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GEOMETRIC COONTINUITY AND CUBIC INTERPOLATION

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ABSTRACT

A description and analysis of an interpolatory cubic spline curve is made for use in Computer Aided Geometric Design (CAGD). The cubic pieces are stitched together with a generalized continuity: the parameters in the description of this continuity provide a variety of shape control. This geometric cubic spline provides not only a computationally simple allIternative to the exponential based spline under tension [Cline '74, Preuss' 76, Schweikert'66] and the rational spline methoods [Gregory and Sarfraz'90, Sarfraz'92] but also recovers the well known existing GC^2 or C^1 methods like cubic ν-spline of Nielson [Nielson'86], γ-splines of Boehm [Bohem'85] and weighted ν-spines [Foley'87].

KEYWORDS

Cubic spline, rational spline, interpolation, tension, shape control.

1. INTRODUCTION

This paper is motivated by the geometric continuity concepts (i.e. GC^2 or C^1) of Nielson [Nielson'86] and Foley [Foley'86,87]; Nielson has used GC^2 continuity to join the cubic pieces together whereas Foley utilized the C^1 continuity to stitch the cubic pieces. They achieved some shape controls like *point* and *interval tensions* due to the parameters in the descriptions of their continuity constrains. A most generalized concept of continuity than that of Nielson or Foley has been considered idn this paper which not only allows the user to join the cubic pieces with sufficient smoothness but also permits to play freely with the shape of the interpolatory curve. In addition one can recover easily the cubic ν-spline method of Nielson [Nielson'86] and weighted spline and weighted ν-spline spline methods of Foley [Foley'86,87]. The continuity used, in this paper, has been named as *sigma*(σ) *continuity* for the sake of convinience and the spline so obtained will be called as *sigma*(σ) *spline.* This cubic σ-spline method also provides an alternative to the rational spliens in [Gregory and Sarfraz'90, Sarfraz'92]; these rational spline methods were also constructed to play with the shape of the curve.

The cubic σ-spline is based on a cubic Hermite interpolant which is introduced in Section 2 together with some preliminary analysis. Section 3 describes the cubic σ-splien an analyses its behaviour with respect to shape parameters in each interval. Section 4 explains some special cases of this σ-spline method and Section 5 consists of some illustrative examples.

2. CUBIC INTERPOLANT

Let $F_i \in \mathbb{R}^N$ be given values at knots t_i, $i = 0, \ldots, n-1$, where $t_0 < t_1 <,\ldots,< t_n$, and let $V_i, W_i \in \mathbb{R}^N$, $i = 0, \ldots n-1$.

The general form of a cubic, which interpolates at the knots, is given by

$$P_i(t) = (1-\theta)^3 F_i + 3\theta\,(1-\theta)^2 Vi + 3\theta^2 (1-\theta) W_i + \theta^3 F_{i+1} \qquad (1)$$

where

$$\theta \equiv \theta(t) = (t - t_i) / h_i, \quad h_i = t_{i+1} - t_i. \qquad (2)$$

Obviously $0 \le \theta \le 1$.

Remark 1

The following can be noted:

(*i*) The curve segment (1) lies in the convex hull of the control points $\{F_i, V_i, W_i, F_{i+1}\}$ (see Proposition 2.1 in [Sarfraz'90]).

(*ii*) The curve segment (1) satisfies the variation diminishing property (see Proposition 2.2 in [Sarfraz'90]).

(*iii*) If the pieces $P_i(t)$, $i = 0,\ldots, n-1$, are joined together with any kind of continuity, then the composed curve.

$$P(t) = P_i(t), \ i = 0, \ldots, n-1. \qquad (3)$$

s at least C^0.

(*iv*) The equivalent Hermite representation of (1) is obtained when

$$V_i = F_i + h_i D_i^+/3, \; W_i = F_{i+1} - h_i D_{i+1}^-/3 \tag{4}$$

where

$$\begin{cases} P^{(1)}(t_i^+) = D_i^+ \\ P^{(1)}(t_{i+1}^-) = D_{i+1}^- \end{cases} \tag{5}$$

(*v*) The second derivatives of (1) at the knots ti and t_{i+1}, are obtained as:

$$\begin{cases} P_i^{(2)}(t_i) = 2\{3F_i - 6V_i + 3W_i\}/h_i^2 \\ P_i^{(2)}(t_{i+1}) = 2\{3F_{i+1} - 6W_i + 3V_i\}/h_i^2 \end{cases} \tag{6}$$

3. INTERPOLATORY CUBIC σ-SPLINES

Definition 2

We will call a function $P(t)$ σ-continuous at $t = t_i$ if it satisfies the following constraints.

$$\begin{pmatrix} P(t_{i+}) \\ P^{(1)}(t_{i+}) \\ P^{(2)}(t_{i+}) \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & \sigma_{1,i} & \\ 0 & \sigma_{2,i} & \sigma_{3,i} \end{pmatrix} \begin{pmatrix} P(t_{i-}) \\ P^{(1)}(t_{i-}) \\ P^{(2)}(t_{i-}) \end{pmatrix} \tag{7}$$

Now, we use this generalized form of continuity i.e. σ-continuity (c.f. (7)) to connect the pieces of the cubic (1). The second and third equations of the σ-continuity constraints (7) together with equaions (4), (5) and (6) lead to the system of *consistencey equations*

$$h_i\sigma_{1,i-1}\sigma_{3,i}D_{i-1}^- + \{\frac{h_i h_{i-1}}{2}\sigma_{2,i} + 2h_i\sigma_{3,i} + 2h_{i-1}\sigma_{1,i}\} D_i^- \tag{8}$$

$$+ h_{i-1} D_{i-1}^+ = 3h_i\sigma_{3,i}\,\Delta_{i-1} + 3h_{i-1}\Delta_i, \; i = 1, \ldots, n$$

in unknowns D_i^-, $i = 0, ..., n$, where $\Delta_i = (F_{i+1} - F_i)/h_i$. Hence for appropriate end conditions D_0^- and D_n^- and the constraints

$$\sigma_{1,i} = 1, \sigma_{3,i} > 0, \sigma_{2,i} \geq 0, \ \forall\, i, \tag{9}$$

the system of equations (8) defines a diagonally dominant tridiagonal linear system which can be easily solved using the *LU* decomposition algorithm. Thus a unique cubic interpolatory spline is obtained which is at least C^1. (Since $\sigma_{1,i} = 1$ we have $D_i^- = D_i^+$.)

4. SHAPE CONTROL

Now we look at the effects of the shape parameters on the cubic spline interpolant in the rest of this section.

(*i*) If we vary the $\sigma_{2,i}$'s and keep the others fixed according to (9), then

(*ia*) (**Point tension**) for fixed $i = k$ if we assume $\sigma_{2,k} \to \infty$, then the k^{th} equation of the system of equations (8) results as:

$$\lim_{\sigma 2_{,k} \to \infty} D_k = 0. \tag{10}$$

Thus the curve at the point P_k will appear to have a *corner*.

(*ib*) (**Interval tension**) Similarly as above large values of $\sigma_{2,k}$ and $\sigma_{2,k+1}$ cause D_k and D_{k+1} to approach zero. This behaviour tigtens the curve in the interval $[t_k,t_{k+1}]$.

(*ic*) (**Global tension**) Following in the same way as above, if $\sigma_{2,i} \to \infty$, for all i, then

$$\lim_{\sigma 2_{,i} \to \infty} D_k = 0, \text{ for } i = 1, ..., n-1.$$

Thus the curve is globally tightened in $[t_1, t_{n-1}]$.

(*ii*) (**Biased behaviour**) If we vary the $\sigma_{3,i}$'s and keep the other shape prameters fixed according to (9), then for any i if $\sigma_{3,i} \to \infty$, the following relationship is obtained from the system (8):

$$D_i = \frac{3\Delta_{i-1} - D_{i-1}}{2}.$$

This shows a biased behaviour i.e. the curve is inclined towards a side of the interval $[t_i, t_{i+1}]$. A similar behaviour can be observed when $\sigma_{2,i} = \sigma_{3,i} \to \infty$.

5. SOME SPECIAL CASES

A number of spline methods can be obtained as a result of distinct replacements of the parameters involved in the above construction. For example

A. $\sigma_{1,i} = \sigma_{3,i} = 1, \sigma_{2,i} = 0.$

corresponds to the cubic spline interpolation.

B. The weighted spline [Salkauskas'84] can be obtained by the following replacement:

$$\sigma_{1,i} = 1, \ \sigma_{2,i} = 0, \sigma_{3,i} = \frac{\omega_{i-1}}{\omega_i}, \ \omega_i > 0.$$

C. The ν-spline [Nielson'86] can be obtained with the following choice:

$$\sigma_{1,i} = \sigma_{3,i} = 1, \sigma_{2,i} = \nu_i \geq 0.$$

D. The replacement

$$\sigma_{1,i} = 1, \ \sigma_{2,i} = \frac{\nu_i}{\omega_i}, \sigma_{3,i} = \frac{\omega_{i-1}}{\omega_i},$$

where $\nu_i \geq 0$, $\omega_i > 0$, $\forall\ i$, gives weighted ν-spline interpolation method of Foley [Foley'87]. This also covers the cases **B** and **C**.

6. EXAMPLES

The shape control of the cubic σ-spline interpolants is illustrated by the folloiwng examples for the data sets in $\mathbb{R}^2$. Unless otherwise stated we will assume $\sigma_{1,i} = 1, \sigma_{2,i} = 0, \sigma_{3,i} = 1$ in all the examples.

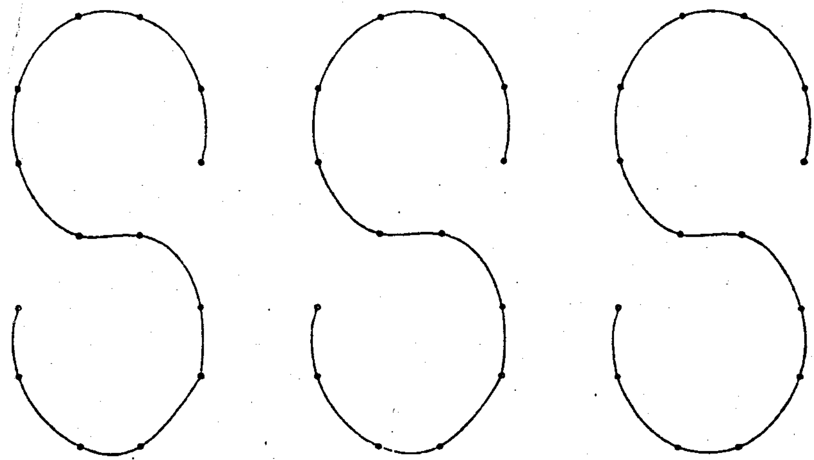

Figure 1. *Interpolatory rational σ-splines with $\sigma_{2,4}$ varying for point tension.*

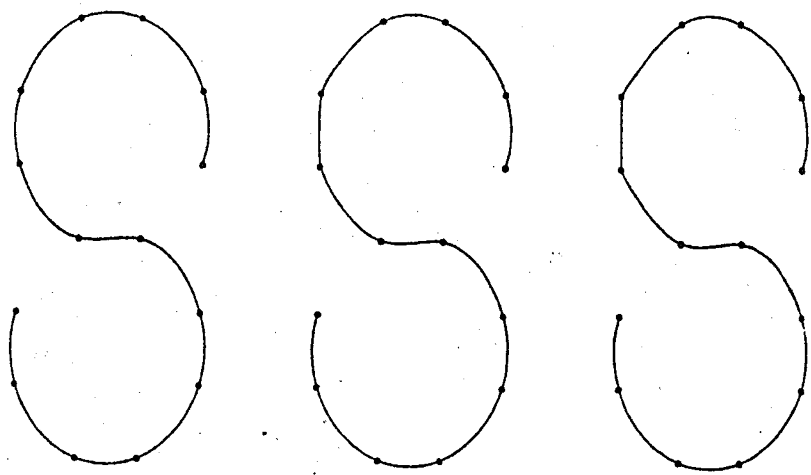

Figure 2. *Interpolatory rational σ-splines with $\sigma_{2,4}$ and $\sigma_{2,5}$ varying for interval tension control.*

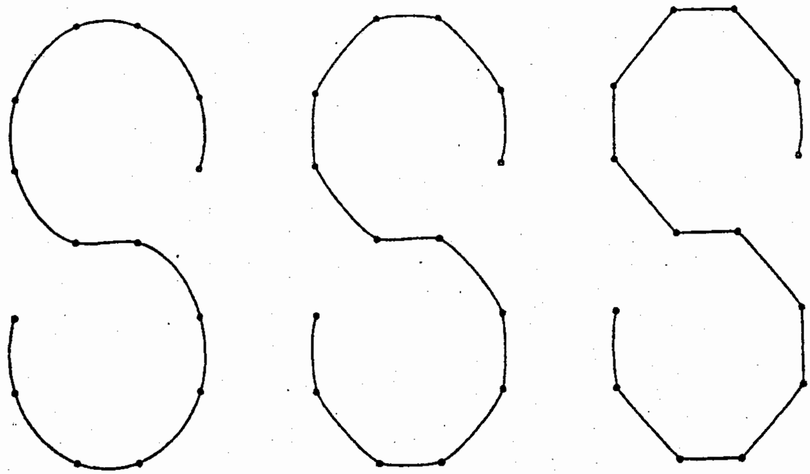

Figure 3. *Interpolatory rational σ-splines with global tension using the shape parameter* $\sigma_{2,i}$.

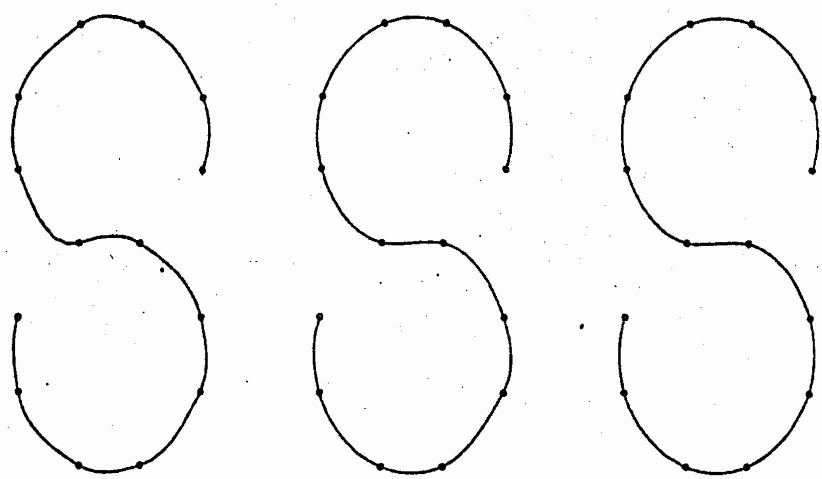

Figure 4. *Interpolatory rational σ-splines with local and global biased behaviour using* $\sigma_{3,i}$.

Figure 1 illustrates the effect of progressively increasing the value of the point tension parameter $\sigma_{2,4}$ at the knot t_4 whilst Figure 2 shows the interval tension effect due to progressive increases in $\sigma_{2,4}$ and $\sigma_{2,5}$. The Figure 3 displays the global tension effect due to progressive increase in $\sigma_{2,i}$. The values of the varying parameters, in each curve of the Figures 1, 2 and 3, are taken as 0, 5 and 50 respectively.

Figure 4 demonstrates the result of Remark 4(ii) regarding local and global biased behaviour; the shape paramter σ_3 is chosen as 1 and 50 in the first and third curves respectively whereas $\sigma_{3,i}$ is 50 for $i = 4$, and 1 elsewhere in the second curve.

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ON A TWO-PIONT NEWTON METHOD IN BANACH SPACES OF ORDER THREE AND APPLICATIONS

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ABSTRACT

Sufficient conditions are given for the convergence of a two-point Newton method to a zero of a nonlinear operator equation in a Banach space. The order of convergence is three.

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AMS (MOS) SUBJECT CLASSIFICATIONS: 65H10, 65J15, 47H17.

KEYWORDS AND PHRASES

Majorant theory, Banach space, Newton's method.

1. INTRODUCTION

In this study we are concerned with the problems of approximating a locally unique zero x^* of the equation

$$F(x) = 0 \tag{1}$$

in a Banach space E_1, where F is a nonlinear operator defined on some convex subset D of E_1 with values in another Banach space E_2.

The convergece of single-step methods, like Newton's method as well as Newton-like methods to a zero x^* of equation (1) has been studied extensively in a Banach space setting, [1]-[20]. But the convergence analysis for multipoint methods is less developed, although the fundamental theory was developed several years ago, see [16], [17] and [19] and the references there. The reason is that the expreslson $F(x)$ cannot easily be dominated by a real scalar function. It is well known, from the efficiency index point of view [17], [19] that multipoint methods are faster than single-step methods.

Here, in particular we consider a two-point newton method of the form

$$y_n = x_n - F'(x_n)^{-1} F(x_n) \tag{2}$$

$$x_{n+1} = y_n - F'(x_n)^{-1} F(y_n) \tag{3}$$

for some arbitrary $x_0 \in D$ and for all $n \geq 0$. The linear operator $F'(x_n)$ is the first Frechet-derivative of F evaluated at $x = x_n$. Note that the evaluation of x_{n+1} requires one inverse $(F'(x_n)^{-1})$ and two function evaluations $(F(x_n)$ and $F(y_n))$, whereas the regular Newton's method required the evaluation of the same inverse and one function evaluation $(F(x_n))$.

Using standard Newton-Kantorovich assumptions we will show that the two-point Newton method (2)-(3) converges to a zero x^* of equation (1) with order three, whereas the regular Newton's method has order only two.

Finally, our results apply to the solution of some nonlinear integral equations appearing in radiative transfer [1], [2], [3], [8].

2. CONVERGENCE ANALYSIS

Let $x_0 \in D$ be arbitrary and for $R > 0$ such that $U(x_0, R) = \{x \in E_1 / ||x - x_0|| \leq R\} \subseteq D$.

We assume that

$$||F''(x)|| \leq M \tag{4}$$

and $$||F'(x) - F'(y)|| \leq K||x - y|| \tag{5}$$

for all $x, y \in D$. It is convenient to introduce the constants

$$\eta \geq ||y_0 - x_0||, \ \beta \geq ||F'(x_0)^{-1}||, \ t_0 = 0, \ h = M\eta\beta \tag{6}$$

$$r_1 = \frac{1 - \sqrt{1 - 2h}}{h} \eta, \tag{7}$$

$$r_2 = \frac{1 + \sqrt{1 - 2h}}{h}\eta, \tag{8}$$

$$\theta = \frac{r_1}{r_2} \tag{9}$$

and the scalar iterations

$$s_n = t_n - \frac{g(r_n)}{g'(t_n)} \tag{10}$$

$$s_{n+1} = s_n - \frac{g(s_n)}{g'(t_n)} \text{ for all } n \geq 0, \tag{11}$$

where

$$g(t) = \frac{M}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}. \tag{12}$$

Note that if $2h \leq 1$, r_1 is the smallest zero of the equation

$$g(t) = 0 \tag{13}$$

We can now prove the main result:

Theorem

Let $F : D \subseteq E_1 \to E_2$ be a nonlinear operator defined on some convex subset D of a Banach space E_1 with values in another Banach space E_2. Assume:

(a) F is twice-Frechet differentiable on $U(x_0, r_1) \subseteq D$ for some $x_0 \in D$, and satisfies (4) – (5);

(b) the inverse of the linear operator $F'(x_0)$, $x_0 \in D$ exists and is bounded;

(c) the following estimates are true:

$$h < \frac{2d}{(d+1)^2} = q, \; d = \sqrt[6]{2} \tag{14}$$

and $$\beta K(3r_1 + r_2) < 2. \tag{15}$$

Then the two-point Newton method generated by (2)-(3) is well defined, remains in $U(x_0, r_1)$ for all $n \geq 0$ and converges to a unique zero x^* of equation $F(x) = 0$ in $U(x_0, r_2)$.

Moreover, the following estimates are true:

$$||x_n - x^*|| \le r_1 - t_n, \tag{16}$$

$$||y_n - x^*|| \le r_1 - s_n \tag{17}$$

and $$r_1 - t_n = \frac{(1-\theta^2)\,\eta\,(d\theta)^{3^n-1}}{1-(d\theta)^{3^n}} \text{ for all } n \ge 0 \tag{18}$$

Proof

We will show that if

$$||y_n - x_n|| \le s_n - t_n, \tag{19}$$

$$||F(x_n)|| \le g(t_n), \tag{20}$$

$$||F(y_n)|| \le g(s_n) \tag{21}$$

and $$||F'(x_n)^{-1}|| \le -g'(t_n)^{-1}, \tag{22}$$

then $$||x_{n+1} - y_n|| \le t_{n+1} - s_n, \tag{23}$$

$$||y_{n+1} - x_{n+1}|| \le s_{n+1} - t_{n+1}, \tag{24}$$

$$||F(x_{n+1})|| \le g(t_{n+1}) \tag{25}$$

and $$||F(x_{n+1})|| \le g(s_{n+1}) \text{ for all } n \ge 0. \tag{26}$$

Using (3), (11), (20) and (22), we get

$$||x_{n+1}-y_n|| \le ||F'(x_n)^{-1}||.||F(y_n)|| \le -g'(t_n)^{-1}g(s_n) = t_{n+1}-s_n.$$

Hence, (23) is true.

From (2)-(4), (19), (22) and the approximation

$$F(x_{n+1}) = F(x_{n+1}) - F(y_n) - F'(y_n)(x_{n+1}-y_n) + \big(F'(y_n)-F'(x_n)\big)(x_{n+1}-y_n)$$

$$= \int_0^1 F''(y_n + t(x_{n+1}-y_n))(1-t)dt(x_{n+1}-y_n)^2$$

$$+ \int_0^1 F''(x_n + t(y_n - x_n))(y_n - x_n)dt(x_{n+1}-y_n).$$

we obtain

$$||F(x_{n+1})|| \leq \frac{M}{2}||x_{n+1}-y_n||^2 + M||y_n-x_n|| \cdot ||x_{n+1}-y_n||$$

$$\leq \frac{M}{2}(t_{n+1}-s_n)^2 + M(s_n-t_n)(t_{n+1}-s_n) = g(t_{n+1})$$

Hence, (25) is true. Also by (2), (22) and (25) we get

$$||y_{n+1}-x_{n+1}|| \leq F'(x_{n+1})^{-1}|| \cdot ||F(x_{n+1})|| \leq -g'(t_{n+1})^{-1}g(t_{n+1})$$

$$= s_{n+1}-t_{n+1}$$

Hence, (24) is also true.

Similarly, from (2), (4), (24) and the approximation

$$F(y_{n+1}) = F(y_{n+1}) - F(x_{n+1})(y_{n+1} - x_{n+1})$$

$$= \int_0^1 F''(x_{n+1} + t(y_{n+1} - x_{n+1})(1-t)\,dt(y_{n+1}-x_{n+1})^2,$$

we get

$$||F(y_{n+1})|| \leq \frac{M}{2}||x_{n+1}-y_{n+1}||^2 \leq \frac{M}{2}(s_{n+1}-t_{n+1})^2 = g(s_{n+1}).$$

Hence, (26) is also true.

We have also the estimates

$$||x_{n+1}-x_0|| \leq ||x_{n+1}-y_0|| + ||y_0-x_0|| \leq ||x_{n+1}-y_n|| + ||y_n-y_0|| + ||y_0-x_0||$$

$$\leq \ldots\ldots \leq (t_{n+1}) - s_n) + (s_n - s_0) + s_0 \leq t_{n+1} \leq r_1, \qquad (27)$$

and

$$||y_{n+1}-x_0|| \leq ||y_{n+1}-y_0|| + ||y_0-x_0|| \leq ||y_{n+1}-x_{n+1}|| + ||x_{n+1}-y_n||$$

$$+ ||y_n - y_0|| + ||y_0 - x_0|| \leq \ldots \leq (s_{n+1} - t_{n+1}) + (t_{n+1}-s_n)$$

$$+ (s_{n+1} - s_n) + (s_n - s_0) + s_0 \leq s_{n+1} \leq r_1. \qquad (28)$$

Moreover, from (4), (6), (27) and the estimate

$$||F'(x_0)^{-1}|| \cdot ||F'(x_n)-F'(x_0)|| \leq ||F'(x_0)^{-1}|| \cdot \int_0^1 ||F''(x_0+t(x_n-x_0))$$

$$(x_n-x_0)dt|| \leq \beta M\, ||x_n - x_0|| \leq \beta M(t_n - t_0) \leq \beta M r_1 \leq 1.$$

It follows from the Banach lemma on invertible operators that $F'(x_n)^{-1}$ exists and for all $n \geq 1$.

$$||F'(x_n)^{-1}|| \leq \frac{||F'(x_0)^{-1}||}{1-||F'(x_0)^{-1}||.||F'(x_n)-F'(x_0)||} \leq \frac{\beta}{1-\beta M t_n} = -g'(t_n)^{-1} \tag{29}$$

Hence, the iterates (2)–(3) are well defined for all $n \geq 0$.

It now follows that the sequence $\{x_n\}$ is Cauchy in a Banach space and as such it coverges to some $x^* \in U(x_0, r)$, which by taking the limit as $n \to \infty$ in (2) becomes a zero of F since $F(x^*) = 0$. Moreover by (27) and (28) $x_n, y_n \in U(x_0, r_1)$. The estimates (16) and (17) follow easily from

$$||x_{n+1}-x_n|| \leq ||x_{n+1}-y_n|| + ||y_n-x_n|| \leq t_{n+1}-s_n+s_n-t_n=t_{n+1}-t_n$$

and

$$||y_{n+1}-y_n|| \leq ||y_{n+1}-x_{n+1}||+||x_{n+1}-y_n|| \leq s_{n+1}-t_{n+1}+t_{n+1}-s_n = s_{n+1}-s_n$$ for all $n \geq 0$.

Moreover, from (8)-(12) we get

$$r_1 - s_n = \frac{(r_1 - t_n)^2}{r_1 - t_n + r_2 - t_n}$$

$$r_1 - t_{n+1} = \frac{(r_1 - s_n)(r_1 - t_n + s_n - t_n)}{r_1 - t_n + r_2 - t_n}$$

and $$r_2 - t_{n+1} = \frac{(r_2 - s_n)(r_2 - t_n + s_n - t_n)}{r_1 - t_n + r_2 - t_n}$$

Hence, we get

$$\frac{r_1 - t_{n+1}}{r_2 - t_{n+1}} = \left(\frac{r_1 - t_n}{r_2 - t_n}\right)^3 \left(\frac{1 + \frac{s_n - t_n}{r_1 - t_n}}{1 + \frac{s_n - t_n}{r_2 - t_n}}\right) \leq \left(c\frac{r_1 - t_n}{r_2 - t_n}\right)^3, c = \sqrt[3]{2}$$

$$\leq \ldots \leq \left(d\frac{r_1 - t_0}{r_2 - t_0}\right)^{3^{n+1}} = (d\theta)^{3^{n+1}} \tag{30}$$

Note that $d\theta < 1$, by (14).

Since,

$$r_2 - t_n = r_1 - t_n + (1 - \theta^2)\frac{\eta}{\theta} \tag{31}$$

the result (18) follows from (30) and (31).

To show uniqueness, let us assume that there exists another zero y^* of equation (1) in $U(x_0, r_2)$. Then from (5) and (29) we obtain

$$||F'(x^*)^{-1}|| \int_0^1 ||F'(x^* + t(y^* - x^*)) - F'(x^*)||\, dt$$

$$\leq \frac{\beta K}{1 - \beta K r_1} ||x^* - y^*|| \int_0^1 t\, dt \leq \frac{1}{2}\frac{\beta K}{1 - \beta K r_1}(r_1 + r_2) < 1 \text{ by (15)}$$

(since $||x^* - y^*|| \leq ||x^* - x_0|| + ||y^* - x_0|| \leq r_1 + r_2$)

It now follows from the above inequality that the linear opeator

$$\int_0^1 F'(x^* + t\,(y^* - x^*))\, dt$$

is invertible. From this and the approximation

$$F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, dt\, (y^* - x^*),$$

it follows that $x^* = y^*$.

That completes the proof of the theorem.

Remarks

(a) From the estimates

$$||x_n - y_0|| \leq ||x_n - y_n|| + ||y_n - y_0|| \leq t_n - s_n + s_n - s_0 \leq t_n - \eta \leq r_1 - \eta$$

and

$$||y_{n+1} - y_0|| \leq ||y_{n+1} - x_{n+1}|| + ||x_{n+1} - y_n|| + ||y_n - y_0|| \leq s_{n+1} - t_{n+1} + t_n - 1 - s_n + s_n - s_0 \leq s_{n+1} - \eta \leq r_1 - \eta,$$

it follows that $x_n, y_n \in U(y_0, r_1 - \eta)$ for all $n \geq 0$.

(b) We can use the two-point method to approximate nonlinear equations with nondifferentiable operators. Indeed consider the equation

$$F_1(x) = 0, \tag{32}$$

where $F_1(x) = F(x) + Q(x)$

with F as before and Q satisfying an estimate of the from

$$||Q(x) - Q(y)|| \leq K_1 \, ||x - y|| \text{ for all } x \in D. \tag{33}$$

Note that the differentiability of Q is not assumed here. Replaec F in (2) and (3) by F_1 and leave the Frechet-derivatives as they are. Define the sequences $\{\bar{t}_n\}$ and $\{\bar{s}_n\}$ as the corresponding $\{t_n\}$ and $\{s_n\}$ given by (10) and (11) respectively. The only change will be an extra term of the form $K_1(s_n - t_n)$ and $K_1(t_{n+1} - s_n)$ added at the right hand sides of (10) and (11) respectively and multiplied by the corresponding fractions. Define also g_1 and g in (12) but add the term $K_1 t$. Then following the proof of the above theorem step by step we can show a similar theorem with identical hypotheses and conclusions, but holding for equation (32). (See, also [5]).

(c) Similar theorems can be proved if relation (5) is replaced by a weaker older estimate of the form

$$||F'(x) - F'(y)|| \leq K \, ||x - y||^p \text{ for all } x, y \in D$$

and some $p \in [0, 1]$, [5].

(d) Many times tue computation of the inverse of the lienar operator $F'(x_n)$ for all $n > 0$ is a very difficult or an impossible task. That is why we then recommand the two-point modified Newton's method given by

$$w_n = v_n - F'(x_0)^{-1} F(v_n)$$

$$v_{n+1} = w_n - F'(x_0)^{-1} F(w_n) \quad \text{for all } n \geq 0.$$

By just replacing (14) with

$$2h \leq 1$$

in the hypothesis of the theorem we can state and prove a similar theorem, but holding for the above mentioned. The scalar iterations (10) and (11) are replaced by

$$b_n = a_n - \frac{g(a_n)}{g'(t_0)}, \quad t_0 = 0$$

$$a_{n+1} = b_n - \frac{g(b_n)}{g'(t_0)} \quad \text{for all } n \geq 0$$

The estimates (16)-(18) now become

$$\|v_n - x^*\| \leq r_1 - a_n$$

$$\|w_n - x^*\| \leq r_1 - b_n \quad \text{for all } n \geq 0$$

and $$r_1 - a_n \leq \frac{(1-\theta^2)\,\eta\theta^{2^n}}{1-\theta^{2^n}} \quad \text{for all } n \geq 0.$$

That is, the order of convergence of the abvoe method is two, but we invert the linear foperator involved only once. Note that the order of convergence of the regular Newton's method is also two, but then we have to invert $F'(x_n)$ for all $n \geq 0$, [15].

(e) Note that using the approximation

$$\int_0^1 [F'(x^*+t(y^*-x^*))-F'(x^*)]\,dt = \int_0^1 \int_0^{x^*+t(y^*-x^*)} f''(z)dz$$

we can show that (15) can be replaced by $\beta M(3r_1 + r_2) < 2$, which may be useful, especially when $M \leq K$.

(f) Uniqueness can also be established in the ball $U(y_0, r_2 - \eta)$. We just need to replace in the above proofs r_1 by $r_1 - \eta$ and r_2 by $r_2 - \eta$. Then our conditions become

$$\beta K(3r_1 + r_2 - 4\eta) < 1$$

or $$\beta M(3r_1 + r_2 - 4\eta) < 1.$$

3. APPLICATIONS

In this section we use the theorem to suggest new approaches to the solution of quadratic intègral equations of the form

$$x(s) = y(s) + \lambda x(s) \int_0^1 q(s, t)\, x(t)\, dt \qquad (34)$$

in the space $E_1 = C[0, 1]$ of all functions continuous on the interval [0, 1], with norm

$$||x|| = \max_{0 \le s \le 1} |x(s)|$$

Here we assume that λ is a real number called the "albedo" for scattering and the kernel $q(s, t)$ is a continuous function of two variable s, t with $0 < s, t < 1$ and satisfying

(i) $0 < q(s, t) < 1,\ 0 \le s, t \le 1;$

(ii) $q(s, t) + q(t, s) = 1,\ 0 \le s, t \le 1.$

The function $y(s)$ is a given continuous function defined on [0, 1], and finally $x(s)$ is the unknown function sought in [0, 1].

Equations of this type are closely related with the work of S. Chandrasekhar [8]. (Nobel prize winner of physics for 1983), and arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gasses, [1], [2], [3], [8].

There exists an extensive literature on equations like (34) under various assumptions on the kernel $q(s, t)$ and λ is a real or complex number. One can refer to the recent work in [1], [2], [3] and the references there. Here we demonstrate that the theorem via the iterative procedure (2)-(3) provides existence results for (34). Moreover the iterative procedure (2)-(3) converges faster to the solution than all the previous known ones. Futhermore a better information on the location of the solutions is given. Note that the computational cost is not higher than the corresponding one of previous methods.

For simiplicity (without loss of generality) we will assume that

$$q(s, t) = \frac{s}{s + t} \text{ for all } 0 \le s, t \le 1.$$

Note that $q(s, t)$ so defined satisfies (i) and (ii) above.

Let us now choose $\lambda = .25$, $y(s) = 1$ for all $s \in [0, 1]$; and define the operator F on E_1 by

$$F(x) = \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)\, dt - x(s) + 1.$$

Note that every zero of the equation $F(x) = 0$ satisfies the equation (34).

Set $x_0(s) = 1$, use the definition of the first and second Frechet-derviatives of the operator F to obtain using and the theorem,

$$K = M = 2|\lambda| \max_{0\le s\le 1} \left| \int_0^1 \frac{s}{s+t} dt \right| = 2\ |\lambda| \ell n\, 2 = .34657359,$$

$$\beta = ||F'(1)^{-1}|| = 1.53039421,$$

$$\eta = ||F'(1)^{-1} F(1)|| \geq \beta\lambda \ell n^2 = .265197107,$$

$$q = .494902504,$$

$$h = .140659011 < q,$$

$$r_1 = .28704852, \quad r_2 = 3.4837317$$

$$\theta = .08239685$$

and $\quad 2\beta K r_1 = .304497749 < 1,$

which shows that x^* is unique in $U(x_0, r_1)$ and not in $U(x_0, r_2)$; since (15) is violated. (For detailed computations, see also [1], [2] and [3].)

Therefore according to the theorem equation (34) has a solution x^* and the two-point Newton method (2)-(3) converges to x^* faster than any other method used so far according to (16) and (18). (See also, [1], [2], [3], [8]). Moreover the information on the location of the solution given here is better than the ones given before.

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ON A TWO-POINT NEWTON METHOD IN BANACH SPACES OF ORDER FOUR AND APPLICATION

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ABSTRACT

Sufficient conditions are given for the convergence of a two-point Newton method to a zero of a nonlinear operator equation in a Banach space. The order of convergence is four.

C.R. CATEGORIES

5.1, 5.15.

AMS (MOS) SUJECT CLASSIFICATIONS

65H10, 65J15, 47H17.

KEY WORDS AND PHRASES

Majorant theory, Banach space, Newton's method.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique zero x^* of the equation

$$F(x) = 0 \tag{1}$$

in a Banach space E_1, where F is a nonlinear operator defined on some convex subset D of E_1 with values in another Banach space E_2.

The convergence of single-step methods, like Newton's method as well as Newton-like methods to a zero x^* of equation (1) has been studied extensively in a Banach space setting [1]-[19]. But the convergence analysis for multipoint methods is less developed, although the fundamental theory was developed several years ago, [15], [16] and [18]. The reason is that the expression $F(x)$ cannot easily be dominated by a real scalar function. It is well known, from the efficiency index point of view [15] that multipoint methods are faster than single-step methods.

Here, in particular we consider a Newton two-step method of the form

$$y_n = x_n - F'(x_n)^{-1} F(x_n) \tag{2}$$

$$x_{n+1} = y_n - F'(y_n)^{-1} F(y_n) \tag{3}$$

for some arbitrary $x_0 \in D$ and for all $n \geq 0$. The linear operator $F'(x_n)$ is the Frechet-derivative of F evaluated at $x = x_n$.

Using standard Kantorovich assumptions we will show that the two-point Newton method (2)-(3) converges to a zero x^* of equation (1) with order 4.

2. CONVERGENCE ANALYSIS

Let $x_0 \in D$ be arbitrary and for $R > 0$ such that $U(x_0, R) = \{x \in E_1 / ||x - x_0|| \leq R\} \subseteq D$.

We assume that

$$||F''(x)|| \leq M \tag{4}$$

and $$||F'(x) - F'(y)|| \leq K||x - y|| \tag{5}$$

for all $x, y \in D$. It is convenient to introduce the constants

$$\eta \geq ||y_0 - x_0||,\ \beta \geq ||F'(x_0)^{-1}||,\ t_0 = 0,\ h = M\eta\beta \tag{6}$$

$$r_1 = \frac{1 - \sqrt{1 - 2h}}{h}\eta, \tag{7}$$

$$r_2 = \frac{1 + \sqrt{1 - 2h}}{h}\eta, \tag{8}$$

$$\theta = \frac{r_1}{r_2}, \tag{9}$$

and the scalar iterations

$$s_n = t_n \frac{g(t_n)}{g'(t_n)} \tag{10}$$

$$t_{n+1} = s_n - \frac{g(s_n)}{g'(s_n)} \text{ for all } n \geq 0, \tag{11}$$

where

$$g(t) = \frac{M}{2} t^2 - \frac{1}{\beta} t + \frac{\eta}{\beta} \tag{12}$$

Note that r_1 is the smallest zero of the equation

$$g(t) = 0 \quad (\text{if } 2h \leq 1). \tag{13}$$

Theorem

Let $F : D \subseteq E_1 \to E_2$ be a nonlinear operator defined on some convex subset D of a Banach space E_1 with values in another Banach space E_2. Assume:

(a) F is twice-Frechet differentiable on $U(x_0, r_1) \subseteq D$ for some $x_0 \in D$, and satisfies (4)-(5).

(b) the inverse of the linear operator $F'(x_0)$, $x_0 \in D$ exists and is bounded;

(c) the following estimates are true

$$h \leq \frac{1}{2} \tag{14}$$

and

$$\beta K (3r_1 + r_2) < 2. \tag{15}$$

Then the two-point Newton method generated by (2)-(3) is well defined, remains in $U(x_0, r_1)$ for all $n \geq 0$ and converges to a unique zero x^* of equation $F(x) = 0$ in $U(x_0, r_2)$.

Moreover, the following estimates are true:

$$||x_n - x^*|| \leq r_1 - t_n, \tag{16}$$

$$||y_n - x^*|| \leq r_1 - s_n, \tag{17}$$

and

$$r_1 - t_n = \frac{(1 - \theta^2)\, \eta \theta^{4^n - 1}}{1 - \theta^{4^n}} \text{ for all } n \geq 0. \tag{18}$$

Proof

We will show that if

$$||y_n - x_n|| \le s_n - t_n, \; ||F(x_n)|| \le g(t_n), \; ||F(y_n)|| \le g(s_n), \quad (19)$$

$$||F'(x_n)^{-1}|| \le -g'(t_n)^{-1}, \quad (20)$$

and $$||F'(y_n)^{-1}|| \le -g'(s_n)^{-1}, \quad (21)$$

then $$||x_{n+1} - y_n|| \le t_{n+1} - s_n, \quad (22)$$

$$||y_{n+1} - x_{n+1}|| \le s_{n+1} - t_{n+1}, \quad (23)$$

$$||F(x_{n+1})|| \le g(t_{n+1}) \quad (24)$$

and $$||F(y_{n+1})|| \le g(s_{n+1}) \text{ for all } n \ge 0. \quad (25)$$

Using (3), (19) and (21), we get

$$||x_{n+1} - y_n|| \le ||F'(y_n)^{-1}||.||F(y_n)|| \le -g'(s_n)^{-1} g(s_n) = t_{n+1} - s_n.$$

Hence, (22) is true.

From (2)-(4), (22) and the approximation

$$F(x_{n+1}) = F(x_{n+1}) - F(y_n) - F'(y_n)(x_{n+1} - y_n)$$

$$= \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\, dt (x_{n+1} - y_n)^2$$

we obtain

$$||F(x_{n+1})|| \le \frac{1}{2} M ||x_{n+1} - y_n||^2 \le \frac{1}{2} M (t_{n+1} - s_n)^2 = g(t_{n+1})$$

Hence, (24) is true.

Also by (2), (20) and (25) we get

$$||y_{n+1} - x_{n+1}|| \le ||F'(x_{n+1})^{-1}||.||F(x_{n+1})|| \le -g'(t_{n+1}) g(t_{n+1}) = s_{n+1} - t_{n+1}$$

Hence, (23) is also true.

Similarly, from (2), (4), (23) and the approximation

$$F(y_{n+1}) = F(y_{n+1}) - F(x_{n+1}) - F'(x_{n+1})(y_{n+1} - x_{n+1})$$

$$= \int_0^1 F''(x_{n+1} + t(y_{n+1} - x_{n+1}))(1-t)\, dt (y_{n+1} - x_{n+1})^2$$

we get

$$||F(y_{n+1})|| \leq \frac{1}{2}M||x_{n+1}-y_{n+1}||^2 \leq \frac{1}{2}M(s_{n+1}-t_{n+1})^2 = g(s_{n+1}).$$

Hence, (25) is also true.

We have also the estimates

$$\begin{aligned}||x_{n+1}-x_0|| &\leq ||x_{n+1}-y_0||+||y_0-x_0|| \leq ||x_{n+1}-y_n|| \\ &+ ||y_n-y_0|| + ||y_0-x_0|| \qquad (26)\\ &\leq \ldots \leq (t_{n+1}-s_n) + (s_n-s_0) + s_0 \leq t_{n+1} \leq r_1,\end{aligned}$$

and

$$\begin{aligned}||y_{n+1}-x_0|| &\leq ||y_{n+1}-y_0||+||y_0-x_0|| \leq ||y_{n+1}-x_{n+1}||+||x_{n+1}-y_n||+ \\ &||y_n-y_0||+||y_0-x_0||\end{aligned}$$

$$\leq \ldots \leq (s_{n+1}-t_{n+1})+(t_{n+1}-s_n)+(s_{n+1}-s_n)+(s_n-s_0)+s_0 \leq s_{n+1} \leq r_1. \quad (27)$$

Moreover, from (4), (6), (26) and the estimate

$$||F'(x_0)^{-1}||\ ||F'(x_n)-F'(x_0)|| \leq ||F'(x_0)^{-1}|| \int_0^1 ||F''(x_0+t(x_n-x_0))\ (x_n-x_0)\ dt||$$

$$\leq \beta M\ ||x_n-x_0|| \leq \beta M(t_n-t_0) \leq \beta M r_1 < 1, \qquad (28)$$

If follows from the Banach lemma on invertible operators that $F'(x_n)^{-1}$ exists and for all $n \geq 1$

$$||F'(x_n)^{-1}|| \leq \frac{||F'(x_0)^{-1}||}{1-||F'(x_0)^{-1}||.||F'(x_n)-F'(x_0)||} \leq \frac{\beta}{1-\beta M t_n} = -g'(t_n)^{-1} \quad (29)$$

Similarly, we obtain

$$||F'(y_n)^{-1}|| \leq -g'(s_n)^{-1} \quad \text{for all } n \geq 0. \qquad (30)$$

Hence, the iterates (2)-(3) are well defined for all $n \geq 0$.

It now follows that the sequence $\{x_n\}$ is Cauchy in a Banach space and as such it converges to some $x^* \in U(x_0, r_1)$, which by taking the limit as $n \to \infty$ in (2) becomes a zero of F since $F(x^*) = 0$. Moreover by (26) and (27) $x_n, y_n \in U(x_0, r_1)$. The estimates (16) and (17) follow easily from

$$||x_{n+1}-x_n|| \leq ||x_{n+1}-y_n|| + ||y_n-x_n|| \leq t_{n+1}-s_n+s_n-t_n=t_{n+1}-t_n$$

and

$$||y_{n+1}-y_n|| \le ||y_{n+1}-x_{n+1}|| + ||x_{n+1}-y_n|| \le s_{n+1}-t_{n+1}+t_{n+1}-s_n$$
$$= s_{n+1} - s_n \text{ for all } n \ge 0.$$

Moreover, using (7), (8) and (11) we get

$$r_1 - s_n = \frac{K\beta}{2(1 - K\beta t_n)}(r_1 - t_n)^2,$$

$$r_1 - t_{n+1} = \frac{K\beta}{2(1 - K\beta s_n)}(r_1 - s_n)^2,$$

and $$r_1 - t_{n+1} = \frac{K\beta}{2(1 - K\beta s_n)}(r_2 - s_n)^2.$$

Hence, we get

$$\frac{r_1 - t_{n+1}}{r_2 - t_{n+1}} = \left(\frac{r_1 - s_n}{r_2 - s_n}\right)^2 = \left(\frac{r_1 - t_n}{r_2 - t_n}\right)^4 = \dots = \left(\frac{r_1 - t_0}{r_2 - t_0}\right)^{4^{n+1}} = \theta^{4^{n+1}}. \quad (31)$$

Since,

$$r_2 - t_n = r_1 - t_n + (1 - \theta^2)\frac{\eta}{\theta} \quad (32)$$

the result (18) follows from (31) and (32).

To show uniqueness, let us assume that there exists another zero y^* of equation (1) in $U(x_0, r_2)$. Then from (5) and (24) we obtain

$$||F'(x^*)^{-1}|| \int_0^1 ||F'(x^* + t(y^* - x^*)) - F'(x^*)|| \, dt$$

$$\le \frac{\beta K}{1 - \beta K r_1} ||x^* - y^*|| \int_0^1 t \, dt \le \frac{1}{2}\frac{\beta K}{1 - \beta K r_1}(r_1 + r_2) < 1 \text{ by (12)}$$

(since $||x^* - y^*|| \le ||x^* - x_0|| + ||y^* - x_0|| \le r_1 + r_2$).

It now follows from the above inequality that the linear operator

$$\int_0^1 F'(x^* + t(y^* - x^*)) \, dt$$

is invertible. From this and the approximation

$$F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, dt\, (y^* - x^*)$$

it follows that $x^* = y^*$.

That completes the proof of theorem.

Remarks

(a) from the estimates

$$||x_n - y_0|| \leq ||x_n - y_n|| + ||y_n - y_0|| \leq t_n - s_n + s_n - s_0 \leq t_n - \eta \leq r_1 - \eta$$

and

$$||y_{n+1} - y_0|| \leq ||y_{n+1} - x_{n+1}|| + ||x_{n+1} - y_n|| + ||y_n - y_0||$$
$$\leq s_{n+1} - t_{n+1} + t_{n+1} - s_n + s_n - s_0 \leq s_{n+1} - \eta \leq r_1 - \eta.$$

It follows that $x_n, y_n \in U(y_0, r_1 - \eta)$ for all $n \geq 0$.

(b) We can use the two-point method to approximate nonlinear equations with nondifferentiable operators. Indeed consider the equations

$$F_1(x) = 0, \qquad (33)$$

where $F_1(x) = F(x) + Q(x)$

with F as before and Q satisfying an estimate of the form

$$||Q(x) - Q(y)|| \leq K_1 ||x - y|| \quad \text{for all } x \in D. \qquad (34)$$

Note that the differentiability of Q is not assumed here. Replace F in (2) and (3) by F_1 and leave the Frechet-derivatives as they are. Define the sequences $\{\bar{t}_n\}$ and $\{\bar{s}_n\}$ as the corresponding $\{t_n\}$ and $\{s_n\}$ given by (10) and (11) respectively. The only change will be an extra term of the form $K_1(s_n - t_n)$ and $K_1(t_{n+1} - s_n)$ added at the right hand sides of (10) and (11) respectively and mutliplied by the corresonding fractions. Define also g_1 as g in (12) but add the term $K_1 t$ at the numerator. Then following the proof of the above theorem step by step we can show a similar theorem with identical hypotheses and conclusions, but holding for equation (27). (See, also [4]).

(c) Similar theorems can be proved if relation (5) is replaced by a weaker Holder estimate of the form

$$||F'(x) - F'(y)|| \leq K \, ||x - y||^p \quad \text{for all } x, y \in D$$

and some $p \in [0, 1]$, [5].

(d) Note that using the approximation

$$\int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x^*)] \, dt = \int_0^1 \int_{x^*}^{x^* + t(y^* - x^*)} F''(z) \, dz$$

we can show that (15) can be replaced by $M\beta(3r_1 + r_2) < 2$, whih may be useful, especially when $M \leq K$.

(e) Uniqueness can also be established in the ball $U(y_0), r_2 - \eta)$. We just need to replace in the above proofs r_1 by $r_1 - \eta$ and r_2 by $r_2 - \eta$. Then our conditions become

$$\beta K (3r_1 + r_2 - 4\eta) < 1$$

or $\quad \beta M (3r_1 + r_2 - 4\eta) < 1.$

3. APPLICATIONS

In this section we use the theorem to suggest new approaches to the solution of quadratic integral equations of the form

$$x(s) = y(s) + \lambda x(s) \int_0^1 q(s, t) \, x(t) \, dt \tag{35}$$

in the space $E_1 = C[0, 1]$ of all functions continuous on the interval [0, 1], with norm

$$||x|| = \max_{0 \leq s \leq 1} x(s).$$

Here we assume that λ is a real number called the "albedo" for scattering and the kernel $q(s, t)$ is a continuous function of two variables s, t with $0 < s, t < 1$ and satisfying

(i) $0 < q(s, t) < 1, \; 0 \leq s, t \leq 1;$

(ii) $q(s, t) + q(t, s) = 1, \; 0 \leq s, t \leq 1.$

The function $y(s)$ is a given continuous function defined on [0, 1], and finally $x(s)$ is the unknown funciton sought in [0, 1].

Equations of this type are closely related with the work of S. Chandrasekhar [7], (Nobel prize of physics 1983), and arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gasses, [1], [2], [7].

There eixsts an extensive literature on equations like (35) under various assumtions on the kernel $q(s, t)$ and λ is a real or complex number. One can refer to the recent work in [1], [2] and the references there. Here we demonstrate that the theorem via the two-point Newton method (2)-(3) provides existence results for (35). Moreover the two-point Newton iterative method (2)-(3) converges faster to the solution than all the previous known ones. Furthermore a better information on the location of the solutions is given. Note that the computational cost is not higher than the corresponding one of previous methods.

For simplicity (without loss of generality) we will assume that

$$q(s, t) = \frac{s}{s + t} \quad \text{for all } 0 \leq s, t \leq 1.$$

Note that q so defined satisfied (i) and (ii) above.

Let us now choose $\lambda = .25$, $y(s) = 1$ for all $s \in [0, 1]$; and define the operator F on E_1 by

$$F(x) = \lambda x(s) \int_0^1 \frac{s}{s + t} x(t)\, dt - x(s) + 1.$$

Note that every zero of the equation $F(x) = 0$ satisfies the equation (35).

Set $x_0(s) = 1$, use the definition of the first and second Frechet-derivatives of the operator F to obtain using and the theorem,

$$K = M = 2\,|\lambda| \max_{0 \leq s \leq 1} \left| \int_0^1 \frac{s}{s + t}\, dt \right| = 2\,|\lambda| \text{ In}2 = .34657359,$$

$$\beta = \| F'(1)^{-1} \| = 1.53039421$$

$$\eta \geq \| F'(1)^{-1} F(1) \| \geq \beta \lambda \text{ In}2 = .265197107,$$

$$h = .140659011 < \frac{1}{2},$$

$$r_1 = .28704852, \quad r_2 = 3.4837317$$

$$\theta = .08239685$$

and $$2\beta K r_1 = .304497749 < 1,$$

which shows that x^* is unique in $U(x_0, r_1)$ and not in $U(x_0, r_2)$; since (15) is violated. (For detailed computations, see also [1], [2].)

Therefore according to the theorem equation (35) has a solution x^* and the two-point Newton method (2)-(3) converges to x^* faster than any other method used so far according to (14) and (16). (See also, [1], [2], [7]). Moreover the information on the location of the solution given here is better than the ones given before.

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ON NEWTON'S METHOD AND NONLINEAR OPERATOR EQUATIONS

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ABSTRACT

We use Newton's method to find "small" and "large" solutions of polynomial operator equations. Under a natural assumption we generalize Vietta relations in a Banach space.

KEY WORDS AND PHRASES: Bilinear operator, Newton's method.

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1. INTRODUCTION

Consider the quadratic equation

$$x = y + B(x, x) \tag{1.1}$$

in a Banach space X, where $y \in X$ is fixed and B is a bounded symmetric bilinear operator on X [7], [8].

Most of the results obtained up to now with the exception of the work in [6], guarantee the existence of a "small" solution which is unique in a sphere centered at $0 \in X$. The hypothesis is that

$$1 - 4\,||B||\,.\,||y|| > 0 \tag{1.2}$$

It is well known however that the real quadratic equation

$$x = c + dx^2$$

has two solutions x_1, x_2 which satisfy the Vietta relations

$$d(x_1 + x_2) = 1$$

$$dx_1\,x_2 = c \tag{1.3}$$

We prove that if there exists an $x_0 \in X$ such that

$$B(x_0) = 1 \tag{1.4}$$

where 1 is the identity on X and x_1 is a solution of (1.1) then

$$x_2 = x_0 - x_1$$

is a solution of (1.1) also and the Vietta relations (1.3) generalize in X as

$$B(x_1 + x_2) = 1$$

$$B(x_1, x_2) = y \quad (1.5)$$

An application when $X = \mathbf{R}^2$ is given. The results obtained here can easily be generalized to include polynomial equations of the form

$$x = M_0 + M_1(x) + M_2(x, x) + \dots + M_k(x, x, \dots, x)$$

where $M_j, j = 1, 2, \dots, k$ is a j-linear operator on X [7] and $M_0 \in X$ is fixed.

We now state the Newton-Kantorovich theorem as it appears in [5].

Theorem 1

Let X, Y be Banach spaces, $U \subset X$ and suppose $F : U \to Y$. Assume that on an open set $U_0 \subset U$. F is Frechet differentiable and that

$$||F'(x) - F'(y)|| \leq \ell ||x - y|| \quad x, y \in U_0$$

Moreover, assume that $\Gamma_0 = [F'(x_0)]^{-1}$ exists on all of Y for some $x_0 \in U_0$. Let $||\Gamma_0|| \leq b$ and $||\Gamma_0 F(x_0)|| \leq \eta$. Suppose $h = nb\ell \leq 1/2$ and set

$$r_1 = \frac{1}{b\ell}(1 - \sqrt{1 - 2h})$$

$$r_2 = \frac{1}{b\ell}(1 + \sqrt{1 - 2h})$$

and suppose $S = \{x / ||x - x_0|| \leq r_1\} \subset U_0$. Then the iterates

$$x_{k+1} = x_k - \Gamma_0 F(x_k) \quad k = 0, 1, 2, \dots,$$

are well-defined, lie in S and converge to a solution x^* of $F(x) = 0$ which is unique in $U_0 \cap \{x \mid ||x - x_0|| < r_2\}$.

Let P denote an operator on X defined by

$$P(x) = B(x, x) - x + y \quad (1.6)$$

2. EXISTENCE THEOREMS FOR THE QUADRATIC EQUATION

We are going to apply Theorem 1 to this equation and for this we need the Frechet derivative of P. This may be written as

$$P'(x) = 2B(x) - 1 \tag{2.1}$$

and we can easily find that

$$||P'(x) - P'(y)|| \leq 2||B|| \cdot ||x - y||$$

for all x, $y \in X$. This gives $2||B||$ as a bound for ℓ needed in Theorem 1.

We now prove the existence of a "small" solution of P.

Theorem 2

Let P be defined as in (1.6). Suppose $h = 2||B|| \cdot ||y|| \leq \frac{1}{2}$, and define

$$r_1 = \frac{1}{2||B||}(1 - \sqrt{1 - 4||B||\ ||y||})$$

$$r_2 = \frac{1}{2||B||}(1 + \sqrt{1 - 4||B|| \cdot ||y||}).$$

Then $P(x) = 0$ has a solution in the sphere $S_1 = \{x / ||x|| \leq r_1\}$ which is unique in the sphere $S_2 = \{x / ||x|| < r_2\}$.

We apply Theorem 1 to P with $x_0 = 0$. Then $b = 1$, $\eta = ||y||$ and $\ell = 2||B||$. The result now follows from Theorem 1.

We now prove the existence of a "large" solution of P.

Theorem 3

Let P be defined as in (1.6) and assume that there exists $x_0 \in X$ such that

$$B(x_0) = I$$

Suppose $\quad h = 2||B|| \cdot ||y|| \leq \frac{1}{2} \quad (2.2)$

and $$r_1 = \frac{1}{2||B||}(1 - \sqrt{1 - 4||B|| \cdot ||y||})$$

$$r_2 = \frac{1}{2||B||}(1 + \sqrt{1 - 4||B|| \cdot ||y||}).$$

Then $P(x) = 0$ has a solution in the sphere $S_1 = \{x \mid ||x - x_0|| \le r_1\}$ which is unique in the sphere $S_2 = \{x \mid ||x - x_0|| \le r_2\}$.

Proof

We have $P(x_0) = B(x_0, x_0) - x_0 + y = x_0 - x_0 + y = y$, $\Gamma_0 = [2B(x_0) - I]^{-1} = I$. Therefore $b = 1$, $\eta = ||y||$ and $\ell = 2||B|$. The result now follows from Theorem 1.

Theorem 4

Suppose that the hypotheses of Theorem 3 are satisfied, then the roots x_1, x_2 of P of Theorem 2 and 3 are such that

(a) $x_1 \neq x_2$ if (2.2) is a strict inequality

(b) the linear operator $B(x_2)$ is invertible

(c) the linear operator $B(x_1 - x_2)$ is invertible and

$$||(B(x_1 - x_2))^{-1}|| \le \frac{1}{\sqrt{1 - 4||B|| \cdot ||y||}}$$

Proof

(a) We have by (1.4)

$$1 = ||I|| = ||B(x_0)|| \le ||B|| \cdot ||x_0||$$

and $$2r_1 < \frac{1}{||B||} \le ||x_0||$$

Therefore $x_1 \neq x_2$.

(b) Now,

$$||I - B(x_2)|| = ||B(x_0 - x_2)|| \le ||B|| \cdot ||x_0 - x_2||$$

$$\le ||B|| \frac{1 - \sqrt{1 - 4||B|| \cdot ||y||}}{2||B||} < 1.$$

The Banach lemma then implies that $B(x_2)$ is invertible and

$$||B(x_2)^{-1}|| \leq \frac{1}{1-||I-B(x_2)||} \leq \frac{2}{1+\sqrt{1-4||B||\,.\,||y||}}$$

(c) We finally have using (b)

$$||B(x_1)B(x_2)^{-1}|| < ||B||.\frac{1-\sqrt{1-4||B||.||y||}}{2||B||}.\frac{2}{1+\sqrt{1-4||B||.||y||}}$$

$<1.$

The Banach lemma now implies that $B(x_1 - x_2)$ is invertible and

$$||B(x_1-x_2)^{-1}|| \leq ||B(x_2)^{-1}||\,.\,\frac{1}{1-||B(x_2)^{-1}B(x_1)||} \leq \frac{1}{\sqrt{1-4||B||.||y||}}$$

The theorem is now proved.

We note that the spectrum of $B(x_2)$ strictly dominates the spectrum of $B(x_1)$. If L is a linear invertible operator and $||L^{-1}|| \leq c$ then $\lambda \in \sigma(L)$ implies $\frac{1}{c} \leq |\lambda| \leq ||L||$. Then the bound on $||B(x_2)^{-1}||$ implies that if $\lambda \in \sigma(B(x_2))$, then

$$|\lambda| \geq \frac{1+\sqrt{1-4\,||B||\,.\,||y||}}{2||B||} \geq ||B(x_1)|| \geq \sup\{|k| : k \in \sigma(B(x_1))\}.$$

Proposition 1

Let x_0 be as in (1.4) and x_1, x_2 be solutions of (1.1) with

$$x_1 + x_2 = x_0$$

then

$$\left.\begin{aligned} B(x_1 + x_2) &= I \\ B(x_1, x_2) &= y \end{aligned}\right\}$$

The above are called Vietta relations in Banach space.

Proof

We have

$$B(x_1 + x_2) = B(x_0) = I$$

also $$x_1 + x_2 = 2y + B(x_1, x_1) + B(x_2, x_2)$$
$$= 2y + B(x_1 + x_2, x_1 + x_2) - 2B(x_1, x_2)$$
$$= 2y + x_1 + x_2 - 2B(x_1, x_2)$$

So $B(x_1, x_2) = y$

Proposition 2

Let x_1 be a solution of (1.1) then $x_2 = x_0 - x_1$ is a solution also.

Proof

We have

$$y + B(x_0-x_1, x_0-x_1) = y + B(x_0,x_0) - 2B(x_1,x_0) + B(x_1,x_1)$$
$$= y + x_0 - 2x_1 + B(x_1, x_1)$$
$$= y - y + (x_0 - x_1)$$
$$= x_2$$

Proposition 3

Let x_1, x_2, x_3, x_4 be solutions of (1.1) and x_0, x_0' be such that $B(x_0) = B(x_0') = I$. The following are true:

(a) If $x_1 + x_2 = x_0$

$x_2 + x_3 = x_0'$

then $x_1 = x_3$

and $x_0 = x_0'$

(b) If $x_1 + x_2 = x_0$

$x_3 + x_4 = x_0'$

$B(x_1 + x_3) = I$

then $x_2 = x_3$

$x_1 = x_4$

and $x_0 = x_0'$

(c) If $B(x_1, x_3) = y$

$x_1 + x_2 = x_0$

$x_3 + x_4 = x_0'$

and $B(x_3)$ is invertible, then

$x_1 = x_4$

$x_2 = x_3$

and $x_0 = x_0'$

(d) If $x_1 + x_2 = x_0$

$x_1 + x_2 + x_3 = x_0'$

then $x_3 = 0$

$y = 0$

$x_0 = x_0'$.

(e) h is a solution of $h = -y - B(h, h)$ if and only if $x_0 + h$ is a solution of $x = y + B(x, x)$.

(f) If $B(x) = 0$ implies $x = 0$ then $x_0 - x_0'$.

Proof

(a) We have $x_1 - x_3 = x_0 - x_0'$ or $B(x_1) = B(x_3)$

Now,

$$x_1 - y = x_1 - B(x_3, x_2) = B(x_3, x_1)$$

$$\Rightarrow \quad x_1 = B(x_3, x_1 + x_2) = B(x_0, x_3) = x_3$$

$$\Rightarrow \quad x_0 = x_0'.$$

(b) By hypothesis $B(x_1 + x_2) = B(x_1 + x_3) = I$

$$\Rightarrow \quad B(x_2) = B(x_3)$$

$$x_2 - y = B(x_3, x_2)$$

$\Rightarrow \quad x_2 = B(x_3, x_2) + B(x_3, x_1) = B(x_3, x_1 + x_2) = x_3$

$\Rightarrow \quad x_2 = x_3$ and by (a) $x_1 = x_4$ and $x_0 = x_0'$

(c) $B(x_1, x_3) = B(x_3, x_4)$

$\Rightarrow \quad B(x_3, x_1 - x_4) = 0$ and since $B(x_3)$ is

invertible $x_1 = x_4$ and by (a) $x_2 = x_3$ and $x_0 = x_0'$.

(d) We have,

$$0 = B(x_0' - x_0, x_0' - x_0) = B(x_0', x_0') - 2B(x_0', x_0) + B(x_0, x_0)$$
$$= x_0' - 2x_0 + x_0 = x_0' - x_0.$$

Similarly

$$0 = x_0 - x_0' \text{ so,}$$

$$\Rightarrow \quad x_0 = x_0' \Rightarrow x_3 = 0 \Rightarrow y = 0.$$

(e) If $x_0 + h$ is a solution of (1.1) then

$$\begin{aligned} \Leftrightarrow \quad x_0 + h &= y + B(x_0 + h, x_0 + h) \\ &= y + B(x_0, x_0) + 2B(x_0, h) + B(h,h) \\ &= y + x_0 + 2h + B(h,h) \end{aligned}$$

$$\Leftrightarrow \quad h = -y - B(h,h)$$

(f) We have $B(x_0) = I = B(x_0')$

$$\Rightarrow \quad B(x_0 - x_0') = 0 \Rightarrow x_0 = x_0'$$

Proposition 4

If $\quad x_1 + x_2 = x_0$

$\quad x_3 + x_4 = x_0'$

then $\quad x0 = x_0'$

Proof

We have

$$x_1 = y + B(x_1, x_1) = y + B(x_3 + x_4 - x_2 + z_1 - z_2, x_3 + x_4 - x_2 + z_1 - z_2)$$
$$= .... =$$

$$= x_2 + x_4 + x_2 - 2B(x_2, x_3 + x_4)$$

$$= x_3 + x_4 - x_2$$

$$\Rightarrow \quad x_1 + x_2 = x_3 + x_4 \Rightarrow x_0 = x_0'$$

3. APPLICATIONS

Example

Let X = R^2 and define a bilinear operator B on X by

$$B(w, v) = \left\{ (w_1, w_2) \begin{pmatrix} b_{111} & b_{112} \\ b_{121} & b_{122} \\ \hline b_{211} & b_{212} \\ b_{221} & b_{222} \end{pmatrix} \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} b_{111}W_1 + b_{121}W_2 & b_{112}W_1 + b_{122}W_2 \\ \\ b_{211}W_1 + b_{221}W_2 & b_{212}W_1 + b_{222}W_2 \end{pmatrix} \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} b_{111}W_1V_1 + b_{121}W_2V_1 + b_{112}W_1V_2 + b_{122}W_2V_2 \\ \\ b_{211}W_1V_1 + b_{221}W_2V_1 + b_{212}W_1V_2 + b_{222}W_2V_2 \end{pmatrix}$$

Consider the quadratic equation on X given by $z - y + B(z,z)$, where

$$y = \begin{pmatrix} y_1 \\ \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{48} \\ -\frac{1}{48} \end{pmatrix}, b_{111} = -3, b_{112} = 1, b_{121} = 1, b_{122} = -1$$

$b_{212} = -1, b_{211} = 1, b_{221} = -1$ and $b_{222} = -1$

or $$z_1 = \frac{1}{48} - 3z_1^2 + 2z_1z_2 - z_2^2$$

$$z_2 = -\frac{1}{48} + z_1^2 - 2z_1z_2 - z_2^2 \tag{3.1}$$

For a bilinear operator $B = (b_{ijk})$, $i, j, k = 1, 2, \ldots, m$ in $L(R_\infty^m, R_\infty^m)$, one as in [7] has

$$||B|| = \sup_{||x||=1} \max_{(i)} \sum_{j=1}^{m} \left| \sum_{k=1}^{m} b_{ijk} \xi_k \right|, \qquad (3.2)$$

where $x = (\xi_1, \xi_2, ..., \xi_m)$. We now apply (3.2) for $m = 2$ to easily obtain that

$$||B|| = 6.$$

Now, $1 - 4.||B||.||y|| = 1 - 4.||B||.||y|| = 1 - 4.6.\frac{1}{48} = \frac{1}{2} > 0.$

Therefore according to Theorem 2 there exists a unique solution $z \in S = \{x \mid ||x|| \leq r_1 = .02440776\}$. This solution can be found using Newton's iteration and it turns out to be $z^1 \in S$ such that

$$z_1' = .0200308$$

$$z_2' = .0200308$$

It is now easily seen that if $z = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ then $B(z) = 1$. By Proposition 1

$$z_1^2 = -.5 - .0200308 = -.5200308$$

$$z_2^2 = -.5 - .0200308 = -.5200308$$

The same solution can be found if we apply theorem for $x_0 = z$.

Finally, the other two solutions of (3.1) are

$$z_1^3 = -.25$$

$$z_2^3 = .1318813$$

and $$z_1^4 = -.25 = -.5 - z_1^3$$

$$z_2^4 = -.6318812 = -.5 - z_2^3$$

Also, note that $B(z^1, z^2) = y$, $B(z^3, z^4) = y$, $B(z^1 + z^2) = B(z^3+z^4) = I$.

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ON UNEQUAL PROBABILITY SAMPLING

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ABSTRACT

This paper explains the use of unequal probability sampling in finite populations Cochran (1977). Comparison among the well known selection procedures have been carried out. Variances of the estimated population total from all possible samples and exact variances have been calculated. Numerically it has been shown that for population where either the mean per element is proportional to y_i/p_i or rises as size increases with high correlation, the unequal probability sampling method is better than equal probability method.

KEY WORDS

Unequal probability sampling, selection procedures.

1. INTRODUCTION

In general, the sample survey involves sampling from finite population. There are two important components to any sampling plan. The first component consists of the selection procedure in which the sampling units are to be selected according to some well defined procedure. The second component covers methods of estimation which prescribes how inference to be made from the sample to the population as a whole. These inferences may be either enumerative or analytical.

Enumerative inference only to describe the particular finite population under study, analytical inference in same sense to explain it. Analytical and enumerative inferences proceed entirely different paths. For analytical inference the model used its own probability structure and for enumerative inference a quite different probability structure. The co-existence of these two methods of inference is a matter influences very sharply on samples drawn with unequal probability.

The use of unequal probability sampling was first suggested by Hansen & Hurwitz (1943). They demonstrated that the use of unequal selection probabilities within a stratum frequently made far more efficient estimators of total than did equal probability sampling. The probability of selecting ith unit is the sum of probabilities that it is selected either at first or the second draw.

2. BASIC THEORY

Hansen and Hurwitz (1943) presented the general theory for sampling with unequal probabilties and with relacement. They suggested an unbiased estimator for population Y

$$y'_{HH} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{p_i} \tag{2.1}$$

where p_i is the probability of selection of the ith unit.

The variance of y'_{HH} is

$$\text{Var}(y'_{HH}) = \frac{1}{n}\left(\sum_{i=1}^{N}\frac{Y^2{}_i}{P_i} - Y^2\right) \tag{2.2}$$

The variance estimator of (2.2) is

$$\text{var}(y'_{HH}) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(\frac{y_i}{p_i} - y'_{HH}\right)^2 \tag{2.3}$$

Horivtz and Thompson (1952) presented general theory for samling with unequal probabilities and without replacement. They suggested an unbiased estimator for population total Y

$$y'_{HT} = \sum_{i=1}^{n}\frac{y_i}{\pi_i}, \tag{2.4}$$

where π_i is the probability of inclusion of the ith unit in the sample. The variance of y'_{HT} is

$$\text{Var}(y'_{HT}) = \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\sum (\pi_i\pi_j - \pi_{ij})\left(\frac{Y_i}{\pi_i} - \frac{Y_j}{\pi_j}\right)^2 \tag{2.5}$$

The variance estimator of (2.5) is

$$var(y'_{HT}) = \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{n}\sum\left(\frac{(\pi_i\pi_j-\pi_{ij})}{\pi_{ij}}\right)\left(\frac{y_i}{\pi_i}-\frac{y_j}{\pi_j}\right)^2 \qquad (2.6)$$

For the purpose of comparison, Brewer's (1963) selection will be used for $n = 2$. Two special estimators are also included:

(i) Murthy's estimator: They used Yates and Grundy draw by draw procedure and suggested unordered estimator following the work by Raj (1956). For $n = 2$ viz.

$$t_{symm} = \frac{1}{2-p_1-p_2}\left[\frac{y_1}{p_1}(1-p_2)+\frac{y_2}{p_2}(1-p_1)\right] \qquad (2.7)$$

The variance of t_{symm} is

$$\mathrm{Var}(t_{symm}) = \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\sum p_ip_j\frac{1-P_i-P_j}{2-P_i-P_j}\left(\frac{y_i}{p_i}-\frac{y_j}{p_j}\right)^2 \qquad (2.8)$$

The variance estimator of (2.8) for $n = 2$ is

$$var(t_{symm}) = \frac{(1-p_1)(1-p_2)(1-p_1-p_2)}{(1-p_1-p_2)^2}\left(\frac{y_1}{p_1}-\frac{y_2}{p_2}\right)^2 \qquad (2.9)$$

(ii) Rao-Hartley-Cochran estimator: Rao, Hartley and Cochran suggested their own selection procedure and suggested an unbiased estimator for population total

$$y'_{RHC} = \sum_{i=1}^{n}\frac{y_{it}\,\pi_i}{p_{it}}, \qquad (2.10)$$

where p_{it} is the sample value of normal measure of size $P_{it}, \pi_i = \sum_{i=1}^{N_i} P_{it}$

and $\sum_{i=1}^{n}\pi_i = 1$. The variance of (2.10) is

$$\mathrm{Var}(y'_{RHC}) = \frac{n\sum_{i=1}^{n}N_i^2-N}{N(N-1)}\left[\sum_{i=1}^{n}\sum_{t=1}^{N_i}\frac{y^2_{it}}{nP_{it}}-\frac{Y^2}{n}\right] \qquad (2.11)$$

An unbiased estimator of (2.11) is

$$var(y'_{\text{RHC}}) = \frac{\sum_{i=1}^{n} N_i^2 - N}{N^2 \sum_{i=1}^{n} N_i^2} \sum_{i=1}^{n} \pi_i \left(\frac{y_{it}}{p_{it}} - y'_{\text{RHC}}\right)^2 \qquad (2.12)$$

3. COMPARISON

We will compare these method with simple random sample without replacement, Ratio to size without replacement and probability proportional to size with replacement. Their estimators and variances are not stated here as they are well known to the readers. For this purpose, the following three different artificial populations (Cochran 1977) are considered.

Population-A: In population-A the mean per element, which is proportional to Y_i/Z_i, is uncorrelated with relative size and correlation between mean per element and relative size is Zero.

Population-B: In this population the mean per element rises as the size increase. They are highly correlated and correlation is exactly one.

Population-C: In population "C" the unit total has little relation to the relative size. They have negative correlation which is −0.852.

Following populations were used. For simplicity, we took the same relaive measure of size.

Table 3.1

Relative size, Z_i	Population		
	A	B	C
0.1	0.3	0.3	0.7
0.1	0.5	0.3	0.6
0.2	0.8	0.8	0.4
0.3	0.9	1.5	0.9
0.3	1.5	1.5	0.6

Table 3.2

Estimator	Variance of the estimated population total					
	A^*	B^*	C^*	A^+	B^+	C^+
y'_{SRS}	1.575	2.715	0.248	1.575	2.715	0.248
y''(Ratio)	0.344	0.351	1.442	0.375	0.207	0.778
y'_{PPS}	0.300	0.300	1.650	0.400	0.240	1.480
y'_{HT}	0.300	0.300	1.650	0.246	0.248	1.252
y'_{symm}	0.304	0.304	1.540	0.267	0.236	1.130
y'_{RHC}	0.371	0.341	1.693	0.300	0.180	1.110

* *Using all possible samples.*

\+ *Using the variance expression.*

Table 3.2 shows the variance of these estimates for population A, B & C. The computation involves to generate all possible samples from different selection procedures to calculate the estimate of each sample and their corresponding variances.

CONCLUSION

First, consider the simple random sample estimator with equal probability without replacement. The variance of the estimates of population A and B are greater than the variances of the other methods. While in the case of population 'C' it has smaller variance than the others. So, we can say that in simple random sampling without replacement the estimate is found to be of poor precision for the population like A and B. While for the population of the type 'C' this estimate is found to be more precise.

The Ratio to size estimate without replacement has smaller variances for the population type A & B. And has greater variance for the population C. It performs slightly similar to the variance of unequal probability sampling method of estimating population total. One should be careful to use ratio to size estimate because it has bias for small sample size.

Probability proportional to size with replacement also gives the good estimate of the population total as compared to unequal probability sampling estimation. The method is preferable for small population size.

In the methods of estimation of population total using unequal probability sampling without replacement, the sample variances of estimated population total show that they are precise than other methods like simple random sampling, Ratio to size with replacement except population C (It is best estimated by simple random sampling without replacement). Among unequal probability sampling, the Brewer's method is a little ahead to the others in precision for population A and B. While in population C the Murthy's method is better than the other two.

In other words, it is better to use unequal probability methods of estimation of population total (without replacement) for population A and B having small strata, where $n = 2$ cluster of units has been drawn from each stratum. In the population C, in which the units total bears little relation to the sizes, simple random sampling with equal probabilities is much better.

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A CHARACTERISATION OF HADAMARD HYPERNETS

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ABSTRACT

A property of Hadamard hypernets is described. This property is shown to be enjoyed by all those Hadamard hypernets which are obtained from generalised Hadamard matrices. Conversely it is established that if a Hadamard hypernet with class size 3 has this property then it is obtained from a generalised Hadamard matrix.

INTRODUCTION

A generalised Hadamard matrix GH($m\mu$, G) is an $m\mu \times m\mu$ matrix H = $[h_{ij}]$ all of whose entries are from a group G of ordered m such that if $i \neq j$ then each element of G occurs exactly μ times as a difference $h_{ik}-h_{jk}$. If $m = 2$ then H is an ordinary Hadamard matrix.

A Hadamard hypernet H(m, μ), of class size m and index μ, is a symmetric affine resolvable design with $m^2\mu$ points and $m\mu$ points on each block such that its dual is also affine resolvable. Such designs have been studied in many equivalent forms: Hadamard systems in [5], symmetric nets in [1] and Hadamard hypernet in [3]. For surveys see [2] or [4].

If H = $[h_{ij}]$ is a GH($m\mu$, G) matrix, we can construct a Hadamard hypernet H(m, μ) as follows. Label points by pairs (α, i) and blocks by pairs [α, i] where $\alpha \in$ G and i = 1, 2, ..., $m\mu$. Define a point (α, i) and block [β, j] to be incident if and only if $\alpha\beta^{-1} = h_{ij}$. Then it is easy to verify that this incidence structure is a H(m, μ). However given a H(m, μ) there does not exist a general algorithm for constructing a GH matrix from it. In what follows we shall describe a property which must be satisfied by a H(m, μ) if it is at all possible to construct a GH matrix from it. Finally we shall show that it is always possible to construct a GH matrix from a H(3, μ) if it has this property.

A translation α of a H(m, μ) D is an automorphism of D which is either the identity or else it is fixed-point-free but fixes every block class as a set. Thus if α is a translation every block is parallel to its image under α and α fixes no point, unless $\alpha = 1$. If α also induces a translation of the dual of D then α is called a bitranslation of D. The bitranslations of D form a group Γ(D), of order at most m, which is normal in the full automorphism group.

If the order of Γ(D) is exactly m then Γ(D) is regular on each point class and each block class. That is, for any two parallel blocks (or points) there is a unique bitranslation mapping one onto the other. We shall call a H(m, μ) D to be class regular if order of Γ(D) is m.

Consider a class regular H(m, μ) D. Choose distinct representatives $\{p_1, p_2, ..., p_{m\mu}\}$ from the $m\mu$ point classes and $\{B_1, B_2, ..., B_{m\mu}\}$ from the $m\mu$ block classes. Then the matrix $H = [h_{ij}]$, where h_{ij} is the unique bitranslation in Γ(D) which maps p_i onto the unique parallel point incident with B_j, is a GH($m\mu$, G). Thus if a Hadamard hypernet is class regular then it is possible to construct a GH matrix from it.

THE Ω PROPERTY

A Hadamard hypernet is said to satisfy the Ω property if given any two non-parallel blocks X_1, Y_1 and any block X_2 parallel to X_1, there exists a unique block Y_2 parallel to Y_1 such that every point in $X_1 \cap Y_1$ is also parallel to a point in $X_2 \cap Y_2$.

Lemma 1

A H(m, μ) hypernet D has the Ω property if and only if the following property holds. Whenever X, X′ and Y, Y′ are two parallel block pairs from different block classes, then either every one or none of the points in $X \cap Y$ is parallel to a point in $X' \cap Y'$.

Proof

Suppose that D has the Ω property. Then there is a unique block Y_0 parallel to Y such that every point in $X \cap Y$ is parallel to a point in $X' \cap Y_0$. This implies that if $X \cap Y$ has a point, a say, parallel to a point a' in $X' \cap Y'$, then a' is in $X' \cap Y_0$ and in $X' \cap Y'$, since X' contains only one point parallel to a (namely a'). Thus the parallel blocks Y' and Y_0 have a common point, a', and so $Y' = Y_0$. The proof of the converse is straightforward.

Lemma 2

If an $H(m, \mu)$ has the Ω property then so does its dual.

Proof

Let x_1 and y_1 be parallel points and let x_2, y_2 be another pair of parallel points from a different parallel class. Also suppose that

$$x_1, x_2 \in A_1 \cap \ldots \cap A_\mu$$
$$\text{and} \qquad y_1, y_2 \in B_1 \cap \ldots \cap B_\mu$$

where A_1 is parallel to B_1. We shall prove that each A_i is parallel to some B_j; $i, j = 2, \ldots, \mu$.

Let C be the block parallel to A_i which contains the point y_1, and let

$$y_1, z_2, \ldots, z_\mu \in B_1 \cap C$$

Then, because of the Ω property, x_2 is parallel to z_k for some $k=2,\ldots,\mu$. Since y_2 and z_k are both parallel to x_2 and are both incident with B_1, we have $y_2 = z_k$. But $B_1, \ldots, B_\mu$ are the μ blocks containing y_1 and y_2. Hence $C = B_j$ for some $j = 2, \ldots, \mu$.

Lemma 3

Any $H(m, \mu)$ obtained from a $GH(m, \mu)$ has the Ω property.

Proof

For $\sigma \in G$ note that the mapping $\phi_\sigma : H(m, \mu) \rightarrow H(m, \mu)$ defined by

$$\phi_\sigma : (i, \alpha) \rightarrow (i, \alpha\sigma) \qquad \phi_\sigma : [i, \alpha] \rightarrow [i, \alpha\sigma]$$

is a bitranslation. Let A_1, A_2 and B_1, B_2 be two pairs of parallel blocks. Suppose that a point (i, α) in $A_1 \cap B_1$ is parallel to a point (i, β) in $A_2 \cap B_2$. Then clearly the bitranslation $\phi_{\alpha\beta}$ which maps (i,α) onto (i, β) maps every point in $A_1 \cap B_1$ onto a point in $A_2 \cap B_2$. Hence every point in $A_1 \cap B_1$ is parallel to some point in $A_2 \cap B_2$.

The following theorem establishes that if a Hadamard hypernet with class size 3 has the Ω property then it is obtained from a GH matrix.

Theorem

If a Hadamard hypernet with class size 3 satisfies the Ω property then each of its bitranslation is either of order 1 or 3.

Proof

Let $\{A_1, A_2, A_3\}$ be any block class of a $H(3, \mu)$hypernet D. Then the points of any point class $\{x_1, x_2, x_3\}$ can be labelled so that x_i is on A_i; $i = 1, 2, 3$.

Define a mapping f of the points of D onto themselves by $f(x_i) = x_{i+1}$; all suffixes being considered module 3. It is clear that f maps no points onto itself and maps any block onto a subset disjoint from the block. We shall show that f maps blocks onto blocks and, since f is clearly bijective, it will follow that f is an automorphism of D.

Let X be any block. We show that $f(X)$ is a block parallel to but different from X. If $X = A_i$, then clearly $f(X) = A_{i+1}$; so assume that X is not one of the A_i.

Since D has the Ω property, $f(X \cap A_i) = X_i \cap A_{i+1}$ $(i = 1, 2, 3)$ for some block X_i parallel to but different from X. Therefore, since there are only two possibilities for each X_i, X_1, X_2, X_3 are not all different. Suppose $X_2 = X_3$. We shall prove that $X_1 = X_2$.

If $X_1 \neq X_2$ then $\{X, X_1, X_2\}$ is a block class of D. We then have $f(X \cap A_1) = X_1 \cap A_2$, $f(X \cap A_2) = X_2 \cap A_3$ and $f(X \cap A_3) = X_2 \cap A_1$. Now $f(X_1 \cap A_2) = X' \cap A_3$ (by the Ω property), where X' $(\neq X_1)$ is parallel to X_1. But $X' \neq X_2$, because $f(X \cap A_2) = X_2 \cap A_3$ and $X \cap A_2$, $X_1 \cap A_2$ are disjoint since X and X_1 are distinct parallel blocks. Similarly $X' \neq X_3$ and so X' is non of X, X_1 and X_2; a contradiction. Hence $X_1 = X_2$. Therefore $f(X \cap A_i) = X_1 \cap A_{i+1}$ $(i = 1, 2, 3)$ and, since $Y = YA_1 \cup YA_2 \cup YA_3$ for any block Y, it follows that $f(X) = X_1$ where X_1 is parallel to but different from X.

It is now evident that f induces a bitranslation of order 3 of D. As the order of the group of bitranslations of D is at most 3, any of its bitranslation is either of order 1 or 3.

Corollary

A Hadamard hypernet $H(3, \mu)$ is obtained from a $GH(3\mu, G)$ if and only if it satisfies the Ω property.

Proof

As noted earlier the group $\Gamma(D)$ of bitranslations of a $H(m,\mu)$ hypernet D has order at most m. Since a $H(3, \mu)$ with Ω property has a bitranslation of order 3, the order of the group of bitranslations in this case is exactly 3. Now we can construct a $GH(3\mu, G)$ from this hypernet using the algorithm described at the end of the introduction. The converse follows from Lemma 2.

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A REALIABILITY PROBLEM OF A SYSTEM WITH SPARES AND REPAIR FACILITIES

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ABSTRACT

In this paper we consider a single unit system with N spares and K repair facilities. The system fails when the spares are emptied. It is assumed that repair tie and failure time have the exponential distributions with parameters μ and λ respectively. We are interested in determining the expected time to system failure, the long-run availability of the system and also to know the number of spares and repair facilities required to achieve a preassigned long-run availability for given values of λ and μ.

1. INTRODUCTION

Consider an equipment or system which consists of a single unit. When the unit fails one unit from the spare pool is used to repalce it immediately. The failed unit is sent to a repair facility for repair. The repaired unit is kept in the spare pool after repair completion. The system is considered failed when a unit failed and the spare is emptied. That is all the spares are in the repair facilities waiting for repair. The failure times of individual unit has an exponential distribution with parameter λ. The time taken to complete repair on a failed unit is also exponential. It is assumed that (i) the repairs are carried out in the order in which the units fail and that replacement of a failed unit with a spare one does not take any time and also there is no switch-over time.

2. EXPECTED AND VARIANCE OF TIME TO SYSTEM FAILURE

Let $N(t)$ be the number of failed units in the system at time t. The system fails when $N(t) = N$. Let $P_j(t) = P\{N(t) = j\}$. $N(t)$ is considered as continuous time discrete state stochastic process.

Let $$T_{ON} = T_{01} + T_{12} + \ldots\ldots + T_{N-1,N}$$

then, T_{ON} is the time to failure starting from state 0, and $T_{N-1,N}$ is the time the system will be working after it has jut been restored through a repair completion from failure. $\{T_{i,i+1}\}$ is a sequence of independent random variables and $E[T_{01}] = \lambda^{-1}$.

Let $f_{i,i+1}(t)$ be the passage time density from state i to state $i+1$.

Then $$f_{0,N}(t) = f_{0,1} * f_{12} * f_{23} * \ldots * f_{N-1,N}(t)$$

The time spent in any state i has an exponential distribution with parameter $\gamma_i = \lambda + \mu_i$, where μ_i is the repair rate when in state i and $f_{i,N}(t) = \lambda f_{i,N-1}(t)$. It is easily seen that

$$f_{i,i+1}(t) = \frac{\lambda}{\gamma_i}\gamma_i e^{-\gamma_i t} + \frac{i\mu}{\gamma_i}\gamma_i e^{-\gamma_i t} * f_{i-1,i}(t) * f_{i,i+1}(t)$$

Using Laplace transform, we have

$$g_{i,i+1}(s) = \frac{\lambda}{s + \lambda_i} + \frac{i\mu}{s + \gamma_i} g_{i-i,i}(s)\, g_{i,i+1}(s)$$

$$= \frac{\lambda}{s + i - \mu_i g_{i-1,i}(s)} \;;\; i \geq 1$$

differentiating, we have

$$E[T_{i,i+1}] = -g'_{i,i+1}(0)$$

$$= \frac{1}{\lambda}[1 + \mu_i E(T_{i-1,i})]$$

$$= \frac{1}{\lambda\theta_i}\sum_{j=0}^{i}\theta_j$$

where $$\theta_i = \frac{\lambda^i}{\prod_{j=1}^{i}\mu_j} = \prod_{j=1}^{i}\left(\frac{\lambda}{\mu_j}\right)$$

hence $$E[T_{ON}] = \sum_{i=0}^{N-1}\frac{1}{\lambda\theta_i}\sum_{j=0}^{i}\theta_j$$

If $k = 1$, the number of repair facilities,

$$\mu_i = \mu \text{ for all } i$$

$$\rho = \frac{\lambda}{\mu}; \theta_j = \rho^j$$

then $$E[T_{ON}] = \sum_{i=0}^{N-1} \frac{1}{\lambda\rho^i} \sum_{j=0}^{i} \rho^j$$

$$= \frac{\rho}{\lambda(1-\rho)} \left(\frac{1-\rho^{-N}}{\rho - 1} - N \right)$$

By differentiating $g_{i,i+1}(s)$ twice, it is seen that

$$\text{Var}(T_{i,i+1}) = \{E(T_{i-1,i})\}^2 + \frac{\mu_i}{\lambda} \text{Var}(T_{i-1,i})^2$$

hence $$\text{Var}(T_{ON}) = \sum_{i=0}^{N} \left(\{E(T_{i-1,i})\}^2 + \frac{\mu_i}{\lambda} \text{Var}(T_{i-1,i})^2 \right)$$

3. THE STATIONARY AVAILABILITY OF THE SYSTEM

The Stationary availability of the system is the probability that the system is working after a long period of time and is defined as P_N.

Where P_N = $\dfrac{\text{Expected time to system failure}}{\text{Expected time to system failure + Expected time to system reactivation after failure.}}$

So that

The stationary unavailability is given by

$1 - P_N$ = $\dfrac{\text{Expected time to system reactivation after failure}}{\text{Expected time to system failure + Expected time to system reactivation after failure.}}$

$$P_N = \frac{E[T_{M-1,N}]}{E[T_{N-1,N}] + \frac{1}{\mu_N}}$$

$$= \frac{\frac{1}{\lambda\theta_{N-1}}\sum_{j=0}^{N-1}\theta_j}{\lambda\theta_{N-1}\sum_{j=0}^{N-1}\theta_j + \frac{1}{\mu_N}} = \frac{\sum_{j=0}^{N-1}\theta_j}{\sum_{j=0}^{N-1}\theta_j}.$$

hence $$1 - P_N = \frac{\theta_N}{\sum_{j=0}^{N}\theta_j}.$$

When $k = 1$, the stationary unavailability is

$$\frac{\rho^N}{\sum_{j=0}^{N}\rho^j} = \frac{(1-\rho)\rho^N}{1-\rho^{N+1}} \text{ if } \rho \neq 1 \text{ and } \frac{1}{N+1} \text{ if } \rho = 1.$$

For various values of N, λ, μ and k (the number of repair facilities), a table of stationary availabilities can be constructed to reveal their relationship.

Hence a choice of additional number of spares and or additional repair facilities can be made to achieve the same stationary availability based upon their relative costs.

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A GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS III

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ABSTRACT

Let $T_p(a, b, A, B)$ denote the class of functions $f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z_{p+n}$ $(a_{p+n} \geq 0;\ p \in N = \{1, 2, ..., \})$ which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$ and satisfy the condition

$$\left| \frac{z^{1-p} f'(z) - p}{Bz^{1-p} f'(z) - [pB + (A - B)(p - a)]} \right| < b, z \in U,$$

for $0 \leq a < p$, $0 < b \leq 1$, $-1 \leq A < B \leq d$ and $0 < B \leq 1$. Further $f(z)$ is said to belong to the class $C_p(a, b, A, B)$ if and only if $p^{-1} z f'(z) \in T_p(a, b, A, B)$.

In this paper we obtain for these classes sharp results concerning coefficient estimates, distortion theorems, closure theores, modified Hadamard products and some distortion theorems for the fractional calculus.

Keywords: AMS (1991)

1. INTRODUCTION

Let S_p denote the class of functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \ (p \in N = \{1, 2, ... \}) \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z: |z| < 1\}$. We say that $f(z)$ belongs to the class $S_p(a, b, A, B)$ if $f(z) \in S_p$ satisfies the condition

$$\left|\frac{z^{1-p} f'(z) - p}{Bz^{1-p} f'(z) - [pB + (A-B)(p-a)]}\right| < b, z \in U \qquad (1.2)$$

for $0 \leq a < p$, $0 < b \leq 1$, $-1 \leq A < B \leq 1$ and $0 < B \leq 1$. Further $f(z)$ is said to belong to the class $K_p(a, b, A, B)$ if and only if $p^{-1}zf'(z) \in S_p(a, b, A, B)$. Recently Aouf [4] showed a distortion theorem, coefficient estimates and radius of convexity for the class $S_p(a,b,A,B)$.

In particular, $S_p(a, b, -1, 1) = S_p(a, b)$, is the class of functions $f(z) \in S_p$ satisfying the condition

$$\left|\frac{p^{-1} z^{1-p} f'(z) - 1}{p^{-1} z^{1-p} f'(z) + 1 - 2p^{-1} a}\right| < b, (z \in U, 0 \leq p^{-1}a < 1, 0 < b \leq 1) \qquad (1.3)$$

The class $S_p(a, b)$ was studied by Owa [13]. Also $S1(0, b, -1, 1)$ was studied by Padanbhan [18] and later by Caplinger and Causey [5]. Furthermore Owa [11] studied the class $S_1(a, b, -1, 1)$.

Let Tp denote the subclass of Sp consisting of functions analytic and p-valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (a_{p+n} \in 0; p = N) \qquad (1.4)$$

We denote by $T_p(a, b, A, B)$ and $C_p(a, b, A, B)$ the classes obtained by taking intersections of the classes $S_p(a, b, A, B)$ and $K_p(a, b, A, B)$ with T_p, respectively.

We note that $T_p(a, 1, A, B)$ is studied by Aouf [3] and $T_p(a, b, -1, 1)$ and $C_p(a, b, -1, 1)$, $0 \leq p^{-1} a < 1$, $0 < b \leq 1$, are studied by Owa [12]. Also $T_p(0, 1, A, B)$ was studied by Shukla and Dashrath [21], for $A = \alpha$ and $B = \beta$, $T_p(0, 1, \alpha, \beta)$ and $S_p(0, 1, \alpha, \beta)$, $-1 \leq \alpha < \beta \leq 1$, $0 < \beta \leq 1$, are studied by Owa and Srivastava [17].

In 1976. Gupta and Jain [8] studied the class $T_1(a, b, -1, 1)$. Moreover Silverman [22], Silveran and Silvia [23], [24], Ahuja and Jain [1] and Owa and Aouf [14], [15] have studied certain subclasses of univalent functions with negative coefficients. For other classes of analytic p-valent functions with negative coefficients, Goel and Sohi [7], Srivastava and Owa [26], [27] and Aouf [2] showed some results.

2. COEFFICIENT ESTIMATES

Theorem 1

A function $f(z)$ defined by (1, 4) is in the class $T_p(a, b, A, B)$ if and only if

$$\sum_{n=1}^{\infty} (p + n)(1 + bB)\, a_{p+n} \leq (B - A)\, b(p - a)$$

This result is sharp, the extremal function being

$$f(z) = z^p - \frac{(B - A)\, b(p - a)}{(p + n)(1 + bB)} z^{p+n}, \quad (n \geq 1) \tag{2.1}$$

The proof of Theorem 1 follows on the lines of the proof of Theorem 1 in [3]. The details are omitted.

Corollary 1

Let the function $f(z)$ defined by (1.4) be in the clazss $T_p(a, b, A, B)$. Then we have

$$a_{p+n} \leq \frac{(B - A)\, b(p - a)}{(p + n)(1 + bB)}, \qquad n \geq 1.$$

The result is sharp with the extremal function given in (2.1).

Theorem 2

A function $f(z)$ defined by (1.4) is in the class $C_p(a, b, A, B)$ if and only if

$$\sum_{n=1}^{\infty} (p + n)^2 (1 + bB)\, a_{p+n} \leq (B - A)\, b(p - a)\, p.$$

This result is sharp, the extremal function being

$$f(z) = z^p - \frac{(B - A)\, b(p - a)p}{(p + n)^2 (1 + bB)} z^{p+n} \ (n \geq 1). \tag{2.2}$$

Proof

The function $f(z)$ is in the class $C_p(a, b, A, B)$ if and only if $zf'(z)/p \in T_p(a, b, A, B)$. Now, since

$$\frac{zf'(z)}{p} = n^p - \sum_{n=1}^{\infty} \frac{p+n}{p} a_{p+n} z^{p+n},$$

by replacing a_{p+n} by $\frac{p+n}{p} a_{p+n}$ in Theorem 1, we have the theorem.

Corollary 2

Let the functin $f(z)$ defined by (1.4) be in the class $C_p(a,b,A,B)$. Then we have

$$a_{p+n} \leq \frac{(B-A)\, b(p-a)p}{(p+n)2\,(1+bB)}, \qquad n \geq 1.$$

The result is sharp with the extremal function given in (2.2).

3. DISTORTION THEOREMS

Theorem 3

Let a function $f(z)$ defined by (1.4) be in the class $T_p(a,b,A,B)$. Then we have

$$|f(z)| \geq |z|^p - \frac{(B-A)\, b(p-a)p}{(p+n)^2\,(1+bB)} |z|^{p+1}$$

and
$$|f(z)| \leq |z|^p + \frac{(B-A)\, b(p-a)p}{(p+1)^2\,(1+bB)} |z|^{p+1}$$

for $z \in U$. Further

$$|f'(z)| \geq p|z|^{p-1} - \frac{(B-A)\, b(p-a)}{(1+bB)} |z|^p$$

and
$$|f'(z)| \leq p|z|^{p-1} + \frac{(B-A)\, b(p-a)}{(1+bB)} |z|^p$$

for $z \in U$. These estimates are sharp and are attained for the function

$$f(z) = z^p - \frac{(B-A)\, b(p-a)}{(p+1)^2\,(1+bB)} z^{p+1}$$

The proof of Theorem 3 follows on the lines of the proof of Theorem 2 in [3]. The details are omitted.

Corollary 3

Under the hypotheses of Theorem 3, $f(z)$ is included in the disc with center at the origin and radius $1 + \dfrac{(B-A)\,b(p-a)}{(p+1)^2(1+bB)}$.

Further $f'(z)$ is included in the disc with center at the origin and radius $\dfrac{p + b\,[pB + (B-A)(p-a)]}{(1+bB)}$.

Theorem 4

Let a function $f(z)$ defined by (1.4) be in the class $C_p(a,b,A,B)$. Then we have

$$|f(z)| \geq |z|^p - \frac{(B-A)\,b(p-a)}{(p+1)^2(1+bB)}\,|z|^{p+1}$$

and
$$|f(z)| \leq |z|^p + \frac{(B-A)\,b(p-a)p}{(p+1)^2(1+bB)}\,|z|^{p+1}$$

for $z \in U$. Further

$$|f'(z)| \geq p|z|^{p-1} - \frac{(B-A)\,b(p-a)p}{(p+1)^2(1+bB)}\,|z|^p$$

and
$$|f'(z)| \geq p|z|^{p-1} - \frac{(B-A)\,b(p-a)p}{(p+1)^2(1+bB)}\,|z|^p$$

for $z \in U$. If $p \in N - \{1\}$, then we have

$$|f''(z)| \geq p(p-1)\,|z|^{p-2} - \frac{(B-A)\,b(p-a)p}{(1+bB)}\,|z|^{p-1}$$

and
$$|f''(z)| \leq p(p-1)\,|z|^{p-2} + \frac{(B-A)\,b(p-a)p}{(1+bB)}\,|z|^{p-1}$$

for $z \in U$. The estimates for $f(z)$ and $f'(z)$ are sharp and are attained for the function

$$f(z) = z^p - \frac{(B-A)\,b(p-a)p}{(p+1)^2(1+bB)}\,z^{p+1}$$

Proof

By using Theorem 2, we obtain

$$(p+1)^2(1+bB)\sum_{n=1}^{\infty} a_{p+n} \le \sum_{n=1}^{\infty} (p+n)^2\,(1+bB)\,a_{p+n} \le (B-A)\,b(p-a)p$$

which implies that

$$\sum_{n=1}^{\infty} a_{p+n} \le \frac{(B-A)\,b(p-a)p}{(p+1)^2\,(1+bB)}$$

Consequently we have

$$|f(z)| \ \ge\ |z|^p - \sum_{n=1}^{\infty} a_{p+n}\,|z|^{p+n}$$

$$\ge\ |z|^p - |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}$$

$$\ge\ |z|^p - \frac{(B-A)\,b(p-a)p}{(p+1)^2\,(1+bB)}\,|z|^{p+1}$$

and

$$|f(z)| \ \le\ |z|^p + \sum_{n=1}^{\infty} a_{p+n}\,|z|^{p+n}$$

$$\le\ |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}$$

$$\le\ |z|^p + \frac{(B-A)\,b(p-a)p}{(p+1)^2\,(1+bB)}\,|z|^{p+1}$$

for $z \in U$.

In order to show the second half of the theorem, by using

$$\sum_{n=1}^{\infty} (p+n)\,a_{p+n} \le \frac{(B-A)\,b(p-a)p}{(p+1)^2\,(1+bB)},$$

we obtain

$$|f'(z)| \ \ge\ p|z|^{p-1} - \sum_{n=1}^{\infty} (p+n)\,a_{p+n}\,|z|^{p+n-1}$$

$$\geq p|z|^{p-1} - |z|^p \sum_{n=1}^{\infty} (p+n)\, a_{p+n}$$

$$\geq p|z|^{p-1} \frac{(B-A)\, b(p-a)p}{(p+1)^2 (1+bB)} \; |z|^p$$

and $$|f'(z)| \leq p|z|^{p-1} + \sum_{n=1}^{\infty} (p+n)\, a_{p+n}\, |z|^{p+n-1}$$

$$\leq p|z|^{p-1} + |z|^p \sum_{n=1}^{\infty} (p+n)\, a_{p+n}$$

$$\leq p|z|^{p-1} + \frac{(B-A)\, b(p-a)p}{(p+1)(1+bB)} \; |z|^p$$

for $z \in U$. Furthermore, for $p \in N - \{1\}$ and $z \in U$, we have

$$|f''(z)| \leq p(p-1)\; |z|^{p-2} - \sum_{n=1}^{\infty} (p+n)(p+n-1)\, a_{p+n}\; |z|^{p+n-2}$$

$$\leq p(p-1)\; |z|^{p-2} - |z|^{p-1} \sum_{n=1}^{\infty} (p+n)^2\, a_{p+n}$$

$$\leq p(p-1)\; |z|^{p-2} - \frac{(B-A)\, b(p-a)p}{(1+bB)} \; |z|^{p-1}$$

and

$$|f''(z)| \leq p(p-1)\; |z|^{p-2} + \sum_{n=1}^{\infty} (p+n)(p+n-1)\, a_{p+n}\; |z|^{p+n-2}$$

$$\leq p(p-1)\; |z|^{p-2} + |z|^{p-1} \sum_{n=1}^{\infty} (p+n)^2\, a_{p+n}$$

$$\leq p(p-1)\; |z|^{p-2} + \frac{(B-A)\, b(p-a)p}{(1+bB)} \; |z|^{p-1}$$

by using theorem 2.

Corollaory 4

Under the hypotheses of Theorem 4, $f(z)$ is included in the disc with center at the origin and radius $1 + \frac{(B-A)\, b(p-a)p}{(p+1)^2\,(1+bB)}$, and $f'(z)$ is included in the disc with center at the origin and radius $p + \frac{(B-A)\, b(p-a)p}{(p+1)^2\,(1+bB)}$.. Further $f''(z)$ is included in the disc with center at the origin and radius $p(p-1) + \frac{(B-A)\, b(p-a)p}{(1+bB)}$.

4. SOME PROPERTIES OF THE CLASSES $T_p(a, b, A, B)$ AND $C_p(a, b, A, B)$

In this section we derive some useful properties of the classes $T_p(a, b, A, B)$ and $C_p(a, b, A, B)$ by employing Theorems 1 and 2.

Theorem 5

Let $-1 \le A_1 \le A_2 < B_1 \le B_2 \le 1$, $0 < B_1 \le B_2 \le 1$, $0 \le a_1 \le a_2 < p$ and $0 < b_1 \le b_2 \le 1$. The we have

$$T_p(a_1, b_2, A_1, B_2) \supset T_p(a_2, b_1, A_2, B_1)$$

Theorem 5 is an immediate consequence of the definition of $T_p(a, b, A, B)$.

Theorem 6

Let $-1 \le A_1 \le A_2 < B_1 \le B_2 \le 1$, $0 < B_1 \le B_2 \le 1$, $0 \le a_1 \le a_2 < p$ and $0 < b_1 \le b_2 \le 1$. Then we have

$$C_p(a_1, b_2, A_1, B_2) \supset C_p(a_2, b_1, A_2, B_1)$$

Proof

Let the function $f(z)$ defined by (1.4) be in the class $C_p(a_2, b_1, A_2, B_1)$, $B_2 = B_1 + \varepsilon$ and $b_2 = b_1 + \delta$. Then, by Theorem 2, we get

$$\sum_{n=1}^{\infty} (p+n)^2\,(1 + b_1 B_1)\, a_{p+n} \le (B_1 - A_2)\, b_1 (p - a_2)\, p.$$

Hence

$$\sum_{n=1}^{\infty} (p + n)^2 (1 + b_2 B_2)\, a_{p+n}$$

$$= \sum_{n=1}^{\infty} (p + n)^2 [1 + (b_1 + \delta)(B_1 + \varepsilon)]\, a_{p+n}$$

$$= \sum_{n=1}^{\infty} (p + n)^2 (1 + b_1 B_1)\, a_{p+n} + \delta B_1 \sum_{n=1}^{\infty} (p + n)^2 a_{p+n}$$

$$+ \varepsilon b_1 \sum_{n=1}^{\infty} (p + n)^2 a_{p+n} + \varepsilon\delta \sum_{n=1}^{\infty} (p + n)^2 a_{p+n}$$

$$\leq (B_1 - A_2)\, b_1 (p - a_2)\, p + \delta B_1 \frac{(B_1 - A_2)\, b_1 (p - a_2) p}{1 + b_1 B_1}$$

$$+ \varepsilon b_1 \frac{(B_1 - A_2)\, b_1 (p - a_2) p}{1 + b_1 B_1} + \varepsilon\delta \frac{(B_1 - A_2)\, b_1 (p - a_2) p}{1 + b_1 B_1}$$

$$\leq (B_1 - A_2)\, b_1 (p - a_2)\, p + \delta (B_1 - A_2)(p - a_2)\, p$$

$$+ \varepsilon b_1 (B_1 - A_2)(p - a_2) p + \varepsilon\delta (B_1 - A_2)(p - a_2) p$$

$$= (B_1 - A_2)(b_1 + \delta)(p - a_2)\, p + \varepsilon (B_1 - A_2)(b_1 + \delta)(p - a_2)\, p$$

$$= (B_1 - A_2)\, b_2 (p + a_2)\, p + \varepsilon (B_1 - A_2)\, b_1 (p - a_2)\, p$$

$$\leq (B_1 - A_2)\, b_2 (p + a_1)\, p$$

which gives that $f(z) \in C_p(a_1, b_2, A_1, B_2)$.

Theorem 7

Let a function $f(z)$ defined by (1.4) be in the class $C_p(a,b,A,B)$. Then $f(z)$ belongs to the classes:

(i) $T_p\left(\frac{p(1 + a)}{p + 1}, b, A, B\right)$, that is

$$C_p(a, b, A, B) \subset T_p\left(\frac{p(1 + a)}{p + 1}, b, A, B\right),$$

(ii) $T_p\left(a, b, \frac{B + pA}{p + 1}, B\right)$, that is

$$C_p(a, b, A, B) \subset T_p\left(a, b, \frac{B + p^A}{p + 1}, B\right)$$

Proof

To prove the first part (i), since $f(z) \in C_p(a, b, A, B)$, we have

$$\sum_{n=1}^{\infty} (p + n)\,(1 + bB)\,a_{p+n} \le \frac{(B - A)\,b\,(p - a)\,p}{p + 1}$$

$$= (B - A)\,b\left(p - \frac{p(1 + a)}{p + 1}\right)$$

with the aid of Theorem 2. Further

$$0 \le \frac{p(1 + a)}{p + 1} < p$$

for $0 \le a < p$ and $p \in N$. Conequently we have the first part (i) with Theorem 1.

Similarly, since $f(z) \in C_p(a, b, A, B)$, we have

$$\sum_{n=1}^{\infty} (p + n)\,(1 + bB)\,a_{p+n} \le \frac{(B - A)\,b\,(p - a)\,p}{p + 1}$$

$$= \left(B - \frac{B + pA}{p + 1}\right) b(p - a).$$

Note also that

$$-1 \le \frac{B + pA}{p + 1} < B \text{ for } -1 \le A < B \le 1,\ 0 < B \le 1$$

and for $p \in N$, and the second part (ii) follows at once.

5. CLOSURE THEOREMS

Let the functions $f_i(z)$ be defined, for $i = 1, \ldots, m$, by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{i,p+n}\, z^{p+n} \quad (a_{i,p+n} \ge 0; p \in N) \tag{5.1}$$

We shall prove the following results for the closure of functions in $T_p(a, b, A, B)$ and $C_p(a, b, A, B)$.

Theorem 8

Let the functions $f_i(z)$ defined by (5.1) be in the class $T_p(a_i, b_i, A_i, B_i)$, for $i = 1, \ldots, m$. Then the function $h(z)$ defined by

$$h(z) = z^p - \frac{1}{m} \sum_{n=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{i,p+n} \right) z^{p+n} \tag{5.2}$$

is in the class $T_p(a, b, A, B)$, where

$$a = \min_{1 \le i \le m} \{a_i\}, \; b = \min_{1 \le i \le m} \{b_i\}, \; A = \min_{1 \le i \le m} \{A_i\} \text{ and } B = \min_{1 \le i \le m} \{B_i\} \tag{5.3}$$

Proof

Since $f_i(z) \in T_p(a_i, b_i, A_i, B_i)$ for each $i = 1, \ldots, m$, we have

$$\sum_{n=1}^{\infty} (p + n)(1 + b_i B_i)\, a_{i,p+n} \le (B_i - A_i)\, b_i\, (p - a_i)$$

by Theorem 1. Hence we obtain

$$\sum_{n=1}^{\infty} (p + n) \left(\frac{1}{m} \sum_{n=1}^{\infty} a_{i,p+n} \right) = \frac{1}{m} \sum_{n=1}^{\infty} \left\{ \sum_{n=1}^{\infty} (p + n)\, a_{i,p+n} \right\}$$

$$\le \frac{1}{m} \sum_{n=1}^{\infty} \left\{ \frac{(B_i - A_i)\, b_i (p - a_i)}{1 + b_i B_i} \right\} \le \frac{(B - A)\, b\, (p - a)}{1 + bB},$$

because

$$\frac{(B_1 - A_1)\, b_1 (p - a_1)}{1 + b_1 B_1} \ge \frac{(B_2 - A_2)\, b_2 (p - a_2)}{1 + b_2 B_2}$$

for $a_1 \le a_2$, $b_1 \ge b_2$, $A_1 \le A_2$ and $B_1 \ge B_2$. Thus we get

$$\sum_{n=1}^{\infty} (p + n)(1 + bB) \left(\frac{1}{m} \sum_{n=1}^{\infty} a_{i,p+n} \right) \le (B - A)\, b\, (p - a),$$

which shows that $h(z) \in T_p(a, b, A, B)$, completing the proof of Theorem 8.

Theorem 9

Let the functions $f_i(z)$ defined by (5.1) be in the class $C_p(a_i, b_i, A_i, B_i)$, for each $i = 1, 2, \ldots, m$. Then the function $h(z)$ defined by (5.2) is in the class $C_p(a, b, A, B)$, where a, b, A, and B are defined by (5.3).

The proof of Theorem 9 is obtained by using the same technique as in the proof of Theorem 8 with the aid of Theorem 2.

Theorem 10

Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \quad (b_{p+n} \geq 0; \quad p \in N) \tag{5.4}$$

be in the classes $T_p(a, b, A, B)$ and $C_p(a, b, A, B)$, respectively. Then the function $k(z)$ defined by

$$k(z) = z^p - \left(\frac{p+1}{2p+1}\right) \sum_{n=1}^{\infty} [a_{p+n} + b_{p+n}] z^{p+n}$$

is in the class $T_p(a, b, A, B)$.

Proof

Since $f(z) \in T_p(a, b, A, B)$ and $g(z) \in C_p(a, b, A, B)$, by using Theorem 1 and Theorem 2, we get

$$\sum_{n=1}^{\infty} (p+n)(1+bB)\, a_{p+n} \leq (B-A)\, b\, (p-a)$$

and $$\sum_{n=1}^{\infty} (p+n)(1+bB)\, b_{p+n} \leq \frac{(B-A)\, b\, (p-a)\, p}{p+1}.$$

Therefore, we have

$$\left(\frac{p+1}{2p+1}\right) \sum_{n=1}^{\infty} (p+n)(1+bB)\, [a_{p+n} + b_{p+n}] \leq (B-A)\, b(p-a)$$

which implies that $k(z) \in T_p(a, b, A, B)$, and the proof of Theorem 10 is thus completed.

Theorem 11

Let

$$f_p(z) = z_p \qquad (p \in N)$$

and

$$f_{p+n}(z) = z^p - \frac{(B - A)\, b\, (p - a)}{(p + n)\,(1 + bB)}\, z^{p+n} \qquad (p \in N)$$

for $n = 1, 2, \ldots$.. Then $f(z)$ belongs to the class $T_p(a, b, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n}\, f_{p+n}(z),$$

where $\lambda_{p+n} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n} = 1.$

Theorem 12

Let $f_p(z) = z^p \qquad (p \in N)$

and

$$f_{p+n}(z) = z^p - \frac{(B - A)\, b\, (p - a)\, p}{(p + n)^2\,(1 + bB)}\, z^{p+n} \qquad (p \in N)$$

for $n = 1, 2, \ldots$. Then $f(z)$ belongs to the class $C_p(a, b, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n}\, f_{p+n}(z),$$

where $\lambda_{p+n} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n} = 1.$

Proofs of Theorems 11 and 12 follow on the lines of the proof of Theorem 8 in [3]. The details are omitted.

6. THEOREMS INVOLVING MODIFIED HADAMARD PRODUCTS

Let $f(z)$ be defined by (1.4), and let $g(z)$ be defined by (5.4). For the modified Hadamard product of $f(z)$ and $g(z)$ defined here by

$$f * g(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n} \qquad (6.1)$$

we first prove

Theorem 13

Let the function $f(z)$ defined by (1.4) and the function $g(z)$ defined by (5.4) be in the classes $T_p(a_1, b_1, A_1, B_1)$ and $T_p(a_2, b_2, A_2, B_2)$, respectively. Then the modified Hadamard product $f * g(z)$ defined by (6.1) belongs to the class $T_p(a(2p - a)/p, b, A(2B - A), B_2)$, where

$a = \min\{a_1, a_2\}$, $b = \max\{b_1, b_2\}$, $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$. (6.2)

Proof

Since $f(z) \in T_p(a_1, b_1, A_1, B_1)$ and $g(z) \in T_p(a_2, b_2, A_2, B_2)$, by using Theorem 1, we have

$$\sum_{n=1}^{\infty} (p + n)(1 + bB^2) a_{p+n} b_{p+n}$$

$$\leq \sum_{n=1}^{\infty} (p + n)(1 + bB) a_{p+n} b_{p+n}$$

$$\leq (B - A) b (p - a) \frac{(B_0 - A) b_0 (p - a) p}{(p + 1)(1 + b_0 B_0)},$$

$$\leq (B - A)^2 b \left(p - a\left(\frac{2p - a}{p}\right)\right)$$

$$= [B^2 - A(2B - A)] b \left(p - a\left(\frac{2p - a}{p}\right)\right),$$

where a, b, A and B are given by (6.2) and (for convenience $b_0 = \min \{b_1, b_2\}$ and $B_0 = \min \{B_1, B_2\}$.

Now observe that

$$-1 \le A(2B - A) < B^2 \le 1 \text{ and } 0 < B^2 \le 1$$

for $\quad -1 \le A < B \le 1$ and $0 < B \le 1$

and $\quad 0 \le a\left(\frac{2p - a}{p}\right) < p$

$$0 \le a < p.$$

Consequently the Hadamard product $f * g(z)$ is in the class $T_p\left(a\left(\frac{2p-a}{p}\right), b, A(2B - A), B^2\right)$ by Theorem 1.

In a similar manner we can prove

Theorem 14

Let the function $f(z)$ defined by (1.4) and the function $g(z)$ defined by (5.4) be in the classes $C_p(a_1,b_1,A_1,B_1)$ and $C_p(a_2,b_2,A_2,B_2)$, respectively. Then the modified Hadamard product $f * g(z)$ defined by (6.1) belongs to the class $C_p\left(a\left(\frac{2p-a}{p}\right), b, A(2B - A), B^2\right)$ where a, b, A and B are given by (6.2).

Theorem 15

Let the function $f(z)$ defined by (1.4) and the function $g(z)$ defiend by (5.4) be in the same class $T_p(a, b, A, B)$. Then the modified Hadamard product $f * g(z)$ defiend by (6.1) belongs to the class $C_p\left(a\left(\frac{2p-a}{p}\right), b, A(2B - A), B^2\right)$.

Proof

Since $f(z) \in T_p(a, b, A, B)$ and $g(z) \in T_p(a, b, A, B)$, by using Theorem 1, we have

$$\sum_{n=1}^{\infty} (p + n)^2 (1 + bB)^2 a_{p+n} b_{p+n}$$

$$\leq \sum_{n=1}^{\infty} (p + n)^2 (1 + bB)^2 a_{p+n} b_{p+n}$$

$$\leq (B - A)^2 b^2 (p - a)^2$$

$$\leq [B^2 - A(2B - A)] b \left[p - a\left(\frac{2p - a}{p}\right)\right]$$

Note also that

$$-1 \leq A(2B - A) < B^2 \leq 1 \text{ and } 0 < B^2 \leq 1$$

for $\quad -1 \leq A < B \leq 1$ and $0 < B \leq 1$

and $\quad 0 \leq a\left(\frac{2p - a}{p}\right) < p$

for $\quad 0 \leq a < p$

and Theorem 15 follows immediately.

Theorem 16

Let the function $f(z)$ defined by (1.4) and the function $g(z)$ defined by (5.4) be in the same class $T_p(a, b, A, B)$. Then the modified Hadamard product $f * g(z)$ defined by (6.1) belongs to the class $C_p\left(a\left(\frac{2p - a}{p}\right), b, \frac{(1 + 2bA)B - bA^2}{1 + bB}, B\right)$.

Proof

It follows from Theorem 1 that

$$\sum_{n=1}^{\infty} (p + n)^2 (1 + bB)^2 a_{p+n} b_{p+n} \leq \frac{(B + A)^2 b^2 (p - a)^2}{(1 + bB)}$$

$$= \left[\frac{(1 + 2bA)B - bA^2}{1 + bB}\right] b \left[p - a\left(\frac{2p - a}{p}\right)\right]$$

Observe also that

$$-1 \leq \frac{(1 + 2bA)B - bA^2}{1 + bB} < B \leq 1$$

for $\quad -1 \leq A < B \leq 1$ and $0 < B \leq 1$,

and the proof of Theorem 16 is completed.

7. FRACTIONAL CALCULUS

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature rather extensively (*cf.*, e.g., [6, Chapter 13], [9], [16], [19], [20], [25, p. 28 et seq.], and [28]. We find it to be convenient to restrict ourselves to the following definitions used recently by Owa [10] (and by Srivastava and Owa [26]).

Definition 1 (Fractional Integral Operator)

The fractional interal of order k is defined, for a function $f(z)$, by

$$D_z^{-k} f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\xi)\, d\xi}{(z-\xi)^{1-k}} \qquad (k > 0),$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z - \xi)^{k-1}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

Definition 2 (Fractional Derivative Operator)

The fractional derivative of order k is defined, for a function $f(z)$, by

$$D_z^{k} f(z) = \frac{1}{\Gamma(1-k)} \frac{d}{dz} \int_0^z \frac{f(\xi)\, d\xi}{(z-\xi)^{k}} \qquad (0 \le k > 1),$$

where $f(z)$ is constrained, and the multiplicity of $(z - \xi)^{-k}$ is removed, as in Definition 1.

Definition 3 (Extended Fractional Derivative Operator)

Under the hypotheses of Definition 2, the fractional derivative of order $n + k$ is defined, for a function $f(z)$, by

$$D_z^{n+k} f(z) = \frac{d^n}{dz^n} D_z^{k} f(z) \quad (0 \le k < 1;\, n \in \mathrm{N} \cup \{0\}),$$

where, as also in (1.1) and (1.4), N denotes the set of natural numbers.

Theorem 17

Let a function $f(z)$ defined by (1.4) be in the class $T_p(a,b,A,B)$. Then we have

$$|D_z^{-k} f(z)| \geq \frac{\Gamma(p+1)}{\Gamma(p+1+k)} |z|^{p+k} \left\{1 - \frac{1}{(p+1+k)} \cdot \frac{(B-A)\, b\, (p-a)\, |z|}{(1+bB)}\right\} \tag{7.1}$$

and

$$|D_z^{-k} f(z)| \geq \frac{\Gamma(p+1)}{\Gamma(p+1+k)} |z|^{p+k} \left\{1 + \frac{1}{(p+1+k)} \cdot \frac{(B-A)\, b\, (p-a)\, |z|}{(1+bB)}\right\} \tag{7.2}$$

for $0 < k < 1$ and $z \in U$. The bounds (7.1) and (7.2) are sharp and are attained by the function $f(z)$ defined by

$$|D_z^{-k} f(z)| \geq \frac{\Gamma(p+1)}{\Gamma(p+1+k)} z^{p+k} \left\{1 - \frac{1}{(p+1+k)} \cdot \frac{(B-A)\, b\, (p-a)}{(1+bB)}\right\},$$

or $$f(z) = z^p - \frac{(B-A)\, b\, (p-a)\, p}{(p+1)^2\, (1+bB)} z^{p+1}$$

The proof of Theorem 17 follows on the lines of the proof of Theorem 9 in [3]. The details are omitted.

Corollary 5

Under the hypotheses of Theorem 17. $D_z^{-k} f(z)$ is included in the disc with center at the origin and radius

$$\frac{\Gamma(p+1)}{\Gamma(p+1+k)} \left\{1 + \frac{1}{(p+1+k)} \cdot \frac{(B-A)\, b\, (p-a)\, p}{(1+bB)}\right\},$$

Using Theorem 2, we have

Theorem 18

Let a function $f(z)$ defined by (1.4) be in the class $C_p(a,b,A,B)$. Then we have

$$D_z^{-k} f(z) \geq \frac{\Gamma(p+1)}{\Gamma(p+1+k)} |z|^{p+k} \left\{1 - \frac{1}{(p+1+k)} \cdot \frac{(B-A)\,b\,(p-a)\,p}{(p+1)(1+bB)} |z| \right\}$$

and

$$D_z^{-k} f(z) \leq \frac{\Gamma(p+1)}{\Gamma(p+1+k)} |z|^{p+k} \left\{1 + \frac{1}{(p+1+k)} \cdot \frac{(B-A)\,b\,(p-a)\,p}{(p+1)(1+bB)} |z| \right\}$$

for $0 < k < 1$ and $z \in U$. The result is sharp for the function

$$f(z) = z^p - \frac{(B-A)\,b\,(p-a)\,p}{(p+1)(1+bB)}$$

rollary 6

Under the hypotheses of Theorem 18, $D_z^{-k} f(z)$ is included in the disc with center at the origin and radius

$$\frac{\Gamma(p+1)}{\Gamma(p+1+k)} \left\{1 + \frac{1}{(p+1+k)} \cdot \frac{(B-A)\,b\,(p-a)\,p}{(p+1)(1+bB)} \right\},$$

Theorem 19

Let a function $f(z)$ defined by (1.4) be in the class $C_p(a,b,A,B)$. Then we have

$$|D_z^{k} f(z)| \geq \frac{\Gamma(p+1)}{\Gamma(p+1-k)} |z|^{p-k} \left\{1 - \frac{1}{(p+1-k)} \cdot \frac{(B-A)\,b\,(p-a)\,p}{(p+1)(1+bB)} |z| \right\}$$

and

$$|D_z^{k} f(z)| \leq \frac{\Gamma(p+1)}{\Gamma(p+1-k)} |z|^{p-k} \left\{1 + \frac{1}{(p+1-k)} \cdot \frac{(B-A)\,b\,(p-a)\,p}{(p+1)(1+bB)} |z| \right\}$$

for $0 \leq k < 1$ and $z \in U$. Then result is sharp.

Proof

Let $$G(z) = \frac{\Gamma(p+1-k)}{\Gamma(p+1)} z^k D_z^k f(z)$$

$$= z^p - \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\,\Gamma(p+1-k)}{\Gamma(p+n+1-k)\,\Gamma(p+1)} a_{p+n} z^{p+n}$$

$$= z^p - \sum_{n=1}^{\infty} (p+n)\ B(n)\ a_{p+n} z^{p+n}$$

where

$$B(n) = \frac{\Gamma(p+n)\,\Gamma(p+1-k)}{\Gamma(p+n+1-k)\,\Gamma(p+1)} \qquad (n \geq 1).$$

Nothing

$$0 < B(n) \leq B(1) = \frac{1}{(p+1-k)}$$

with Theorem 2, we have

$$|G(z)| \geq |z|^p - B(1)\ |z|^{p+1} \sum_{n=1}^{\infty} (p+n)\, a_{p+n}$$

$$\geq |z|^p - \frac{1}{(p+1-k)} \cdot \frac{(B-A)\, b\, (p-a)\, p}{(p+1)\,(1+bB)}$$

and $$|G(z)| \leq |z|^p - B(1)\ |z|^{p+1} \sum_{n=1}^{\infty} (p+n)\, a_{p+n}$$

$$\leq |z|^p + \frac{1}{(p+1-k)} \cdot \frac{(B-A)\, b\, (p-a)\, p}{(p+1)\,(1+bB)} |z|^{p+1}$$

which give the inequalities of Theorem 19. Since equalities are attained for the function $f(z)$ defined by

$$D_z^k f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-k)} z^{p-k} \left\{1 - \frac{1}{(p+1-k)} \cdot \frac{(B-A)\, b\, (p-a)\, p}{(p+1)\,(1+bB)} z\right\}$$

that is, by

$$f(z) = z^p - \frac{(B-A)\, b\, (p-a)\, p}{(p+1)^2 (1+bB)} z^{p+1},$$

we complete the assertion of Theorem 19.

Corollary 7

Under the hypotheses of Theorem 19. $D_z^k f(z)$ is included in the disc with center at the origin and radius

$$\frac{\Gamma(p+1)}{\Gamma(p+1-k)} \left\{ 1 + \frac{1}{(p+1-k)} \cdot \frac{(B-A)\, b\, (p-a)\, p}{(p+1)(1+bB)} \right\}$$

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SOME FIXED POINT THEOREMS WITHOUT CONTINUITY

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ABSTRACT

In this paper we establish some common fixed point theorems without using the continuity for mappings in a normed space and in a metric space satisfying some contractive conditions.

1. INTRODUCTON

Let T be a self-mapping on a Banach space B. For any x_0 in B we shall consider the Mann iterates process $\{x_n\}$ [3] as:

$$x_{n+1} = (1 - c_n)\, x_n + c_n T x_n \text{ for } n \geq 0.$$

where $\{c_n\}$ satisfies

$$\text{(i) } c_0 = 1, \text{ (ii) } 0 < c_n < 1, \text{ for } n \geq 0, \text{ (iii) } \lim_{n\to\infty} c_n > 0.$$

As stated in [5], we assume the following requirement

$$\text{(iv) } \lim_{n\to\infty} \sup c_n > 0 \text{ instead of (iii).}$$

In 1988, H.K. Pathak [3] established the following two fixed point theorems:

Theorem A

Let K be a closed, convex subset of a normed space N and let T be a continuous self-mapping on K such that

$$\text{(I) } \|Tx-Ty\| \leq q \max \left\{ \|x-y\|, \frac{\|x-Tx\|\,[1-\|x-Tx\|]}{1+\|x-Tx\|}, \frac{\|x-Tx\|\,[1-\|x-Tx\|]}{1+\|x-Tx\|}, \right.$$

$$\left. \frac{\|Tx-y\|\,[1-\|y-Ty\|]}{1+\|Tx-y\|}, \frac{\|y-Ty\|\,[1-\|Tx-y\|]}{1+\|y-Ty\|} \right\}$$

for all x, y in K where $0 < q < 1$, $\{x_n\}$ the sequence of Mann iterates associated with T be same as in (1), where $\{c_n\}$ satisfy (i), (ii) & (iii), if $\{x_n\}$ converges in K, then it converges to a fixed point of T.

Theorem B

Let K be a closed, convex subet of a normed space N and let T_1 and T_2 be two continuous self maps on K such that

$$\text{(II)}\ \|T_1x-T_2y\| \le q \max\left\{ \|x-y\|, \frac{\|x-T_1x\|\,[1-\|x-T_2y\|]}{1+\|x-T_1y\|}, \frac{\|x-T_2y\|\,[1-\|x-T_1y\|]}{1+\|x-T_2y\|}, \frac{\|T_1x-y\|\,[1-\|y-T_2y\|]}{1+\|T_1x-y\|}, \frac{\|y-T_2y\|\,[1-\|T_1x-y\|]}{1+\|y-T_2y\|} \right\}$$

for all x, y in K where $0 < q < 1$, $\{x_n\}$ the sequence of Mann iterates associated with T_1 and T_2 are given below:

For $x_0 \in K$, set

$$\left.\begin{aligned} x_{2n+1} &= (1-c_n)\,x_{2n} + c_n T_1 x_{2n}, \\ x_{2n+2} &= (1-c_n)\,x_{2n+1} + c_n T_2 x_{2n+1} \text{ for } n = 0, 1, 2, \ldots\ldots \end{aligned}\right\}$$

where $\{c_n\}$ satisfies (i), (ii) and (iii). If $\{x_n\}$ converges to u in K and if u is a fixed point of either T_1 or T_2 then u is the common fixed point of T_1 and T_2.

In 1990, P.P. Murthy and H.K. Pathak [2], improved the result of Paliwal [4] and they proved the following theorem.

Theorem C

Let (X, d) be a metric space, T_1 and T_2 be self maps on X such that:

$$\text{(III)}\ d(T_1^r x, T_1^s y) \le \alpha \frac{d(x, T_1^r x)\, d(y, T_2^s y)}{d(x, y) + d(y, T_1^r x) + d(x, T_2^s y)} + \beta\, d(x, y).$$

for all x, y in X, $x \neq y$, where $r, s > 0$ are integers and α and β are non-negative real numbers such that $\alpha + \beta < 1$. If for $x_0 \in X$, the sequence $\{x_n\}$ consisting of the points.

$x_{2n+1} = T_1^r x_{2n}$, $x_{2n+2} = T_1^r x_{2n+1}$, has a subsequence $\{x_{n_k}\}$ converging to a point $u \in X$, then T_1 and T_2 have a unique common fixed point u.

In this paper we remove the continuity hypothesis of the mappings T and T_1, T_2 from Theorems A and B and we give examples to justify our theorems. Further we extend Theorem C to a more generalized contractive condition. Finally we prove a common fixed point theorem on a complete metric space.

2. MAIN RESULTS

Theorem 2.1

Let K be a closed, convex subset of a normed space N and let T be a self-mapping on K satisfying condition (I), $\{x_n\}$ the sequence of Mann iterates associated with T as in (1), where $\{c_n\}$ satisfy (i), (ii) and (iv). If $\{x_n\}$ converes to a point u in K, then u is a fixed point of T.

Proof

From (iv), we infer that

$\lim_{k\to\infty} c_{n_k} = p > 0$ for a subsequence $\{c_{n_k}\}$ of $\{c_n\}$.

Thus, we obtain

$$Tx_{n_k} = \frac{x_{n_k+1} - (1 - c_{n_k})\,x_{n_k}}{c_{n_k}} \to \frac{pu}{p} = u \text{ as } k \to \infty.$$

In (I) puting $x = x_{n_k}$, $y = u$ we get

$$||Tx_{n_k} - Tu|| \le q \max\left\{ ||x_{n_k} - u||, \frac{||x_{n_k} - Tx_{n_k}||\,[1 - ||x_{n_k} - Tu||]}{1 + ||x_{n_k} - Tx_{n_k}||}, \right.$$
$$\frac{||x_{n_k} - Tu||\,[1 - ||x_{n_k} - Tx_{n_k}||]}{1 + ||x_{n_k} - Tx_{n_k}||}, \frac{||Tx_{n_k} - u||\,[1 - ||u - Tu||]}{1 + ||Tx_{n_k} - u||}$$
$$\left. \frac{||u - Tu||\,[1 - ||Tx_{n_k} - u||]}{1 + ||u - Tu||} \right\}$$

As $k \to \infty$, we get

$$||u - Tu|| \leq q \max\left\{ 0, 0, \frac{||u - Tu||}{1 + ||u - Tu||}, 0, \frac{||u - Tu||}{1 + ||u - Tu||} \right\}$$

i.e. $$||u - Tu|| \leq q \frac{||u - Tu||}{1 + ||u - Tu||}$$

i.e. $$||u - Tu||^2 \leq -(1 - q) ||u - Tu|| \leq 0$$

Hence $u = Tu$, i.e., u is the fixed point of T.

Theorem 2.2

Let K be a closed, convex subset of a normed space N and let T_1 and T_2 be satisfies condition (II), $\{x_n\}$ the sequence of Mann iterates associated with T_1 and T_2 as given by (2) where $\{c_n\}$ satisfy (i), (ii) and (iv). If $\{x_n\}$ converges to u in K then u is the common fixed point of T_1 and T_2.

Proof

$\{x_n\}$ is a sequence of Mann iterates associated with T_1 and T_2 and converges to u.

From (iv), we infer that

$$\lim_{k\to\infty} c_{n_k} = p > 0 \text{ for a subsequence } \{c_{n_k}\} \text{ of } \{c_n\}.$$

Thus, we obtain

$$T_1 c_{2n_k} = \frac{x_{2n_k+1} - (1 - c_{n_k}) x_{2n_k}}{c_{n_k}} \to \frac{pu}{p} = u \text{ as } k \to \infty.$$

Similarly, one can show that $T_2 x_{2n_k+1} \to u$, as $k \to \infty$.

Put $x = u, y = x_{2n_k+1}$ in (II) we get

$$||T_1u - T_2x_{2n_k+1}|| \leq q\max\left\{ ||u - x_{2n_k+1}||, \frac{||u-T_1u||[1-||u-T_2x_{2n_k+1}||]}{1 + ||u_k - T_1u||}, \right.$$

$$\frac{||u-T_2x_{2n_k+1}||[1-||u-T_1u||]}{1 + ||u - T_2x_{2n_k+1}||}, \frac{||T_1u-x_{2n_k+1}||[1-||x_{2n_k+1}-T_2x_{2n_k+1}||]}{1 + ||T_1u-x_{2n_k+1}||}$$

$$\left. \frac{||x_{2n_k+1} - T_2x_{2n_k+1}|| [1 - ||T_1u - x_{2n_k+1}||]}{1 + ||x_{2n_k+1} - T_2x_{2n_k+1}||} \right\}$$

as $k \to \infty$, we get

$$||T_1u - u|| \le q \max\left\{ 0, \frac{||u - T_1u||}{1 + ||u - T_1u||}, 0, \frac{||T_1u - u||}{1 + ||T_1u - u||} \right\}$$

$$||T_1u - u|| \le q \frac{||u - T_1u||}{1 + ||u - T_1u||}$$

i.e. $\quad ||T_1u - u||^2 \le -(1 - q)\ ||u - T_1u|| \le 0$

Hence $T_1u = u$, i.e., u is the fixed point of T_1.

Similarly, we can prove that $T_2u = u$.

i.e. u is the common fixed point of T_1 and T_2. This completes the proof.

By suitable extension of Theorem (C) we establish the following theorem:

Theorem 2.3

Let T_1 and T_2 be two selfmaps on a metric space (x, d) such that

(IV) $$d(T_1^r x, T_1^s y) \le \alpha \frac{d(x, T_1^r x)\, d(y, T_2^s y)}{d(x, y) + d(x, T_2^s y) + d(y, T_1^r x)}$$

$$+ \beta \frac{d(x, y)\,[d(x, T_2^s y) + d(x, T_1^r y)]}{d(x, T_1^r x) + d(y, T_2^s y) + d(x, y)} + \gamma\, d(x, y)$$

for all $x, y \in X$, $x \ne y$ where r, s are positive integers and α, β and γ are non neagative real numbers such that

$$\alpha + \beta + \gamma < 1.$$

If for $x_0 \in X$, the sequence $\{x_n\}$ consisting of points

$$x_{2n+1} = T_1^r x_{2n},\ x_{2n+2} = T_2^s x_{2n+1}, \qquad (3)$$

has a subsequence $\{x_{n_k}\}$ converging to a point $u \in X$, then T_1 and T_2 have a unique common fixed point u.

Proof

Using condition (IV), we have

$$d(x_{2n+1}, x_{2n+2}) = d(T_1^r x_{2n}, T_2^s x_{2n+1})$$

$$\leq \alpha \frac{d(x_{2n}, T_1^r x_{2n})\, d(x_{2n+1}, T_2^s x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, T_2^s x_{2n+1}) + d(x_{2n+1}, T_1^r x_{2n})}$$

$$+ \beta \frac{d(x_{2n}, x_{2n+1})\, [d(x_{2n}, T_2^s x_{2n+1}) + d(x_{2n}, T_1^r x_{2n})]}{d(x_{2n}, T_1^r x_{2n}) + d(x_{2n+1}, T_2^s x_{2n+1}) + d(x_{2n}, x_{2n+1})}$$

$$+ \gamma\, d(x_{2n}, x_{2n+1})$$

$$= \alpha \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}$$

$$+ \beta \frac{d(x_{2n}, x_{2n+1})\, [d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1})]}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1})}$$

$$+ \gamma\, d(x_{2n}, x_{2n+1})$$

$$\leq (\alpha + \beta + \gamma)\, d(x_{2n}, x_{2n+1})$$

$$\therefore \quad d(x_{2n+1}, x_{2n+2}) \leq r d(x_{2n}, x_{2n+1}),$$

where $r = \alpha + \beta + \gamma < 1$.

Similarly, we can show that $d(x_{2n}, x_{2n+1}) \leq r d(x_{2n-1}, x_{2n})$ proceeding in the similar manner, we have

$$d(x_{2n+1}, x_{2n+2}) \leq r\, d(x_{2n}, x_{2n+1}) \leq r^2\, d(x_{2n-1}, x_{2n})$$

$$\leq r^{2n+1}\, d(x_0, x_1)$$

This shows that $\{x_n\}$ is a Cauchy sequence.

Since the subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ converges to u, then

$$\lim_{n\to\infty} x_n = u \in x$$

Hence, we have

$$d(T_1^r u, x_{2n_k}) = d(T_1^r u, T_2^s x_{2n_k-1})$$

$$\leq \alpha \frac{d(u,T_1^r u)\, d(x_{2n_k-1}, T_2^s x_{2n_k-1})}{d(u, x_{2n_k-1}) + d(u,T_2^s x_{2n_k-1}) + d(x_{2n_k-1},T_1^r u)}$$

$$+ \beta \frac{d(u, x_{2n_k-1})\, [d(u, T_2^s x_{2n_k-1}) + d(u, T_1^r u)]}{d(u, T_1^r u) + d(x_{2n_k-1},T_2^s x_{2n_k-1}) + d(u, x_{2n_k-1})} + \gamma\, d(u, x_{2n_k-1})$$

Letting $k \to \infty$, we get $d(T_1^r u, u) = 0$

Then $T_1^r u = u$. Similarly $T_2^s u = u$.

Hence T_1^r and T_2^s have a common fixed point $u \in x$. Now, for the uniqueness of u, if possible, let $v(v \neq u)$ be another common fixed point of T_1^r and T_2^s, then

$$d(u, v) = d(T_1^r u, T_2^s v) \leq \alpha \frac{d(u,T_1^r u)\, d(v, T_2^s v)}{d(u, v) + d(u,T_2^s v) + d(v, T_1^r u)}$$

$$+ \beta \frac{d(u, v)\, [d(u, T_2^s v) + d(u, T_1^r u)]}{d(u, T_1^r u) + d(v,T_2^s v) + d(u, v)} + \gamma\, d(u, v)$$

$$\leq (\beta + \gamma)\, d(u, v) < d(u, v)\ (\beta + \gamma < 1).$$

This is a contradiction. Hence u is the unique common fixed point of T_1^r and T_2^s.

Now, we prove that u is a fixed point of T_1 and T_2.

We have

$$T_1^r T_1 u = T_1^{r+1} u = T_1 T_1^r u = T_1 u$$

i.e. $T_1 u$ is a fixed point of T_1^r. But u is a unique fixed point of T_1^r.

$\therefore \quad T_1 u = u$, similarly $T_2 u = u$.

To prove the uniqueness of u, let $w(w \neq u)$ be another fixed point of T_1 and T_2. Then

$$d(u, w) = d(T_1u, T_2w) = d(T_1^r u, T_2^s w)$$

$$\leq \alpha \frac{d(u, T_1^r u)\, d(w, T_2^s w)}{d(u, w) + d(u, T_2^s w) + d(w, T_1^r u)}$$

$$+ \beta \frac{d(u, w)\,[d(u, T_2^s w) + d(u, T_1^r u)]}{d(u, T_1^r u) + d(w, T_2^s w) + d(u, w)} + \gamma\, d(u, w) \leq (\beta + \gamma)\, d(u, w)$$

i.e., $d(u, w) \leq (\beta + \gamma)\, d(u, w)$, $(\beta + \gamma < 1)$.

Which implies that $u = w$.

Hence u is the common fixed point of T_1 and T_2.

Remark 2.1

(i) If we put $\beta = 0$ in Theorem 2.3, we obtain Theorem C.

(ii) If we put $\beta = 0$, $T_1 = T_2$ and $r = s = 1$ in Theorem 2.3, we obtain a result of Jaggi and Dass [1].

Now, we give examples of some discontinuous mappings which are satisfying Theorem 2.1 and Theorem 2.2 and have a fixed points.

3. EXAMPLES

Example 3.1

Let N = R, the set of all real numbers regarded as a normed space. Let K = [0, 1] and define a mapping T of K into itself such that

$$Tx = \begin{cases} x/2 & , 0 \leq x < 1 \\ 0 & , x = 1 \end{cases}$$

It is clear that K = [0, 1] is closed and convex also T is discontinuous mapping on K and satisfies the condition (I) with $1/2 \leq q < 1$ and 0 is the unique fixed point of T.

Example 3.2

Let X = [0, 1], T_1 and T_2 be two self-maps on x let $r = s = 1$ and

$$T_1 x = \begin{cases} x/(x+3) & , 0 \leq x < 1/2 \\ x/6 & , 1/2 \leq x \leq 1 \end{cases}$$

and $$T_2 x = \begin{cases} y/(y+1) & , 0 \leq y < 1 \\ 1/6 & , y = 1 \end{cases}$$

both T_1 and T_2 are discontinuous and satisfies all conditions of theorem 2.2 also 0 is the common fixed point of T_1 and T_2.

Now we prove the following theorem on a complete metric space.

Theorem 2.4

Let E and F be two selfmaps on a coplete metric space (x, d) and there exists a positive integers $p(x)$ and $q(x)$ such that

$$\text{(v) } d(E^{p(X)} x, F^{q(Y)} y) \leq \alpha \frac{d(x, E^{p(x)} x)\, d(y, F^{q(y)} y)}{d(x, y) + d(x, F^{q(y)} y) + d(Y, E^{p(x)} x)}$$

$$+ \beta \frac{d(x, y)\, [d(x, F^{q(y)} y) + d(x, E^{p(x)} x)]}{d(x, E^{p(x)} x) + d(y, F^{q(y)} y) + d(x, y)} + \gamma\, d(x, y)$$

for all x, y in X, where α, β and γ are non-negative real numbers such that $\alpha + \beta + \gamma < 1$.

Then E and F have a unique common fixed point in X.

Proof

Let $x_0 \in X$, and define the sequence $\{x_n\}$ by

$$x_{2n+1} = E x_{2n}^{p(x_{2n})}, \quad x_{2n+2} = F x_{2n+1}^{q(x_{2n+1})}$$

If $x_{2n+1} = x_{2n+2}$, then $\{x_n\}$ is a Cauchy sequence.

Let $x_{2n+1} \neq x_{2n+2}$ for each $p \neq q$, then

$$d(x_{2n+1}, x_{2n+2}) = d(Ex_{2n}^{p(x_{2n})}, Ex_{2n+1}^{q(x_{2n+1})})$$

$$\leq \alpha \frac{d(x_{2n}, Ex_{2n}^{p(x_{2n})})\, d(x_{2n+1}, Fx_{2n+1}^{q(x_{2n+1})})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, Fx_{2n+1}^{q(x_{2n+1})}) + d(x_{2n+1}, Ex_{2n}^{p(x_{2n})})}$$

$$+ \beta \frac{d(x_{2n}, x_{2n+1})\, [d(x_{2n}, Fx_{2n+1}^{q(x_{2n+1})}) + d(x_{2n}, Ex_{2n}^{p(x_{2n})})}{d(x_{2n}, Ex_{2n}^{p(x_{2n})}) + d(x_{2n+1}, Fx_{2n+1}^{q(x_{2n+1})}) + d(x_{2n}, x_{2n+1})}$$

$$+ \gamma\, d(x_{2n}, x_{2n+1})$$

$$\leq (\alpha + \beta + \gamma)\, d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n+1}); (\alpha + \beta + \gamma < 1).$$

Thus, we have

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$$

Proceeding in the similar manner, we have

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n}) < \ldots < d(x_0, x_1)$$

$\{x_n\}$ is monotonically decreasing sequence which converges to a real number. Thus $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = u \in X$.

Now, we shall prove that $E^{p(u)}\, u = F^{q(u)}\, u = u$.

$$d(x_{2n+1}, F^{q(u)}\, u) = d(Ex_{2n}^{p(x_{2n})}, F^{q(u)}\, u)$$

$$\leq \alpha \frac{d(x_{2n}, Ex_{2n}^{p(x_{2n})})\, d(u, F^{q(u)}\, u)}{d(x_{2n}, F^{q(u)}\, u) + d(u, Ex_{2n}^{p(x_{2n})}) + d(x_{2n}, u)}$$

$$+ \beta \frac{d(x_{2n}, u)\, [d(x_{2n}, F^{q(u)}\, u) + d(x_{2n}, Ex_{2n}^{p(x_{2n})})]}{d(x_{2n}, Ex_{2n}^{p(x_{2n})}) + d(u, F^{q(u)}\, u) + d(x_{2n}, u)} + \gamma\, d(x_{2n}, u)$$

Letting $n \to \infty$, we have $d(u, F^{q(u)}\, u) = 0$

i.e. $u = F^{q(u)}$. Similarly $E^{p(u)}\, u = u$.

Hence $E^{p(u)}$ and $F^{q(u)}$ have a common fixed point $u \in X$.

Let v also be a periodic point of E and F

i.e. $E^{p(v)}\, v = F^{q(v)}\, v = v$,

then $d(u, v) = d(E^{p(u)} u, F^{q(v)} v)$

$$\leq \alpha \frac{d(u, E^{p(u)} u)\, d(v, F^{q(v)} v)}{d(u, F^{q(v)} v) + d(v, E^{p(u)} u) + d(u, v)}$$

$$+ \beta \frac{d(u, v)\,[d(u, F^{q(v)} v) + d(u, E^{p(u)})]}{d(u, E^{p(u)} u) + d(v, F^{q(v)} v) + d(u, v)} + \gamma d(u,v) = (\beta + \gamma) d(u,v)$$

$\Rightarrow \quad d(u, v) < d(u, v) \cdot (\beta + \gamma < 1)$

a contradiction. Which implies $u = v$.

Now, we prove that u is a fixed point of E and F

since $u = E^{p(u)} u \Rightarrow E^{p(u)}\, Eu = E\, E^{p(u)}\, u = E\, u$.

Thus E u is a periodic point of E.

Now from the uniqueness of u, $Eu = u$.

Similarly $Fu = u$. Hence u is a unique common fixed point of E and F in X. This ends the proof.

Remark 2.2

If we put $\beta = 0$ in theorem 2.4 we get Theorem 2 of Murthy and Pathak [2].

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FIXED POINTS OF MAPPINGS IN L-SPACES

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ABSTRACT

In this paper, we continue investigating fixed points of mappings in L-spaces.

1. INTRODUCTION

In 1975, Kasahara ([8], [9]) gave a very useful treatment for finding fixed points in general spaces (called L-spaces) in which all the metric properties, in particular, the triangle inequality, are not used. So far a few results have appeared in the literature (see K. Iseki [6]). In [1], We presented some results on L-spaces using root-condition (a contractive condition by M.S. Khan [7]).

In this paper, we continue investigating some more results of fixed points of mappings in L-spaces. Throughout X denotes a d-complete L-space [6], where d is non-negative real valued function defined on $X \times X$ and d satisfieds : $d(x, y) = 0$ implies $x = y$, unless otherwise stated. For the preliminaries, we refer to K. Iseki [6] and Kasahara ([8], [9]).

Let $T{:}X \to X$ be a mapping such that for each x in X $T^{n_i}x \to a$ implies $T(T^{n_i}x) \to Ta$. Then T is called an orbitrally continuous mapping [2].

K. Iseki ([4], [5]) proved some results concerning fixed points of certain mappings in metric spaces. We prove theorem 1 and 2 using similar contractive conditions respectively and obtain fixed point theorems in d-complete L-spaces.

Theorem 1

If S and T are orbitrally continuous self maps on X such that for all x, y in X

$$d(Sx, Ty) \leq a\, d(Sx, x) + b\, d(Ty, y) + c\, d(x, y)$$

where $a, b, c > 0$ and $a + b + c < 1$, then S and T have a common fixed point.

Proof

Let x_0 in X be arbitrary. Define a sequence (x_n) in X as:

$x_{2n+1} = Sx_{2n}, x_{2n} = Tx_{2n-1}$. Then

$$d(x_{2n+1}, x_{2n}) = d(Sx_{2n}, Tx_{2n-1})$$

$$\leq a\, d(Sx_{2n+1}, x_{2n}) + b\, d(Tx_{2n-1}, x_{2n-1}) + c\, d(x_{2n}, x_{2n-1})$$

$$= a\, d(x_{2n+1}, x_{2n}) + b\, d(x_{2n}, x_{2n-1}) + c\, d(x_{2n}, x_{2n-1}) \text{ or}$$

$$d(x_{2n+1}, x_{2n}) \leq q_1\, d(x_{2n}, x_{2n-1}), \text{ where } q_1 = \frac{b + c}{1 - a} < 1$$

Similarly $\quad d(x_{2n}, x_{2n-1}) \leq q_2 d(x_{2n-1}, x_{2n-2})$, where $q_2 = \dfrac{a + c}{1 - b} < 1$

In general, $d(x_{n+1}, x_n) \leq q d(x_n, x_{n-1}), q = \max\{q_1, q_2\} < 1$ or

$d(x_{n+1}, x_n) \leq q^n d(x_1, x_0)$. Consequently, $\Sigma\, d(x_{n+1}, x_n) < \infty$. From the d-completeness, $\{x_n\}$ has a limit a in X, that is $\lim_{n\to\infty} x_n = a$. Since S is orbitrally continuous, we have $\lim_{n\to\infty} x_{2n} = a$ implies $\lim_{n\to\infty} Sx_{2n} = Sa$. This gives $Sa = a$. Similarly, we have $Ta = a$. This completes the proof.

Theorem 2

If T is an orbitrally continuous self mapping on X such that

$d(Tx, Ty) \leq a\, [d(y, Ty)\, \{1 + d(x, Tx)\}/1+d(x, y)] + b\, \{d(x, Tx) + d(y, Ty)\} + cd(x, y)$ for all x, y in X, $a, b, c > 0$, $a + 2b + c < 1$, then T has a fixed point.

Proof

Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X as: $x_{n+1} = Tx_n$. Then

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a\ [d(x_n, Tx_n)\ \{1 + d(x_{n-1}, Tx_{n-1})\}/1+ d(x_{n-1}, x_n)] + b\ \{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) + c\ d(x_{n-1}, x_n)$

$= a[d(x_n, x_{n+1})\ \{1 + d(x_{n-1}, x_n)\}/1 + d(x_{n-1}, x_n)] + b\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} + cd(x_{n-1}, x_n)$ or $d(x_n, x_{n+1}) \leq q\ d(x_{n-1}, x_n)$, where $q=b+c/1-a-b < 1$.

Similarly, $d(x_{n-1}, x_n) \leq q\ d(x_{n-2}, x_{n-1})$. Continuing till we have

$d(x_n, x_{n+1}) \leq ... \leq q^n d(x_0, x_1)$ or $d(x_n, x_{n+1}) \leq q^n d(x_0, x_1)$. Consequently $\Sigma d(x_n,x_{n+1})<\infty$. By the d-completeness of $X,\{x_n\}\to a$. Since T is orbitrally continuous, therefore, $\lim_{n\to\infty} x_n = a$ implies $\lim_{n\to\infty} Tx_n = Ta$. This gives that $Ta = a$. This completes the proof.

If $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing and upper semi-continuous from the right with $\phi(t) < t$, $t > 0$, and $\phi(0) = 0$, then J. Matkowski [10] proved that $\phi^n(t) \to 0$ as $n \to \infty$.

Hicks [3] proved theorem 1 in which he established the necessary and sufficient conditions for a certain mapping to have a unique fixed point in complete metric spaces. We generalize this in d-complete L-spaces as:

Theorem 3

Let S, T be orbitrally continuous self maps of X such that $d(Sx, TSy) \leq \phi d(x, Sy)$, for all x, y in X. Then S and T have a common fixed point.

Proof

Let x_0 be an arbitrary point of X. Define a sequence $\{x_n\}$ in X as: $x_{2n+1} = Sx_{2n}, x_{2n} = Tx_{2n-1}$. Then

$d(x_{2n+1},x_{2n}) = d(Sx_{2n},TSx_{2n-2}) \leq \phi\ d(x_{2n},Sx_{2n}-2) = \phi\ d(x_{2n},x_{2n-1})$ or

$d(x_{2n+1},x_{2n}) \leq \phi\ d(x_{2n},x_{2n-1})$. Similarly

$d(x_{2n}, x_{2n-1}) \leq \phi\ d(x_{2n-1}, x_{2n-2})$. In general, we have

$d(x_{n+1}, x_n) \leq \phi\ d(x_n, x_{n-1})$. Continuing till we have

$d(x_{n+1}, x_n) \leq \phi^n d(x_1, x_0)$. Using lemma [5], we conclude that

$\Sigma\ d(x_{n+1}, x_n) < \infty$. Then by d-completeness of $X_n, \{x_n\} \to x$ in X

Suppose S is orbitrally continuous, then $\lim_{n\to\infty} x_{2n} = x$ gives $\lim_{n\to\infty} Sx_{2n} = Sx$ or $x = Sx$. Similarly we can have $x = Tx$. This completes the proof.

M.S. Khan [7] proved a result (see theorem 2) concerning a fixed point of a self mapping in metric spaces, we generalize this result in L-spaces. Moreover, this theorem classifies L-spaces in which each x in X is a fixed point.

Theorem 4

Let X be a L-space and T be a self map of X. Let $F : X\times X \to R_+$ be a mapping such that $F(x, y) = 0$ iff $x = y$ and

$F(Tx, Ty) \geq \{F(x, Tx)\, F(y, Ty)\}^{1/2}$, for all x, y in X. Then each x in X is a fixed point of T.

Proof

Let x in X be arbitrary. Then

$0 = F(Tx, Tx) \to \{F(x, Tx)\, F(x, Tx)\}^{1/2} = F(x, Tx)$ or $F(x, Tx) = 0$ implies $x = Tx$. This proves that each x in X is a fixed point of T.

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FIXED POINT THEOREMS IN SAKS SPACES

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ABSTRACT

In this paper we continue investigating some more fixed point theorems in Saks sapces.

INTRODUCTION

Okada [12], Singh-Virendra [24], Kulshrestha [9], and Naimpally, Singh, Whitefield [11] extended Goebel's coincidence theorem [4] to L-spaces, 2-metric spaces, metric spaces and multivalued contraction mappings on metric spaces, respectively. Park [17] generalized Goebel's coincidence theorem using Meir-Keeler's contraction condition [10]. Later Jungck's theorem [5] was improved by Singh [19] for a pair of commuting mappings satisfying some conditions to have a unique fixed point. Then results regarding a triplet and a quadruplet of mappings were also established (see [1], [3], [6]-[9], [17]-[25]).

In 1986, Y.J. Cho et-al [2] proved a coincidence theorem for three mappings on an arbitrary set having values in Saks spaces and derived a fixed point theorem for a triplet of mappings not necessarily continuous. These results are, in fact, extension of findings of Khan [7], Singh-Singh [23], Yeh [25].

In this paper, we present some more fixed point theorems in Saks spaces satisfying an SR-condition (a condition due to M.S. Khan [7]) and rational inequalities. Theorems 2-5 also give classification of Saks spaces in which each x in X is a fixed point. For classification of complete metric spaces and fixed point theorems satisfying a rational inequality, we refer to [8] and [18], respectively.

In [14] Orlicz has proved the following lemma:

Lemma

Let $(X_s, d) = (X, N_1, N_2)$ be a Saks space. Then the following statements are equivalent:

(1) N_1 is equivalent to N_2 on X.

(2) (X, N_1) is a Banach space and $N_1 \leq N_2$ on X.

(3) (X, N_2) is a Frechet space and $N_2 \leq N_1$ on X.

For the preliminaries and general information of Saks spaces, we refer to [2], [13]-[15].

The following theorem gives a fixed point of a sequence of self maps in Saks sapces satisfying an SR-condition.

Theorem 1

Let $(X_s, d) = (X, N_1, N_2)$ be a Saks space with N_1 equivalent to N_2 on X. Let $\{S_n\}$ and $\{T_n\}$ be sequences of self maps such that for all x, y in X and for every positive integers m, n, $0 < h < 1$, we have

$$N_2(S_m x - T_n y) \leq h\{N_2(S_m x - x)\, N_2(T_n y - y)\}^{1/2}$$

Then $\{S_n\}$ and $\{T_n\}$ have a unique common fixed point.

Proof

Define a sequence $\{x_n\}$ in X as: $x_{2n} = T_n x_{2n-1}$, $x_{2n+1} = S_{n+1} x_{2n}$. Then

$$N_2(x_{2n+1} - x_{2n}) = N_2(S_{n+1}x_{2n} - T_n x_{2n-1})$$

$$\leq h\{N_2(S_{n+1}x_{2n} - x_{2n})\, N_2(T_n x_{2n-1} - x_{2n-1})\}^{1/2}$$

$$= h\{N_2(x_{2n+1} - x_{2n})\, N_2(x_{2n} - x_{2n-1})\}^{1/2}$$

or $N_2(x_{2n+1} - x_{2n}) \leq q N_2(x_{2n} - x_{2n-1})$, $q = h^2 < 1$. Similarly $N_2(x_{2n} - x_{2n-1}) \leq q N_2(x_{2n-1} - x_{2n-2})$. In general $N_2(x_{n+1} - x_n) \leq q^n N_2(x_1 - x_0)$. Since $N_1 \leq N_2$, this implies $N_1(x_{n+1} - x_n) \leq q^n N_2(x_1 - x_0)$.

This shows that $\{x_n\}$ is a Cauchy sequence with respect to N_1. By lemma, (X, N_1) is a Banach space. Therefore $\{x_n\} \to x$ in X. Now

$$N_2(x - T_m x) \leq N_2(x - x_{2n+1}) + N_2(x_{2n+1} - T_m x)$$
$$= N_2(x - x_{2n+1}) + N_2(S_{n+1}x_{2n} - T_m x)$$
$$\leq N_2(x - x_{2n+1}) + h\{N_2(S_{n+1}x_{2n} - x_{2n})\, N_2(T_m x - x)\}^{1/2}$$
$$= N_2(x - x_{2n+1}) + h\,\{N_2\,(x_{2n+1} - x_{2n})\, N_2\,(T_m x - x)\}^{1/2}$$

Since $N_2 \leq N_1$, we have

$$N_2(x - T_m x) \leq N_1(x - x_{2n+1}) + h\{N_1(x_{2n+1} - x_{2n})\, N_1(T_m x - x)\}^{1/2}.$$

When $n \to \infty$ we obtain $N_2(x - T_m x) = 0$ for all m or $x = T_m x$ for all m. Similarly $N_2(S_m x - x) \leq N_2(S_m x - x_{2n}) + N_2(x_{2n} - x)$ gives $x = S_m x$ for all m. This proves that x is a common fixed point of $\{S_n\}$ and $\{T_n\}$. The uniqueness is trivially obtained.

Let $\varnothing : [0, \infty) \to [0, \infty)$ be a non-decreasing and continuous on the right such that $\varnothing(t) < t$, for $t > 0$. Clearly $\varnothing(0) = 0$.

The following theorems 2–4 classify Saks spaces in which each x in X is a fixed point.

Theorem 2

Let $(X_s, d) = (X, N_1, N_2)$ be a Saks space and T a self mapping of X satisfying

$$\varnothing(N_2(Tx - Ty)) \geq \{N_2(Tx - x)\, N_2(Ty - y)\}^{1/2}$$

for all x, y in X. Then for every x, $x = Tx$.

Proof

Let x in X be an arbitrary point. Then

$$0 = \varnothing(0) = \varnothing(N_2(Tx - Tx) \geq \{N_2(Tx - x)\, N_2(Tx - x)\}^{1/2} = N_2(Tx - x)$$

or $N_2(Tx - x) = 0$ implies $x = Tx$ for all x. This completes the proof.

Corollary

Let $(X_s, d) = (X, N_1, N_2)$ be a Saks space. Let $T : X \to X$ satisfy for all x, y in X, $h > 1$,

$$N_2(Tx - Ty) \geq h\{N_2(Tx - x)\, N_2(Ty - y)\}^{1/2}$$

Then each x in X is a fixed point.

Proof

Define $\varnothing : [0, \infty) \to [0, \infty)$ by $\varnothing(t) = t/h$. Then using the above theorem, we obtain the required result.

Theorems 3 and 4 are similar and are left to the reader.

Theorem 3

Let (X_s, d) be a Saks space and $T : X \to X$ be such that for all x, y in X.

$$\varnothing(N_2(Tx-Ty)) \geq \{N_2^2(y-Tx) + N_2^2(x - Ty)\}\ \{N_2(y-Tx)+N_2(x - Ty)\}^{-1}$$

where $N_2(y - Tx) + N_2(x - Ty) \neq 0$. Then each x in X is a fixed point of T.

Theroem 4

Let (X_s, d) be a Saks space. If a self map $T : X \to X$ satisfies

$$\varnothing(N_2^m(Tx-Ty) \geq \{N_2^p(y-Tx)+N_2^p(x-Ty)\}\{N_2^{p-m}(y-Tx)+N_2^{p-m}(x-Ty)\}^{-1}$$

for all x, y in X, where $N_2^{p-m}(y - Tx) + N_2^{p-m}(x - Ty) \neq 0$, $m \geq 1$, $p \geq 2$, $m < p$, then each x in X is a fixed point of T.

Theorem 5

Let $(X_s, d) = (X, N_1, N_2)$ be a Saks space with N_1 equivalent to N_2. Let S and T be self maps such that for all x, y in X,

$$N_2^m(Sx-TSy) \leq c\{N_2^p(Sy-Sx)+N_2^p(x-TSy)\}\{N_2^{p-m}(Sy-Sx)+N_2^{p-m}(x-TSy)\}^{-1}$$

where $N_2^{p-m}(Sy - Sx) + N_2^{p-m}(x - TSy) \neq 0$, $m \geq 1$, $p \geq 2$, $m < p$ and $0 < c < 1/2$. Then S and T have a unique common fixed point.

Proof

Define a sequence $\{x_n\}$ in X as: $x_{2n} = Tx_{2n-1}, x_{2n+1} = Sx_{2n}$. Then $N_2^m(x_{2n+1} - x_{2n}) = N_2^m(Sx_{2n} - TS_{2n-2})$

$\leq c\ \{N_2^p(Sx_{2n-2} - Sx_{2n}) + N_2^p\ (x_{2n} - TSx_{2n-2})\}\ \{N^{p-m}(Sx_{2n-2} - Sx_{2n}) +$
$N_2^{p-m}(x_{2n} - TSx_{2n-2})$

$= c\ \{N_2^p(x_{2n-1} - x_{2n+1}) + N_2^p(x_{2n} - x_{2n})\}\ \{N_2^{p-m}(x_{2n-2} - x_{2n+1}) +$
$N_2^{p-m}(x_{2n} - x_{2n})\}$

$= c\ N_2^m\ (x_{2n-1} - x_{2n+1})$. Thus we have

$N_2^m(x_{2n+1} - x_{2n}) \leq cN_2^m\ (x_{2n-1} - x_{2n+1})$

or $N_2(x_{2n+1} - x_{2n}) \leq hN_2(x_{2n-1} - x_{2n+1})$

$\leq h\{N_2(x_{2n-1} - x_{2n}) + N_2(x_{2n} - x_{2n+1})\},\ h = c^{1/2} < 1$

or $\quad N_2(x_{2n+1} - x_{2n}) \leq qN_2(x_{2n} - x_{2n-1})$, where $q = h/1-h < 1$.

Similarly,

$N_2(x_{2n} - x_{2n-1}) \leq qN\ (x_{2n-1} - x_{2n+1})$.

In general

$N_2(x_{n+1} - x_n) \leq qN_2(x_n - x_{n-1}) \leq ... \leq q^nN_2(x_1 - x_0)$

or

$N_2(x_{n+1} - x_n) \leq q^nN_2(x_1 - x_0)$.

Since $N_1 \leq N_2$, this gives

$N_1(x_{n+1} - x_n) \leq N_2(x_1 - x_0)$.

This shows that $\{x_n\}$ is a Cauchy sequence with respect to N_1. By lemma, (X, N_1) is a Banach space and therefore $\{x_n\}$ converges to some x in X. Now

$N_2^m(Sx - x_{2n}) = N_2^m(Sx - TSx_{2n-2})$

$\leq c\{N_2^p(Sx_{2n-2} - Sx) + N_2^p(x - TSx_{2n-2})\}\ \{N_2^{p-m}(Sx_{2n-2} - Sx) + N_2^{p-m}$ $(x - TSx_{2n-2})\}$

$= c\{N_2^p(x_{2n-1} - Sx) + N_2^p(x - x_{2n})\}\ \{N_2^{p-m}(x_{2n-1} - Sx) + N_2^{p-m}(x - x_{2n})\}$

when $n \to \infty$, we have $N_2^m(Sx - x) \leq cN_2^m(x - Sx)$ or $(1 - c)\, N_2^m(Sx - x)$ ≤ 0 or $N_2^m(Sx - x) = 0$ or $N_2(Sx - x) = 0$ or $x = Sx$.

Now

$N_2^m(x - Tx) = N_2^m(Sx - TSx) \leq c\{N_2^p(Sx - Sx) + N_2^p(x - TSx)\}\ \{N_2^{p-m}$ $(Sx - Sx) + N_2^{p-m}(x - TSx)$

$N_2^m(x - Tx) \leq cN_2^m(x - Tx)$ implies $N_2(x - Tx) = 0$ or $x = Tx$. The uniqueness is easy to obtain. This completes the proof.

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A RELATED FIXED POINT THEOREM ON TWO METRIC SPACES

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ABSTRACT

A new related fixed point theorem on two metric spaces is given.

Key Words

Fixed point, metric space.

Classification

54H25.

The following related fixed point theorem was proved in [1].

Theorem 1

Let (X, d) and (Y, ρ) be complete metric spaces, let T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities

$$\rho(Tx, TSy) \leq c \max \{d(x, Sy), \rho(y, Tx), \rho(y, TSy)\},$$

$$\rho(Sy, TSx) \leq c \max \{\rho(y, Ty), d(x, Sx), d(x, STx)\},$$

for all x in X and y in Y, where $0 \leq c < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

We now prove the following related fixed point theorem.

Theorem 2

Let (X, d) *and* (Y, ρ) *be complete metric spaces, let* T *be a mapping of* X *into* Y *and let* S *be a mapping of* Y *into* X *satisfying the inequalities*

$$d(Sy, STx) \leq c\,\phi(x, y), \qquad (1)$$

$$\rho(Tx, TSx) \leq c\,\psi(x, y), \qquad (2)$$

For all x *in* X *and* y *in* Y *for which*

$$g(x, y) \neq 0 \neq h(x, y),$$

where $0 \leq c < 1$,

$$\phi(x, y) = \frac{f(x, y)}{g(x, y)}, \quad \psi(x, y) = \frac{f(x, y)}{h(x, y)}$$

and

$$f(x,y) = \max\{d(x,Sy)\rho(y,Tx), d(x,STx)\rho(y,TSy), d(Sy,STx)\rho(y,Tx)\}$$

$$g(x,y) = \max\{d(x,STx), \rho(y,TSy), d(x, Sy)\},$$

$$h(x,y) = \max\{d(x,STx), \rho(y,TSy), d(y, Tx)\}.$$

Then ST *has a unique fixed point* z *in* X *and* TS *has a unique fixed point* w *in* Y*. Further,* $Tz = w$ *and* $Sw = z$.

Proof

Let x be an arbirary point in X. We define sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$(ST)^n x = x_n, \; T(ST)^{n-1} x = y_n$$

for $n = 1, 2, \ldots$

We will assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n, otherwise, if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n, we could put $x_n = z$ and $y_n = w$. Applying inequality (1) we get

$$d(x_n, x_{n+1}) = d(Sy_n, STx_n)$$

$$\leq \frac{cd(x_n, x_{n+1})\,\rho(y_n, y_{n+1})}{\max\{d(x_n, x_{n+1}), \rho(y_n, y_{n+1})\}}$$

from whch it follows that

$$d(x_n, x_{n+1}) \leq c\rho(y_n, y_{n+1}) \qquad (3)$$

Applying inequality (2) we get

$$\rho(y_n, y_{n+1}) = \rho(Tx_{n-1}, TSy_n)$$
$$\leq \frac{cd(x_{n-1}, x_n)\, \rho(y_n, y_{n+1})}{\max\{d(x_{n-1}, x_n), \rho(y_n, y_{n+1})\}}$$

from which it follows that

$$\rho(y_n, y_{n+1}) \leq cd(x_{n-1}, x_n).$$

It now follows from inequalities (3) and (4) that

$$d(x_n, x_{n+1}) \leq c\rho(y_n, y_{n+1}) \leq c^{2n}d(x, x_1),$$

for $n = 1, 2, \ldots$ Since $c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y.

Applying inequality (1) we have

$$d(Sw, x_n) = d(Sw, STx_{n-1})$$
$$\leq c \max\{d(x_{n-1}, Sw)\, \rho(w, y_n), d(x_{n-1}, x_n)\, \rho(w, TSw),$$
$$d(Sw, x_n)\, \rho(w, y_n)\} \times$$
$$\times [\max\{d(x_{n-1}, x_n), \rho(w, TSw), d(x_{n-1}, Sw)\}]^{-1}.$$

Letting n tend to infinity, we have $d(Sw, z) \leq 0$ and so $Sw = z$.

Applying inequality (2), we have

$$\rho(Tz, y_n) = \rho(T_z, TSy_{n-1})$$
$$\leq c \max\{d(z, x_{n-1})\, \rho(y_{n-1}, Tz), d(z, STz)\, \rho(y_{n-1}, y_n),$$
$$d(x_{n-1}, STz)\, \rho(y_{n-1}, Tz)\} \times$$
$$\times [\max\{d(z, STz), \rho(y_{n-1}, y_n), \rho(y_{n-1}, Tz)\}]^{-1}.$$

Letting n tend to infinity, we have

$$d(w, Tz) \leq cd(z, STz) \qquad (5)$$

Applying inequality (1) again we have

$$\rho(z_n, STz) = d(Sw, STz)$$
$$\leq \frac{cd(z, STz)\, \rho(w, Tz)}{\max\{d(z, STz), \rho(w, Tz)\}}$$

and it follows that

$$d(z, STz) \leq c\rho(w, Tz). \tag{6}$$

It now follows from inequalities (5) and (6) that

$$\rho(w, Tz) \leq cd(z, STz) \leq c^2\rho(w, Tz)$$

and so

$$STz = Sw = z, TSw = Tz = w.$$

To prove uniqueness, suppose that ST has a second distinct point z'. Then it follows easily from inequalities (1) and (2) that

$$\begin{aligned} d(z', z) &= d(STz', STz) \leq c\rho(Tz', Tz) \\ &= c\rho(Tz', TSTz) \leq c^2 d(z', z), \end{aligned}$$

a contradiction. The fixed point z must therefore be unique.

The proof that w is the unique fixed point of TS follows similarily.

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STRONG IDEALS, ASSOCIATIVE IDEALS AND P-IDEALS IN BCI-ALGEBRAS

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ABSTRACT

In this paper, at first, we show that strong ideals are p-ideals and p-ideal is strong iff it is closed. Next, we investigate the relation between strong ideals and associative ideals. We also show that p-ideal, strong ideal and associative ideal coincide in quasi-associative BCI-algebra. Finally, we give some examples of strong ideals.

1. INTRODUCTION

In [1] the concept of strong ideals was introduced. M. Daoji [3], gave the notion of a regular ideal in BCI-algebras. In [4], it was shown that strong ideals and regular ideals coincide in BCI-algebras. In [10], associative ideal was discussed and the following results were obtained: a BCI-algebra X is associative iff evey ideal of X is associative ideal; X/A is associative BCI-algebra iff A is associative ideal of X. In [7] the concept of p-ideal was introduced and some interesting results were obtained. In this paper, we will study the relation among strong, associative and p-ideals. We also give some examples of strong ideals.

Definition 1 [1]:

An ideal A in a BCI-algebra X is called a strong ideal if for $a \in A, x \in X-A, a * x \in X - A$.

Definition 2 [3]:

An ideal A in a BCI-algebra X is called regular if $x * y \in A$, $x \in A$ imply $y \in A$.

Definition 3 [8]:

An ideal A in a BCI-algebra X is called a closed ideal if $0 * a \in A$ for $a \in A$. Note that a closed ideal is a sub-algebra.

Definition 4 [7]:

A nonempty subset A in a BCI-algebra X is called a p-ideal of X, if it satisfies: (i) $0 \in A$, (ii) $(x * z) * (y * z) \in A$ and $y \in A$ imply $x \in A$.

(1) Let X be a BCI-algebra and $A \subseteq X$ an ideal in X. Then followings are equivalent: (i) A is strong, (ii) A is regular ([4]).

(2) An ideal A of a BCI-algebra X is a p-ideal iff $x \in X$ and $0 * (0 * x) \in A$ imply $x \in A$ ([7]).

(3) Let A be an ideal of a BCI-algebra X, then $x \in A$ implies $0 * (0 * x) \in A$ ([7]).

Definition 5 [10]:

A nonempty subset A in a BCI-algebra X is called an associative ideal of X, if it satisfies: (i) $0 \in A$, (ii) $(x * y) * z \in A$ and $y * z \in A$ imply $x \in A$, where $x, y, z \in X$.

Lemma 1 [10]:

An ideal A of a BCI-algebra X is an associative ideal iff $x, y \in X$ and $(x * y) * y \in A$ imply $x \in A$.

Let X be a BCI-algebra, then

(4) $0 * (x * y) = (0 * x) * (0 * y)$, for all $x, y \in X$ ([7]).

(5) $0 * [0 * (0 * x)] = 0 * x$, for all $x \in X$ ([7]).

(6) $0 * [0 * (x * y)] = (0 * y) * (0 * x)$, for all $x, y \in X$ ([7]).

(7) $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$, $0 * (0 * x^n) = 0 * (0 * x)^n$, $0 * [0 * (0 * x)]^n = 0 * x^n$, for all $n \in$ N and all $x, y \in$ X. Where

$$0 * x^n = (\dots((0 * x) * x) \dots) * x \quad ([11]).$$

Theorem 1

Let X be a BCI-algebra and A $\subseteq$ X be a strong ideal. Then A is a p-ideal.

Proof

For all $x \in$ X, we have

$$[0 * (0 * x)] * x = 0 \in A.$$

Suppose that $0 * (0 * x) \in$ A, using (1), Definition 2 and $[0 * (0 * x)] * x \in$ A, we get that $x \in$ A. Hence A is p-ideal (according to (2)).

The following example shows that p-ideal may not be necessarily strong and may not be a sub-algebra.

Example 1 [4]:

Let Z be the set of integers and "−" the minus operation, then (Z,−,0) is a p-semisimple BCI-algebra. Let us consider A = {0,1,2,3,....}, then A is a p-ideal (according to Theorem 3.3 of [7]). But A is not strong (see [4]). Also note that A is not sub-algebra.

Theorem 2

In BCI-algebra a p-ideal A is strong iff it is closed.

Proof

Since every strong ideal in BCI-algebra is closed ([2]), therefore necessity is true.

Conversely, let A be a p-ideal and be closed, we show that A is regular. Suppose $x * y \in$ A and $x \in$ A. Since A is closed, therefore

$0 * x \in$ A, $0 * (x * y) \in$ A, $0 * [0 * (x * y)] \in$ A. By (6), we have

$$(0 * y) * (0 * x) = 0 * [0 * (x * y)] \in A.$$

Now, $0 * x \in A$ and $(0 * y) * (0 * x) \in A$. Using the definition of ideal, we get that $0 * y \in A$. Thus $0 * (0 * y) \in A$, because A is closed. Using (2) and A is a p-ideal, we see that $y \in A$. Hence A is regular, i.e. A is strong (by(1)). This completes the proof.

The following corollary is obvious.

Corollary 1

In a BCI-algebra a p-ideal A is a sub-algebra iff it is strong.

Theorem 3

Let X be a BCI-algebra and $A \subseteq X$ be an associative ideal. Then A is strong.

Proof

Assume that $x * y \in A$, $x \in A$. Since $x \in A$ and A is an associative ideal (Obviously, every associative ideal is an ideal), therefore $0 * (0 * x) \in A$ (by [3]). Thus

$$[(0 * x) * (0 * x)] * (0 * x) = 0 * (0 * x) \in A,$$

$$(0 * x) * (0 * x) = 0 \in A.$$

Using definition 5 it follows that $0 * x \in A$. Then

$$(y * x) * y = (y * y) * x = 0 * x \in A,$$

$$x * y \in A.$$

By definition 5, we have $y \in A$. Hence A is a regular idea, i.e. a strong ideal.

The following example shows that a strong ideal may not be necessarily an associative ideal.

Example 2

Let X = {0, 1, 2, 3}, and the operation * be defined by the table

*	0	1	2	3
0	0	3	0	1
1	1	0	1	3
2	2	3	0	1
3	3	1	3	0

Then (X, *, 0) is a BCI-algebra, A = {0, 2} is a strong ideal in X. But A is not an associative ideal, because (1 * 3) * 3 = 0 $\in$ A, 3 * 3=0$\in$A, 1 $\notin$ A.

X*i* Changchang [12] introduced a new class of BCI-algebras, called quasi-associative BCI-algebra.

Definition 6 [12]:

BCI-algebra X is quasi-associative BCI-algebra, if it satisfies

$$(x * y) * z \leq x * (y * z), \text{ for all } x, y, z \in X.$$

Lemma 2 [12]:

A BCI-algebra X is quasi-associative iff it satisfies

$0 * (0 * x) = 0 * x$, for all $x \in X$.

Theorem 4

Let X be a quasi-associative BCI-algebra and A be an ideal of X, then the followings are equivalent:

(i) A is associative,

(ii) A is strong,

(iii) A is p-ideal.

Proof

By Theorem 3 and Theorem 1, we get that (i) implies (ii) and (ii) implies (iii). Now, we show that (iii) implies (i). Let A be a p-ideal. If $(x * y) * y \in A$, then

$$0 * \{0 * [(x * y) * y]\} \in A. \qquad \text{(by (3))}$$

But $0 * \{0 * [(x * y) * y]\}$

$$= (0 * y) * [0 * (x * y)] \qquad \text{(by (6))}$$

$$= \{0 * [0 * (x * y)]\} * y$$

$$= [0 * (x * y)] * y \qquad \text{(by Lemma 2)}$$

$$= [(0 * x) * (0 * y)] * y \qquad \text{(by (4))}$$

$$= \{[0 * (0 * y)] * x\} * y$$

$$= \{[0 * (0 * y)] * y\} * x$$

$$= 0 * x$$

$$= 0 * (0 * x) \qquad \text{(by Lemma 2)}$$

Therefore $0 * (0 * x) \in A$. Using (2) we have $x \in A$. Thus $(x*y)*y \in A$ implies $x \in A$. By Lemma 1, we get that A is associative ideal. This completes the proof.

Finally we give the following results.

Theorem 5

Let X be a BCI-algebra. For all $n \in N$, we define

$$T_n(X) = \{x \in X \mid 0 * x^n = 0\}$$

Then $T_n(X)$ are strong ideals (not necessarily associative) for all $n \in N$.

Proof

Obviously, $0 \in T_n(X)$. We claim that $T_n(X)$ is an ideal in X. If $x * y, y \in T_n(X)$, then

$$0 * (x * y)^n = 0, \quad 0 * y^n = 0$$

By (7), we have

$$0 * x^n = (0 * x^n) * 0$$
$$= (0 * x^n) * (0 * y^n)$$
$$= 0 * (x * y)^n$$
$$= 0$$

Thus $x \in T_n(X)$ and $T_n(X)$ is an ideal of X.

Now, we claim that $T_n(X)$ is closed. If $x \in T_n(X)$, then $0 * x^n = 0$. By (7), we have

$$0 * (0 * x)^n = 0 * (0 * x^n) = 0 * 0 = 0$$

Thus $0 * x \in T_n(X)$ and $T_n(X)$ is closed.

Suppose $0 * (0 * x) \in T_n(X)$, then $0 * [0 * (0 * x)]^n = 0$. By (7), we have

$$0 * x^n = 0 * [0 * (0 * x)]^n = 0$$

Thus $x \in T_n(X)$. Using (2), $T_n(X)$ is p-ideal.

Hence $T_n(X)$ is p-ideal and closed. Using Theorem 2, we get that A is a strong ideal.

In Example 2, $T_1(X) = \{0, 2\}$ is a strong ideal, but it is not associative ideal.

This completes the proof.

Corollary 2

Let X be a BCI-algebra, then $X/T_n(X)$ is p-semisimple BCI-algebra for all $n \in N$.

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ON P-IDEALS OF A BCI-ALGEBRA

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ABSTRACT

In this paper we introduce the concept of a P-ideal in a BCI-algebra and characterize P-semi simple BCI-algebras by P-ideals.

1. INTRODUCTION

In 1980, K. Iseki [5], introduced the concept of an ideal in a BCI-algebra. In [1], the concepts of strong ideals, obstinate ideals and weak ideals were introduced and it was shown that if A is a strong ideal in a BCI-algebra X, then X/A is a p-semisimple algebra. Further, it was proved that the BCK-part denoted by B(X), of a BCI-algebra X is a strong ideal. M. Daoji [3], introduced another class of ideals namely, regular ideals in BCI-algebras and proved some of its properties. S.A. Bhatti [2], proved that regular ideals and strong ideals coincide. In this paper we introduce a new class of ideals namely, p-ideals and study some of its properties.

2. PRELIMINARIES

A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$.

(1) $((x * y) * (x * z)) * (z * y) = 0$

(2) $(x * (x * y)) * y = 0$

(3) $x * x = 0$

(4) $x * y = 0 = y * x \Rightarrow x = y$

(5) $x * 0 = 0 \Rightarrow x = 0$

(6) $x * y = 0 \Leftrightarrow x \leq y$. [5]

In a BCI-algebra X, the set $B(X) = \{x \in X : 0 * x = 0\}$ is a sub-algebra of X and known as BCK-part of X. B(X) is an ideal in X. If $B(X) = \{0\}$, then X is known as p-semisimple [6].

(7) If X is a BCI-algebra, then followings are equivalent:

(i) X is p-semisimple,

(ii) $0 * (0 * x) = x$,

(iii) $x * y = 0 \Rightarrow x = y$, for all $x, y \in X$. [3]

In a BCI-algebra X, for $x, y, z \in X$.

(8) $(x * y) * z = (x * z) * y$. [5]

(9) $x * 0 = x, \quad x \in X$. [5]

(10) $x \leq y \Rightarrow z * y \leq z * x$ and $x * z \leq y * z$ [5].

(11) Let X be a BCI-algebra and A a subset of X. A is an ideal in X if,

(i) $0 \in A$

(ii) $x * y \in A, \quad y \in A$ imply $x \in A$. [5]

(12) Let X be a BCI-algebra and $x, y, z \in X$, then

(i) $x * (x * (x * y)) = x * y$

(ii) $0 * (0 * (0 * x)) = 0 * x$ [2].

3. P-IDEALS

Definition 3.1

A non-empty subset A in a BCI-algebra X is called a p-ideal of X, if it satisfies:

(i) $0 \in A$;

(ii) $(x * z) * (y * z) \in A$ and $y \in A$ imply $x \in A$.

Theorem 3.1

A p-ideal is an ideal.

Proof

Let A be a p-ideal of a BCI-algebra X. By Definition 3.1 (i) we have $0 \in A$. If $x * y \in A$ and $y \in A$, then we have $(x * 0) * (y * 0) \in A$ and $y \in A$, since $x * 0 = x$, $y * 0 = y$. By definition 3.1 (ii), it follows that $x \in A$. Therefore A is an ideal of X. This completes the proof.

Theorem 3.2

Let A be a p-ideal of a BCI-algebra X, then

(13) $0 * (0 * x) \in A$ implies $x \in A$.

Proof

If $0 * (0 * x) \in A$, then $(x * x) * (0 * x) \in A$ since $x * x = 0$. Note that $0 \in A$, because A is a p-ideal. By definition 3.1 (ii), $x \in A$. This completes the proof.

Remark 3.1

By (7) (iii), in a p-semisimple BCI-algebra X, $(x * z) * (y * z) = x * y$ holds for $x, y, z \in X$.

Theorem 3.3

A BCI-algebra X is p-semisimple iff every ideal of X is a p-ideal.

Proof

Let A be an ideal of a p-semisimple BCI-algebra X. Suppose that $(x * z) * (y * z) \in A$ and $y \in A$. By Remark 3.1, $x * y = (x * z) * (y * z) \in A \Rightarrow x * y \in A$. Now $x * y \in A$, $y \in A$ and A being an ideal imply $x \in A \Rightarrow$ A is a p-ideal.

Conversely, let X be a BCI-algebra in which every ideal is a p-ideal. Then, especially, $\{0\}$ is a p-ideal of X. For $x \in X$, we have

$$((x * (0 * (0 * x))) * x) * (0 * x) = ((x * x) * (0 * (0 * x))) * (0 * x)$$

$= (0 * (0 * x)) * (0 * (0 * x)) = 0 \in \{0\}$. By definition 3.1 (ii), $x * [0 * (0 * x)] \in \{0\} \Rightarrow x * [0 * (0 * x)] = 0$. Also, $(0*(0*x))*x=0$. Hence, $0 * (0 * x) = x$. By (7) (ii), X is p-semisimple. This completes the proof.

Corollary 3.1

A BCI-algebra $X = (X, *, 0)$ is p-semisimple iff $\{0\}$ is a p-ideal.

4. CHARACTERIZATION OF P-IDEALS

Theorem 4.1

An ideal A of a BCI-algebra X is a p-idreal iff

(14) $(x * z) * (y * z) \in A$ implies $x * y \in A$, for $x, y, z \in X$.

Proof

Assume that A is a p-ideal of X and $(x * z) * (y * z) \in A$, for $x, y, z \in X$. By (1) and (8), $0 = ((x * z) * (x * y)) * (y * z) = ((x * z) * (y * z)) * (x * y) \Rightarrow [(x * z) * (y * z)] * (x * y) = 0$. Let us consider

(15) $[(x * y) * (x * y)] * \{[(x * z) * (y * z)] * (x * y)\} = 0*0=0 \in A$.

By definition 3.1 (ii), $x * y \in A$.

Conversely, suppose that A is an ideal in X and (14) holds. If $(x * z) * (y * z) \in A$ and $y \in A$, then by (14), $x * y \in A$. By definition 3.1, A is a p-ideal. This completes the proof.

Lemma 4.1

Let X be a BCI-algebra, then for $x, y \in X$.

(16) $0 * (x * y) = (0 * x) * (0 * y)$.

Proof

Since $[(0 * (0 * x)) * y] * (x * y) = [(0 * y) * (0 * x)] * (x * y) = 0$,

then $0 = ((0 * y) * (0 * x)) * (x * y)$

$0 * x = \{[(0 * y) * (0 * x)] * (x * y)\} * x$

$0 * x = \{[(0 * (0 * x)) * y\} * (x * y)) * x$

$0 * x = \{[(0 * (0 * x)) * x] * y\} * (x * y)$

$0 * x = ((0 * x) * (0 * x) * y) * (x * y)$

$0 * x = (0 * y) * (x * y)$.

$(0 * x) * (0 * y) = ((0 * y) * (x * y)) * (0 * y)$

$(0 * x) * (0 * y) = ((0 * y) * (0 * y)) * (x * y)$

$(0 * x) * (0 * y) = 0 * (x * y)$. This completes the proof.

Theorem 4.2

Let X be a BCI-algebra, then $x, y, z \in X$

(17) $0 * \{0 * [(x * z) * (y * z)]\} = (0 * y) * (0 * x)$.

Proof

$0 * \{0 * [(x * z) * (y * z)]\}$

$= 0 * \{[0 * (x * z)] * [0 * (y * z)]\}$ (by (16))

$= \{0 * [0 * (x * z)]\} * \{0 * [0 * (y * z)]\}$ (by (16))

$= \{0 * \{0 * [0 * (y * z)]\} * [0 * (x * z)]$ (by (8))

$= [0 * (y * z)] * [0 * (x * z)]$ (by (12))

$= \{0 * [0 * (x * z)]\} * (y * z)$ (by (8))

$= \{0 * (0 * x) * (0 * z)]\} * (y * z)$ (by (16))

$= \{[0 * (0 * x)] * [0 * (0 * z)]\} * (y * z)$ (by (16))

$= \{\{0 * [0 * (0 * z)]\} * (0 * x)\} * (y * z)$ (by (8))

$= [(0 * z) * (0 * x)] * (y * z)$ (by (12))

$= ([0 * (y * z)] * z\} * (0 * x)$ (by (8))

$= \{[(0 * y) * (0 * z)] * z\} * (0 * x)$ (by (8))

$= \{\{[0 * (0 * z)] * z\} * y\} * (0 * x)$ (by (16))

$= \{\{[(0 * z) * (0 * z)] * y\} * (0 * x)$ (by (8))

$= (0 * y) * (0 * x).$

Corollary 4.1

Let X be a BCI-algebra, then $x, y \in X \Rightarrow$

(18) $0 * (0 * (x * y)) = (0 * y) * (0 * x).$

Proof

(18) follows from (17) by taking $z = 0$.

Lemma 4.2

Let A be an ideal of a BCI-algebra X, then $x \in A$ implies $0 * (0 * x) \in A$.

Proof

$0 = (0 * x) * (0 * x) = (0 * 0 * x) * x \Rightarrow (0 * (0 * x)) * x = 0$ implies $[0 * (0 * x)] * x = 0$. Now $(0 * (0 * x)) * x \in A$ and $x \in A$ and A being an ideal implies $0 * (0 * x) \in A$. This completes the proof.

Theorem 4.3

An ideal A of a BCI-algebra X is a p-ideal iff

(19) $0 * (0 * x) \in A$ implies $x \in A$, for $x \in X$.

Proof

If A is a p-ideal of X and $0 * (0 * x) \in A$. By theorem 3.2, $x \in A$.

Conversely, let A be an ideal of X and suppose that (19) holds. If $(x * z) * (y * z) \in A$, then by lemma 4.2 $0 * \{0 * [(x * z) * (y * z)]\} \in A$. Now, using Corollary 4.1 and Theorem 4.2, we get that $0 * [0 * (x * y)) = (0 * y) * (0 * x) = 0 * \{0 * [(x * z) * (y * z)]\} \in A$ implies $0 * (0 * (x * y)) \in A$. Since (19) holds for A, so $x * y \in A$. Thus $(x * z) * (y * z) \in A \Rightarrow x * y \in A$. By Theorem 4.1, A is a p-ideal. This completes the proof.

5. P-SEMISIMPLE QUOTIENT ALGEBRAS

Theorem 5.1

Let X be a BCI-algebra and A be an ideal of X, then X/A is a p-semisimple BCI-algebra iff A is a p-ideal of X.

Proof

Suppose that A is a p-ideal. For $x \in X$ we have

$0 * \{0 * \{x * [0 * (0 * x)]\}\}$

$= 0 * \{(0 * x) * \{0 * [0 * (0 * x)]\}\}$ (by (16))

$= 0 * [(0 * x) * (0 * x)]$ (by (12))

$= 0 \in A.$

According to Theorem 4.3, we have $x * [0 * (0 * x)] \in A$ which implies $C_0 * (0 * x) = C_x$ or $C_0 * (C_0 * C_x) = C_x$. Here, by C_x, we mean the equivalence class which contains x. By (7) (ii), X/A is a p-semisimple BCI-algebra.

Conversely, if X/A is p-semisimple, then $C_0 * (C_0 * C_x) = C_x$ implies $x * [0 * (0 * x)] \in A$ for any $x \in X$. Assume that $0*(0*x) \in A$, then it follows that $x \in A$. Now $x * [0 * (0 * x)] \in A$ implies $x \in A$ and A is an ideal. By Theorem 4.3 A is a p-ideal. This completes the proof.

Theorem 5.2

Let X be a BCI-algebra, then B(X) is a p-ideal of X.

Proof

For $x \in X$, suppose that $0 * (0*x) \in B(X)$, then $0*(0*(0*x)=0$. On the other hand, by (12) (ii), $0 * x = 0 * (0 * (0 * x)$, hence $0*x=0$. Thus $x \in B(X)$. By Theorem 4.3, B(X) is a p-ideal. This completes the proof.

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THE PROJECTIVE MODULAR CHARACTERS OF THE HIGMAN-SIMS SIMPLE GROUP

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1. INTRODUCTION AND NOTATIONS

In this work, the Brauer characters of the projective (faithful) irreducible characters of the covering group $\hat{H}$ of the Higman-sims simple group H over an algebraically clsoed field of characteristics p an odd prime, are determined. In [*hu*3], the modular irreducible characters of H have already been determined and their free use is made.

Among the others, M_{22} and $P\Sigma U_3(5)$, the extension of the simple group $PSU_3(5)$ by its field automorphism are the maximal subgroups of H. The inverse images in $\hat{H}$ of these subgroups are, respectively $\hat{M}$, the 2-fold proper cover of M_{22} and the group denoted by $\hat{P}$ and their character tables are given in [Hu2] (or [Ru]) and in [Sa2] respectively.

To determine the modular character, besides the other techniques, the general method used in James and Kerber [JK, section 6.3] will also be used without further reference.

Ordinary, projective and modular irreducible characters are named by their degree, with a subscript if there is more than one of the same degree. Projective principal indecomposable characters are denoted by *d*'s and the corresponding Projective irreducible Brauer characters are denoted by ϕ(least degree character occur in the column d's). A bar denotes the complex conjugate characters. But the set of irreducibles, *p*-modular irreducibles and principal *p*-indecomposable characters of a group G, for a prime *p* are, respectively denoted by irr(G), irm(G) and idec(G). In case, the characters are projective, these notation are replaced with irr($\hat{G}$), irm($\hat{G}$) and indec(G). We will omitt the prime *p* when it will be clear.

Furthermore, $\uparrow G$ (respectively $\downarrow G$) denote the induction from (respectively restriction to) a subgroup of G.

2. THE 3-MODULAR CHARACTERS OF $\hat{H}$

The group $\hat{H}$ has 13-regular classes. Then there are 13-elements in irm(H). The elements 1980*a*, 1980*b*, 2304*a*, 2304*b*, 2520*a* and 2520*b* belonging to irr($\hat{H}$) have degrees divisible by 3^2, hence these all are in irm($\hat{H}$) and in indec ($\hat{H}$) as well.

The 3-block of defect 1 contains only three elements namely 924*a*, 924*b* and 1848 of irr($\hat{H}$) and thus its Brauer tree is trivially known.

All the remaining elements of irr($\hat{H}$) lies in the same 3-block. Furthermore, this block contains only 5-elements of irm($\hat{H}$).

If now we take $c_1 = (56 + 2.440) \uparrow \hat{H}$; $c_2 = (22 \otimes 1980a)$; $\tilde{c}_2 = (120 + 330) \uparrow \hat{H}$; $c_3 = (22 + 1980b) \uparrow \hat{H}$; $c_4 = [(56 \otimes 693)/2] \uparrow \hat{H}$ and $c_5 = (126) \uparrow \hat{H}$. Then the matrix $R_3(\hat{H})$ is the first approximation toward the decomposition matrix of $\hat{H}$.

$R_3(\hat{H})$

	c_1	$\tilde{c}_2$	c_2	c_3	c_4	c_5
56	1					
176a	1	1	1			
176b	1	1				
616a	3	1	2	1	1	
616b	3	1	1	2	1	
1000						1
1232a	2	1	1			1
1232b	2	1		1		1
1792	4	2	2	2	1	1

Clearly; $\tilde{c}_2$, in addition to c_2 contains c_3. But comparing the cofficient of c_2 and c_3 with $\tilde{c}_2$ on 616a yields that c_4 is a direct summand of c_2 and c_3. Hence subtracting off c_4 from c_2 and c_3, the columns we obtained are the elements of indec(H) and are denoted by d_2 and d_3 respectively.

Now, by expressing the tensor product $56 \otimes 1386$ of the element 1386 of indec(H) with 56 in terms of elements of irr(H), we have

56+3.1232a+3.1232b+2.1000 + 2.1792 +(characters of other blocks)

Then $56 \otimes 1386 = c_1 - d_2 - d_3 - 2.d_4 + 2.d_5$; thus $c_1 - d_2 - d_3 - 2.d_4$ exists and belongs to indec(H), we denote it by d_1. This also completely determine the decomposition matrix $D_3(\hat{H})$, for $p = 3$.

$D_3(\hat{H})$

	d_1	d_2	d_3	d_4	d_5
56	1				
176a		1			
176b			1		
616a		1		1	
616b			1	1	
1000					1
1232a	1	1			1
1232b	1		1		1
1792		1	1	1	1

3. The 5-MODULAR CHARACTER OF $\hat{H}$

In this case all but the elements $1000 \in \mathrm{irr}(\hat{H})$, which also belongs to indec(H), are contained in a 5-block. The group $\hat{H}$ has 11 5-regular classes and hence there remains 10 more elements of $\mathrm{irm}(\hat{H})$ to be determined. On 5-regular classes of $\hat{H}$, the following relations between the elements of $\mathrm{irr}(\hat{H})$ holds;

$$1848 = 1792 + 56$$

$$2520a = 2520b = 1232a + 1232b + 56 \qquad (3.2)$$

The elements of $\mathrm{irm}(\hat{M})$ yields the only 5-block of nonzero defect determined by Humphrey [Mu2], with Brauer tree given as:

$$126 \text{ —— } 154 \text{ —— } 56 \text{ —— } 154' \text{ —— } \overline{126}$$

(i) Each of the element (56 + 154), (56 + 154') of $\mathrm{indec}(\hat{M})$ induces to $\hat{H}$ has the following decompositions in terms of elements of $\mathrm{irr}(\hat{H})$:

56+176a+176b+616a+616b+924b+1848+1980a+2.1980b+2304a+ 2304b+2520a+2520b (3.3)

56+176a+176b+616a+616b+924a+1848+2.1980a+1980b+2304a+ 2304b+2520a+2520b (3.4)

Later on, we see that (3.3) and (3.4) are in indec($\hat{H}$).

Moreover, in terms of 5-modular irreducibles of $\hat{M}$, we have

$$56{\downarrow}\hat{M} = \phi\,(28) + \overline{\phi\,(28)}$$

$$924_a{\downarrow}\hat{M} = \overline{\phi\,(28)} + \phi\,(120) + \phi\,(440) + \phi\,(210) + \phi\,(126)$$

$$924_a{\downarrow}\hat{M} = \phi\,(28) + \phi\,(120) + \phi\,(440) + \phi\,(210) + \overline{\phi\,(126)}$$

It therefore follows that the elements of irm($\hat{H}$) associated with (3.3) and (3.4) are of degree 28 each and are also complex conjugate under an outer automorphism as $\phi(28)$, $\overline{\phi\,(28)} \in$ irm($\hat{M}$) has this property under such an outer automorphism when restricted to $\hat{M}$.

Now first of all we calculate the values of the elements denoted by $\phi(56)$, $\overline{\phi(56)}$ of degree 28 each of irm($\hat{H}$), corresponding to (3.3) and (3.4) respectively, explicitly. The explicit values of elements of irm($\hat{M}$) required in the calculation can be found in [Hu2].

Clearly, $\phi(56){\downarrow}\hat{M} = \phi(28)$ and $\overline{\phi(56)}{\downarrow}\hat{M} = \overline{\phi(28)}$.

The values of $\phi(28)$, $\overline{\phi(28)} \in$ irm($\hat{M}$) on 5-regular classes is:

Class names	1A	2A	3A	4A	7A	B^*	8A	6A	11C	C^*
$\phi(28)$	28	4	-1	0	0	0	$2i$	1	$(1-i\sqrt{11})/2$	$(1+i\sqrt{11})/2$
$\overline{\phi(28)}$	28	4	-1	0	0	0	$-2i$	1	$(1+i\sqrt{11})/2$	$(1-i\sqrt{11})/2$

Now, since $\phi(28)\otimes\overline{\phi(28)} = 1 + 55 + 210 + 385 + 133$ (5-mod irre of M, see [Ja]). Furthermore $1{\downarrow}M = 1$; $55{\downarrow}M = 55$; $210{\downarrow}M = 210$ and $518{\downarrow}M = 385 + 133$. Then it follows that $\phi(56)\otimes\overline{\phi(56)} = 1+55+210+518$. Therefore, the values of the characters of $\phi(56)\otimes\overline{\phi(56)}$ on 5-regular classes of $\hat{H}$ are:

Class names	1A	11A	B*	7A	3A	6A	2A	8A	6B	12A	4C
Values	784	3	3	1	0	1	16	4	9	1	64

Thus the values of $\phi(56)$, $\overline{\phi(56)}$ on these classes are:

$\phi(56)$	28	$(1-i\sqrt{11})/2$	$(1+i\sqrt{11})/2$	0	1	1	4	$2i$	$-3i$	$-i$	$-8i$
$\overline{\phi(56)}$	28	$(1-i\sqrt{11})/2$	$(1-i\sqrt{11})/2$	0	1	1	4	$-2i$	$3i$	i	$8i$

Thus, by denoting $\tilde{c}_3$, c_6, c_7 and $\tilde{c}_8$, the inductionof the elements 120, ($\overline{126}$ + 154′), (126 + 154), 330 $\in$ indec($\hat{M}$), respectively to $\hat{H}$ and by c_4, c_5; the induction of 125^+, $125^- \in$ indec($\hat{P}$) to $\hat{H}$, we obtain the matrix $R_5^1(\hat{H})$ as a part of the decomposition matrix for the elements of irr($\hat{H}$):

$R_5^1(\hat{H})$

	c_1	c_2	$\tilde{c}_3$	c_4	c_5	$\tilde{c}_6$	c_6	c_7	$\tilde{c}_8$
56	1	1							
176a	1	1	1						
176b	1	1	1						
616a	1	1	1	1					
616b	1	1	1		1				
924a		1	1	1		1	1		
924b	1		1		1	1		1	
1232a				1		1	1	1	1
1232b					1	1	1	1	1
1792	1	1		1	1		1	1	2
1848	2	1		1	1		1	1	2
1980a	1	2	1	2	1			1	1
1980b	1	1	1	1	2		1		1
2304a	1	1	1	1	1	1	1	1	2
2304b	1	1	1	1	1	1	2	2	2
2520a	1	1		1	1	2	2	2	2
2520b				1	1	2	2	2	2

where $\tilde{c}_6$ = (21⊗1000), c_1 and c_2 are as given in (3.3) and (3.4), respectively.

(ii) Next, we note that $176a\downarrow\hat{M} = 56 + 120 = 176b\downarrow\hat{M}$. This implies that 56 is contined in $176a$ and $176b$ as well; hence $\phi(176a) := 176a - 56$ and $\overline{\phi(176a)} := \phi(176b) := 176b - 56$, leaves distinct projective Brauer characters of degree 120 each and these when restricted to $\hat{M}$ belongs to $\mathrm{irm}(\hat{M})$. Hence $\phi(176a)$, $\overline{\phi(176a)} \in \mathrm{irm}(\hat{H})$.

Furthermore, $22 \otimes \phi(56) = 616a - 176b + 176a - \phi(56)$ (3.5)

and

$$22 \otimes \overline{\phi(56)} = 616b - 176a + 176b - \overline{\phi(56)} \tag{3.6}$$

Now, (3.5) implies that the characters $\overline{\phi(176a)}$ and $\phi(56)$ are contained in $616a$. Similarly, (3.6) implies that $\phi(176a)$ and $\overline{\phi(56)}$ occur in $616b$. Hence it follows that $\tilde{c}_3$ is the sum of two elements of $\mathrm{indec}(\hat{H})$ and thus it must splits to give the columns c_3 and c_3 corresponding to $\phi(176a)$ and $\overline{\phi(176a)}$, respecitvely. Obviously, we have eight possibilities for the splitting of $\tilde{c}_3$ that is,

$$176a + 616b + x.924a + (1-x).924b + y.1980a + (1-y).1980b + z.2304a + (1-z).2304b + 2520a + 2520b. \tag{3.7}$$

$$176b + 616a + (1-x).924a + x.924b + (1-y).1980a + y.1980b + (1-z).2304a + z.2304b + 2520a + 2520b. \tag{3.8}$$

where $0 \le x, y, z \le 1$.

We shall now consider (3.7) only and then (3.8) can be deduced. For convenience, we denote the (3.7) obtained by putting the tripplet (0,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1) and (0,1,1) by T_1, T_2, T_3, T_4, T_5, T_6, T_7 and T_8, respectively. Then we note that the inner product T_1, T_2, T_3, T_4, T_5, T_6, T_7 and T_8 with the projective Brauer character $[(1056-55-133_1-133_2-518) \otimes \overline{\phi(56)}]$; $[55 \otimes \overline{\phi(56)}]$; $[(1408-22-98-210-280-518) \otimes \overline{\phi(56)}]$; $[\phi(56)]$; $[175 \otimes \overline{\phi(56)}]$;

$[55 \otimes \overline{\phi(56)}]$ and $[(1056-55-133_1-133_2-518) \otimes \phi(56)]$, respectively, is a negative integer. Hence the sixth possibility holds. This yields that (3.7) and hence (3.8) are as follows:

$$c_3 = 176a + 616b + 924a + 1980b + 2304a.$$

$$\text{and} \quad c_3' = 176b + 616a + 924b + 1980a + 2304b.$$

However no subsum of (c_3) and (c_3') has degree congruent to 0 modulo 125, thus c_3 and c_3' are elements of indec($\hat{H}$) so we denote them by d_3 and d_3' respectively.

(iii) Now, from the above calculation, we have

$$\phi(924a) := 924a - \overline{\phi(56)} - \phi(176a) - x.\phi(616a)$$

and $\overline{\phi(924a)} = \phi(924b) := 924b - \phi(56) - \overline{\phi(176a)} - x.\overline{\phi(616a)}$,

with $x = 0$ or 1 and clearly they are distinct [see $R_5^1(\hat{H})$].

It therefore follows that $\tilde{c}_6$ must break up to give the new elements d_6 and d_7 of indec($\hat{H}$) associated with $\phi(924a)$ and $\overline{\phi(924a)}$, respectively. Since $\phi(924a)$ and $\overline{\phi(924a)}$ are complex conjugate under an outer automorphism, whereas $2520a = 2520b$, remains invariant under such an outer automosphism, so there are four possible ways in which $\tilde{c}_6$ can split into sum of two elements of indec($\hat{H}$), and they are;

$$924a + x.1232a + (1-x).1232b + y.2304a + (1-y).2304b + 2520a + 2520b \quad (3.9)$$

and

$$924b + (1-x).1232a + x.1232b + (1-y).2304a + y.2304b + 2520a + 2520b \quad (3.10)$$

where $0 \le x, y \le 1$

We shall verify these possibilities only for (3.9). We note that if $(x, y) = (0, 0)$ or $(1, 0)$, then the restriction of (3.9) to $\hat{M}$ contians $\phi(28) \in \text{irm}(\hat{M})$ with negative multiplicity and the inner product of $(1056-55-133_1-133_2-518) \otimes \phi(56)$ with (3.9) is -2 when $(x, y) = (0, 1)$.

Thus (3.9) and hence (3.10) are of the form:

$$c_6' := 924a + 1232a + 2304a + 2520a + 2520b.$$

and

$$c_7' := 924b + 1232b + 2304b + 2520a + 2520b.$$

Clearly, we can take $d_6 = c_6'$ and $d_7 = c_7'$.

(iv) The tensor product $[1750 \otimes \phi(56)]$, with $1750 \in \text{indec}(H)$, gives

$$c_{31}' := 176a + 616a + 616b + 924a + 924b + 1232b + 1792 + 1848 + 2.1980a + 2.1980b + 2304a + 2304b + 2520a + 2520b.$$

as a sum of elements of $\text{indec}(\hat{H})$. Furthermore, $c_{31} = d_3 + c_4 - d_6 + d_7$. It follows that d_6 is a direct summand of c_4 and subtracting this off from c_4, we get that $d_4 \in \text{indec}(\hat{H})$, by the usual argument.

Similarly, the tensor product $[1750 \otimes \overline{\phi(56)}]$, yields that $c_{31}' := d_3' + c_5 + d_6 - d_7$. Hence $d_5 = c_5 - c_7 \in \text{indec}(\hat{H})$.

At this stage, we have found 8 out of 10 elements of $\text{irm}(\hat{H})$ and two elements of irm(H) yet remains to be determined.

(v) Next, firstly we note that

$$d_6 + d_7 = \tilde{c}_6 = c_6 + c_7 - \tilde{c}_8 \qquad (3.11)$$

Now, since

$\phi(1232a) := 1232a - \phi(924a)$ and $\overline{\phi(1232a)} = \phi(1232b) := 1232b - \phi(924b)$,

are distinct elements of $\text{irm}(\hat{H})$, then $\tilde{c}_8$ must split to give c_8, c_8' corresponding to $\phi(1232a)$ and $\overline{\phi(1232a)}$, respectively. Moreover $\phi(1232a)$ and $\overline{\phi(1232a)}$ contains the real characters 1792, 1848, 2520a and 2520b with equal multiplicity. We thus have six possiblities namely; (0, 0), (1, 2), (0, 1), (1, 0), (1, 1) and (0, 2) for the splitting of the column $\tilde{c}_8$ that is

$$1232a + 1792 + 1848 + x.1980a + (1 - x).1980b + y.2304a + (1-y).2304b + 2520a + 2520b \qquad (3.12)$$

1232b + 1792 + 1848 + $(1-x)$.1980a + x.1980b + $(1-y)$.2304a + y.2304b + 2520a + 2520b (3.13)

where $0 \leq x, y \leq 1$.

The first four possibilities for (3.12) are not true, since (3.12) when restricted to $\hat{M}$ contains $\phi(28)$ (respectively, $\overline{\phi(28)}$) with negative multiplicity if (x, y) = (0, 0) (respectively, (1, 2); while the inner product of (3.12) [693 – 55 – 518)] ⊗ $\overline{\phi(56)}$ (respectively, $\phi(56)$) is a negative integer for 3rd and 4th conditions. Let c_{81} & c_{82} denotes (3.12) obtained form 5th & 6th possibilities respectively. However, the R.H.S. of (3.11) implies that $(c_6 - c_{82})$ does not contain 2304a which is contrary to the fact that 6 contains 2304a. Hence $c_8 = c_{81}$, then $d_8 = c_8$, by usual subsum argument.

Similarly, $d'_8 = c'_8 = c'_{81} \in \text{indec}(\hat{H})$, corresponds to (3.13).

Thus we obtain a better approximation $R_5^2(\hat{H})$ to the decomposition matrix of $\hat{H}$, for $p = 5$.

In fact, $R_5^2(\hat{H}) = D_5(\hat{H})$, since none of the di's is contained in c_1 or c_2.

$$R_5^2(\hat{H}) = D_5(\hat{H})$$

	c_1	c_2	d_3	d'_3	d_4	d_5	d_6	c_7	d_8	d'_8
56	1	1								
176a	1	1	1							
176b	1	1		1						
616a	1	1		1	1					
616b	1	1	1			1				
924a		1	1				1			
924b	1			1				1		
1232a							1		1	
1232b								1		1
1792					1	1			1	1
1848	1	1			1	1			1	1
1980a	2	1		1	2	1			1	
1980b	1	2	1		1	2				1
2304a	1	1	1			1	1		1	
2304b	1	1		1	1			1		1
2520a	1	1					1	1	1	1
2520b	1	1					1	1	1	1

4. THE 7-MODULAR CHARACTERS OF $\hat{H}$

Each of the elements 56, 616*a*, 616*b*, 924*a*, 924*b*, 1232*a*, 1232*b*, 1792, 1848, 2520*a* and 2520*b* of irr($\hat{H}$) is in irm($\hat{H}$) and in indec($\hat{H}$) as well, since each of them forms its own block of defect zero.

The remaining 7-block of full defect contains the elements vizly; 176*a*, 176*b*, 1000, 1980*a*, 1980*b*, 2304*a* and 2304*b* of irr($\hat{H}$). Then $126\uparrow\hat{H}$ and $\overline{126}\uparrow\hat{H}$ shows that 2304*a* and 2304*b* are connected with 1000. Moreover

$$56\uparrow\hat{H} = 176a + 176b + 1980a + 1980b + \text{(characters of other blocks)}$$

The elements 1980*a*, 1980*b* $\in$ irr($\hat{H}$) are the only characters which have nonzero value on the class (8*a*) for this block. Furthermore, we have

$$21 \otimes 924a = 176a + 1980a + 1000 + 2304a + 2304b + \text{(characters of zero defect)}$$

Since there are always ambiguities of sign associated with projective characters if a group, therefore without any loss of generality we will assume that the elements of irr($\hat{H}$) are indexed so that the Brauer tree is as follows:

176*a* — 1980*a* — 2304*a* — 1000 — 2304*b* — 1980*b* — 176*b*

5. THE 11-MODULAR CHARACTER OF $\hat{H}$

All but the elements 56, 1000, 1792, 2520*a*, 2520b, 2304*a* and 2304*b*, the later two being exceptional for $p = 11$; belongs to irm($\hat{H}$) and these elements form an 11-block with full defect. Furthmore,

$$22 \otimes 616a = 56 + 2304a + \text{(characters of zero defect)}.$$

$$20^{+}\downarrow\hat{H} = 1000 + 2520b.$$

$$20^{-}\uparrow\hat{H} = 1000 + 2520a.$$

$$22 \otimes 924a = 1000 + 2304a + 2520a + 2520b.$$

$$330\uparrow\hat{H} = 2.1792 + 2.2304a + 2.2520a + 2.2520b + \text{(characters of other blocks)}.$$

where 20^{+} $20^{-} \in$ indec($\hat{P}$), sice 11 does not divide the order of $\hat{P}$.

From above it follows that the Brauer tree of tue 11-block is:

56 — 2304*a* — 2520*a* — 1000 — 2520*b* — 1792

6. THE 2-MODULAR CHARACTER OF $\hat{H}$

Since 2 divides the order of the Schur multiplier of the group H therefore the 2-modular characters of $\hat{H}$ are precisely those of the group H (see [Hul, ∫6]).

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