)} \tag{30}$$

Replace hypotheses (b), (f) and (g) by (b′), (f′), and (g′) respectively as follows

(b′) the inverse of the linear operator $I - F'(x_0)$ exists;

(f′) there exists a minimum number $r_1^* \in (0, R]$ such that

$$T_1(r_1^*) \leq r_1^* \; ; \tag{31}$$

(g′) the following estimates are true

$$v(r_1^*) < 1 , \tag{32}$$

$$\bar{\alpha} + v(\bar{\alpha})\,\bar{\alpha} + v_2(\bar{\alpha}) < r_1^* \tag{33}$$

and $w(||x - y||) < ||x - y||$ for all $x, y \in U(x_0, r_1^*)$

Finally, replace $I - F'(F(x_0))$ by $I - F'(x_0)$ in (3) and assume (3) remains true then. Then exactly as in Theorem 1 we can show:.

Theorem 3

Let $F : E_2 \to E_1$ be a nonlinear operator. Assume hypotheses (a), (b′), (c), (d), (f′) and (g′) are true. Then

(i) the scalar sequence $\{q_n\}$, $n \geq 0$ defined by (30) is nonnegative,nondecreasing and converges to some $t_2^* \in [0, r_1^*]$;

(ii) the iteration $\{z_n\}$, $n \geq 0$ generated by (25) is well defined, remain in $U(x_0, r_1^*)$ and converges to a unique fixed point z^* of equation (1) in $U(x_0, r_1^*)$.

Moreover the following estimates are true

$$||z_{n+1} - z_n|| \leq q_{n+1} - q_n \qquad (34)$$

and $$||z_n - z^*|| \leq t_1^* - q_n \text{ for all } n \geq 0 \qquad (35)$$

Remark 2

(a) A theorem similar to Theorem 2 can be immediately produced for the modified Newton's method

$$m_{n+1} = m_n - (P(I - F(m_0)))^{-1} (m_n - F(m_n)), \; m_0 = x_0, \; n \geq 0 \qquad (36)$$

(b) Estimates similar to the ones obtained in Remark 1 can also follow immediately for Newton's as well as the modified Newton's method.

(c) Note that due to uniqueness of the solution in the corresponding balls centered at x_0, $x^* = y^* = z^*$.

We can now compare estimates (15) - (16) with (34) - (35).

Theorem 4

Let $F : E_2 \to E_1$ be a nonlinear operator. Assume

(a) hypotheses (a) - (g), (b′) , (f′) and (g′) are true.

(b) the following estimates are true

$$\alpha \leq \frac{1}{\beta (1 + v_0 w (r^*)) + 1} \bar{\alpha}, \qquad (37)$$

$$v_1 (t) \leq v_2 (t) \qquad (38)$$

and $$w (t) \leq t \text{ for all } t \in [0, R] \qquad (39)$$

(i) $$t_{n+1} - t_n \leq \frac{1}{\beta (1 + v_0 w (t_n)) + 1} (q_{n+1} - q_n) \leq q_{n+1} - q_n$$

for all $n \geq 0$; and (40)

(ii) $$t^* - t_n \leq \frac{1}{\beta (1 + v_0 w (t_n)) + 1} (t_2^* - q_n) \text{ for all } n \geq 0 \qquad (41)$$

Proof:

We will only show (i), since (41) can follow easily from (40). (i) Estimate (40) is true for $n = 0$, by (37) – (39) and the estimate $q_0 = t_0 \leq r^*$. Let us assume that (40) is true for $k = 0, 1, 2, \ldots\ldots\ldots, n$, then by (13) using (38), (39) and (40) we get

$$t_{k+2}-t_{k+1} \leq \frac{v(q_{k+1}-q_k)}{1-v(q_{k+1})} \frac{(q_{k+1}-q_k)+v_2(q_{k+1}-q_k)}{\beta(1+vow(r^*))+1}$$

$$\leq \frac{1}{\beta(1+vow(q_{k+1}))+1}(q_{k+2}-q_{k+1})$$

The induction is now complete. That ends the proof of the theorem. Moreover our results can be extended to include equations of the form.

$$x = F_2(x) \tag{42}$$

where $F_2(x) = F(x) + F_1(x)$

with F as before and F_1 satisfying an estimate of the form (h) $||(P(I-F' \ (F(x_0))))^{-1}(F_1(x)-F_1(y))|| \leq v_3(||x-y||)$ for all x, $y \in U(x_0, r^*)$ (43), where v_3 is a nonnegative, nondecreasing function with $v_3(0) = 0$.

In particular, let us define the iteration $\{\bar{x}_n\}$, $n \geq 0$ by

$$\bar{x}_{n+1} = \bar{x}_n - (P(I-F'(F(\bar{x}_n))))^{-1}(\bar{x}_n-(F(\bar{x}_n)+F_1(\bar{x}_n))),\ n \geq 0 \tag{44}$$

and the scalar iteration $\{c_n\}$, $n \geq 0$ by

$$c_0 = 0,\ c_1 \geq ||(P(I-F'(F(x_0))))^{-1}(x_0-(F(x_0)+F_1(x_0)))|| \tag{45}$$

and $c_{n+2}=c_{n+1}$ +

$$[v[(\beta(1+v_3)(c_n)+\alpha_1+vow\,(c_n))+1)\,(c_{n+1}-c_n)]\,(c_{n+1}-c_n)+v_3\,(c_{n+1}-c_n)]/(1-vow)$$

$$n \geq 0, \tag{46}$$

where $\alpha_1 = ||(P(I - F'(F(x_0))))^{-1}F_1(x_0)||$

Then exactly as in theorem 1 we can show

Theorem 4

Let $F_2 : E_2 \rightarrow E_1$ be a nonlinear operator. Assume that hypotheses (a)-(h) are true. Then

(i) The scalar sequence $\{c_n\}$, $n \geq 0$ defined by (45)-(46) is nonnegative, nondecreasing and converges to some $t_3^* \in [0, r^*]$;

(ii) The iteration $\{\bar{x}_n\}$, $n \geq 0$, generated by (44) is well defined, remains in $U(x_0, r^*)$ and converges to a unique fixed point x^* of equation (42), in $U(x_0, r^*)$.

Moreover the following estimates are true

$$||\bar{x}_{n+1} - \bar{x}_n|| \leq c_{n+1} - c_n$$

and $$||\bar{x}_n - \bar{x}^*|| \leq t_3^* - c_n \text{ for all } n \geq 0$$

Furthermore, we note that similar results can be obtained for Modified Sterling's, Newton's, modified Newton's method and equation (42). (When P = I or not).

Remark 3

(a) By setting $w(t) = d_1 t$ and $v(t) = d_2 t$ for some $d_1, d_2 \geq 0$ one can easily obtain the results in [13] and [14] concerning Stirling's and the modified Stirling's method. The results in [2] − [8] can also be obtained as special cases of theorems 3 or 4. Note also that with the above choices of the functions 2 and v the quadratic convergence can be established for iterations (2) and (25).

(b) Finally, by setting $w(t) = e_1(t)t$ and $v(t) = e_2(t)t$ where e_1, e_2 are nonnegative, nondecreasing functions with $e_1(0) = e_2(0) = 0$ the results in [6], [9]-[12] and [15] can be obtained as special cases of theorems 3 and 4.

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ON THE APPROXIMATE CONSTRUCTION OF IMPLICIT FUNCTIONS AND PTAK ERROR ESTIMATES

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ABSTRACT

Using the Newton-Kantorovich method we construct implicit functions. We also provide error estimates by means of the majorant theory. In particular, these error estimates generalize the ones in [5] and [8].

Key Words and Phrases: Banach space, implicit function, Newton-Kantorovich method.

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1. INTRODUCTION

Let E, Λ be Banach spaces and let $F(x, \lambda)$ be a nonlinear operator with values in E defined for all $x \in E$, $\lambda \in \Lambda$ with

$$x \in \overline{U}(x_0, R) = \{x \in E / \|x - x_0\| \leq R\} \text{ and } \lambda \in \overline{U}(\lambda_0, S)$$

Consider the equation

$$F(x, \lambda) = 0 \tag{1}$$

We seek solutions $x^*(\lambda)$ of equation (1) which are close to x_0 for values of λ close to λ_0. In order to solve approximately this problem of constructing an implicit function the method of successive approximations has been used, as well as the basic and modified Newton-Kantorovich methods under various assumptions [1], [3], [4], [6]. We will use the same symbol for the norms in both spaces.

We will define successive approximation $x_{n+1}(\lambda)$ to $x^*(\lambda)$ from the equations

$$x_{n+1}(\lambda) = x_n(\lambda) - F'(x_n, \lambda)^{-1} F(x_n, \lambda), n \geq 0 \qquad (2)$$

We assume that F is Frechet differentiable for all $x \in \overline{U}(x_0,R)$ and $\lambda \in \overline{U}(\lambda_0, S)$. Moreover we assume that the linear operator $F'(x_0,\lambda_0)$ is invertible and that the following Nguen-Zabrejko type assumptions are satisfied:

$$||F'(x_0, \lambda)^{-1} (F'(x, \lambda) - F'(y, \lambda))|| \leq K_1(r, s) ||x - y|| \qquad (3)$$

$$||F'(x, \lambda)^{-1} (F'(x_0, \lambda) - F'(x_0, \lambda_0)|| \leq K_2(s) ||\lambda - \lambda_0|| \qquad (4)$$

for all $x, y \in \overline{U}(x_0,r) \subset \overline{U}(x_0,R)$ and $\lambda \in \overline{U}(\lambda_0,s) \subset \overline{U}(\lambda_0,S)$, where K_1 and K_2 are non-decreasing functions on $[0, R] \times [0, S]$ respectively. By x_0 we mean $x_0(\lambda)$. That is x_0 depends on the λ used in (2). The above assumptions generalize the ones given by Potra-Ptak [5], Nguen-Zabrejko [8] and Yamamoto [7] in this case (if G = 0 in [7], [8]).

We will provide sufficient conditions for the convergence of (2) to a solution $x^*(\lambda)$ of equation (1) as well as various error estimates on the distances $||x_{n+1}(\lambda) - x_n(\lambda)||$ and $||x_n(\lambda) - x^*(\lambda)||$, $n \geq 0$.

We need to define the functions

$$a_s = K(s) ||F'(x_0, \lambda_0)^{-1} F(x, \lambda)||, (s = 0 \text{ if } \lambda = \lambda_0),$$

$$w_s(r) = \int_0^r K_1(t,s)dt, \; K_3(s) = \int_0^s K_2(t)dt, \; K(s) = (1 - K_3(s))^{-1}$$

provided that

$$K_3(s) < 1 \text{ and } \varphi_s(r) = a + K(s) \int_0^r w_s(t)\, dt - r$$

2. CONVERGENCE RESULTS

The following theorem can now be proved as Theorem 1 in [8].

Theorem 1

Suppose that the function $\varphi_s(r)$ has a unique zero $\rho^* = \rho^*_s$ in the interval [0, R] and $\varphi_s(R) \leq 0$. Then equation (1) admits a solution $x^*(\lambda)$ in $\overline{U}(x_0, \rho^*)$, this solution is unique in the ball $\overline{U}(x_0, R)$, and the approximations (2) defined for all n, belong to $\overline{U}(x_0, \rho^*)$, and satisfy the estimates

$$||x_{n+1}(\lambda) - x_n(\lambda)|| \leq \Delta^{(n)}(a) = \rho_{n+1} - \rho_n, n \geq 0$$

and $$||x_n(\lambda) - x^*(\lambda)|| \leq w(\Delta^{(n)}(a)) = \rho^* - \rho_n, n \geq 0$$

where $\Delta(r) = u(r+v(r)), v(r) = u^{(-1)}(r)$ on $[0,a]$, $u(r) = -\varphi_s(r)\varphi'_s(r)^{-1}$,

$$\Delta^{(0)}(r) = r, \Delta^{(n+1)}(r) = \Delta(\Delta^{(n)}(r)), n \geq 0, w(r) = \sum_{n=0}^{\infty} \Delta^{(n)}(r),$$

and the sequence ρ_n which is monotonically increasing and converges to ρ^*, is defined by

$$\rho_{n+1} = \rho - \frac{\varphi_s(\rho_n)}{\varphi'_s(\rho_n)}, \quad n \geq 0, \rho_0 = 0$$

Similar results can be obtained for the modified Newton-Kantorovich method

$$y_{n+1}(\lambda) = y_n(\lambda) - F'(x_0, \lambda)^{-1} F(y_n, \lambda), n \geq 0, y_0 = x_0$$

(see e.g. [8, p. 674]).

We will now obtain some *a posteriori* error bounds for iteration (2). Let $r_{n,s} = a_n = ||x_{n+1}(\lambda) - x_n(\lambda)||$, $h_{n,s}(r) = h_n(r) = K_1(r_n + r, s)$ for $r \in [0, R - r_n]$ and set $a_{n,s} = a_n = ||x_{n+1}(\lambda) - x_n(\lambda)||$, $b_{n,s} = b_n = K(s)(1 - K(s)w_s(r_n))^{-1}$.

The notation $r_{n,s} = r_n$, mean that once λ is fixed then we take $s = ||\lambda - \lambda_0||$ and denote $r_{n,s}$ by r_n. Without loss of generality, we may assume that $a_n > 0$. Then as in [7, p. 989], we can show that the equation

$$r = a_n + b_n \int_0^r (r - t)\, h_n(t)\, dt$$

has a unique positive zero $\rho^*_{n.s} = \rho^*_n$ in the interval $[0, R - r_n]$ and

$$||x^*(\lambda) - x_n(\lambda)|| \leq \rho^*_n, n \geq 0 \text{ with } \rho^*_0 = \rho^*$$

The proof of the following theorem now follows as the proof of Theorem 2 in [7, p. 989].

Theorem 2

Under the assumptions of Theorem 1, we have

$$\begin{aligned} ||x_n(\lambda) - x^*(\lambda)|| &\leq \rho_n^*, n \geq 0 \\ &\leq (\rho^* - \rho_n)\, a_n/\Delta\rho_n, n \geq 0 \\ &\leq (\rho^* - \rho_n)\, a_{n-1}/\Delta\rho_{n-1}, n \geq 1 \\ &\leq \rho^* - \rho_n, n \geq 0 \end{aligned}$$

where $\Delta\rho_n = \rho_{n+1} - \rho_n, n \geq 0$

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CHARACTERIZATION OF BCI-ALGEBRAS OF ORDER FIVE

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ABSTRACT

In this paper we investigate proper BCI-algebras of order five and show that there are only seventy such algebras. Their tables have also been constructed.

1. INTRODUCTION

K. Iseki [6] introduced the concept of BCI-algebras and established certain properties. Unlike finite order groups, the problem of characterizating finite order BCI-algebras has not been investigated so far. In [4], S.K. Goel, as a first step, characterized completely BCI-algebras of order 3 and partially BCI-algebras of order 4. In this paper we show, regarding isomorphic BCI-algebras as equal that there are only 70 distinct proper BCI-algebras of order 5. We also construct their tables.

2. PRELIMINARIES

A BCI-algebra is an algebra $(x, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \varepsilon X$:

(1) $(x*y) * (x*z) \leq z*y$,

(2) $x*(x*y) \leq y$,

(3) $x \leq x$,

(4) $x \leq y, y < x$ implies $x = y$,

(5) $x \leq 0$ implies $x = 0$,

(6) $x \leq y$ iff $x*y = 0$.

In a BCI-algebra X, the set $M = \{x \in X : 0*x = 0\}$ is a subalgebra and is called the BCK-part of X. A BCI-algebra X is called proper if $X - M \neq \phi$. Moreover, in a BCI-algebra the following hold:

(7) $(x*y) * z = (x*z) * y$

(8) $x*0 = x$

(9) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$ [6].

Further we know that in a BCI-algebra X,if $m \in M$, $x \in X-M$, then $x*m$, $m*x \in X-M$ ([6]). If $M = \{0\}$, then X is called a p-semisimple BCI-algebra. Further every BCI-algebra X is a partially ordered set with respect to the relation $\leq$.

DEFINITION 1. [1]

Let X be BCI-algebra and $x, y \in X$. Then x, y are said to be comparable iff $x*y = 0$ or $y*x = 0$. Further, we shall say that x proceeds y or y succeeds x iff $x*y = 0$ and denote it by $x \rightarrow y$ or $x \leq y$.

DEFINITION 2. [1]

Let X be a BCI-algebra. We choose an element $x_0 \in X$ such that there does not exist any $y \neq x_0$ satisfying $y*x_0 = 0$ and define

$$A(x_0) = \{x \in X : x_0*x = 0\}$$

We note that $A(x_0)$ consists of all those elements of X which succeed x_0. The element x_0 is known as the initial element of $A(x_0)$ as well as X. Let I denote the set of all initial elements of X. We call it the centre of X.

DEFINITION 3. [1]

Let X be a BCI-algebra. If $0*x \neq x$ for all $x \in X-\{0\}$, then X is called fully non-associative. If $0*x = x$ for all $x \in X$, then X is called associative. If $0*x = x$ for some $x \in X$, $x \neq 0$ and $0*y \neq y$ for some $y \neq 0$, $y \in X$, then X is called a neutral BCI-algebra.

In the sequel we shall need the following results:

(10) Let X be a BCI-algebra and $A(x_0) \subseteq X$. Then $x, y \in A(x_0)$ imply $x*y$, $y*x \in M$ ([1]).

(11) Let X be a BCI-algebra and A(x_0), A(y_0) $\subseteq$ X where $x_0 \neq y_0$. Let x ε A(x_0), y ε A(y_0), then $x*y$, $y*x$ ε X$-$M ([1]).

(12) Let X be a BCI-algebra with I as its centre. Then I is a p-semi-simple sub-algebra of X ([2]).

(13) Let X be a BCI-algebra, then following are equivalent:

(i) X is p-semi-simple,

(ii) $x*y = 0$ implies $x = y$,

(iii) $x*(x*y) = y$,

(iv) $x*a = x*b$ implies $a = b$,

(v) $a*x = b*x$ implies $a = b$,

(vi) $0*(x*y) = y*x$,

(vii) $x*(0*y) = y*(0*x)$,

for all x, y, z ε X ([5], [7])

(14) Let X be a BCI-algebra with I as its centre. Then $\bigcup_{x_0 \varepsilon I} A(x_0) =$ X and $\bigcap_{x_0 \varepsilon I} A(x_0) = \phi$. Further each x ε X is contained in a unique A(x_0) $\subseteq$ X for x_0 ε I. Moreover, if for x, y ε X, $x \leq y$, then x, y are contained in the same A(z_0) $\subseteq$ X for some z_0 ε I ([1]).

(15) If X is p-semi-simple then X = I and A(x) = $\{x\}$ for all x ε I ([2]).

(16) Let X be a BCI-algebra with A(x_0) $\subseteq$ X. Then for all x,yA(x_0), $0*x = 0*y = 0*x_0$ ([2]).

(17) Let X be a p-semi-simple algebra. Let A be a sub-algebra of X. Then 0(A) divides 0(X) ([7]).

(18) Let X = $\{0, x, y\}$ be a p-semi-simple algebra, then $0*x = y$, $0*y = x$, $x*y = y$, $y*x = x$ ([4]).

(19) Let X be an associative p-semi-simple BCI-algebra then 0(X) is even ([2]).

(20) Let X be a BCI-algebra and $G = \{x \varepsilon X : 0^*x = x\}$, then G is a sub-algebra of $I \subseteq X$. G is called the BCI-G-part of X ([3]).

1. BCI-ALGEBRAS OF ORDER FIVE WITH 0(M) = 1

In this section we prove that there is only one p-semi-simple BCI-algebra of order five.

Theorem 1

Let X be a p-semi-simple algebra, such that 0(X) is odd, then X is fully non-associative.

Proof:

Let X be not fully non-associative p-semi-simple BCI-algebra. Thus there is an $x \varepsilon X$ such that $x \neq 0$ and $0^*x = x$. Further it is known that X is abelian group under the operation $x+y = x^*(0^*y)$ and the inverse $-x$ is 0^*x (see [5] and [7]). Thus x is an element of order 2 in the abelian group X. Hence 2 divides 0(X), a contradiction. This completes the proof.

Remark:

In [2], we proved that if X is a fully non-associative p-semi-simple BCI-algebra, then 0(X) is odd. We combine it with theorem 1, and have the following result.

Theorem 2

Let X be a p-semi-simple algebra. Then X is fully non-associative iff 0(X) is odd.

Let X be BCI-algebra of order 5 with 0(M) = 1, thus M = {0} and consequently X is p-semi-simple. Since p-semi-simple BCI-algebras are precisely abelian groups (see [5] and [7]), therefore isomorphism classes of p-semi-simple BCI-algebras are the isomorphism classes of abelian groups. Since there is only one abelian group of order 5, therefore there is only one p-semi-simple BCI-algebra of order 5. Its table is given below:

*	0	a	b	c	d
0	0	d	c	b	a
a	a	0	d	c	b
b	b	a	0	d	c
c	c	b	a	0	d
d	d	c	b	a	0

(Table 1)

Thus we must have the following result:

Theorem 3

Let X be a BCI-algebra with M as its BCK-part. Let $0(X) = 5$ and $0(M) = 1$, then there is only one such BCI-algbera.

2. BCI-ALGEBRAS OF ORDER FIVE WITH 0(M) = 2:

In this section we show that there are five distinct BCI-algebras of order five with $0(M) = 2$.

Lemma 1

Let X be a BCI-algebra with M as its BCK-part such that $0(X)=5$ and $0(M) = 2$. If each pair $x, y \in X-M$ is incomparable, then number of such BCI-algebras is 2.

Proof:

Let $X = \{0, a, b, c, d\}$. Without any loss of generality, we suppose that $M = \{0, a\}$ and $X-M = \{b, c, d\}$. Since each pair of $X-M$ is incomparable, therefore $A(b) = \{b\}$, $A(c) = \{c\}$, $A(d) = \{d\}$ and $I = \{0, b, c, d\}$. Now corresponding to $x \in X-M = I - \{0\}$, we have the following three possibilities for $0*x$:

(I) $0*x = x$ for all $x \in X-M$,

(II) $0*x \neq x$ for some $x \in X-M$ and $0*y = y$ for some $y \in X-M$

(III) $0*x \neq x$ for all $x \in X-M$

Case I: Let $0*x = x$ for all $x \in X-M$. Then $0*b = b$, $0*c = c$, $0*d = d$.

By (12), I is a p-semi-simple algebra. Now we compute $a*b$, $a*c$, $a*d$, $b*a$, $b*c$, $b*d$, $c*a$, $c*b$, $c*d$, $d*a$, $d*b$, $d*c$.

Computation of $a*b$: $0 \leq a \Rightarrow 0*b \leq a*b \Rightarrow b \leq a*b$, which gives $a*b \varepsilon A(b) = \{b\}$. Thus $a*b = b$.

Computation of $a*c$: $0 \leq a \Rightarrow 0*c \leq a*c \Rightarrow c \leq a*c$, which gives $a*c \varepsilon A(c) = \{c\}$. Thus $a*c = c$.

Computation of $a*d$: $0 \leq a \Rightarrow 0*d \leq a*d \Rightarrow d \leq a*d$, which $a*d \varepsilon A(d) = \{d\}$. Thus $a*d = d$.

Computation of $b*a$: $0 \leq a \Rightarrow b*a \leq b \Rightarrow b*a = b$, because $b \varepsilon$ I.

Computation of $c*a$: $0 \leq a \Rightarrow c*a \leq c \Rightarrow c*a = c$, because $c \varepsilon$ I.

Computation of $d*a$: $0 \leq a \Rightarrow d*a \leq d \Rightarrow d*a = d$, because $d \varepsilon$ I.

Computation of $b*c$: $b, c \varepsilon$ I and I is closed, therefore $b*c \varepsilon$ I.

We claim that $b*c = d$, suppose $b*c = 0$, then by (13), $b = c$, a contradiction. Suppose $b*c = b = b*0$. By (13), $c = 0$, a contradiction. Suppose $b*c = c = 0*c$. By (13), $b = 0$, a contradiction. Hence $b*c = d$.

Computation of $b*d$: $b*c = d \Rightarrow b*d = c$.

Computation of $c*b$: $c*b = 0*(b*c) = 0*d = d$ or $c*b = d$.

Computation of $d*b$: $d*b = 0*(b*d) = 0*c = c$ or $d*b = c$.

Computation of $d*c$: Since $c*b = d$, therefore $c*d = b$.

Computation of $d*c$: $d*c = 0*(c*d) = 0*b = b$ or $d*c = b$.

Consequently, multiplication table for the BCI-algebra is given in the following table:

*	0	a	b	c	d
0	0	0	b	c	d
a	a	0	b	c	d
b	b	b	0	d	c
c	c	c	d	0	b
d	d	d	c	b	0

(Table 2)

Case II: Suppose, $0*b = b$, and $0*c \neq c$. Then $0*c \neq b$, because otherwise $0*c = b \Rightarrow 0*b = c$ and $c = 0*b = b \Rightarrow b = c$, a contradiction. But $0*c \varepsilon$ I; hence $0*c = d$. Using the similar argument as used in Case I, we can easily compute the multiplication table for this case which is given below:

*	0	a	b	c	d
0	0	0	b	d	c
a	a	0	b	d	c
b	b	b	0	d	d
c	c	c	d	0	b
d	d	d	c	b	0

(Table 3)

Similarly if $0*c = c$, or $0*d = d$, there will be two more such BCI-algebras whose tables are given by:

*	0	a	b	c	d
0	0	0	d	c	b
a	a	0	d	c	b
b	b	b	0	d	c
c	c	c	b	0	d
d	d	d	c	b	0

(Table 4)

*	0	a	b	c	d
0	0	0	c	b	d
a	a	0	c	b	d
b	b	b	0	d	c
c	c	c	d	0	b
d	d	d	b	c	0

(Table 5)

But algebras of tables 3, 4 and 4, 5 are isomorphic through the isomorphisms defined by $f(0) = 0$, $f(a) = a$, $f(b) = c$, $f(c) = d$, $f(d) = b$ and $g(0) = 0$, $g(a) = a$, $g(b) = c$, $g(c) = d$ and $g(d) = b$, respectively. Consequently for Case II, there is only one BCI-algebra.

Case III: We claim that case III is not possible, because if $0^*x \neq x$ for all $x \in X-M$. Then $\{0, b, c, d\} = I$, is a fully non-associative p-semi-simple algebra of order 4, which is a contradiction of theorem 2.

Hence from case I, II and III, it follows that there exist two such BCI-algebras. This completes the proof.

Corollary 1

Let X be a p-semi-simple algebra with $0(X) = 4$, then number of such BCI-algebras is 2.

Lemma 2:

Let $X = \{0, a, x, y, z\}$ be a BCI-algebra with $M = \{0, a\}$ and for $x, y, z \in X - M$, $x \leq y$ and z is incomparable with x and y. Then X is unique.

$$0 \rightarrow a \ ; \ x \rightarrow y \ . \ z$$

Proof:

It follows from the given hypothesis that $A(0) = \{0, a\} = M$, $A(x) = \{x, y\}$, $A(z) = \{z\}$ and $I = \{0, x, z\}$.

By (18), $0^*x = z$, $0^*z = x$, $x^*z = z$, $z^*x = x$

Further $0^*a = 0$, $a^*0 = a$, $x^*0 = x$, $y^*0 = y$, $z^*0 = z$

Computation of 0^*y: $x \leq y \Rightarrow 0^*y \leq 0^*x = z$. Thus $0^*y = z$, because $z \in I$

Computation of a^*x: $0 \leq a \Rightarrow 0^*x \leq a^*x$. But $0^*x = z$. Thus $z \leq a^*x$ or $z^*(a^*x) = 0$ implies $a^*x \in A(z) = \{z\}$. Hence $a^*x = z$.

Computation of a^*y: $x \leq y \Rightarrow a^*y \leq a^*x = z$. Thus $a^*y = z$, because $z \in I$.

Computation of x^*y: $x \leq y \Rightarrow x^*y = 0$.

Computation of y^*x: $y \in A(x)$. By (10), $y^*x \in M$. Thus $y^*x = 0$ or a. We claim that $y^*x \neq 0$, because otherwise $y^*x = 0 = x^*y \Rightarrow x = y$, a contradiction, hence $y^*x = a$.

Computation of x^*a: $0 \leq a \Rightarrow x^*a \leq x^*0 = x$ or $x^*a \leq x$ or $x^*a = x$, because $x \in I$.

Computation of y^*z: $x \leq y \Rightarrow x^*z \leq y^*z$ or $z \leq y^*z$

Computation of a^*z: $a^*y = (y^*x)^*z = (y^*z)^*x = z^*x = x$

Computation of y^*a: $y^*a = y^*(y^*x) \leq x$. Thus $y^*a = x$, because $x \in I$

Computation of $z*a$: $0 \leq a \Rightarrow z*a \leq z*0 = z$. Thus $z*a = z$, because $z \in I$.

Computation of $z*y$: $x \leq y \Rightarrow z*y \leq z*x = x$. Thus $z*y = x$, because $x \in I$.

Hence the multiplication table of the BCI-algebra with $M = \{0,a\}$, $x \leq y$ and z is incomparable with x and y is:

*	0	a	x	y	z
0	0	0	z	z	x
a	a	0	z	z	x
x	x	x	0	0	z
y	y	x	a	0	z
z	z	z	x	x	0

(Table 6)

Lemma 3

Let $X = \{0, a, x, y, z\}$ be a BCI-algebra with $M = \{0, a\}$ and $x \leq y \leq z$. Then X is unique.

Proof:

It follows from the given hypothesis that $A(0) = \{0, a\}$, $A(x) = \{x, y, z\}$ and $I = \{0, x\}$. Since I is p-semi-simple therefore $0*x = x$. Further $0*a = 0$, $a*0 = a$, $x*0 = x$, $y*0 = y$, $z*0 = z$.

Computation of $0*y$ *and* $0*z$: $0 \in M$ and $y \in X-M = A(x)$ give $0*y \in X-M$; that is, $0*y \in A(x)$. By (16), $0*y = 0*x = x$. Similarly $0*z = x$.

Computation of $x*a$: $0 \leq a \Rightarrow x*a \leq x*0 = x$. Thus $x*a = x$, because $x \in I$.

Computation of $x*y$, $x*z$ *and* $y*z$: $x \leq y \Rightarrow x*y = 0$, similarly $x \leq z \Rightarrow x*z = 0$. Also $y*z = 0$, because $y \leq z$.

Computation of $y*x$, $z*y$ *and* $z*x$: By (10), $y, x \in A(x)$ simply $y*x \in M$. We claim that $y*x \neq 0$, because otherwise $y*x = 0 = x*y$ implies $x = y$, a contradiction. Hence $y*x = a$. Similarly $z*y = z*x = a$. Thus $y*x = z*y = z*x = a$.

Computation of $z*a$: $z*a = z*(z*x) \leq x$. implies $z*a = x$, because $x \in I$.

Computation of $y*a$: $y*a = y*(y*x) \leq x$. implies $y*a = x$, because $x \varepsilon \mathrm{I}$.

Computation of $a*z$, $a*y$, $a*x$: $a*z = (z*x)*z = (z*z)*x = 0*x = x$. Further $a*y = (y*x)*y = (y*y)*x = 0*x = x$. Also $a*x = (z*y)*x = (z*x)*y = a*y = x$. Thus $a*x = a*y = a*z = x$.

Hence the multiplication table of the only possible BCI-algebra with M = $\{0, a\}$, $x \leq y \leq z$, is given below:

*	0	a	x	y	z
0	0	0	x	x	x
a	a	0	x	x	x
x	x	x	0	0	0
y	y	x	a	0	0
z	z	x	a	a	0

(Table 7)

Lemma 4

Let X = $\{0, a, x, y, z\}$ be a BCI-algebra with M = $\{0, a\}$, $x \leq y$, $x \leq z$; and y and z are incomparable. Then X is unique.

Proof:

It follows from the given hypothesis that $\mathrm{A}(0) = \{0, a\} = \mathrm{M}$, $\mathrm{A}(x) = \{x, y, z\} = \mathrm{X} - \mathrm{M}$ and $\mathrm{I} = \{0, x\}$. Since I is p-semi-simple, therefore $0*x = x$. Further $0*a = 0$, $a*0 = a$, $x*0 = x$, $y*0 = y$ and $z*0 = z$.

Computation of $0*y$, $0*z$: $x \leq z \Rightarrow 0*y \leq 0*x = x$. Thus $0*y = x$, because $x \varepsilon \mathrm{I}$. Similarly $0*z = x$.

Computation of $x*a$: $0 \leq a \Rightarrow x*a \leq x*0 = x$. Thus $x*a = x$, because $x \varepsilon \mathrm{I}$.

Computation of $x*y$, $x*z$: $x \leq y \Rightarrow x*y = 0$ and $x \leq z \Rightarrow x*z = 0$.

Computation of $y*x$, $y*z$, $z*x$, $z*y$: $y, x \varepsilon \mathrm{A}(x)$. By (10) $y*x \varepsilon \mathrm{M}$, that is, $y*x = 0$ or a. We claim that $y*x \neq 0$; because if $y*x = 0$, then $x*y = 0 = y*x \Rightarrow x = y$, a contradiction. By (10) $y, z \varepsilon \mathrm{A}(x)$ imply $y*z \varepsilon \mathrm{M}$, that is, $y*z = 0$ or a. If $y*z = 0$, then $y \leq z$, but $y \leq z$, a contradiction. Thus $y*z = a$. Similarly $z*x = a$ and $z*y = a$. Thus $y*x = y*z = z*x = z*y = a$.

Computation of $y*a$, $z*a$: $y*a = y*(y*x) \leq x$. Thus $y*a = x$, because $x \varepsilon I$. Now $z*a = z*(z*x) \leq x$. Hence $z*a = x$, because $x \varepsilon$ I.

Computation of $a*z$, $a*y$, $a*x$: $a*z = (z*x)*z = (z*z)*x = 0*x = x$, or $a*z = x$. Further

$$a*y = (y*x)*y = (y*y)*x = 0*x = x \text{ and}$$

$$a*x = (z*y)*x = (z*x)*y = a*y = x.$$

Thus $\quad a*x = a*y = a*z = x.$

Hence the multiplication table of the only possible BCI-algebra with M = $\{0, a\}$, $x \leq y$, $x \leq z$ and y, z are incomparable, is as follows:

*	0	a	x	y	z
0	0	0	x	x	x
a	a	0	x	x	x
x	x	x	0	0	0
y	y	x	a	0	0
z	z	x	a	a	0

(Table 8)

Theorem 4

Let X be a BCI-algebra with 0(X) = 5. Let M be the BCK-part of X with 0(M) = 2. Then number of all such BCI-algebras is 5.

Proof:

Let X = $\{0, a, b, c, d\}$ be a BCI-algebra. Without any loss of generality, we can take M = $\{0, a\}$ and X$-$M = $\{b, c, d\}$. Then there are following possibilities for the elements of X$-$M:

(i) b, c, d are all incomparable with each other,

(ii) $b \leq c$ and d is incomparable with b and c,

(iii) $b \leq c$, $b \leq d$ and c, d are incomparable,

(iv) $b \leq c \leq d$.

By lemma 1, case (i) gives two BCI-algebra.

By lemma 2, case (ii) gives a unique BCI-algebra.

By lemma 3, case (iii) gives a unique BCI-algebra.

By lemma 4, case (iv) gives a unique BCI-algebra.

Hence combining all of them together, we get 5 BCI-algebras with O(X) = 5 and O(M) = 2. This completes the proof.

3. BCI-ALGEBRAS OF ORDER FIVE WITH O(M) = 3

In this section we prove that there are 23 distinct BCI-algebera of order five with O(M) = 3.

Lemma 5

Let X be a BCI-algebra with O(X) = 5. Let M be its BCK-part such that O(M) = 3 and for $0, x, y \in M$, $0 \leq x \leq y$, and for $c, d \in X - M$, c, d are comparable, then there are 16 such BCI-algebras.

Proof:

Let $X = \{0, a, b, c, d\}$ be a BCI-algebra. Without any loss of generality, let $M = \{0, a, b\}$ and $c, d \in X - M$, then we have $0 \leq a \leq b$ and $c \leq d$. Thus $A(0) = M\ \{0, a, b\}$, $A(c) = \{c, d\}$ and hence $I = \{0, c\}$. Thus $0*a = 0$, $0*b = 0$, $0*c = c$, $a*0 = a$, $b*0 = b$, $c*0 = c$, $d*0 = d$, $a*b = 0$, $c*d = 0$.

Computation of $b*a$*:* Since $b, a \in M$ and M is closed, therefore, $b*a \in M$, that is, $b*a = 0$ or a or b.

But $\quad b*a \neq 0$, because otherwise

$$b*a = 0 = a*b \Rightarrow a = b, \text{ a contradiction.}$$

Thus $\quad b*a = a$ or b

Further $c, d \in A(c)$. By (10), $d*c \in M$. We claim that $d*c \neq 0$, because otherwise $c*d = 0 = d*c \Rightarrow c = d$, a contradiction. Thus $d*c = a$ or b. Thus combining $b*a = a$ or b and $d*c = a$ or b; we have the following sub-cases:

Case (i): $\quad b*a = a, d*c = a,$

Case (ii): $\quad b*a = a, d*c = b,$

Case (iii): $\quad b*a = b, d*c = a,$

Case (iv): $\quad b*a = b, d*c = b.$

Case (i): Let $b*a = a$, $d*c = a$. We compute $c*a$, $c*b$, $0*d$, $d*a$, $a*d$, $b*c$, $a*c$, $d*b$, and $b*d$.

Computation of c^*a*:* $0 \leq a \Rightarrow c^*a \leq c^*0 = c$. Thus $c^*a = c$, because $c \varepsilon \mathrm{I}$.

Computation of c^*b*:* $0 \leq b \Rightarrow c^*b \leq c^*0 = c$. Thus $c^*b = c$, because $c \varepsilon \mathrm{I}$.

Computation of 0^*d*:* $0 \leq d \Rightarrow 0^*d \leq 0^*c = c$. Thus $0^*d = c$, because $c \varepsilon \mathrm{I}$.

Computation of d^*a*:* $d^*a = d^*(d^*c) \leq c$. Thus $d^*a = c$, because $c \varepsilon \mathrm{I}$.

Computation of a^*d*:* $a^*d = (d^*c)^*d = (d^*d)^*c = 0^*c = c$.

Computation of b^*c*:* Since $b \varepsilon \mathrm{M}$, $c \varepsilon \mathrm{X-M}$, therefore, $b^*c \varepsilon \mathrm{X-M}$, which implies $b^*c = c$ or d.

Computation of a^*c*:* $a^*c = (b^*a)^*c = (b^*c)^*a$. When $b^*c = c$, then $a^*c = (b^*c)^*a$, becomes $a^*c = d^*a = c$. When $b^*c = d$, then $a^*c = (b^*c)^*a$, becomes $a^*c = d^*a = c$. Thus $a^*c = c$.

Computation of d^*b*:* $a < b \Rightarrow d^*b < d^*a = c$. Thus $d^*b = c$, because $c \varepsilon \mathrm{I}$.

Computation of b^*d*:* $b \varepsilon \mathrm{M}$, $d \varepsilon \mathrm{X-M}$. Thus b^*d $\mathrm{X-M}$, that is, $b^*d = c$ or d.

Since $b^*c = c$ or d and $b^*d = c$ or d, it follows that there exist four distinct BCI-algebras whose multiplication tables are as follows:

*	0	*a*	*b*	*c*	*d*
0	0	0	0	*c*	*c*
a	*a*	0	0	*c*	*c*
b	*b*	*a*	0	◇	®
c	*c*	*c*	*c*	0	0
d	*d*	*c*	*c*	*a*	0

(Table 9)

where ◇ = *c* or *d* and ® = *c* or *d*

Similarly in each of the case (ii), (iii) and (iv) we get four distinct BCI-algebras whose multiplication tables are given by:

*	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	◇	c
b	b	a	0	®	c
c	c	c	c	0	0
d	d	d	c	b	0

(Table 10)

where ◇ = *c* or *d* and ® = *c* or *d*

*	a	a	b	c	d
0	0	0	0	c	c
a	a	0	0	®	c
b	b	b	0	◇	◇
c	c	c	c	0	0
d	d	c	c	a	0

(Table 11)

where ® = *c* or *d* and ◇ = *c* or *d*

*	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	◇	c
b	b	b	0	®	c
c	c	c	c	0	0
d	d	d	c	b	0

(Table 12)

where ◇ = *c* or *d* and ® = *c* or *d*

From cases (i), (ii), (iii) and (iv), it follows that there exist 16 such BCI-algebras. This completes the proof.

Lemma 6

Let X be a BCI-algebra with 0(X) = 5 and M as its BCK-part. If 0(M) = 3, and 0(X−M) = 2 such that x, y ε X−M are incomparable. The number of such BCI-algebras is 3.

Proof:

Since M contains three elements, without loss of generality, let M = $\{0, a, b\}$ and X−M = $\{x, y\}$. It can easily be seen that there are total three distinct BCK-algebras of order 3. One of them correspond to the case: a and b are incomparable. Two correspond to the case: $0 \leq a \leq b$. The BCK-algebras corresponding to the case: $0 \leq b \leq a$ are isomorphic to the BCK-algebras corresponding to the case: $0 \leq a \leq b$, because they involve only interchange of the symbols a and b.

Since $x, y \in X-M$ are incomparable therefore $I = \{0, x, y\}$, $A(x)=\{x\}$ and $A(y) = \{y\}$. Further (18) gives that $0*x = y$, $0*y = x$, $x*y=y$ and $y*x = x$. We compute $m*x$, $m*y$, $x*m$, $y*m$ for $m \in M$.

Now $0 \leq m \Rightarrow 0*x \leq m*x$, that is, $y \leq m*x$.

Hence $m*x \in A(y) = \{y\}$ which gives $m*x = y$. Again $0 \leq m \Rightarrow x*m \leq x$. Thus $x*m = x$, because $x \in I$. Similarly $m*y = x = y*m$ for all $m \in M$. Hence there are three such distinct BCI-algebras. This completes the proof.

Lemma 7

Let X be a BCI-algebra with $0(X) = 5$. Let M be its BCK-part. Let $0(M) = 3$ and $0 \neq a, b \in M$ are incomparable and $x, y \in X-M$ are comparable. Then there exist 4 such BCI-algebras.

Proof:

Let X = $\{0, a, b, c, d\}$ be a BCI-algebra. Without any loss of generality we take M = $\{0, a, b\}$ such that a and b are incomparable. Then X−M = $\{c, d\}$. Since c and d are comparable, therefore, either

(I) $c \leq d$ or (II) $d \leq c$

It is sufficient to discuss case (I) because (II) involves only interchange of symbols c and d.

Case I: Since $c \leq d$, therefore $c*d = 0$. Obviously $A(0) = \{0, a, b\}$, $A(c)=\{c, d\}$, $I = \{0, c\}$. Thus $0*a = 0*b = 0$ and $0*c = c$.

*Computation on 0*d:* $c \leq d \Rightarrow 0*d \leq 0*c = c \Rightarrow 0*d = c$, because $c \in I$.
*Computation of a*b, b*a:* Since $a*b \leq a$, therefore $a*b = a$ or 0. But $a*b \neq 0$, because if $a*b = 0$, then $a \leq b \Rightarrow a$ is comparable with b, a contradiction. Thus $a*b = a$. Similarly $b*a = b$.

Computation of $c*a$, $c*b$, $c*d$: $0 \leq a \Rightarrow c*a \leq c \Rightarrow c*a = c$, because $c \varepsilon \mathrm{I}$. Similarly $c*b = c$, since $c \leq d$, therefore $c*d = 0$.

Computation of $d*c$: $d, c \varepsilon \mathrm{A}(c)$. By (10), $d*c \varepsilon \mathrm{M} \Rightarrow d*c = a$ or b or 0. But $d*c \neq 0$, because otherwise $c*d = 0 = d*c \Rightarrow c = d$, a contradiction. Thus d*c = a or b, that is,

(i) $d*c = a$,

(ii) $d*c = b$.

Case I (i): Let $d*c = a$

Computation of $d*a$: Since $d*c = a$, therefore $(d*c)*a = 0 \Rightarrow (d*a)*c = 0 \Rightarrow d*a < c \Rightarrow d*a = c$, because $c \varepsilon \mathrm{I}$.

Computation of $d*b$: $0 \leq b \Rightarrow d*b \leq d \Rightarrow d*b = c$ or d. We claim that $d*b \neq c$ because if $d*b = c$, then $(d*b)*c = 0 \Rightarrow (d*c) \leq b$. But $d*c = a$, thus $d*c \leq b \Rightarrow a \leq b$, a contradiction. Thus $d*b = d$.

Computation of $a*d$: $a*d = (d*c)*d = (d*d)*c = 0*c = c$.

Computation of $b*d$: $b*d \varepsilon \mathrm{X} - \mathrm{M} \Rightarrow b*d = c$ or d. But $b*d \neq d$, because if $b*d = d$, then $b*d = d \Rightarrow (b*d)*a = d*a = c \Rightarrow (b*a)*d = c$, or $b*d = c \Rightarrow d = c$, a contradiction. Thus $b*d = c$.

Computation of $b*c$: $b*c \varepsilon \mathrm{X} - \mathrm{M} \Rightarrow b*c = c$ or d. We claim that $b*c \neq d$. Let $b*c = d$, then $(b*c)*b = d*b \Rightarrow (b*b)*c = d*b$, or $0*c = d*b \Rightarrow c = d$, a contradiction. Thus $b*c = c$.

Computation of $a*c$: $c < d \Rightarrow a*d < a*c$, which gives $c < a*c \Rightarrow a*c = c$ or d.

Thus there exist two distinct BCI-algebras for case I(i), with multiplication table as follows:

*	0	a	b	c	d
0	0	0	0	c	c
a	a	0	a	◇	c
b	b	b	0	c	c
c	c	c	c	0	0
d	d	c	d	a	0

(Table 13)

where ◇ = c or d

Similarly for case I(ii), there exist two distinct BCI-algebras with multiplication table as follows:

*	0	a	b	c	d
0	0	0	0	c	c
a	a	0	a	c	c
b	b	b	0	$\diamond$	c
c	c	c	c	0	0
d	d	c	c	b	0

(Table 14)

where $\diamond = c$ or d

From cases I(i) and I(ii) it follows that there exist four distinct BCI-algebras. This completes the proof.

Theorem 5

Let X be a BCI-algebra with O(X) = 5. Let M be its BCK-part with 0(M) = 3. Then number of all such BCI-algebras is 23.

Proof:

It follows from lemmas 5, 6 and 7.

4. BCI-ALGEBRAS OF ORDER FIVE WITH 0(M) = 4:

In this section we show that there are 41 distinct BCI-algebras with order five and O(M) = 4.

Lemma 8

Let X be a BCK-algebra with O(X) = 4, and each pair a, b ε X is incomparable, then X is unique.

Proof:

Let X = $\{0, x, y, z\}$ be a BCK-algebra. W constract the multiplication table for X under the given hypothesis:

Computation of $x*y$, $x*z$, $y*x$, $y*z$, $z*x$, $z*y$: Since $x*y \leq x$, we claim that $x*y = x$. Suppose $x*y = 0$, then $x \leq y \Rightarrow x$ and y are comparable, a contradiction. Similarly $x*z = x$, $y*x = y$, $y*z = y$, $z*x = z$, $z*y = z$ and multiplication table turns out to be as follows:

*	0	x	y	z
0	0	0	0	0
x	x	0	x	x
y	y	y	0	y
z	z	z	z	0

(Table 15)

Lemma 9

Let X be a BCK-algebra with O(X) = 4 and for $0 \neq a, b, c \,\varepsilon$ X, $a \leq b$ and c is not comparable with a and b, then number of such BCK-algebras is 4.

Proof:

Let X = $\{0, a, b, c\}$ be a BCK-algebra. Without any loss of generality, let $a \leq b$ and c is incomparable with a and b. We compute, a^*c, b^*c, c^*a, c^*b, b^*a.

Now $a^*c \leq a \Rightarrow a^*c = a$ or 0. We claim that $a^*c = a$, because if $a^*c = 0$, then $a \leq c$ implies a is comparable with c, a contradiction. Now $b^*c < b \Rightarrow b^*c = 0$ or a or b. But $b^*c \neq 0$. Thus $b^*c = a$ or b. Also $c^*a \leq c \Rightarrow c^*a = 0$ or c. We claim that c*a = c, because otherwise we get $c \leq a$, a contradiction. Also $c^*b \leq c \Rightarrow c^*b = c$, $b^*a \leq b \Rightarrow b^*a = a$ or b.

Thus there exist 4 distinct BCK-algebras whose multiplication table is shown as follows:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	◇	0	◇
c	c	c	c	0

(Table 16)

where ◇ = a or b

Lemma 10

Let X be a BCK-algebra with O(X) = 4 and for $0 \neq x, y, z \varepsilon X$. $x \leq y$, $x \leq z$, but y, z are incomparable. Then there are 16 such BCK-algebras.

Proof:

Obviously $0*x = 0*y = 0*z = x*y = x*z = 0$. We compute $y*x$, $y*z$, $z*x$, $z*y$.

$y*x \leq y \Rightarrow y*x = 0$ or x or y. But $y*x \neq 0$, because otherwise $y*x = 0 = x*y \Rightarrow x = y$, a contradiction. Thus $y*x = x$ or y.

$y*z \leq y \Rightarrow y*z = 0$ or x or y. But $y*z \neq 0$, because otherwise $y*z = 0 \Rightarrow y \leq z$, a contradiction. Thus $y*z = x$ or y.

$z*x \leq z*x = 0$, x or z. But $z*x \neq 0$, because otherwise $z*x = 0 = x*z \Rightarrow x = z$, a contradiction. Thus $z*x = x$ or z.

Similarly $z*y = x$ or z.

The multiplication table is shown as follows:

*	0	x	y	z
0	0	0	0	0
x	x	0	0	0
y	y	◇	0	◇
z	z	®	®	0

(Table 17)

where ◇ = x or y and ® = x or z

Thus we see that there exist 16 such distinct BCK-algebras. This completes the proof.

Lemma 11

Let X be a BCK-algebra with O(X) = 4 and $0 \neq x, y, z$, $0 \leq x \leq z$, $0 \leq y \leq z$, where x and y are not comparable. Then there exist two such BCK-algebras.

Proof:

We compute the multiplication table for X under the given hypothesis. Obviously $0*x = 0*y = 0*z = x*z = y*z = 0$, $x*0 = x$, $y*0=y$, $z*0 = z$, $x*x = 0$, $y*y = 0$, $z*z = 0$.

Computation of $x*y$, $y*x$: $x*y \le x$ implies $x*y = 0$ or x. But $x*y \neq 0$, because otherwise $x*y = 0$ implies $x \le y$ which gives that x is comparable with y, a contradiction. Thus $x*y = x$. Similarly $y*x = y$.

Computation of $z*x$: We know that $y \le z$ and $x \le z$. Thus $y \le z \Rightarrow y*x \le z*x$. But $y*x = y$. Thus $y*x \le z*x$ implies $y \le z*x$, which gives (i) $z*x = y$ or (ii) $z*x = z$.

Computation of $z*y$: *Case (i):* Let $z*x = y$, then $(z*x)*y = 0$, $(z*y)*x = 0 \Rightarrow z*y \le x$. Thus $z*y \le x \Rightarrow z*y = 0$ or x. But $z*y \neq 0$, because otherwise $z*y = 0 = y*z$ implies $z = y$, a contradiction. Thus $z*y = x$.

Case (ii): Let $z*x = z$. Then we claim that $z*y \neq 0, x, y$. Let $z*y = 0$, then $z*y = 0 = y*z \Rightarrow z = y$, a contradiction. Let $z*y = x$. Then $(z*y)*x = 0$ or $(z*x)*y = 0$ or $z*x < y \Rightarrow z*x = y$. But $z*x = z$. Thus $z=y$, a contradiction. Let $z*y = y$. Since $x \le z$ gives $x*y \le z*y$ and $x*y=x$. Thus $x*y \le z*y$ becomes $x \le y$, a contradiction. Thus $z*y \neq 0, x, y$. Thus $z*y \,\varepsilon\, X$ implies $z*y = z$.

Thus case (i) and (ii) give distinct BCK-algebras, respectively, in each case. Their multiplication tables are given in the following:

*	0	x	y	z
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
z	z	y	x	0

(Table 18)

*	0	x	y	z
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
z	z	z	z	0

(Table 19)

Lemma 12

Let X be a BCK-algebra with O(X) = 4 and for $0 \neq x, y, z \,\varepsilon$ X, $x \le y \le z$, then there exist 18 such BCK-algebras.

Proof:

Obviously $0*x = 0*y = 0*z = x*y = x*z = y*z = 0$. We compute $y*x$, $z*y$, $z*x$.

$y*x \leq y \Rightarrow y*x = 0$ or x or y. But $y*x \neq 0$, because otherwise $y*x = 0 = x*y \Rightarrow x = y$, a contradiction. Thus $y*x = x$ or y.

Also $z*x \leq z \Rightarrow z*x = 0$ or x or y or z. But $z*x \neq 0$, because otherwise $z*x = 0 = x*z \Rightarrow x = z$, a contradiction. Thus $z*x = x$ or y or z.

Now $z*y \leq z \Rightarrow z*y = 0$ or x or y or z. But $z*y \neq 0$, because otherwise $y*z = 0 = z*y \Rightarrow y*z$, a contradiction. Thus $z*y = x$ or y or z. The following table gives us the required result.

*	0	x	y	z
0	0	0	0	0
x	x	0	0	0
y	y	Δ	0	0
z	z	®	◇	0

(Table 20)

where $\Delta = x$ or y, ® $= x$ or y or z and ◇ $= x$ or y or z

Theorem 6

Let X be a BCI-algebra with O(X) = 5 and M be its BCK-part. If O(M) = 4, then there exist 41 such distinct BCI-algebras.

Proof:

Let X = $\{0, a, b, c, d\}$. Without any loss of generality, we take M = $\{0, a, b, c\}$ and X−M = $\{d\}$. The elements of M has the following possibilities:

(i) For each pair $0 \neq x, y \in$ M, x, y are not comparable and $\{d\}$=X−M.

(ii) For $a, b, c \in$ M, $a \leq b$ and c is not comparable with a and b, respectively, and $\{d\}$ = X−M.

(iii) For $a, b, c \in$ M, $a \leq c$, $a \leq b$, b and c are not comparable and X−M = $\{d\}$.

(iv) $0 \leq a \leq b \leq c$ and $X-M = \{d\}$.

(v) $0 \leq a \leq b$, $0 \leq c \leq b$, where a and c are not comparable, and $X-M = \{d\}$.

Case (i): By lemma 8, there is unique BCK-algebra. Again for $x \varepsilon M$, $d \varepsilon X-M$, x^*d, $d^*x \varepsilon X-M = \{d\}$ imply $x^*d = d^*x = d$, which implies there exists a unique BCI-algebra.

Case (ii): By lemma 9, there exist 4 BCK-algebras. Since for $x \varepsilon M$, $d \varepsilon X-M$, x^*d, $d^*x \varepsilon X-M = \{d\}$, therefore $x^*d = d^*x = d$. Thus there exist 4 such BCI-algebras.

Case (iii): By lemma 10, there exist 16 BCK-algebras. Thus there exist 16 distinct BCI-algebras in this case.

Case (iv): By lemma 12, there exist 18 BCK-algebras, which implies there exist 18 BCI-algebras in this case.

Case (v): By lemma 11, there exist two BCK-algebras, which implies there exist two BCI-algerbas in this case.

Thus there exist $1+4+16+18+2 = 41$ distinct BCI-algebras. This completes the proof.

Let X be a proper BCI-algebra with $O(X) = 5$. Then there are following possibilities for the BCK-part M:

(i) $O(M) = 1$ (ii) $O(M) = 2$

(iii) $O(M) = 3$ (iv) $O(M) = 4$

We have seen in Theorem 3, 4, 5 and 6 that there exist 1, 5, 23 and 41 proper BCI-algebras in each case, respectively. Thus we have the following theorem:

Theorem 7

There are 70 proper BCI-algebras of order five. We now state the following open problem:

Problem:

How many proper BCI-algebras of order n exist ?

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BEST APPROXIMATION AS A FIXED POINT IN LOCALLY BOUNDED TOPOLOGICAL VECTOR SPACES

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1. INTRODUCTION

Several authors have applied fixed point theorems to obtain interesting results in approximation theory in the setting of normed linear spaces. (see for example [4], [8] and [9]). In this paper we use the technique of Dotson [2] to extend some recent results on best approximation as a fixed point by Habiniak and Sahab to the case of locally bounded topological vector spaces which are not necessarily locally convex.

2. DEFINITIONS AND PRE-REQUSITES

Let (Y, d) be a metric space and M be a non empty subset of Y. For a fixed $x \varepsilon Y$ we define the set of best approximations of x from M by

$$P_M(x) = \{z \varepsilon M : d(x, z) = d(x, M)\}, \text{ where}$$

$$d(x, M) = \inf_{m \varepsilon M} d(x, m)$$

A mapping $T : Y \to Y$ is said (i) non expansive if $d(Tx, Ty) \leq d(x, y)$, $x, y \varepsilon Y$ (ii) Compact if for any bounded subset B of Y, the set $T(B)$ is compact (iii) to leave the set M invarient provided $T(M) \subseteq M$ (iv) to have $x \varepsilon Y$, a fixed point if $T(x) = x$. The set of all fixed points of T will be denoted by $F(T)$. Let E be a vector space. A subset K of E is said to be star-shaped with respect to (w.r.t.) $p \varepsilon K$ if $x \varepsilon K$ and $0 < t < 1$, then $tx + (1 - t)p \varepsilon K$. Clearly every convex set is star-shaped w.r.t. each of its point.

A mapping $|| \cdot ||_p : E \to R$ is called a p-norm if it has the following properties for $0 < p \leq 1$.

(i) $||x||_p \geq 0$ and $||x||_p = 0$ if and only if $x = 0$

(ii) $||\alpha x||_p = |\alpha|^p \, ||x||_p$ for all scalars α

(iii) $||x + y||_p \leq ||x||_p + ||y||_p$ for all $x, y \,\varepsilon\, E$

A topological vector space E is called locally bounded if it has a bounded neighbourhood of the origin. The topology of every Hausdorff locally bounded topological vector space is given by some p-norm, $0 \leq p \leq 1$. (see [7], p.161).

In the rest of this paper E denotes a locally bounded topological vector space equipped with a p-norm $|| \cdot ||_p$.

3. GENERALIZATION OF DOTSON'S THEOREMS

Theorem 3.1

Let C be a compact and star-shaped subset of E. If T is non expansive self map on C, then T has a fixed point in C.

Proof:

Obviously C is a complete subset of E. Since C is a star-shaped set so for a z in C,

$$tz + (1 - t)\, x \,\varepsilon\, C \text{ for all } x \text{ in } C \text{ and } 0 < t < 1$$

Define $T_n x = \frac{z}{n} + \left(1 - \frac{1}{n}\right) Tx$ for all x in C.

$$||T_n x - T_n y||_p = \left|\left| \left(1 - \frac{1}{n}\right)(Tx - Ty)\right|\right|_p$$

$$= \left(1 - \frac{1}{n}\right)^p ||Tx - Ty||_p \leq r_n \, ||x - y||_p$$

where $0 < r_n = \left(1 - \frac{1}{n}\right)^p < 1.$

It follows that each T_n is a contraction on C. By the Banach Contraction Principle each T_n has a unique fixed point $x_n \,\varepsilon\, C$. The sequence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ converging to $x \,\varepsilon\, C$. It follows by the continuity of T that $Tx_{n_j} \to Tx$. Hence

$$x_{n_j} = Tx_{n_j} = \frac{z}{n} + \left(1 - \frac{1}{n}\right) Tx_{n_j} \to Tx$$

By the uniqueness of limit of a sequence in C, we have $Tx = x$.

Let X be a metrizable topological vector space whose topology is generated by an F-norm q. (Cf. [6]). Suppose that S is a subset of X. Let $P = \{f_\alpha\}_{\alpha \varepsilon S}$ be a family of functions from [0, 1] into S with the property that for each $\alpha \varepsilon S$, we have $f_\alpha(1) = \alpha$.

We shall require the following pair of definitions.

Definition 3.2

The family P is said to be contractive provided there exists a function $\phi : (0, 1) \to (0, 1)$ such that for all α, β in S and for all $t \varepsilon (0,1)$, we have

$$q\ (f_\alpha t - f_\beta t) \leq \phi\ (t)\ q\ (\alpha - \beta) \qquad \text{.... (1)}$$

Definition 3.3

The family P is said to be jointly continuous if $f_\alpha t \to f_{\alpha_0} t_0$ in S whenever $t \to t_0$ in [0, 1] and $\alpha \to \alpha_0$ is S.

Theorem 3.4

Let S be a compact subset of (X, q). Suppose the family P of functions associated with S as given above be contractive and jointly continuous. Then any non expansive self mapping T of S has a fixed point in S.

Proof:

For each $n = 1, 2, 3, \ldots$, let $k_n = \dfrac{n}{n+1} \to 1$ for large n and let $T_n : S \to S$ be defined by $T_n x = f_{Tx} k_n$ for all x in S. Since $T(S) \subseteq S$ and $0 < k_n < 1$, we have each T_n is well defined and maps S into S.

Consider $q\ (T_n x - T_n y) = q\ (f_{Tx} k_n - f_{Ty} k_n) \leq \phi\ (k_n)\ q\ (T_x - T_y) \leq \phi\ (k_n)\ q\ (x - y)$ for all x, y in S. The contractions T_n have a unique fixed point x_n in S by the Banach Contraction Principle. By reasoning as in the proof of theorem 3.1, we will have a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging to $x \varepsilon$ S and $Tx_{n_j} \to Tx$. Since $T_{n_j} x_{n_j} = x_{n_j}$ so it follows that $T_{n_j} x_{n_j} \to x$. This gives by joint continuity of P, $T_{n_i} x_{n_i} = f_{Tx_{n_i}} x_{n_i} \to f_{Tx} = Tx$. The uniqueness of the limit of a sequence in X gives that $Tx = x$.

Remark 3.5

(i) Theorem 3.1 does not hold in case of a metrizable topological vector space because we are unable to convert T_n into a contraction by means of an F-norm.

(ii) Theorem 3.4 holds in a non convex setting in contrast to theorem 3.1 which applies to star-shaped sets.

(iii) Theorems 3.1 and 3.4 give generalizations of ([2], theorem 1) and ([3], theorem 1) respectively for p-normed spaces.

4. BEST APPROXIMATION AS A FIXED POINT

In this section we generalize Brosowaki type theorems due to Singh [9] and Sahab [8] to the case of locally bounded topological vector spaces.

Theorem 4.1

Let T be a non expansive operator on E. Let M be a T-invarient subset of E and $x \varepsilon F(T)$. If the set $P_M(x)$ is non empty compact and star-shaped, then the set $P_M(x)$ contains a fixed point of T.

Proof:

If $y \varepsilon P_M(x)$, then $y \varepsilon M$ and $||x - y||_p = \inf_{m \varepsilon M} ||x - m||_p$.

By the T-invariance of M, $Ty \varepsilon M$.

Consider $||Ty - x||_p = ||Ty - Tx||_p \leq ||x - y||_p = \inf_{m \varepsilon M} ||x - m||_p$

we conclude from the definition of infimum that

$$||Ty - x||_p = \inf_{m \varepsilon M} ||x - m||_p \text{ and hence } Ty \varepsilon P_M(x).$$

The star-shaped condition on $P_M(x)$ with respect to say, b in it enables us to define;

$T_n : P_M(x) \to P_M(x)$ by

$$T_n a = \frac{b}{n} + \left(1 - \frac{1}{n}\right) T_a \text{ for all } a \in P_M(x).$$

The rest of the proof of this theorem is same as that of theorem 3.1 and is therefore omitted.

Definition 4.2

An operator T on E is said to be non expansive with respect to another operator I if $||Tx - Ty||_p \leq ||Ix - Iy||_p$ for all x, y in E.

Remark 4.3

If in definition 4.2, I is continuous, then T is also continuous.

Theorem 4.4

Let (Y, d) be a compact metric space and T, I:Y → Y be two commuting mappings such that $T(Y) \subseteq I(Y)$, I is continuous and $d(Tx, Ty) < d(Ix, Iy)$ whenever $Ix \neq Iy$. Then $F(T) \cap F(I)$ is singleton.

Proof:

See Jungck [5].

Theorem 4.5

Let T, I : E → E be operators, C be a subset of E and ∂C denote the boundary of C. Suppose that T is defined from ∂C into C and $x \in F(T) \cap F(I)$, T is non expansive with respect to I on $D = P_M(x) \cup \{x\}$ and I is linear and continuous on $P_M(x)$. Further I and T commute on $P_M(x)$. If $P_M(x)$ is non empty, compact and star-shaped with respect to a point $q \in F(I)$ and $I(P_M(x)) = P_M(x)$, then $P_M(x) \cap F(T) \cap F(I) \neq \phi$.

Proof:

Let $y \in P_M(x)$. Since $I(P_M(x)) = P_M(x)$ so $Iy \in P_M(x)$. Further if $y \in \partial C$, then $Ty \in C$ by the definition of T.

Now $$||Ty - x||_p = ||Ty - Tx||_p \leq ||Iy - Ix||_p = ||Iy - x||_p = \inf_{m \in M} ||x - m||_p$$

If follows as in the proof of theorem 4.1 that T is a self map on $P_M(x)$. Since $P_M(x)$ is star-shaped with respect to q so each T_n given by $T_n a = k_n Ta + (1 - k_n) q$, where $0 < k_n < 1$ and converges to 1, is a self map on $P_M(x)$. Since I is linear and commutes with T on $P_M(x)$ so we have,

$$\begin{aligned} IT_n a &= I(k_n Ta + (1 - k_n)q) \\ &= k_n ITa + (1 - k_n) Iq \\ &= k_n ITa + (1 - k_n) Iq = T_n Ia \quad \text{for all } a \in P_M(x) \end{aligned}$$

This implies that I commutes with T_n on $P_M(x)$ for each n. We note that $T_n(P_M(x)) = I(P_M(x))$

Now $$\begin{aligned} \|T_n a - T_n b\|_p &= (k_n)^p \|Ta - Tb\|_p \qquad 0 < (k_n)^p < 1 \\ &\leq (k_n)^p \|Ia - Ib\|_p \\ &< \|Ia - Ib\|_p \text{ provided } Ia \neq Ib \end{aligned}$$

It is given that $P_M(x)$ is compact and I is continuous. It follows from theorem 4.4 that $F(T_n) \cap F(I) = \{x_n\}$. By the compactness of $P_M(x)$ we can find a subsequence $\{x_{n_j}\}$ converging to $z \in P_M(x)$. Therefore, we get

$$x_{n_j} = T_{n_j} x_{n_j} = k_{n_j} Tx_{n_j} + (1 - k_{n_j})q$$

By the continuity of T and the argument used in the proof of theorem 3.1, we obtain $z = Tz$.

Also $$Iz = I(\lim_{j\to\infty} x_{n_j}) = \lim_{j\to\infty} Ix_{n_j} = \lim_{j\to\infty} x_{n_j} = z$$

Hence $z \in P_M(x) \cap F(T) \cap F(I)$ as required.

We continue the theme of best approximation as a fixed point and now present a generalization of a recent result of Habiniak ([4], theorem 8).

Definition 4.6

A self map T on a metric space Y is called a Banach mapping if $d(Tx, Tx^2) < kd(x, Tx)$, $x \in Y$, $0 \leq k < 1$.

Lemma 4.7

If S is a closed subset of a metric space Y and T is a continuous Banach mapping on Y and $\overline{T(S)}$ is compact, then T has a fixed point.

Proof:

For any $x \varepsilon S$, it is easy to show $\{T^n(x)\}$ is a cauchy sequence in S. Using compactness of $\overline{T(S)}$ the result follows immediately.

Theorem 4.8

Let T be a non expansive Banach operator on a metrizable topological vector space X with fixed point a and M be a closed T-invariant subset of X. If restriction of T to M is compact, then point 'a' has a best approximation b in M which is also a fixed point of T.

Proof:

It is easy to prove that $P_M(a)$ is non empty and closed subset of X. By reasoning as in the proof of theorem 4.1, T is a self map on $P_M(a)$. Clearly T is continuous on $P_M(a)$. Since T leaves M-invarient and $P_M(a)$ is a bounded subset of M so $\overline{T(P_M(a))}$ is compact. Thus by lemma 4.7, T has a fixed point in $P_M(a)$ as desired.

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ON THE SOLUTION OF QUADRATIC INTEGRAL EQUATIONS

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ABSTRACT

We suggest a new iteration for finding "large" solutions of quadratic equations in Banach algebras and Banach spaces. Our results can apply to quadratic integral equations arising in the theories of radiative transfer, neutron transport and in the kinetic theory of gases.

Key Word and Phrases:

Quadratic integral equation, Banach algebra, Banach space, compact operator.

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1. INTRODUCTION

In the theories of radiative transfer (c.f: [4], [5]) and neutron transport (c.f: [4], [8]) an important role is played by nonlinear integral equations of the form

$$H(s) = 1 + H(s) \int_0^1 f(s, t)\, \varphi(t)\, H(t)\, dt, \tag{1}$$

where $\varphi(t)$ is a given function on [0, 1]. Multiplying this equation by φ, one obtains the equation,

$$x(s) = \varphi(s) + x(s) \int_0^1 f(s, t)\, x(t)\, dt, \tag{2}$$

for the function $x(s) = \varphi(s) H(s)$. Equations similar to (2) for functions on the half-line $[0, \infty)$ arise in the kinetic theory of gases.

One of the questions considered here is the following: when can x be found by solving (2) by iteration?

The question has been answered under certain positivity assumptions on φ and f [3], [8]. It has also been answered for quite more general φ than above [10], [11].

In many cases of physical importance, however, φ may become negative. The methods of [1], [2], [4], [11] then provide the best known results.

In almost all the above cases however the solution x obtained is such that

$$||x|| \leq \frac{1 - \sqrt{1 - 4\,||\varphi||\,.\,||\tilde{f}\,||}}{2\,||\tilde{f}\,||} = d \tag{3}$$

$$\text{where}\quad ||\tilde{f}\,|| = \sup_{0 \leq s \leq 1} \int_0^1 |f(s, t)|\, dt \tag{4}$$

$$\text{provided that}\quad 4\,||\varphi||\;\;||\tilde{f}\,|| < 1, \tag{5}$$

$$\text{with}\quad ||\varphi|| = \max_{0 \leq s \leq 1} |\varphi(s)|\,.$$

Under the above assumption however it is known that the corresponding real quadratic equation has two solutions. We wonder if this can be true in a Banach space X. It turns out that this is true under certain assumptions. What we really need to do (assuming that (5) holds) is to introduce a convergent iteration $\{x_n\}$, $n = 0, 1, 2, \ldots$, which guarantees that if $||x_0|| \geq d$ then $||x_n|| \geq d$ and therefore the solution x is such that $||x|| \geq d$.

Equation (2) can be considered as a special case of the equation;

$$x = y + x\,\tilde{K}\,(x) \tag{6}$$

where $\tilde{K}$ is a linear operator on a Banach algebra XA and $y \in X_A$ is fixed. Here we suggest the iteration,

$$x_{n+1} = (L(x_n))\;\;(K(x_n)) \tag{7}$$

for solving (2), where

$$L(x) = x - y \quad\text{and}\quad K(x) = \frac{1}{\tilde{K}(x)}$$

provided that K(x) is well defined and L(x) $\neq$ 0 on U (z, r) = $\{x \varepsilon X_A \mid ||x-z|| < r\}$ for some $z \ \varepsilon \ X_A$ and $r > 0$.

Using a theorem of Darbo [6], we provide conditions for the convergence of (7) to a solution x of (6) such that $||x|| \geq d$, if $||x_0|| \geq d$.

Finally we generalize our results to solve the abstract quadratic equation

$$x = y + B\ (x, x) \tag{8}$$

where B is a bounded bilinear operator on a Banach space X and $y \ \varepsilon$ X is fixed, using the iteration

$$x_{n+1} = B\ (x_n)^{-1}\ (x_n - y),\ n = 0, 1, 2, \ldots \tag{9}$$

for some $x_0 \ \varepsilon$ U(z, r), where $B(x_n)$ is a linear operator on X such that $B(x_n)\ (y) = B\ (x_n, y)$ for all $y \ \varepsilon$ X.

I. Basic Results

We denote by C[0, 1] the Banach space of all real continuous functions on [0, 1] with the maximum norm,

$$||x||_C = \max_{0 \leq s \leq 1} |x\ (s)| \tag{10}$$

We let $C^p[0, 1]$, $0 < p < 1$, denote the Banach space of all real continuous functions on [0, 1] such that

$$\sup_{0 \leq t.s \leq 1} \frac{|x(t) - x(s)|}{|t-s|^p} < \infty, \text{ with the norm}$$

$$||x||_{C^p} = \max_{0 \leq s \leq 1} |x(s)| + \sup_{0 \leq t.s \leq 1} \frac{|x(t) - x(s)|}{|t-s|^p} \tag{11}$$

We note that the spaces X_A^1 = C [0, 1] and x_A^p = C^p [0, 1] with norms given by (10) and (11) respectively are Banach algebras.

Following Kuratowshi [9] we define the measure of noncompactness M_M(D) of a bounded set D of a metric space M to be

M_M(D) = inf $\{\varepsilon > 0 |$ D can be covered by a finite number of sets of diameter less than or equal to $\varepsilon\}$.

Suppose g maps M continuously into a metric space N, and suppose g takes bounded sets to bounded sets. If for some $e \varepsilon [0, \infty)$

$$M_M(g(D)) \leq e M_M(D)$$

for every bounded subset D of M, we say that g is an e-set contraction. If $e < 1$, then g is a strict set contraction.

We will use the following theorem [6].

Theorem 1

Let D be a subset of a Banach algebra X_A and suppose that $T : D \rightarrow X_A$ is of the form $T(x) = x_0 + L(x) K(x)$, where

(1) $x_0 \varepsilon X_A$;

(2) $L : D \rightarrow X_A$ satisfies $||L(x) - L(y)|| < b \, ||x - y||$ for some $b > 0$ and all $x, y \varepsilon D$;

(3) $K : D \rightarrow X_A$ is compact;

(4) $a = \sup\limits_{x \varepsilon D} ||K(x)|| < \infty$.

If $ab < 1$, then T is a strict set contraction.

Moreover if T leaves a closed bounded convex subset D of a Banach algebra invariant, then T has a fixed point in D.

From now on except for the norm of $\tilde{f}$ and related operators, $||\cdot||$ will denote that norm.

We will need the following proposition:

Proposition 1

Let $f(s,t)$ be a continuous function on $[0,1] \times [0,1]$ for which

$$\sup_{0 \leq s_1 \cdot s_2 \leq 1} \frac{1}{|s_1 - s_2|^p} \int_0^1 \left| (f(s_1, t) - f(s_2, t) \right| dt \leq M \quad (12)$$

for some p, $0 < p < 1$, and some $M < \infty$. Define a linear operator $\tilde{f}$ on X_A^1 by

$$(\tilde{f} x)(s) = \int_0^1 f(x, s)\, x(t)\, dt,$$

and assume there exists $z \varepsilon X_A^1$ such that $\mathcal{F}(z)$ never vanishes on [0,1]. Let $||\mathcal{F}||$ be defined as in (4) and fix $r_1 \varepsilon (0, r_0)$ where

$$r_0 = \frac{1}{||\mathcal{F}|| \, ||\mathcal{F}(z)^{-1}||}.$$

Then the operator K on U (z, r_1) given by

$$K(x) = \frac{1}{\mathcal{F}(x)} \tag{13}$$

is well defined, satisfies

$$a = \sup_{x \in (z_1, r_1)} ||K(x)|| \leq \frac{||\mathcal{F}(z)^{-1}||}{1 - ||\mathcal{F}|| \, ||\mathcal{F}(z)^{-1}|| \cdot r_1}, \tag{14}$$

and is compact on U (z, r_1)

Proof:

(a) The result follows immediately from the Banach lemma for invertible operators [12] the choice of r_2 and the identity,

$$\mathcal{F}(x) = \mathcal{F}(z) [1 + \mathcal{F}^{-1}(z) \, \mathcal{F} \, (x - z)]$$

(b) We will use a result in [7], namely if, for every uniformly bounded sequence $\{x_n\}$ in a subset of X_A^1, there is a p, $0 < p < 1$, such that $K(x_n) \, \varepsilon \, X_A^p$ for every n and $\{K(x_n)\}$ is bounded in the norm of X_A^p, then K is a compact operator.

Now let x_n belong to $U(z, r_1)$ such that $||x_n|| < M_1$ for some $M_1 > 0$. Consider

$$h_n(s) = \int_0^1 f(s, t) \, x_n(t) \, dt$$

Then for p as in (12)

$$\sup_{0 \leq s_1, s_2 \leq 1} \frac{|h_n(s_1) - h_n(s_2)|}{|s_1 - s_2|^p} =$$

$$\sup_{0\leq s_1, s_2\leq 1} \frac{|h_n(s_1)-h_n(s_2)|}{|s_1 - s_2|^p} \left| \int_0^1 (f(s_1, t) - f(s_2, t))\, x_n (t)\, dt \right| \quad (15)$$

By (14), there exists $q > 0$ such that

$$\left| \int_0^1 (f(s, t)\, x_n (t)\, dt \right| \geq q \tag{16}$$

From (13), (15) and (16)

$$||K(x_n)||_{X_A^p} \leq \frac{1}{q} + \frac{MM_1}{q^2}. \tag{17}$$

From (17) we obtain that K is a compact operator on $U(z_z, r_1)$. That completes the proof of the proposition.

Let $z, f, \bar{f}$, K and r_0 be as in Proposition 1. Next, fix $y \neq z$ and denote by P (y, z) the constant $y + z\bar{f}(z) - z$. Assume the constants c_1, c_2 and Δ given by

$$c_1 = 1 - ||\bar{f}(z)^{-1} (I - z\bar{f})||$$

$$c_2 = 1 - ||\bar{f}(z)^{-1}||, \text{ and}$$

$\Delta = (||\bar{f}(z)^{-1}(I - z\bar{f})|| - 1)^2 - 4||\bar{f}(z)^{-1}||.||\bar{f}||.||\bar{f}(z)^{-1}p(z)||$ are positive. Define the constants $r_2 - r_5$ by

$$r_2 = \frac{1 - ||\bar{f}(z)^{-1}||}{||\bar{f}||\,.\,||\bar{f}(z)^{-1}||},$$

$$r_3 = \frac{1 - ||\bar{f}(z)^{-1}||\,(I - z\bar{f})|| - \sqrt{\Delta}}{2||\bar{f}||\,.\,||\bar{f}(z)^{-1}||},$$

$$r_4 = \frac{1 - ||\bar{f}(z)^{-1}||\,(I - z\bar{f})|| + \sqrt{\Delta}}{2||\bar{f}||\,.\,||\bar{f}(z)^{-1}||},$$

and $\quad r_5 = ||z - y||$

Theorem 2

Let $z, f, \bar{f}$, K, y and the constants c_1, c_2, Δ, r_0 and $r_2 - r_5$ be as above. Assume:

the following condition is satisfied:

The $z, f, \mathcal{f}$, K are as in Proposition 1 and $y \varepsilon X_A^1$ is fixed; the following are true:

$c_1 < 0, c_2 > 0, \Delta > 0, r_3 < \min(r_2, r_4, r_5)$

and choose $r_6 \varepsilon (r_3, \min(r_2, r_4, r_5))$.

Define the operator T on $\overline{U}(z, r_6)$ by $T(x) = (L(x))(K(x))$ where L, K are operators given by $L(x) = x - y$

and $$K(x) = \frac{1}{\mathcal{f}(x)},$$

Then the operator T has fixed point in $\overline{U}(z, r_6)$.

Proof:

One only needs to show that the hypotheses of theorem 1 are satisfied. Note that the hypotheses (1), with $b = 1$, and (2) are obvious and that (3) follows from proposition 1 since $r_6 < r_2 < r_0$. We now prove the claims:

Claim 1 *T is a strict set contraction on $\overline{U}(z, r_6)$.*

Since, $b = 1$ it is enough to show that

$$a = \sup_{x \in U(z, r_6)} ||K(x)|| < 1$$

or by (15) $$\frac{||\mathcal{f}(z)^{-1}||}{1 - ||\mathcal{f}|| \cdot ||\mathcal{f}(z)^{-1}|| \cdot r_6}$$

which is true by the choice of r_6.

Claim 2 *T maps $\overline{U}(z, r_6)$ into $\overline{U}(z, r_6)$*

Let $x \in \overline{U}(z, r_6)$ then $T(x) \in \overline{U}(z, r_6)$ if

$$||T(x) - z|| = ||\mathcal{f}(x)^{-1}[(I - z\mathcal{f})(x - z) - P(z)]||$$
$$\leq \frac{||\mathcal{f}(z)^{-1}(I - z\mathcal{f})|| \cdot r_6 + ||\mathcal{f}(z)^{-1}P(z)||}{1 - ||\mathcal{f}(z)^{-1}|| \cdot ||\mathcal{f}|| \cdot r_6} \leq r_6$$

which is also true because $r_3 < r_6 \leq r_4$.

That completes the proof of the theorem.

Note that results similar to those in Proposition 1 and theorem 2 can easily be proved if we work in the space X_A^P.

We will now extend our results to include equation (8) and iteration (9).

II. Extension - Remarks

From now on we assume that X is a banach space and that B in (8) is a nontrivial bounded symmetric bilinear operator [11], [12]. The operator B is assumed to be symmetric without loss of generality since B can always be replaced by the mean $\overline{B}$ of B defined by

$$\overline{B}(x, y) = \frac{1}{2}(B(x, y) + B(y, x)), x, y \varepsilon X$$

We have $\overline{B}(x, x) = B(x, x)$ for all $x \varepsilon X$.

Denote by $B(x)$, $x \varepsilon X$, the linear operator on X defined by

$$B(x)(y) = B(x, y), x, y \varepsilon X.$$

We are now going to show that iteration $\{x_n\}$ given by (9) (or (6) in case of convergence to a solution x of (9) (or (5)) is such that $||x|| \geq \frac{1}{2||B||}$ $\left(\text{or } ||x|| \geq \frac{1}{2||T||} \right)$ under certain assumptions.

Proposition 2

(1) the iteration

$$x_{n+1} = B(x_n)^{-1}(x_n - y)$$

is well defined for all $n = 0, 1, 2, \ldots$ for some $x_0 \varepsilon X$ and converges to a solution x of (9),

(2) $1 - 4\,||B||\,.\,||y|| > 0$, and (18)

(3) $p \varepsilon [p_1, p_2]$, where p_1, p_2 are the solutions of the equation

$$||B||\,p^2 - p = ||y|| = 0.$$

If $||x_0|| > p$,

then $||x_n|| \geq p, n = 0, 1, 2, \ldots$

and $||x|| \geq p$

Proof:

We use induction on n. Since $||x_0|| > p$ by assumption, suppose $||x_k|| \geq p$ for $k = 0, 1, ..., n$. To show $||x_{n+1}|| \geq p$, note that from

$$B(x_n, x_{n+1}) = x_n - y$$

or $$||x_n - y|| = ||B(x_n, x_{n+1})|| \leq ||B|| \cdot ||x_n|| \cdot ||x_{n+1}||,$$

we have $$||x_{n+1}|| \geq \frac{||x_n - y||}{||B|| \cdot ||x_n||} \geq \frac{||x_n - ||y||}{||B|| \cdot ||x_n||}.$$

Since $||x_n|| \geq p \geq ||y||$ to show $||x_{n+1}|| \geq p$, it is enough to show $\frac{||x_n - ||y||}{||B|| \cdot ||x_n||} \geq p$ or $||x_n|| \geq \frac{||y||}{1 - p\,||B||}$. Finally it suffices to show $p \geq \frac{||y||}{1 - p\,||B||}$ or $||B||\,p^2 - p + ||y|| \leq 0$ which is true for $p \, \varepsilon \, [p_1, p_2]$.

That completes the proof of the proposition. Not that

$$p = \frac{1}{2\,||B||} \, \varepsilon \, [p_1, p_2]$$

Using the Banach lemma for the invertibility of linear operators [12] and working as in the proof of Proposition 1 (part (a)) we can easily show the following result.

Proposition 3

Let $z \, \varepsilon \, X$ be such that the linear operator $B(z)$ is invertible. Then $B(x)$ is also invertible for all $x \, \varepsilon \, U(z, R_0)$, where

$$R_0 = \frac{1}{||B|| \cdot ||B(z)^{-1}||}$$

We will need the definition:

Let $z \, \varepsilon \, X$ be such that the linear operator $B(z)$ is invertible. Let $R_0 > 0$ be fixed and choose R with $0 < R < R_0$.

The operators $\tilde{P}$, $\tilde{T}$ given by

$$\tilde{P}(x) = B(x, x) + y - x$$

and $\tilde{T}(x) = (B(x))^{-1}(x - y)$

are then well-defined on U(z, R).

Define the real functions F_1 and F_2 on $\mathbb{R}^+$ by

$$F_1(R) = e_1 R^2 + e_2 R + e_3$$

and $$F_2(R) = e_4 R^2 + e_5 R + e_6,$$

where $$e_1 = (||B|| \,.\, ||B(z)^{-1}||)^2,$$

$$e_2 = -2\,||B|| \,.\, ||B(z)^{-1}||,$$

$$e_3 = 1 - ||B(z)^{-1}|| - ||B|| \,.\, ||B(z)^{-1}||^2\, ||z - y||,$$

$$e_4 = ||B|| \,.\, ||B(z)^{-1}||,$$

$$e_5 = ||B(z)^{-1}(I - B(z)|| - 1$$

and $$e_6 = ||B(z)^{-1}\tilde{P}(z)||\,.$$

Working as in theorem 2 and using Proposition 2, we can easily show the following consequence of the contraction mapping principle [12].

Theorem 3

Let $y \neq z$, $\tilde{P}$, $\tilde{T}$, B, $r_1 - r_6$, R_0, F_1 and F_2 be as above. Assume

(1) There exists $z \in X$ such that the linear operator B(z) is invertible.

(2) The following are true:

$$e_3 > 0,\ e_5 < 0,$$

and $$e_5^2 - 4\, e_4\, e_6 > 0$$

(3) There exists R > 0 such that

$$F_1(R) > 0,\ F_2(R) \leq 0$$

and $$R < ||z - y||,$$

Then,

(a) The operator $\tilde{T}$ given by

$$\tilde{T} = B(x)^{-1}(x - y)$$

is well-defined and it has a unique fixed point $x \varepsilon \overline{U}(x, R)$.

(b) The iteration

$$x_{n+1} = B(x_n)^{-1}(x_n - y), n = 0, 1, 2, ...$$

is well-defined and it converges to x for any $x_0 \varepsilon \overline{U}(z, r)$.

Moreover, if $1 - 4 \|B\| . \|y\| > 0$ and for $p = \frac{1}{2\|B\|}$ say,

$$\|x_0\| \geq \frac{1}{2\|B\|}, \text{ then by proposition 2}$$

$$\|x\| \geq \frac{1}{2\|B\|}.$$

Remarks:

(a) If the hypotheses of theorem 3 are true, then equation (8) (or (5)) has two solutions x_1 and x_2 such that

$$\|x_1\| \leq \frac{1}{2\|B\|} \text{ and } \|x_2\| \geq \frac{1}{2\|B\|}$$

(b) If $X = X_A^1$ or $X = X_A^P$ then the hypotheses of theorem 2 can easily be verified. If X is a Banach space then the conditions of theorem 3 may be difficult to verify since the invertibility of the linear operator $B(z)$ may be almost impossible to ascertain. Moreover z has to be chosen close to the solution.

However the other two popular methods for solving (8), namely Newton's method

$$x_{n+1} = x_n - (2B(x_n) - I)^{-1}(\tilde{P}(x_n)), n = 0, 1, 2, ... \quad (19)$$

and the method of successive substitutions

$$x_{n+1} = y + B(x_n, x_n), n = 0, 1, 2, ... \quad (20)$$

share similar difficulties.

In particular Newton's method also requires z to be "close" to the solution and the invertibility of the operator $I - 2B(x_n)$ at each step (or the invertibility of $(I - 2B(x_0))$ if we are referring to the modified Newton's method).

Although the method of successive substitution makes on use of the invertibility of the linear operator B(z), z must still be close to the solution and

$$||z|| < \frac{1}{2\ ||B||},$$

under hypothesis (18) [1], [2], [11]. Therefore it cannot be used to find a solution x such that

$$||x|| > \frac{1}{2\ ||B||},$$

since the solution obtained then satisfies

$$||x|| \leq \frac{1}{2\ ||B||},$$

Finally note that in a general Banach space X neither (19) nor (20) has the property of keeping the iterates away from zero as iteration (9) (or (6)) does. Therefore iteration (9) if applicable can be used to find the "large" solutions of (8) (if they exist) under hypothesis (18).

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ON A UNIVERSAL PROPERTY OF BCI-ALGEBRA

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ABSTRACT

In this paper we solve an open problem of [6] and show that epimorphisms are precisely onto BCI-homomorphisms. Some properties of epimorphisms are proved.

INTRODUCTION

In [7] it is shown that onto BCI-homomorphisms are epimorphisms. In [5] some properties of category BCI, category with BCI-algebras as its objects and BCI-homomorphisms as its morphisms, have been investigated. The following problem regarding epimorphisms was posed. Problem: Let X, Y be BCI-algebras and f : X → Y an epimorphism. Is f : X → Y onto? In [3] and [4], the problem has been solved under certain conditions. However, we show that epimorphisms are precisely onto BCI-homomorphims and prove some properties of epimorphisms.

We shall follow standard definitions. Our categorical concepts shall be those of standard text [9]. Our notions of BCI-algebras shall be as developed in [1], [2], [6], [8] and [10].

We recall that a BCI-homomorphism $f : X \to Y$ means that $f(x*y) = f(x) * f(y)$, for $x, y \varepsilon$ X. We note that $f(0) = 0$. Further, we denote by BCI, the category of BCI-algebras as its objects and BCI-homomorphisms as its morphisms. |BCI| denotes the objects class and set of morphisms from an object X to an object Y is denoted by

BCI (X, Y). An $f \varepsilon$ BCI (X, Y) is an epimorphism if for all $Z \varepsilon |BCI|$, g, $h \varepsilon$BCI (Y, Z) and satisfying $gof = hof$ implies $g = h$.

DEFINITION 1 [6]:

An ideal which is also a sub-algebra is called a closed ideal. In [4], it is called as a regular ideal. An ideal A in a BCI-algebra X is closed ideal if and only if $0 * a \varepsilon A$, for $a \varepsilon A$. It is interesting to note that such ideals do exist which are not closed. Consider the following example.

Example 1:

Let X be the set of all integers with (−) operation. Then (X, −, 0) is a p-semi-simple algebra. Define $A = \{x \varepsilon X : x \geq 0\}$ and $B = \{x \varepsilon X : x \leq 0\}$. Note that A and B are ideals in X, but both are not closed. Thus regular and non-regular ideals exist in BCI-algebras.

DEFINITION 2 [1]:

Let X be a BCI-algebra. Let $x_0 \varepsilon X$ be such that for each $y \varepsilon X$ and satisfying $y * x_0 = 0$ implies $y = x_0$. Then x_0 is called a special element of X. Let I denote the set of all special elements, we call it the centre of X. For some $x_0 \varepsilon I$, we define

$$A(x_0) = \{x \varepsilon X : x_0 * x = 0\}$$

We call $A(x_0)$ a special section of X. The following are used in the sequel.

1. Let X be a BCI-algebra with I as its centre. Then $\cup_{x_0 \varepsilon I} A(x_0) = X$ and $A(x_0) \cap A(y_0) = 0$, for $x_0, y_0 \varepsilon I$ ([1]).

2. The centre I of a BCI-algebra X is a p-semi-simple algebra ([2]).

3. Let X be a BCI-algebra with I as its centre. Let $0 \varepsilon N \subseteq I$ and $H = \bigcup_{x_0 \varepsilon I} A(x_0)$. H is a closed ideal in X if and only if N is a closed ideal in I ([2]).

4. Every sub-algebra in a p-semi-simple algebra is a closed ideal ([11]).

5. Let X, Y be BCI-algebras with I_X, I_Y as their centres respectively. $f : X \rightarrow Y$ be a BCI-homomorphism. Then $f(I_X) \subseteq I_Y$. ([3])

6. Let X be a BCI-algebra and A, B are closed ideals in X. If A an ideal in B, then $X/B \cong X/A \,/\, B/A$ ([19]).

In [4], Z Chen an H Wang proved the following result for regular ideals.

Result:

Let X, Y be BCI-algebras and $f : X \rightarrow Y$ an epimorphism. If B is a regular ideal in Y, then $f^{-1}(B)$ is a regular ideal in X. However, we generalize it for ideals as well as homomorphisms in as follows.

Lemma 1

Let X, Y be BCI-algebra and $f : X \rightarrow Y$ a BCI-homomorphism. Let B be an ideal in Y, then $f^{-1}(B)$ is an ideal in X.

Proof:

Let X, Y be BCI-algebras and $f : X \rightarrow Y$ a BCI-homomorphism. Let B be an ideal in Y. Take $A = f^{-1}(B)$; we show that A is an ideal in X.

$f(0) = 0 \in B$ implies $0 \in f^{-1}(B) = A$. Let $y * x \in A$ and $x \in A$ for $x, y \in X$. $y * x \in f^{-1}(B)$ implies $f(y*x) \in B$.

$f(y*x) \in B$ implies $f(y)*f(x) \in B$. $f(y)*f(x) \in B$, $f(x) \in B$ and B being an ideal implies $f(y) \in B$. Thus $y \in f^{-1}(B) = A$ or $y \in A$. Hence A is an ideal in X.

Lemma 2

Let X, Y be BCI algebras with I_X, I_Y as their centres and $f{:}X \rightarrow Y$ a BCI-homorphism. Let $N \subseteq I_X$ and $H = \bigcup_{x_0 \in N} A(x_0)$, then $f(H) \subseteq \bigcup_{y_0 = f(x_0) \in f(N)} A(y_0) = \mathbf{Z}$.

Proof:

Let $x_0 \in N$ and $x \in A(x_0) \subseteq H$. $x_0 \leq x$ implies $f(x_0) \leq f(x)$ implies $f(x) \in A(f(x_0))$. Or $f[A(x_0)] \subseteq A(f(x_0))$. Or $f[\bigcup_{x_0 \in N} A(x_0)]$

$\subseteq \cup f_{(x_0)\in f(N)}$ $A(f(x_0)) = \cup_{y_0} = f_{(x_0)\in(N)} =$ Z. or f[H] $\subseteq$ Z. This competes the proof.

We restate theorem 5 of [3] as result 1 and use it in what follows.

Result 1:

Let X, Y $\in$ |BCI|, $f \in$ BCI (X, Y) be an epimorphism such that f(X) is an ideal in Y, then f is onto ([3]).

Theorem 1

Let X, Y $\in$ |BCI|, $f \in$ BCI (X, Y) be an epimorphism, then f is onto.

Proof:

Let X, Y be BCI-algebras with I_X, I_Y as their centres respectively. By (1), X = $\cup_{x_0 \in I_X} A(x_0)$. By (5), $f(I_X) \subseteq I_Y$. Also, $f(I_X)$ is a sub-algebra. By (4), $f(I_X)$ is a closed ideal in I_Y. By (3), Z = $\cup_{y_0 \in f(I_X)} A(y_0)$ is a closed ideal in Y.

By lemma 2, f(X) $\subseteq$ Z. By lemma 1, f^{-1}(Z) a closed ideal in X, is contained in X. So Z $\subseteq$ f(X). But f(X) $\subseteq$ Z. Hence Z = f(X) is an ideal in Y.

Further $f : X \to Y$ is an epimorphism. By result 1, f is onto. This completes the proof.

NON-REGULAR IDEALS AND QUOTIENT ALGEBRAS

Quotient Algebra X/A of a BCI-algebra X by a closed ideal A has been studied in [4], [10] and [11]. But quotient algebra of a BCI-algebra X by a non-regular ideal still remains un-explored. We undertake it and show that it is a proper BCI-algebra.

DEFINITION 3:

Let A be a non-regular ideal in a BCI-algebra X. We define

R(A) = $\{a \in A: 0*a \in A\}$ and NR(A) = $\{a \in A: 0*a \in A^C\}$.

R(A) and NR(A) are called regular and non-regular parts of A respectively. It follows from the definition that if A is a regular ideal then R(A) = A and $C_0 = \{x \in X: 0*x, x*0 \in A\} = \{x \in A: 0*x \in A\}$ = R(A).

Theorem 2

Let X be a BCI-algebra and A a non-regular ideal in X, then $\{X/A, *, C_0\}$ is a BCI-algebra.

Proof:

Let X be a BCI-algebra and A an ideal in X. We define a relation R on X by:

$R = \{(x, y) : x, y \in X \text{ and } x*y, y*x \in A\}$,

Routine calculations show that R is an equivalence relation on X and it partitions X into equivalence classes $(C_x^R : x \in X)$, where $C_x^R = (y \in X: (x, y) \in R\} = (y \in X: x*y, y*x \in A\} = C_x^A = C_x$.

We note that $(x, y) \in R$ if and only if $C_x = C_y$.

Let us consider $X/A = (C_x : x \in X)$. For $C_x.C_y \subseteq X/A$, define $C_x * C_y = C_{x*y}$.

We show that * is well-defined in X/A. For $C_x = C_u$ and $C_y = C_v$, we show that $C_{x*y} = C_{u*v}$.

$C_x = C_u$ implies $(x, u) \in R$ and $C_y = C_v$ implies $(y, v) \in R$. Thus $(x, u), (y, v) \in R$ imply $x*u, u*x, y*v, v*y \in A$, for $x, y, u, v \in X$. $(x*y) * (x*v) \leq v*y$ implies $(x*y) * (x*y) \in A$. Consequently, $(x*y, x*v) \in R$. (a)

Again, $(x*v) * (u*v) < x*u$ implies $(x*v) * (u*v) \in A$ and $(u*v) * (x*v) < u*x$ implies $(u*v) * (x*v) \in A$ which implies that $(x*v, u*v) \in R$. (b)

From (a) and (b) $(x*y, u*v) \in R$ which implies that

$$C_{x*y} = C_{u*v}$$

Routine calculations show that $(X/A, * C_0)$ is a BCI-algebra. This completes the proof.

Lemma 3

Let X be a BCI-algebra and A non-regular ideal in X. Then R(A) is a regular ideal in X.

Proof:

Let X be a BCI-algebra and A a non-regular ideal in X. We consider $(X/A, *, C_0)$ and define $f : X \to X/A$ by $f(x) = C_x$.

We note that f is onto. We show that $\ker(f) = R(A)$.

$$\begin{aligned} \ker(f) &= \{x \in X : f(x) = C_0\} \\ &= \{x \in X : C_x = C_0\} \\ &= \{x \in X : 0*x, x*0 \in A) \\ &= \{x \in A : 0*x \in A\} = R(A) \end{aligned}$$

By lemma 1, $f^{-1}(C_0) = \{x \in X : f(x) = C_0\} = \ker(f)$, is an ideal in X, because C_0 is an ideal in X/A. Hence R(A) is a regular ideal in X. This completes the proof.

Theorem 3

Let $f : X \to Y$ be an epimorphism and A an ideal in X such that $\ker(f) = \{0\}$, then $X/R(A) \cong Y$.

Proof:

By theorem 1, f is onto. By lemma 3, R(A) is a regular ideal in X and X/R(A) is a quotient algebra. We define $F : X/R(A) \to Y$ by $F(C_x) = f(x)$. F is onto, because f is onto. We show that F is one-one. Suppose $f(x) = f(y)$, for $x, y \in X$. Then $f(x) * f(y) = f(y) * f(x) = 0$ or $f(x*y) = f(y*x) = 0$ or $x*y, y*x \in \ker(f) = \{0\}$. Thus $x = y$ which implies $C_x = C_y$.

Again, $F(C_x*C_y) = F(C_{x*y}) = f(x*y) = f(x)*f(y) = C_x*C_y$ which implies that F is a BCI-homomorphism. Hence $X/R(A) \cong Y$. This completes the proof.

COROLLARY 1 [4]:

Let $f : X \to Y$ be an onto BCI-homomorphism such that $\ker(f) = \{0\}$. If A is a regular ideal in X, then $X/A \cong Y$.

Theorem 4

Let $X, Y \in |BCI|$, $f \in BCI(X, Y)$. If f is an onto homomorphism and B an ideal in Y, then $X/f^{-1}(B) \cong Y/B$.

Proof:

From lemma 1, if B is an ideal in Y, then $f^{-1}(B)$ is an ideal in X. Take $f^{-1}(B) = A$ and consider X/A, Y/B as quotient algebras of X and Y by A and B respectively. We define F : X/A → Y/B by

$$F(C_x) = C_{f(x)}$$

F is well-defined. Let $C_x = C_y$, for $x, y \in X$. Then $C_x * C_y = C_0 = C_y * C_x$ Or $C_{x*y} = C_0 = C_{y*x}$ implies $F(C_{x*y}) = F(C_0) = F(C_{y*x})$ implies $C_{f(x*y)} = C_{f(0)} = C_{f(y*x)}$. Or $f(x) * f(y), f(y)*f(x) \in C_0 \subseteq B \Rightarrow f(x)*f(y), f(y)*f(x) \in B$. Or $C_{f(x)} = C_{f(y)} \Rightarrow F(C_x) = F(C_y)$; that if F is well-defined.

F is a BCI-homomorphism. Let us consider;

$F(C_x*C_y) = F(C_{x*y}) = C_{f(x*y)} = C_{f(x)} * C_{f(y)} = F(C_x) * F(C_y)$, which gives that F is a BCI-homomorphism.

F is one-one. Suppose $F(C_x) = F(C_y)$, for $C_x, C_y \in X/A$, $x, y \in X$. We show that $C_x = C_y$. $F(C_x) = F(C_y) \Rightarrow C_{f(x)} = C_{f(y)}$. Or $C_{f(x)*f(y)} = C_0 = C_{f(y)*f(x)}$. Or $C_{f(x*y)} = C_0 = C_{f(y*x)}$ implies $f(x*y)$, $f(y*x) \in C_0 \subseteq B \Rightarrow f(x*y); f(y*x) \in B$ and $x*y, y*x \in f^{-1}(B) = A$ $\Rightarrow C_x = C_y$.

F is onto. f is onto. Thus for each $y \in Y$ there exists a $x \in X$ such that $y = f(x)$. Hence each $C_y \in Y/B$ implies $C_y = C_{f(x)} = F(C_x)$; that $C_y = F(C_x)$. Hence $X/A \cong Y/B$. This completes the proof.

COROLLARY 2 [4]:

Let f : X → Y be a epimorphism and B a regular ideal in Y, then $X/A \cong Y/B$, where $A = f^{-1}(B)$.

The third isomorphism theorem of [10] does not hold for non-regular ideals, because a non-regular ideal itself is not a BCI-algebra and if A and B are ideals in X such that $B \subseteq A$, then A/B is not defined. However, we give the next best thing in the sequel.

Lemma 4

Let A, B be ideals in a BCI-algebra X such that $B \subseteq A$, then B is an ideal in A.

Proof:

Since B is an ideal in X, so $x*b \in X-B$, for $x \in X-B$ and $b \in B$. Note that $A-B \subseteq X-B$ implies B is an ideal in A. This completes the proof.

Theorem 5

Let K be an ideal in a BCI-algebra X and H an ideal in K, then $X/R(K) \cong X/R(H)/R(K)/R(H)$.

Proof:

Since H is an ideal in K, so $0 \in H$. Let $y*x, x \in H$ and $y \in X$. We show that $y \in H$. Since $H \subseteq K$, so $y*x, x \in K$, with $y \in X$. But K is an ideal in X, so $y \in K$. Again $y \in H$, because H is an ideal in K. Thus H is an ideal in X. Further, $H \subseteq K$ implies $R(H) \subseteq R(K)$. By lemma 3, R(H), R(K) are ideals in X. By lemma 4, R(H) is an ideal in R(K). By (6), the result follows. This completes the proof.

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