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CONTENTS

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CERTAIN METABELIAN NILPOTENT LIE ALGEBRAS OF MAXIMAL CLASS

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ABSTRACT

In this paper we describe the structure of certain metabelian nilpotent Lie algebras of maximal (nilpotency) class over the field F; and note that, if $5 \leq n = \dim (L)$ and F is infinite, then the infinite number of non isomorphic such Lie algebras starts from $n \geq 7$.

INTRODUCTION

A nilpotent Lie algebra L of dimension $n \geq 3$ over the field F is said to be a Lie algebra of maximal (nilpotency) class if the dim $(L/L^2)=2$ and dim $(Li/L^{i+1})=1$ for i=2,...,,n-1. If, in addition, $(L^2)^2=0$, then L is said to be a metabelian nipotent Lie algebra of maximal class.

In this context we mention two papers [1] and [3]. However the above definition is an analogue of the definition of metabelian finite p-groups of maximal class (see, for example, [2]).

Unless otherwise stated, L is a meetabelian nilpotent Lie algebra of maximal class and dimension $n \geq 3$. The present work is divided into three sections. In section 1 we describe the structure of such Lie algebras, in section 2 we discuss the classification problem and section 3 contains the main results.

In this section we describe the structure theorem of such Lie algebras. We need the following.

Lemma 1.1. Let L be a Lie algebra with dim $(L)= \geq 4$. Then (i). $L_1 = CL(L^2/L^4)$ is a maximal characteristic proper ideal in L.

(ii). $[L_1, Li] \leq L^{i+2}$, for i = 2,...,n-3.

Proof. (i). Note that for each x in L and Y in $L^2 \backslash L^3$, the mapping ϕ x: $L^2/L^3 \rightarrow L^3/L^4$ defined as ϕ x $(y+L^3) = [x, y] + L^4$ is a linear. Since $L(L^2/L^3, L^3/L^4)$ is a one dimensional vector space of such linear maps, there exists a mapping $\phi: L \rightarrow L$ with ϕ_x for all x in L, which is linear and onto such that ker (ϕ) = L_1 with co-dimension 1. Hence L_1 is a maximal ideal in L. Further if $\alpha \in$ Aut (L), then $\alpha (L^i) \leq L^i$ for i = 2,...,n-1; and if $x \in L_1$, then $\alpha([x, L^2]) \leq \alpha(L^2) \leq L^2$, so that $\alpha(x)$, L_1. This proves (i).

(ii). Follows by induction on i.

Note that for $n \geq 5$, L_1 is both abelian and nonabelian.

The following result describes the structure of L.

Proposition 1.2. Let L be a Lie algebra with $\dim(L) = n \geq 4$. Then there exists a maximal characteristic proper ideal L_1 in L such that $[L_1, L^2] \leq L^4$; and if $Xo \in L \backslash L_1$, $x_1 \in L_1/L^2$ and $[x_{i-1}, x_0] = x_i$, i = 2,...,n-1 , then

(1) $L = < x_0, x_1, ..., x_{n-1} >$ is a vector space over F.

(2) $[x_i, x_j] = 0$ for $2 \leq i \leq j$.

(3) $[x_1, x_i] = c_4 x_{i+2} + c_5 x_{i+3} + ... + c_{n-i+1} x_{n-1}$, where c_j's $\in F$.

Proof. The first part is proved in Lemma 1.1 (i). Further, since $L/L^2 = < x_0 + L^2, x_1 + L^2 >$, we have $L^2 = <[\alpha x_0 + \beta x_1, \gamma x_0 + \delta x_1] > = < \gamma\beta - \alpha\delta) [x_1, x_0] >$, and, therefore, $[x_1, x_0]$ is the basis for L^2 (mod $\underline{L}^3$) as L is a metabelian nipotent Lie algebra of maximal class, so (1) is proved. Further (2) follows from the metabelian property of L; whereas (3) is a consequence of Lemma 1.1 (ii).

Thus in view of above Proposition, L can be defined with the help of a set of parameters $\{c_4, ..., c_{n-1}\}$, where c_j's $\in F$; and every element x in L can be written uniquely as $x = \sum_{i=0}^{n-1} \alpha_i x_i'$ where α_i 's ε F. Conversly.

Proposition 1.3. For each set of parameters $\{c_4,...,c_{n-1})$, $c_j \varepsilon F$, there exists a metabelian nilpotent Lie algebra $L = L(c_1,...,c_{n-1})$ of maximal class with $\dim(L) = n \geq 4$ such that L contains a maximal characteristic proper ideal L_1 with $[L_1 , L^2] \leq L^4$. Further if $x_0 \varepsilon L \backslash L_1$, $x_1 \varepsilon L_1 \backslash L^2$ and $[x_{i-1} , x_0] = x_i$ for $i = 2,..., _{n-1}$, then

(1) $L = < x_0,x_1,...,x_{n-1} >$ is a vector space over F.

(2) $[x_i , x_j] = 0$ for $2 : i , j \leq n-1$.

(3) $[x_1 , x_i] = c_4 x_{i+2} + ... + c_{n-i+1} x_{n-1}$, $2 \leq i \leq n-3$.

Proof. Let $M = <x_2,...,x_{n-1}>$ be an abelian Lie algebra of dimension n-2, where $n \geq 4$, over F. Define a vector space L over F such that $L/M = < x_0+M, x_1+M >$, $[x_{i-1}, x_0] = x_i$, $i = 2,...,n-1$ and $[x_1, x_i] = c_4 x_{i+2} + ... + c_{n-i+2} x_{n-1}$, for $i = 2,...,n-3$. Then it is not difficult to prove that L is a metabelian nilpotent Lie algebra of dimension $n \geq 4$ over F such that $L^2 = M$, with $L_1 = < x_1, M >$ - maximal characteristic proper ideal in L. Since $\dim(L/L^2) = 2$ and $\dim(Li/L^{i+1}) = 1$, $i = 2,...,n-1$, L is a metabelian nilpotent Lie algebra of maximal class.

ξ 2. Now we consider the classification of such Lie algebras. Let $L = L(c_4,...,c_{n-1})$ and $L' = L(c'_4,...,c'_{n-1})$ be two Lie algebras of same dimension $n \geq 4$. If $\phi : L' \to L$ is an isomorphism, then

$$(1) \quad \begin{cases} \phi(x'_0) = \lambda x_0 + \sum_{i-1}^{n-1} a_i x_i \\ \phi(x'_1) = \mu x_1 + \sum_{j=2}^{n-1} b_j x_j \end{cases}$$

where λ, μ, a_i, b_j εF with $\lambda\mu \neq 0$; and

$$\phi(x'_2) = \phi([x'_1, x'_0]) = \lambda(\mu x_2 + \sum_{j-2}^{n=1} b_j x_j + 1) + \sum_{i-2}^{n=3} (\mu a_i -$$

$a_1 b_i)\ [x_1, x_i]$.

To calculaate images of other generators and verify the defining relations, we introduce the following vectors

$$\bar{a} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-2} \end{pmatrix} \qquad \bar{b} \begin{pmatrix} b_1 = \mu \\ \cdot \\ \cdot \\ \cdot \\ b_{n-2} \end{pmatrix}$$

of height n-2, row basis vector $\bar{x} = (x_2, \ldots, x_{n-1})$ and (n-2)x(n-2) matrices

$$C = \begin{pmatrix} 0\ 0\ 0\ 0 \ldots 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \ldots 0\ 0.0 \\ 0\ c_4 0\ 0 \ldots 0\ 0\ 0 \\ 0\ c_5 c_4 0 \ldots 0\ 0\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0\ c_{n-1} \ldots c_5 c_4 0 \end{pmatrix} = \begin{pmatrix} 0\ 0 \ldots 0\ 0\ 0 \\ 1\ 0 \ldots 0\ 0.0 \\ 0\ 1 \ldots 0\ 0\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0\ 0 \ldots 1\ 1\ 0 \end{pmatrix}$$

Note that $IC \neq CI$, but $(IC)\,\bar{y} = (CI)\,\bar{y}$, where

$$\bar{y} = \begin{pmatrix} 0 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \end{pmatrix}$$

Then one can compute

$$\phi(x'_2) = \bar{x}\,(\lambda\,\bar{b} + C\,(\mu\,\bar{a} - a_1\bar{b})),$$

and

$\phi(x'_1) = \bar{x}\,(\lambda E - a_1 C)^{i-1} I^{i-2}\bar{b} + \mu\, C(\lambda E - a_1 C)^{i-2} I^{i-2}\bar{a}$, where E is the identity (n-2)x(n-2) matrix.

The matrix I^{n-3} with unique nonzero element in the left cornor, after multiplying with $(\lambda E - a_1 c)^{n-1}$, Yields unique nonzero element at the same place which is annahilated after multiplying with matric C. Consequently we obtain

$$\phi(x'_{i+1}) = \bar{x}\,(\lambda E - a_1 C)^{\,i} I^{i-1}\,\bar{b} + \mu\, C(\lambda E - a_1 C)^{\,i-1} I^{i-1}\bar{a}$$

•

•

•

$$\phi(x'_{n+2}) = \bar{x}\,(\lambda E - a_1 C)^{\,n-3} I^{n-4}\bar{b} + \mu\, C(\lambda E - a_1 c)^{\,n-4} I^{n-4}\bar{b}$$

$$\phi(x'_{n-1}) = \bar{x}\,(\lambda E - a_1 C)^{\,n-2} I^{n-3}\bar{b} + \mu\, C(\lambda E - a_1 C)^{\,n-3} I^{n-3}\bar{a}$$

$$= \bar{x}\,(\lambda E - a_1 C)^{\,n-2} I^{n-3}\bar{b} = \lambda^{n-2}\,\mu x_{n-1}$$

Further

$$\phi\,([x'_1 , x'_2]) = \bar{x}\,(\lambda\mu\, CI\bar{b}\,(\mu I\bar{a} - a_1 I\bar{b})).$$

Since the co-efficients for the image of $|x|$, $|x'_2|$ do not depend upon $c_4, c_5, \ldots, c_{n-1}$ linearly, we confine ourselves to the case when the co-efficients of the image of $[x'_1 , x'_2]$ depend upon $c_4 , c_5 , \ldots , c_{n-1}$ linearly. The sufficient condition for such case (which is a consequence of simple matrices multiplication) is given by the followingt

Lemma 2.1. Let K be a fixed integer such that $n\text{-}1 \geq K > \left[\frac{n+2}{2}\right]$; and $c_4 = c_5 = \ldots = c_{k-1} = 0$. Then c^2 is zero matrix.

ξ 3. From now on all Lie algebras under consideration are $L = L(c_k, \ldots , c_{n-1})$, where $n\text{-}1 \geq k > \left[\frac{n+2}{2}\right]$. Then $[x_1 , x_{k-1}] = 0$; and CI^{k-2} is a zero matrix. Hence

$$\phi(x'_k) = \bar{x}\,(\lambda^{k-1\;k-2}\bar{b})$$

•

•

•

$$\phi(x'_{n+1}) = \bar{x}\,(\lambda^{n-2} I^{n-3}\bar{b}) = \lambda^{n-2}\mu\, x^{n-1};$$

and

$$\phi([x'_1, x'_2]) = \bar{x}(\lambda\mu\, CI\bar{b}).$$

Lemma 3.1. Let $\theta : L \to L$ be a mapping defined on the generators x_0, x_1 of L by the following formula:

$$\theta(x_0) = x_0 + \sum_{i=2}^{n-1} a_i x_i,$$

$$\theta(x_1) = x_1 + \sum_{i=2}^{n-1} b_j x_j, \; a_i, b_j \in F.$$

Then θ can be extended to an automorphism of L.

Lemma 3.2. Every isomorphism $\phi: L' \to L$ (where $L' = L(c'_k, \ldots, c'_{n-1})$ and $L = L(c_k, \ldots, c_{n-1})$; defined by the formula (I) in ξ 2) is a composition of the automorphism θ from Lemma 3.1 and the isomorphism $\phi: L' \to L$ defined as

$$\text{(II)} \quad \begin{cases} (\psi(x'_0) = \lambda x_0\, a_1 x_1\, a_i x_i) \\ \psi(x'_1) = \mu x_1 \end{cases}$$

where $\lambda, \mu, a_1 \in F$ with $\lambda\mu \neq 0$.

Theorem 3.3. For $n \geq 5$ and $K > [\frac{n+2}{2}]$, Lie algebra L' is isomorphic to L if and only if there exist $\lambda, \mu \in F$ with $\lambda\mu \neq 0$ such that $\mu c_i = \lambda^{1-2} c'_i$, $i = k, \ldots, n-1$.

Proof. .By Lemma 3.2 it is sufficient to consider the isomorphism ψ defined by the formula (II). Then

$$\psi([x'_1, x'_2]) = \psi(c'_k x'_k + \ldots + c'_{n-1} x'_{n-1})$$

implies

$$\bar{x}(\lambda\mu\, CI)\,\bar{b} = \bar{x}(c'_k \lambda\, k^{k-1} I^{k-2} + c'_{k+1} \lambda^{k} I^{k-1} + \ldots + c'_{n-1} \lambda^{n-2} I^{n-3})\bar{b}.$$

Note that in case of ψ we take $\bar{b} = \begin{pmatrix} \mu \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$

Hence

$$\bar{x} \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0^{k-1} \\ \mu\, c'_k \lambda^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ \mu c'_{n-1} \lambda^{n-2} \end{pmatrix} = \bar{x}\, \lambda\mu \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \mu c_k \\ \cdot \\ \cdot \\ \cdot \\ \mu c'_{n-1} \end{pmatrix}.$$

Thus we have $\mu c_i = \lambda^{1-2} c'_i$ for $i = k, k+1, \ldots, n-1$.

Conversly if $\mu c_i = \lambda^{1-2} c'_i$, $i = K, \ldots, n-1$, then it is not difficult to prove that he mapping $\sigma: \rightarrow L$ defined as

$\sigma(x'_0) = \lambda x_o$

$\sigma(x'_1) = \mu x_1$, where $\lambda, \mu \in F$ s.t. $\lambda\mu \neq 0$, is n isomorphism.

If L_1 is abelian i.e $c_k = c_{k+1} = \ldots = c_{n-1} = 0$, then, by above. Theorem, $L = L(0,0, \ldots, 0)$ is unique upto isomorphism; and hence

Corollary 1. For $n \geq 4$ there exists a unique upto isomorphism, metabelian nilpotent Lie algebra of maximal class with abelian maximal characteristic proper ideal.

However L_1 is nonabelian if at least one of the c_i's is nonzero, for $k \leq i \leq n-1$, $n \geq 5$. In this case we have.

Corollary 2. (i). All Lie algebras of the type $L = L(0,\ldots,0,^{cn-1})$, where $c_{n-1} \neq 0$, are, pairwise, isomorphic.

(ii). All Lie algebras of type $L = L(0,...,0, c_{n-2}, c_{n-1})$, where $c_{n-2} \neq 0$, are divided into two classes; namely, $L = L(0,...,0,1,0)$ and $L = L(0,...,0,1,1)$.

To classify the Lie algebras $L = L(c_k,...,c_{n-1})$, where $k \leq n-3$ and $c_k \neq 0$, we consider non-negtive integer h and the sequence q(0), q(1), ... , q(h), whre $K = q(0) < q(1) < ... < q(h) \leq n-1$. Then the total number of the sequences of such type equals to 2^{n-k-1}.

Let $h = |q|$ - number of nonzero terms in q and $C(q)$ be the colection of all Lie algebras which are defined with the help of a set of parameters $\{c_k,...,c_{n-1}\}$ in the folowing way: algebra $L=L(c_k,...,c_{n-1})$ belongs to the class $c(q)$, if $\{c_{q(0)},...,c_{q(h)}\}$ is the set of nonzero parameters of L. Then the Lie algebras belonging to different classes are different upto isomorphism. Moreover.

Proposition 3.4. Let $L = L(c_{q(0)},...,c_{q(h)})$ and $L' = L'(c'_{q(0)},...,c'_{q(h)})$ be two Lie algebras in $C(q)$. Then L is isomorphic to L' if and only if $\mu\, c_{q(i)} = \gamma^{q(i)-2} c'q(i)$, $i = 0,1,...,h$. In particular if $c_{q(0)} = c'_{q(0)}$, then $\mu = \gamma^{q(1)-2}$; and $c_{q(i)} = \gamma^{q(i)-q(0)}\, c'_{q(i)}$, $i = 1,2,...,h$.

Corollary 3.(i). If $h = 0$, then all Lie algebras in $c(q)$ are, pairwise, isomorphic.

(ii). If $h \geq 1$, then the number of isomorphism classes in $c(q)$ depends upon the order of the field F. In particular if F is infinite and $h > 1$, then $c(q)$ consists of infinite number of non isomorphic classes of such Lie algebras.

Remark. It is important to note that infinite number of non isomorphic Lie algebras in $c(q)$ starts from $n \geq 7$.

These results are analogue of certain results of [2] on finite p-groups.

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CENTRAL DECOMPOSITIONS OF FINITE p-GROUPS WITH ABELIAN SECOND CENTRE AND THE CENTRE OF ORDER p.

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ABSRACT

In the present note we study the central decompositions of those finite p-groups G such that $G = A \curlyvee B$, where A is extra-special and B is a finite p-groups with $Z_2(B)$ abelian, so that $Z_2(G)$ is nonabelian with $Z(G)$ of order p; and prove that if

$$G = U_1 \curlyvee U_m \curlyvee V_1 \curlyvee \ldots \curlyvee V_n$$

is an unrefinable central decomposition of G with each $Z_2(U_i)$ of order p^2 and each V_j extra-special of order p^3 (so that $Z_2(G)$, which is generalised directly decomposable into $Z_2(U_j)'$ s and V_j' s, is nonabelian), then G has

$$(1/n!).p^{\binom{m}{2}+2mn+n(n-1)}.(p^{2n}-1)\ldots(p^4-1)/(p^2-1)^{n-1}$$

unrefinable central decompositions, which are, pairwise, isomorphic if p is odd; and if p = 2, then all of such decompositions are not necessarily (pairwise) ismorphic.

INTODUCTION

The present work is in continuation of author's earlier works of [1] and [2] in which the central decompositions of extra-special p-groups and the finite p-groups with abelian second centre and centre of order p, were discussed, respectively.

In the present note we study the central decompositions of those finite p-groups G such that $G = A \curlyvee B$, where A is extra-special and B is a finite p-group with $Z_2(B)$ abelian, so that $Z_2(G)$ is nonabelian with $Z(G)$ of order p; and prove thaat if $G = U_1 \ldots \curlyvee U_m$

$\Upsilon V_1 \Upsilon \ldots \Upsilon V_n$ is an unrefinable central decomposition of G with each $Z_2(U_i)$ of order p^2 and each V_j extra-special of order p^3 (so that $Z_2(G)$, which is generalised directly decomposable into $Z_2(U_i)$ s and V_j' s, is nonabelian), then G has

$$(1/n!).P^{\binom{m}{2} + 2mn + n(n-1)}.(p^{2n}-1)\ldots(p^4-1)/(p^2-1)^{n-1}$$

unrefinable central decompositions, which are, pairwise, isomorphic if p is odd; and if p = 2, then all of such decompositions are not necessarily (pairwise) isomorphic.

This work is divided into three sections. In section 1 we define basic concepts and study the elementar propesties of central decomposition of a group. In section 2, we describe, briefly, the results of [1] and [2] as they will be used in section 3 for finding out the exact (formal) number of the central decompositions of such groups; and to classify them upto isomorphism.

ξ1. A finite set $\{U_1,\ldots,U_r)$ of proper subgroups of a group G is said to be a *generalised direct decomposition* (in short g.d.d) if

i) G is generated by $U_1,\ldots,U_r$; i.e $G = <U1,\ldots,U_r>$,

ii) for $i \neq j$, U_i and U_j commute element wise; i.e $[U_i, U_j] = \{e\}$; and we write G g.d.d $\{U_1,\ldots,U_r\}$.

Such decomposition of G is said to be a *central decomposition* (in short c.d) if $Z(G) \leq U_i$, for all $i = 1,\ldots,r$. In that case each U_i, is called *central factor* of G and we write $G = U_1 \Upsilon \ldots \Upsilon U_r$.

If G possesses such decomposition, then G is said to be *centrally decomposable*, otherwise G is *centrally indecomposable.*

Note that a c.d $G = U_1 \Upsilon \ldots . \Upsilon U_r$ induces a direct decomposition of the factor group $G/Z(G)$; i.e $G/Z(G) = U_1/Z(U_1) x \ldots x U_r/Z(U_r)$.

A c.d $G = U_1 \Upsilon \ldots \Upsilon U_r$ of G is said to be *unrefinable centrad decomposition* (in short u.c.d), if each U_1 is centrally indecomposable.

Two u.c.d, say, $G = U_1 \Upsilon \ldots \Upsilon U_r$ and $G = V_1 \Upsilon \ldots \Upsilon V_s$ of a group G are said to be *isomorphic* if r = s and for each i = 1,...,r there exists j in {1,...,s} such that U_i is isomorphic to U_j.

Let U and V be two subgroups of a group G. Then U is said to be *centrally isomorphic* to V, if there exists an isomorphism $\rho:U \to V$ such that $(u\rho)u^{-1}$ belongs to $Z(G)$.

Note that if U is centrally isomorphic to V, then $UZ(G) = VZ(G)$.

Proposition 1.1. (Remak-Krull-Schmidt theorem (see, for example, [5], pp120)). If a group G has princpal series, then any two direct decompositions of G with directly indecomposable factors are centrally isomorphic.

If G is centrally decomposable, then, in general, the central factors of a u.c.d of G are not determined upto isomorphism and hence two u.c.d of G are not necessarily isomorphic. Even more, Tang [0] proved that two u.c.d of a group may not have equal number of central factors. However, an immediate consequence of Remak-Krull-Schmidt theorem, proves that

Proposition 1.2. If $G = U_1 \Upsilon .... \Upsilon U_r$ and $G = V_1 \Upsilon ... \Upsilon V_s$ are two u.c.d of G such that each $U_i/Z(U_i)$ and each $V_j/Z(V_j)$ is directly indecomposable, then $r = s$.

Again, if $G = U_1 \Upsilon ... \Upsilon U_r$ and $G = V_1 \Upsilon ... \Upsilon V_s$ are two u.c.d of G such that $r = s$, even then they need not to be isomorphic. For example if

$$D = < a,b\ ;\ a^{2^{n-1}} = e = b^2,\ b^{-1}ab = a^{-1} >,$$

$$Q = < c,d\ ;\ c^{2^{n-1}} = e,\ d^2 = c^{2^{n-2}},\ d^{-1}\,ed = c^{-1} >$$

are Dihedral and Generalised Quaternions, respectively, 2-groups of maximal class and order 2^n, where $n \geq 4$, then $G = D\Upsilon D$ and $G = Q\Upsilon Q$ are two u.c.d of G which are not isomorphic.

However, as we see in the following Lemma, if two c.d of a group G have a common central factor, then the other central factors are determined by the centralliser of the common central factor.

Lemma 1.3. Let $G = A\Upsilon B$ and $G = A\Upsilon C$ be two c.d of G. Then $B = C_G(A) = C$.

Proof. Obviously $B \leq C_G(A)$. Conversly let $c \in C_G(A)$. Then there exist a in A and b in B such that $c = ab$. But then $x = x^c = x^{ab} = x^a$

for all x in A; which implies $a \varepsilon Z(A) = Z(B)$. Hence $c = ab$ is an element of B. Replacing B by C proves $C_G(A) = C$.

ξ2. Unless otherwise stated, all groups under consideration are finite p-groups with center of order p.

A non abelian group G is said to be an *extra-special*, if the derived subgroup $G' = Z(G) = \phi(G)$-the Frattini subgroup of G, has order p.

It is well known (see, for example, [3] theorem 5.5.1) that a non-abelian group G of order p^3 is an extra-special and is isomorphic to one of the following groups:

$$M = < x,y,z\,;\,[x\,,y] = z,\ x^p = y^p = z^p = e = [x\,,z] = [y\,,z]>,$$

$$N = < u,v\,;\,u^{p^2} = v^p = e\,,\ v^{-1}uv = u^{-1} > \text{ for } p \geq 3; \text{ and}$$

$$D_3 = < x,y\,;\,x^{2^2} = y^2 = e\,,\ y^{-1}xy = x^{-1} >,$$

$$Q_3 = < u,v\,;\,u^{2^2} = e\,,\ u^2 = v^2,\ v^{-1}uv = u^{-1} > \text{ for } p = 2.$$

Further a nonabelian group of order p^3 is centrally indecomposable and an extra-special group G has a u.c.d into nonabelian subgroups each of order p^3; and, therefore, G has order p^{2r+1}, $r \geq 2$ (see, for example, [1].

Theorem 2.1. ([1] theorem 6). Let G be extra-special group of order p^{2r+1}, $r \geq 2$. Then G has exactly

$(1/r!).p^{r(r-1)}\,(p^{2r}-1)\ldots\,(p^4-1)\,/\,(p^2-1)^{(r-1)}$ u.c.d.

Proposition 2.2. ([3] theorem 5.5.2). Let G be extra-special of order p^{2r+1}, $r \geq 2$. If G has exponent p, then all of the u.c.d of G are, pairwise, isomorphic. In fact each of the u.c.d of G is isomorphic to $M^r = M\Upsilon \,..\, {}_r \,..\, \Upsilon M$. If G has not exponent p, then each of the u.c.d of G is isomorphic to either N^tM^{r-t}, for $p \geq 3$ or $D^t{}_3Q_3{}^{r-t}$, for $p = 2$, where $1 \leq t. \leq r$. Moreover N^tM^{r-t} is isomorphic to NM^{r-1} if $t \geq 1$ and M^r is not isomorphic to NM^{r-1} for $p \geq 3$; and $D^t{}_3Q_3{}^{r-t} \cong D_3Q_3{}^{r-1}$, if $t \geq 1$ and $Q_3^r \neq D_3Q_3^{r-1}$ for $p = 2$. In particular, $N^2 \cong NM$ and $M^2 \neq NM$; and $D_3^2 \cong Q_3^2$, and $D_3Q_3 \neq D_3^2$.

Lemma 2.3. Let G be a group with $Z_2(G)$ of order p^2. Then G is centrally in-decomposable and $K = C_G(Z_2(G))$ is a maximal subgroup of G.

Proof. Suppose G is centrally decomposable and $G = A \curlyvee B$, where A and B, both, are nonabelian. Then $G/Z(G) = A/Z(A) \times B/Z(B)$, and $Z(G/Z(G)) = Z(A/Z(A)) \times Z(B/Z(B))$. Since $Z(G/Z(G))$ has order p, either $Z(A/Z(A))$ or $Z(B/Z(B))$ has trivial order. In that case either A or B is abelian, a contradiction.

Further obviously $K = C_G(Z_2(G))$ is a proper normal subgroup of G and G/K is isomorphic to a p-subgroup of $\mathrm{Aut}(Z_2(G))$. Since $|\mathrm{Aut}(Z_2(G))| = p(p-1)$, G/K has order p, and, therefore, K is a maximal subgroup of G.

In [2] we consider the central decompositions of those groups G such thaat $G = U_1 \curlyvee \ldots \curlyvee U_r$, where each $Z_2(U_i)$ has order p^2, is a u.c.d of G so that $Z_2(G) = $ g.d.d $\{ Z2(U_i) ; i = 1,\ldots,r \}$ is abelian of order p^{r+1}, $r \geq 2$. Because otherwise there exist p-groups with $Z(G)$ of order p, $Z_2(G)$ abelian of order p^3 and $Z2(G) = $ g.d.d $\{A,B\}$, where both A and B has order p^2, yet G is centrally indecomposable.

Example 2.4. Let

$$G = UT(4,p) = < \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_4 & a_5 \\ 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; a_i \varepsilon Z_p >. \text{ Then}$$

$$Z(G) = < \begin{pmatrix} 1&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} >$$ has order p; and $Z_2(G) = < x_1,x_2,x_3 >$, where

$$x_1 = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, x_2 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, x_3 = \begin{pmatrix} 1&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}.$$

Note that $Z_2(G)/Z(G) = < x_1Z(G) , x_2Z(G) >$, where $x_1Z(G) \cap x_2Z(G) = Z(G)$; and, therefore, $Z_2(G) = $ g.d.d $\{ < x_1,x_3 > , < x_2,x_3 > \}$, where each $<x_1,x_3>$ and $<x_2,x_3>$ has order p^2.

Suppose $G = A\Upsilon B$, where A, B, both, are nonabelian subgroups. Then $G/Z(G) = A/Z(A) x B/Z(B)$ and $Z(G/Z(G)) = Z(A/Z(A)) x Z(B/Z(B))$. Since $|G/Z(G)| = p^5$, either $|A/Z(A)| = p^3$ and $|B/Z(B)| = p^2$, or vise versa. If $|B/Z(B)| = p^2$, then B is extra-special of order p^3 and $B/Z(B) = Z(G/Z(B)) = Z(G/Z(G))$ has order p^2. In that case $Z(A/Z(A))$ is trivial, which is not possible. Hence G is centrally indecomposable.

Lemma 2.5. ([2] Lemma 2). Let $G = U_1\Upsilon \ldots \Upsilon U_r$ be a u.c.d of G with each $Z_2(U_i)$ of order p^2. If $G = V_1\Upsilon \ldots \Upsilon V_s$ is a c.d of G is a c.d of G with each $Z_2(V_j)$ of order p^2, then

i) $r = s$ and $G = V_1\Upsilon \ldots \Upsilon V_s$ is a u.c.d of G.

ii) For $i = 1,\ldots,r$, $U_i Z_2(G) = V_j Z_2(G)$, where $1 \leq j \leq s$.

iii) $M_i = C_{U_i}(Z_2(U_i))$ is a maximal subgroup in V_j for some $1 \leq j \leq s$.

Theorem 2.6 ([2] Theorem 2). Let $G = U_1 \Upsilon \ldots \Upsilon U_r$ be a u.c.d of G with $Z_2(U_i)$ of order p^2, $i = 1,\ldots,r$. Then G has exactly $p^{\binom{r}{2}}$ u.c.d; and if $Z_2(G)$ is elementary abelian (as it is for $p \geq 3$, see [4], Theorem 3.7.7), then all u.c.d of G are pairwise isomorphic.

If $p = 2$, then $Z_2(G)$ is not necessarily elementary abelian. Examples of such groups are the 2-groups of maximal nilpotency class, namely, D-the Dihedral group, Q-the Generalised Quaternions and S-the Semidihedral group; each of order 2^n, $n \geq 4$. In this case (as we see in the following Theorem) all u.c.d of G are not necessarily isomorphic.

Theorem 2.7 ([2] theorem 3). Let $G = U_1 \Upsilon \ldots \Upsilon U_r$ be a u.c.d of G, where each $U_i \varepsilon \{ S,D,Q \}$. Then each u.c.d of G is isomrphic to either $S^{r-t}D^t$, $0 \leq t \leq r$ or QD^{r-1}..Moreover

$$S^sD^dQ^q \cong \begin{cases} D^{q+d-1}Q \text{ if } s = 0 \text{ and } q \text{ odd} \\ S^sD^{q+d} \text{ otherwise} \end{cases}$$

ξ3. In this section we study the central decompositions of those groups G such that $G = U_1 \Upsilon \ldots \Upsilon_1 \Upsilon V_s$ is a u.c.d of G, where each $Z_2(U_i)$ has order p^2, and each V_j is extra-special of order p^3;

and, therefore, $Z_2(G) = \text{g.d.d}\ \{ Z_2(U_i)\ ,\ V_j \}$ is non abelian. The following Lemma is an analogue of Lemma 2.5.

Lemma 3.1. Let $G = U_1 \Upsilon \ldots \Upsilon U_r \Upsilon V_1 \Upsilon \ldots \Upsilon V_s$ be a u.c.d of G with each $Z_2(U_i)$ of order p^2 and each V_j- nonablian of order p^3. If $G = A_1 \Upsilon \ldots A_m \Upsilon B_1 \ldots \Upsilon B_n$ is a c.d of G such that each $Z_2(A_k)$ has order p^2 and each B_j is nonabelian of order p^3, then

i) $r = m$ and $s = n$; and, therefore, $G = A_1 \Upsilon \ldots \Upsilon A_m \Upsilon B_1 \Upsilon \ldots \Upsilon Bn$ is a u.c.d of G.

ii) For each $i = 1,\ldots,r$, $U_i Z_2(G) = A_k Z_2(G)$ for $k \in \{1,\ldots,m\}$.

iii) $M_i = C_{U_i}(Z2(U_i))$ is a maximal subgroup in A_k for $1 \leq k \leq m$.

Theorem 3.2. Let $r,s \geq 1$ and $G = U_1 \Upsilon \ldots \Upsilon U_r \Upsilon V_1 \Upsilon \ldots \Upsilon V_s$ be a u.c.d of G, where each $Z_2(U_i)$ has order p^2 and each V_j is nonabelian of order p^3. Then G has exactly

$$(1/s!).p^{\binom{r}{2} + 2rs + s(s-1)}.(p^{2s}-1)\ldots(p^4-1)/(p^2-1)^{s-1}$$

u.c.d.

Proof. Let $U = U_1 \Upsilon \ldots \Upsilon U_r$ and $V = V_1 \Upsilon \ldots \Upsilon V_s$ so that $G = U \Upsilon V$ is a c.d of G. Then the number b of all u.c.d of G is given as $b = b_1.b_2.b_3$, where

b_1 = number of u.c.d of the type $U = U_1 \Upsilon \ldots \Upsilon U_r$, each $Z_2(U_1)$ of order p^2,

b_2 = number of u.c.d of the type $V = V_1 \Upsilon \ldots \Upsilon V_s$, each V_j- nonabelian of order p^3.

b_3 = number of c.d of the type $G = U \Upsilon V$.

Since b_1, b_2 are already known (Theorem 2.1 and Theorem 2.6), we calculate b_3. Note that $A = Z_2(G)/Z(G)$ is elementary abelian; and, therefore, can be considered as a vector space over Z_p, which has skew-symmetric bilinear form β: $AxA \rightarrow Z_p$ such that $\beta(\bar{x},\bar{y}) = z^*([x,y])$, where $\bar{x}$, $\bar{y} \in A$, x,y are the pre-image of $\bar{x}$, $\bar{y}$ (respectively) and z is the generator of $Z(G)$, where $Z(G)$ has order p. Then

$$A_0 = \text{Ker}\beta = \{\bar{x} \in A;\ \beta(\bar{x},\bar{y}) = 0 \text{ for all } \bar{y} \in A\} = Z_2(U)/Z(U).$$

and, therefore, $A = A_0 \oplus A_1$, where A_1 is a subspace of A with nonddegenerating bilinear form β. Since the pre-image V' of the subspace A_1 is extra-special isomorphic to V; and $C_G(V') = C_V(V) = U$, where $U \cap V' = Z(G)$, we have $G = U \curlyvee V'$ another c.d of G. Consequently the number of c.d of G of the type $G = U \curlyvee V$ is equal to the number of decompositions of A of the type $A = A_0 \oplus A_1$.

Note that for any bilinear form of A, defined as above, the kernel A_0 is fixed. Thus if $A = A_0 \oplus A_1$ and $A = A_0 \oplus A_2$ are any two decompositions of A, then $A_1 \cong A/A_0 \cong A_2$, and if α is such isomorphism, then $\alpha(a_1) = a_0 + a_2$, where $a_i \in A_i$ for $i = 0,1,2$. Since the pre-image of A_2 is extra-special isomorphic V; and V has 2s generators and $a_0 \in A_0 = Z_2(U)/Z(U)$ has p^r choices, we have, the number of choices for $a_1 \in A_1$ is p^{2rs}. Hence the number of decompositions of A of the type $A = A_0 \oplus A_1$ is p^{2rs}, i.e $b_3 = p^{2rs}$. Putting the values of b_1,b_2 and b_3, we have required result.

Corollary 3.3. Let $G = U_1 \curlyvee \ldots \curlyvee U_r \curlyvee V_1 \ldots \curlyvee V_s$ be a u.c.d of G as in Theorem 3.2. Then, if p is odd, then all u.c.d of G are pairwise isomorphic. In particular, if $Z_2(G)$ is of exponent p, then each u.c.d of G is isomorphic to $G = U_1 \curlyvee \ldots \curlyvee U_r \curlyvee M^s$: and if $Z_2(G)$ has not exponent p, then each u.c.d of G is isomorphic to $G = U_1 \curlyvee \ldots \curlyvee U_r \curlyvee NM.^{s-1}$

Proof. Let $U = U_1 \curlyvee \ldots \curlyvee U_r$ and $V = V_1 \curlyvee \ldots \curlyvee V_s$ so that $G = U \curlyvee V$ and $Z_2(G)$ = g.d.d { $Z_2(U)$, V }. If $Z_2(G)$ has exponent p, then $Z_2(U)$ is elementary abelian and V has exponent p. Now Theorem 2.6 and Proposition 2.2 yield the result.

If p = 2, then, of course, $Z_2(G)$ has not exponent p; and all of u.c.d of G are not necessarily isomorphic. Examples of such groups are those 2 groups G such that $G = U_1 \curlyvee \ldots \curlyvee U_r \curlyvee V_1 \curlyvee \ldots \curlyvee V_s$ is a u.c.d of G with each $U_i \in \{S,D,Q\}$ and each $V_j \in \{D_3,Q_3\}$. In this case we have the following result:

Theorem 3.5. Let $G = U_1 \curlyvee \ldots \curlyvee U_r \curlyvee V_1 \curlyvee \ldots \curlyvee V_s$ be a u.c.d of G such that each $U_i \in \{S,D,Q\}$ and each $V_j \in \{D_3,Q_3\}$. Then each u.c.d of G isomorphic to either $S^{r-t}D^tQ^s_3$, $0 \le t \le r$ or $D^rQ^s_3 \cong Q^rD^s_3$ or $D^rD^s_3 \cong Q^rQ^s_3$. Moreover

$$S\,{}^{s}D^{d}Q^{q}{}_{3} \cong \begin{cases} Q^{d}D^{q}{}_{3} \text{ if } s = 0 \\ S^{s}Q^{d}D^{q}{}_{3} \cong S^{s}Q^{d}D^{q}{}_{3} \cong S^{s}Q^{d}D^{q}{}_{3} \text{ otherwise.} \end{cases}$$

Proof. The above assertion follows from the fact that $DD_3 \cong QQ_3$, $DQ_3 \cong QD3$ and $SDQ_3 \cong SQQ_3 \cong SDD_3$. For this let

$U = < u,m\ ;\ m^{2^{n-1}} = e\ ,\ u^2 = m^{2^{(n-2)\alpha}}\ ,\ u^{-1}mu = m^{-1+2^{(n-2)\beta}}$ $>$, where $0 \leq \alpha,\beta \leq 1$. Then

$$U \cong \begin{cases} D \text{ if } \alpha = \beta = 0 \\ Q \text{ if } \alpha = 1\ \beta = 0 \\ S \text{ if } \alpha = 0\ \beta = 1. \end{cases}$$

Let $V = < x,y,z\ ;\ [x\ ,\ y] = z,\ z^2 = x^4 = e,\ y^2 = z^{\Upsilon}\ \&\ y^{-1}xy = x^{-1} >$, where $0 \leq \gamma \leq 1$. Then $V \cong D_3$ if $\gamma \cong 0$ and $V \cong Q_3$, if $\gamma = 1$.

If $G = U\Upsilon V$, where $Z_2(U)$ has order p^2 and $V \in \{D_3,Q_3\}$, then, by Theorem 3.2, G has exactly 4 u.c.d; and if $G = A\Upsilon B$ is one of them, then, by Lemma 3.1, $K = C_G(Z_2(U)) = < m >$ is a maximal subgroup in, say, A; and, therefore $A = < ux^ay^b, K >$, where $0 \leq a,b \leq 1$. Similarly $B = < xm^{2^{(n-3)c}}, ym^{2^{(n-3)d}} >$, where $0 \leq c,d \leq 1$. Since A and B commute elementwise, we have $a = d\ \&\ b = c$. It is easy to check that, if $U \cong S,D,Q$, then $A \cong S,Q,D$, respectively; if $V \cong D_3,Q_3$, then $B \cong Q_3,D_3$, respectively.

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ON INVARIANT SUBSETS OF CERTAIN QUADRATIC FIELDS UNDER MODULAR GROUP ACTION

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ABSTRACT

The paper is concerned with the determination of integers, units, and primes in $Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \frac{a^2-n}{c}$ is a rational integer and $(a, \frac{a^2-n}{c}, c) = 1\}$ which is invariant under the action of the Modular group $G = \langle x,y : x^2 = y^3 = 1 \rangle$. The number of ambiguous integers, ambiguous units, non ambiguous units, and ambiguous primes in $Q^*(\sqrt{n})$ are also found.

1. INTRODUCTION

For any two rational integers a and b, (a, b) denotes the greatest common divisor of a and b.

For any non square positive rational integer n, let $Q^*(\sqrt{n}) = \left\{\frac{a+\sqrt{n}}{c} : \frac{a^2-n}{c}\text{ is a rational integer and } (a, \frac{a^2-n}{c}, c) = 1\right\}$.

An element $\alpha = a + b\sqrt{n}$ $a, b \in Q$, $b \neq 0$, in $Q(\sqrt{n})$ is said to be an ambiguous number if α and its conjugate $\bar{\alpha} = a - b\sqrt{n}$, as real numbers, have different sings.

Mushtaq [5] has proved that $Q^*(\sqrt{n})$ is invariant under the group action of $G = \langle x,y : x^2 = y^3 = 1 \rangle$, where x: $C' \rightarrow C$ are the Mobius transformations defined by:

$x(z) = -\frac{1}{z}$, $y(z) = \frac{(z-1)}{z}$ and C′ is the set of non zero complex numbers.

He has further shown that $Q^*(\sqrt{n})$ contains only a finite number of ambiguous numbers and those occurring in a particular orbit of $Q^*(\sqrt{n})$ form a unique closed path in the coset diagram under the action of G on $Q^*(\sqrt{n})$.

The exact number of ambiguous numbers in $Q^*(\sqrt{n})$ has been determined in [4], as a function of n.

In this paper we determine the integers, units, and primes in $Q^*(\sqrt{n})$ (c.f.[2] for definitions) and also find the number of ambiguous integers, ambiguous units, non ambiguous units, and ambigous primes of $Q^*(\sqrt{n})$.

The notation is standard and we follow [1], [2], [3], [4] and [5]. In particular for any real number x, $[x]$ denotes the largest rational integer not greater than x.

2. INTEGERS IN $Q^*(\sqrt{n})$

This section is concerned with the determination of integers of $Q^*(\sqrt{n})$. The number of ambiguous integers of $Q^*(\sqrt{n})$ is also found.

In contrast to the integers of $Q^*(\sqrt{n})$ found in [1] with n as a square free positive rational integer we, throughout this paper, assume that $n=k^2$m, where m is a square free positive rational integer and k is any non zero rational integer. Since either $m \equiv 1 \pmod 4$ or $m \equiv 2, 3 \pmod 4$, we discuss these cases separately.

Theorem 2.1

Let $n=k^2m$. Then:

1. If $m \equiv 2$ or $3 \pmod 4$ with k any rational integer or if $m \equiv 1 \pmod 4$ and k is even then the integers of $Q^*(\sqrt{n})$ are all of the form $\frac{a+\sqrt{n}}{\pm 1}$.

2. If $m \equiv 1 \pmod 4$ and k is odd then the integers of $Q^*(\sqrt{n})$ are either of the form $\frac{a+\sqrt{n}}{\pm 1}$ or $\frac{a+\sqrt{n}}{\pm 2}$.

Proof:

Let $n=k^2m$.

1. Suppose that $m \equiv 2$ or $3 \pmod 4$. Then, by [1], $\frac{a+\sqrt{n}}{\pm c} \in Q^*(\sqrt{n})$ is an integer if and only if $\frac{a+\sqrt{n}}{c} = \frac{a+k\sqrt{m}}{c} = \alpha + \beta$ are rational integers. That is so if and only if $c \mid a$, and $c \mid k$. But then $c^2 \mid (a^2 - mk^2)$ so that $c \mid \frac{(a^2-mk^2)}{c}$ *and* $(a, \frac{(a^2-mk^2)}{c}, c) = c$ Since $(a, \frac{(a^2-mk^2)}{c}, c) = 1$, $c = \pm 1$. Hence the integers of $Q^*(\sqrt{n})$ are all of the form $\frac{a+\sqrt{n}}{\pm 1}$.

Next let $m \equiv 1 \pmod 4$. Then, by [1], $\frac{a+\sqrt{n}}{c} = \frac{a+k\sqrt{m}}{c} \in Q^*(\sqrt{n})$ is an integer if and only if $\frac{a}{b} = \alpha \frac{\beta}{2}, \frac{k}{c} = \frac{\beta}{2}$, where α, β are rational integers. But then $\alpha = \frac{a+k}{c}$, $\beta = \frac{2k}{c}$ so that $c \mid (a+k)$, $c \mid 2k$. Let $(c, k) = d$. Then $d \mid k$ so that, as $c \mid (a+k)$, $d \mid (a+k)$. But then $d \mid a$. So $d = (a, d) = (a, (c, k)) = (a, c, k)$ But $(a, c, k) = 1$. Hence $d = 1$. Since $(c, k) = 1$ and $c \mid 2k$, $c \mid k$, $c \mid 2$ so that we have $c = \pm 1$ or ± 2.

Case (1) Suppose that k is even. Then as $(c, k) = 1$, $c = \pm 1$. So $\frac{a+\sqrt{n}}{\pm 1}$ is an integer of $Q^*(\sqrt{n})$.

Case (2) Suppose that k is odd.

(i) If $c = \pm 1$, then obviously $(a, \frac{(a^2-mk^2)}{c}, c) = 1$ and $\frac{(a^2-mk^2)}{c}$ is a rational integer, so $\frac{a+\sqrt{n}}{\pm 1}$ is an integer of $Q^*(\sqrt{n})$.

(ii) If $c = \pm 2$, then, as k is odd and $c \mid (a+k)$, *a is odd*. Let $a = 2r + 1$, $k = 2t + 1$, where $r, \in Z$. As $m \equiv 4s + 1$ for some $s \in Z$.

So $(a^2 - mk^2) = 4\,[r^2 + r - s\,(4t^2 + 4t + 1) - t^2 - t]$ and, consequently, $4 \mid (a^2 - mk^2)$.

Hence $\dfrac{(a^2 - mk^2)}{\pm 2}$ is an even rational integer. Since a is odd, $\left(a, \dfrac{(a^2 - mk^2)}{\pm 2}, \pm 2\right) = 1$. Thus $\dfrac{a + \sqrt{n}}{\pm 2}$ is also an integer of $Q^*(\sqrt{n})$.

This completes the proof.

Theorem 2.2

Let $n = k^2 m$. Then:

1. If $m \equiv 2$ or $3 \pmod 4$ with k any rational integer or if $m \equiv 1 \pmod 4$ and k is even then the number of ambiguous integers in $Q^*(\sqrt{n})$ is $2+4\,[\sqrt{n}]$.

2. If $m \equiv 1 \pmod 4$ and k is odd then the number of ambiguous integers in $Q^*(\sqrt{n})$ is $2+6\,[\sqrt{n}]$ or $4+6\,[\sqrt{n}]$, according as $[\sqrt{n}]$ is even or odd respectively.

Proof:

Let $n = k^2 m$.

1. If $m \equiv 2$ or $3 \pmod 4$ with k any rational integer or if $m \equiv 1 \pmod 4$ and k is even then, by theorem 2.1, the only integers of $Q^*(\sqrt{n})$ are all of the form $\dfrac{a + \sqrt{n}}{\pm 1}$. If $\alpha = \dfrac{a + \sqrt{n}}{\pm 1}$ is an ambiguous integer of $Q^*(\sqrt{n})$, then $\alpha\,\bar{\alpha} = a^2 - n < 0 \Rightarrow a^2 < n$. So the possible values of a such that $a^2 < n$ are

$$0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm [\sqrt{n}].$$

Hence the number of ambiguous integers of the form $\dfrac{a + \sqrt{n}}{\pm 1}$ of $Q^*(\sqrt{n})$ is $2+4[\sqrt{n}]$.

2. If $m \equiv 1 \pmod 4$ and k is odd then, by theorem 2.1, the only integers of $Q^*(\sqrt{n})$ are either of the form $\dfrac{a + \sqrt{n}}{\pm 1}$ or $\dfrac{a + \sqrt{n}}{\pm 2}$. But

$\frac{a+\sqrt{n}}{\pm 2}$ is an ambiguous integer of $Q^*(\sqrt{n})$ if and only if $a^2 < n$.

The possible choices for such an a are:

$$0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm [\sqrt{n}]. \qquad (*)$$

Moreover $\frac{a^2-n}{2}$ is rational integer if and only if $2|\,(n-a^2)$. An n is odd, a is odd. So such possible values of a in (*) is $[\sqrt{n}]$ or $[\sqrt{n}]+1$ according as $[\sqrt{n}]$ is even or odd respectively. Hence the number of ambiguous integers of the form $\frac{a+\sqrt{n}}{\pm 2}$ of $Q^*(\sqrt{n})$ is $2[\sqrt{n}]$ or $2([\sqrt{n}]+1)$ according as $[\sqrt{n}]$ is even or odd respectively. As proved above the number of ambiguous integers of the form $\frac{a+\sqrt{n}}{\pm 1}$ of $Q^*(\sqrt{n})$ is $2+4[\sqrt{n}]$. Thus the total number of ambiguous integers in $Q^*(\sqrt{n})$ is $2+4[\sqrt{n}]$ or $4+6\,[\sqrt{n}]$ according as $[\sqrt{n}]$ is even or odd respectively.

3. UNITS AND PRIMES IN $Q^*(\sqrt{n})$

In this section we investigate the units and primes in $Q^*(\sqrt{n})$ and determine the number of ambiguous units, non ambiguous units and ambiguous primes in $Q^*(\sqrt{n})$.

It is mentioned that an ambiguous unit (respectively prime) is a unit (respectively prime) which is an ambiguous number in $Q^*(\sqrt{n})$.

Theorem 3.1

Let $n=k^2m$. Then:

1. If $m \equiv 2$ or $3 \pmod 4$ with k any rational integer or if $m \equiv 1 \pmod 4$ and k is even then an integer $\frac{a+\sqrt{n}}{\pm 1}$ in $Q^*(\sqrt{n})$ is a unit if and only if $n \pm 1$ is a perfect square and these units are $\frac{\pm\sqrt{n \pm 1}+\sqrt{n}}{\pm 1}$.

2. If $m \equiv 1(\text{mod}4)$ and k is odd then an integer $\frac{a+\sqrt{n}}{\pm 1}$ or $\frac{a+\sqrt{n}}{\pm 2}$ in $Q^*(\sqrt{n})$ is a unit if and only if $n \pm 1$ or $n \pm 4$ is a perfect square and these units are $\frac{\pm\sqrt{n \pm 1} + \sqrt{n}}{\pm 1}$ or $\frac{\pm\sqrt{n \pm 4} + \sqrt{n}}{\pm 2}$.

Proof:

By [2], we know that an integer $\alpha = \frac{a+\sqrt{n}}{\pm 1}$ in $Q^*(\sqrt{n})$ is a unit $\Leftrightarrow N(\alpha) = \pm 1 \Leftrightarrow \alpha\bar{\alpha} = a^2 - n = \pm 1 \Leftrightarrow a^2 = n \pm 1 \Leftrightarrow n \pm 1$ is a perfect square.

Likewise $\alpha = \frac{a+\sqrt{n}}{\pm 2}$ is a unit $\Leftrightarrow N(\alpha) = \pm 1$

$\Leftrightarrow \alpha\bar{\alpha} = \frac{a^2 - n}{4} = \pm 1 \Leftrightarrow a^2 = n \pm 4 \Leftrightarrow a^2 = n \pm 4 \Leftrightarrow n \pm 4$ is a perfect square.

Theorem 3.2

Let $n=k^2m$. Then:

1. If $m \equiv 2$ or 3 (mod4) with k any rational integer or if $m \equiv 1$(mod 4) and k is even then an integer $\frac{a+\sqrt{n}}{\pm 1}$ in $Q^*(\sqrt{n})$ is a prime if $n \pm p$ is a perfect square for some rational prime p and these primes are $\frac{\pm\sqrt{n \pm p} + \sqrt{n}}{\pm 1}$.

2. If $m \equiv 1$(mod 4) and k is odd then an integer $\frac{a+\sqrt{n}}{\pm 1}$ or $\frac{a+\sqrt{n}}{\pm 2}$ in $Q^*(\sqrt{n})$ is a prime if $n \pm p$ or $n \pm 4p$ is a perfect square for some rational prime p and these primes are $\frac{\pm\sqrt{n \pm p} + \sqrt{n}}{\pm 1}$ or $\frac{\pm\sqrt{n \pm 4} + \sqrt{n}}{\pm 2}$.

Proof:

By [2], we know that an integer $\alpha = \frac{a+\sqrt{n}}{\pm 1}$ in $Q^*(\sqrt{n})$ is a prime if

$N(\alpha) = \alpha\,\bar{\alpha} = a^2 - n = \pm p$, where p is a rational prime. So $a^2 = n \pm p$.

That is if $n \pm p$ is a perfect square. Then $a \pm\sqrt{n}$, where $a = \pm\sqrt{n \pm p}$, are primes of $Q^*(\sqrt{n})$.

Similarly an integer $\alpha = \frac{a+\sqrt{n}}{\pm 2}$ in $Q^*(\sqrt{n})$ is a prime if $N(\alpha) = \alpha\,\bar{\alpha} = \frac{a^2-n}{4} = \pm p$, where p is a rational prime. So α is a prime if $a^2 = n \pm 4p$.

Hence $\frac{a+\sqrt{n}}{\pm 2}$ is a prime if $a = \pm\sqrt{n \pm 4p}$.

The converse is false; for example the eight numbers $\frac{\pm\sqrt{10 \pm 6} + \sqrt{10}}{\pm 1}$ in $Q^*(\sqrt{10})$ are all primes whereas 6 is not a rational prime.

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PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT

Let Q_λ^* (α , δ , k) be the class of analytic functions f in the unit disc with $f(0)=0, f^1(0)=1$ and satisfying

$$[(1-X)\frac{zf'(z)}{g(z)}+\lambda\frac{(zf'(z))'}{g'(z)}]\ \varepsilon\ P_k(\alpha), \qquad k\geq 2$$

$z\varepsilon E$, where $P_k(\varepsilon)$ is a generalized form of the class P of functions of positive real part. We discuss some properties of this class.

KEY WORDS AND PHRASES: analytic, close to convex function, star like, convolution.

1991 AMS SUBJECT CLASSIFICATION: 30C45.

1. INTRODUCTION

Let $P_k(\alpha)$ be the class of functions p analytic in the unit disc E={z: $|z|<1$}, satisfying the properties $p(0)=1$ and

$$\int_0^{2\pi} \left|\frac{\text{Re}\, p(z)-\alpha}{1-\alpha}\right| d\theta \leq k\pi$$

where $z=re^{i\theta}$, $k\geq 2$ and $0\leq \alpha<1$ see [4]. We note that, for $\alpha=0$, we obtain the class $P_k(\alpha)$ defined by Pinchuk [5] and for $\alpha=0$, $k=2$, we have the class P of functions with positive real part. The case $k=2$ gives us the class $P(\alpha)$ of functions with positive real part greater than α.

$$p(z)=\frac{1}{2}\int_0^{2\pi} \left|\frac{1+(1-2\alpha)\, ze^{-it}}{1-ze^{-it}}\right| d\mu(t)$$

where $\mu(t)$ is a function with bounded variation on $[0,2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \le k$$

The class $P_k(\alpha)$ is a convex set and for $p \varepsilon P_k(\alpha)$, we can write

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) P_2(z) \quad P_1 \,\&\, P_2 \,\varepsilon\, P_k(\alpha)$$

Let f and g be analytic in E with:

$f(z)=\sum_{n=0}^{\infty} a_n z_n$, $g(z)=\sum_{n=0}^{\infty} b_n z_n$, then the convolution (Hadamard) product of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Definition 1.1

Let f given by

$$f(z)= z + \sum_{n=2}^{\infty} a_n z^n,$$

be analytic in E and for λ complex with Re $\lambda \ge 0$, let

$$(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda \frac{(zf'(z))'}{g'(z)}] \,\varepsilon\, P_k(\alpha), \quad (z \varepsilon E),$$

$0 \le \alpha \le 1$, $g \varepsilon S^*(\delta)$, $0 \le \delta \le 1$. Then $f \varepsilon Q_\lambda(\alpha, \delta, k)$, $(z \varepsilon E)$. We note that

(i) $Q_0(\alpha, \delta, 2) = K(\alpha, \delta)$, the class of close to convex functions of order α type δ[2].

(ii) $Q_0(0,0,2) = K$, is the well-known class of close to convex functions introduced by Kaplan [1].

2. PRELIMINARY RESULTS

Lemma 2.1 [3]: Let $u=u_1+iu_2$ and $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex valued function satisfying the conditions.

(i) (u,v) is continuous in a domain $D \subset c^2$

(ii) $(1,0) \in D$ and $\Psi(1,0)>0$.

(iii) Re $\Psi(iu_2,v_1) \leq 0$ whenever $(iu_2,v_1) \varepsilon D$ and

$v_1 \leq -\frac{1}{2}(1+u_2^2)$. If $h(z) = 1+\sum_{m=2}^{\infty} c_m z^m$ is a function, analytic in E, such that $[h(z), zh'(z)] \varepsilon D$ and Re $\Psi[h(z), zh'(z)]> 0$ for $z\varepsilon E$, then Re $h(z) > 0$ in E.

Lemma 2.2 [6]: if $p(z)$ is analytic in E with $p(0)=1$

and if λ is a complex number satisfying Re $\lambda \geq 0$, then

Re $[p(z) + \lambda zp'(z)] > \alpha$, $(0\leq\alpha<1)$ implies Re $p(z)>\alpha+(1-\alpha)(2\gamma-1)$ where γ is given by

$$\gamma = \gamma(\text{Re}\,\lambda) = \int_0^1 (1+t^{\text{Re}\lambda})^{-1}\, dt, \quad (2.1)$$

which is an increasing function of (Re λ) and $\frac{1}{2} \leq \alpha < 1$. The estimate is sharp in the sence that the bound can not be improved.

Lemma 2.3 [7]: If $p(z)$ is analytic in E ,$p(0)=1$ and Re $p(z)>\frac{1}{2}$, $z\varepsilon E$, then for any function F, analytic in E, the function $P_* F$ takes values in the convex hull of the image of E under F.

3. MAIN RESULTS

Theorem 3.1

Let $f\varepsilon Q(\alpha, \delta, k)$, $\lambda\geq 0$

Then $\frac{zf'(z)}{g(z)} \varepsilon P_k(\alpha)$, where

$$\alpha = \frac{2\alpha + \lambda\delta_1}{2 + \lambda\delta_1} \quad , \text{Re } h = \text{Re } \frac{Zg'}{g}$$

and

$$\delta_1 = \frac{\text{Re } h(z)}{|h(z)|^2} \quad , h(z) \, \varepsilon \, p(\delta) \quad , 0 \leq \delta_1 \leq 1.$$

Proof:

For $g \, \varepsilon S^*(\delta)$ let

$$\frac{zf'(z)}{g(z)} = [(1-\gamma)\, p(z) + \gamma]$$

where

$$p(z) = (\frac{k}{4} + \frac{1}{2})P_1 - (\frac{k}{4} - \frac{1}{2})P_2 \, , P_1, P_2 \, \varepsilon \, P, p_1(0)=1 \, , p_2(0)=1$$

$$\frac{zf'(z)}{g(z)} = (1-\gamma)\, [(\frac{k}{4} + \frac{1}{2})\, p_1 - (\frac{k}{4} - \frac{1}{2}) p_2] + \gamma$$

$$= (\frac{k}{4} + \frac{1}{2})\, [(1-\gamma)p_1 + \gamma] - (\frac{k}{4} - \frac{1}{2})\, [(1-\gamma)p_2 + \gamma]$$

So,

$$(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda \frac{(zf'(z))'}{g'(z)} - \alpha = [(1-\gamma)\, p(z) + (\gamma-\alpha) + \lambda(1-\gamma)\, \frac{zp'(z)}{h(z)}]$$

$$= (\frac{k}{4} + \frac{1}{2})\, [(1-\gamma)p_1 + (\gamma-\alpha) + \lambda(1-\gamma)\, \frac{zp'1}{h(z)})] - (\frac{k}{4}\, \frac{1}{2})\, [(1-\gamma)p_2 + (\gamma-\alpha) + \lambda(1-\gamma)\, \frac{zp'(z)}{h(z)}] \tag{3.2}$$

The right hand side of (3.2) belongs to $P_k(\alpha)$, since $f \, \varepsilon Q_\lambda(\alpha, \delta, k)$.

Put

$P_i(z) = u$, $zp_i'(z) = v$, then

$$(u, v) = [(1-\gamma)p_i + (\gamma - \alpha) + \lambda(1-\gamma)\, \frac{zp_i'}{h}]$$

$$\gamma(u, v) = (1-\gamma)u + (\gamma-\alpha) + \lambda\, (1-\gamma)\frac{v}{h}.$$

$$\text{Re}\,\Psi(iu_2, v_1) = \text{Re}\,[(1-\gamma)iu_2 + (\gamma-\alpha) + \lambda(1-\gamma)\frac{v1}{h}]$$

$$= (\gamma-\alpha) + \lambda(1-\gamma)\, v_1\, \text{Re}\,\frac{1}{h}$$

$$\text{Re}\,\Psi(iu_2, v_1) = (\gamma-\alpha) + \lambda(1-\alpha)v_1 \frac{\text{Reh}}{|h|^2},$$

$$\text{Re}\,\Psi(iu_2, v_1) = (\gamma-\alpha) + \lambda(1-\gamma)v_1\,\delta_1,\quad \frac{\text{Reh}}{|h|^2} = \delta_1$$

Now, for $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we have

$$\text{Re}\,\Psi(iu_2, v_1) \leq (\gamma-\alpha) - \frac{1}{2}\lambda\,(1-\gamma)\,\delta_1\,(1+u_2^2)$$

$$= \frac{1}{2}[\{(2\gamma - 2\alpha) - \lambda\,(1-\gamma)\delta_1\} \quad -\lambda\,(1-\gamma)\delta_1 u_2^2]$$

$$= \frac{1}{2}(A + Bu_2^2)$$

where

$$A = 2(\gamma-\alpha) - \lambda(1-\gamma)\delta_1 \quad ,$$

$$B = -\lambda(1-\gamma)\delta_1 \leq 0.$$

Re $\Psi(iu_2, v_1) \leq 0$ if $A \leq 0$ and this gives us λ value of γ as defined by (3.1). We now apply lemma (2.1) to conclude that Re$p(z)>0$ in E and hence $f \varepsilon K$ (γ, δ)

Theorem 3.2.

Let $(1-\lambda) f' + \lambda(Zf')' \,\varepsilon P_k(\alpha)$. (3.3)

Then $f' \varepsilon\, P_k(\gamma)$ where γ is defined by (2.1)

Proof:

Let $f'(z) = p(z)$, $p(0) = 1$.

$$f'(z) + \lambda z f''(z) = p(z) + \lambda\ z\ p'(z).$$

$$p(z) + \lambda z\, p'(z)\, \varepsilon P_k\,(\alpha) \qquad (3.4)$$

Let $f'(z) = p(z)$, $p(0) = 1$

$$p = [(\frac{k}{4}+\frac{1}{2})\, p_1 - (\frac{k}{4}-\frac{1}{2})p_2] \quad (3.5)$$

We have to show that p_1 , p_2 ε $P_k(\alpha)$, ($Re p_1 > 0$, $Re p_2 > 0$)

$[p(z)+ \lambda\, z\, p'(z)]\ \varepsilon P_k\ (\alpha)$.

From (3.5) we can write

$$p(z) + \lambda z\, p'(z) = (\frac{k}{4}+\frac{1}{2})\, [p_1 + \lambda z\, p'_1(z)] - (\frac{k}{4}-\frac{1}{2})\, [p_2+\lambda z p'_2(z)].$$

------(3.6)

From (3.4), the right hand side of (3.6) belongs to $P_k(\alpha)$. This implies that

$[p_i(z) + \lambda z\, p'_i(z)] \in P(\alpha)$,$i=1,2$

Now, using Lemma 2.2, we deduce that $p_i \in P(\gamma)$,i=1,2, and hence $p \in P_k(\gamma)$. This completes the proof.

Theorem 3.3

Let $\varnothing$ be convex univalent in E and $f \in Q_\lambda$ (γ, 1, k). Then $(\varnothing * f) \in Q_\lambda$ (γ, 1, k) for $z \in E$

Proof

Let $H = (\varnothing * f)$. Then

$$H'(z) + \lambda z\, H''(z) = (\varnothing * f)\ (z) + \lambda\ (\varnothing * f)\ (z)$$

$$= [\frac{\varnothing (z)}{z} * f'\ (z)]' + \lambda\ [\frac{\varnothing (z)}{z} * z f''\ \ (z)]$$

$$= \frac{\varnothing (z)}{z} * [f'\ (z) + \lambda z f''\ (z)]$$

$$Re\ [\frac{\varnothing (z)}{z} * f'\ (z) + \lambda z\, f''\ (z)]\ G P_k(\alpha).$$

$$(1-\lambda)\, \frac{zf'}{z} + \lambda\ (zf)' \in P_k(\alpha)$$

$$(1-\lambda)\, H' + \lambda\ (zH')' \in P_k(\alpha).$$

$$H' - \lambda H'' + \lambda\ (H' + zH'') = H' + \lambda Z\, H'' \in P_k(\alpha)$$

$$H' + \lambda\, z\, H'' = \frac{\varnothing}{z} * [(\frac{k}{4}+\frac{1}{2})p_1 + (\frac{k}{4}-\frac{1}{2})p_2] \in P_k(\alpha)$$

$$H' + \lambda z H'' = (\frac{k}{4}+\frac{1}{2})(\frac{\varnothing}{z} * p_1) - (\frac{k}{4}-\frac{1}{2})(\frac{\varnothing}{z} * p_2).$$

but

$(\frac{\varnothing}{z} * p_1) \in P(\alpha)$, since $p_1, p_2 \in P(\alpha)$, by using Lemma 2.3

$$[\frac{\varnothing}{z} * P_2] \in P(\alpha)$$

$$[(\frac{k}{4}+\frac{1}{2})(\frac{\varnothing}{z} * P_1) - (\frac{k}{4}-\frac{1}{2})(\frac{\varnothing}{z} * P_2)] \in P_k(\alpha).$$

In fact a more general form of theorem 3.3 can be obtained as follows.

Theorem 3.4

Let h be analytic in E, $h(0)=0$, $h'(0)=1$ satisfy the condition $\mathrm{Re}\frac{h(z)}{z} > \frac{1}{2}$, $Z \in E$.

Let $f \in Q\lambda$ $(\alpha, 1, k)$, then $(f_* h) \in Q_\lambda$ $(\alpha, 1, k)$.

As an application of this theorem we have the following.

Theorem 3.5

Let $f \in Q_\lambda$ $(\alpha, 1, k)$. Then Q_λ $(\alpha, 1, k)$ is invariant under the following integral operators

i) $f_1(z) = \int_0^z \frac{f(t)}{z} dt$

ii) $f_2(z) = \frac{2}{Z} - \int_0^z f(t)\ dt$ Libra's operator.

iii) $f_3(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt$, $|x| \le 1$, $x \neq 1$.

iv) $f_4 = \frac{1+c}{z^c} \int_0^z t^{c-t} f(t)\ dt$, $\mathrm{Re}c \ge 0$.

$f_1(z)=f(z)*\varnothing_1(z)$.

$f_2(z)=f(z)*\varnothing_2(z)$.

$f_3(z)=f(z)*\varnothing_3(z)$.

$f_4(z)=f(z)*\varnothing_4(z)$.

$\varnothing_i$, $i=1,2,3,4$, are convex and

$$\varnothing_1(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n.$$

$$\varnothing_2(z)= \frac{-2(\log(1-z)+z)}{z} = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n.$$

$$\varnothing_3(z)= \frac{1}{1-x} \log\left(\frac{1-xz}{1-z}\right) = \sum_{n=1}^{\infty} \frac{(1-x^n)}{(1-x)^n} z^n.$$

$|x| \leq 1, x \neq 1.$

$$\varnothing n(z)= \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n \quad , \text{Re } c \geq 0.$$

Let μ_1, μ_2 be linear operators defined as

$\mu_1 [f(z)] = z f'(z)$

$\mu_2 [f(z)] = [f(z) + zf'(z)]/2$

Theorem 3.6

Let $f \varepsilon Q_\lambda (\alpha, 1, k)$,. Then

$$h_1 * f = \left[\frac{z}{(1-z)^2}\right] * f = zf' \quad \varepsilon Q_\lambda (\alpha, 1, k).$$

for $|z| < 2-\sqrt{3}$ and

$$h_2 * f = \left(\frac{(z-z^2)/2}{(1-z)^2}\right) * f = (zf)'/2 \quad \varepsilon Q_\lambda (\alpha, 1, k)$$

for $|z| < \frac{1}{2}$.

The proof follows at once when we use theorem 3.3.

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E.F. RINGS

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ABSTRACT

Rings in which every module is embedded in a flat module are characterized.

1. INTRODUCTION

We consider all rings to be associative with nonzero identity 1 and all modules unitary. R–mod and mod–R will mean the category of abelian groups, category of left R–modules and category of right R–modules respectively. (A, B) (respectively $[A, B]$ denote, $\text{Hom}_z(A, B)$ [respectively $\text{Hom}_R(A, B)$] where $A, B \in$ Ab (respectively $A, B \in$ R–mod or mod–R). If S is any other ring then R–mod–S represents the category of left R, right S bimodules. We write Hom and ⊗ for Hom_R and ⊗ for Hom_R and $\otimes_R$ respectively. Unless stated otherwise all modules are considered to be in R–mod. → and → will mean monomorphism arrow and epimorphism arrow respectively. Integral domain means a commutative ring with no nonozero zero divisor. ${}_R\mathfrak{I}$ (respectively $\mathfrak{I}_R$) represents the class of modules in R–mod (respectively in mod–R) which are embedded in flat modules in R–mod (respectively in mod–R).

A module A is said to be in (left) E.F. module if it can be embedded in a flat module. Similar definition for E.F. modules in mod–R holds. For the commutative case E.F. modules were introduced by Enochs [3]. In the general case Colby [2] has introduced an I.F. ring in which every injective module is flat. Thus an I.F. ring is a ring in which every module is an E.F. module. Any regular ring is an I.F. ring, Kaplansky [5], so every module of a regular ring is an E.F. module. E.F. modules are a more natural generalization of torsion free modules and for an integral domain the two notions coincide, i.e., a module A is torsion free if and only if A is an E.F. module. However it is not necessarily true in the general case. Ansari [1] proved that if R is a two sided Ore domain then the two notions coincide in the non-commutative case also.

If I is an integral domain and $\bar{I}$ the classical field of quotients of I then for a module A the following sequence is exact.

$$A/T(A) \rightarrow \bar{I} \otimes_I A \rightarrow \bar{I}/I \otimes_I A \quad (1.1)$$

where $T(A)$ is the torsion submodule of A, Matlis [7]. Ansari [1] obtained a generalization of sequence (1.1) in the general case utilizing E.F. modules.

E.F. modules give rise to E.F. rings such as injective flat modules give rise to I.F. rings. Colby [2] has provided various characterizations of I.F. rings. The aim of this note is to show that some of these characterizations can also be used to characterize E.F. rings.

2. LAZARD'S LEMMA

A ring R is said to be a left (respectively right) E.F. ring if every module in R–mod (respectively in mod–R) belongs to $_R\mathfrak{I}$ (respectively $\mathfrak{I}_R$).

A module is said to be finitely presented if it is the homorphic image of a finitely generated module F such that the kernel of the associated epimorphism is also finitely generated.

Let E be the injective envelope of one copy of each simple module in R–mod then for M in R–mod $(M, E) = M^*$ belonging to mod–R is called the Matlis dual of M and there exists a natural monomorphism from M to M^{**}. Further, M^* is injective in mod–R if and only if M is flat in R–mod. Matlis dual in mod–R is in R–mod and N^* is defined in a similar way and for N belonging to mod–R, N^* is in R–mode and N^* is injective in R–mod if and only if N is flat in mod–R.

For modules M and N there exists a natural homomorphism $\sigma_{M,N}:[M,R] \otimes N \rightarrow [M, N]$ defined by $\sigma_{M,N}(f \otimes n)(m) = f(m)n$ for f belonging to $[M, R]$, m in M and n in N. $\sigma_{M,N}$ is an isomorphism if M is finitely generated and projective or M is finitely presented and N is flat.

Lemma 2.1 (Lazard [6]). For a module M the following conditions are equivalent:

(1) M is flat.

(2) For every finitely presented module P the map.

$$\sigma_{P,M}: [P,R] \otimes M \rightarrow [P,M]$$

is an isomorphism.

(3) For every finitely presented module P, the map

$\sigma_{P,M}: [P,R] \otimes M \to [P,M]$

is an epimorphism.

(4) For every finitely presented module P and α belonging to $[P,M]$ there is a finitely generated free module F, β belonging to $[P,F]$ r belonging to $[F,M]$ such that $\alpha = \beta_r$.

3. CHARACTERIZATION OF E.F. RINGS

Now we come to our main result for the characterization of E.F. rings. The proof is a modified version of theorem 1, section 2 of Colby [2] for I.F. rings.

Proposition 3.1.

For a ring R the following conditions are equivalent:

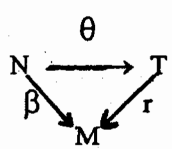
(1) R is left E.F.

(2) Every finitely presented module is a submodule of a free module.

(3) For any free F in mod–F, F^* is flat.

Proof:

(1) implies (2): Let N be a finitely presented module and θ the embedding monomorphism from N to a flat module T. Then by lemma 2.1 there is a finitely generated free module M and homomorphisms β, r such that the following diagram comutes with β monomorphism.



(2) implies (3): Let F be free in mod–R then F^* is injective in R–mod. Let P be finitely presented in R–mod and L be finitely generated free in R–mod containing P. Then the following diagram is commutative

$$\begin{array}{ccc} (L,R) \otimes F^* & \xrightarrow{\sigma_{L,F^*}} & (L,F^*) \\ \alpha\downarrow & & \downarrow\beta \\ (P,R) \otimes F^* & \xrightarrow[\sigma_{P,F^*}]{} & (P,F^*) \end{array}$$

Since F^* is injective, β is epimorphism and $\sigma_{L,F}{}^*$ is an isomorphism, since L is finitely generated and free so that $\sigma_{P,F*}$ is an epimorphism. Thus F^* is flat by lemma 2.1.

(3) **Implies (1):** Let M be in R–mod and $I(M)$ be its injective envelope then $(I(M))^*$ is in mod–R. There is a free F in mod–R and an epimorphism α from F to $(I(M))^*$ which implies that the sequence $F \to (I(M))^*$ is exact which in turn implies that the sequence $[(I(M))^*, E] \to [F,E]$ is exact where E is the injective envelope of direct sum of one copy of each simple module as described in the introduction, that is the sequence $(I(M))^{**} \to F^*$ is exact. Since F^* is flat and $I(M) \subseteq (I(M))^{**}$, $I(M)$ is a direct summand of F^* so $I(M)$ is flat. Thus M is embedded in a flat module. Hence R is left E.F.

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INVARIANTS OF ENERGY FUNCTION W($\underline{E}$, $\underline{S}$, $\underline{A}$, η) OVER THE CRYSTAL CLASS 3m

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ABSTRACT

In this paper, by employing a group-theoretical procedure the invariants (elements of integrity basis) of the function W($\underline{E}$, $\underline{S}$, $\underline{A}$, η) are determined over the arbitrarily chosen crystal class 3m. This function physically corresponds to the internal energy function in the nonlinear theory of perfectly elastic materials with couple-stresses and can be used to derive the related constitutive equations for stress and couple-stress.

1. INTRODUCTION

In the theory of continuous media, there are two sets of principles: The Basic Principles (Field Equations) which are valid for all materials and the Constitutive Principles by which the structures of different materials are taken into account [1, 2]. The second set of principles places certain restrictions on linear and nonlinear constitutive equations.

In this study the restrictions placed on the internal energy function W($\underline{E}$, $\underline{S}$, $\underline{A}$, η) , by one of the constitutive principles (the Material Invariance Principle) will be inspected over the arbitrarily chosen crystal class 3m. This energy function in the nonlinear theory of perfectly elastic materials with couple-stresses has been developed by Toupin [3] and used to derive general constitutive equations for stress and couple-stress.

In W($\underline{E}$, $\underline{S}$, $\underline{A}$, η) , $\underline{E}$ is the strain tensor (a symmetric second order true tensor) defined by

$$E_{ij} = \frac{1}{2}\left(\frac{\partial x_p}{\partial X_i}\frac{\partial x_p}{\partial X_j} - \delta_{ij}\right), \tag{1}$$

$\underline{S}$ and $\underline{A}$ are the symmetric and antisymmetric parts of $\underline{D}$ which is defined as the curl of strain and given by

$$D_{ij} = \frac{1}{3}\varepsilon_{ikm} \frac{\partial^2 x_p}{\partial x_j \partial X_k} \frac{\partial x_p}{\partial X_m}, \tag{2}$$

where x_i and X_j are the rectangular Cartesian coordinates of a material point in the deformed and undeformed states respectively, and ε_{ikm} is the alternating symbol. $\underline{S}$ and $\underline{A}$ are second order traceless pseudo tensors given by

$$S_{ij} = \frac{1}{2}(D_{ij} + D_{ji}), \tag{3}$$

$$A_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \tag{4}$$

Finally, η is the entropy density (a constant).

The material invariance principle states that constitutive equations must be form-invariant under the set of symmetry transformations describing the symmetry properties of the material considered. On the other hand, it is known that symmetry transformations of crystalline materials form groups (point groups {S}) [4,5,6,7]. Thus, these two remarks lead us to suggest a group-theoretical procedure for the solution of this problem.

2. FURTHER REMARKS ON THE PROBLEM AND THE METHOD

In accordance with the form-invariance requirement mentioned above and omitting the constant η, we may write

$$\mathbf{W}(\underline{\mathbf{E}}, \underline{\mathbf{S}}, \underline{\mathbf{A}}) = \mathbf{W}(\mathbf{E}', \mathbf{S}', \mathbf{A}') \tag{5}$$

which should be satisfied under each symmetry element $\underline{S}_k \in \{S\}$. Here $\mathbf{E}'$, $\mathbf{S}'$ and $\mathbf{A}'$ are the transformed forms of $\underline{\mathbf{E}}$, $\underline{\mathbf{S}}$ and $\underline{\mathbf{A}}$ respectively. The effect of (5) is to enforce the argument tensors to form certain combinations $\mathbf{I}_m$ (**E**, **S**, **A**) (m=1,2,....) which remain invariant under the symmetry group of the material considered. These invariant combinations are said to form an "integrity basis" for the function **W(E, S, A)**, which then can be expressed as a polynomial in $\mathbf{I}_m$'s. If none of the invariants $\mathbf{I}_m$ is expressible as a polynomial in the remaining elements, then the set $\mathbf{I}_m$ form an irreducible integrity basis.

The typical multilinear elements of integrity bases in terms of general symbols representing basic quantities (which are linear combinations of the independent components of argument tensors) are available in the literature [8] for most of the crystal classes. Thus, the problem reduces to the actual computation of the basic quantities in terms of the argument tensors involved in (5).

In [8], it is noted that considering transformations of linear combinations of the independent components of a tensor is advantageous than those of its individual components. Let $T_1, T_2, \ldots T_s$ be the s independent components of an argument tensor $\underline{T}$ Consider 5 linear combinations $U_i = L_{ij}\, T_j$ $(i, j = 1,2,\ldots,s)$ of these components. The set of the quantities $U_1, U_2, \ldots, U_s$ may be split up into the subsets [8].

$$(U_1,\ldots,U_k),\ (U_{k+1},\ldots,U_n),\ (U_{n+1},\ldots,U_s) \tag{6}$$

such that each subset forms the carrier space of one of the irreducible representations Γ_i of the point group considered. The quantities (6) are referred to as "basic quantities" and may be obtained in terms of the independent components of argument tensors explicitly with the application of the following formula [8, 9].

$$U_i(p) = \sum_{k=1}^{g} D^*_{1i}(p)(S_k) D_{qr}(S_k) T_r (i = 1,\ldots,d_p; r = 1,\ldots,S), \tag{7}$$

where $U_i(p)$ is the i-th component of the basic quantity $\underline{U}(p)$ associated with the irreducible representation $\Gamma_p = \{\underline{D}(p)(S)\}$ (of dimension d_p) of $\{S\}$, g is the order of $\{S\}$, $*$ denotes complex conjugate, and $\{D(S)\}$ is a reducible representation of $\{S\}$.

In (7), we proceed by taking $i = 1$, $q = 1$ and compute $U_1(p)$. If $U_1(p)$ is nonzero we determine the remaining components $U_2(p)$, $U_3(p), \ldots U^{(p)}_{dp}(p)$ which together with $U_1(p)$ form the carrier space for Γp. If $U_1(p)$ happens to be zero; we take $q = 2$ and repeat the process until we obtain all basic quantities needed.

It is also to be noted that the computation of the basic quantities may be simplified further by first decomposing the argument tensors over $\{S\}$ considered. We recall that a tensor $\underline{t}$ $<\lambda>$ of the basic symmetry type $<\lambda>$ forms the carrier space for the irreducible representation $<\lambda>$ of the general linear group $GL(3)$ [4,5]. If we pass from $GL(3)$ to $\{S\}$, which is a subgroup of $GL(3)$, the irreducible representation $<\lambda>$ becomes reducible [4] over $\{S\}$, i.e.,

$$<\lambda> = \sum_j n_j \Gamma_j \tag{8}$$

where Γ_j's are the irreducible representations of { S} and available in the literature [8]. The coefficients n_j's are evaluated+ by the well-known formula [4,5]

$$n_j = \frac{1}{g} \sum_{k=1}^{g} \chi<\lambda> (\underline{S}_k)\, \chi_j^* (\underline{S}_k) \tag{9}$$

where $\chi_j^*(\underline{S}_k)$ is the complex conjugate of the character $\chi_j(\underline{S}_k)$ of Γ_j and $\chi<\lambda>(\underline{S}_k)$ is the character of the representation $<\lambda>$. The decomposition (8) simplifies the actual computation of the basic quantities. Since the coefficients n_j's give the numbers of the corresponding basic quantities and the irreducible representation Γ_j's reveal the symmetry types and the dimensions (numbers of components) of the basic quantities to be computed.

3. DETERMINATION OF INVARIANTS

As noted in the previous section, it is advantageous to decompose the argument tensors $\underline{E}'$, $\underline{S}'$, $\underline{A}'$ in (5) over the point group {S} considered. For ease of reference, the irreducible representations of 3m and the relevant character systems obtained are given in Table I and II respectively.

Table I. Irreducible Representations of 3m and Associated Basic Quantities

Γ_p \ 3m	$\underline{I}$	$\underline{S}_1$	$\underline{S}_2$	$\underline{R}_1$	$\underline{R}_1\underline{S}_1$	$\underline{R}_1\underline{S}_2$	*B.Q.*
Γ_1	1	1	1	1	1	1	Φ, Φ', ..
Γ_2	1	1	1	−1	−1	−1	Ψ, Ψ',..
Γ_3	$\underline{E}$	$\underline{A}$	$\underline{B}$	$\underline{E}$	$\underline{G}$	$\underline{H}$	$\underline{a}, \underline{b}$,...

In Table I, $\underline{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\underline{A} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$, $\underline{B} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$,

$\underline{F} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\underline{G} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$, $\underline{H} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$

Table II. Relevant Character System of 3m

χ 3m	$\underline{I}$	$\underline{S}_1$	$\underline{S}_2$	$\underline{R}_1$	$\underline{R}_1\underline{S}_1$	$\underline{R}_1\underline{S}_2$
$\chi\Gamma_1$	1	1	1	1	1	1
$\chi\Gamma_2$	1	1	1	−1	−1	−1
$\chi\Gamma_3$	2	−1	−1	0	0	0
$\chi\Gamma_{<2>}$	6	0	0	2	2	2
$\chi\Gamma_{(2)}$	5	−1	−1	1	1	1
$\chi\Gamma_{<11>}$	3	0	0	−1	−1	−1

Using (9) and Table I, II and keeping in mind that $\underline{S}$ is traceless [10], the argument tensors are decomposed over 3m as

$$\begin{aligned} \underline{E}: <2> &= 2\Gamma_1 + 2\Gamma_3 \\ \underline{S}: (2) &= \Gamma_1 + 2\Gamma_3 \\ \underline{A}: <11> &= \Gamma_2 + \Gamma_3 \end{aligned} \tag{10}$$

Then referring to (10) and Table I, it is observed that the basic quantities (11) given below are to be determined explicitly.

$$\left.\begin{array}{ll} \text{Associated with } \Gamma_1: & \\ \Phi, \Phi' & (\text{in terms of components of } \underline{E}) \\ \Phi'' & (\text{in terms of components of } \underline{S}) \\ \text{Associated with } \Gamma_2: & \\ \Psi & (\text{in terms of components of } \underline{A}) \\ \text{Associated with } \Gamma_3: & \\ \underline{a}, \underline{b} & (\text{in terms of components of } \underline{E}) \\ \underline{c}, \underline{d} & (\text{in terms of components of } \underline{S}) \\ \underline{e} & (\text{in terms of components of } \underline{A}) \end{array}\right] \tag{11}$$

Next, using the formula (7), the basic quantities specified in (11) are computed explicitly. The results are given in Table III below:

Table III. Explicit Forms of Basic Quantities

Γ_j	$\underline{E}$	$\underline{S}$	$\underline{A}$
Γ_1	$E_{11}+E_{22}, E_{33}$	$S_{11}+S_{22}$	
Γ_2			A_{12}
Γ_3	$\begin{bmatrix} 2E_{12} \\ E_{11}-E_{22} \end{bmatrix}, \begin{bmatrix} E_{31} \\ E_{23} \end{bmatrix}$	$\begin{bmatrix} S_{11}+S_{22} \\ 2S_{12} \end{bmatrix}, \begin{bmatrix} S_{23} \\ A_{31} \end{bmatrix}$	$\begin{bmatrix} A_{23} \\ A_{31} \end{bmatrix}$

Now using the general results provided in [8], the following typical multilinear elements of the integrity basis for $W(\underline{E}, \underline{S}, \underline{A})$ over 3m are listed:

$$\left.\begin{array}{l} \text{degree 1}: \Phi \\ \text{degree 2}: a_1b_1+a_2b_2, \Psi\Psi' \\ \text{degree 3}: a_2b_2c_2-a_1b_1c_2-b_1c_1a_2-c_1a_1b_2, \Psi(a_1b_2-a_2b_1) \\ \text{degree 4}: \Psi(a_1b_1c_1-a_2b_2c_1-b_2c_2a_1-c_2a_2b_1) \end{array}\right] \quad (12)$$

Thus, using (11), (12) and Table III, the invariants of the polynomial function $W(\underline{E}, \underline{S}, \underline{A})$ over 3m are obtained as follows:

degree 1 : (3 elements)

$E_{11}+E_{22}, E_{33}, S_{11}+S_{22}$

degree 2 : (16 elements)

$4E^2{}_{12}+(E_{11}-E_{22})^2, E^2{}_{31}+E^2{}_{23}, (S_{11}+S_{22})^2+4S^2{}_{12}, S^2{}_{23}+S^2{}_{31}, A^2{}_{23}+A^2{}_{31}, A^2{}_{12}, 2E_{12}E_{31}+E_{23}(E_{11}-E_{22}), E_{12}(S_{11}+S_{23})+S_{12}(E_{11}-E_{22}), 2E_{12}S_{23}-S_{31}(E_{11}-E_{22}), 3E_{12}A_{23}+A_{31}(E_{11}-E_{22}), E_{31}(S_{11}+S_{22})+2E_{23}S_{12}, E_{31}S_{23}-E_{23}S_{31}, E_{31}A_{23}+E_{23}A_{31}, S_{23}(S_{11}+S_{22})-2S_{12}S_{31}, A_{23}(S_{11}+S_{22})+2S_{12}A_{31}, S_{23}A_{23}-S_{31}A_{31}$

degree 3 : (45 elements)

(i) (5 elements)

$(E_{11} - E_{22})^3 - 12E^2_{12}(E_{11} - E_{22})$, $E^3_{23} - 3E^2_{31}E_{23}$, $4S^3_{12} - 3S_{12}(S_{11} + S_{22})^2$, $S^3_{31} + 3S^2_{23}S_{31}$, $A^3_{31} - 3A^2_{23}A_{31}$

(ii) (20 elements)

The 20 elements obtained upon substituting two different vectors from the set

$\underline{a} = (2E_{12}, E_{11} - E_{22})$, $\underline{b} = (E_{31}, E_{23})$, $\underline{c} = (S_{11} + S_{22}, 2S_{12})$,

$\underline{d} = (S_{23}, -S_{31})$, $\underline{e} = (A_{23}, A_{31})$

in all possible ways in typical element

$a_2b_2c_2 - a_1b_1c_2 - b_1c_1a_2 - c_1a_1b_2$

(iii) (10 elements)

The 10 elements obtained upon substituting three different vectors from the set given in (ii) in all possible ways in typical element

$a_2b_2c_2 - a_1b_1c_2 - b_1c_1a_2 - c_1a_1b_2$

(iv) (10 elements)

The 10 elements obtained upon substituting two different vectors from the set given in (ii) in all possible ways in typical element $(a_1b_2 - a_2b_1)$ and then multiplying the resulting expressions by $\Psi = A_{12}$.

degree 4 : (35 elements)

The elements to be listed here are obtained from the 35 elements constituting (i), (ii) and (iii) of degree 3 as follows:

(a) interchange first and second components of $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$, $\underline{e}$, in the 35 elements mentioned above.

(b) multiply the resulting expressions by $\Psi = A_{12}$.

4. CONCLUSION

The results given here are in good agreement with those given by Huang [11] who obtained the invariants of degree three or less applying the theorems in the theory of invariants [12] by using classical methods. We further note that our results consist of the invariants of degree four or less which is a complete integrity basis for the problem under consideration. Finally, it is to be noted that the group-theoretical procedure employed here may easily be extended to determine the integrity bases for **W(E̲, S̲, A̲)** over all the crystal classes.

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DECOMPOSITION OF TRACELESS TENSORS OVER CRYSTALLOGRAPHIC POINT GROUPS

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ABSTRACT

Traceless tensors from the irreducible representations of the full orthogonal group 0(n). These irreducible representations are different from those of GL(n). In this paper, the decomposition of traceless tensors over the single coloured crystallographic (classical) point groups { $\underline{S}$} is discussed. Then, restricting the problem to 3-dimensional space, traceless symmetric tensors of order $r \leq 4$ are decomposed over the triclinic, monoclinic, rhombic crystal systems and the results are listed in a table.

1. INTRODUCTION

It is known that crystallographic point groups {$\underline{S}$} representing the geometric symmetry properties of the classical crystal classes [1-3]. Since symmetry is a tensor property, those tensors with particular symmetry will be transformed among themselves under the general linear group GL(n). This suggests that the whole linear space of r-th order tensors is reducible to the tensor $\underline{t}<\lambda>$ of the basic symmtry type $<\lambda>$, [4–6]. In other words the tensor $\underline{t}<\lambda>$ form the irreducible representations of GL(n) which are named by the same symbols $<\lambda>$.

In [4], it is shown that the "operation of contraction" commutes with the orthogonal transformations $\underline{S}$ which satisfy $S_{ij}\, S_{ik} = S_{ji}\, S_{ki} = \delta_{jk}$. For example, consider an r-th order general tensor $t_{i1 \cdots\cdots ir}$. By setting the first two indices equal to each other and summing over all values of $i_1 = i_2$, the (12)– trace (contraction)

$$t_{i3}{}^{(12)}{}_{i4\cdots\cdots ir} = t_{ii\; i3\cdots\cdots ir} = \delta_{i1\; i2}\, t_{i1\; i2\; i3\cdots\cdots ir} \tag{1}$$

of the tensor $\underset{\sim}{t}$ is obtained, which is obviously a new tensor of rank (r−2). Now consider its transformed form under an orthogonal transformation $\underset{\sim}{S}$, i.e.,

$$t'_{i3}{}^{(12)}{}_{i4\cdots ir} = t'_{ii\,i3\cdots ir} = \underbrace{S_{i1j1}\, S_{i2j2}}_{\delta_{j1j2}},\, S_{i3j3}\cdots\cdots S_{irjr}\, t_{j1j2\cdots\cdots j_r} \tag{2}$$

$$= S_{i3j3}\cdots\cdots S_{irjr}\;\; t_{j3}{}^{(12)}{}_{\cdots\cdots jr}$$

$$= t_{j3}{}^{(12)'}{}_{j4\cdots\cdots jr}$$

$$= t_{i3}{}^{(12)'}{}_{i4\cdots\cdots ir}$$

From (2), we conclude that the operations of contraction and tensor transformation commute. Secondly, if we conclude the r-th rank tensors for which all pair-traces are zero, from (2) it is observed that they are transformed among themselves under the transformations induced by $0(n)$. In other wòrds, traceless tensors of given symmetry types form irreducible representations of $0(n)$. These irreducible representations are different from those of GL(n). Since the process of repeated contraction provides traceless tensors of order r, r-2, r-4,......; the irreducible representations of $0(n)$ may be obtained by a regular removal of two boxes in a repeated manner, from the corresponding Young frames [4] of the irreducible representations of GL(n). When we go from $0(n)$ to the crystallographic point groups {$\underset{\sim}{S}$}, the irreducible representations of $0(n)$ to the crystallographic point groups {$\underset{\sim}{S}$}, the irreducible representations of $0(n)$ will be reducible. The next section is devoted to this reduction problem when n = 3.

2. REDUCTION OF 0(3) OVER CRYSTALLOGRAPHIC POINT GROUPS

Traceless tensors of given symmetry types which from irreducible representations of $\theta(n)$ can be obtained explicitly by applying the corresponding Young symmetry operator [4] to the traceless tensors of rank r. Thus, an irreducible representations of $0(n)$ is associated with a Young frame $<\lambda> = <\lambda_1\ \lambda_2\ldots\ldots\ \lambda_n>$, where λ_i's are non-negative integers satisfying $\lambda_1+\lambda_2+\ldots\ldots\ldots+\lambda_n = r$; $\lambda_1 \geq \lambda_2 \geq\ldots\ldots\geq \lambda_n$. However, not all Young frames are admissible. There is a general theorem in the literature (see, for example, [4] , P. 394) stating that "the traceless tensors corresponding to Young frames in which the sum of the lengths of the first two columns exceeds the dimension n of the space, are identically zero". Further, the admissible frames (diagrams) can be paired into "associate diagrams" Y and Y' as follows: The length "a" of the first column in Y is a

$\leq n/2$, the length of the first column in Y' is $(n-a)$; and all other columns in Y and Y' have the same length. For example, if $n = 3$, in general $<r>$ and $<r,1>$ are associate diagrams.

In order to describe the irreducible representations of $0(n)$ formed by the r-th rank traceless tensors, a symbol $(\mu) = (\mu_1,....,\mu_p)$ and its associate are used [4], where

$$\mu_1 + \mu_2 +.......\mu_p = r, \quad \mu_1 \geq \mu_2 \geq......\geq \mu_p \geq 0,$$

$$P = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n-1/2 & \text{if } n \text{ is odd} \end{cases} \tag{3}$$

when $n = 3$, from (3) we see that $p = 1$, $\mu_1 = r$ and one of the irreducible representations of $0(3)$ is $(\mu_1) = (r)$, and its associate will be $(\mu_1, 1)$. But these two representations are not independent [4] and in particular

$$(\mu_1, 1) = \det(\underline{S}_k)\,(\mu_1), \qquad \underline{S}_k \in \{\underline{S}\} \tag{4}$$

Hence, the irreducible representations of $0(3)$ may be denoted by a single label (μ_1), i.e., they are furnished by the traceless completely symmetric tensors of order $r = \mu_1$.

The number of independent components of a traceless tensor $\underline{t}(\mu_1)$ with symmetry type (μ_1), gives the dimension of the irreducible representation (μ_1) and is found to be

$$\chi_{(\mu 1)}(\underline{I}) = 2\mu_1 + 1 \tag{5}$$

Now, if we go from $0(3)$ to any point group $\{\underline{S}\}$, the irreducible representation (μ_1) of $0(3)$ will be reducible, i.e.,

$$(\mu_1) = \sum_i n_j \Gamma_j \tag{6}$$

where, Γ_j*'s* are the irreducible representations of $\{S\}$. The numbers n_j*'s* are obtained by the formula (see, for example, [4]).

$$n_j = \frac{1}{g} \sum_k \chi_{(\mu 1)}(\underline{S}_k)\, \chi_j^*(\underline{S}_k) \tag{7}$$

where, g is the order of $\{\underline{S}\}$, $\chi_{(\mu 1)}(\cdot)$ is the character of the representation (μ_1) and $\chi_j^*(\cdot)$ is the complex conjugate of the character of Γ_j. The character $\chi_{(\mu 1)}(\underline{S}_k)$ is evaluated by the following formula ([7], P.230).

$$\chi_{(\mu 1)}(\underset{=}{S}_k) = q_{\mu 1}(\underset{=}{S}_k) - q_{\mu 1-2}(\underset{=}{S}_k) \tag{8}$$

The simple characters q_m $(\underset{=}{S}_k)$ in (8) are obtained from the m-th order determinant [7,8]

$$q_m(\underset{=}{S}_k) = \frac{1}{m!}\begin{vmatrix} S_1 & -1 & 0 & 0 & \cdot & \cdot & 0 \\ S_2 & S_1 & -2 & 0 & \cdot & \cdot & 0 \\ S_3 & S_2 & S_1 & -3 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ S_m & S_{m-1} & S_{m-2} & S_{m-3} & \cdot & \cdot & S_1 \end{vmatrix} \tag{9}$$

where, S_β = trace $(\underset{=}{S}_{\beta k})$

Using (8) and (9), the character $\chi_{(\mu 1)}(\cdot)$ are evaluated first and then by (7) the traceless tensor under consideration is decomposed over any point group $\{\underset{=}{S}\}$.

3. APPLICATION

As an application, we consider the decomposition of the space (μ_1) = (4) of a traceless completely symmetric 4-th order tensor over the rhombic-dipyramidal class mmm (D_{2h}). For ease of reference, the symmetry elements and the irreducible representations of mmm are listed in Table I.

TABLE I. Irreducible Representations of mmm.

mmm	$\underset{=}{I}_1$	$\underset{=}{D}_1$	$\underset{=}{D}_2$	$\underset{=}{D}_3$	$\underset{=}{C}$	$\underset{=}{R}_1$	$\underset{=}{R}_2$	$\underset{=}{R}_3$
Γ_1	1	1	1	1	1	1	1	1
Γ_2	1	1	−1	−1	1	1	−1	−1
Γ_3	1	−1	1	−1	1	−1	1	−1
Γ_4	1	−1	−1	1	1	−1	−1	1
Γ_1'	1	1	1	1	−1	−1	−1	−1
Γ_2'	1	1	−1	−1	−1	−1	1	1
Γ_3'	1	−1	1	−1	−1	1	−1	1
Γ_4'	1	−1	−1	1	−1	1	1	−1

From (8), we write

$$\chi_{(4)}(\underline{S}_k) = q_4(\underline{S}_k) - q_2(\underline{S}_k) \tag{10}$$

and using (9) evaluate $q_m(\cdot)$ (m = 1,2,3,4) in terms of S_β (S_β = trace ($\underline{S}_{\beta k}$)) as

$$\begin{aligned} q_1 &= S_1 \\ q_2 &= \frac{1}{2}(S_1^2 + S_2) \\ q_3 &= \frac{1}{3!}(S_1^3 + 3S_1S_2 + 2S_3) \\ q_4 &= \frac{1}{4!}(S_1^4 + 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 + 6S_4) \end{aligned} \tag{11}$$

Next, the traces S_β = (β = 1,2,3,4) needed in (11) and the relevant characters are evaluated for mmm (D_{2h}). The results are listed in Table II.

TABLE II. Relevant Characters for mmm.

mmm	S_1	S_2	S_3	S_4	q_1	q_2	q_3	q_4	$\chi_{(1)}$	$\chi_{(2)}$	$\chi_{(3)}$	$\chi_{(4)}$
$\underline{I}$	3	3	3	3	3	6	10	15	3	5	7	9
$\underline{D}_1,\underline{D}_2,\underline{D}_3$	−1	3	−1	3	−1	2	−2	3	−1	1	−1	1
$\underline{C}$	−3	3	−3	3	−3	6	−10	15	−3	5	−7	9
$\underline{R}_1,\underline{R}_2,\underline{R}_3$	1	3	1	3	1	2	2	3	1	1	1	1

Using (7), Table I and Table II we get the numbers n_j's as

$$n_1=3, \quad n_2=n_3=n_4=2 \quad n_1'=n_2'=n_3'=n_4'=0 \tag{12}$$

Hence the reduced form of $(\mu_1) = (4)$

$$(4) = 3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + 2\Gamma_4 \tag{13}$$

is obtained. This reduction means that the nine independent components of a traceless completely symmetric 4-th rank tensor form the carrier spaces of the irreducible representations Γ_j of mmm which appear in (13).

By straight forward application of the procedure described above. are obtained the decomposition of completely symmetric traceless tensors of order r ≤ 4 over the triclinic, monoclinic, rhombic systems and list them in Table III. All such problems can be treated in the same manner.

TABLE III. Decomposition of Traceless Tensors of order r ≤ 4 over certain classes

(μ)	Crystal Classes						
	$1(C_i)$	$2\,(C_2)$	$m\,(C_s)$	$2/m\,(C_{2h})$	$222\,(D_2)$	$2mmm\,(C_{2v})$	$mmm\,(D_{2h})$
(1)	$3\Gamma_2$	$\Gamma_1+2\Gamma_2$	$2\Gamma_1+\Gamma_2$	$\Gamma_2+2\Gamma_3$	$\Gamma_2+\Gamma_3+\Gamma_4$	$\Gamma_1+\Gamma_3+\Gamma_4$	$\Gamma_2'+\Gamma_3'+\Gamma_4'$
(2)	$5\Gamma_1$	$3\Gamma_1+2\Gamma_2$	$3\Gamma_1+2\Gamma_2$	$3\Gamma_1+2\Gamma_4$	$2\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4$	$2\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4$	$2\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4$
(3)	$7\Gamma_2$	$3\Gamma_1+4\Gamma_2$	$4\Gamma_1+3\Gamma_2$	$3\Gamma_2+4\Gamma_3$	$\Gamma_1+2\Gamma_2+2\Gamma_3+2\Gamma_4$	[illegible] $+\Gamma_2+2\Gamma_3+2\Gamma_4$	$\Gamma_1'+2\Gamma_2'+2\Gamma_3'+2\Gamma_4'$
(4)	$9\Gamma_1$	$5\Gamma_1+4\Gamma_2$	$5\Gamma_1+4\Gamma_2$	$5\Gamma_1+4\Gamma_4$	$3\Gamma_1+2\Gamma_2+2\Gamma_3+2\Gamma_4$	$3\Gamma_1+2\Gamma_2+2\Gamma_3+2\Gamma_4$	$3\Gamma_1+2\Gamma_2+2\Gamma_3+2\Gamma_4$

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AN EXTENSION OF THE BINOMIAL THEOREM WITH APPLICATION TO STABILITY THEORY

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ABSTRACT

We show how it is possible to put different stability types such as Routh-Hurwitz and Schur-Cohn on common grounds by establishing direct links between them. In the process, we obtain natural and elegant extensions of both Pascal's rule and the binomial theorem, which prove useful in establishing our main results.

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1. INTRODUCTION:

A linear system of differential equations is said to be Routh-Hurwitz (Schur-Cohn) stable if and only if all of its eigenvalues lie in the left half of the complex plane(within the unit circle). The problem of locating the eigenvalues of a system of differential equations has fascinated mathematicians for decades, and the literature is full of ingenious methods, analyses of these methods, and discussions of their merits. Over the last forty years or so they had tremendous impact on various areas of control theory. In case of real systems, the theory of stability is well developed. These are results which mathematicians and engineers are familiar with and they can be readily applied to theoretical problems in differential equations and linear algebra as much as to practical problems in electrical engineering and electronics, see for example [4], [6], [11] and [13] to mention just a few. The case of complex coefficients has received much less attention in t'e past, but recently a flurry of results has been reported, among many others see [1], [2], [5], [7], [9] and [14]. Many fresh attempts were made to put stability criteria of different natures such as Routh-Hurwitz and Schur-Cohn on common grounds, by invoking the intimate relationships that might prevail between these various stability types, for some very recent works in this direction see [3], [8], [10] and [12].

This paper is basically a contribution to the mainstream of bringing together these two important types of stability. It is structured as follows: In section 2 we give the necessary definitions and notations. In section 3 natural extensions of Pascal's rule and the binomial theorem are obtained which are then applied in section 4 to prove the main results.

2. DEFINITIONS AND NOTATIONS:

By induction, define the following sequence of sets:

$$Z^{(n)} = \{ z_1 , z_2 , \ldots\ldots , z_n \}$$

for all $n \geq 1$, where for any positive integer j, z_j is a real or complex number.

A j-subset of $Z^{(n)}$ is a set consisting of j elements of $Z^{(n)}$ having different subscripts.

$C_j^{(n)}$ denotes the set of all j-subsets of $Z^{(n)}$.

If $1 \leq k \leq \binom{n}{j}$ where $\binom{n}{j}$ is the binomial coefficient, let $P_{jk}^{(n)}$ be the product of all j elements of the k[th] subset of $C_j^{(n)}$

$$\text{Let } S_j^{(n)} = \sum_{k=1}^{\binom{n}{j}} P_{jk}^{(n)} \text{ for } j = 1, \ldots\ldots, n \text{, and } S_0^{(n)} = 1 \text{ for all } n \geq 1.$$

For any j = 1 , ... , n , let $w_j = \frac{z_j - 1}{z_j + 1}$ which is equivalent to $z_j = \frac{1 + w_j}{1 - w_j}$. Similarly, let $W^{(n)} = \{ w_1 , w_2 , \ldots\ldots , w_n \}$, and let $D_j^{(n)}$ be the set of all j-subsets of $W^{(n)}$. If $1 \leq k \leq \binom{n}{j}$, $Q_{jk}^{(n)}$ denotes the product of all j

elements of the k^{th} subset of $D_j^{(n)}$. Let $T_j^{(n)} = \sum_{k=1}^{\binom{n}{j}} Q_{jk}^{(n)}$ for all $j = 1, \ldots\ldots,$ n, and $T_0^{(n)} = 1$ for all $n \geq 1$.

A given linear system of differential equations is said to be Routh-Hurwitz (Schur-Cohn) stable if and only if all its eigenvalues lie in the left-half plane (inside the unit circle).

If A is an nxn real or complex matric , and X(t) is an n-dimensional column vector function of t , let X′ = A.X be a system of differential equations , with eigenvalues z_1 , z_2 , ... , z_n . Then the characteristic polynomial of this system may be written in both factored and expanded forms as follows:

$f(z) = \prod_{j=1}^{n} (z - z_j) = \sum_{j=0}^{n} a_j z^{n-j}$ where $a_0 = 1$ by definition. Similarly if X′ = B.X is a system with eigenvalues w_1 , w_2, ..., w_n (where w_j is related to z_j of the previous system by $w_j = \dfrac{z_j - 1}{z_j + 1}$), then its characteristic polynomial is

$$g(w) = \prod_{j=1}^{n} (w - w_j) = \sum_{j=0}^{n} b_j w^{n-j}, \text{ with } b_0 = 1.$$

3. BASIC RESULTS:

The following results are needed later.

Lemma 3.1 $\quad S_j^{(n)} + S_{j-1}^{(n)} \cdot z_{n+1} = S_j^{(n+1)}$ for all $j = 1$, ... , n

Proof. If $1 \leq j \leq n$, let $C = C_{j-1}^{(n)}$ x { z_{n+1} } be the cartesian product of the two sets $C_{j-1}^{(n)}$ and { z_{n+1} }.

If $\left\{ \phi_i \right\}_{i=1}^{\binom{n}{j-1}}$ is the family of all subsets forming $C_{j-1}^{(n)}$, then it is clear that C is in a one-to-one correspondence with the set $\Phi = \left\{ \phi_i \cup \right.$

$\{ z_{n+1} \}$, $i = 1, \ldots\ldots, \binom{n}{j-1} \}$, and card C = card $\Phi = \binom{n}{j-1}$ where card X denotes the number of elements of a set X.

Now ($C_j^{(n)} \cup \Phi) \subset C_j^{(n+1)}$ and $C_j^{(n)} \cap \Phi = \varnothing$, since no j-subsets of $Z^{(n)}$ contain z_{n+1} . Hence

card ($C_j^{(n)} \cup \Phi$) = card $C_j^{(n)}$ + card $\Phi = \binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j}$ = card $C_j^{(n+1)}$.

Therefore $C_j^{(n)} \cup \Phi = C_j^{(n+1)}$, from which it follows automatically that $S_j^{(n)} + S_{j-1}^{(n)} \cdot z_{n+1} = S_j^{(n+1)}$ for all $j = 1, \ldots\ldots, n$.

Lemma 3.1 is an extension of the famous Pascal's rule.

Theorem 3.1 $\quad f(z) = \sum_{j=0}^{n} (-1)^j S_j^{(n)} z^{n-j}$

Proof. We proceed by induction on n.

$z - z_1 = z - S_1^{(1)}$, hence our proposition is true for n = 1.

Suppose $f(z) = \sum_{j=0}^{n} (-1)^j S_j^{(n)} z^{n-j}$ then

$$f(z) \cdot (z - z_{n+1}) = \sum_{j=0}^{n} (-1)^j S_j^{(n)} z^{n-j+1} - \sum_{j=0}^{n} (-1)^j S_j^{(n)} z^{n-j} z_{n+1}, \text{ but}$$

$$\sum_{j=0}^{n} (-1)^j S_j^{(n)} z^{n-j} z_{n+1} = \sum_{j=1}^{n} (-1)^{j-1} S_{j-1}^{(n)} z^{n-j+1} z_{n+1} + (-1)^n S_{n+1}^{(n+1)}$$

since $S_{n+1}^{(n+1)} = S_n^{(n)} Z_{n+1}$. Hence

$$f(z) \cdot (z - z_{n+1}) = z^{n+1} + \sum_{j=1}^{n} (-1)^j z^{n-j+1} (S_j^{(n)} + S_{j-1}^{(n)} z_{n+1}) + (-1)^{n+1}$$

$S_{n+1}^{(n+1)} = \sum_{j=0}^{n+1} (-1)^j S_j^{(n+1)} z^{n-j-1}$, by Lemma 3.1 and the proof is complete.

Theorem 3.1 is an extension to the well-known binomial theorem. The following is now clear,

Corollary 3.1 $S_j^{(n)} = (-1)^j a_j$ and $T_j^{(n)} = (-1)^j b_j$ for all $j = 0,1,\ldots,n$.

The intimate relationship between Routh-Hurwitz and Schur-Cohn types of stability could best be expressed by the following:

Theorem 3.2 The system $X' = A.X$ is Schur-Cohn stable if and only if $X' = B.X$ is Routh-Hurwitz stable.

Proof. Suppose $z = \dfrac{1+w}{1-w}$ or equivalently $w = \dfrac{z-1}{z+1}$ where z and w are complex numbers. The following relationships can easily be established

$w + \bar{w} = \dfrac{2(z\bar{z} - 1)}{|z+1|^2}$ and $z \,.\, \bar{z} - 1 = \dfrac{2(w + \bar{w})}{|1-w|^2}$, from either of which it follows that $|z| < 1$ if and only if $\mathrm{Re}\, w < 0$.

4. ROUTH-HURWITZ IN TERMS OF SCHUR-COHN:

If r and s are non-negative integers , define:

$$\binom{s}{r} = \begin{cases} \dfrac{s!}{r!(s-r)!} & if\ s \geq r \\ 0. & if\ s < r \end{cases}$$

For technical purposes we also define:

$$\binom{-1}{-1} = 1 \text{ and } \binom{s}{-1} = 0 \text{ for any integer } s \geq 0.$$

If $X' = A.X$ and $X' = B.X$ are the two systems defined in section 2 with their corresponding characteristic polynomials, then

Theorem 4.1

$$b_p = \frac{\sum_{t=0}^{n} \sum_{s=0}^{t'} \sum_{r=0}^{s} (-1)^{\min(p,t)+t+r} \binom{s-1}{r-1} \binom{n-t'}{|p-t|+r} a_t}{\sum_{r=0}^{n} (-1)^r a_r},$$

for all $p = 1, \ldots\ldots, n$ and where $t' = \begin{cases} t & \text{if } t \leq p \\ n-t & \text{if } t > p \end{cases}$

Proof. Let $1 \leq p \leq$ n. By corollary 3.1 $(-1)^p b_p = \sum_{k=1}^{\binom{n}{p}} Q^{(n)}_{pk}$.

We bring all terms $Q^{(n)}_{pk}$ to a common denominator

$D_p = \prod_{r=1}^{n} (z_r + 1)$. Call N_p the numerator, hence $(-1)^p b_p = \frac{N_p}{D_p}$.

A typical element in the sum appearing in N_p is

$$t_p = \begin{cases} \prod_{r=1}^{n} (z_r - 1) . \prod_{s=p+1}^{n} (z_s + 1) & \text{if } p < n \\ \prod_{r=1}^{n} (z_r - 1) & \text{if } p = n \end{cases}$$

All elements of N_p can be produced from t_p by considering all possible positions of the p minus signs of t_p into the n factors of t_p. It is clear that the constant term in N_p is $(-1)^p \binom{n}{p}$ and

$$D_p = \sum_{r=0}^{n} (-1)^r a_r .$$

Let $1 \leq t \leq p$. We propose to calculate the coefficient of

$S_t^{(n)} = \sum_{k=1}^{\binom{n}{t}} P^{(n)}_{tk}$ appearing in N_p . But first we note the following:

If we consider the product of any t factors chosen from the set $\{ z_r - 1, 1 \leq r \leq p \} \cup \{ z_s + 1, p+1 \leq s \leq n \}$ if $p < n$ and from the set $\{ z_r - 1, 1 \leq r \leq n \}$ if $p = n$, this product clearly shows up in exactly

$\binom{n-t}{p-t}$ of the elements forming N_p . This leads to the fact that the arrangement of the factors in such products is not significant. Therefore all $P^{(n)}_{t\,k}$ for $1 \leq k \leq \binom{n}{t}$ have the same coefficient, which is that of $S_t^{(n)}$.

Hence, it suffices to calculate the coefficient of $P^{(n)}_{t1}$ where naturally $P^{(n)}_{t1} = \prod_{r=1}^{t} z_r$.

Next we explain our strategy in producing all terms of N_p from t_p : As a first step , consider the terms of N_p corresponding to all possible positions of the (p − t) minus signs into the (n − t) different positions indicated in

$$t_p = \prod_{j=1}^{t} (z_j - 1) . \underbrace{\prod_{k=t+1}^{p} (z_k - 1) .}_{p-t} \prod_{m=p+1}^{n} (z_m + 1) .$$
$$\underline{\hspace{10em}}$$
$$n - t$$

where we suppose $t < p < n$. If c_0 is the coefficient of $P^{(n)}_{t1}$ calculated among these terms , then $c_0 = (-1)^{p-t}\binom{n-t}{p-t}$. The cases $t < p = n$, $t = p < n$ and $t = p = n$ lead to the same conclusion.

Next we go back to t_p and consider the block of ($p - t + 1$) minus signs appearing in the product $\prod_{k=t}^{p} (z_k - 1)$ of t_p which we shift one step to the right to get to the position:

$$(1) \qquad \prod_{j=1}^{t-1} (z_j - 1) . (z_t + 1) . \underbrace{\prod_{k=t+1}^{p+1} (z_k - 1) .}_{P-t+1} \prod_{m=p+2}^{n} (z_m + 1).$$
$$\underline{\hspace{10em}}$$
$$n - t$$

Then consider the terms which arise from all possible positions of the ($p - t + 1$) minus signs into the ($n - t$) factors shown in (1). If c_1 is the coefficient of $P^{(n)}_{t1}$ calculated among these terms, then $c_1 = (-1)^{p-t+1} \binom{n-t}{p-t+1}$.

In general if $1 \le s \le t$, shift the block of ($p - t + s$) minus signs of the product $\prod_{k=t-s+1}^{p} (z_k - 1)$ one step to the right to obtain the position.

$$(2) \quad \prod_{j=1}^{t-s} (z_j - 1).(z_{t-s+1} + 1). \underbrace{\underbrace{\prod_{k=t-s+2}^{p+1} (z_k - 1)}_{P-t+s} . \prod_{m=p+2}^{n} (z_m + 1)}_{n-t+s-1}.$$

Let c_s be the coefficient of $P^{(n)}_{t1}$ calculated among the terms of N_p which correspond to all possible positions of the ($p - t + s$) minus signs into the ($n - t + s - 1$) different positions shown in (2).

We claim that $c_s = \prod_{r=1}^{s} (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r}$ for all s, $1 \le s \le t$. We proceed by induction on s. Our claim is true when $s = 1$. Suppose it is true for s where $1 \le s < t$. In t_p, we move the $(p - t + s + 1)$ minus signs appearing in the product $\prod_{k=t-s}^{p} (z_k - 1)$ one step to the right.

So we are in the position:

$$(3) \quad \prod_{j=1}^{t-s-1} (z_j - 1).(z_{t-s} + 1). \underbrace{\underbrace{\prod_{k=t-s+1}^{p+1} (z_k - 1)}_{P-t+s+1} . \prod_{m=p+2}^{n} (z_m + 1)}_{n-t+s}.$$

Since $\prod_{k=t-s+1}^{p+1}(z_k-1)=(z_{t-s+1}-1).\prod_{k=t-s+2}^{p+1}(z_k-1)$, the product $\prod_{k=t-s+2}^{p+1}(z_k-1)$ corresponds to shifting in t_p the $(p-t+s)$ minus signs showing up in the product $\prod_{k=t-s+1}^{p}(z_k-1)$ one step to the right to get the position already shown in (2). If c_s as defined above, then by our induction assumption.

$$c_s=\sum_{r=1}^{s}(-1)^{p-t+r}\binom{s-1}{r-1}\binom{n-t}{p-t+r}.$$

Once this done, we go back to $\prod_{k=t-s+1}^{p+1}(z_k-1)$ in (3) and shift the $(p-t+s+1)$ minus signs one further step to the right to get the position:

$$(4)\qquad \prod_{j=1}^{t-s-1}(z_j-1).(z_{t-s}+1).(z_{t-s+1}+1)\underbrace{\underbrace{\prod_{k=t-s+2}^{p+2}(z_k+1)}_{P-t+s+1}\prod_{m=p+3}^{n}(z_m+1)}_{n-t+s-1}$$

Call c'_s the coefficient of $P^{(n)}_{t1}$ calculated among the terms of N_p corresponding to all possible positions of the $(p-t+s+1)$ minus signs into the $(n-t+s-1)$ different positions shown in (4). If we compare (4) to (2), we realize that in (4) we are dealing with the product $\prod_{k=t-s+2}^{p+2}(z_k-1)$ whereas in (2) we dealt with $\prod_{k=t-s+2}^{p+1}(z_k-1)$. therefore we may obtain c'_s by replacing $p + 1$ by $p + 2$ or equivalently p by $p + 1$ in c_s. Therefore

$$c'_s = \sum_{r=1}^{s} (-1)^{p-t+r+1} \binom{s-1}{r-1} \binom{n-t}{p-t+r+1}.$$

It is clear that $c_{s+1} = c_s + c'_s$. Hence

$$C_{s+1} = \sum_{r=1}^{s} (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r} + \sum_{r=1}^{s} (-1)^{p-t+r+1} \binom{s-1}{r-1} \binom{n-t}{p-t+r+1}$$

$$= (-1)^{p-t+r} \binom{s-1}{o} \binom{n-t}{p-t+1} + \sum_{r=1}^{s-1} (-1)^{p-t+r+1} \binom{s-1}{r} \binom{n-t}{p-t+r+1}$$

$$+ \sum_{r=1}^{s-1} (-1)^{p-t+r+1} \binom{s-1}{r-1} \binom{n-t}{p-t+r+1} + (-1)^{p-t+s+1} \binom{s-1}{r-1} \binom{n-t}{p-t+s+1}$$

$$= (-1)^{p-t+1} \binom{s}{0} \binom{n-t}{p-t+1} + \sum_{r=1}^{s-1} (-1)^{p-t+r+1} \left[\binom{s-1}{r} + \binom{s-1}{r-1}\right] \binom{n-t}{p-t+r+1}$$

$$+ (-1)^{p-1+s+1} \binom{s}{s} \binom{n-t}{p-t+s+1}$$

Since $\binom{s-1}{r} + \binom{s-1}{r-1} = \binom{s}{r}$ and by shifting indices, we get

$$C_{s+1} = \sum_{r=1}^{s+1} (-1)^{p-t+r} \binom{s}{r-1} \binom{n-t}{p-t+r} \text{ proving our claim.}$$

The coefficient of $S_t^{(n)}$ is therefore

$$\sum_{s=0}^{t} C_s = (-1)^{p-t} \binom{n-t}{p-t} + \sum_{s=1}^{t} \sum_{r=1}^{t} (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r}$$

$$= \sum_{s=0}^{t} \sum_{r=0}^{s} (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r}$$

Now let $p+1 \leq t \leq n$ and reconsider

$$(5) \qquad t_p = \underbrace{\prod_{j=1}^{p} (z_j - 1) \cdot \underbrace{\prod_{k=p+1}^{t} (z_k + 1)}_{t-p}}_{t} \cdot \prod_{m=t+1}^{n} (z_m + 1),$$

where we suppose $t < n$.

Again we propose to calculate the coefficient of $S_t^{(n)}$. First consider all terms of N_p arising from all possible positions of the $(t - p)$ plus signs into the t different positions shown in (5). Let c'_0 be the coefficient of $P_{t1}^{(n)}$ calculated among these terms, then $c'_0 = \binom{t}{t-p}$. If $t = n$, we are clearly lead to the same conclusion.

In general if $1 \leq s \leq n - t$, in (5) above we shift the block of $(t - p + s)$ plus signs of the product $\prod_{k=p+1}^{t+s} (z_k + 1)$ one step to the left to get:

$$(6) \qquad \underbrace{\prod_{j=1}^{p-1} (z_j - 1) \cdot \underbrace{\prod_{k=p}^{t+s-1} (z_k + 1)}_{t-p+s}}_{t+s-1} \cdot (z_{t+s} - 1) \cdot \prod_{m=t+s+1}^{n} (z_m + 1).$$

If c'_s is the coefficient of $P_{t1}^{(n)}$ calculated among all terms of N_p corresponding to all possible positions of the $(t - p + s)$ plus signs into the $(t + s - 1)$ different positions shown in (6). By an induction similar to the previous one, we show that $c'_s \sum_{r=1}^{s} (-1)^r \binom{s-1}{r-1}\binom{t}{t-p+r}$ for all s, $1 \le s \le n-t$.

The coefficient of $S_t^{(n)}$ is therefore

$$\sum_{s=0}^{n-t} c'_s = \sum_{s=0}^{n-t}\sum_{r=0}^{s} (-1)^r \binom{s-1}{r-1}\binom{t}{t-p+r}$$

So if $(-1)^p b_p = \frac{N_p}{D_p}$, then

$$N_p = \sum_{t=0}^{p}\sum_{s=0}^{t}\sum_{r=0}^{s} (-1)^{p-t+r}\binom{s-1}{r-1}\binom{n-t}{t-p+r} S_t^{(n)}$$

$$+ \sum_{t=p+1}^{n}\sum_{s=0}^{n-t}\sum_{r=0}^{s} (-1)^r \binom{s-1}{r-1}\binom{t}{t-p+r} S_t^{(n)}$$

Easy to see how b_p can be brought to the form stated in the theorem.

5. SCHUR-COHN IN TERMS OF ROUTH-HURWITZ:

The converse of theorem 4.1 states the following:

Theorem 5.1 If t' as defined before, then

$$a_{n-p} = \frac{\sum_{t=0}^{n}\sum_{s=0}^{t'}\sum_{r=0}^{s} (-1)^{\text{mix}(p,t)+n+r}\binom{s-1}{r-1}\binom{n-t'}{|p-t|+r} b_t}{\sum_{r=0}^{n} b_r},$$

for all $p = 0, 1, \ldots\ldots, n-1$.

Proof. Suppose $1 \le P \le n-1$, them

$$(-1)^{n-p} a_{n-p} = S_{n-p}^{(n)} = \sum_{k=1}^{\binom{n}{n-p}} P_{(n-p)k}^{(n)}.$$

From both sides of this relation we cancel out the factor $(-1)^{n-p}$ and we bring all terms in the right-hand side to a common denominator $D'_p = \prod_{r=1}^{n} (w_r - 1)$. call N'_p the numerator . Hence $a_{n-p} = \frac{N'_p}{D'_p}$.

It is clear that $D'_p = (-1)^n \prod_{r=0}^{n} b_r$. A typical element in the sum appearing in N'_p is

$$\underbrace{\prod_{r=1}^{n-p} (w_r + 1)}_{n-p} . \underbrace{\prod_{s=n=p+1}^{n} (w_s - 1)}_{p}.$$

This element is entirely similar to t_p of theorem 4.1 except that the $w's$ replace the $z's$. Therefore ,

$$N'_p = \sum_{t=0}^{p} \sum_{s=0}^{t} \sum_{r=0}^{s} (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{t-p+r} T_t^{(n)}$$

$$+ \sum_{t=p+1}^{n} \sum_{s=0}^{n-t} \sum_{r=0}^{s} (-1)^r \binom{s-1}{r-1} \binom{t}{t-p+r} T_t^{(n)}, \text{ and}$$

(7) $$a_{n-p} = \frac{\sum_{t=0}^{n} \sum_{s=0}^{t'} \sum_{r=0}^{s} (-1)^{\text{mix}(p,t)+n+r} \binom{s-1}{r-1} \binom{n-t'}{|p-t|+r} b_t}{\sum_{r=0}^{n} b_r},$$

for $1 \leq p \leq n-1$.

Let N'_0 be the numerator of the right-hand side of (7) corresponding to p = 0. Then

$$N'_0 = (-1)^n + \sum_{t=1}^{n} \sum_{s=0}^{n-t} \sum_{r=0}^{s} (-1)^{t+n+r} \binom{s-1}{r-1} \binom{t}{t+r} b_t,$$ which reduces to

$$N'_0 = (-1)^n + \sum_{t=1}^{n} (-1)^{t+n} b_t = \sum_{t=0}^{n} (-1)^{t+n} b_t \text{, and}$$

$$(-1)^n a_n = S_n^{(n)} = \prod_{t=1}^{n} z_t = (-1)^n \prod_{t=1}^{n} \left(\frac{w_t + 1}{w_t - 1}\right) \text{. Therefore}$$

$$a_n = \frac{\sum_{t=0}^{n} (-1)^t b_t}{(-1)^n \sum_{r=0}^{n} b_t} = \frac{\sum_{t=0}^{n} (-1)^{t+n} b_t}{\sum_{r=0}^{n} b_r},$$

We conclude that relation (7) covers the case p = 0 and the proof is complete.

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LOCALLY CONSTRUCTED C^2 SPLINE

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ABSTRACT

The classical rational cubic Bezier function in its most general form is used to construct a C^2 interpolatory spline. The spline is local in the sense that changing data at one point requires the processing to be performed in the neighbouring intervals only. Further, the free parameters embeded in the description of the spline are used to determine a C^2 monotone interpolant through monotone data.

Keyword: Globally C^2. Rational cubic spline, Monotonicity.

1. INTRODUCTION:

In two recent papers by Sarfraz (1992, 1994) the classical rational cubic Bezier function in its most general form is used to construct an interpolatory rational cubic spline. This spline provides a C^2 alternative to GC^2 or C^1 spline methods such as ν-spline. Nielson (1984). β-spline, Barsky (1981), γ-spline, Boehm (1985), and weighted spline, Foley (1986). This spline is a useful addition to CAGD literature due to its ability to provide a variety of interesting shape controls such as biased, point, and interval tensions. Further it recovers the cubic B-spline, de Boor (1978), and the rational cubic spline with tension, Gregory and Sarfraz (1990) as a special case. However, the spline is not local in the sense that a change in the value of even one coordinate of a point, requires the entire processing to be done again.

In this paper, we construct a C^2 spline locally by using the rational cubic Bezier function in its most general form. We also extend this locally constructed C^2 spline to preserve the monotonicity of the data.

The rest of the paper is organized as follows: In Section 2 we introduce the rational cubic Bezier function and use it to construct a C^2 spline locally. Both slopes and curvature remain free and two methods have been included in Section 3 to estimate them. Finally, in Section 4, we extend the C^2 spline to ensure the monotonicity of the data.

2. RATIONAL CUBIC BEZIER FUNCTION:

Let $\{(t_i, f_i) : i = 0, 1,\dots,n\}$ be given values where the t–values define a partition of the interval $[t_0, t_n]$, that is, $t_0 < t_1 < \dots < t_n$. Further, if d_i denotes the derivative value at data point (t_i, f_i), $i = 0, 1,\dots,n$ then in each interval $[t_i, t_{i+1}]$, a rational cubic Bezier function in most general for may be defined as below:

$$S_i(t) = \frac{p_i(t)}{q_i(t)}$$

where

$$p_i(t) = r_i f_i (1-\theta)^3 + A_i \theta (1-\theta)^2 + B_i \theta^2 (1-\theta) + w_i f_{i+1} \theta^3,$$

$$q_i(t) = r_i (1-\theta)^3 + u_i \theta (1-\theta)^2 + v_i \theta^2 (1-\theta) + w_i \theta^3,$$

$$A_i = u_i f_i + r_i h_i d_i,$$

$$B_i = v_i f_i{+1} - w_i h_i d_i{+1},$$

with $h_i = t_{i+1} - t_i$, $\theta = (t - t_i)/h_i$ and r_i, u_i, v_i, w_i are free parameters. The piecewise rational cubic spline so constructed is C^1 and is called the rational cubic Bezier spline. It has the Hermite interpolation properties, that is,

$$S_i(t_i) = f_i,\ S_i^{(1)}(t_i) = d_i,\quad i = 0, 1,\dots,n.$$

Further, the rational cubic Bezier spline may be modified by changing the values of free parameters. For example, the restrictions $r_i = w_i = 1$ and $u_i = v_i = 3$ reduce this spline to standard Hermite cubic spline. Also $r_i > 0$, $u_i \geq 0$, $v_i \geq 0$, and $w_i > 0$ ensure a positive denominator in (2.1).

Sarfraz (1994) has extended (2.1) to a C^2 spline, but the methods are not local as a change in data at one point requires the entire processing to be performed all over again. In following theorem, we gives formulae to extend (2.1) to a locally constructed C^2 spline.

Theorem 2.1: Let $\{(t_i, f_i) : i = 0, 1, \dots, n\}$ be given data values with $t_0 < t_1 < \dots < t_n$. If d_i and m_i, respectively, denote the first and the second derivative values at the data points then the C^1 spline (2.1) is in $C^2[t_0, t_n]$ if and only if, in each interval $[t_i, t_{i+1}]$, the free parameters r_i and w_i are determined as below:

$$r_i = \frac{4d_{i+1}\delta_i - 2\beta_i\gamma_i}{\alpha_i\beta_i - 4d_i d_{i+1}},$$

$$w_i = \frac{4d_i\gamma_i - 2\alpha_i\delta_i}{\alpha_i\beta_i - 4d_i d_{i+1}},$$

where $\alpha_i = h_i m_i - 2d_i$, $\beta_i = h_i m_{i+1} + 2d_{i+1}$, $\gamma_i = v_i\Delta_i - u_i d_i$,

$\delta_i = u_i\Delta_i + v_i d_{i+1}$, $\Delta_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$.

Proof: we begin by calculating $S_i^{(2)}(t_i)$ and $S_i^{(2)}(t_{i+1})$ which, after simplifications, are as follows:

$$S_i^{(2)}(t_i) = \frac{2}{h_i r_i}(d_i r_i - d_{i+1}w_i + \gamma_i),$$

$$S_i^{(2)}(t_{i+1}) = \frac{2}{h_i w_i}(-d_i r_i - d_{i+1}w_i + \delta_i),$$

where $\gamma_i = v_i\Delta_i - u_i d_i$ and $\delta_i = u_i\Delta_i + v_i d_{i+1}$. Now $S_i(t) \in C^2[t_i, t_{i+1}]$ if and only if:

$$m_i = \frac{2}{h_i r_i}(d_i r_i - d_{i+1}w_i + \gamma_i),$$

$$m_{i+1} = \frac{2}{h_i w_i}(-d_i r_i - d_{i+1}w_i + \delta_i),$$

which may be arranged into the following simultaneous equations:

$$\alpha_i r_i + 2d_{i+1} w_i = 2\gamma_i ,$$
$$2d_i r_i + \beta_i w_i = 2\delta_i .$$

The solution of simultaneous equations (2.4) and (2.5) produces r_i and w_i as in (2.2) and (2.3) respectively. Hence the proof.

3. ESTIMATION OF FIRST AND SECOND DERIVATIVES:

Thus for, we have assigned values to free parameters r_i and w_i such that $S_i(t) \in C^2$ $[t_0, t_n]$. We have so far taken first and second derivatives arbitrarily. In this section, we give some practical choices of d_i and m_i. The first choice is to assign d_i a value using three-points difference formula, Brodlie (1985), that is,

$$d_i = \Delta_1 + \frac{\Delta_1+\Delta_2}{h_i+h_2},$$

$$d_i = \frac{h_i\Delta_{i-1}+h_{i-1}\Delta_i}{h_{i-1}+h_i}, \; i = 2, 3, ..., n-1$$

$$d_n = \Delta_{n-1} + \frac{\Delta_{n-1}+\Delta_{n-2}}{h_{n-1}+h_{n-2}},$$

and then repeating this three-point formula for m_i using d_i, that is,

$$m_i = D_1 + \frac{D_1+D_2}{h_1+h_2},$$

$$m_i = \frac{h_iD_{i-1}+h_{i-1}D_i}{h_{i-1}+h_i}, \; i = 2, 3, ..., n-1$$

$$m_n = D_{n-1} + \frac{D_{n-1}+D_{n-2}}{h_{n-1}+h_{n-2}},$$

where $D_i = (d_{i+1} - d_i)/h_i$, $i = 1,2,. . . n-1$.

Next scheme is somewhat complex. Here, to find derivative d_i at (t_i, f_i), a cubic polynomial is determined which passes through the three points (t_{i-1}, f_{i-1}), (t_i, f_i) and (t_{i+1}, f_{i+1}) and which gives a least-square fit to the two neighbouring points (t_{i-2}, f_{i-2}) and (t_{i+2}, f_{i+2}), Ellis and Mclain (1977). In order to take account of variation in the spacing of data, these two points are weighted in proportional to the square of their distances from their neighbours t_{i-1} and t_{i+1} respectively. The derivative of this cubic at its central point (t_i, f_i) is then assigned to d_i. To estimate m_i, we calculate the second derivative of this cubic at the central point (t_i, f_i).

4. MONOTONICITY CONTROL:

Let $\{(t_i, f_i) : i = 0, 1, ..., n\}$ be given data values where the t-values define a partition of the interval $[t_0, t_n]$, that is, $t_0 < t_1 < ... < t_n$ and f_i, and f_i, $i = 0, 1, ..., n$ is monotone, that is

$$f_i \begin{cases} < f_{i+1} \ \forall i = 0\ 1 \dots n & \text{(monotone increasing)} \\ \geq f_{i+1} \ \forall i = 0\ 1 \dots n & \text{(monotone decreasing)} \end{cases}$$

This kind of monotone data arise in various situations as mentioned in Butt(1991). For example, in probability distribution, if mutually exclusive events $\{(t_i, f_i) : i = 0, 1, \dots, n\}$ constitute a sample and P is the probability function, then

$$P(A_1 \cup A_2 \cup \dots \cup A_i) = P(A_1) + P(A_2) + \dots + P(A_i),$$

where $i = 3, 4, \dots, n$, that is, the probability occurance of an event is directly proportional to the size of the sample taken. This generates a monotonically increasing data.

The C^2 spline developed in Theorem 2.1 does not preserve monotonicity, in general. In this section we extend it to construct a monotone C^2 spline through monotone data. In case $f_i = f_{i+1}$, then the derivative assignments $d_i = d_{i+1} = 0$ reduce $S_i(t)$ to a straight line, that is,

$$S_i(t) = f_i \text{ for all } t \in [t_i, t_{i+1}]$$

and so the monotonicity is preserved. For rest of the study, we assume that $f_i \neq f_{i+1}$ for all $i = 0, 1, \dots, n-1$. The following theorem presents conditions which ensure monotonicity. We have considered monotonically increasing data only. The case of monotonicity decreasing can be developed in a similar way.

Theorem 4.1: Let $\{(t_i, f_i) : i = 0, 1, \dots, n\}$ be a monotone increasing data, that is, $f_i < f_{i+1}$ for each $i = 0, 1, \dots, n-1$ and $d_i > 0$, $i = 1, \dots, n$. Then the C^2 spline of Theorem 2.1 is monotone increasing if the free parameters u_i, v_i satisfy the following conditions:

$$v_i > \frac{w_i d_{i+1}}{\Delta_i} \text{ if } r_i > 0 \text{ otherwise } v_i < \frac{w_i d_{i+1}}{\Delta_i} \qquad 4.2$$

$$u_i > \frac{r_i d_i}{\Delta_i} \text{ if } w_i > 0 \text{ otherwise } u_i < \frac{r_i d_i}{\Delta_i} \qquad 4.3$$

$$k_{i,1} u_i v_i + k_{i,2} u_i + k_{i,3} v_i > k_{i,4} \qquad 4.4$$

$$k_{i,1} u_i v_i + k_{i,5} u_i + k_{i,6} v_i > k_{i,7} \qquad 4.5$$

with $k_{i,1} = \Delta_i$, $k_{i,2} = d_{i+1} w_i$, $k_{i,3} = r_i(2\Delta_i - d_i)$, $k_{i,4} = r_i w_i(2d_{i+1} - 3\Delta_i)$, $k_{i,5} = w_i(2\Delta_i - d_{i+1})$, $k_{i,6} = -d_i r_i$, $k_{i,7} = r_i w_i(2d_i - 3\Delta_i)$.

Proof: We begin by calculating $S_i^{(1)}(t)$ which after simplification is given by:

$$S_i^{(1)}(t) = \frac{\alpha_i(1-\theta)^5+\beta_i\theta(1-\theta)^4+\gamma_i\theta^2(1-\theta)^3+\delta_i\theta^3(1-\theta)^2+\mu_i\theta^4(1-\theta)+\nu_i\theta^5}{\{q_i(\theta)^2\}}$$

where

$$\alpha_i = r^2_i d_i,$$

$$\beta_i = \alpha_i + 2r_i(v_i\Delta_i - w_i d_{i+1}),$$

$$\gamma_i = \beta_i - \alpha_i + \chi_i,$$

$$\delta_i = \mu_i - \nu_i + \chi_i,$$

$$\mu_i = \nu_i + 2w_i(u_i\Delta_i - r_i d_i),$$

$$\nu_i = w_i^2 d_{i+1},$$

with $\chi_i = (3r_iw_i + u_iv_i)\Delta_i - (u_iw_id_{i+1} + r_iv_id_i)$.

$S_i(t)$ is monotonically increasing if and only if $S_i^{(1)}(t) \geq 0$, $\forall\ t \in [t_0, t_n]$. The sufficient conditions for monotonicity will be:

$$\alpha_i \geq 0,\ \beta_i \geq 0,\ \gamma_i \geq 0,\ \delta_i \geq 0,\ \mu_i \geq 0,\ \nu_i \geq 0.$$

Both $\alpha_i \geq 0$, $\nu_i \geq 0$ are non negative due to the necessary conditions for monotonicity, *viz.* $d_i \geq 0$ $i = 0, 1,...n$. However,

$$\beta_i \geq 0 \text{ if } 2r_iw_id_{i+1} \leq r^2_i d_i + 2r_iv_i\Delta_i,$$

which after ignoring the non-negative term $r^2_i d_i$ reduces to the condition (4.2). Similarly, $\mu_i \geq 0$ reduces to condition (4.3). Next, we consider $\gamma_i \geq 0$, which after rearrangements, can be expressed in the form (4.4). Similarly, $\delta_i \geq 0$ reduces to condition (4.5) and hence the proof.

5. CONCLUSION:

In this paper, we have used the rational cubic Bezier function in its most general form to construct a C^2 spline locally. We have also extended it to preserve the monotonicity of the data. Future work will extend this spline to preserve other shapes such as Convexity and Positively.

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A NOTE ON THE ESTIMATION OF SAMPLE SIZE FOR SIMPLE RANDOM SAMPLING

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ABSTRACT

In this note a new expression for estimation of Sample Size has been obtained which works well when the prevalence of disease or proportion of any characteristic in the population is known. It also ensures at the same time that a specified number of units having this characteristic are in the sample.

1. INTRODUCTION

In all the standard text books on survey sampling, generally two formulas for the estimation of Sample Size for simple random sampling are given. The first one is:

$$n = [Z^2_{1-\Omega} P(1-P)] / d^2,$$

where P is the prevalence of disease or proportion of any characteristic present in the population, Z is the value of standard normal distribution and d is the absolute precision required on either side of proportion (in percentage points). If we take $Z_{1-\Omega}$ = 1.96 (5% level of significance) or 2.58 (1% level of significance), d = .05 then for different value of P the Sample Size is:

P	.01	.02	.03	.04	.05	.1	.15	.2	.25	.3	.35	.4	.45	.5
n(1.96)	15	30	45	59	73	138	196	246	288	323	350	369	380	384
n(2.58)	26	52	77	102	126	240	339	426	499	559	606	639	659	666

Looking at the above table, we see as the prevalence rate is low the sample size is also low where as if the prevalence rate is high the sample size is also high. In fact the case must be opposite i.e., if the disease is rare the sample should be large and must be small if the disease is common. To cover this point an other formula was developed.

$$n = [Z^2_{1-\Omega} (1-P)] / \in^2 P,$$

where $\in$ denotes the relative precision i.e., if P = 50% then 10% relative precision mean 10% of 50% (note that d = 10% × 50% = 0.05). If we take $\in$ = 0.05 or 0.10 then for different value of P at 5% level of significance with Z = 1.96 the sample size will be:

P	0.05	0.10	.15	.20	.225	.30	.35	.40	.45	.50
n(=0.05)	29196	13830	8708	6147	4610	3585	2854	2305	1878.	1537
n(=0.10)	7299	3457	2177	1537	1152	896	713	576	470	384

P	.55	.60	.65	.70	.75	.80	.85	.90	.95
n(=0.05)	1257	1024	827	659	512	384	188	119	56
n(=0.10)	314	256	207	165	128	96	68	43	20

A natural course of action is that we should look for the formula which cover the above point i.e. size of the sample must be large if the disease is rare and at the same time make sure that minimum or more units must having this attribute likely to be in the sample. In the next section a formula for the estimate of sample size is derived using normal distribution as an approximation to be binomial distribution.

2. DERIVATION OF NEW FORMULA:

We know that the normal approximation to the binomial

$$Z = (x-np)/\sqrt{np\ (1-p)}$$

After simplification we get

$$np + Z\sqrt{p\ (1-p)}\ n^{1/2} - x = 0$$

Solving for quadratic in $n^{1/2}$

$$n^{1/2} = \frac{-Z\sqrt{P\ 1-P} \pm \sqrt{Z^2 P\ 1-P + 4Px}}{2P}$$

OR

$$n = \left[\frac{-Z\sqrt{P(1-P)} \pm \{Z^2 P(1-P) + 4Px\}^{1/2}}{2P} \right]^2$$

We impose the condition that <u>x < nP</u> to get the <u>maximum saple</u> size

$$n = \left[\frac{-Z\sqrt{P(1-P)} \pm \{Z^2 P(1-P) + 4Px\}^{1/2}}{2P} \right]^2$$

For Z=1.96 and for Z = 1.645 and for various values of P. The Sample Size has been calculated and given in Table 1 and 2.

This formula works well when the prevalence of disease or proportion of any characteristic in the population is known and at the same time it also ensures that a specified number of units having this characteristic are in the sample.

TABLE 2: Sample Size for Z = 1.96 and for different values of P

Z = 1.96

P X	5	10	15	20	25	30	35	40	45	50
0.01	1166	1836	2469	3083	3684	4276	4869	5438	6012	6582
0.02	581	915	1231	1538	1838	2134	2426	2715	3002	3287
0.03	385	608	819	1023	1223	1420	1614	1807	1998	2188
0.04	288	455	613	766	916	1063	1209	1353	1496	1639
0.05	229	363	489	611	731	849	965	1081	1195	1309
0.06	190	301	406	508	608	806	803	899	995	1089
0.07	162	257	347	435	520	604	687	770	851	932
0.08	142	225	303	379	454	528	600	672	744	915
0.09	125	119	269	337	403	468	533	597	660	723
0.10	112	179	241	302	362	421	479	536	493	650
0.11	102	162	219	274	328	382	434	487	438	590
0.12	93	148	200	251	300	349	397	445	493	540
0.13	85	136	184	231	277	322	388	410	454	598
0.14	79	126	171	214	256	298	340	380	421	462
0.15	73	117	159	199	239	278	316	355	392	430

P X	5	10	15	20	25	30	35	40	45	50
0.16	68	109	148	186	223	216	296	332	367	403
0.17	64	103	139	171	210	244	278	312	345	378
0.18	60	97	131	165	198	230	262	294	325	357
0.19	57	91	124	156	187	218	248	278	308	338
0.20	54	86	117	148	177	206	235	264	292	320
0.21	51	82	112	140	168	196	223	251	278	304
0.22	48	78	106	134	160	187	213	239	265	290
0.23	46	74	101	127	153	178	203	228	253	277
0.24	44	71	97	122	146	171	194	218	242	265
0.25	42	68	93	117	140	163	186	209	232	254
0.30	34	56	76	96	116	135	154	173	191	210
0.35	29	47	74	81	98	114	131	147	163	179
0.40	24	40	55	70	85	99	113	127	141	155
0.45	21	35	48	61	74	87	99	112	124	136
0.50	18	31	43	54	16	77	88	100	111	122
0.55	16	27	38	49	59	69	79	89	100	109
0.60	14	24	34	44	53	63	72	81	90	99
0.65	13	22	31	40	48	57	65	47	82	91
0.70	11	20	28	36	44	52	60	68	75	83
0.75	10	18	26	33	41	48	55	62	69	77
0.80	9	16	24	30	37	44	51	57	64	71
0.85	8	15	21	28	34	41	47	53	59	65
0.90	7	14	20	26	31	37	43	49	55	61
0.95	6	12	18	23	29	34	40	45	51	56

TABLE 2: **Sample Size for Z = 1.645 and for different values of P**

Z = 1.645

P X	5	10	15	20	25	30	35	40	45	50
0.01	1024	1669	2282	2878	3463	4040	4611	5178	5740	6299
0.02	510	832	1139	1436	1729	2017	2303	2586	2867	3146
0.03	339	553	557	956	1151	1343	1533	1121	1909	2095
0.04	253	414	567	716	862	1006	1148	1289	1430	1569
0.05	202	330	453	571	688	803	9177	1030	1142	1254
0.06	168	275	376	475	572	668	763	857	951	1044
0.07	143	235	322	407	490	572	653	734	814	894
0.08	125	205	281	355	428	500	471	641	711	781
0.09	111	182	249	315	380	443	407	569	631	793
0.10	99	163	224	28[illegible]	341	399	445	512	568	623
0.11	90	148	203	25[illegible]	310	362	413	464	515	566
0.12	82	135	186	234	283	331	378	424	472	618
0.13	75	124	171	216	281	305	349	392	435	478
0.14	70	115	158	201	242	283	323	363	403	443
0.15	65	107	148	187	225	264	301	339	376	413
0.16	61	100	138	175	211	247	282	317	342	387
0.17	57	94	130	164	198	232	265	298	321	363
0.18	53	89	122	155	187	219	250	281	312	343
0.19	50	84	115	146	177	207	236	266	295	324
0.20	48	79	109	139	168	196	224	252	280	308
0.21	45	75	104	132	159	186	213	240	266	293
0.22	43	72	99	122	152	178	203	229	254	279

P X	5	10	15	20	25	30	35	40	45	50
0.23	41	68	94	120	145	170	194	218	243	267
0.24	39	65	90	115	139	162	186	209	232	255
0.25	37	63	86	110	133	156	178	200	223	245
0.30	31	51	71	91	110	128	147	166	184	202
0.35	26	43	60	77	93	109	125	141	157	172
0.40	22	37	52	66	81	95	108	122	136	150
0.45	19	33	46	58	71	83	96	108	120	132
0.50	17	29	40	52	63	74	85	96	107	118
0.55	15	26	36	47	57	67	77	87	96	106
0.60	13	23	33	42	51	60	70	79	88	97
0.65	12	21	30	38	47	55	63	72	80	88
0.70	11	19	27	35	43	51	58	66	74	81
0.75	10	17	25	32	39	46	54	61	68	75
0.80	9	16	23	29	36	43	50	56	63	69
0.85	8	14	21	27	33	40	46	52	58	64
0.90	7	13	19	25	31	37	42	48	53	59
0.95	6	12	17	23	28	34	39	45	50	55

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FINITE ELEMENT APPROXIMATION FOR OPTIMAL SHAPE DESIGN

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ABSTRACT

In this paper we present some results concerning the optimal shape design problem governed by the variational inequalities of the fourth-order. This problem can be considered as a model example for the design of the shapes for elastic-plastic problem. The computations are done by finite element method, and the performance criterion is minimized by the material derivative method. We also discuss the error estimates in the appropriate norm and present some numerical results.

1. INTRODUCTION:

The purpose of this paper is to develop an optimal shape design which governed by the variational inequalities of the fourth-order for the design of the shape for elastic-plastic problem. Since we know that much work has been done in the optimal shape design for systems described by partial differential equations ([7]), the main idea of this paper is to obtain the optimal shape design for the systems described by differential inequalities by introducing penalized differential equations and then taking limits of the equations resulting from the penalized or differential approximation. We develop the material derivative method ([10]) for the optimal shape design of an elastic-plastic problem. The problem arises when studying the torsion of a cylindrical bar of the section $\Omega \subset \Re^2$, the bar is made of an elastic plastic material, f is a constant proportional to the angle of twist of the end section of the bar which is not clamped. Several authors have proposed and studied the elastic plastic problem (see, [4], [5], [8], [9]). For the formulation of the problem we consider a domain Ω, which consists of two regions Ω_e and Ω_p (the elastic region and plastic region, respectively). Then the stress potential φ satisfies the fourth-order partial differential equation on Ω,

$$\mu\varphi - \Delta^2\varphi = f, \text{ in } \Omega_e \text{ (situated in the interior of } \Omega) \tag{1}$$

$$|\Delta\varphi| = 1, \text{ in } \Omega_p,$$

with boundary conditions

$$\varphi = \partial\varphi/\partial n = 0, \text{ on } \Gamma, \tag{2}$$

where μ is a non negative constant. Consider the Sobolev space

$$H^2{}_0(\Omega) = \left\{ \phi | \phi \in H^2(\Omega), \phi|\Gamma = \frac{\partial\phi}{\partial n}|\Gamma = 0 \right\}; \tag{3}$$

(where $H^2(\Omega)$ is the set of square integrable functions with squareable second derivatives) since the domain Ω is bounded and Γ is sufficiently regular the mapping

$$\phi \to || \Delta\phi ||_{L^2(\Omega)},$$

defines on $H^2{}_0(\Omega)$ a norm which is equivalent to that induced by $H^2(\Omega)$. It is well known that (1) admits one and only one solution in $H^2{}_0(\Omega)$; this solution is also the unique solution of the variational equation (of order 4), for all $\varphi \in H^2{}_0(\Omega)$

$$\int_\Omega (\Delta\varphi\Delta\phi + \mu\varphi\phi)\, dx = < f, \phi >, \forall\phi \in H^2{}_0(\Omega) \tag{4}$$

which is also the solution of he minimization problem

$$\min_{\phi \in H_0^2(\Omega)} \left[1/2 \int_\Omega |\Delta\phi|^2 dx - < f, \phi > \right. \tag{5}$$

In (4) and (5), $< , >$ represents the bilinear form of the duality between $H^{-2}(\Omega)$ and $H^2{}_0(\Omega)$, and we thus have

$$< f, \phi > = \int_\Omega f\phi dx.$$

Finally, let K be the closed convex subset of $H^2(\Omega)$, defined by

$$K = \{\phi | \phi \in H_0^2(\Omega), |\Delta\phi| \le 1 \text{ a.e. in } \Omega\} \tag{6}$$

The problem we want to consider consists of finding the solution φ such that

$$J = \min_{\varphi \in K} \Big[1/2 \int_{\Omega} \left(|\Delta\phi|^2 - 2f\phi \right) dx. \tag{7}$$

Since the functional J is continuous and strictly convex on $H^2{}_0(\Omega)$, with

$$\lim_{\|\varphi\| \to \infty} J(\varphi) = +\infty,$$

then by using the known theorem ([4]), which implies the existence and uniqueness of a solution for (7). We denote this solution by φ which is also the unique solution of the variational inequality

$$\varphi \in K \; (A\varphi, \phi - \varphi) + \mu(\varphi, \phi - \varphi) \geq (f, \phi - \varphi) \; \forall \; \phi \in K \tag{8}$$

We note that the bilinear form in (8) is elliptic, that is

$$(A\phi, \phi) \geq \alpha \|\phi\|^2, \; \alpha > 0, \; \forall \; \phi \in H_0^2(\Omega) \tag{9}$$

In particular ([4]) the optimal solution of (7) is also the solution of the following:

$$J = \min_{\varphi \in K'} \Big[1/2 \int_{\Omega} \left(|\Delta\phi|^2 - 2f\phi \right) dx. \tag{10}$$

where

$$K' = \{\phi | \phi \in H_0^2(\Omega), \; |\Delta\phi| \leq \delta(x, \Gamma) \; a.e.\}, \tag{11}$$

$\delta(x, \Gamma)$ is the distance from x to Γ. Also, it is shown ([4]) that if Ω is a bounded open domain in $\mathfrak{R}^n$ with sufficiently regular boundary Γ, then the solution of (7) is also a solution of (10), and of the equivalent variational inequality:

$$\varphi \in K' \; (A\varphi, \phi - \varphi) + \mu(\varphi, \phi - \varphi) \geq (f, \phi - \varphi) \; \forall \; \phi \in K' \tag{12}$$

the bilinear form being defined as in (9). To solve the problem for the systems described by the variational inequalities, instead of (1), we assume another problem which is penalized form of (1), $\in > 0$, a penalty parameter, that is, we shall introduce our first penalized equation,

$$A\varphi_{\in} + \mu\varphi_{\in} + F(\varphi_{\in}) = f, \; \forall \; \varphi_{\in} \in H_0^2(\Omega), \tag{13}$$

(where $F(\varphi_{\in}) = 1/\in (\varphi_{\in}^{-})$, and $\varnothing^{-} = -\sup(-\varnothing, 0)$, and $A : V = H^2{}_0(\Omega) \rightarrow V'$ is a continuous and symmetric operator satisfying the coercivity condition (9), and for instance, we may take, $A = -\Delta.\Delta$) whose solution

$\varphi_\in$ tends to the solution of (8) when $\in$ tends to zero. For the existence and uniqueness of the solution of (13), see [1]. It is well explained ([2]) that $\phi \to \phi^-$ is not differentiable at $\phi = 0$, we have defined $F'(0) = 0$. This choice turns out to be unimportant because $\phi_\in > 0$, on Ω, with expection of a (zero-measure) subset of Γ.

2. COMPUTATION OF DERIVATIVES:

The material derivative method of shape sensitivity analysis ([10]) is used here for solving optimal shape problem for our differential inequality (8) by taking limit, as $\in$ tends to zero, of the equation resulting from the penalized approximation introduced in section 1. Here we shall explain how this method works for the systems described by the differential equation, which will be helpful in solving our problem, that is, the optimal shape problem for the systems described by a differential inequality.

The main idea of the material derivative method is to make the derivative of the cost function as negative as possible by selecting a suitable value of a vector field V to be defined below. This vector field is used for the perturbation of the domain Ω into Ω_t at time t, and the value of the derivative of the cost function depends only on the value of this vector field at the boundary. To minimize the cost function $J(\Omega_t)$, we shall take the vector field V, at $t = 0$, in the opposite direction to a vector G, the gradient of the cost function. By using this method, various formulae have been obtained ([10]) for the derivative with respect to shape. Let Ω be a smooth bounded open set in $\Re^n$, $n \geq 2$, and V be a regular, n-dimensional vector field defined on $[0, 1] \times U$ is an open neighbourhood of Ω. Suppose that he mapping $x \to V(t, x)$ for any $t \in [0, 1]$, has space derivatives which are continuous and the mapping $t \to V(t, x)$ is continuous for the topology given by uniform convergence of these derivatives on any compact subset of U. In this approach the deformation of the set Ω_t is given by the solution of the ordinary differential equation

$$dx(t)/dt = V(t, x(t)), \tag{14}$$

with the initial condition $x(0) = X \in \Omega$. Let F_t be the transformation, which depends on V and is defined by the differential equation (14),

$$X \to x = x(t, X) = F_t(V).X, \tag{15}$$

since X does not depend on t. The domain Ω_t and its boundary Γ_t can now be defined as:

$$\Omega_t \equiv F_t(V)(\Omega) = \{x \in \Re^n : \exists X \in \Omega \text{ such that}$$
$$x = x(t) \text{ with } x'(s) = V(x(s), s),\ 0 < s \leq t,\ x(0) = X\}, \tag{16}$$

$\Gamma_t \equiv F_t(V)(\Gamma)$; of course $\Omega_0 = \Omega$ and $\Gamma_0 = \Gamma$, see figure. 1.

The cost function can therefore be considered as a function of Ω_t as follows:

$$J(\Omega_t) = 1/2 \int_{\Omega t} |\Delta\varphi_{\epsilon,t}|^2 dx - \int_{\Omega t} f\varphi_{\epsilon,t} dx. \tag{17}$$

where $\phi_{\epsilon,t}$ is the solution of the differential equation (13) on Ω_t. It can also be considered as a function of t, by the mapping

$$t \rightarrow \Omega_t \rightarrow \varphi_{\epsilon,t} \rightarrow J(\Omega_t). \tag{18}$$

We wish to define the derivative $J'(\Omega_t)$ of the cost function $J(\Omega_t)$ at Ω_t, which depends on the vector field V; which will be chosen so as to obtain the value of the derivative as negative as possible. Let ψ be a smooth function of x, which may depend also smoothly on t. Then we know ([10]) that

$$d/dt \int_{\Omega_t} \psi(t,x)dx = \int_{\Omega_t} \partial\psi/\partial t dx + \int_{\Omega_t} div(\psi(t,x)V))dx,$$

by applying Green's formula, we obtain

$$d/dt \int_{\Omega_t} \psi(t,x)dx = \int_{\Omega_t} \partial\psi/\partial t dx + \int_{\Gamma t} \psi(t,x) < V, n > d\Gamma, \tag{19}$$

where div is the divergence operator, n is the unit exterior normal to Γ_t, V is the velocity field, and $<,>$ is the scalar product in $\Re^n$. It is emphasized that in this formula the derivative of ψ with respect to t is taken with x fixed (not depending on t). In [10] there are many examples of such derivatives. Now we again return to our main problem and write the equation (13) in variational form:

$$\int_{\Omega_t} (\Delta\varphi_{\epsilon,t}\Delta w + \mu\varphi_{\epsilon,t} w + F(\varphi_{\epsilon,t})\, w - fw)dx = 0, \forall\, w \in H^2_0(\Omega_t) \tag{20}$$

the unknown is denoted by $\varphi_{\epsilon,t} \in H^2_0(\Omega_t)$. In order to eliminate certain problems related to the definition of w in Ω, which is variable with t, suppose that w is the restriction to Ω_t of a function $w \in H^2(\Re^n)$. We denote by $\phi'_{\epsilon,t} = \partial | \partial t(\phi_{\epsilon,t})$. Taking the derivative of the left hand side of (20) by using (19), we have

$$\int_{\Omega_t} (\Delta\varphi'_{\epsilon,t}\Delta w + \mu\varphi'_{\epsilon,t} w + F'(\varphi_{\epsilon,t})\, \varphi'_{\epsilon,t}\, w)dx +$$

$$\int_{\Gamma_t} (\Delta\varphi_{\epsilon,t}\Delta w + F(\varphi_{\epsilon,t})\, w - fw) < V, n > d\Gamma = 0, \tag{21}$$

(because $\varphi_{\in,t} = 0$, on Γ_t) where $F'(\varphi_{\in,t}) = 1/\in(d/d\varphi F(\varphi_{\in,t}))$. In similar way, the derivative of the cost function J is:

$$J'(\Omega_t) = \int_{\Omega_t} (\Delta\varphi'_{\in,t}\Delta\varphi_{\in,t} - f\varphi'_{\in,t})dx + 1/2 \int_{\Gamma_t} |\Delta\varphi_{\in,t}|^2 <V,n> d\Gamma, \quad (22)$$

Now we shall find the value of the first integral of (22). We define the adjoint state $P_{\in,t} \in H^2{}_0(\Omega_t)$; for this we introduce our second penalized differential equation as follows:

$$AP_{\in,t} + \mu P_{\in,t} + F'(\varphi_{\in,t})P_{\in,t} = f_1, \quad (23)$$
$$P_{\in,t} = 0, \; on \; \Gamma_t,$$

where $f_1 = (\Delta.(\Delta\varphi_{\in,t}) - f)$; the state $P_{\in,t}$ is needed to compute the derivative of the cost function $J(\Omega_t)$. The variational form of (23) is

$$\int_{\Omega_t} (\Delta P_{\in,t}\Delta w + \mu P_{\in,t} w + F'(\varphi_{\in,t})P_{\in,t} w)\, dx = \int_{\Omega_t} f_1 w dx, \quad (24)$$

for all $w \in H^2{}_0(\Omega_t)$. Taking $w = P_{\in,t}$ in (21) and $w = \varphi'_{\in,t}$ in (24), we get:

$$\int_{\Omega_t} (\Delta\varphi_{\in,t}\Delta\varphi'_{\in,t}) - f\varphi'_{\in,t})dx = -\int_{\Gamma_t} (\Delta\varphi_{\in,t}\Delta P_{\in,t}) < V,n > d\Gamma. (25)$$

because $P_{\in,t} = 0$ on Γ_t. Thus

$$J'(\Omega_t) = 1/2 \int_{\Gamma_t} (|\Delta\varphi_{\in,t}|^2 - 2\Delta\varphi_{\in,t},\Delta P_{\in,t}) < V,n > d\Gamma\,, \quad (26)$$

or,

$$J'(\Omega_t) = \int_{\Gamma_t} C_{\Gamma_t \in,t} < V,n > d\Gamma\,, \quad (27)$$

where

$$C_{\in,t} = (1/2|\Delta\varphi_{\in,t}|^2 - \Delta\varphi_{\in,t}\Delta P_{\in,t}) \quad (28)$$

and n is called the normal field on Γ_t (n is taken going out of Ω_t). We sha'! choose $t = 0$ throughout, and write (27) as

$$J'(\Omega_t) = < G, V >_{L_2}, \text{ of course, } G = (C_{\in,t=0}.n); \quad (29)$$

this is called the gradient of $J(\Omega_t)$ at $t = 0$; it is distribution with support on the boundary Γ_t. Several different presentations of the gradient have been developed; see ([3]). Since we know that the derivative of the cost function J depends only on the value of V so as to make the derivative of the cost function as negative as possible. The corresponding value of the vector field V, to be considered unitary, that is $||V||_{L2} = 1$, as explained below, is of course determined by the following relations:

$$|<G, V>| \leq ||G|| \, ||V|| = ||G||; \tag{30}$$

the value of V must be:

$$V = -G / ||G||. \tag{31}$$

Note that the choice $||V||_{L2} = 1$ is justified, since V is simply a direction of optimum descent. We consider now the problem associated with the variational inequality.

3. FORMULATION OF VARIATIONAL INEQUALITY:

Here we shall discuss the optimal shape problem for the systems described by the differential inequality by taking the limits, as $\in$ tends to zero, of the corresponding expression in the previous section 2; we shall use the material derivative for solving it. The optimal shape is found by success approximation starting form initial guess Ω^0; the algorithm is then developed by means of a material derivative method. As we noted in the previous section 2, the problem has been discretized, so that the shape Ω_t is defined by the co-ordinates of the nodes; then, the expression for the cost function J is:

$$J(\Omega_t) = 1/2 \int_{\Omega_t} |\Delta\varphi_{\in,t}|^2 dx - \int_{\Omega_t} f\varphi_{\in,t} dx, \tag{32}$$

where $\phi_{\in,t}$ is solution of (13). Now we shall take the limit as $\in$ tends to zero, of these quantities. First we shall find the value of the limit of the cost function J, as $\in$ tends to zero. Since we already know from the existence solution of our first penalized equation ([6]) that

$$\varphi_{\in,t} \to \varphi_t \text{ in } H_0^2(\Omega) \text{ weakly, as } \in \to 0,$$

and also

$$\varphi_{\in,t} \to \varphi_t \text{ in } L^2(\Omega) \text{ strongly, as } \in \to 0,$$

so by taking the limit as $\in$ tends to zero, on both sides of (32) and since $\varphi_{\in,t} \to \varphi_t$ in $L^2(\Omega)$ strongly, so

$$\int_{\Omega_t} (|\Delta\varphi_{\in,t}|^2 - f\varphi_{\in,t}) dx \to \int_{\Omega_t} |\Delta\varphi_t|^2 - f\varphi_t) dx,$$

since the functional

$$\varphi_t \to \int_{\Omega_t} (|\Delta\varphi_t|^2 - f\varphi_t) dx.$$

is continuous in $L^2(\Omega_t)$; thus, we have

$$J(\Omega_t) = 1/2\int_{\Omega_t} |\Delta\varphi_t|^2 dx - \int_{\Omega_t} f\varphi_t dx, \tag{33}$$

which is the required value of the cost function $J(\Omega_t)$. Now we shall find the derivative of the cost function as $\in$ tends to zero. Taking limit (as $\in \rightarrow 0$) on the both sides of (26), we get

$$\lim_{\in\rightarrow 0} J'(\Omega_t) = \lim_{\in\rightarrow 0} 1/2 \int_{\Gamma_t} (|\Delta\varphi_{\in,t}|^2 - 2\Delta\varphi_{\in,t}\Delta P_{\in,t}) < V, n > d\Gamma. \tag{34}$$

Now we shall need to find the limit of the vector $P_{\in,t}$ as $\in$ tends to zero; in the appendix we prove the following theorem, which shows that this limit, P_t, is itself the solution of a variational inequality:

Theorem A: As $\in \rightarrow 0$, $P_{\in,t} \rightarrow P_t$ in K, P_t being the solution of the variational inequality, for $P_t \in K$,

$$(AP_t, w - P_t) + \mu(P_t, w - P_t) \geq (f_1, w - P_t), \forall\, w \in K, \tag{35}$$

where K is closed convex set. Now, we shall find the value of the derivative of the cost function when $\in$ tends to zero by using (34). Since we know that

$$\lim_{\in\rightarrow 0}\int_{\Omega_t}(\Delta\varphi_{\in,t}\Delta\varphi'_{\in,t}) - f\varphi'_{\in,t})dx = -\lim_{\in\rightarrow 0}\int_{\Gamma_t}(\Delta\varphi_{\in,t}\Delta P_{\in,t}) < V, n > d\Gamma.,$$

since

$$\varphi'_{\in,t} \rightarrow \varphi'_t \text{ in } L^2(\Omega) \text{ weakly, as } \in \rightarrow 0,$$

then

$$\varphi_{\in,t} \rightarrow \varphi_t \text{ in } L^2(\Omega) \text{ strongly, as } \in \rightarrow 0,$$

so

$$\lim_{\in\rightarrow 0}\int_{\Omega_t}(\Delta\varphi_{\in,t}\Delta\varphi'_{\in,t}) - f\varphi'_{\in,t})dx = \int_{\Omega_t}(\Delta\varphi_t\Delta'_t) - f\varphi'_t)dx,$$

since the functional

$$\varphi'_t \rightarrow \int_{\Omega_t}(\Delta\varphi_t\Delta\varphi'_t - f\varphi'_t)dx,$$

is continuous in $L^2(\Omega_t)$, so the above equation becomes:

$$\int_{\Gamma_t}(\Delta\varphi_{\in,t}\Delta P_{\in,t}) < V, n > d\Gamma = -\int_{\Omega_t}(\Delta\varphi_{\in,t}\Delta\varphi'_{\in,t}) - f\varphi'_{\in,t})dx,$$

and (34) gives rise to

$$J'(\Omega_t) = 1/2 \int_{\Gamma_t} (|\Delta\varphi_t|^2 - 2\Delta\varphi_t \Delta P_t) < V, n > d\Gamma \qquad (36)$$

which is the required value of the derivative of the cost function $J(\Omega_t)$. As before, the derivative of the cost function can be explicitly computed and used to minimize $J(\Omega)$.

$$J'(\Omega_t) = \int_{\Gamma_t} C_t < V, n > d\Gamma \qquad (37)$$

where

$$C_t = 1/2(|\Delta\varphi_t|^2 - 2\Delta\varphi_t \Delta P_t), \qquad (38)$$

also, we can write (38) in this way:

$$J'(\Omega_t) = < G, V >_{L_2}, \text{ where, } G = (C_t, n). \qquad (39)$$

As before, we choose $V = -G/||G||$.

4. FINITE ELEMENT APPROXIMATION:

The finite element method is easily adapted to produce optimal control of the partial differential equations because we simply have to replace all the respective spaces with their finite dimensional approximations. We briefly review this method. We consider an approximation of (8) by 2nd-order finite elements. Let us consider a triangulation τ_h of Ω_t and T_k is called the triangle, $UT_k = \Omega_{h,t}$. The parameter h is the size of the largest side or edge. Then V and K, respectively, are approximate by

$$V_h = \{v_h \mid v_h \in C^0, v_h = 0, \text{ on } \Gamma_h, v_h \mid T \in P_2, \forall\, T_k \in \tau_h\}, \qquad (40)$$

and

$$K_h = K \cap V_h, \qquad (41)$$

giving the approximate problem:

Find $\varphi_h \in K_h$ such that

$$\int_{\Omega_h} (\Delta\varphi_h \Delta w_h + \mu\varphi_h w_h) dx \geq \int_{\Omega_h} f(w_h - \varphi_h) dx, \ \forall\, w_h \in K_h, \qquad (42)$$

then problem (42) admits a unique solution φ_h. The optimal shapes will be found by successive approximations starting with an initial guess; the algorithm is then developed by means of a material derivative method. We note that the problem has been discretized, so that the shape is defined by the co-ordinates of the nodes; then, the expression for the function J is now

$$J(\Omega_{h,t}) = \int_{\Omega_{h,t}} 1/2\, |\Delta\varphi_{h,t}|^2\, dx - \int_{\Omega_{h,t}} f\varphi_{h,t} dx \tag{43}$$

where φ_h is solution of (42) on Ω_h. The derivative of the cost function is

$$J'(\Omega_{h,t}) = 1/2 \int_{\Gamma_{h,t}} (|\Delta\varphi_{h,t}|^2 - 2\Delta\varphi_{h,t}\Delta P_{h,t} < V, n> d\Gamma,$$

where $P_{h,t}$ is solution of variational inequality (35).

5. ERROR ESTIMATES:

Now we assume that $f \in L^p$ for $p \geq 2$. We consider a one-dimensional problem (8) and derive on $O(h)$ error estimate in an appropriate norm.

One-Dimensional Case

Suppose that $\Omega =]0,1[$ and, let we have $f \in L^2(\Omega)$. Then problem (8) can be written as

Find $\varphi \in K$ such that

$$\int_0^1 \varphi'' (\phi'' - \varphi'')dx + \int_0^1 \mu\varphi\, (\phi - \varphi)dx \geq \int_0^1 f(\phi - \varphi)dx\ \forall\ \phi \in K, \tag{44}$$

were $\varphi'' = d^2\varphi / dx^2$. Let N be an integer > 0 and let $h = 1/N$; we consider $x_i = ih$ for $i = 1,2,\ldots,N$, and $e_i = [x_{i-1} - x_i]$, $i = 1,2,\ldots,N$. We then approximate $H^2{}_0(\Omega)$ and K respectively by

$$H_{oh} = \{\phi_h \in C^1[0, 1],\ \phi_h(0) = \phi_h(1) = 0,\ \phi_h|_{ei} \in P_2,\ i = 1,2,\ldots, N\} \tag{45}$$

and

$$K_h = K \cap H_{oh} = \{\phi_h \in H_{oh},\ |\phi'_h(xi) - \phi'_h(x_{i-1}]| \leq h,\ i = 1,2,\ldots, N\} \tag{46}$$

with P_2 space of polynomial of degree ≤ 2. Then approximate problem is then defined by,

Find $\varphi_h \in K_h$ such that

$$\int_0^1 \varphi''_h(\phi''_h - \varphi''_h)dx + \int_0^1 \mu\varphi_h\, (\phi_h - \varphi_h)dx \geq \int_0^1 f(\phi_h - \varphi_h)dx,\ \ \forall\ \phi_h \in Kh. \tag{47}$$

The problem (47) clearly admits a unique solution. Now for finding the approximation error $||\varphi_h - \varphi||_{H_0^2(0,1)}$, we prove the following theorem:

Theorem B Let φ and φ_h be the respective solutions of variational inequalities (44) and (47). If $f \in L^2(0, 1)$, then we have

$$||\varphi_h - \varphi||_{H_0^2(0,1)} = O(h). \tag{48}$$

Proof

Since φ_h $K_h \subset K$, then it follows from (44) that

$$(A\varphi, \phi_h - \varphi) + \mu(\varphi, \phi_h - \varphi) \geq \int_0^1 f(\phi_h - \varphi)dx. \tag{49}$$

Then by adding (47) and (49), that $\forall\ \phi_h \in K_h$

$$\begin{aligned}(A(\varphi_h - \varphi), \varphi_h - \varphi) &+ \mu(\varphi_h - \varphi, \varphi_h - \varphi) \\ &\leq (A(\phi_h - \varphi), \varphi_h - \varphi) + \mu(\phi_h - \varphi, \varphi_h - \varphi) \\ + (A\varphi, \varphi_h - \varphi) &+ \mu(\varphi, \varphi_h - \varphi) - \int_0^1 f(\phi_h - \varphi)dx,\end{aligned} \tag{50}$$

and hence, for all $\phi_h \in K_h$

$$\begin{aligned}C||\varphi_h - \varphi||^2_{H_0^2(0,1)} &\leq C||\phi_h - \varphi||^2_{H_0^2(0,1)} + \mu(\varphi, \varphi_h - \varphi) \\ &+ \int_0^1 \varphi''(\phi''_h - \varphi'')dx - \int_0^1 f(\phi_h - \varphi)dx;\end{aligned} \tag{51}$$

since $f \in L^2(0, 1)$ implies that $\varphi \in H^4(0, 1) \cap K$, we have

$$\begin{aligned}\int_0^1 \frac{d^2\varphi}{dx^2}\frac{d^2}{dx^2}(\phi_h - \varphi)dx &= \int_0^1 \frac{d^4\varphi}{dx^4}(\phi_h - \varphi)dx \\ &\leq ||\frac{d^4\varphi}{dx^4}||_{L^2(0,1)}||\phi_h - \varphi||_{L^2(0,1)}.\end{aligned} \tag{52}$$

Since we also know that

$$||\frac{d^4\varphi}{dx^4}||_{L^2(0,1)} \leq (||\mu||\ ||\varphi + ||f||)_{L^2(0,1)}, \tag{53}$$

so, by using (51) and (52), implies $\forall\ \phi_h \in K_h$

$$C||\varphi_h - \varphi||^2_{H_0^2(0,1)} \leq C||\phi_h - \varphi||^2_{H_0^2(0,1)}$$

$$+2(||\mu||\ ||\varphi||+||f||)_{L^2(0,1)}||\phi_h-\varphi||_{L^2(0,1)}. \tag{54}$$

Let $\phi \in K$; we define the interpolate $\tau_h\phi$ by

$$\tau_h\phi \in H_{oh},$$
$$(\tau_h\phi\,(x_i) = \phi(xi), \qquad i = 1,2,\ldots, N. \tag{55}$$

We have

$$\left|\frac{d^2}{dx^2}(\tau_h\phi)\right| =_{ei} \frac{\phi'(x_i)-\phi'(x_{i-1})}{h} = 1/h \int_{x_{i-1}}^{x_i}\phi''\,dx, \tag{56}$$

and hence

$$\left|\frac{d^2}{dx^2}(\tau_h\phi)\right|_{ei} \leq 1, \text{ since } |\phi''| \leq 1 \text{ a.e. on}]0,1[; \tag{57}$$

we thus have

$$\tau_h\phi \in K_h, \forall\, \phi \in K.$$

Replacing ϕ_h by $\tau_h\varphi$ in (54), we have

$$C||\varphi_h-\varphi||^2_{H_0^2(0,1)} \leq C||\tau_{h\varphi}-\varphi||^2_{H_0^2(0,1)}$$
$$+2(||\mu||\ ||\varphi||+||f||)_{L^2(0,1)}||\tau_{h\varphi}-\varphi||_{L^2(0,1)}. \tag{58}$$

Then the regularity property $\varphi \in H^4(0, 1)$ and (53) imply

$$C||\tau_{h\varphi}-\varphi||_{H_0^2(0,1)} \leq Ch||\varphi||_{H^4(0,1)} \leq Ch(||\mu||\ ||\varphi||+||f||)_{L^2(0,1)}, \tag{59}$$

and

$$C||\tau_{h\varphi}-\varphi||_{L_0^2(0,1)} \leq Ch^2||\varphi||_{H^4(0,1)} \leq$$
$$Ch^2(||\mu||\,||\varphi||+||f||)_{L^2(0,1)}, \tag{60}$$

where C denotes various constants, independent of φ and h. The estimates (48) then follows trivially from (58)-(60).

We can now define an algorithm to solve the optimal shape problem for the variational inequalities:

Algorithm

0. Choose Ω_h^o, that is, $\{q^{k,0}\}$.

1. Compute $\varphi_h^{m'}$.

2. Compute $P_h^{m'}$.

3. Compute G.

4. Compute the vector field V.

5. Let $q^{k,m'}(\rho) = q^{k,m'} + \rho^m$, an approximation of

$$\arg \min_{0<\rho>\rho_{max}} J((q^{k,m'}(\rho))).$$

This step involves a one-dimensional optimization in the direction of the gradient; hence ρ_{max} is an appropriate value.

6. Set $q^{k,m'+1} = q^{k,m'}(\rho)$.

7. Perform a terminal check, if necessary go back to step 1.

6. DESCRIPTION OF THE PROGRAM AND ALGORITHM USED:

1. A module for solving the direct problem (or state problem). We give again the formulation of this problem:

Find $\varphi_h \in K_h$ such that

$$\int_{\Omega_t} (\Delta\varphi_h.\Delta w_h + \mu\varphi_h w_h - f w_h)dx \geq 0, \ \forall\, w_h \in K_h, \tag{61}$$

or find $\varphi_h \in K_h$ such that

$$I(\varphi_h) \leq I(w_h), \ \forall\, w_h \in K_h, \tag{62}$$

where $I(\varphi_h)$ is given by

$$I(\varphi_h) = 1/2 \int_{\Omega_t} (|\Delta\varphi_h|^2 + 2\mu|\varphi_h|^2 - 2f(\varphi_h)dx, \tag{63}$$

minimized over the convex set K_h. The function $I(\varphi_h)$, which may be written $I(\varphi_1, \varphi_2, \ldots\ldots, \varphi_{N(h)})$ to emphasize the dependence of φ_h on the coefficients in $\varphi_h = \sum_{i=1}^{N} \varphi_i \varphi^i$. The problem can be solved by the relaxation method, which is iterative in nature; let,

$$\varphi_h^0 = (\varphi_1, \varphi_2, \ldots\ldots, \varphi_{N(h)}), \text{ given in } K_h,$$

with φ_h^n known, φ_h^{n+1} is determined co-ordinate by co-ordinate, further iterations in the algorithm being given by

$$\varphi^{n+1} = \varphi^n + w(\varphi^{n+1/2} - \varphi^n); \tag{64}$$

w is the relaxation parameter $0 < w < 2$. The process described above is stopped when

$$\sum_{i=1}^{Nh} |\varphi_i^{n+1} - \varphi_i^n| / | \sum_{i=1}^{Nh} |\varphi_i^{n+1}| \leq \in_r.$$

(in our computational experiments we took $\in_r = 10^{-4}$).

2. A module for solving the adjoint-state problem, whose solution is needed to compute the derivative of the cost function J (the vector G). The adjoint state P_h is given by the solution of the following variational inequality; find $P_h \in K_h$ such that

$$\int_{\Omega_t} (\Delta P_h . \Delta w_h + \mu P_h w_h - f_1 w_h) dx \geq 0, \ \forall\, w_h \in K_h, \tag{65}$$

where φ_h is given and is solution of (61). For this problem, we use the same method as we used in the case of the state problem. In Butt ([1]), we showed that this variational inequality has a solution which minimized the following functional:

$$I(P_h) = 1/2 \int_{\Omega_t} (|\Delta P_h|^2 + 2\mu |P_h|^2 - 2 f_1 P_h) dx, \tag{66}$$

over the closed function K_h.

3. A module for the computation of the derivative of the cost function J when we know the solution φ_h of the state problem and the solution P_h of the adjoint-state problem. In the formula we must account for the variability of he criterion domain.

4. A module for the computation of the vector field V when we know the projected vector G, which we can get from the derivative of the cost function J. For each moving node we can compute the real vector if a displacement direction is imposed.

5. A module minimizing the criterion functional when we know a vector field V. We used a one-dimensional optimmization routine with optimal choice of step length ρ and eventually projection.

6. A drawing module for the plotting of the results related to a given geometry. This is convenient for quickly analyzing computational results.

The finite element method (on triangles, using second order polynomials) was used to solve the (61), (66) and (36) with $f = 11$, and $\mu = 1$. We discussed two different domains for the problem. The triangulation is composed of 289 nodes and 512 triangles (Square domain), and 217 nodes with 384 triangles (equilateral triangle domain), are shown respectively, by figures 2 and 3. The shaded regions are the plastic regions.

7. CONCLUSIONS:

We have developed a method for the optimal shape design for the shape of an elastic-plastic problem. The work has been helped by the fact that our system is governed by a variational inequality, with all its strong properties, which make the approximation much simpler. The main theoretical result – Theorem A in section 3– shown that vector which eventually defines the search direction for a minimum, is itself the solution of an associated variational inequality.

APPENDIX

The main purpose of this appendix is to sketch the proof of he Theorem A. Let the function P_ϵ minimizes the function

$$I_\epsilon(P_\epsilon) = 1/2 \int_\Omega (|\Delta P_\epsilon|^2 + 2\mu|P_\epsilon|^2 - 2f_1 P_\epsilon)dx, \qquad (67)$$

over the convex set K; the function $f_1 = (\Delta(\Delta\varphi_\epsilon) - f)$. Then it is the unique solution of (35). As $\in \to 0$, P_ϵ tends to P, a function which minimizes the functional

$$I(P) = 1/2 \int_\Omega (|\Delta P|^2 + 2\mu|P|^2 - 2f1P)dx, \qquad (68)$$

over the convex set K. We note that such a minimizer is the unique solution of the variational inequality of the Theorem A in section 3. By taking limits as $\in \to 0$ of both sides of (67), we can show ([1]) that I_ϵ tends to I. The Theorem A follows from the fact that the corresponding minimizers of these functionals are unique.

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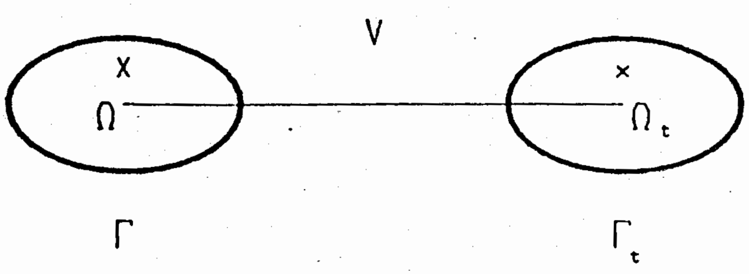

Figure 1 Shows the deformation of the domain Ω into Ω_t

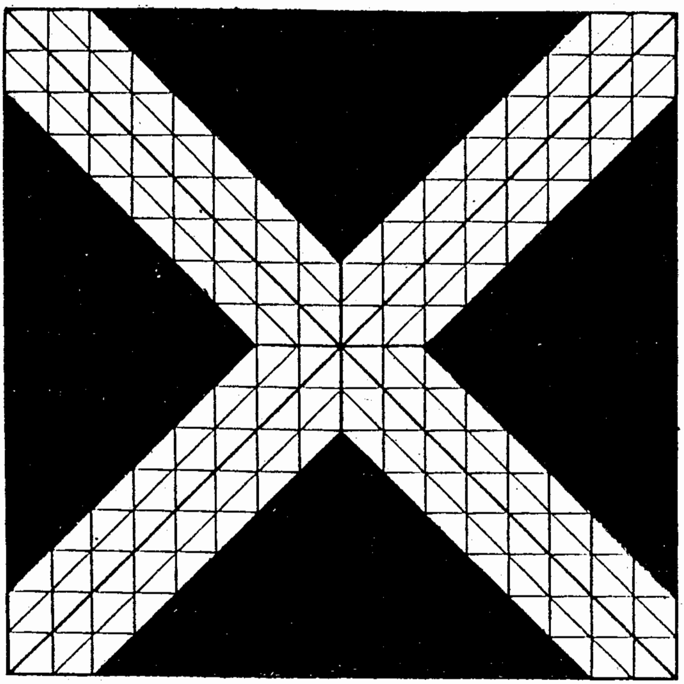

Figure 2

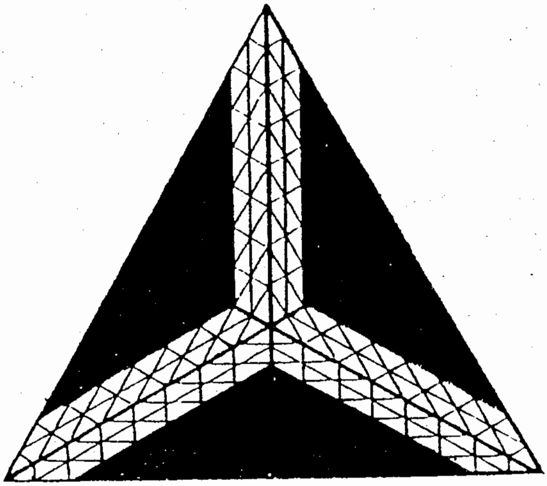

Figure 3

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ON THE MAXIMUM LIKELIHOOD REGULARIZATION OF INVERSION OF LAPLACE TRANSFORM

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ABSTRACT

A regularization method is presented in this paper for numerical inversion of Laplace transform. Regularization is affected by Maximum Likelihood technique using trigonometric polynomial approximation.

We propose to obtain an optimal amount of smoothing by calculating regularization parameter, optimally. Numerical examples are given.

1. INTRODUCTION:

The inversion of Laplace transform is a topic of fundamental importance in many areas of applied physics and mathematics. In the terminology of ill-posed problems, the Laplace transform inversion is a severely ill-posed problem; therefore considerable effort is required to obtain an accurate numerical value of the inverse for a specified value of the argument.

Attention has been paid by mathematicians, engineers, physicists and others to find ways and means of evaluating the inverse. Early methods e.g. Widder [26]; Triconi [19] and Sohat [16] involved expansion of the inverse in series of Laguerre functions, whereas Salzar [15] and Nordan [11] used orthogonal polynomials.

Lanczos [7] and Papoulis [12] have described methods in which the inverse transform is obtained as series expansions in terms of trigonometric functions, Legendre and Laguerre polynomials. For a detailed bibliography the reader is referred to Piessen [13], Piessen and Branders [14]. McWhirter and Pike [9] used eigenfunction expansion for Laplace transform inversion.

Davies and Martin [5] and Talbot [17] have given a fairly comprehensive survey of methods of numerical Laplace transform inversion, claiming that no single method gives optimum results for all purposes and all occasions.

2. LAPLACE TRANSFORM INVERSION AS A FIRST KIND EQUATION:

Varah [21] has discussed four methods for dealing with linear discrete ill-posed problems including Laplace transform inversion. In some of his methods he has converted the ill-posed problem to a well-posed problem by means of regularization. We shall compare our methods with McWhirter and Pike's method and Varah's methods on the same test examples.

The Laplace transform under consideration is denoted by $g(s)$ and is related to the (unknown) original function $f(t)$ by

$$\int_0^\infty e^{-st} f(t)dt = g(s). \qquad (1)$$

The data function $g(s)$, $s \geq 0$ is given and we desire to find $f(t)$ when $t \geq 0$ and $f(t) = 0$ for $t < 0$, so that (1) holds.

Frequently, $g(s)$ is measured at certain points. We assume $g(s)$ is given analytically with known $f(t)$, so that we can measure the error in the numerical solution.

We shall convert the Laplace transform into first kind integral equation of convolution type. The convolution equation is an ill-posed problem and the positivity constraints supply much needed information.

In order to convert the Laplace transform inversion into a Fredholm integral equation of the first kind of convolution type, we make the following substitution in equation (1)

$$s = a^x \quad \text{and} \quad t = a^{-y} \qquad \text{where } a \geq e \qquad (2)$$

Then

$$g(a^x) = \int_{-\infty}^{\infty} \log a \, e^{-a^{x-y}} f(a^{-y}) \, a^{-y} \, dy \tag{3}$$

Multiplying both sides of (3) by a^x we obtain the convolution equation.

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \quad -\infty \leq x \leq \infty \tag{4}$$

where

$$\left.\begin{aligned} G(x) &= a^x g(a^x) = sg(s) \\ K(x) &= \log a.\ a^x.e^{-a^x} = \log a.s.e^{-s} \\ F(y) &= f(a^{-y}) = f(t) \end{aligned}\right\} \tag{5}$$

In order that we can apply our deconvolution method to equation (1), it is necessary that $G(x)$ has essentially compact support, i.e., $G(x) \to 0$ as $x \to \pm\infty$, which is a property, we demand from our data function $G(x)$.

We need to choose two number namely $x_{\min}$ and $x_{\max}$ such that $|G(x)| \leq \in$, whenever $x \geq x_{\min}$ and $x \leq x_{\max}$ with $\in = 10^{-4} \cdot |G(x)|$. We calculate $x_{\min}$ and $x_{\max}$ as the smallest and largest solution of non-linear equation $|G(x)| = \in$. We then pose the deconvolution problem on the interval $[0, T]$ where $T = x_{\max} - x_{\min}$.

We shall use maximum likelihood unconstrained method of second order regularization (i.e. p = 2) in T_N (trigonometric polynomial of degree N) to solve equation (4). The Fourier transforms of $F(x)$, $G(x)$ and $K(x)$ in (5) clearly must depend on `a' in (2). It is evident that `a' plays the role of smoothing parameter in the numerical solution of (4) in addition to the usual regularization parameter λ.

Let $G_{a,n} = G(x_n) = G(nh)$, $n = 0, 1, 2, \ldots N-1$ denote the data on $[0, T]$. Then we have the DFT

$$\hat{G}_{a,q} = \sum_{n=0}^{N-1} G_{a,n} \exp\left(-\frac{2\pi i n q}{N}\right), \; q = 0, 1, \ldots, N-1. \tag{6}$$

Similarly for the kernel function

$$\hat{K}_{a,q} = \sum_{n=0}^{N-1} K_{a,n} \exp\left(\frac{2\pi i}{N} nq\right),\ q = 0, 1, \ldots, N-1. \tag{7}$$

where $K_{a,n} = \log a \exp(-a^{x_n})\, a^{x_n}$.

3. THE FILTER FUNCTION:

Now consider the functional

$$C(F; \lambda) = ||KF - G||_2 + \lambda || F^{(2)} ||_2^2 \tag{8}$$

which is minimized over the subspace $H^p \subset L_2$.

Both norms in (8) are L_2. $F^{(2)}$ denotes the second derivative of F and λ the regularization parameter.

The minimizer of (8) in H^p is given by

$$F_\lambda(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(w; \lambda, a) \cdot \frac{\hat{G}_{a\,q\,w}}{\hat{K}_{a\,q\,w}} \exp(iw_q x) \tag{9}$$

F in (9) is approximated by

$$F_{n,a,\lambda}(x) = \sum_{q=0}^{n} Z_{q;\lambda,a} \frac{\hat{G}_{N\,a\,q}}{\hat{K}_{N\,a\,q}} \exp(iw_q x) \tag{10}$$

(where '$\wedge$' stands for fast Fourier transform). Where

$$Z_{q;\lambda,a} = \frac{|\hat{K}\,N\,q\,a|^2}{|\hat{K}\,N\,q\,a|^2 + \lambda N^2 \tilde{w}_q^4} \tag{11}$$

and

$$\tilde{w}_q = \begin{cases} w_q & 0 \le q < N/2 \\ w_{N-q} & \frac{1}{2}N \le q \le N-1 \end{cases} \tag{12}$$

From equation (10) we know that the filtered solution $F_{N,\lambda,a}(x) \in T_N$ which minimizes

$$\sum_{n=0}^{N} [(K_n * F)(x_n) - G_n]^2 + \lambda || F^{(2)}(x) ||_2^2 \tag{13}$$

is

$$F_{N,\lambda,a}(x) = \frac{1}{N} \sum_{q=0}^{n} \hat{F}_{a,\lambda,q} \exp(2\pi iqx) \tag{14}$$

where

$$\hat{F}_{a,\lambda,q} = Z_{a,\lambda,q} \frac{\hat{G}_{a\ q}}{\hat{K}_{a\ q}}. \tag{15}$$

The optimal λ in $Z_{a,\lambda,q}$ is still to be determined by maximum likelihood method. $Z_{a,\lambda,q}$ is our filter function and K is instrument function in (11).

4. DETERMINATION OF OPTIMAL λ (the Regularization Parameter). MAXIMUM LIKELIHOOD (ML) METHOD (unconstrained case):

In this paper we construct maximum likelihood (unconstrained and constrained) methods, which determine λ optimally. Our construction of the methods is a simple extension of the ideas of Anderssen and Bloomfield [1, 2] and Wahba [25, 24].

Here we relate the second order convolution filter (11) to certain spectral densities which play a role in the ML optimization of λ. Assume that the data g_n are noisy and that there is an underlying function $U_N \in T_N$ such that

$$G_{n,a} = U_N(x_n) + \in_n = U_n + \in_n$$

We identify both u_n and $\in_n$ with independent stationary stochastic processes, since in general the expectation $E(U_n)$ is not zero.

Now consider $F_n = F_N(x_n) \in T_N$ defined by

$$(FK)_n = U_n, \quad n = 0, 1, 2, \ldots N$$

where

$$K = \psi \operatorname{diag}(h\hat{K}_{N,a,q})\psi^H \tag{16}$$

and ψ is the unitary matrix with elements $\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} rs\right)$,

$r,s = 0, 1, 2, \ldots, N-1$.

Now,

$$F_n = \sum_{m=0}^{N} \left\{ (K)^{-1}_{m\,n} \int_0^1 \exp(2\pi\, imn) ds_n(w) \quad \text{(Davies [4].)} \right\}$$

$$F_n = \int_0^1 [\hat{K}(w)]^{-1} \exp(2\pi i m_n) ds_n(w)$$

where

$$\hat{K}_{N,a}(w) = \sum_{n=0}^{N-1} K_{n,a} \exp(-2\pi i w_n) \tag{17}$$

Assume that F_n is estimated by $\sum_{n=0}^{N-1} l_m g_{n-m}$ where l_m is a filter which we shall relate to $Z_{q;\lambda,a}$ and $[g_n]$ is periodically continued for $n \in [0, N]$, Then the error,

$$F_n - \sum_{n=0}^{N-1} l_m G_{n-m} \tag{18}$$

is given by

$$\int_0^1 \exp(2\pi i w_n) \quad \left[(\hat{K}_{n,a}(w))^{-1} - \hat{l}_N(w) ds_n(w) \right]$$

$$- \int_0^1 \exp(2\pi i w_n)\, \hat{l}_N(w) ds_{\in}(w) \tag{19}$$

Using Davies [4], we have the variance of this error is

$$\int_0^1 | (\hat{K}_{N,a}(w))^{-1} - \hat{l}(w) |^2 P_U(w) dw \left\{ \int_0^1 | \hat{l}_N(w) | \right\} P_{\in}(w) dw \tag{20}$$

which is minimized when

$$\hat{l}_N(w) \hat{K}_{N,a}(w) = P_U(w) / [P_U(w) + P_{\in}(w)] \tag{21}$$

(where $P_U(w)$ is the spectral density).

Since the discrete Fourier coefficients of the filtered solution must satisfy

$$\begin{aligned} \hat{F}_{a;q,\lambda} &= h \hat{l}_{N,q} \hat{G}_{N,q,a} \\ &= Z_{q;\lambda,q} \hat{G}_{a,q} (h \hat{K}_{a,q})^{-1} \end{aligned} \tag{22}$$

we find

$$Z_{q;\lambda,q} = h^2 \hat{l}_{N,q} \hat{K}_{N,q,a} \tag{23}$$

In order to find optimal λ, using Iqbal [6] and Wahba [24], we have to minimize

$$V_{ML}(\lambda,a) = \frac{1}{2} N \log \left[\sum_{q=1}^{N-1} |\hat{G}_{a,q}|^2 (1 - Z_{a;\lambda,q})^2 \right]$$

$$- \frac{1}{2} \sum_{q=1}^{N-1} \log (1 - Z_{a,\lambda,q}) \tag{24}$$

w.r.t λ and a both, where $a \geq e$. We find that the overall minimum of $V(\lambda, a)$ gives the values of λ and a for which the L_∞ error of the regularized solution is least.

5. MAXIMUM LIKELIHOOD (ML) METHOD (Constrained Case):

In this section our main interest is to develop a method for choosing optimal λ and 'a', suitable for non-negativity constrained case for regularization using maximum likelihood method using trigonometric approximation.

We propose an extension of the ML method of the previous section to the constrained case. The performance of ML regularization in the constrained case is dramatically superior as compared to the unconstrained case and it is too expensive to compute.

From the cross-validation (CV) constrained regularization method discussed in Iqbal [6] and Wahba [23], we conclude that the indicator set I obtained through quadratic programming plays a key role in the algorithm. It affects the filter function and ultimately affects the expression for $V_{ML}(\lambda, a)$.

Our second order unconstrained filter is given by equation (11) and $V_{ML}(\lambda, a)$ by equation (24). If I is the indicator set underlying the matrix E (see Iqbal [6]) i.e. the set of inactive constraints indices I and L is the number of inactive constraints.

We approximate the constrained filter by

$$Z^*_{q,\lambda,a} = \begin{cases} Zq\,\lambda\,a & q \in I \\ 0 & q \notin I \end{cases} \tag{25}$$

and $V_{ML}(\lambda, a)$ in the constrained case is

$$V_{ML}(\lambda, a) = \frac{1}{N}\log\left[\sum_{q\in I}(1 - Z^{*}_{q,\lambda,a})\,|\hat{G}_{q,a}|\,|^{2}\right]$$

$$+ \sum_{q\notin I}|\hat{G}_{q,a}|^{2} - \sum_{q\in I}\log(1 - Z^{*}_{q,\lambda,a}) \,. \qquad (26)$$

To minimize (26) we used linear search technique for each λ evaluation in the minimization process. Since $V_{ML}(\lambda, a)$ is not necessarily a continuous function of λ, we made a linear search in order to find optimal λ, corresponding to the least value of $V_{ML}(\lambda, a)$. We noted the corresponding solution vector $\underline{F}_{\lambda}$, then by suitable change of variable we found $\underline{f}_{\lambda},(t)$.

6. PROBLEMS AND NUMERICAL RESULTS:

In this section we shall discuss two problems by McWhirter and Pike [9, 10], and Varah [21]. In these problems all data functions have the property $g(s) = 0(s)^{-1}$ and no noise is added apart from the machine rounding error. In both problems we have taken $N = 64$ data points.

Problem 1 (McWhirter and Pike [10]).

$$\int_{0}^{\infty} e^{-st} f(t)dt = g(s)$$

where

$$f(t) = te^{-t}$$

$$g(s) = \frac{1}{(1+s)^{2}}$$

This problem has been taken from McWhirter and Pike [9]. We have employed ML unconstrained method on this problem and results obtained are shown in Table 1 and Diag (1).

Problem 2 (Varah [21]).

$$\int_{0}^{\infty} e^{-st} f(t)dt = g(s)$$

Here

$$f(t) = \begin{cases} 1.0 & t > 2 \\ 0 & t \leq 2 \end{cases}$$

and

$$g(s) = \frac{e^{-2s}}{s} .$$

This problem has been taken from Varah [21], we used two methods on this problem

(i) unconstrained ML method.

(ii) constrained ML method.

Since this is a problem in which f(t) is discontinuous, we have experienced Gibb's phenomenon very acutely in this problem. The Fourier series does not converge uniformly. The failure of Fourier series to converge at discontinuities is given the name "Gibb's phenomenon".

(a) We have tried this problem using unconstrained ML method and the results are demonstrated in Table 2 and Diag (2). There are some negative lobes which need positivity constraints.

(b) We also tried this problem using constrained ML method. The results are better and reasonable as shown in Table 3 and Diag (3).

Table 1 (Problem 1)

(ML Unconstrained Method)

a	T_a	H_a	λ	$V_{ML}(\lambda a)$	L_∞ norm
$a = e$	20.96	0.32750	0.6620×10^{-10}	0.63927×10^{-10}	2.24×10^{-4}
$a = 5.0$	12.0	0.1875	0.1920×10^{-11}	0.39826×10^{-10}	2.08×10^{-4}
$a = 10.0$	9.00032	0.14063	0.3220×10^{-12}	0.37123×10^{-10}	2.02×10^{-4}
$a = 15.0$	7.61	0.11875	0.1411×10^{-12}	0.57920×10^{-10}	2.19×10^{-4}
$a = 20.0$	6.96	0.10875	0.7920×110^{-13}	0.52321×10^{-10}	2.17×10^{-4}

Table 2 (Problem 2)

(ML Unconstrained Method)

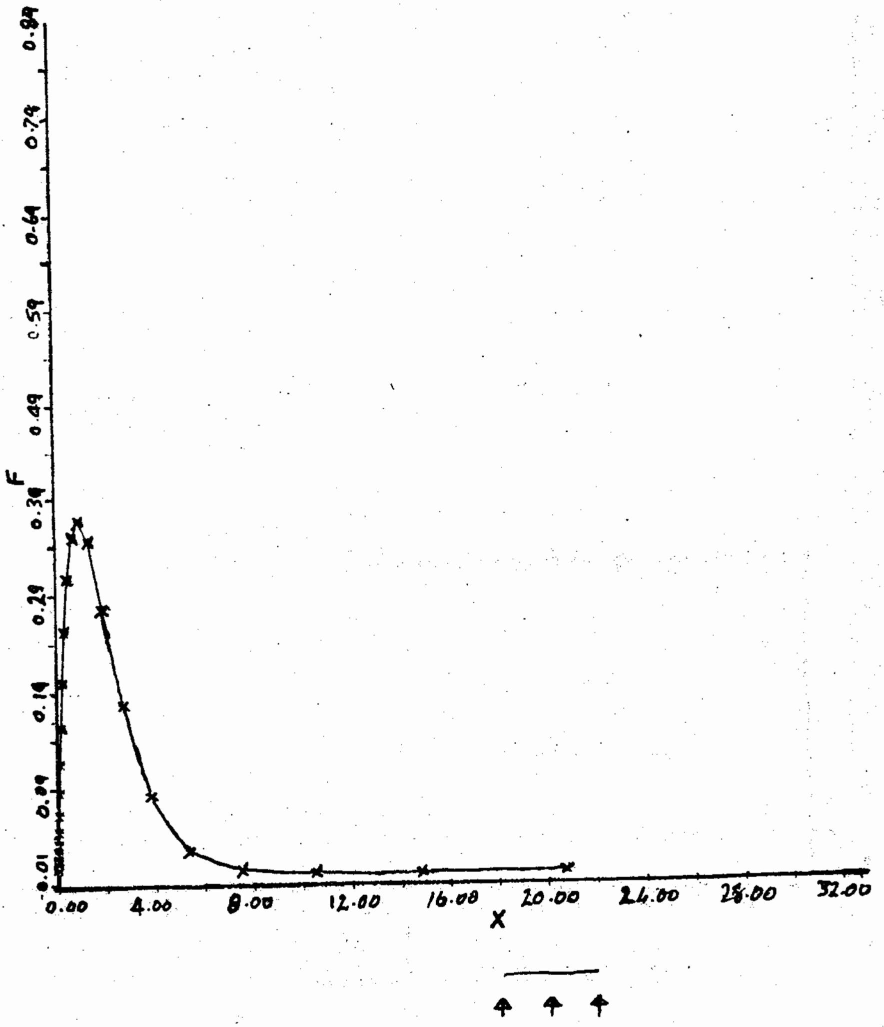
a	T_a	H_a	λ	$V_{ML}(\lambda a)$	L_∞ norm
$a = e$	16.10	0.25156	0.3020×10^{-9}	0.57231	0.63
$a = 5.0$	9.0	0.14063	0.3021×10^{-10}	0.57226	0.63
$a = 10.0$	6.72	0.1050	0.521×10^{-11}	0.57196	0.61
$a = 15.0$	5.80	0.0963	0.221×10^{-11}	0.57182	0.59
$a = 20.0$	5.28	0.08250	0.1163×10^{-11}	0.57149	0.53
$a = 25.0$	4.94	0.7719	0.793×10^{-12}	0.57189	0.57

Table 3 (Problem 2)

(ML Constrained Method)

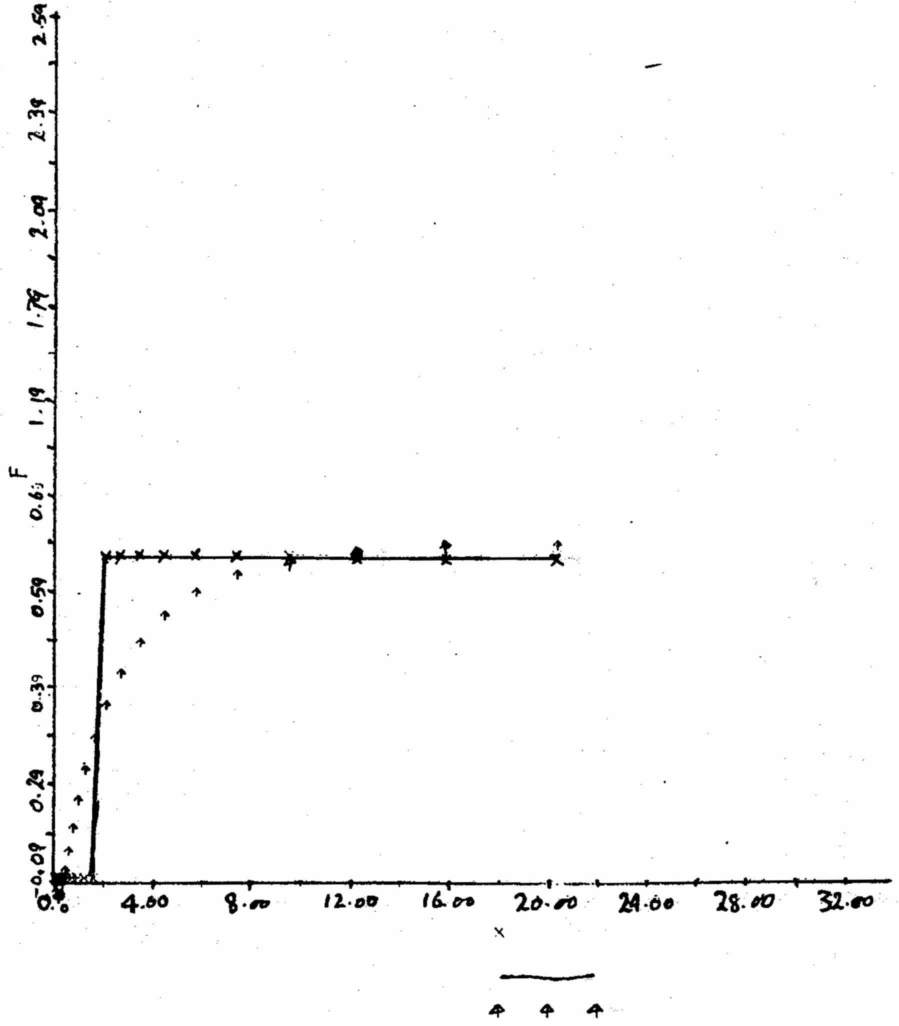
a	T_a	H_a	λ	$V_{ML}(\lambda a)$	L_∞ norm
$a = e$	16.10	0.25156	0.5623×10^{-11}	0.63215	0.590
$a = 5.0$	9.0	0.14063	0.5673×10^{-11}	0.63209	0.577
$a = 10.0$	6.72	0.1050	0.9321×10^{-12}	0.62932	0.537
$a = 15.0$	5.80	0.963	0.7632×10^{-12}	0.63125	0.571
$a = 20.0$	5.28	0.08250	0.7612×10^{-12}	0.61399	0.570
$a = 25.0$	4.94	0.07719	0.7912×10^{-12}	0.63091	0.581

DIAG (1) MCWHIRTERS' PROBLEM (1)
LAP. TRANSFORM. SOL. BY M. L. METHOD



TRUE SOL.
SOL. FOR A = 20

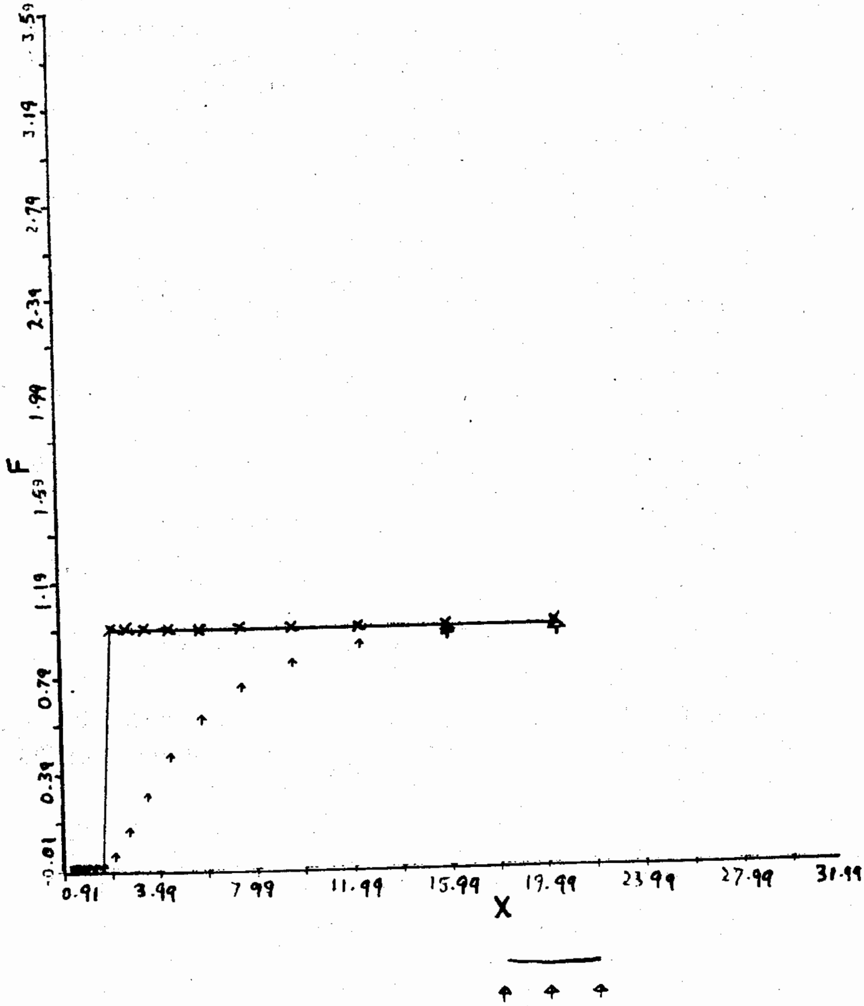
DIAG (2) VARAHS' PROBLEM (4)
LAP. TRANSFORM. SOL. BY M. L. METHOD



TRUE SOL.
SOL. FOR A = 20

DIAG (3) VARAHS' PROBLEM (4)

LAP. TRANSFORM. SOL. BY M. L. NON - NEGATIVITY



TRUE SOL.

SOL. FOR A = 20

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