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ON SOME INTEGRAL OPERATIONS WHICH PRESERVE THE UNIVALENCE

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1. INTRODUCTION

Let S be the class of regular and univalent functions $f(z) = z + a_2z^2 + \dots$ in the unit disc $U = \{z: | \, |z| < 1\}$.
The aim of this work is to prove new univalence criteria for some integral operators.
Many authors studied the problem of some integral operators which preserve the class S. In this sense, important results are due to Y.J. Kim, E.P. Merkes [3] and N.N. Pascu, V. Pescar [6].

Theorem A [3]

If the function f(z) belongs to the class S then, for any complex number γ, $|\gamma| \leq 1/4$ the function.

$$F_\gamma(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\gamma d\zeta$$

is in S.

Theorem B [6]

Let α, β, γ be complex numbers and the function $h \in S$.

If i) $\operatorname{Re} \beta \leq \operatorname{Re} \alpha > 0$ and

ii) $|\gamma| \le \frac{Re\,\alpha}{2}$ for Re $\alpha \in \left(0, \frac{1}{2}\right)$.

iii) $|\gamma| \le \frac{1}{4}$ for Re $\alpha \in \left[\frac{1}{2}, \infty\right)$

then the function

$$G_{\beta,\alpha}(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{h(u)}{u}\right)^r du\right]^\beta \tag{2}$$

belongs to the class S.

2. PRELIMINARIES

Theorem C [1]

If the function f is regular in unit disc U, $f(z) = z + a_2 z^2 + \ldots$ and

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \tag{3}$$

for all $z \in U$, then the function f is univalent in U.

Theorem D [5]

Let α be a complex number, $Re\ \alpha > 0$ and $f(z) = z + a_2 z^2 + \ldots$ be a regular function in U. If

$$\frac{1 - |z|^{2 Re\,\alpha}}{Re\,\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \tag{4}$$

for all $z \in U$, then for any complex number β, $Re\ \beta \ge Re\ \alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\, du\right]^{1/\beta} \tag{5}$$

is in the class S.

Theorem [2],[4]

If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\zeta \in U$ and $z \in U$ the following inequalities hold

$$\left| \frac{g(\zeta) - g(z)}{1 - \overline{g(z)}g(\zeta)} \right| \leq \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right| \tag{6}$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2} \tag{7}$$

the equalities hold only in case $g(z) = \epsilon \dfrac{z+u}{1+\bar{u}z}$ where $|\epsilon| = 1$ and $|u| < 1$.

REMARK A[2],(319-322)

For $z = 0$, from inequality (6) we obtain for every $\zeta \in U$

$$\left| \frac{g(\zeta) - g(0)}{1 - \overline{g(0)}.g(\zeta)} \right| \leq |\zeta| \tag{8}$$

and, hence

$$|g(\zeta)| \leq \frac{|\zeta| + |g(0)|}{1 + |g(0)||\zeta|} \tag{9}$$

Considering $g(0) = a$ and $\zeta = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|.|z|} \tag{10}$$

for all $z \in U$.

3. MAIN RESULTS

Theorem 1

Let γ be a complex number and the function $h \in S$, $h(z) = z+a_2z^2+\dots$

If
$$\left|\frac{zh'(z)-h(z)}{zh(z)}\right| \le 1 \tag{11}$$

for all $z \in U$ and the constant $|\gamma|$ satisfies the condition

$$|\gamma| \le \frac{1}{\max\limits_{|z|\le 1}\left[(1-|z|^2)|z|\frac{|z|+|a_2|}{1+|a_2||z|}\right]} \tag{12}$$

then the function

$$F_\gamma(z) = \int_0^z \left(\frac{h(u)}{u}\right)^\gamma du \tag{13}$$

belongs to the class S.

Proof

The function h is regular and univalent in U and, hence, $\frac{h(z)}{z} \neq 0$ for all $z \in U$. We can choose the regular branch of the function $\left(\frac{h(z)}{z}\right)^\gamma$ which is equal to 1 at the origin.

The function F_γ defined by (13) is regular in U. Let's consider the function

$$g(z) = \frac{1}{|\gamma|} \frac{F_\gamma''(z)}{F_\gamma'(z)} \tag{14}$$

where the constant $|\gamma|$ satisfies the inequality (12).

The function g is regular in U and we have

$$g(z) = \frac{\gamma}{|\gamma|} \left| \frac{zh'(z) - h(z)}{zh(z)} \right| \tag{15}$$

From (15) and (11) we obtain

$$|g(z)| < 1 \tag{16}$$

for all $z \in U$.

Using (15) and the definition of function h we obtain

$$g(0) = \frac{\gamma}{|\gamma|} a_2 \tag{17}$$

and, hence $|g(0)| = |a_2|$.

Applying **REMARK A [2]** of the function g we obtain

$$\frac{1}{|\gamma|} \left| \frac{F_\gamma''(z)}{F_\gamma'(z)} \right| \leq \frac{|z| + |a_2|}{1 + |a_2||z|} \tag{18}$$

for all $z \in U$.

From (18) it results that

$$\left| (1-|z|^2) z \frac{F_\gamma''(z)}{F_\gamma'(z)} \right| \le |\gamma| (1-|z|^2)|z| \frac{|z|+|a_2|}{1+|a_2||z|} \tag{19}$$

for all $z \in U$.

Let's consider the function $H:[0,1] \to \mathfrak{R}$

$$H(x) = (1-x^2)x\frac{x+|a_2|}{1+|a_2|x}; \; x = |z|$$

Because $H\left(\frac{1}{2}\right) > 0$ it results that

$$\max_{x \in [0,1]} H(x) > 0$$

Using this result and from (19) we have

$$\left| (1-|z|^2) z \frac{F_\gamma''(z)}{F_\gamma'(z)} \right| \le |\gamma| \max_{|z|\le 1} \left[(1-|z|^2)|z| \frac{|z|+|a_2|}{1+|a_2||z|} \right] \tag{20}$$

From (20) and (21) we obtain

$$(1-|z|^2) \left| \frac{zF_\gamma''(z)}{F_\gamma'(z)} \right| \le 1$$

for all $z \in U$. From Theorem C, it results that the function F_γ defined by (13) belongs to the class S.

Theorem 2

Let α, β, γ be complex numbers and the function $h \in S$, $h(z) = z + a_2z^2 + \dots$

If
$$\left| \frac{zh'(z) - h(z)}{zh(z)} \right| \leq 1 \tag{21}$$

for all $z \in U$,

Re $\beta \geq$ Re $\alpha > 0$ and (22)

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2\,Re\,\alpha}}{Re\,\alpha}|z|\frac{|z|+|a_2|}{1+|a_2||z|}\right]} \tag{23}$$

then the function

$$G_{\beta,\gamma}(z) = \left[\beta\int_0^z u^{\beta-1}\left(\frac{h(u)}{u}\right)^{\gamma} du\right]^{1/\beta} \tag{24}$$

belongs to the class S.

Proof

Let's consider the function

$$f(z) = \int_0^z \left(\frac{h(u)}{u}\right)^{\gamma} du \tag{25}$$

The function f is regular in U. Let the function

$$p(z) = \frac{1}{|\gamma|}\frac{f''(z)}{f'(z)} \tag{26}$$

where the constant $|\gamma|$ satisfies the inequality (23). The function p is regular in U. From (26) and (25) we obtain

$$p(z) = \frac{\gamma}{|\gamma|}\left[\frac{zh'(z) - h(z)}{zh(z)}\right] \tag{27}$$

and using the inequality (21) we have $|p(z)| < 1$ for all $z \in U$.

From (27) we obtain $|p(0)| = |a_2|$ and applying **REMARK A[2]** we have

$$\left|\frac{1}{|\gamma|}\frac{f''(z)}{f'(z)}\right| \leq \frac{|z| + |a_2|}{1 + |a_2||z|} \tag{28}$$

for all $z \in U$.

From (28) it results that

$$\frac{1 - |z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\frac{1 - |z|^{2Re\,\alpha}}{Re\,\alpha}|z|\frac{|z| + |a_2|}{1 + |a_2||z|} \tag{29}$$

Let's consider the function $Q:[0,1] \to \mathbb{R}$

$$Q(x) = \frac{(1 - x^{2Re\,\alpha})}{Re\,\alpha}x\frac{x + |a_2|}{1 + |a_2|x}; x = |z|$$

Because $Q\left(\frac{1}{2}\right) > 0$ it results that

$$\max_{x \in [0,1]} Q(x) > 0$$

Using this result and from (29) we conclude

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le |\gamma|\max_{|z|\le 1}\left[(1-|z|^2)\,|z|\,\frac{|z|+|a_2|}{1+|a_2|\,|z|}\right] \tag{30}$$

From (30) and (23) we obtain

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\cdot\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{31}$$

for all $z \in U$.

From (31) and Theorem D we obtain that the function

$$T_\beta(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)\,du\right]^{1/\beta} \tag{32}$$

belongs to the class S.

From (32) and (25) we conclude that the function $G_{\beta,\gamma}$ defined by (24), belongs to the class S.

Observation 1

From $\beta = 1$, $Re\alpha = 1$, from Theorem 2 we obtain Theorem 1.

Observation 2

In this work we obtain the conditions of univalence which use the coefficient a_2 too.

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BEHAVIOUR OF AMBIGUOUS AND TOTALLY POSITIVE OR NEGATIVE ELEMENTS OF $Q^*(\sqrt{n})$ UNDER THE ACTION OF THE MODULAR GROUP

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ABSTRACT: In this paper we obtain a set of criteria for the elements of $Q^*(\sqrt{n})$ to be ambiguous, totally positive and totally negative. Using these criteria we determine the behaviour of the elements of $Q^*(\sqrt{n})$ under the action of the modular group.

1. INTRODUCTION

For any non-square natural number n, the set

$$Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \in Z, \frac{a^2-n}{c}$$

is a rational integer and $\left(a, \frac{a^2-n}{c}, c\right) = 1\}$ is invariant under the action of the modular group G with presentation $G = \langle x,y : x^2 = y^3 = 1 \rangle$.

For $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$, its conjugate $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ may or may not have the same sign as that of α. If α and $\bar{\alpha}$ have different signs, then α is called ambiguous number. If they are both positive then α is called totally positive. However, if they are both negative, we call α totally negative.

Mushtaq [3]; Aslam, Husnine and Majeed [1] and [2] have discussed various aspects of $Q^*(\sqrt{n})$, and Modular group action on $Q^*(\sqrt{n})$. In the following section we determine the conditions on a, b and c such that $\alpha = \frac{a+\sqrt{n}}{c}$, where $b = \frac{a^2-n}{c}$ is totally positive, totally negative of ambiguous number, and the action of group G on α, where α is totally positive, totally negative or ambiguous number. For this purpose we have the following results.

2. PROPERTIES OF QUADRATIC FIELD $Q^*(\sqrt{n})$

Lemma 2.1

An $\alpha = \frac{a+\sqrt{n}}{c}$ is a totally positive number if and only if

either, $c>0$ and $a>0, b>0$ (1)

or $c<0$ and $a<0, b<0$ (2)

Proof

Suppose α is a totally positive number. Then α and $\overline{\alpha}$ both are positive,

that is, $\frac{a+\sqrt{n}}{c}>0$ and $\frac{a-\sqrt{n}}{c}>0$. Let $c>0$, then $a+\sqrt{n}>0, a-\sqrt{n}>0$.

So we have $a>0$. Also $a^2-n>0$, i.e. $bc>0$, This forces that $b>0$. Since, $c>0$. Hence $a>0, b>0, c>0$.

Next, let $c<0$. Then $a+\sqrt{n}<0$ and $a-\sqrt{n}<0$. This implies $a<0$

Also, $a^2-n>0$, Implies $b = \dfrac{a^2-n}{c}<0$.

Hence, $c<0$, $a<0$, $b<0$, as required.

Conversely, suppose that (1) or (2) is satisfied. In case of (1), $c>0$,

then $b = \dfrac{a^2-n}{c}>0$ implies $a^2-n>0$, This force that either $a>\sqrt{n}$

or, $a<-\sqrt{n}$. But $a>0$, so we must have $a>\sqrt{n}$

Hence, $\alpha = \dfrac{a+\sqrt{n}}{c}>0$, $\bar{\alpha} = \dfrac{a-\sqrt{n}}{c}>0$

This α is a totally positive number.

In case (2), $c<0, b = \dfrac{a^2-n}{c}<0$ implies $a^2-n>0$. This forces that

either $a>\sqrt{n}$ or, $a<-\sqrt{n}$. But $a<0$. So we must have $a<-\sqrt{n}$ that

is, $\alpha = \dfrac{a+\sqrt{n}}{c}>0$ and $\bar{\alpha} = \dfrac{a-\sqrt{n}}{c}>0$. Thus α is a totally positive

number.

Lemma 2.2

An $\alpha = \dfrac{a+\sqrt{n}}{c}$ is totally negative number if and only if

either, $c>0$ and $a<0, b>0$ (1)

or, $c<0$ and $a>0, b<0$ (2)

Proof

Suppose α is a totally negative number. Then α and $\bar{\alpha}$ both are negative, that is, $\frac{a+\sqrt{n}}{c}<0$ and $\frac{a-\sqrt{n}}{c}<0$. Let $c>0$, then $a+\sqrt{n}<0$, and $a-\sqrt{n}<0$. So we have $a<0$. Also $a^2-n>0$, i.e. $bc>0$. This forces that $b>0$, since $c>0$. Hence, $c>0$, $a<0$, $b>0$.

Next let $c<0$, then $a+\sqrt{n}>0$ and $a-\sqrt{n}>0$. This implies $a>0$. Also $a^2-n>0$, implies $b = \frac{a^2-n}{c}<0$. Hence, $c<0$, $a>0$, $b<0$ as required.

Conversely, suppose that (1) or (2) is satisfied. In case (1), $c>0$, $b = \frac{a^2-n}{c}>0$, imply $a^2-n>0$. This forces that either, $a>\sqrt{n}$ or $a<-\sqrt{n}$. But $a<0$. So, we must have $a<-\sqrt{n}$.

Hence $\alpha = \frac{a+\sqrt{n}}{c}<0$ and $\bar{\alpha} = \frac{a-\sqrt{n}}{c}<0$

Thus α is a total negative number.

In case (2), $c<0$, $b = \frac{a^2-n}{c}<0$, imply $a^2-n>0$. This force that either $a>\sqrt{n}$ or $a<-\sqrt{n}$. But $a>0$. So we must have $a>\sqrt{n}$.

Hence $\alpha = \frac{a+\sqrt{n}}{c}<0$ and $\bar{\alpha} = \frac{a-\sqrt{n}}{c}<0$

Thus α is a totally negative number.

Lemma 2.3

An $\alpha = \frac{a+\sqrt{n}}{c}$ is an ambiguous number if and only if one of the following two conditions hold.

1) $c>0,\ b<0$

2) $c<0,\ b>0$

Proof

Suppose α is an ambiguous number. Then α and $\bar{\alpha}$ have different signs

that is $\alpha\bar{\alpha}<0$.

And $\alpha\bar{\alpha} = \frac{a+\sqrt{n}}{c}\frac{a-\sqrt{n}}{c} = \frac{a^2-n}{c^2} = \frac{a^2-n}{c}\frac{1}{c} = \frac{b}{c},<0.$

Hence, we have either, $c>0,\ b<0$

or $c<0,\ b>0$

Note that a can be both positive or negative.

Conversely, suppose that (1) or (2) hold.

In case (1), $c>0,\ b<0$.

Then $$\frac{b}{c} = \frac{a^2-n}{c}\frac{1}{c} = \frac{a^2-n}{c^2}<0$$

or $$\frac{a+\sqrt{n}}{c}\ \frac{a-\sqrt{n}}{c}<0$$

This implies that α and α have different signs. Hence α is an ambiguous number.

In case (2), $c<0$, $b>0$.

Then $\frac{b}{c}<0$ or $\frac{a^2-n}{c^2}<0$, so

or $$\frac{a+\sqrt{n}}{c}\,\frac{a-\sqrt{n}}{c}<0$$

Again α is an ambiguous number.

3. GROUP ACTION AND QUADRATIC FIELD

The transformation $y:C^* \to C^*$ defined by $y(z) = \frac{z-1}{z}$, $z \in C^*$ shall denoted by y. It is easy to see that $y^3 = 1$, the identity transformation.

For each $\alpha \in Q^*(\sqrt{n})$, $y(\alpha)$, $y^2(\alpha)$ and $y^3(\alpha)$ form the vertices of a triangle. If α is a totally negative number, what can we say about $y(\alpha)$ and $y^2(\alpha)$?. The following theorem gives an answer to his question. This result shows that under the transformation y we have a triangle with one vertex totally negative and the other two vertices totally positive.

Lemma 3.1

Let $\alpha = \frac{a+\sqrt{n}}{c}$ be a totally negative number. Then the images $y(\alpha)$ and $y^2(\alpha)$ both are totally positive.

Proof

Since $\alpha = \frac{a+\sqrt{n}}{c}$ is a totally negative number, therefore by lemma 2.2 we have the following two cases:

$$c>0,\ a<0,\ b>0 \qquad (1)$$

$$c<0,\ a>0,\ b<0 \qquad (2)$$

Now, $y(\alpha) = \frac{\alpha - 1}{\alpha} = \frac{(-a+b)+\sqrt{n}}{b} = \frac{a_1+\sqrt{n}}{c_1}$ (Say)

Thus $a_1 = -a+b$, $c_1 = b$ and $b_1 = \frac{a_1^2-n}{c_1} = b-2a+c$

And $y^2(\alpha) = \frac{1}{1-\alpha} = \frac{(-a+c)+\sqrt{n}}{b-2a+c} = \frac{a_2+\sqrt{n}}{c_2}$ (Say)

Thus $a_2 = -a+c$, $c_2 = b\text{-}2a+c$, $b_2 = \frac{a_2^2-n}{c_2} = c$

In case (1), $a<0$, $b>0$ i.e. $-a>0$, $b>0$ implies that $-a+b>0$. Also, $a<0$, $b>0$ and $c>0$ implies $b\text{-}2a+c>0$. So $c_1 = b>0$, $a_1 = -a+b>0$, $b_1 = b\text{-}2a+c>0$.

Hence, by lemma 2.1 y(α) is totally positive.

And, $a<0$, $c>0$ i.e. $-a>0$, $c>0$ implies $-a+c>0$. Also, $a<0$, $c>0$, $b>0$ implies that $b\text{-}2a+c>0$. Thus, we have, $c_2 = b\text{-}2a+c>0$, $a_2 = -a+c>0$, $b_2 = c>0$. Again, by lemma 2.1 y^2(α) is also totally positive.

In case (2), $a>0$, $b<0$ i.e. $a>0$, $-b>0$ implies that $a\text{-}b>0$ or $-a+b<0$. And $a>0$, $c<0$ i.e. $a>0$, $-c>0$ implies $a\text{-}c>0$ or $-a+c<0$, $-a+b<0$ and $-a+c<0$ forces that $(-a+b)+(-a+c)<0$ or $b\text{-}2a+c<0$. We have,

$$c_1 = b<0,\ a_1 = -a+b<0,\ b_1 = b\ \text{-}2a+c<0$$

$$c_2 = b\text{-}2a+c<0,\ a_2 = -a+c<0,\ b_2 = c<0$$

Hence by lemma 2.1 y(α) and y^2(α) are totally positive.

The following theorem explains the action of transformation $x: C^* \to C^*$ given by $x(z) = \frac{-1}{z}$, $z \in C^*$ on the totally positive or totally negative elements of $Q*(\sqrt{n})$

Lemma 3.2

For any $\alpha = \frac{\alpha + \sqrt{n}}{c}$ in $Q*(\sqrt{n})$, x transforms a totally positive (respectively totally negative) α into totally negative (respectively totally positive), element of $Q*(\sqrt{n})$

Proof

Let $\alpha = \frac{a+\sqrt{n}}{c}$ be a totally positive number, then by lemma 2.1, we have,

$$\text{either,} \quad c>0,\ a>0,\ b>0 \tag{1}$$

$$\text{or} \quad c<0,\ a<0,\ b<0 \tag{2}$$

$$x(\alpha) = \frac{-1}{\alpha} = \frac{-a+\sqrt{n}}{b} = \frac{a_3+\sqrt{n}}{c_3} \text{ (say)}$$

Thus, $a_3 = -a, b_3 = \frac{a_3^2-n}{c_3} = c, c_3 = b$

Let $c>0$, $a>0$, $b>0$

Then $a_3 = -a<0$, $b_3 = c>0$, $c_3 = b>0$.

Again, by lemma 2.2 x(α) is totally negative.

Now suppose $c<0,\ a<0,\ b<0$

Then $a_3 = -a>0,\ b_3 = c<0,\ c_3 = b<0$

Again, by lemma 2.2, x(α) is total negative.

Next, suppose that $\alpha = \dfrac{a+\sqrt{n}}{c}$ is a totally negative number.

Then by lemma 2.2 we, have.

Either $\quad c>0,\ a<0,\ b>0 \quad$ (1)

or, $\quad c<0,\ a>0,\ b<0 \quad$ (2)

In case (1)', $a_3 = -a>0,\ b_3 = c>0,\ c_3 = b>0$

In case (2)', $a_3 = -a<0,\ b_3 = c<0,\ c_3 = b<0$

Hence, by lemma 2.1, x(α) is totally positive in both cases.

Illustration

Let $\alpha = \dfrac{28+\sqrt{2}}{46}, b = \dfrac{(28)^2-2}{46} = 17$

Here, $a = 28>0,\ b = 17>0,\ c = 46>0$

So, α is a totally positive number.

Now $x(\alpha) = \dfrac{-1}{\alpha} = \dfrac{-28+\sqrt{2}}{17} = \dfrac{a_1+\sqrt{2}}{c_1}, b_1 = \dfrac{(-28)^2-2}{17} = 46$

$a_1 = -28<0,\ b_1 = 46>0,\ c_1 = 17>0$

$c_1>0,\ a_1<0,\ b_1>0$

Hence x(α) is totally negative

Again we take $\alpha = \dfrac{-74+\sqrt{2}}{119}, b = \dfrac{(-74)^2-2}{119} = 46>0$

$$a = -74<0,\ b = 46>0,\ c = 119>0.$$

which is a totally negative number.

And, $x(\alpha) = \dfrac{-1}{\alpha}, = \dfrac{74+\sqrt{2}}{46} = \dfrac{a_1+\sqrt{2}}{c_1}, b_1 = \dfrac{(74)^2-2}{46} = 119>0$

$$a_1 = 74>0,\ b_1 = 119>0,\ c_1 = 46>0$$

This shows that x(α) is totally positive. If we take.

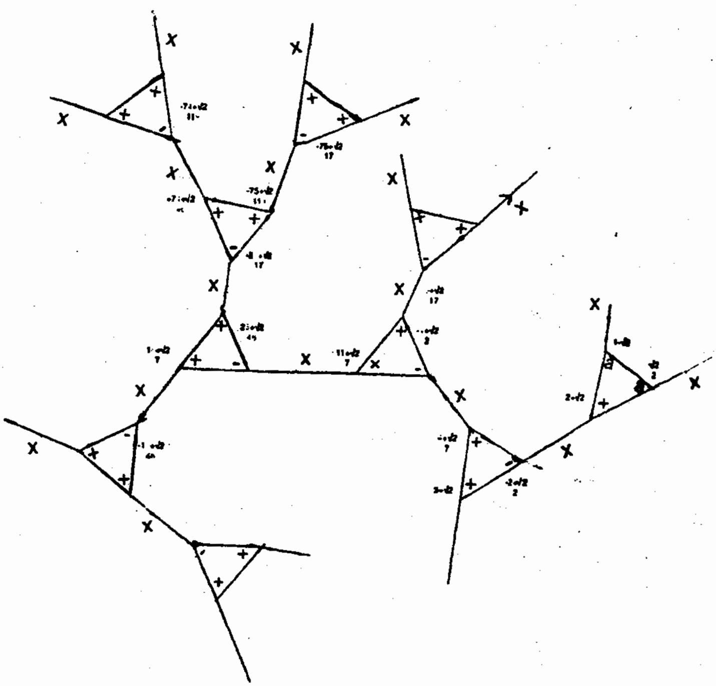

Fig.3.1

$$\alpha = \frac{-28+\sqrt{2}}{-46}, b = \frac{(-28)^2-2}{-46} = -17<0$$

$$a = -28<0,\ b = -17<0,\ c = -46<0.$$

So, α is a totally negative number.

$$x(\alpha) = \frac{-1}{\alpha} = \frac{28+\sqrt{2}}{-17} = \frac{a_1+\sqrt{2}}{c_1}, b = \frac{(28)^2-2}{-17} = -46<0$$

$$a_1 = 28>0,\ b_1 = -46<0,\ c_1 = -17<0.$$

So, $x(\alpha)$ is a totally negative number. Again if we take

$$\alpha = \frac{74+\sqrt{2}}{-119}, b = \frac{(74)^2-2}{-119} = -46<0$$

$$a = 74>0,\ b = -46<0,\ c = -119<0.$$

Which is a totally negative number.

And, $$x(\alpha) = \frac{-1}{\alpha} = \frac{-74+\sqrt{2}}{-46} = \frac{a_1+\sqrt{2}}{c_1}, b_1 = \frac{(-74)^2-2}{-46} = -119$$

$$a_1 = -74<0,\ b_1 = -119<0,\ c_1 = -46<0.$$

This shows that $x(\alpha)$ is a totally positive.

These remarks are explained in [Fig.3.1].

The theorem given below states the action of transformation y defined by $y(z) = \frac{z-1}{z}$ and of y^2 on the totally positive element of $Q^*(\sqrt{n})$

Theorem 3.3

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally positive number, then one of the following conditions hold:-

1) If $c>0$, $a>0$, $b>0$ and $y(\alpha)$ is totally negative then $a>b$, $a<c$ and $y^2(\alpha)$ is totally positive. However, if $y(\alpha)$ is totally positive then $a<b$, $a>c$ and $y^2(\alpha)$ is totally negative.

2) If $c<0$, $a<0$, $b<0$ and $y(\alpha)$ is totally negative then $|a|>|b|$, $|a|<|c|$ and $y^2(\alpha)$ is totally positive, however, if $y(\alpha)$ is totally positive then $|a|<|b|$, $|a|>|c|$ and $y^2(\alpha)$ is totally negative.

Proof

Since $\alpha = \dfrac{a+\sqrt{n}}{c}$ is a totally positive number, therefore by lemma 2.1, we have the following two cases.

$$\text{either} \quad c>0,\ a>0,\ b>0 \tag{1}$$

$$\text{or} \quad c<0,\ a<0,\ b<0 \tag{2}$$

Now $y(\alpha) = \dfrac{\alpha-1}{\alpha} = \dfrac{(-a+b)+\sqrt{n}}{b} = \dfrac{a_1+\sqrt{n}}{c_1}$ (Say)

and $y^2(\alpha) = \dfrac{1}{1-\alpha} = \dfrac{(-a+c)+\sqrt{n}}{b-2a+c} = \dfrac{a_2+\sqrt{n}}{c_2}$ (Say)

Thus $a_1 = -a+b, c_1 = b, b_1 = \dfrac{a_1^2-n}{c_1} = b-2a+c$

And $a_2 = -a+c, c_2 = b-2a+c, b_2 = \dfrac{a_2^2-n}{c_2} = c$

Suppose y(α) is totally negative, then by lemma 2.2, we have

either $c_1>0,\ a_1<0,\ b_1>0$ (A)

or $c_1<0,\ a_1>0,\ b_1<0$ (B)

Since y(α) is totally negative, by lemma 3.1, $y^2(\alpha)$ is totally positive and by lemma 2.1, we have

either $c_2>0,\ a_2>0,\ b_2>0$ (C)

or $c_2<0,\ a_2<0,\ b_2<0$ (D)

Now, suppose y(α) is totally positive, then by lemma 2.1, we have

either $c_1>0,\ a_1>0,\ b_1>0$ (P)

or $c_1<0,\ a_1<0,\ b_1<0$ (Q)

Since y(α) is totally positive, so by lemma 3.1, $y^2(\alpha)$ is totally negative and by lemma 2.2, we have

either $c_2>0,\ a_2<0,\ b_2>0$ (R)

or $c_2<0,\ a_2>0,\ b_2<0$ (S)

(1)' Let $c>0,\ a>0,\ b>0$

Then in this case (B), (D), (Q) and (S) cannot hold because in these cases $c_1 = b<0$ and $b_2 = c<0$ but we have taken $c>0$ and $b>0$. So we left with (A), (C), (P) and (R). That is,

$c_1 = b>0,\ a_1 = -a+b<0,\ b_1 = b\text{-}2a+c>0$ (A) and $c_2 = b\text{-}2a+c>0,\ a_2 = -a+c>0,\ b_2 = c>0$ (C)

Now $-a+b<0,\ -a+c>0$ implies that $a>b,\ a<c$

Again $c_1 = b>0,\ a_1 = -a+b>0,\ b_1 = b\text{-}2a+c>0$ (P)

$c_2 = b\text{-}2a+c>0,\ a_2 = -a+c>0,\ b_2 = c>0$ (R)

$-a+b>0,\ -a+c<0$, together implies that $a<b,\ a>c$.

(2)' Now let $c<0,\ a>0,\ b<0$.

Then in this case (A), (C), (P) and (R) cannot hold because in thus case $c_1 = b>0$ and $b_2 = c>0$ but we have taken $b<0$ and $c<0$. So we left with (B), (D), (Q) and (S). That is

$c_1 = b<0,\ a_1 = -a+b>0,\ b_1 = b\text{-}2a+c<0$ (B)

$c_2 = b\text{-}2a+c<0,\ a_2 = -a+c<0,\ b_2 = c<0,$ (D)

Now $-a+b>0$ and $-a+c<0$, implies that $|a| > |b|,\ |a| < |c|$.

And $c_1 = b<0,\ a_1 = -a+b<0,\ b_1 = b-2a+c<0,$ (Q)

$c_2 = b\text{-}2a+c<0,\ a_2 = -a+c>0,\ b_2 = c<0,$ (S)

So, $-a+b<0$ and $-a+c>0$ implies that $|a| < |b|,\ |a| > |c|$.

Remarks:

The converse of theorem 3.3 is not true, that is if $a>b,\ a<c$ or $|a| > |b|,\ |a| < |c|$, then $y(\alpha)$ may not be totally negative as is evident from the following example.

Example

Let $\alpha = \frac{4+\sqrt{11}}{5}, b = \frac{16-11}{5} = 1$

which is a totally positive number.

and $a = 4$, $c = 5$, $b = 1$, $a>b$, $a<c$.

But $y(\alpha) = \frac{-3+\sqrt{11}}{1}, = \frac{a_1+\sqrt{11}}{c_1}, b_1 = 9-11 = -2$

$c_1 = 1>0$, $a_1 = -3<0$, $b_1 = -2<0$

$y(\alpha)$ is not totally negative.

Again, if we take $\alpha = \frac{-4+\sqrt{11}}{-5}, b = \frac{16-11}{-5} = -1$

$a = -4<0$, $c = -5<0$, $b = -1<0$

So α is totally positive.

But $y(\alpha) = \frac{3+\sqrt{11}}{-1}$

$a = -4$, $c = -5$, $b = -1$

$|a| = 4$, $|c| = 5$, $|b| = 1$

$|a| > |b|$, $|a| < |c|$

Here again, $y(\alpha)$ is not totally negative, because

$a_1 = 3>0$, $c_1 = -1<0$, $b_1 = 2>0$

Apart of the coset diagram depicting the action of G and $Q*(\sqrt{11})$ is given below.

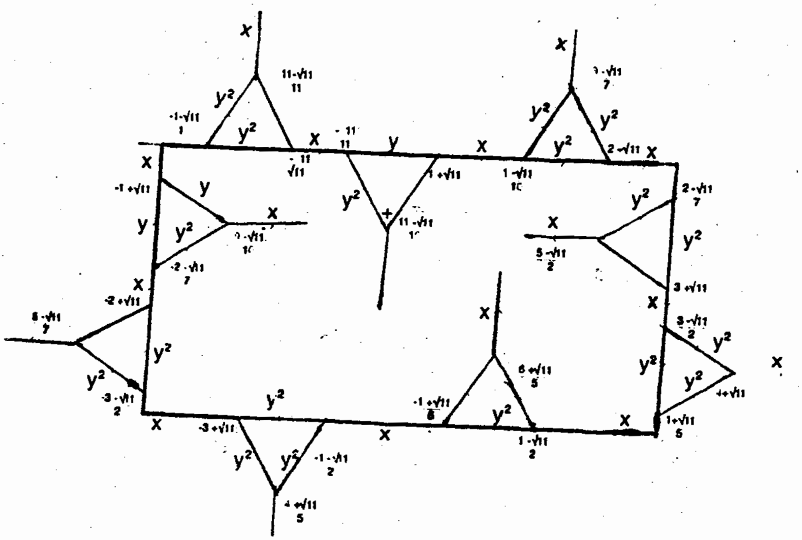

Fig.3.2

Remark

The converse of the above theorem 3.3 is not true that is, if $a<b$, $a>c$ of $|a|<|b|$, $|a|>|c|$ then $y(\alpha)$ is not totally positive.

Example

Let $\alpha = 4+\sqrt{11}$, $b = 16\text{-}11 = 5$

$a = 4>0$, $b = 5>0$, $c = 1>0$

So, α is totally positive and $a<b$, $a>c$

$$\text{But } y(\alpha) = \frac{1+\sqrt{11}}{5} = \frac{a_1+\sqrt{n}}{c_1}, b_1 = \frac{1-11}{5} = \frac{-10}{5} = -2$$

$$a_1 = 1>0,\ b_1 = -2<0,\ c_1 = 5>0$$

which is not totally positive.

Again, $\dfrac{-4+\sqrt{11}}{-1}$

$$a = -4<0,\ b = \frac{16-11}{-1} = \frac{+5}{-1} = -5<0,\ c = -1<0$$

This is α totally positive

$$|a| = 4,\ |b| = 5,\ |c| = 1.$$

$$|a| < |b|,\ |a| > |c|$$

$$\text{But } y(\alpha) = \frac{-1+\sqrt{11}}{-5} = \frac{a_1+\sqrt{11}}{c_1}$$

$$a_1 = -1<0,\ c_1 = -5<0,\ b_1 = \frac{1-11}{-5} = \frac{-10}{-5} = 2>0$$

which is not totally positive.

A part of coset diagram depicting the action of G on $Q*(\sqrt{n})$ is given below:

See page # 34

Fig.3.3

Theorem 3.4

Let $\alpha = \frac{a+\sqrt{n}}{c}$ be an ambiguous number then x(α) is always an ambiguous number.

Proof

Let $\alpha = \frac{a+\sqrt{n}}{c}$ be an ambiguous number then by lemma 2.3, we have the following two cases.

(1) $c>0,\ b<0$

(2) $c<0,\ b>0$

Now $$x(\alpha) = \frac{-1}{\alpha} = \frac{-1}{\frac{a+\sqrt{n}}{c}} \frac{-c}{a+\sqrt{n}} \frac{a-\sqrt{n}}{a-\sqrt{n}} = c\frac{-a+\sqrt{n}}{a^2-n} = c\frac{-a+\sqrt{n}}{bc}$$

$$x(\alpha) = \frac{-a+\sqrt{n}}{b} = \frac{a_1+\sqrt{n}}{c_1} \text{ (Say)}$$

Thus $a_1 = -a,\ c_1 = b.\ b_1 = \frac{a_1^2-n}{c_1} = \frac{(-a)^2-n}{b} = \frac{a_1^2-n}{b} = c$

Case (1), Let $c>0,\ b<0$

Then $c_1 = b<0,\ b_1 = c>0$

Hence, by lemma 2.3, x(α) is an ambiguous number.

Case (2), let $c<0,\ b>0$

Then $c_1 = b>0,\ b_1 = c<0$

Again, by lemma 2.3 $x(\alpha)$ is an ambiguous number.

Now we show that if $\alpha = \frac{a+\sqrt{n}}{c}$ is an ambiguous number then under the transformation y we have a triangle with one vertex totally positive and the other two vertices ambiguous.

Theorem 3.5

Let $\alpha = \frac{a+\sqrt{n}}{c}$ be an ambiguous number then one of $y(\alpha)$ and $y^2(\alpha)$ is ambiguous while the other is totally positive.

Proof

Let $\alpha = \frac{a+\sqrt{n}}{c}$ be an ambiguous number then by lemma 2.3, we have the following two cases:

(1) $c>0,\ b<0$

(2) $c<0,\ b>0$

Now $y(\alpha) = \frac{\alpha-1}{\alpha} = \frac{(-a+b)+\sqrt{n}}{b} = \frac{a_2+\sqrt{n}}{c_3}$ (Say)

Thus $a_2 = -a+b$, $c_2 = b$, $b_2 = \frac{a_2^2-n}{c_2} = \frac{(-a+b)^2-n}{b} = b-2a+c$

Any $y^2(\alpha) = \frac{1}{1-\alpha} = \frac{(-a+c)+\sqrt{n}}{b-2a+c} = \frac{a_3+\sqrt{n}}{c_3}$ (Say)

So, $a_3 = -a+c$, $c_3 = b\text{-}2a+c$, $b_3 = \frac{a_3^2-n}{c_3} = c$

In case (1), $c>0$, $b<0$ implies that $\frac{b}{c}<0$ i.e. $\frac{a^2-n}{c^2}<0$ or a^2-$n<0$

So a can be both positive or negative.

First we take $a<0$,

$a<0$, $c>0$ implies that $-a+c>0$.

$a<0$, $b<0$ implies either $-a+b>0$ or $-a+b<0$.

If $-a+b>0$ then $-a+c>0$ forces that $(-a+b)+(-a+c)>0$

or $b-2a+c>0$

So, $c_2 = b<0,\ a_2 = -a+b>0,\ b_2 = b-2a+c>0$

$c_3 = b-2a+c>0,\ a_2 = -a+c>0,\ b_3 = c>0$

By lemma 2.3 lemma 2.1, $y(\alpha)$ is ambiguous and $y^2(\alpha)$ totally positive.

If $-a+b<0$ then $-a+c>0$ forces that either $(-a+b)+(-a+c)<0$ or $(-a+b)+(-a+c)>0$ i.e. either $b-2a+c<0$ or $b-2a+c>0$.

Let $b-2a+c<0$

Then $c_2 = b<0,\ a_2 = -a+b<0,\ b_2 = b-2a+c<0$,

$c_3 = b-2a+c<0,\ a_3 = -a+c>0,\ b_3 = c>0$

This shows that $y(\alpha)$ is totally positive and $y^2(\alpha)$ is ambiguous $b-2a+c>0$ implies that

$c_2 = b<0,\ a_2 = -a+b<0,\ b_2 = b-2a+c>0$,

$c_3 = b-2a+c>0,\ a_3 = -a+c>0,\ b_3 = c>0$

Hence $y(\alpha)$ is ambiguous and $y^2(\alpha)$ totally positive.

Now we take $a>0$

Then $a>0$, $b<0$ implies that $-a+b>0$.

$a>0$, $c>0$, implies either $-a+c<0$ or $-a+c>0$.

If $-a+c<0$, then $-a+b<0$ forces that $(-a+b)+(-a+c)<0$ or $b-2a+c<0$.

So we have

$$c_2 = b<0,\ a_2 = -a+b<0,\ b_2 = b-2a+c<0,$$

$$c_3 = b-2a+c<0,\ a_3 = -a+c<0,\ b_3 = c>0$$

Hence $y(\alpha)$ is totally positive and $y^2(\alpha)$ ambiguous.

Again, we take $-a+c>0$ then $-a+b<0$ forces that

either $(-a+c)+(-a+b)<0$ or $(-a+b)+(-a+c)>0$

That is $b-2a+c<0$ or $b-2a+c>0$

Let $b-2a+c<0$ then

$$c_2 = b<0,\ a_2 = -a+b<0,\ b_2 = b-2a+c<0,$$

$$c_3 = b-2a+c<0,\ a_3 = -a+c>0,\ b_3 = c>0$$

This shows that $y(\alpha)$ is totally positive and $y^2(\alpha)$ ambiguous.

If $b-2a+c>0$, then

$$c_2 = b<0,\ a_2 = -a+b<0,\ b_2 = b-2a+c>0,$$

$$c_3 = b-2a+c>0,\ a_3 = -a+c>0,\ b_3 = c>0$$

By lemma 2.3 and lemma 2.1 $y(\alpha)$ is ambiguous and $y^2(\alpha)$ totally positive.

Case (2)

$$c<0,\ b>0 \text{ implies that } \frac{b}{c}<0 \text{ i.e. } \frac{a^2-n}{c^2}<0 \text{ or } a^2-n<0$$

So a can be both positive or negative. First we take $a<0$

Then $a<0$, $b>0$ implies that $-a+b>0$

$a<0$, $c<0$ implies either $-a+c>0$ or $-a+c<0$

If $-a+c>0$, then $-a+b>0$ forces that

$(-a+c)+(-a+b>0$ or $b-2a+c>0$

So $c_2 = b>0,\ a_2 = -a+b>0,\ b_2 = b-2a+c>0$

$c_3 = b-2a+c>0,\ a_3 = -a+c>0,\ b_3 = c<0$

Hence by lemma 2.1 and lemma 2.3 $y(\alpha)$ is totally positive and $y^2(\alpha)$ ambiguous.

If $-a+c<0$, then $-a+b>0$ forces that either $(-a+b)+(-a+c)<0$ or $(a+b)+(-a+c)>0$, that is $b-2a+c<0$ or $b-2a+c>0$.

Let $b-2a+c<0$, then

$c_2 = b>0,\ a_2 = -a+b>0,\ b_2 = b-2a+c<0$

$c_3 = b-2a+c<0,\ a_3 = -a+c<0,\ b_3 = c<0$

So $y(\alpha)$ is ambiguous and $y^2(\alpha)$ totally positive.

Again we taken $b\text{-}2a+c>0$, then

$$c_2 = b>0,\ a_2 = -a+b>0,\ b_2 = b\text{-}2a+c>0$$

$$c_3 = b\text{-}2a+c>0,\ a_3 = -a+c>0,\ b_3 = c>0$$

So y(α) is totally positive and $y^2(\alpha)$ ambiguous.

Now we take a>0.

Then $a>0$, $c<0$ implies that $a\text{-}c>0$ or $-a+c<0$.

And $a>0$, $b>0$ implies either $-a+b<0$ or $-a+b>0$.

Let $-a+b<0$, then $-a+c<0$ forces that $(-a+b)+(-a+c)<0$ or $b\text{-}2a+c<0$

So $$c_2 = b>0,\ a_2 = -a+b<0,\ b_2 = b\text{-}2a+c<0$$

$$c_3 = b\text{-}2a+c<0,\ a_3 = -a+c<0,\ b_3 = c<0$$

So, y(α) is ambiguous and $y^2(\alpha)$ totally positive.

Now we take $-a+b>0$ then $-a+c<0$ forces that either

$$(-a+b)+(-a+c)<0 \text{ or } (-a+b)+(-a+c)>0$$

That is $b\text{-}2a+c<0$, $b\text{-}2a+c>0$,

Let $b\text{-}2a+c<0$,

$$c_2 = b>0,\ a_2 = -a+b>0,\ b_2 = b\text{-}2a+c<0$$

$$c_3 = b\text{-}2a+c<0,\ a_3 = -a+c<0,\ b_3 = c<0\text{n}$$

So, y(α) is ambiguous and y^2 (α) totally positive.

If $b\text{-}2a+c>0$, then

$$c_2 = b>0,\ a_2 = -a+b>0,\ b_2 = b-2a+c>0$$

$$c_3 = b-2a+c>0,\ a_3 = -a+c<0,\ b_3 = c<0$$

Again by lemma 2.1 and lemma 2.3. $y(\alpha)$ is totally positive and $y^2(\alpha)$ ambiguous.

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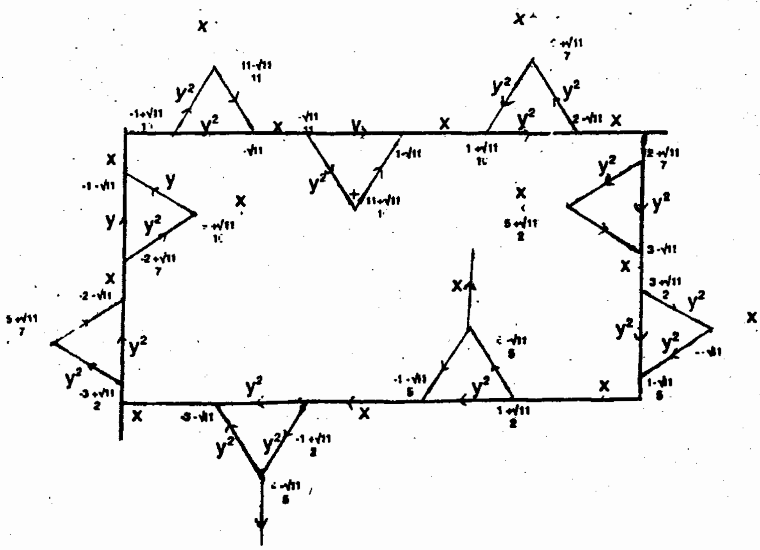

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SOME PROPERTIES OF HADAMARD HYPERNETS WITH CLASS SIZE 2

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ABSTRACT: The designs considered are such that the design and its dual are symmetric affine resolvable, with each parallel class consisting of two blocks and any two non-parallel blocks meeting in μ points. It is shown that any three mutually non-parallel blocks meet in $\mu/2$ points. Some properties of these designs are thus established that could be used for classifying the designs.

1. INTRODUCTION

There is a natural correspondence between Hadamard matrices of order 2μ and affine $1-(4\mu,2\mu,2\mu)$ designs whose duals are also affine; μ being the number of points common to any two non-parallel blocks. These designs have been studied in various guises, for example: Hadamard systems in [7], symmetric nets in [4] and Hadamard hypernets in [6]. We denote these designs by $H_2(\mu)$ in this paper, and note that any three mutually non-parallel blocks of an $H_2(\mu)$ meet in $\mu/2$ points. An unordered quadruple of mutually non-parallel blocks of an $H_2(\mu)$ is defined to be a *D-quadruple* if the intersection of any three of them is the same $\mu/2$ tuple of points. In this article we discuss some properties of D-quadruples and thus develope a characterization for these designs.

2. BACKGROUND

A *t-design* Π with parameters $t-(v,k,\lambda)$ where $v>k>0$ and $t\geq 1$, is an arrangement of v objects, called points, into subsets called blocks, so that each block consists of k points and any subset of t points is contained in exactly λ blocks. Normally, the total number of blocks is denoted by b. A design Π is said to be symmetric if $b = v$.

The *dual design* Π^* of Π has the blocks of Π as its points and then a block of Π^* is defined for each point of Π to consist of all the subset of blocks containing that point.

Π *is resolvable* if its blocks can be partitioned into subsets, called parallel classes, such that each parallel class partitions its set of points. Clearly each parallel class has size $m = v/k$. In this case two blocks are said to be parallel if they are in the same parallel class and non-parallel otherwise. If Π is resolvable so that any two of its non-parallel blocks meet in a constant number of points, say μ, Π is said to be *affine resolvable*. It is easy to see that $\mu = k/m$.

If Π is an affine resolvable $2-(v,k,\lambda > 0)$ design with m blocks in each parallel class and μ points common to any two non-parallel blocks, then the parameters of Π are completely determined by the integers m and μ [1]. We shall denote such designs by $A_m(\mu)$.

Lemma 1 [5]

Let a and c be two non parallel blocks of an $A_m(\mu)$ Π; then there exist at most $m + 1$ blocks containing the μ points common to a and c.

We call an affine $1-(v,k,r)$ design a *net* (m,r,μ), where $m = v/k$ is the number blocks in a parallel class and $\mu = k/m$ is a constant such any two non-parallel blocks intersect in μ points. Note that nets are equivalent to the orthogonal arrays of strength two of Bush and Bose [2].

An (m,r,μ) net whose dual is also a net is called a hypernet. In this case [3], $r = \mu m = k$ and so a hypernet is symmetric. Since r is determined by μ, m, we shall refer to such a hypernet as an $H_m(\mu)$ and say it is Hadamard if $m = 2$. Hadamard hypernets and Hadamard matrices are closely related (see [4], [6] or [7]).

Lemma 2 [3]

Let a be a block of an $H_2(\mu)$ Π, $\mu > 1$. The design Π_a obtained by taking the points of a as points and the intersections with a of blocks not parallel to it as blocks is an $A_2(\mu/2)$.

3. D-QUADRUPLES

Lemma 3

Any three mutually non-parallel blocks of an $H_2(\mu)$ meet in $\mu/2$ points for $\mu > 1$.

Proof

Let Π be an $H_2(\mu)$ and let d be a block of Π. Consider the affine 2-design Π_d. Since any two non-parallel blocks of Π_d meet in $\mu/2$ points, any three mutually non-parallel blocks meet in $\mu/2$ points.

Lemma 4

There exist at most four blocks containing the $\mu/2$ points common to any three mutually non-parallel blocks of an $H_2(\mu)$, $\mu > 1$.

Proof

Let a, b and c be any three mutually non-parallel blocks of an $H_2(\mu)$ Π. By lemma 1 there exist at most three blocks containing the $\mu/2$ points common to the non-parallel blocks $a \cap b$ and $a \cap c$ of Π_a, which is an $A_2(\mu/2)$. Hence in Π there exist at most four blocks containing the $\mu/2$ points of $a \cap b \cap c$.

Definition 5

An unordered quadruple (a,b,c,d) of mutually non-parallel blocks of an $H_2(\mu)$ Π is called a D-quadruple if the intersection of any three of them is the same $\mu/2$ tuple of points.

The number α of such D-quadruples in Π is called the characteristic number of Π.

If d is a block of Π then the degree of Π is the number of D-quadruples of Π in which d occurs.

Notation

Let a be a block of an H_2 (μ)Π. Then the block parallel to a is denoted by a′ and the parallel class {a,a′} is denoted by ā.

The following is easy to verify.

Lemma 6

Let a,b,c,d be four mutually non-parallel blocks of an $H_2(\mu)$ Π. Then (a,b,c,d) is a D-quadruple of Π if and only if (a,b,c′,d′) is a d-quadruple of Π.

Corollary 7

The characteristic number of any $H_2(\mu)$ is a multiple of 8.

Definition 8

A block a of an $H_2(\mu)$ Π is called a hyperplane if for any two blocks b and c, where a, b and c are mutually non-parallel, there exists a block d such that (a,b,c,d) is a D-quadruple.

A hadamard hypernet $H_2(\mu)$ is said to be complete if for any three mutually non-parallel blocks a,b and c there exists a fourth block d such that (a,b,c,d) is a D-quadruple.

Lemma 9

Let Π the be an $H_2(\mu)$. Then:

i) a block of Π is hyperplane if and only if it is of degree $4(\mu-1)(2\mu-1)/3$.

ii) Π is complete if and only if the characteristic number of Π is $4(\mu-1)(2\mu-1)/3$.

Proof

(i). Let a be a hyperplane and let (a,b,c,d) be a D-quadruple containing a. Since there are $4\mu - 2$ blocks in Π non-parallel to a, we have $4\mu - 2$ choices for b. After selecting b we have $4\mu - 4$ choices for c, since there are $4\mu - 4$ blocks non-parallel to both a and b. Hence for a given hyperplane a the unordered triple (a,b,c) can be chosen in $(4\mu - 2)(4\mu - 4)/3! = 4(\mu - 1)(2\mu - 1)/3$ ways. Now by Lemma 4 we have only one choice for d, such that (a,b,c,d) is a D-quadruple. Hence the degree of a is $4(\mu - 1)(2\mu - 1)/3$.

(ii) Clearly Π is complete if and only if every block of Π is *a* hyperplane. Thus Π is complete if and only if every block of Π appears in $4(\mu-1)(2\mu-1)/3$ D-quadruples. Since there are 4μ blocks in Π and 4 blocks in a D-quadruple, therefore Π is complete if and only if the characteristic number of Π is

$$4\mu\ 4(\mu-1)(2\mu-1)\ (3.4) = 4\mu(\mu-1)(2\mu-1)/3$$

Corollary 10

There cannot exist huperplane in a $H_2(\mu)$ with μ a multiple of 3.

Theorem 11

If a is a hyperplane in an $H_2(\mu)$, $\mu > 1$, then the $A_2(\frac{1}{2}\mu)$ induced (as in Lemma 2) on a is isomorphic to the affine design formed by the pointes and hypernets of $AG(n,2)$, where $\mu = 2^{n-1}$.

Proof

It follows, since a is a hyperplane, that in the $A_2(\frac{1}{2}\mu)$ the intersection of any two non-parallel blocks is contained in a third block. Therefore by Norman's theorem proved in [5] the $A_2(\frac{1}{2}\mu)$ is as asserted in as asserted (since all its blocks are good).

Corollary 12

If an $H_2(\mu)$ has a hyperplane, then μ is a power of 2.

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ON A CHARACTERIZATION OF $F_4(2)$ THE REE-EXTENSION OF THE CHEVALLEY GROUP $F_4(2)$..IV

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ABSTRACT: In this paper we obtain a result which improves the result of Husnine [5] and takes us nearer to the characterization of groups with a noncentral involution whose centralizer is isomorphic to the centralizer of central involution in the Chevalley Group $F_4(2)$.

1. INTRODUCTION

The centre of a Syllow 2-subgroup, S, of the Chevalley Group $F_4(2)$ is the four-group. Following [3], we denote the three involutions of this centre by x_{21}, x_{24}, x_{21}, x_{24}.

According to section 2 of [3], we have $S = \Pi S_i$ $i = 1, \ldots, 24$; $M = D_{10} = \Pi S_i$, $i \neq 10$; $D_5 = \Pi S_i$, $i \neq 5$ where, S_i, $= <x_i>$ is a subgroup of order 2. The centralizer of x_{21} in $F_4(2)$ is generated by the x_i, s, w_1, w_2 and w_5 subject to the action of w_i, s, on the x_j s given in Table-1, the commutator relations between x_i and x_j, given in Table-2 and the relations $w_1^2 = w_2^2 = w_5^2 = (w_1 w_2)^4 = (w_2 w_5)^3 = (w_1 w_5)^2 = 1$ as stated in [3]. In Husnine [5] the following result has been proved:

Theorem D

Let G be a finite group with a noncentral involution y_1, such that $C = C_G(y_1)$ is isomorphic to the centralizer of x_{21} in $F_4(2)$. We identify C with this centralizer. Then the following hold:

i) There is an element u in the normalizer of S in G such that u acts on Z_5 (S) as the graph automorphism of $F_4(2)$, permutes x_8 and x_{12}, $x_7^u = x_{18}\, x_{21}\, (\alpha)\, x_{24}\, (\beta)$ and $x_{18}^u = x_7\, x_{21}\, (\beta)\, x_{24}\, (\alpha)$; $\alpha\, \beta \in \{0, 1\}$.

ii) There is an element $N_G(M)$, such that w acts upon $Z_5(S)$ as w_{10} in $F_4(2)$.

We refer the reader to [3], for the description of the group $F_4(2)$. All notations are standard and follow [1] and [2]. Throughout the remainder of this paper, we shall refer to the Table 1 & 2 of [3], Table-3 and Table-4 of [4] and Table-5 of [5] by their numbers without referring to the papers.

2. ACTION OF $N_G(S)$ ON Z_7 (S)

In this section we prove the following Theorem.

Theorem E

Let G be a finite group with a noncentral involution y_1, such that $C = C_G(y_1)$ is isomorphic to the centralizer of x_{21} in $F_4(2)$. We identify C with this centralizer. Then the following hold:

(i) There is an element u in the normalizer of S in G such that u acts on $Z_5(S)$ as the graph automorphism of $F_4(2)$, permutes x_8 and x_{12}, $x_7^u = x_{18}\, x_{21}\, (\alpha)\, x_{24}\, (\beta)$ and $x_{18}^u = x_7\, x_{21}\, (\beta)\, x_{24}\, (\alpha)$; $\alpha\, \beta \in x_{12}$, $x_7^u = x_{18}\, x_{21}\, (\alpha)\, x_{24}\, (\beta)$ and $x_{18}^u = x_7\, x_{21}\, (\beta)\, x_{24}\, (\alpha)$; $\alpha\, \beta \in \{0, 1\}$.

ii) There is an element w in $N_G\, (M)$, such that w acts upon $Z_5\, (S)$ as w_{10} in $F_4(2)$.

iii) $x_3^u \in x_4\, S_{21}\, S_{24}$, $x_4^u \in x_3\, S_{21}\, x_{24}$

iv) $x_6^u \in x_{11}\, S_{17}\, S_{24}$, $x_{11}^u \in x_6\, S_{21}\, S_{23}$

We complete the proof of this theorem in a sequence of lemmata.

Lemma 2.1

$$Z_7(S) = S_3\, S_4\, S_6\, S_{11}\, Z_6\,(S)$$
$$Z_6(D_5) = S_3\, S_7\, S_{11}\, Z_5\,(D_5)$$
$$Z_6(M) = S_4\, S_6\, S_{18}\, Z_5\,(M)$$

Proof

This is directly verified by Table 2.

Lemma 2.2

There is an element u in $N_G(S)$ such that u satisfies theorem D, $x_4^u \in x_3 S_{21}\, S_{24}$. $x_3^u \in x_4\, S_{21}\, S_{24}$.

Proof

Let u be an element of $N_G(S)$ such that u satisfies theorem D. Then u normalizes $Z_7(S)$ and permutes $Z_5(D_5)$ and $Z_5(M)$. So, $x_3^u = x_4\, z \in Z_6(S)$.

If z involves x_{13}, then $x_3^u = x_4\, x_{13}\, z_1$, $z_1 \in z_6\,(S)$.

Thus, $[x_3^u, x_{14}] = [x_4\, x_{13}\, z_1, x_{14}] = x_{17}\, x_{21}\, x_{24}$.

We apply u^{-1} on both sides to obtain $\left[x_3, x_{14}^{u^{-1}}\right] = x_{21}x_{23}x_{34}$. This implies that x_3 is conjugate to $x_3\, x_{21}\, x_{23}\, x_{24}$ which is conjugate to $x_3\, x_{24}$ under x_{19} in S. This contradicts Table 3. So $x_3^u = x_4\, z_1$, z_1 does not involve x_{13}. If x_{14} appears in z_1, then $x_3^u = x_4\, x_{14}\, z_2$. So, $[x_3^u\, x_{13}] = [x_4\, x_{14}\, z_2, x_{13}] = x_{21}$.

This implies $\left[x_3, x_{13}^{u^{-1}}\right] = x_{24}$ which implies x_3 is conjugate in S to $x_3\, x_{24}$. This contradicts Table 3.

If x_{12} appears in x_3^u, then $[x_3^u, x_{15}] = x_{21}$. Conjugating both sides by u^{-1} we find that x_3 is conjugate in S to $x_3\, x_{24}$, again a contradiction to Table 3. If x_8 appears in x_3^u, then $[x_3^u, x_{20}] = x_{23}\, x_{24}$. Thus conjugating both sides by u^{-1} we find that x_3 is conjugate in S to $x_3\ x_{17}\ x_{21}$ which is conjugate to $x_1\ x_{21}$ violating Table 3.

Let x_3^u involve x_{15}. Then $[x_3^u, x_{12}] = x_{12}$. This implies x_3 is conjugate in S to $x_3\ x_{24}$, violating Table 3 again. So far we have proved $x_3^u = x_3, z$, $z \in Z_6\ (S)$ and z does not involve x_8, x_{12}, x_{13}, x_{14} and x_{15}.

Now, if x_{16} appears in x_3^u, $[x_3^u, x_{11}] = x_{21}$ which implies as above that x_3 is conjugate to $x_3\ x_{24}$. If x_9 appears x_3^u, $[x_3^u, x_{19}] = x_{16}\ x_{21}$. Conjugation by u^{-1} yields that x_3 is conjugate to $x_3\ x_{16}\ x_{21} = (x_3 x_{21})^{14^z}$. By Table 3 we conclude that x_{16} and x_9 can not appear in x_3^u.

If x_3^u involves x_{18}, then $[x_3^u, x_7] = x_{21}\ x_{24}$ thus conjugation By u^{-1} implies that x_3 is conjugate to $x_3\ x_{21}\ x_{24}$ in S, a contradiction to Table 3.

If x_{19} appears in x_3^u, then $[x_3^u, x_9] = x_{24}$ which implies x_3 is conjugate to $x_3\ x_{21}$ in S.

If x_{17} appears in x_3^u, then $[x_3^u, x_{10}] = x_{13}\ x_{18}\ x_{21}$ which implies x_3 is conjugate to $x_3\ x_9\ x_{24}\ x_7\ x_{21}\ (\alpha)\ x_{24}\ (\beta)$, which is conjugate to $x_3\ x_7\ x_{21}$ in S. This contradicts Table 3.

Thus x_{19} and x_{17} can not appear in x_3^u.

Let x_{20} appear in x_3^u. Then $x_3^u = x_4\ x_{20}\ z$, z is in $S_{21}\ S_{22}\ S_{23}\ S_{24}$. Thus $[x_3^u, x_8] = x_{24}$, which implies x_3 is conjugate to $x_3\ x_{21}$ in S. So x_{20} is not involved in x_3^u. Let x_{20} appear in x_3^u. Then $[x_3^u, x_6] = x_9\ x_{24}$. This implies x_3 is conjugate to $x_3\ x_{13}\ x_{21}$ which is conjugate to $x_3\ x_{21}$ under x_{11}. This violates Table 3. Thus x_{20} and x_{22} do not occur in x_3^u. Thus $x_3^u = x_4\ x_{21}\ (\alpha)\ x_{23}\ (\beta)\ x_{24}\ (\gamma)$. So $[x_3^u, x_5] = x_8\ x_{24}\ (\beta)$.

Now conjugation by u^{-1} yield that x_3 is conjugate to $x_3\ x_{12}\ x_{21}\ (\beta)$ which is conjugate to $x_3\ x_{21}\ (\beta)$ by x_{10}. Now Table 3 forces, $\beta = 0$.

Hence, x_3^u belongs to $x_4\ S_{21}\ x_{24}$ and x_4^u belongs to $x_3^{u^2}\ S_{21}\ x_{24}$. Since, u^2 satisfies theorem D, x_4^u belongs to $x_3\ S_{21}\ S_{24}$. This proves the lemma.

Lemma 2.3

There is an element u in $N_G(S)$ such that u satisfies lemma 2.2, $x_6^u \in x_{11}\ S_{17}\ S_{24}$ and $x_{11}^u \in x_6\ S_{21}\ S_{23}$.

Proof

Let u be as in lemma 2.2. Now u permutes $Z_6\ (M)$ and $Z_6\ (D_5)$. So, $x_6^u = x_{11}\ z$, z belongs to $Z_5\ (D_5) \cap Z_6\ (S)$. If x_8 appears in z, then $[x_6^u, x_{20}] = x_{24}$. Conjugation by u^{-1} yields that x_6 is conjugate in S to $x_6\ x_{21}$, contradicting Table 3. If x_9 appears in z, then $[x_6^u, x_{19}] = x_{24}$, again leading to a contradiction to Table 3. Thus x_8 and x_9 do not occur in x_6^u.

Let $x_6^u = x_{11}\ x_{18}\ (\alpha)\ x_{19}\ (\beta)\ z$. Then $[x_6^u, x_9] = x_{17}\ x_{22}\ x_{23}\ (\alpha)\ x_{24}\ (\beta)$.

Now conjugating both sides by u^{-1} and using Table 2, we find that x_6 becomes conjugate to $x_6\ x_{17}\ (\alpha)\ x_{21}\ (\beta)$. Now Table 3 forces $\alpha = \beta = 0$.

So x_{18}, x_{19} can not appear in x_6^u.

Let $x_6^u = x_{11}\ x_{12}\ (\alpha)\ x_{13}\ (\beta)\ z$. Then $[x_6^u, x_7] = x_{15}\ x_{16}\ (\alpha)\ x_{17}\ (\beta)\ x_{21}\ (\alpha)$. Now conjugation by u^{-1} yield, x_6 is conjugate in S to $x_6\ x_{22}\ (\alpha)\ x_{23}\ (\beta)$. Now $x_6\ x_{22}\ x_{23}$ is conjugate in S under $x_2\ x_{18}\ x_1\ x_{18}$ to $x_6\ x_{21}$ which is not conjugate to x_6 by Table 3, $x_6\ x_{22}$ is conjugate in S to $x_6\ x_{21}$ and $x_6\ x_{23}$ is conjugate to $x_6\ x_{16}$ in S. They all violate Table 3. So x_{12}, x_{13} can not appear in x_6^u.

Let $x_6^u = x_{11}\ x_{20}\ z$. Then $[x_6^u, x_8] = x_{22}\ x_{24}$. This implies x_6 is conjugate in S to $x_6\ x_{21}$. But x_6 and $x_6\ x_{21}$ belong to different conjugacy classes in S by Table 3. So x_{20} can not occur in x_6^u. Let $x_6^{*} = x_{11}\ x_{14}\ (\alpha)\ x_{15}\ (\beta)\ z$. Then $[x_6^u, x_3] = x_{13}\ x_{16}\ (\alpha)\ x_{17}\ (\beta)\ x_{21}\ (\alpha)$. Now conjugating both sides by u^{-1}, we find that x_6 becomes conjugate in S to $x_6\ x_9\ x_{22}\ (\alpha)\ x_{23}\ (\beta)\ x_{24}\ (\alpha)$. The last expression is conjugate under $x_4\ x_{22}$ to $x_6\ x_{22}\ (\alpha)\ x_{23}\ (\beta)$. But in S, $x_6 x_{22}$ is conjugate to $x_6 x_{21}$, $x_6 x_{23}$ is conjugate to $x_6 x_{16}$ and $x_6 x_{22} x_{23}$ is conjugate to $x_6 x_{21}$. Since x_6, $x_6 x_{21}$ and $x_6 x_{16}$ are in distinct conjugacy classes of S, x_{14}, x_{15} can not occur in x_6^u. If x_{16} appear in x_6^u, $[x_6^u, x_1] = x_{17}$. This implies x_6 is conjugate to $x_6 x_{23}$ which is conjugate $x_6 x_{16}$ in S. This contradiction to Table 3 shows that x_{16} does not occur in x_6^u.

Thus we have $x_6^u = x_{11} x_{17} (\alpha) x_{21} (\beta) x_{24} (\gamma)$ and $x_{11}^u = x_6 x_{21} (\gamma) x_{23} (\alpha) x_{24} (\beta) x_{24} (\epsilon)$. Writing u for $u x_{22} (\beta+\epsilon)$, We get $x_{11}^u = x_6 x_{21} (\gamma) x_{23} (\alpha)$.

Thus $x_{11}^{u^2} = x_6^u x_{17}(\alpha) x_{24}(\gamma)$. This forces $x_6^u = x_{11} x_{17} (\alpha) x_{24} (\gamma)$ and $x_{11}^u = x_6 x_{21} (\gamma) x_{23} (\alpha)$. This completes the proof of lemma 2.3 and there by theorem E is established.

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ACTION OF THE GROUP h = $<t,y:t^3 = y^3 = 1>$ ON THE QUADRATIC FIELD

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ABSTRACT: In this paper we study the action of the group H = $<t,y:t^3 = y^3 = 1>$ on $Q^*(\sqrt{n})$ and establish the existence of an alternating sequence of totally positive and totally negative numbers of $Q^*(\sqrt{n})$ that comes to an end when an ambiguous number shows up.

1. INTRODUCTION

The Modular group G has the presentation G = $<x,y:x(z) = -1/z, y(z) = \frac{z-1}{z}>$. Put t = xyx. Then $t(z) = -1/(z+1)$ and $x^2 = y^3 = t^3 = 1$.

The group H = $<t,y>$ is thus a subgroup of G. Various properties of $Q^*(\sqrt{n})$ and Modular group action on it have been discussed in Mushtaq [4], Aslam, Husnine and Majeed [1] & [2], and Imrana, Husnine and Majeed [3]. Here we study the action of H on $Q^*(\sqrt{n}) = \{(a+\sqrt{n})/c : a,c \in z \; ; \; b = (a^2-n)/c$ is a rational integer, and $(a,b,c) = 1\}$. We recollect that $\alpha \in Q^*(\sqrt{n})$ is called ambiguous if α and its conjugate have different signs. If they are both positive we call them totally positive and they are both negative they are called totally negative.

For each $\alpha \in Q^*(\sqrt{n})$ $t(\alpha)$, $t^2(\alpha)$ and $t^3(\alpha) = \alpha$ form the vertices of a triangle. If α is a totally positive number, what can we say about $t(\alpha)$ and $t^2(\alpha)$. The following theorem gives an answer to this question. This result shows that under the transformation t we have a triangle with one vertex totally positive and the other two vertices totally negative.

In the following sections we shall be using lemma 2.1, lemma 2.2 and lemma 2.3 of [3] without mentioning the paper.

2. THE ACTION OF H

We prove the following theorems on the action of H on $Q^*(\sqrt{n})$

Theorem 2.1

If $\alpha = \dfrac{a+\sqrt{n}}{c}$ is a totally positive number then $t(\alpha)$ and $t^2(\alpha)$ are both totally negative.

Proof

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally positive number.

$$t(\alpha) = \frac{-1}{\alpha+1} = \frac{(-a-c)+\sqrt{n}}{2a+b+c} = \frac{a_1+\sqrt{n}}{c_1} \text{ (say)}$$

Thus we have, $a_1 = -a-c, b_1 = \dfrac{a_1^2-n}{c_1} = c, c_1 = 2a+b+c$

and $t^2(\alpha) = \frac{(-a-b)+\sqrt{n}}{b} = \frac{a_2+\sqrt{n}}{c_2}$ (say)

where $a_2 = -a-b, b_2 = \frac{a_2^2-n}{c_2} = 2a+b+c, c_2 = b.$

Since, $\alpha = \frac{a+\sqrt{n}}{c}$ is a totally positive number so by lemma 2.1. we have.

either $c>0, a>0, b>0$ (1)

or $c<0, a<0, b<0$ (2)

In case (1), $a>0, c>0$ implies $a+c>0$ or $-a-c<0$

$a>0, b>0$ implies $a+b>0$ or $-a-b<0$

So we have, $c_1 = 2a+b+c>0, a_1 = -a-c<0, b_1 = c>0$

$c_2 = b>0, a_2 = -a-b<0, b_2 = 2a+b+c>0$

By lemma 2.2 $t(\alpha)$ and t^2 (α) are totally negative.

In case (2), $c_1 = 2a+b+c<0, a_1 = -a-c> 0, b_1 = c<0$

$c_2 = b<0, a_2 = -a-b>0, b_2 = 2a+b+c<0$

Again by lemma 2.2. $t(\alpha)$ and $t^2(\alpha)$ are totally negative.

The theorem given below states the action of transformation t defined by $t(z) = \frac{-1}{z+1}$ and of t^2 on the totally negative element of $Q*(\sqrt{n})$

Theorem 2.2

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally negative number, then one of the following conditions hold.

(1) If $c>0$, $a<0$, $b>0$ and $t(\alpha)$ is totally negative then $|a|>b$, $|a|<c$ and $t^2(\alpha)$ is totally positive. However, if $t(\alpha)$ is totally positive then $|a|<b$, $|a|>c$ and $t^2(\alpha)$ is totally negative.

(2) If $c<0$, $a>0$, $b<0$ and $t(\alpha)$ is totally negative then $a>|b|$, $a<|c|$ and $t^2(\alpha)$ is totally positive. However, if $t(\alpha)$ is totally positive then $a<|b|$, $a>|c|$ and $t^2(\alpha)$ is totally negative.

Proof

Since $\alpha = \dfrac{a+\sqrt{n}}{c}$ is a totally negative number, therefore by lemma 2.2, we have;

$$\text{either} \quad c>0,\ a<0,\ b>0 \qquad (1)$$

$$\text{or} \quad c<0,\ a>0,\ b<0 \qquad (2)$$

Now $$t(\alpha) = \frac{-1}{\alpha+1} = \frac{(-a-c)+\sqrt{n}}{2a+b+c} = \frac{a_1+\sqrt{n}}{c_1} \text{ (say)}$$

and $$t^2(\alpha) = \frac{(-a-b)+\sqrt{n}}{b} = \frac{a_2+\sqrt{n}}{c_2} \text{ (say)}$$

Thus $$a_1 = -a-c, c_1 = 2a+b+c, b_1 = \frac{a_1^2-n}{c_1} = c$$

$$a_2 = -a-b, c_2 = b, b_2 = \frac{a_2^2-n}{c_2} = 2a+b+c,$$

Suppose $t(\alpha)$ is totally negative, then by lemma 2.2, we have

either $c_1>0,\ a_1<0,\ b_1>0$ (A)

or $c_1<0,\ a_1>0,\ b_1<0$ (B)

Since, $t(\alpha)$ is totally negative, by theorem 2.1. $t^2(\alpha)$ is totally positive, So by lemma 2.1 we have

either $c_2>0,\ a_2>0,\ b_2>0$ (C)

or $c_2<0,\ a_2<0,\ b_2<0$ (D)

Now suppose $t(\alpha)$ is totally positive then by lemma 2.1, we have

either $c_1>0,\ a_1>0,\ b_1>0$ (P)

or $c_1<0,\ a_1<0,\ b_1<0$ (Q)

Since $t(\alpha)$ is totally positive, so by theorem 2.1, $t^2(\alpha)$ is totally negative and by lemma 2.2 we have

either $c_2>0,\ a_2<0,\ b_2>0$ (R)

or $c_2<0,\ a_2>0,\ b_2<0$ (S)

(1) Let $c>0,\ a<0,\ b>0$

Then in this case (B), (D), (Q) and (S) cannot hold, because in these cases $b_1 = c<0$, $c_2 = b<0$ but we have taken $b>0$ and $c>0$. So we are left with (A), (C), (P) and (R), that is

$$c_1 = 2a+b+c>0,\ a_1 = -a-c<0,\ b_1 = c>0 \quad \text{(A)}$$

$$c_2 = b>0,\ a_2 = -a-b>0,\ b_2 = b+2a+c>0 \quad \text{(C)}$$

Now $-a-c<0$ and $-a-b>0$ implies that $|a|>b$ $|a|<c$.

And $$c_1 = 2a+b+c>0,\ a_1 = -a-c>0,\ b_1 = c>0 \quad \text{(P)}$$

$$c_2 = b>0,\ a_2 = -a-b<0,\ b_2 = b+2a+c>0 \quad \text{(R)}$$

$-a-b<0$ and $-a-c>0$ implies that $|a|<b, |a|>c$.

(2) Let $c<0,\ a>0,\ b<0$

Then in this case (A), (C), (P) and (R) cannot hold, because in these cases $b_1 = c>0$, $c_2 = b>0$ but we have taken $b<0$ and $c<0$. So we left with (B), (D), (Q) and (S)

That is $$c_1 = 2a+b+c<0,\ a_1 = -a-c>0,\ b_1 = c<0 \quad \text{(B)}$$

$$c_2 = b<0,\ a_2 = -a-b<0,\ b_2 = 2a+b+c<0, \quad \text{(D)}$$

$-a-b<0$ and $-a-c>0$ implies that $a>|b|$, $a<|c|$ since $c<0, b<0$

And $$c_1 = 2a+b+c<0,\ a_1 = -a-c<0,\ b_1 = c<0 \quad \text{(Q)}$$

$$c_2 = b<0,\ a_2 = -a-b>0,\ b_2 = 2a+b+c<0, \quad \text{(S)}$$

$-a-b>0$ and $-a-c<0$ implies that $a<|b|$, $a>|c|$.

Now we show that if $\alpha = \dfrac{a+\sqrt{n}}{c}$ is an ambiguous number then under the transformation t we have a triangle with one vertex totally negative and the other two vertices ambiguous.

Theorem 2.3

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be an ambiguous number. Then one of $t(\alpha)$ and $t^2(\alpha)$ is ambiguous and the other is totally negative.

Proof

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be an ambiguous number. Then by lemma 2.3 we have the following two cases

(1) $c>0,\ b<0$

(2) $c<0,\ b>0$

Now $$t(\alpha) = \frac{-1}{\alpha+1} = \frac{(-a-c)+\sqrt{n}}{2a+b+c} = \frac{a_1+\sqrt{n}}{c_1} \text{ (say)}$$

and $$t^2(\alpha) = \frac{(-a-b)+\sqrt{n}}{b} = \frac{a_2+\sqrt{n}}{c_2} \text{ (say)}$$

Thus $$a_1 = -a-c,\ b_1 = \frac{a_1^2-n}{c_1} = c, c_1 = 2a+b+c$$

$$a_2 = -a-b,\ b_2 = \frac{a_2^2-n}{c_2} = 2a+b+c, c_2 = b$$

In case (1), $c>0$, $b<0$ imply that $\dfrac{b}{c}<0$ i.e. $\dfrac{a^2-n}{c^2}<0$ or $a^2-n<0$.

So a can be both positive or negative.

First we take $a<0$

$$a<0,\ b<0 \text{ imply that } a+b<0, \text{ so } -a-b>0$$

$$a<0,\ c>0 \text{ imply that either } a+c<0 \text{ or } a+c>0$$

If $a+c<0$, then $a+b<0$, forces that

$$(a+b)+(a+c) = 2a+b+c<0$$

We have

$$c_1 = 2a+b+c<0,\ a_1 = -a-c>0,\ b_1 = c>0$$

$$c_2 = b<0,\ a_2 = -a-b>0,\ b_2 = 2a+b+c<0,$$

By lemma 2.3 and by lemma 2.2. $t(\alpha)$ is ambiguous and $t^2(\alpha)$ is totally negative.

If $a+c>0$, then $a+b<0$ implies that either $(a+b)+(a+c) = 2a+b+c<0$ or $2a+b+c>0$

Let $2a+b+c<0$

Then $c_1 = 2a+b+c<0,\ a_1 = -a-c<0,\ b_1 = c>0$

$$c_2 = b<0,\ a_2 = -a-b>0,\ b_2 = 2a+b+c<0$$

Again by lemma 2.3 and lemma 2.2. $t(\alpha)$ is ambiguous and $t^2(\alpha)$ is totally negative.

If $2a+b+c>0$

Then $c_1 = 2a+b+c>0,\ a_1 = -a-c<0,\ b_1 = c>0$

$$c_2 = b<0,\ a_2 = -a-b>0,\ b_2 = 2a+b+c>0$$

Imply that $t(\alpha)$ is totally negative and $t^2(\alpha)$ is ambiguous.

Now we take $a>0$

$$a>0,\ c>0 \text{ imply that } a+c>0 \text{ or } -a-c<0$$

$$a>0,\ b<0 \text{ imply that either } a+b>0 \text{ or } a+b<0$$

If $a+b>0$, then $a+c>0$ forces that $(a+b)+(a+c) = 2a+b+c>0$

So $\quad c_1 = 2a+b+c>0,\ a_1 = -a-c<0,\ b_1 = c>0$.

$$c_2 = b<0,\ a_2 = -a-b<0,\ b_2 = 2a+b+c>0,$$

Thus $t(\alpha)$ is totally negative and $t^2(\alpha)$ ambiguous.

If $a+b<0$, then $a+c>0$ forces that either $(a+b)+(a+c) = 2a+b+c<0$ or $2a+b+c>0$

Let $2a+b+c<0$

Then $\quad c_1 = 2a+b+c<0,\ a_1 = -a-c<0,\ b_1 = c>0$

$$c_2 = b<0,\ a_2 = -a-b>0,\ b_2 = 2a+b+c<0$$

So $t(\alpha)$ is ambiguous and $t^2(\alpha)$ is totally negative.

If $2a+b+c>0$

Then $\quad c_1 = 2a+b+c>0,\ a_1 = -a-c<0,\ b_1 = c>0$

$$c_2 = b<0,\ a_2 = -a-b>0,\ b_2 = 2a+b+c>0$$

Imply that $t(\alpha)$ is totally negative and $t^2(\alpha)$ is ambiguous.

In case (2), $c<0$, $b>0$ imply that $\frac{b}{c}<0$ or $\frac{a^2-n}{c^2}<0$ or $a^2-n<0$.

So a can be both positive or negative.

First we take $a<0$

$$a<0,\ c<0 \text{ imply that } a+c<0 \text{ or } -a-c>0$$

$$a<0,\ b>0 \text{ imply that either } a+b<0 \text{ or } a+b>0$$

If $a+b<0$ then $a+c<0$ forces that $(a+b)+(a+c) = 2a+b+c<0$

So $c_1 = 2a+b+c<0,\ a_1 = -a-c>0,\ b_1 = c<0$

$$c_2 = b>0,\ a_2 = -a-b>0,\ b_2 = 2a+b+c<0$$

Thus $t(\alpha)$ is totally negative and $t^2(\alpha)$ ambiguous.

Now if $a+b<0$, then $a+c<0$ forces that either $(a+b)+(a+c) = 2a+b+c<0$ or $2a+b+c>0$.

Let $2a+b+c<0$

Then $c_1 = 2a+b+c<0,\ a_1 = -a-c>0,\ b_1 = c<0$

$$c_2 = b>0,\ a_2 = -a-b<0,\ b_2 = 2a+b+c<0$$

Imply that $t(\alpha)$ is totally negative and $t^2(\alpha)$ is ambiguous.

If $2a+b+c>0$

Then $c_1 = 2a+b+c>0,\ a_1 = -a-c>0,\ b_1 = c<0$

$$c_2 = b<0,\ a_2 = -a-b<0,\ b_2 = 2a+b+c>0$$

Hence by lemma 2.3 and lemma 2.2 $t^2(\alpha)$ is ambiguous and $t^2(\alpha)$ is totally negative.

Now we take $a>0$

$a>0$, $b>0$ imply that $a+b>0$ or $-a-b<0$

$a>0$, $c<0$ imply that either $a+c>0$ or $a+c<0$

If $a+c>0$ then $a+b>0$ forces that $(a+b)+(a+c)\ 2a+b+c>0$

So we have

$$c_1 = 2a+b+c>0,\ a_1 = -a-c<0,\ b_1 = c<0$$

$$c_2 = b>0,\ a_2 = -a-b<0,\ b_2 = 2a+b+c>0$$

Thus $t(\alpha)$ is ambiguous and $t^2(\alpha)$ is totally negative.

If $a+c<0$ then $a+b>0$ forces that $(a+b)+(a+c) = 2a+b+c<0$ or $2a+b+c>0$.

Let $2a+b+c<0$

Then $c_1 = 2a+b+c<0,\ a_1 = -a-c>0,\ b_1 = c<0$

$$c_2 = b>0,\ a_2 = -a-b<0,\ b_2 = 2a+b+c<0$$

Thus $t(\alpha)$ is totally negative and $t^2(\alpha)$ is ambiguous.

If we take $2a+b+c>0$

Then $c_1 = 2a+b+c>0,\ a_1 = -a-c>0,\ b_1 = c<0$

$$c_2 = b>0,\ a_2 = -a-b<0,\ b_2 = 2a+b+c>0$$

Hence by lemma 2.3 and lemma 2.2, $t(\alpha)$ is ambiguous and $t^2(\alpha)$ is totally negative.

3. EXISTENCE OF AN ALTERNATIVE SEQUENCE

In this section we prove the existence of an alternate sequence of totally positive and totally negative numbers. We start with a definition.

Definition

For real quadratic irrational number $\alpha \in Q^*(\sqrt{n})$ we define norm of α as

$$\|\alpha\| = |a|, \text{ where } \alpha = \frac{a+\sqrt{n}}{c}$$

Theorem 3.1

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally negative number.

Then (i) $\|y(\alpha)\| > \|\alpha\|, \|y^2(\alpha)\| > \|\alpha\|$

(ii) $\|t(\alpha)\| < \|\alpha\|$, if $t(\alpha)$ is totally positive and
$\|t^2(\alpha)\| < \|\alpha\|$, if $t^2(\alpha)$ is totally positive

Proof

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally negative number.

Then by lemma 2.2, we have

either $c>0, a<0, b>0$

or $c<0, a>0, b<0$

Let us take $c>0, a<0, b>0$

(i) Since α is totally negative so, by lemma 3.1, $y(\alpha)$, $y^2(\alpha)$ both are totally positive.

Now $$y(\alpha) = \frac{(-a+b)+\sqrt{n}}{b} = \frac{a_1+\sqrt{n}}{c_1} \text{ (say)}$$

$$y^2(\alpha) = \frac{(-a+c)+\sqrt{n}}{b-2a+c} = \frac{a_2+\sqrt{n}}{c_2} \text{ (say)}$$

Thus we have,

$$a_1 = -a+b, b_1 = \frac{a_1^2-n}{c_1} = b-2a+c, c_1 = b$$

$$a_2 = -a+c, b_2 = \frac{a_2^2-n}{c_2} = c, c_2 = b-2a+c.$$

And $$t(\alpha) = \frac{(-a-c)+\sqrt{n}}{b+2a+c} = \frac{a_3+\sqrt{n}}{c_3} \text{ (Say)}$$

$$t^2(\alpha) = \frac{(-a-b)+\sqrt{n}}{b} = \frac{a_4+\sqrt{n}}{c_4} \text{ (Say)}$$

We have $$a_3 = -a-c, b_3 = \frac{a_3^2-n}{c_3} = c, c_3 = b+2a+c$$

$$a_4 = -a-b, b_4 = \frac{a_4^2-n}{c_4} = b+2a+c, c_4 = b$$

$$b_1c_1 = b(b-2a+c) = b^2 - 2ab+bc > bc \quad \because a<0$$

$$b_2c_2 = c(b-2a+c) = bc-2ac+c^2 > bc \quad \because a<0$$

Thus $b_1c_1 > bc$, $b_2c_2 > bc$

$$a_1^2-n > a^2-n, \; a_2^2-n > a^2-n$$

$$a_1^2 > a^2, \; a_2^2 > a^2$$

Thus $|a_1| > |a|$ and $|a_2| > |a|$

$$\|y(\alpha)\| > \|\alpha\| \text{ and } \|y^2(\alpha)\| > \|\alpha\|$$

(ii) Since $\alpha = \dfrac{a+\sqrt{n}}{c}$ is totally negative So, by theorem 3.1

either $t(\alpha)$ is totally positive and $|a|<b$, $|a|>c$

or $t^2(\alpha)$ is totally positive and $|a|>b$, $|a|<c$

If $t(\alpha)$ is totally positive then $|a|>c$ implies $-a>c$ since $a<0$ and $-a>c$ imply that $-a>0$, $-a>c$ or $-2a>c$.

So $-2ac>c^2$, $-2ac-c^2>0$ or $2ac+c^2<0$

We have $b_3c_3 = c(b+2a+c) = bc+2ac+c^2<bc$, $b_3c_3<bc$

$$a_3^2-n < a^2-n \text{ or } |a_3| < |a|$$

So that $\|t(\alpha)\| < \|\alpha\|$

Again, if $t^2(\alpha)$ is totally positive then $|a|>b$ imply $-a>b$ since $a<0$, now $a<0$ and $-a>$b implies that $-a>0$, $-a>b$, so $-2a>b$ or $-2ab-b^2>0$, $2ab+b^2<0$

Now $b_4c_4 = b(b+2a+c) = b^2+2ab+bc<bc$

$$b_4c_4<bc$$

$$a_4^2-n<a^2-n$$

$$|a_4|<|a|$$

Thus $\|t^2(\alpha)\| < \|\alpha\|$

Theorem 3.2

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally positive number, then

(i) $\|t(\alpha)\| > \|\alpha\|$, $\|t^2(\alpha)\| > \|\alpha\|$

(ii) $\|y(\alpha)\| < \|\alpha\|$, if $y(\alpha)$ is totally negative.

$\|y^2(\alpha)\| < \|\alpha\|$, if $y^2(\alpha)$ is totally negative.

Proof:

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally positive number. Then by lemma 2.1, we have

either $c>0, a>0, b>0$

or $c<0, a<0, b<0$

Let us take $c>0, a>0, b>0$

(i) Since $\alpha = \dfrac{a+\sqrt{n}}{c}$ is a totally positive number, so by Theorem 3.1 $t(\alpha)$ and $t^2(\alpha)$ are both totally negative.

$b_3c_3 = c(b+2a+c) = bc+2ac+c^2>bc$, because $c>0$, $a>0$, $b>0$

$b_4c_4 = b(b+2a+c) = b^2+2ab+bc>bc$

We have $b_3c_3>bc$ and $b_4c_4>bc$

$a_3^2 - n > a^2 - n$ and $a_4^2 - n > a^2 - n$

$|a_3| > |a|$ and $|a_4| > |a|$

$\|t(\alpha)\| > \|\alpha\|$ and $\|t^2(\alpha)\| > \|\alpha\|$

(ii) Since $\alpha = \frac{a+\sqrt{n}}{c}$ is a totally positive number, So by theorem 3.3[3].

Either $y(\alpha)$ is totally negative and $a>b$, $a<c$

Or $y^2(\alpha)$ is totally negative and $a<b$, $a>c$

If $y(\alpha)$ is totally negative then $a>b$ and $a>0$ imply that $2a>b$, $2ab>b^2$, so $2ab-b^2>0$, $b^2-2ab<0$

$$b_1c_1 = b(b-2a+c) = b^2-2ab+bc<bc, \text{ because } b^2-2ab<0$$

Thus $b_1c_1<bc$.

$$a_1^2-n < a^2-n, \text{ So, } |a_1|<|a| \text{ or } \|y(\alpha)\| < \|\alpha\|$$

Again, if $y^2(\alpha)$ is totally negative then $a>c$, and $a>0$ implies that $2a>c$, $2ac>c^2$.

So, $2ac-c^2>0$, $c^2-2ac<0$

Thus $b_2c_2 = c(b-2a+c) = bc-2ac+c^2<bc$, because $c^2-2ac<0$,

$$b_2c_2<bc$$

$$a_2^2-n<a^2-n, \text{ So, } |a_2|<|a|$$

Hence $\|y^2(\alpha)\| < \|\alpha\|$

Theorem 3.3

Let $\alpha = \frac{a+\sqrt{n}}{c}$ be a totally negative real quadratic irrational number. Then there exists an alternate sequence $\alpha = \alpha_0, \alpha_1, \alpha_2 \ldots\ldots \alpha_m$ of totally negative and totally positive numbers where α_m is an ambiguous number in the coset diagram for the orbit α^H.

Proof

Let $\alpha = \dfrac{a+\sqrt{n}}{c}$ be a totally negative number. Then by lemma 2.2, we have

either $c>0, a<0, b>0$

or $c<0, a>0, b<0$

Let us take $c>0, a<0, b>0$

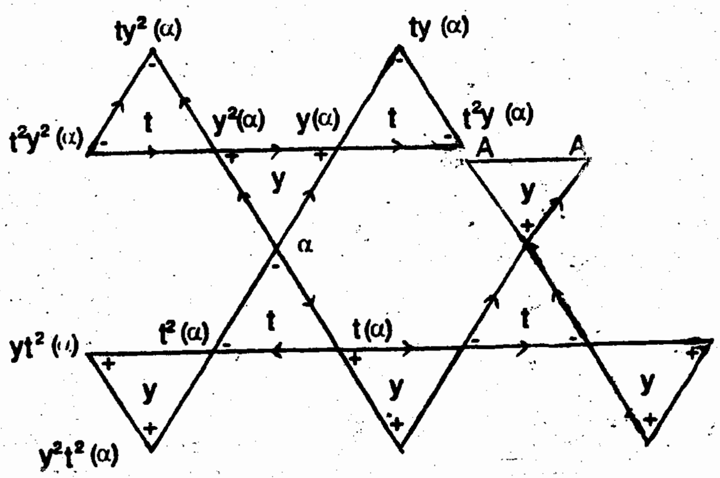

Fig.3.3.1

Since $\alpha = \frac{a+\sqrt{n}}{c}$ is a totally negative number, so by lemma 3.1 $y(\alpha)$ and $y^2(\alpha)$ are both totally positive and by Theorem 3.1 (i)

$$\|y(\alpha)\| > \|\alpha\| \text{ and } \|y^2(\alpha)\| > \|\alpha\| \qquad \text{(A)}$$

Also, since $y(\alpha)$ and $y^2(\alpha)$ are totally positive, therefore by Theorem 2.1. $ty(\alpha)$, $t^2y(\alpha)$, $ty^2(\alpha)$, $t^2y^2(\alpha)$ are totally negative and by Theorem 3.2 (i)

$$\|ty(\alpha)\| > \|y(\alpha)\|, \ \|t^2 y(\alpha)\| > \|y(\alpha)\|$$

and (B)

$$\|ty^2(\alpha)\| > \|y^2(\alpha)\|, \ \|t^2y^2(\alpha)\| > \|y^2(\alpha)\|$$

Combine (A) and (B)

$$\|ty(\alpha)\| > \|y(\alpha)\| > \|\alpha\|$$

$$\|t^2y(\alpha)\| > \|y(\alpha)\| > \|\alpha\|$$

$$\|ty^2(\alpha)\| > \|y^2(\alpha)\| > \|\alpha\|$$

$$\|t^2 y^2(\alpha)\| > \|y^2(\alpha)\| > \|\alpha\|$$

If we take $\|\alpha\| = |a|$, $\|y(\alpha)\| = |a_1|$, $\|ty(\alpha)\| = |a_2|$

Thus we have $|a_2| > |a_1| > |a|$ in each case. so if we continue this process then it is impossible to obtain an ambiguous number, because a become maximum or minimum. On the other had as α is totally negative, So by theorem 2.2

either $t(\alpha)$ is totally positive.

or $t^2(\alpha)$ is totally positive.

Let t(α) be totally positive, then by Theorem 3.1 (ii)

$$\|t(\alpha)\| < \|\alpha\| \tag{C}$$

Since t(α) is totally positive, So by theorem 3.3.

either yt(α) is totally negative.

or $y^2t(\alpha)$ is totally negative.

and be Theorem 3.2 (ii)

$$\|yt(\alpha)\| < \|t(\alpha)\|$$

$$\|y^2t(\alpha)\| < \|t(\alpha)\| \tag{D}$$

Combine (C) and (D)

$$\|yt(\alpha)\| < \|t(\alpha)\| < \|\alpha\|$$

$$\|y^2t(\alpha)\| < \|t(\alpha)\| < \|\alpha\|$$

Similarly, if $t^2(\alpha)$ is totally positive, then

$$\|yt^2(\alpha)\| < \|t^2(\alpha)\| < \|\alpha\|$$

$$\|y^2t^2(\alpha)\| < t^2(\alpha)\| < \|\alpha\|$$

If we take $\|\alpha\| = |a|$, $\|t(\alpha)\| = |a_1|$, $\|yt(\alpha)\| = |a_2|$

Then $|a_2| < |a_1| < |a|$ in each case

$$\text{or } \|\alpha_2\| < \|\alpha_1\| < \|\alpha_0\|$$

So, we have an alternate sequence α_0 (totally negative) α_1 (totally positive) and α_2 (totally negative). Now if we start from the vertex α_2 we get the totally positive vertex α_3 and so, if we continue in this way we have an alternate sequence. $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ ------- of totally negative and totally positive numbers, such that $\|\alpha\| > \|\alpha_1\| > \|\alpha_2\| >$ .... and $\|\alpha_0\|$, $\|\alpha_1\|$, $\|\alpha_2\|$ .... is a decreasing sequence of positive integers which must terminate. After a finite number of steps we have a number α_m such that

$$\|\alpha_m\| < (\sqrt{n}) \quad . \text{ Where } \alpha_m = \frac{\acute{a}+\sqrt{n}}{\acute{c}} \; \|\alpha_m\| = |a| < \sqrt{n}$$

So $a^{2'} - n < 0$ or $(\acute{a}+\sqrt{n})(\acute{a}-\sqrt{n}) < 0$

Hence $\acute{a}+\sqrt{n}$ and $\acute{a}+\sqrt{n}$ are of opposite sings and so, α_m is an ambiguous number.

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THE SEQUENCE SPACE C(s,p) AND RELATED MATRIX TRANSFORMATIONS

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ABSTRACT: In this paper, the main purpose is to define and to investigate the sequence space $C(s,p)$ and to determine the matrices of classes $(C(s,p),\ell_\infty)$ and $(C(s,p),c)$ where ℓ_∞ and c are respectively the space of bounded and convergent complex sequences.

Key Words and Phrases: Kothe-Toeplitz dual, $C^*(s,p)$, Matrix transformation.

1. INTRODUCTION

Let V and W be any two subsets of the space of all sequences of complex numbers and let $A = (a_{n,k})$ be an infinite matrix of complex numbers. We say that the matrix A defines a matrix transformations from V into W and denote it by $A \in (V, W)$ if for every sequence $x = (x_k) \in V$ the sequence $A(x) = (A_n(x))$ is in W, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$$

The main purpose is to define and to investigate the sequence space $C(s,p)$ and to determine the matrices of classes $(C(s,p),\ell_\infty)$ and $(C(s,p),c)$, where ℓ_∞ and c are respectively the spaces of bounded and convergent complex sequences and for $p = (p_r)$ with $\inf p_r > 0$, the space $C(s,p)$ is defined by

$$C(s,p) = \{ x = (x_k) : \sum_{r=0}^{\infty} \left(2^{-r} \Sigma_r k^{-s} |x_k|\right)^{p_r} < \infty,\ s \geq 0 \}$$

where Σ_r denotes a sum over the ranges $2^r \leq k < 2^{r+1}$

Obviously, the sequence space

$$Ces(p) = \{ x = (x_k): \sum_{r=o}^{\infty} \left(2^{-r}\Sigma_r|x_k|\right)^{p_r} < \infty \},$$

where $\inf p_r > 0$, which has been investigated by K.P. Lim ([3],[4]) is a special case of $C(s,p)$ which corresponds to $s = 0$. And $C(s,p) \supseteq Ces(p)$.

Throughout the paper the following well known inequality (See [1]) will be used frequently.

For any $C>0$ and any two complex numbers a, b:

$$|ab| \leq C(|a|^q C^{-q} + |b|^p \tag{1}$$

where $p>1$ and $p^{-1}+q^{-1} = 1$.

To begin with we can show that the space $C(s,p)$ is para-normed by

$$g(x) = \left(\sum_{r=o}^{\infty} \left(2^{-r}\Sigma_r k^{-s}| x_k |\right)^{p_r}\right)^{1/M} \tag{2}$$

if $H = \sup p_r < \infty$ and $M = \max(1,H)$. Clear $g(\theta) = 0$ and $g(x) = g(-x)$, where $\theta=(0,0......)$. Take any x, $y \in C(s,p)$. Since $p_r \leq M$ and $M \geq 1$, using Minkowski's inequality we obtain that g is subaddivite. Finally we have to check the continuity of scalar multiplication. For any complex θ, we have

$$g(\lambda x) = \left(\sum_{r=o}^{\infty} \left(2^{-r}\Sigma_r k^{-s} |\cdot\lambda x_k | \right)^{p_r}\right)^{1/M}$$

$$\leq \sup_r \ |\lambda|^{p_r M} \ g(x)$$

Now let $\lambda \to 0$ for any fixed x with $g(x) \neq 0$. Since

$$0 \sum_{r=o}^{\infty} \left(2^{-r} \Sigma_r k^{-s} \mid x_k \mid\right)^{p_r} < \infty$$

there exists an integer $m_o > 1$, for $|\lambda| < 1$ and $\epsilon > 0$, such that

$$\sum_{r=o}^{\infty} \left(2^{-r} \Sigma_r k^{-s} \mid \lambda x_k \mid\right)^{p_r} < \epsilon \tag{3}$$

Taking $|\lambda|$ sufficiently small such that $|\lambda|^{p_r} < \epsilon / g(x)$ for $r = 0,1,\ldots, m_o - 1$, we then have

$$\sum_{r=o}^{m_o - 1} \left(2^{-r} \Sigma_r k^{-s} \mid \lambda x_k \mid\right)^{p_r} < \epsilon \tag{4}$$

(3) and (4) together implies that $g(\lambda x) \to 0$ as $\lambda \to 0$.

It is quite routine to show that $(C(s,p),d)$ is a metric space with the metric $d(x,y) = g(x-y)$ provided that $x, y \in C(s,p)$, where g is defined by (2). And using a similar method to that in [2] one can show that $C(s,p)$ is complete under the metric mentioned above.

2. β AND CONTINUOUS DUALS

If (X,g) is paranormed space, with paranorm g, then we denote by X^* the continuous dual of X. If E is a set of complex sequences $x = (x_k)$ then E^β will denote generalized Kothe-Toeplitz dual of E,

$$E^\beta = \{a \colon \Sigma\, a_k x_k \text{ converges, for all } x \in E\}.$$

Theorem 1

If $1 < p_r \leq \sup p_r < \infty$ and $p_r^{-1} + q_r^{-1} = 1$, $r = 0,1,2,\ldots$ then

$$C^{\beta}(s,p) = \{ a = (a_k) : \sum_{r=o}^{\infty} (2^r \max_r k^s \mid a_k \mid)^{q_r} N^{-qr} < \infty, \; s \geq 0 \text{ for some)}$$

integer $N > 1\}$

Proof

Let $1 < p_r \leq \sup p_r < \infty$ and $p_r^{-1} + q_r^{-1} = 1$, for $r = 0,1,2,\ldots$. Then take

$$\mu(s,p) = \{ a : \sum_{r=o}^{\infty} (2^r \max_r k^s \mid a_k \mid)^{q_r} N^{-qr} < \infty, \; s \geq 0 \text{ for some integer}$$

$N > 1\}$.

We now want to show that $C^{\beta}(s,p) = \mu(s,q)$. Let $x \in C(s,p)$ $a \in \mu(s,q)$. Therefore using inequality (1), we get

$$\sum_{k=1}^{\infty} \mid a_k x_k \mid = \sum_{r=o}^{\infty} \Sigma_r \mid a_k x_k \mid$$

$$= \sum_{r=o}^{\infty} \Sigma_r k^s \mid a_k \mid k^{-s} \mid x_k \mid$$

$$\leq \sum_{r=o}^{\infty} 2^r \max_r k^s \mid a_k \mid 2^{-r} \Sigma_r k^{-s} \mid x_k \mid$$

$$\leq N \left(\sum_{r=o}^{\infty} (2^r \max_r k^s \mid a_k \mid)^{q_r} N^{-q_r} + \sum_{r=o}^{\infty} (2^{-r} \Sigma_r k^{-s} \mid x_k \mid)^{p_r} \right)$$

So $\Sigma \mid a_k x_k \mid$ is convergent, which implies that $\Sigma\, a_k x_k$ is convergent i.e. a $\in C^{\beta}(s,p)$. In other words $C^{\beta}(s,p) \supseteq \mu(s,q)$. Conversely, let us suppose that $\Sigma\, a_k x_k$ is convergent and $x \in C(s,p)$, but $a \notin \mu(s,q)$. Then we write that

$$\sum_{r=0}^{\infty} (2^r \max_r k^s \mid a_k \mid)^{q_r} N^{-q_r} = \infty \quad \text{for each } s \geq 0$$

and for every $N>1$. So we can find a sequence $0 = n(0) < n(1) < n(2) < \ldots.$ such that for $\nu = 0,1,2,\ldots..$

$$M_\nu = \sum_{r=n(\nu)}^{n(\nu+1)} \left(2^r \max_r k^s \mid a_k \mid\right)^{q_r} (\nu+2)^{-q_r/p_r} > 1$$

Now define a sequence $x \in (x_k)$ as follows: for each ν,

$$x_{N(r)} = 2^{rq_r}\left(N(r)^s \mid a_{N(r)} \mid\right) q_r^{-1} N\cdot\cdot)^s sgn\ a_{N(r)} M_\nu^{-1} (\nu+2)^{-q_r}$$

for $n(\nu) \leq r \leq n(\nu+1)-1$, and $x_k = 0$ for $k \neq N(r)$ where $N(r)$ is such that $N(r)^s a_{N(r)} = \max_r k^s \mid a_k \mid$, the maximum is taken for k in $[2^r, 2^{r+1})$. Therefore.

$$\sum_{r=n(\nu)}^{n(\nu+1)-1} \left(2^r N(r)^s \mid a_{N(r)} \mid\right)^{q_r} (\nu+2)^{-q_r} M_\nu^{-1}$$

$$= M_\nu^{-1} (\nu+2)^{-1} \sum_{r=n(\nu)}^{n(\nu+1)-1} \left(2^r N(r)^s \mid a_{N(r)} \mid\right)^{q_r} (\nu+2)^{-q_r/p_r}$$

$$= (\nu+2)^{-1} \ .$$

It follows that $\sum_{k=1}^{\infty} a_k x_k = \sum_{\nu=o}^{\infty} (\nu+2)^{-1}$ diverges. Moreover

$$\sum_{r=n(\nu)}^{n(\nu+1)-1} \left(2^r k^{-s} \mid x_k \mid\right)^{p_r}$$

$$= \sum_{r=n(\nu)}^{n(\nu+1)-1} \left(2^r N(r)^s \mid a_{N(r)} \mid\right)^{(q_r-1)p_r} (\nu+2)^{-q_r p_r} M_\nu^{-p_r}$$

$$\leq (\nu+2)^{-2} M_\nu^{-1} \sum_{r=n(\nu)}^{n(\nu+1)-1} \left(2^r N(r)^s \mid a_{N(r)} \mid\right)^{q_r} (\nu+2)^{-q_r/p_r}$$

$$= (\nu+2)^{-2}$$

Hence,

$\sum_{r=0}^{\infty}\left(2^{-r}\Sigma_r k^{-s} \mid x_k \mid\right)^{p_r} \leq \sum_{\nu=0}^{\infty} (\nu+2)^{-2} < \infty$ i.e. $x \in C(s,p)$. And this

contradicts our assumption. So $a \in \mu(s,q)$ i.e. $\mu(s,q) \supseteq C^{\beta}(s,p)$. Then combining these two results we get

$$C^{\beta}(s,p) = \mu(s,q)$$

Let us now determine the continuous dual of $C(s,p)$ by the following theorem.

Theorem 2

Let $1 < p_r \leq \sup_r p_r < \infty$. Then $C^{*}(s,p)$ is isomorphic to $\mu(s,q)$ which is defined by (5).

Proof

It is easy to check that each $x \in C(s,p)$ can be written as $x = \sum_{k=1}^{\infty} x_k e_k$

where $e_k = (0,0,\ldots.1,0,0,\ldots..)$ where 1 appears at k-th place. Then for any $f \in C^{*}(s,p)$

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k a_k,$$

where $f(e_k) = a_k$.

By theorem 1, the convergence of $\sum_{k=1}^{\infty} a_k x_k$ for every x in $C(s,p)$ implies that $a \in \mu(s,q)$.

If $x \in C(s,p)$ and if we take $a \in \mu(s,q)$ taken by Theorem 1 $\sum_{k=1}^{\infty} a_k x_k$ converges and clearly defines a linear functional on $C(s,p)$. Using the same kind of argument to that in Theorem 1, it is easy to check that

$$\sum_{k=1}^{\infty} |a_k x_k| \le N\left(\sum_{r=0}^{\infty}\left(2^r \max_r k^s |a_k|\right)^{q_r} N^{-q_r} + 1\right) g(x)$$

whenever $g(x) \le 1$.

Hence $\sum_{k=1}^{\infty} a_k x_k$ define an element of $C^*(s,p)$. Obviously, the map $T: C^*(s,p) \to \mu(s,q)$ given by $T(f) = (a_1, a_2, \ldots..)$ is linear and bijective. Hence $C^*(s,p)$ is isomorphic to $\mu(s,q)$.

3. MATRIX TRANSFORMATIONS

Theorem 3

Let $1 < p_r \le \sup_r p_r < \infty$. Then $A \in (C(s,p), \ell_\infty)$ iff there exists an integer $N > 1$ such that $U(N) < \infty$, where

$$U(N) = \sup_n \sum_{r=0}^{\infty}\left(2^r A_r(n,s)\right)^{q_r} N^{-q_r} < \infty$$

and $p_r^{-1} + q_r^{-1} = 1$, $r = 0,1,2,\ldots..$, where

$A_r(n,s) = \max_r k^s |a_{n,k}|$, where for each n the maximum is taken for k in $[2^r, 2^{r+1}]$.

Proof

Sufficiency. By inequality (1), we have

$$\sum_{k=1}^{\infty} | a_{n,k} x_k | = \sum_{r=0}^{\infty} \Sigma_r | a_{n,k} x_k |$$

$$= \sum_{r=0}^{\infty} \Sigma_r k^s | a_{n,k} | k^{-s} | x_k |$$

$$\leq N \sum_{r=0}^{\infty} \left(2^r A_r(n,s)\right)^{q_r} N^{-q_r} + \sum_{r=0}^{\infty} \left(2^{-r} \Sigma_r k^{-s} | x_k |\right)^{p_r} < \infty$$

Therefore $A \in (C(s,p), \ell_\infty)$.

Necessity. Suppose that $A \in (C(s,p), \ell_\infty)$ but that

$$\sup_n \sum_{r=0}^{\infty} \left(2^r A_r(n,s)\right)^{q_r} N^{-q_r} = \infty \quad \text{for every integer } N > 1.$$

Then $\sum_{k=1}^{\infty} a_{n,k} x_k$ convergences for every n and for every $x \in C(s,p)$ whence $(a_{n,k}), k = 1,2,\ldots.. \in C^{\beta}(s,p)$ for every n.

By Theorem 1, it follows that each A_n defined by $A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$ is an element of $C^*(s,p)$. Since $C(s,p)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $C(s,p)$, by the uniform boundedness principle there exists a number L independent of n and x and a number $\delta < 1$ such that

$$|A_n(x)| \leq L$$

for every $x \in S(\theta,\delta)$ and every n, where $S(\theta,\delta)$ is the closed sphere in $C(s,p)$ with centre the origin and radius δ. Now choose an integer $Q > 1$ such that $Q\delta^M > L$

$$\sup_n \sum_{r=0}^{\infty} \left(2^r A_r(n,s)\right)^{q_r} Q^{-q_r} = \infty,$$

there exists an integer $m_o > 1$ such that

$$R = \sum_{r=o}^{m_o} \left(2^r A_r(n,s)\right)^{q_r} Q^{-q_r} > 1. \tag{7}$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0 \text{ if } k \geq 2^{m_o+1}$$

and $\quad x_{N(r)} = 2^{rq_r} \delta^{M/p_r} \left| N(r)^s \, a_{n,N(r)} \right|^{q_r - 1} \left(QN(r)^s sgn \; a_{n,N(r)}\right) 0^{-q_r/p_r}$

$x_k = 0$ ($k \neq N(r)$, for $0 \leq r \leq m_o$) where $N(r)$ is the smallest integer such that $N(r)^s \; |a_{n,N(r)}| = \max_r k^s \; |a_{n,k}|$. Then one can easily show that $g(x) \leq \delta$ but $|A_n(x)|^r > L$, which contradicts to (6). This completes the proof of the Theorem.

Theorem 4

Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (C(s,p),c)$ iff

(4.1) $\quad a_{n,k} \rightarrow \alpha_k$ ($n \rightarrow \infty$, k fixed)

(4.2) $\quad$ there exists an integer $N > 1$ such that $U(N) < \infty$, where $U(N)$ is defined as in Theorem 3.

Proof

Suppose $A \in (C(s,p),c)$. Then $A_n(x)$ exists for each $n \geq 1$ and $\lim_n A_n(x)$ exists, for every $x \in C(s,p)$. Therefore, by a similar argument to that in Theorem 3, we have condition (4.2). The condition (4.1) is obtained by taking $x = e_k \in C(s,p)$, where $e_k = (0,0,\ldots,0,1,0,0,\ldots)$ where 1 appears at k-th place.

For the sufficiency the condition of the Theorem imply that

$$\sum_{r=0}^{\infty}\left(2^r \max_r k^s \mid \propto_k \mid\right)^{q_r} N^{-q_r} \le U(N) < \infty \tag{8}$$

convergent for each $x \in C(s,p)$. Moreover, for each $x \in C(s,p)$, there exists an integer $m_o \geq 1$, such that

$$g_{m_o}(x) = \sum_{r=m_o}^{\infty}\left(2^r \Sigma_r\, k^{-s} \mid x_k \mid\right)^{p_r} < 1$$

If $g_{m_o}(x) \neq 0$ then by then proof of Theorem 2 and by inequality (1) we have

$$\sum_{k=2}^{\substack{\infty \\ m_o}} \mid a_{n,k} - \alpha_k \mid \mid \alpha_k \mid \le N\left(\sum_{r=m_o}^{\infty} 2^r B_r(n,s)^{q_r}\, N^{-q_r} + 1\right) g_{m_o}(x)^{1/M} \tag{9}$$

where $B_r(n,s) = \max_r k^s \mid a_{n,k,} - \propto_k \mid$.

Clearly (9) holds if $g_{m_o}(x) = 0$. Since

$$\sum_{r=m_o}^{\infty}\left(2^k B_r(n,s)\right)^{q_r} N^{-q_r} \le 2U(N) < \infty$$

from (9), it follows immediately that $\lim_{n\to\infty} \Sigma\, a_{n,k} x_k \quad \Sigma \propto_k x_k$. This shows that $A \in (C(s,p),c)$ which proves the Theorem.

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D^n-TORSION AND D^n-COTORSION MODULES

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ABSTRACT: For a Ring R which is embedded in a ring D relationships of D and D/R with R-modules are discussed. Evolution of the notions of D-torsion and D-cotorsion modules over general rings from the notions of torsion and cotorsion modules over commutative rings is reviewed and finally D-torsion and D-cotorsion modules are generalized as D-torsion modules and D-cotorsion modules. Several existing results for D-torsion and D-cotorsion modules are carried over to D-torsion modules and D-cotorsion modules.

1. TERMINOLOGY AND NOTATION

We will consider all rings to be associative with nonzero identity I and all modules unitary. If R is a ring then we shall suppose that T is a subring if a ring D. Ab R-mod and mod-R will mean the category of abelian groups, category of left R-modules and category of right R-modules respectively. (A.B) (respectively) [A.B]) will denote $Hom_Z(A.B)$ respectively $Hom_R(A.B)$ where A.B $\in$ Ab (respectively) A.B $\in$ R-mod or mod-R). We will write Hom, $\otimes$, Ext. Tor, Ext^n and Tor_n for Hom_R, $\otimes_R$, Ext_R^1, Tor_1^R, Ext_R^n and Tor_n^R respectively. Unless stated otherwise all modules will be considered to be in R-mod. R^* will mean R-{O} and K = D/R. We always have the exact sequence in R-mod (respectively) in mod-R).

$$R \rightarrowtail D \twoheadrightarrow K$$

where $\rightarrowtail$ and $\twoheadrightarrow$ represent the monomorphism arrow and the epimorphism arrow respectively.

Integral domain will mean a commutative ring with no nonzero zero division. An entire ring R (i.e. r, s $\in$ R, rs = 0 if and only if r = 0 or s = 0) is said to be a left Ore domain if Rr $\cap$ Rs $\neq$ 0 for all r, s $\in$ R^*.

2. INTRODUCTION

Much variation exists in literature regarding the definition of torsion, torsion free and divisible modules however, we adopt these concepts as usual that is x $\in$ A in R-mod is said to be a torsion (respectively torsion free respectively divisible) element if rx = 0 for some r $\in$ R^* (respectively rx$\neq$0 for any r $\in$ R^*, respectively there exists, for every r $\in$ R^* an element y $\in$ A such that x = ry). A is said to be a torsion (respectively torsion free, respectively divisible) module if each element of A is torsion (respectively if every nonzero element of A is torsion free, respectively if every element of A is divisible.

Relationships between D the embedding ring of R and modules in R-mod or mod-R exhibit interesting properties. These relationships characterize various classes of modules as well.

If I is an integral domain and $\bar{I}$ the classical field of quotients of I. Then a modules A is torsion if and only if [A,$\bar{I}$] if and only if $\bar{I} \otimes A$ = 0. In the general case the assertion "D $\otimes$ A = 0 if and only if [A,D] = 0" is not necessarily true. However, in Ansari [2] it is proved that in case of a left ore domain, replacing D by the minimal left skew filed Q of R the assertion holds as good as in the commutative case. if R is a ring with R its maximal left quotient ring then Diana Yun-Dee Wie [15] has

defined a modules A to be torsion modules if $[A,\bar{R}]=0$ and a modules

B to be a bad modules if $\bar{R} \otimes B=0$.

If R is any ring and Y $\in$ mod-R then Y is said to be pseudoflat if and only if for any L $\in$ R-mod satisfying y $\otimes$ L$\neq$0 and any monomorphism α;L $\to$ N, $1_y \otimes \alpha$:y$\otimes$L $\to$ y$\otimes$N is nonzero. If Y $\in$ mod-R is pseudoflat then a modules M is said to be torsion in the sense of Wakamatsu [14] if Y $\otimes$ M = 0.

Considering R to be any ring and d any other ring in which R can be embedded we define a modules A to be D torsion (respectively D-reduced, respectively D-cotorsion if $D\otimes A = 0$ (respectively $[D,A] = 0$, respectively $Ext^n(D,A) = 0$ for $n = 0,1$ Ansari [2]. D-torsion modules generalize the concept of a torsion modules and a bad modules whereas, D-reduced and D-cotorsion modules generalize similar concepts appearing in Matlis [10] and Henderson and Orzech [9] for the commutative case. For the noncommutative case reduced modules over left Ore domains are treated in Qureshi [12] and Ansari [6] and over general rings in Ansari [5].

Relationships of $K = D/R$ with modules in R-mod or mod-R are also of considerable importance. A modules A is said to be D-injective modules if $Ext(K,A) = O$ Ansari [1]. This is a generalization of a similar concept appearing in Henderson and Orzech [9] for commutative rings.

If I is an integral domain then $[A,\bar{I}] = 0$ where A is a torsion modules and since $\bar{I}$ is injective, also we have Ext $[A,\bar{I}] = 0$. Thus the concept of a cotorsion modules is as sort of dual to concept of a torsion modules.

In the noncommutative case if R is a left Ore domain then this sort of duality remains restored Ansari [2,6]. D-torsion modules and D-cotorsion modules are discussed in detail in Ansari [2,5]. We will generalize the concepts of D-torsion and D-cotorsion modules to D^n-torsion modules and D^n-cotorsion modules and carry over certain results appearing in Ansari [2,5] regarding D-torsion and D-cotorsion modules to D-torsion modules and D-cotorsion modules.

3. RELATIONSHIPS OF D AND K WITH R-MODULES

In this section we provide some useful results concerning torsion, torsion free divisible modules for the noncommutative case. Relationships of D and K with R-modules are also investigated.

Proofs of the following results are similar to the parallel results for integral domain appearing in Cartain and Eilenberg [7, Chapter VII].

Proposition 3.1

(1) [A,C] = 0 whenever A and C in R-mod are torsion and torsion free respectively.

(2) $B \otimes A = 0$ for all torsion (respectively divisible) modules A in R-mod and divisible (respectively torsion modules B in mod-R.

Obviously we have

Corollary 3.2

If D is divisible in mod-R, then for a torsion modules A.$D \otimes A = 0$.

For a modules A the map $\lambda_r: A \rightarrow A$ defined by $\lambda_r(x) = rx$ for all $r \in R$ is said to be a left multiplication by r. Obviously $\lambda_r \in (A,A)$. Thus

Proposition 3.3

(1) A modules A is torsion free if any only if λ_r is injection for all $r \in R^*$.

(2) A modules A is divisible if and only if λ_r is surjection for all $r \in R^*$.

Corollary 3.4

D is torsion free and divisible if and only if each left multiplication of D is an abelian group automorphism.

Proposition 3.5

For any other ring S, A $\in$ S-mod-R, B $\in$ S-mod we have:

(1) A divisible in mod-R implies that $Hom_S(A,B)$ is torsion free.

(2) If $C \in$ S-mod, $\mu{:}B \to C$ an epimorphism, A torsion free and divisible in mod-R then $Hom_S(A,B)$ and $Hom_S(A.C)/1\ m\mu^*$ are torsion free and divisible where $\mu^* = [1_A,\mu]$: $Hom_S(A,B) \to Hom_S(A,C)$ is defined by (a) $(\mu^*(f)) = ((a)f)\mu$ where $a \in A$.

Proof

(1) Same as Cartan and Eilenberg [7, Chapter VII, Corollary 1.5].

(2) If A is also torsion free in mod-R then for $s \in S$, $a \in A$ and $r \in R^*$ we have

$$sa = (sa)'r \quad (sa, (sa)' \in A \text{ in mod-R}$$

and also

$$\begin{aligned} sa &= s(a'r) \quad (a,a' \in A \text{ in mod-R}) \\ &= (sa')r \end{aligned}$$

i.e. $(sa)' - (sa')r = 0$ which implies that $(sa)' - (sa') = 0$, since A in mod-R is torsion free so that $(sa)' = sa'$.

Furthermore, for a_1, $a_2 \in A$ we have

$$(a_1+a_2) = (a_1+a_2)'r \qquad (a_1+a_2, (a_1+a_2)' \in A \text{ in mod-R})$$

and also

$$\begin{aligned} (a_1+a_2) &= a_1'\,r+a_2'r \qquad (a_1, a_2, a_1'\ a_2' \in \text{in mod-R}) \\ &= (a_1'+a_2')r \end{aligned}$$

i.e. $(a_1+a_2)'r - (a_1'+a_2')r = 0$ which implies that $((a_1+a_2)' - (a_1'+a_2'))r = 0$ so that $(a_1+a_2)' - (a_1'+a_2') = 0$, since A in mod-R in torsion free. Hence $(a_1+a_2)' = a_1'+a_2'$.

Define $g_r:A \to B$ by $(a)g_r = (a')f$ for all $a \in A$ in obvious notations. Then g_r is well defined as well as $g_r \in Hom_S(A,B)$.

Also for $r \in R^*$, $ar = (ar)'r$ implies that $a = (ar)'$ which implies that $(a)(rg_r) = (ar)g_r = (ar)'f = (a)f$ which implies that $rg_r = f$. Hence Hom_S (A.B) is divisible in mod-R.

In a similar way $Hom_S(A,C)$ is also divisible in R-mod and hence also $Hom_S(A,C)/Im\mu^*$.

Consider now $\phi \in Hom_S(A,C)$ and $r \in R^*$. Then $r\bar{\phi} = \bar{0}$ implies that $r\phi = \mu^* f$ for some $f \in Hom_S(A,B)$ which implies that $r\phi = [1_A,\mu](f) = f\mu$.

Further $a \in A$ implies that

$$
\begin{aligned}
(a)\ \phi &= (a'r)\ \phi \qquad (a = a'r \text{ as in (1) above}) \\
&= (a')(r\phi) \\
&= (a')f\mu
\end{aligned}
$$

Thus $(a)\ \phi = (a')\ f\mu$

Now since $Hom_S(A.B)$ is divisible in R-mod, $f = rg$, $f,g \in Hom_S(A,B)$. We have

$$
\begin{aligned}
(a)\ \phi &= (a')\ f\mu \\
&= (a')(rg)\mu \\
&= (a'r)g\mu \\
&= (a)g\mu
\end{aligned}
$$

so that $\phi = g\mu$ which implies that $\phi = [1_A,\mu]g = \mu^* g \in 1m\ \mu^*$. This implies $\bar{\phi} = \bar{0}$. Thus $Hom_S(A,C)/Im\ \mu$ is torsion free.

Recall that we have assumed R to be a subring of a ring D. Now $D \in$ R-mod-R so that for D divisible in mod-R we have

Corollary 3.6

(1) For a module A, [D,A] is torsion free.

(2) If D is also torsion free in mod-R then for A.B $\in$ R-mod and an epimorphism μ:A $\rightarrow$ B, [D,A] and [D,B]/Im μ^* are torsion free and divisible where $\mu^* = [1_D,\mu]$:[D,A] $\rightarrow$ [D.B].

Further we have

Proposition 3.7

D in mod-R is divisible if any and only if every element of R^* is a left unit in D and D is divisible as two sided module if and only if each element of R^* is a unit in D.

3.1, 3.3 and 3.7 remain valid for left replaced by right with obvious changes.

In addition if D is divisible both in mod-R and R-mod then we have

Proposition 3.8

(1) For every B $\in$ Ab, (D,B) is a torsion free and divisible module.

(2) For every module B, [D,B] is a torsion free and divisible submodule of (D.B).

Proof

(1) D $\in$ mod-R implies that (D,B) $\in$ R-mod. Let f $\in$ (D,B) and r $\in R^*$ such that rf = 0. Then (x)(rf) = 0 for all x $\in$ D. D divisible (in mod-R) implies that every element of R^* is a left unit in D so that (xr')(rf) = 0 for all x $\in$ D where r' is the left inverse of r in D, which implies that x(r'rf) = 0 for all x $\in$ D, that is rf = 0 for all r $\in$ D which implies that Df = 0. Hence f = 0 so that (D.B) is torsion free.

Now for $r \in R^*$ for $f \in (D,B)$ define $g: D \rightarrow B$ by $(x)g = (xr")f$ for all $x \in D$ where r" is the right inverse of r. Then $g \in (D,B)$ and

$$\begin{aligned} x(rg) &= (xr)g \\ &= ((xr)r")f \\ &= (x(rr"))f \\ &= (x)f \end{aligned}$$

for all $x \in D$ which implies that $f = rg$ that is f is divisible do that (d,B) is divisible.

(2) Consider a module B and $f \in [D,B]$. Then for r, $s \in R$ and $x \in D$ we have:

$$\begin{aligned} (sx)(rf) &= ((sx)r)\, f \\ &= (s(rx))\, f \\ &= (s((rx)\, f) \\ &= s((x))(rf)) \end{aligned}$$

which implies that $rf \in [D,B]$ so that [D,B] is a submodule of [D,B] which is obviously torsion free.

Defining g as in (1), for $r \in R^*$, $s \in R$ and $x \in D$ we have:

$$\begin{aligned} (sx)g &= (sxr")\, f \\ &= (s((rx")f) \\ &= s(xg) \end{aligned}$$

so that $g \in [D,B]$ and $f = rg$ that is f is divisible. Hence [D,B] is divisible.

Recalling that $K = D/R$ we now investigate the relationship of K with R-modules. In this regard we first mention the following:

Proposition 3.9

If D is divisible both in R-mod and mod-R then for B ∈ Ab (K,B) is a torsion free module.

Proof

Consider the exact sequence

$$R \to D \to K$$

which gives the exact sequence

$$(K,B) \to (D,B) \to (R,B)$$

so that (K,B) is a submodule of (D,B) hence torsion free.

By the above proof it is also evident that if D is such that (D,B) is torsion free then (K,B) is torsion free.

Now we mention a result of Ansari [3] as follows:

Proposition 3.10

For a module M, Tor (K,M) = 0 implies that the sequence

$$M \to D \otimes M \to D \otimes M$$

is exact.

A module A is said to be an e.f. module if it can be embedded in a flat module Ansari [4]. The following result concerning e.f. modules is also reproduced from Ansari [4].

Proposition 3.11

If D is flat in mod-R then $Tor_n(K,A) = 0$ for each module A and all $n \geq 2$ and Tor(K,A) = 0 in case A is an e.f. module.

Some more results regarding the relationship of D and K with R modules the behaviour of R and K in various situations and exactness of certain sequences involving D and K can be found in Ansari [1,3,4]. The properties of D and K in case of a left Ore domain with D replaced by Q the minimal left skew field of R can be found in Gentile [8], Qureshi [11,13] and Ansari [2].

4. D^n-TORSION AND D^n-COTORSION MODULES

In this section we will generalize some concepts and results appearing in Ansari [2,5,6].

A module A is said to be D^n-cotorsion module (respectively D^n-torsion module) if $\mathrm{Ext}^n(D,A) = 0$ (respectively $\mathrm{Tor}_n(D.A) = 0$).

Thus a D^0-cotorsion module is a D-reduced and a module which is both D^0-cotorsion and D^1-cotorsion is a D-cotorsion module (respectively D^0-torsion module is a D-torsion module).

If $\{A_i\}_{i \in I}$ is a family of modules then $\mathrm{Ext}^n(D,\Pi A_i) \cong \Pi \mathrm{Ext}^n(D,A_i)$ and $\mathrm{Tor}_n(D,\coprod A_i) \cong \coprod \mathrm{Tor}_n(D,A_i)$, so we immediately have

Proposition 4.1

(1) Every direct product of D^n-cotorsion modules is a D^n-cotorsion module.

(2) Every direct sum of D^n-torsion modules is a D^n-torsion modules.

Proposition 4.2

Let
$$A \to B \to C$$
be an exact sequence of modules. Then B is D^n-torsion whenever A and C are D^n-torsion.

Proof

We have the exact sequence

$$\ldots \to Tor_n(D.A) \to Tor_n(D.B) \to Tor_n(D.C) \to \ldots$$

$$\ldots \to D \otimes A \to D \otimes B \to D \otimes C \qquad (3.1)$$

from which the result follows.

Proposition 4.3

If $$A \to B \to C$$

is an exact sequence of modules then

(1) If A and C are D^n-cotorsion then D is D^n-cotorsion.

(2) A is D^n-cotorsion whenever B is D^n-cotorsion and C is D^{n-1}-cotorsion.

(3) C is D^n-cotorsion in case B is D^n-cotorsion and A is D^{n+1}-cotorsion.

Proof

We have the exact sequence

$$[D,A] \to [D,B] \to [D,C] \to Ext(D,A)$$

$$\to Ext(D,B) \to \ldots \to Ext^{n-1}(D,C) \to Ext^n(D,A)$$

$$\to Ext^n(D,B) \to Ext^n(D,C) \to Ext^{n+1}(D,A) \to \qquad \ldots..(3.2)$$

from which the results follow.

The following corollaries are immediate consequence of the sequence (3.2):

Corollary 4.4

If C is D^n-cotorsion and D^{n-1}-cotorsion, then

$$Ext^n(D,A) \cong Ext^n(D,B)$$

Corollary 4.5

If B is D^n-cotorsion and D^{n-1}-cotorsion then A is D^n-cotorsion if and only if C is D^{n-1}-cotorsion.

Some results appearing here are generalizations of some results included in the author's doctoral thesis submitted at the University of Karachi and some results appearing here form a part of it. The author wishes to express sincere thanks to Professor M.A. Rauf Qureshi for his kind supervision. N.S.R.D.B., Pakistan and Federal Government Urdu Science College, Karachi, acknowledged for a partial financial support and the grant of study leave respectively.

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RATE OF CONVERGENCE OF MODIFIED BASKAKOV TYPE OPERATORS FOR FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT: In the present paper, we study the rate of convergence of modified Baskakov type operators for functions of bounded variation, using probabilistic approach.

1. INTRODUCTION

The Durrmeyer type integral modification of Baskakov operator was first introduced and studied by Sahai and Prasad [6]. Sinha et al. [7] improved and corrected the results of [6]. Recently Gupta [5] defined another modification of Baskakov operators by taking the weight function of Beta operators on $L_1\ [0,\infty]$ as

$$L_n(f,x) = \sum_{k=0}^{\infty} p_{n,k} \int_0^{\infty} b_{n,k}(t) f(t)dt,\ x \in [0,\infty) \tag{1.1}$$

where $\quad p_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$

and $\quad b_{n,k}(t) = t^k/B(k+1,n)(1+t)^{n+k+1} \qquad (1.2)$

$B(K+1,n)$ being the Beta function given by $k!(n-1)!/(n+k)!$.

Guo ([3], [4]) has established the rate of convergence of the Durrmeyer operators and Meyer Konig and Zeller operators for functions of bounded variation. As the operator $L_n(f,x)$ defined by (1.1) give better approximation than the earlier integral modification of Baskakov operators studied in [1], [6] and [7] etc., this motivated us to study the rate of convergence of $L_n(f,x)$ for functions of bounded variation, using some results of probability theory.

2. AUXILIARY RESULTS

In order to prove the main theorem, we shall need the following results:

Lemma 2.1

[2, p.104 & 110]: (Berry-Esseen Theorem). Let $X_1, X_2,\ldots,X_n$ be n independent and identically distributed (i.i.d) random variables with zero mean and a finite absolute third moment. If $\rho_2 \equiv E(X_1^2)>0$, then

$$\sup_{x\in R} |F_n(x)-\phi(x)| \leq (0.409)\, \ell_{3,n},$$

where F_n is the distribution function of $(n\rho_2)^{-1}(x_1+x_2+\ldots+x_n)$, ϕ is the standard normal distribution function and $\ell_{3,n}$ is the Liapounov ratio given by $\ell_{3,n} = (\rho_3/\rho_2^{3/2})n^{-1/2}$, $\rho_3 = E|X_1|^3$.

Lemma 2.2 [8]

If (ζ_i), $i = 1,2,\ldots$ are independent random variables with same geometric distribution

$$p(\zeta_i = k) = \left(\frac{x}{1+x}\right)^k \frac{1}{1+x},\ x>0;\ i = 1,2,\ldots$$

then $E(\zeta_i) = x$, $\rho_2 = E(\zeta_i - E(\zeta_i))^2 = x(1+x)$

and $n_n = \sum_{i=1}^{n} \zeta_i$ is a random variable with distribution

$$P(\eta_n = k) = \binom{n+k+1}{k} \frac{k}{(1+x)^{n+k}} = p_{n,k}(x) \tag{2.1}$$

Lemma 2.3

If y is a positive valued random variable with a non-degenerate probab..ity distribution, then we have

$$E(y^3) \le [E(y^4)]^{3/4}, \text{ provided } E(y^3),\ E(y^4) < \infty$$

Proof

If f be a convex function, then by Jensen's inequality we have

$$f(E(x)) \le E(f(x)).$$

Now suppose $f(z) = z^{4/3}$, then $f''(z) = \frac{4}{9} z^{-2/3} > 0$ for $z > 0$ and hence

f(z) is convex function and by applying Jensen's inequality, we have

$$(E(z))^{4/3} \le E(z^{4/3}) \text{ i.e. } E(z) \le (E(z^{4/3}))^{3/4}.$$

Letting $z = y^3$, we get the required result.

Lemma 2.4

For every $k \in N$, $x \in (0, \infty)$, we have

$$P_{n,k}^{(x)} \le \frac{1}{2\sqrt{n}} \left[\frac{2\{9x(1+x)+1\}^{3/4} + \{x(1+x)\}^{1/4}}{\{x(1+x)\}^{3/4}} \right] \tag{2.2}$$

Proof

By (2.1) we have

$$p_{n,k}(x) = p(\eta_n = k) = p(k-1 < \eta_n \leq k)$$

$$= P\left(\frac{k-1-nk}{\sqrt{nx(1+x)}} < \frac{\eta_n - nx}{\sqrt{nx(1+x)}} \leq \frac{k-nx}{\sqrt{nx(1+x)}}\right).$$

Using Lemma 2.1 with $a_1 = x$ and $b_1 + \sqrt{x(1+x)}$ (from Lemma 2.2), we have

$$\left|P(P(\eta_n = k)) - \frac{1}{\sqrt{2\pi}} \int_{\frac{k-1-nx}{\sqrt{nx(1+x)}}}^{\frac{k-nx}{\sqrt{nx(1+x)}}} e^{-t^2/2} dt\right| < \frac{2(0.409)\rho_3}{\sqrt{n}\ b_1^3} \tag{2.3}$$

Now, we calculate ρ_3, by an easy computation, we can show that

$$T_o(x) = \sum_{k=0}^{\infty} \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = 1, \quad T_1(x) = \sum_{k=0}^{\infty} k\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = x$$

$$T_2(x) = \sum_{k=0}^{\infty} k^2\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = (1+2x)x \tag{2.4}$$

$$T_3(x) = \sum_{k=0}^{\infty} k^3\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = x[1+6x(1+x)]$$

$$T_4(x) = \sum_{k=0}^{\infty} k^4\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = x[1+14x+36x^2+24x^3]$$

Also if $M_r(x)\left(= \sum_{k=0}^{\infty} (k-x)^r \left(\frac{k}{1+x}\right)^k \frac{1}{1+x}\right)$ stands for the central moment of order r about the mean $T_1(x)$ i.e. x, it is easily checked

$$M_4(x) = \sum_{j=0}^{4}\binom{4}{j}(-1)^j T_{4-j}(x)(T_1(x))^j$$
$$= x(1+10x+18x^2+9x^3) \text{ (using (2.4)).}$$

Next, in view of Lemma 2.3, we have

$$\rho_3 \le (M_4(x))^{3/4} = [x(1+10x+18x^2+9x^3)]^{3/4}$$

So the right hand side of (2.3) is less than $\frac{1}{\sqrt{n}}\left(9+\frac{1}{x(1+x)}\right)^{3/4}$ as $0.818<1$ and

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{k-1-nx}{\sqrt{nx(1+x)}}}^{\frac{k-nx}{\sqrt{nx(1+x)}}} e^{-t^2/2}dt < \frac{1}{\sqrt{2\pi\, nx(1+x)}} < \frac{1}{2\sqrt{nx(1+x)}}$$

Hence,

$$p_{n,k}(x) = P(\eta_n = k) \le \frac{1}{2\sqrt{n}}\left[\frac{2\{9x(1+x)+1\}^{3/4}+\{x(1+x)\}^{1/4}}{\{x(1+x)\}^{3/4}}\right]$$

Lemma 2.5

For $x \in (0,\infty)$, we have

$$(n-1)\int_k^{\infty} p_{n,k}(t)dt = \sum_{j=0}^{k} p_{n-1,j}(x)$$

Using (1.2), we also have $b_{n,k}(x) = np_{n+1,k}(x)$, so we have the following equivalence form

$$\int_x^\infty b_{n,k}(t)dt = \sum_{j=0}^{k} p_{n,j}(x) \tag{2.5}$$

The proof of the above lemma is simple and left for the readers.

Lemma 2.6

Let the m-th order moment for the operator $L_n(f,x)$ be defined by

$$V_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t)(t-x)^m dt$$

then

$$V_{n,0}(x) = 1, V_{n,1}(x) = \frac{1+x}{n-1}, n>1$$

$$V_{n,2}(x) = \frac{2(n+1)x^2+2(n+2)x+2}{(n-1)(n-2)}, n>2.$$

Consequently for each $x \in [0,\infty)$

$$V_{n,m}(x) = 0(n^{-[(m+1)/2]})$$

The proof of the above lemma is simple and easily follows by substituting r = 0 in [5]. In particular, we have $V_{n,2}(x) \sim \frac{2x(1+x)}{n}$

Lemma 2.7

Let $K_n(x,t) = \sum_{k=0}^{\infty} p_{n,k}(x) b_{n,k}(t)$. If n is sufficiently large, then

(i) For $0 \leq Y < x$, we have

$$\int_0^y K_n(x,t)\,dt \leq \frac{2x(1+x)}{n(x-y)^2}.$$

(ii) For $x < z < \infty$, we have

$$\int_z^\infty K_n(x,t)\,dt \leq \frac{2x(1+x)}{n(z-x)^2}.$$

The proof of this lemma follows by using Lemma 2.6 (see e.g. [4]).

3. MAIN THEOREM

In this section, we shall state and prove our theorem:

Theorem 3.1

Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and let $V_a^b(g_x)$ be the total variation of g_x on [a,b]. If $f(t) = 0(t^\alpha)$ for some positive integer $\alpha > 2$ as $t \to \infty$, then for n sufficiently large, we have

$$\left| L_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\} \right| \leq \frac{(6+7x)}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$

$$+ 0(1)\frac{x^{-\alpha-1}(1+x)}{n} + \frac{1}{4\sqrt{n}}|f(x+) - f(x-)|$$

$$\left[\frac{2\{9x(1+x)+1\}^{3/4} + \{x(1+x)\}^{1/4}}{\{x(1+x)\}^{3/4}}\right] \tag{3.1}$$

where $V_a^b(g_x)$ is the total variation of g_x on [a,b] and

$$g_x(t) = \begin{cases} f(t)-f(x+) & x<t<\infty \\ 0 & t=x \\ f(t)-f(x-), & 0\leq t<x \end{cases}$$

Proof

We have

$$\left| L_n(f,x) - \frac{1}{2}\{f(x+)+f(x-)\} \right| \tag{3.2}$$

$$\leq | L_n(g_x,x) | + \frac{1}{2} | f(x+)-f(x-) | \, . \, | L_n(Sign(t-x),x) |$$

In order to obtain the result, we need the estimates for $L_n(g_x,x)$ and L_n (Sign(t−x),x). Now,

$$L_n(Sign(t-x),x) = \int_0^\infty Sign(t-x)K_n(x,t)\,dt$$

$$= \int_x^\infty K_n(x,t)dt - \int_0^x K_n(x,t)\,dt$$

$$= A_n(x) - B_n(x), \quad \text{say.}$$

Using (2.5), we have

$$A_n(x) = \int_x^\infty K_n(x,t)dt = \sum_{k=0}^\infty p_{n,k}(x) \int_x^\infty b_{n,k}(t)dt$$

$$= \sum_{k=0}^\infty \left(p_{n,k}(x) \sum_{j=0}^k p_{n,k}(x) \right)$$

For simplicity, below we shall use the notation p_k instead of $p_{n,k}(x)$.

$$\begin{aligned} A_n(x) &= p_o^2 + p_1(p_o + p_1) + p_2(p_o + p_1 + p_2) + \ldots \\ &= p_o^2 + p_1^2 + p_2^2 + \ldots + p_o(p_1 + p_2 + p_3 + \ldots) \\ &\qquad + p_1(p_2 + p_3 + p_4 + \ldots) + \ldots\ldots \,. \end{aligned}$$

Now, $I \equiv (p_o + p_1 + p_2 + \ldots)(p_o + p_1 + p_2 + \ldots)$.

Hence, $2A_n(x) - I = p_o^2 + p_1^2 + p_2^2 + \ldots$.

By using Lemma 2.4, we get

$$|2A_n(x) - 1| \le \frac{1}{2\sqrt{n}} \left[\frac{2\{9x(1+x)+1\}^{3/4} + \{x(1+x)\}^{1/4}}{\{x(1+x)\}^{3/4}} \right] \sum_{k=0}^{\infty} p_{n,k}(x)$$

$$= \frac{1}{2\sqrt{n}} \left[\frac{2\{9x(1+x)+1\}^{3/4} + \{x(1+x)\}^{1/4}}{\{x(1+x)\}^{3/4}} \right]$$

Now, whereas $A_n(x) + B_n(x) = \int_0^{\infty} K_n(k,t)\,dt = 1$

We have $A_n(x) - B_n(x) = 2A_n(x) - 1$ i.e.

$$|A_n(x) - B_n(x)| \le \frac{1}{2\sqrt{n}} \left[\frac{2\{9x(1+x)+1\}^{3/4} + \{x(1+x)\}^{1/4}}{\{x(1+x)\}^{3/4}} \right]. \qquad (3.3)$$

To estimate $L_n(g_x, x)$, we decompose $[0, \infty)$ interval into three parts as follows:

$$L_n(g_x,x) = \int_0^\infty g_x(t)K_n(x,t)dt$$

$$\int_0^{x-x/\sqrt{n}} g_x(t)K_n(k,t)dt + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)K_n(x,t)dt$$

$$+ \int_{x+x/\sqrt{n}}^{\infty} g_x(t)K_n(x,t)dt$$

$$= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) g_x(t)K_n(x,t)dt$$

$$= R_1 + R_2 + R_3, \quad \text{say.}$$

Suppose $\lambda_n(x,t) = \int_0^t K_n(x,u)du$. We first estimate R_1.

Let $Y = x - x/\sqrt{n}$.

Using partial integration, we get

$$R_1 = \int_o^y g_x(t)K_n(x,t)dt = \int_o^y g_x(t)d_t(\lambda_n(x,t))$$

$$= g_x(Y+)\lambda_n(x,y) - \int_o^y \lambda_n(x,t)d_t(g_x(t))$$

Since $|g_x(y+)| = |g_x(y+)-g_x(x)| \le V_Y^x+(g_x)$, then by using (i) of Lemma 2.7, we have

$$| R_1 | \leq V_{y+}^{x}(g_x)\,\lambda_n(x,y) + \int_{o}^{y} \lambda_n(x,t)\,d_t\left(-V_t^{x}(g_x)\right)$$

$$\leq V_{y+}^{x}(g_x)\frac{2x(1+x)}{n(x-y)^2} + \frac{2x(1+x)}{n}\int_{o}^{y}\frac{1}{(x-t)^2}\,d_t\left(-V_t^{x}(g_x)\right) .$$

Integration by parts leads to the following

$$\int_{o}^{y}\frac{1}{(x-t)^2}\,d_t\left(-V_t^{x}(g_x)\right) = \frac{-V_{y+}^{x}(g_x)}{(x-y)^2} + \frac{V_o^{x}(g_x)}{x^2} + 2\int_{o}^{y}\frac{\hat{V}_t^{x}(g_x)}{(x-t)^3}\,dt \ ,$$

where $\hat{V}_t^x$ is the normalized form of $V_t^2(g_x)$ and $\hat{V}_t^x(g_x) = V_t^x(g_x)$

Consequently, we have

$$| R_1 | \leq V_{y+}^{x}(g_x)\frac{2x(1+x)}{n(x-y)^2} + \frac{2x(1+x)}{n}\left[\frac{-V_{y+}^{x}(g_x)}{(x-y)^2} + \frac{V_o^{x}(g_x)}{x^2} + 2\int_{o}^{y}\frac{V_t^{x}(g_x)}{(x-t)^3}\,d_t\right]$$

$$9 = \frac{2x(1+x)}{n}\left[\frac{V_o^{x}(g_x)}{x^2} + 2\int_{o}^{y}\frac{V_t^{x}(g_x)\,dt}{(x-t)^3}\right]$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\int_{0}^{x-x/\sqrt{n}} V_t^{x}(g_x)\frac{dt}{(x-t)^3} = \frac{1}{2x^2}\int_{1}^{n} V_{x-x/\sqrt{n}}^{x}(g_x)\ dt$$

$$\leq \frac{1}{2x^2}\sum_{k=1}^{n} V_{x-x/\sqrt{n}}^{x}(g_x) .$$

Hence, $\quad |R_1| \leq \frac{2x(1+x)}{n}\left[\frac{V_o^x(g_x)}{x^2} + \frac{1}{x^2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x)\right]$

$$\leq \frac{4(1+x)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x) \quad . \tag{3.4}$$

We now estimate R_2. For $x \in I_2$, we have

$$|g_x(t)| = |g_x(t)-g_x(x)| \leq V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) \quad .$$

Since $\int_a^b d_t\lambda_n(x,t) \leq 1$ for all $(a,b) \subset [0,\infty)$, therefore

$$|R_2| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leq \frac{1}{n}\sum_{k=0}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) \quad . \tag{3.5}$$

Finally, we estimate R_3, setting $z = x+x/\sqrt{n}$;

$$R_3 = \int_z^\infty g_x(t)K_n(x,t)dt = \int_z^\infty g_x(t)\, d_t(\lambda_n(x,t)) \quad .$$

We define $Q_n(x,t)$ on $[0,2x]$ as

$$Q_n(x,t) = 1-\lambda_n(x,t-) \quad , \quad 0 \leq x < 2t$$

$$= 0 \quad , \quad x = 2t$$

Then,

$$R_3 = -\int_z^{2x} g_x(t)d_tQ_n(x,t) - g_x(2x)\int_{2x}^\infty K_n(x,t)dt + \int_{2x}^\infty g_x(t)d_t(\lambda_n(x,t))$$

$$= R_{31} + R_{32} + R_{33} \text{ , say.}$$

Now, integrating partially the first term, we get

$$R_{31} = g_x(z-)Q_n(x,z-) + \int_z^{2x} \hat{Q}_n(x,t)\, d_t g_x(t)$$

where $\hat{Q}_n(x,t)$ is the normalized form of $Q_n(x,t)$.

Since $Q_n(x,z-) = Q_n(x,z)$ and $|g_x(z-)| \le V_x^{z-}(g_x)$, we have

$$|R_{31}| \le V_x^{z-}(g_x)Q_n(x,z) + \int_z^{2x} \hat{Q}_n(x,t)\, d_t V_x^t(g_x)$$

Next, using (i), (ii) of Lemma 2.7 and the fact that $\hat{Q}_n(x,t) \le Q_n(x,t)$ on $[0,2x]$, we have

$$| R_{31} | \le V_x^{z-}(g_x)\frac{2x(1+x)}{n(z-x)^2} + \frac{2x(1+x)}{n}\int_z^{2x-} \frac{1}{(t-x)^2} d_t V_x^t(g_x)$$

$$+ \frac{1}{2}\left[V_{2x-}^{2x}(g_x)\int_{2x}^{\infty} K_n(x,u)\, du\right]$$

$$\le V_x^{z-}(g_x)\frac{2x(1+x)}{n(z-x)^2} + \frac{2x(1+x)}{n}\int_z^{2x-} \frac{1}{(t-x)^2} d_t V_x^t(g_x)$$

$$+ \frac{1}{2}\left[V_{2x-}^{2x}(g_x)\frac{2x(1+x)}{nx^2}\right]$$

$$\le V_x^{z-}(g_x)\frac{2x(1+x)}{n(z-x)^2} + \frac{2x(1+x)}{n}\int_z^{2x} \frac{d_t V_x^t(g_x)}{(t-x)^2}$$

$$= V_x^{z-}(g_x)\frac{2x(1+x)}{n(z-x)^2} + \frac{2x(1+x)}{n}\{\frac{V_x^{2x}(g_x)}{x^2} - \frac{V_x^{z-}(g_x)}{(z-x)^2} + 2\int_z^{2x}\frac{V_x^t(g_x)dt}{(t-x)^3}$$

Thus

$$| R_{31} | \leq \frac{2x(1+x)}{n}\{\frac{V_x^{2x}(g_x)}{x^2} + 2\int_z^{2x}\frac{V_x^t(g_x)}{(t-x)^3}dt\} .$$

Replacing the variable int he last integral by $x+x/\sqrt{n}$, we find that

$$zINT_z^{2x}\frac{V_x^t(g_x)dt}{(t-x)^3} = \frac{1}{2x^2}\int_1^n V_x^{x+x/\sqrt{n}}(g_x)\,dt$$

$$\leq \frac{1}{2x^2}\sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x)$$

Therefore

$$| R_{31} | \leq \frac{2x(1+x)}{nx^2}\{V_x^{2x}(g_x) + \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x)\} \quad (3.7)$$

$$\leq \frac{4(1+x)}{nx}\sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x)$$

Further for evaluating R_{32}, using Lemma 2.7 (ii), we have

$$| R_{32} | \leq g_x(2x)\frac{2x(1+x)}{nx^2} \leq \frac{2(1+x)}{nx}\sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x) \quad (3.8)$$

Finally, using Lemma 2.7(ii) and the assumption that $f(t) = 0(t^{\alpha})$, $\alpha>0$ as $t\to\infty$, we find for n sufficiently large,

$$| R_{33} | \leq M \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} b_{n,k}(t) t^{\alpha} \, dt$$

$$= M \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} \frac{(n+k)!}{k!(n-1)!} \frac{t^{k+\alpha}}{(1+t)^{n+k+1}} \, dt$$

Substituting $k' = k+\alpha$ and $n' = n-\alpha$, we get

$$|R_{33}| \leq M \sum_{k=0}^{\infty} \frac{k'!(n'-1)!]^2 \, x^{-\alpha}}{[(k'-\alpha)!(n'+\alpha-1)!]^2} p_{n',k'}(x) \int_{2x}^{\infty} b_{n',k'}(t) dt$$

$$|R_{33}| \leq M' x^{-\alpha} \frac{2x(1+x)}{(n-\alpha)x^2} . = 0(1) \frac{x^{-\alpha-1}(1+x)}{n} \tag{3.9}$$

Combining (3.6) to (3.9), we have for n sufficiently large

$$|R_3| \leq \frac{6(1+x)}{nx} \sum_{k=1}^{n} v_x^{x+x\sqrt{k}}(g_x) + 0(1) \frac{x^{-\alpha-1}(1+x)}{n} \tag{3.10}$$

Using (3.2) to (3.5) and (3.10), we are lead to (3.1). This completes the proof of the theorem.

Remark

Exactly similarly as in [3] and [4], we can show that the estimate in (3.1) is essentially the best asymptotically.

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A REPRESENTATION OF THE PROLONGATIONS OF A G-STRUCTURE

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ABSTRACT: In this paper, we describe the general group of order two GP_n^2. We prove an arbitrary prolongation of a Lie subgroup of $GL(n,\Re)$ is a direct sum of additive Lie group of the form $\Re^{\tilde{n}}$ and a Lie sub-group of $GL(n,\Re)$. Then we show that an arbitrary prologation of a Lie subalgebra of $Mat(n\times n)$ is a direct sum of an additive Lie subalgebra of the form $\Re^{\tilde{n}}$ and a Lie subalgebra of $Mat(n\times n)$. In conclusion structure group of every k'th order Geometric structure on a given n dimmentinal manifold is isomorphic to an additive standard group $\Re^{\tilde{n}}$, with $0\le \tilde{n}\le k\times \dfrac{n^2(3n-1)}{2}$, and a Lie subgroup of $GL(n,\Re)$.

Key Words: G-structure, Matrix Lie group, Prolongation, Vector bundle. 1991 MSC: 53 C 15.

1. INTRODUCTION

In this paper, all manifolds are finite dimentional paracompact and, all mappings and functions are smooth.

Let M and M' be two manifolds and $\phi{:}M\to M'$ be an immersion, and also assume that $m\in Dom(\phi)$, $\phi(m)=m'$, (x,U) is a chart contains m, and

((x',U') is a chart around of m'. The k'th order jet $j_m^k\phi$ of ϕ at m is denoted by the following coordinates:

$$\left(x^i, x'^j, x'^j_{i_1}, \ldots, x'^j_{i_1,\ldots,i_k}\right)$$

$$x'^j_{i_1,\ldots,i_t} := \frac{\partial^t\left(x' o \phi o x^{-1}\right)}{\partial x_{i_1},\ldots,\partial x_{i_t}}\Big|_{x(m)},$$

where $i, i_1, i_2,\ldots, i_k$ vary in the set $\{1,2,\ldots, dimM'\}$. The $x^j_{i_1,\ldots,ik}$ will not change by any permutations in the lower indices.

Let G be a Lie subgroup of $GL(n,\Re)$ (the general linear group) and G a Lie subalgebra of $Mat(n\times n)$ (Lie algebra of $n\times n$ square matrices with real entries). We denote the *k'th* prolongation of G and G, by $G^{(k)}$ and $G^{(k)}$ respectively.

The group of all invertible *k-jets* with source and target in 0 (the zero of $\Re^n$), is denoted by GP_n^k This is a Lie group which is proved that $GP_n^k \cong [GL(n,\Re)]^k$

By Reinhart's notation, an element of GP_n^k can be represented by an n-tuple $(f_1, f_2,\ldots, f_n)$, where f_i, for $i = 1,2,\ldots,n$, is a polynomial of variables of the form

$$f_i(0) = 0, \ \det\left[\frac{\partial f_i}{\partial x_j}\right] \neq 0$$

In this notation, the operation in GP_n^k is

$$(f_1,\ldots f_n) \bigstar (g_1, \ldots, g_n) = (f_1(g_1,\ldots,g_n),\ldots f_n(g_1,\ldots,g_n)).$$

2. STRUCTURE OF GP_n^2

Proposition 1

Let n be a natural number. Then there exist Lie group isomorphism

$$GP_n^2 \cong \Re^{\frac{n^2(3n-1)}{2}} \oplus GL(n,\Re),$$

where $\Re^{\frac{n^2(3n-1)}{2}}$ *is standard additive Lie group of* $\Re^{\frac{n^2(3n-1)}{2}}$

Proof

Let $M: = \left\{ j_0^2\Phi \in GP_n^2 \mid j_0^1\Phi = [\delta_{ij}] \right\},$

$N: = \left\{ j_0^2\Phi \in GP_n^2 \mid j_0^2\Phi = j_0^1\Phi \right\}.$

We prove this proposition in steps (a) to (g).

a) M is Lie subgroup of GP_n^2.

For, let $j_0^2\phi$ and $j_0^2\psi$ are in M. Then $(j_0^2\phi)*(j_0^2\psi) = j_0^2(\phi o \psi)$ and we have

$$j_0^1(\phi o \psi) = (j_0^1\phi)*(j_0^1\psi) = [S_{ij}]$$

Moreover, since $(j_0^2\phi)^{-1} = j_0^2(\phi^{-1})$ we obtain

$$j_0^1(\phi^{-1}) = (j_0^1\phi)^{-1} = [S_{ij}]$$

Therefore M is a subgroup of GP_n^2, and furthermore with charts

$$M \ni j_0^2\Phi \to \left(\frac{\partial^2 \Phi^j}{\partial x_{i_1} \partial x_{x_2}} \right) \in \Re^{\frac{n^2(3n-1)}{2}},$$

is a Lie subgroup of GP_n^2.

b) M is a normal subgroup in GP_n^2

For, let $j_0^2\phi$ belongs to M and $j_0^2\psi$ be a member of GP_n^2, then

$$(j_0^2\psi^{-1})\star(j_0^2\phi)\star(j_0^2\psi) = j_0^2(\psi^{-1} \circ \phi \circ \psi).$$

On the other hand,

$$\begin{aligned} j_0^1(\psi^{-1}\phi\psi) &= (j_0^1\psi)^{-1})\star(j_0^1\phi)\star(j_0^1\psi) \\ &= (j_0^1\psi^{-1})\star[\delta_{ij}]\star(j_0^1\psi) \\ &= [\delta_{ij}] \end{aligned}$$

therefore $(j_0^2\psi)^{-1}\star(j_0^2\phi)\star(j_0^2\psi)$ belongs to M.

c) M is isomorphic with the additive Lie group $\Re^{\frac{n^2(3n-1)}{2}}$.

For, we define the function $\eta : M \to \Re^{\frac{n^2(3n-1)}{2}}$ as follow:

$$M \ni \left(\delta_{ij}, \frac{\partial^2\phi^j}{\partial x_{i_1}\partial x_{i_2}}|0\right) \to \left(\frac{\partial^2\phi^j}{\partial x_{i_1}\partial x_{i_2}}|0\right) \in \Re^{\frac{n^2(3n-1)}{2}}$$

The smoothness of function is easily proved. Then it is just enough to prove that it is a "group isomorphism".

For this, suppose that

$$\left(\ldots,x_k + a^k_{i_1i_2}x_{i_1}x_{i_2},\ldots\right),\left(\ldots,x_l + b^l_{j_1j_2}x_{j_1}x_{j_2},\ldots\right),$$

two elements of M. Then

$$\eta\left((\ldots, x_k + a^k_{i_1 i_2} x_{i_1} x_{i_2}, \ldots) \star (\ldots, x_1 + b^l_{j_1 j_2} x_{j_1} x_{j_2}, \ldots)\right),$$

$$= \eta\ (\ldots, \left(x_k + b^l_{j_1^k j_2^k} x_{j_1^k} x_{j_2^k}\right) + a^k_{i_1 i_2}\left(x_{i_1} + b^{i_1}_{j_1^{i_1} j_2^{i_1}}\ x_{j_1^{i_1}}\ x_{j_2^{i_1}}\right)$$

$$\times \left(x_{i_2} + b^{i_2}_{j_1^{i_2} j_2^{i_2}}\ x_{j_1^{i_2}}\ x_{j_2^{i_2}}\right), \ldots.$$

$$= \eta\left(\ldots, x_k + b^k_{j_1^{i_1} j_2^{i_2}} + a^k_{i_1 i_2}\ x_{i_1}\ x_{i_2}, \ldots\right)$$

$$= \eta\left(\ldots, x_k + \left(a^k_{i_1 i_2} + b^k_{i_1 i_2}\right) x_{i_1} x_{i_2}, \ldots\right)$$

$$= \left(\ldots, a^k_{i_1 i_2} + b^k_{i_1 i_2}, \ldots\right)$$

$$= \eta\left(\ldots, x_k + a^k_{i_1 i_2} x_{i_1} x_{i_2}, \ldots\right) + \eta\left(\ldots, x_k + b^k_{i_1 i_2} x_{i_1} x_{i_2}, \ldots\right)$$

d) N is a normal Lie subgroup of GP_n^2

For, let $j_0^2\phi$ and $j_0^2\psi$ are in N. Then $(j_0^2\phi)\star(j_0^2\psi) = j_0^2(\phi o \psi)$ and

$$\begin{aligned} j_0^2(\phi o \psi) &= (j_0^2\phi)\star(j_0^2\psi) \\ &= (j_0^1\phi)\star(j_0^1\psi) \\ &= j_0^1(\phi o \psi) \end{aligned}$$

also $(j_0^2\phi)^{-1} = j_0^2(\phi^{-1})$, and

$$j_0^2(\phi^{-1}) = (j_0^2\phi)^{-1} = (j_0^1\phi)^{-1} = j_0^1(\phi^{-1})$$

therefore N is a subgroup of GP_n^2

Let $j_0^2\phi$ belongs to N and $j_0^2\phi$ belongs to GP_n^2. Then

$$(j_0^2\psi)^{-1}\star(j_0^2\phi)\star(j_0^2\psi) = j_0^2(\psi^{-1} \circ \phi \circ \psi).$$

Since $j_0^1\phi = j_0^2\phi$, then $(j_0^1\phi)\star(j_0^2\psi) = (j_0^2\phi)\star(j_0^2\psi)$; but $(j_0^1\phi)\star(j_0^2\psi) = (j_0^1\phi)\star(j_0^1\psi)$, therefore $(j_0^1\phi)\star(j_0^1\psi) = (j_0^2\phi)\star(j_0^2\psi)$. Hence we have

$$(j_0^1\phi)\star(j_0^1\psi) = (j_0^2\psi)\star [(j_0^2\psi^{-1})\star(j_0^2\phi)\star(j_0^2\psi)]$$

$$= (j_0^1\psi)\star [(j_0^2\psi^{-1})\star(j_0^2\phi)\star(j_0^2\psi)].$$

Hence $j_0^1(\psi^{-1} o \phi o \psi) = j_0^2(\psi^{-1} o \phi o \psi)$, and N is normal in GP_n^2. On the other hand the function

$$\eta : N \ni j_0^2\phi \to j_0^1\phi \in GL(n,\Re) \subseteq \Re^{n^2}$$

induced a Lie subgroup structure on N.

e) N is isomorphic with $GL(n,\Re)$.

For, Let η be a function which is defined in (d) step. We have $\eta\Big(\big(\ldots,\, a_i^k x_i, \ldots\big)\star\big(\ldots,\, b_i^k x_i, \ldots\big)\Big) = \eta\Big(\ldots,\, \sum_j a_j^k b_i^j x_i, \ldots\Big)$

$$= \Big[\sum_j a_j^k b_i^j\Big]$$

$$= \big[a_j^k\big]\big[b_j^k\big]$$

$$= \eta\big(\ldots,\, a_i^k x_i,\, \ldots\big)\,\eta\big(\ldots,\, b_i^k x_i,\, \ldots\big)$$

therefore η is a Lie group isomorphism from N onto $GL(n,\Re)$.

f) $M \cap N = \{j_0^2\, id\}$, where id is identity function on $\Re^n$. For, let $j_0^2\phi \in M \cup N$. Then $j_0^2\phi \in M$ and $j_0^1\phi = j_0^1\, id$. On the other hand $j_0^2\phi \in N$ and $j_0^2\phi = j_0^1\phi$. Therefore $j_0^2\phi = j_0^2 id$.

g) GP_n^2 as a Lie group, is isomorphic to $M \oplus N$.

For, let $j_0^2\phi = \left(\ldots,\ a_i^k x_i + a_{i_1 i_2}^2 x_{i_1} x_{i_2}\ ,\ldots\right)$ belongs to GP_n^2, $j_0^2\zeta = (\ldots,\ A_i^k\ x_i,\ \ldots)$ belongs to N and $j_0^2\,\psi = \left(\ldots,\ x_k + A_{i_1 i_2}^k x_{i_1} x_{i_2},\ \ldots\right)$ belongs to M such that

$$j_0^2\phi = \left(j_0^2\psi\right) * \left(j_0^2\zeta\right),$$

then we have

$$A_i^k = a_i^k,\ \sum_{i_1, i_2} A_{i_1 i_2}^k A_j^{i_1} A_l^{i_2} = a_{jl}^k,$$

thus, for all k, l and j

$$\sum_{i_1, i_2} A_{i_1 i_2}^k a_j^{i_1} a_l^{i_2} = a_{jl}^k.$$

Let l be fixed, then

$$\left[\sum_{i_1} A_{i_1 i_2}^k a_j^{i_1}\right] \left[a_l^{i_2}\right] = \left[a_{jl}^k\right],$$

and
$$\sum_{i_1} A_{i_1 i_2}^k a_l^{i_1} = \sum_s a_{js}^k \left(a^{-1}\right)_s^{i_2},$$

now if i_2 is fixed, then

$$\left[A^k_{i_1 i_2}\right]\left[a^{i_1}_j\right]=\left[\sum_s a^k_{js}(a^{-1})^{i_2}_s\right],$$

therefore $$A^k_{i_1 i_2}=\sum_t \sum_s a^k_{ts}(a^{-1})^{i_2}_s(a^{-1})^{i_1}_t.$$

Where $\left[a^i_j\right]^{-1}=\left[(a^{-1})^j_i\right]$. Hence $GP_n^{\,2}$, as an abstract group, is a direct sum of M and N.

Finally by corollary at page 96 of [3], we access which we required.□

Corollary 1

Let G be a Lie subgroup of $GL(n,\Re)$. Then there exists a Lie subgroup $\tilde{G}$ of $GL(n,\Re)$ and an integer $\tilde{n}$ such that $0\leq\tilde{n}\leq\dfrac{n^2(3n-1)}{2}$, and the Lie group $G^{(1)}\cong\Re^{\tilde{n}}\oplus\tilde{G}$, where $\Re^{\tilde{n}}$ is the standard additive Lie group of $\tilde{\Re}^{\tilde{n}}$

3. STRUCTURE OF $GP_n^{\,k}$

Lemma 1

Let G and H be two Lie subalgebras of $Mat(n\times n)$, then

$$(H\oplus G)^{(1)} \cong H^{(1)} \oplus G^{(1)}$$

Proof

We note that (refer to [1])

$$G^{(1)} \cong Hom(\Re^n, G) \cap (\Re^n \otimes S^2(\Re^{n*})) \qquad (1)$$

therefore

$$\begin{aligned}(G \oplus H)^{(1)} &\cong Hom(\Re^n, G \oplus H) \cap (\Re^n \otimes S^2(\Re^{n*})) \\ &\cong (Hom(\Re^n, G) \oplus Hom(\Re^n, H)) \\ &\quad \cap [(\Re^n \otimes S^2(\Re^{n*})) \oplus (\Re^n \otimes S^2(\Re^{n*}))] \\ &= [Hom(\Re^n, G) \cap (\Re^n \otimes S^2(\Re^{n*}))] \\ &\quad \oplus [Hom(\Re^n, H) \cap (\Re^n \otimes S^2(\Re^{n*}))] \\ &= G^{(1)} \otimes H^{(1)}. \square\end{aligned}$$

Example 1

We have proved that $<\Re^n,+>^{(1)} \cong <\Re^n,+>$.

a) It is proved that $L(<\Re^n,+>) \cong (\Re^n,+)$.

b) As a Lie algebra $<\Re^n,+>$ is isomorphic to $\Delta Mat(n \times n)$, where $\Delta Mat(n \times n)$ is the Lie subgroup of all $n \times n$ diagonal matrices of $Mat(n \times n)$.

For we define

$$\Psi : <\Re^{n+}, \times> \rightarrow \Delta Mat(n \times n)$$

$$(x_1, \ldots, x_n) \rightarrow [\delta_{ij} x_i].$$

c) We prove that "as a vector space $<\Re^n,+>^{(1)}$ (prolongation of Lie algebra $<\Re^n,+>$) is isomorphic to $<\Re^n,+>$".

For this let T belongs to $(\Re^n)^{(1)}$. Therefore T is a linear mapping of $\Re^n \times \Re^n$ into $\Re^n$. Let $T(e_i.e_j)\ \sum_{ij} T_{ij}^k\ e_k$, where $\{e_1, \ldots, e_2\}$ is standard basis for $\Re^n$, and by definition, $T_{ij}^k =$

T_{ji}^{k} and $T_{ij}^{k} \in \Delta\, Mat(n\times n)$ for all i,j,k. Thus $T_{ij}^{k} = \delta_{jk}$ and $T_{ij}^{k}\neq 0 \Leftrightarrow i = j = k$. To see the result we define the mapping

$$\Gamma: <\Re^n, +>^{(1)} \ni [T_{ij}^{k}] \to (T_{ii}^{i}) \in <\Re^n, +>$$

e) By [4], if G is a Lie subgroup of $GL(n,\Re)$, then Lie group $G^{(1)}$ is isomorphic with group of all linear mappings of $\Re^n + G$ of the form a_T (for $T\in G^{(1)}$) where

$$a_T(A,v) = (v, A+T(v,.)),\quad A \in G,\ v \in \Re^n$$

Therefore prolongation of Lie group $(\Re^n, +>$ consist of all linear mappings of $\Re^n + L(\Re^n) \cong \Re^n$ of the form a_T (for $T = [T_{ij}^{k}] \in (\Re^n)^{(1)} \cong \Re^n$) where

$$\begin{aligned} a_T(A,v) &= (v, A+[T_{ij}^{k}](v,.)) \\ &= \left(v,\ A + tras\sum \left(T_{ii}^{i} v_i + a_i\right)\vec{e}_i(.)\right) \\ &= \left(\sum v_i\vec{e}_i,\ trans\ \Sigma\left(t_{ii}^{i} v_i + \right)\vec{e}_i(.)\right) \end{aligned}$$

Here the $trans_{\vec{\omega}}$ (.) is translation by $\vec{\omega}$ in $\Re^n$. It completes the proof. □

Lemma 2

Let G and H be two matrix Lie subgroups. Then

$$[H\oplus G]^{(1)} \cong H^{(1)} \oplus G^{(1)}.$$

Proof

Suppose that H be a Lie subgroup of $GL(n, V)$ with $L(H) = \mathcal{H}$ and G be a Lie subgroup of $GL(n, W)$ with $L(G) = G$, and let $(v_1\oplus\omega_1)\otimes(s_1\oplus t_1) = (v_1\otimes s_1)\oplus(\omega_1\otimes t_1)$ be an elements of $(V\oplus W)\otimes S^{k+1}((V\oplus W)^*)$ and $(v_2\oplus w_2)\otimes(s_2\oplus t_2) = (v_2\otimes s_2)\oplus(w_2\otimes t_2)$ be an elements of $(V\oplus W)\otimes S^{k+1}((V\oplus W)^*)$.

Now, we define the bracket of these two elements (denoted by symbol"[,]") as follows:

$$(v_2 \oplus w_2) \otimes D_{(v1 \oplus w1)} (s_2 \oplus t_2) \circ (s_1 \oplus t_1)$$

$$- (v_1 \oplus w_1) \otimes D_{(v1 \oplus w1)} (s_1 \oplus t_1) \circ (s_2 \oplus t_2)$$

$$= [v_2 \otimes D_{v1}\ s_2 \circ s_1 - v_1 \otimes D_{v2}\ s_1 \circ s_2]$$

$$\oplus [w_2 \otimes D_{w1}\ (t_2 \circ t_1) - w_1 \otimes D_{w2}\ (t_1 \circ t_2)].$$

Note that this lies in $(V \oplus W) \otimes S^{k+l+1}((V \oplus W)^*)$.

"[,]" extends a bilinear mapping of $((V \oplus W) \otimes S^{k+1}((V \oplus W)^*) \times (V \oplus W) \otimes S^{l+1}((V \oplus W)^*))$ into $(V \oplus W) \otimes S^{k+l+1}((V \oplus W)^*)$. Recalling (1), this induces a bilinear mapping of $(\mathcal{H} \oplus G)^{(k)} \otimes (\mathcal{H} \oplus G)^{(l)} = (\mathcal{H}^{(k)} \otimes \mathcal{H}^{(l)} \oplus G^{(k)} \otimes G^{(l)})$ into $(V \oplus W) \otimes S^{k+l+1}((V \oplus W)^*)$, which is in fact a bilinear mapping into $(\mathcal{H} \oplus G)^{(k+l)} = \mathcal{H}^{(k+l+1)} \oplus G^{(k+l+1)}$. Moreover "[,]" makes the vector space

$$(V \oplus W) + (\mathcal{H} \oplus \mathrm{G}) + (\mathcal{H} \oplus \mathrm{G})^{(1)} + \ldots$$

$$= [V + \mathcal{H} + \mathcal{H}^{(1)} + \ldots] \oplus [W + \mathrm{G} + \mathrm{G}^{(1)} + \ldots].$$

into a Lie algebra. But, the bracket operation on the Lie algebra $(G \oplus H)^{(1)}$ coincides with the bracket operation already defined on

$$(\mathcal{H} \oplus \mathrm{G}) + (\mathcal{H} \oplus \mathrm{G})^{(1)} + \ldots + (\mathcal{H} \oplus \mathrm{G}) + (\mathcal{H} \oplus \mathrm{G})^{(k)} + (\mathcal{H} \oplus \mathrm{G})^{(k+1)} + \ldots.$$

truncated at degree k (refer to [1]). Thus as a Lie algebra

$$L((\mathcal{H} \oplus G)^{(1)}) \cong L(H)^{(1)} \oplus L(G)^{(1)}.$$

This proves the lemma. □

Lemma 3

Let G be a Lie subalgebra of $Mat(n \times n)$. There exist a Lie subalgebra $\tilde{G}$ of $Mat(n \times n)$ and an integer $\tilde{n}$ such that $0 \le \tilde{n} \le \frac{n^2(3n-1)}{2}$ and the Lie algebra $G^{(1)} \cong \Re^{\tilde{n}} \oplus \tilde{G}$, where $\Re^{\tilde{n}}$ is the standard additive Lie algebra of $\Re^{\tilde{n}}$.

Proof

Let G be a Lie subgroup of $GL(n,\Re)$ where it's Lie algebra is G (in this case we write $L(G)$ = G). Now we have, by corollary 1:

$$G^{(1)} \cong \Re^{n'} \oplus \tilde{G},\ \tilde{G} \le GL(n,\Re),\ 0 \le n' \le \frac{n^2(3n-1)}{2}$$

On the other hand (refer to [1]) we know that $L(G^{(1)}) \cong G \oplus G^{(1)}$; therefore

$$zG \oplus G^{(1)} \cong L(\Re^{n'}) \oplus L(\tilde{G})$$

Hence, there exists a Lie subalgebra G of $L(\tilde{G})$ (and Lie subalgebra of $Mat(n \times n)$ and an integer n' with $0 \le n' \le n$ such that $G^{(1)} \cong \Re^{n'} \oplus (\tilde{G})$.□

Proposition 2

Let n and m are two natural numbers. Then there exists Lie group isomorphism

$$GP_n^m \cong \Re^{m \times \frac{n^2(3n-1)}{2}} \oplus GL(n,\Re),$$

where $\Re^{m\times\frac{n^2(3n-1)}{2}}$ has standard Lie group structure.

Proof

Let m be an integer greater than 2. Assume that result is proved for $m-1$. Then

$$GP_n^m \cong \left(GP_n^{m-1}\right)^{(1)}$$

$$\cong \left[\Re^{\frac{n^2(3n-1)}{2}} \oplus GL(N,\Re)\right]^{(1)}$$

$$\cong \left[\Re^{\frac{n^2(3n-1)}{2}}\right]^{(1)} \oplus [GL(n,\Re)]^{(1)}$$

$$\cong \Re^{\frac{n^2(3n-1)}{2}} \oplus GP_n^{m-1} \qquad \text{(by example 1)}$$

$$\cong \Re^{m-1\times\frac{n^2(3n-1)}{2}} \oplus GL(n,\Re) \qquad \text{(by assumption)}$$

Then by induction Proposition is proved.□

Corollary 2

Let G be a Lie subgroup of $GL(n,\Re)$, and k be an integer. Then there exists a Lie subgroup $\tilde{G}$ of $GL(n\Re)$ and an integer $\tilde{n}$ such that $0\le \tilde{n} \le \frac{n^2(3n-1)}{2}$, and the Lie group $G^{(1)} \cong \Re^{\tilde{n}} \bigoplus \tilde{G}$, where $\Re^{\tilde{n}}$ is the standard additive Lie group of $\Re^{\tilde{n}}$.□

Example 2

We study the $GP_1^3 = \{ax^3+bx^2+cx \mid a,b,c \in \Re,\ c\neq 0\}$ where proved that

$$(ax^3+bx^2+cx) * (Ax^3+Bx^2+Bx^2+Cx)$$

$$= (aC^3+2bBC+cA)x^3 + (bC^2+cB)x^2 + cCx.$$

Let $M = \{ax^3+bx^2+x \mid a,b \in \Re\}$ and $N = \{ax \mid a \in \Re-\{0\}\}$. Then N and M are normal subgroup of GP_1^3 and $GP_1^3 \cong M \oplus N$. On the other hand we have proved that N is isomorphic with multiplicative group $\Re-\{0\}$. For M, assume $T = \{ax^3+x \mid a \in \Re\}$ and $S = \{a^2x^3+ax^2+x \mid a \in \Re\}$. We know that T and S are normal subgroups of M and $M \cong S \oplus T$. But with respect to above operation we have

$$(a^2x^3+ax^2+x)*(A^2x^3+Ax^2+x) = (A+a)^2x^3+(A+a)x^2+x,$$

$$(ax^3+x)*(Ax^3+x) = (A+a)x^2+x$$

Therefore S and T are isomorphic with additive group $\Re$. In conclusion

$$GP_1^3 \cong (\Re^2,+) \oplus (\Re-\{0\}, x)$$

Corollary 3

Structure group of every k'th order geometric structure on a given n dimmentinal manifold is isomorphic with an additive standard group $\Re^m$,

where $0 \leq m \leq k \times \dfrac{n^2(3n-1)}{2}$, and Lie subgroup of $GL(n,\Re)$.□

Proposition 3

Let G be a Lie subalgebra of Mat(n×n), and k be a natural number.

Then there exists a Lie subalgebra $\tilde{G}$ of mat(n×n) and an integer $\tilde{n}$

such that $0 \leq \tilde{n} \leq k \times \frac{n^2(3n-1)}{2}$ and the Lie algebra where $G^{(1)} \cong \Re^{\tilde{n}} \oplus \tilde{G}$

$\Re^{\tilde{n}}$ has the standard Lie algebra structure.□

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A ULTIMATE BOUNDEDNESS RESULT FOR THE SOLUTIONS OF CERTAIN FIFTH ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT: The main purpose of this lecture is to give sufficient conditions, which ensure that all solutions of (1.1) are ultimately bounded.

KEYWORDS: Nonlinear differential equations of the fifth order, boundedness, V-function.

1. INTRODUCTION AND STATEMENT OF THE RESULT

The work deals with the ultimate boundedness of the solutions of the real differential equations of the form:

$$x^{(5)} + \varphi(\dddot{x})x^{(4)} + \psi(\ddot{x}, \dddot{x}) + h(\dot{x}, \ddot{x}) + g(\dot{x}) + f(x)$$

$$= p(t, x, \dot{x}, \ddot{x}, \dddot{x}, vx^{(4)}) \qquad (1.1)$$

in which the functions φ, ψ, h, g, f and p depend at most on the arguments shown in (1.1) and the dots denote differentiation with respect to t.

Further it will be assumed that the functions φ, ψ, h, g, f and p are continuous for all values of their respective arguments and that the derivatives $\frac{\partial}{\partial z}\psi(z, w)$, $\frac{\partial}{\partial y}h(yz)$, $g'(y)$ and $f'(x)$ exist and are continuous for all x, y, z and w.

The problem of interest here is to investigate a specific property of solutions of (1.1), namely the strong boundedness property of solutions in which the bounding constant is independent of solutions. This problem has received considerable attention on various special cases of fourth order non-linear differential equations from a number of authors, for example: Abou-El-Ela [1], Abou-El-Ela & Sadek [2], Asmussen [3], Ezeilo & Tejumola [10], Harrow [12], Tejumola [20], Tiryaki & Tunç [21], and others.

On the other hand, firstly, Chukwu [5] obtained sufficient conditions for the ultimate boundedness of solutions of the equation

$$x^{(5)}+ax^{(4)}+f_2(\dddot{x})+c\ddot{x}+f_4(\dot{x})+f_5(x)=p(t,x,\dot{x},\ddot{x},\dddot{x},x^{(4)})$$

Until this time I have not found out any research on the above problem with in the relevant literature.

Equation (1.1) has an equivalent system

$$\dot{x}=y,\ \dot{y}=z,\ \dot{z}=w,\ \dot{w}=u,$$

$$\dot{u} = -\varphi(w)u-\psi(z,w)-h(y,z)-g(y)-f(x)+p(t,x,y,z,w,u). \qquad (1.2)$$

obtained from (1.1) by setting $\dot{x}=y$, $\ddot{x}=z$, $\dddot{x}=w$ and $x^{(4)} = u$.

The boundedness result to be proved is as follows:

Theorem

Further to the basic assumptions on φ, ψ, h, g, f and p, suppose the existence of arbitrary positive constants a, b, c, d, e and of sufficiently small positive constants ϵ, ϵ_0, ϵ_1, ϵ_2, ϵ_3 such that

(i) $a>0,\ ab-c>0,\ (ab-c)\,c-(ad-e)a>0,$

$$\Delta = (cd-be)(ab-c)-(ad-e)^2>0, e>0. \qquad (1.3)$$

(ii) $$\Delta_1 = \frac{(cd-be)(ab-c)}{(ad-e)} - (ag'(y)-e) > 2\varepsilon b \text{ for all } y \quad (1.4)$$

$$\Delta_2 = \frac{(cd-be)}{ad-e} - \frac{d'(ad-e)}{d(ab-c)} - \frac{\epsilon}{a} > 0 \text{ for all } y \quad (1.5)$$

where $$d' = \frac{g(y)}{y},\quad y \neq 0,\quad g'(0),\quad y=0 \quad (1.6)$$

(iii) $\varphi(w) \geq a$ for all w.

(iv) $\Psi(z,0) = 0$ for all z,

$$0 \leq \frac{\psi(z,w)}{w} - b \leq \epsilon_1 \text{ for all } z \text{ and } w \neq 0,\ \psi_z(z,w) \leq 0 \text{ for}$$

all z and w.

(v) $\frac{h(y,z)}{z} \geq c$ for all y and $z \neq 0$, $h_y(y,z) \leq 0$ for all y and z.

$$\left[\frac{h(y,z)}{z} - c\right]^2 \leq \frac{\Delta_1}{16}[\varphi(w)-a] \text{ for all } y, w \text{ and } z \neq 0, \text{ where}$$

$$[\varphi(w)-a] < \epsilon_0 \equiv \min\left[\frac{\epsilon}{4a^2}, \frac{\epsilon}{4\delta^2}, \frac{\Delta_1(ad-e)^2}{16d^2(ad-c)^2}\right] \text{ for all } w.$$

(vi) $g(0) = 0$, $\frac{g(y)}{y} \geq d, g'(y) - \frac{g(y)}{y} \leq \beta$ for all w.

where β is a positive constant such that $\beta < \frac{e\Delta}{d^2(ab-c)}$,

$|g'(y)-d| \leq \epsilon_2$ for all y.

(vii) $0 \leq e - f'(x) \leq \epsilon_3$ for all x, $f(x)\text{sign } x \to \infty$ as $|x| \to \infty$.

(viii) $|p(t,x,y,z,w,u)| \leq A$ for all t,x,y,z,w and u, where A is a positive constant.

Then there exists a constant K whose magnitude depends only on $a,b,c,d,e,\Delta,\Delta_1,\Delta_2,\epsilon,A$ as well as on the functions φ,Ψ,h,g and f such that every solution $(x(t),y(t),z(t),w(t),u(t))$ of (1.2) ultimately satisfies

$$|x(t)|\leq K,\ |y(t)|\leq K,\ |z(t)|\leq K,\ |w(t)|\leq K,\ |u(t)|\leq K$$

for all sufficiently large t.

Remark 1

When $\varphi(\dddot{x}) = a$ $\psi(\ddot{x},\dddot{x}) = b\dddot{x}$; $h(\dot{x},\ddot{x}) = c\ddot{x}$, $g(\dot{x}) = d\dot{x}$, $f(x) = ex$ and $p(t,x,\dot{x},\ddot{x},\dddot{x},x^{(4)})=0$, equation (1.1) reduces to the linear constant coefficient differential equation

$$x^{(5)} + ax^{(4)} + b\dddot{x}+c\ddot{x}+d\dot{x}+ex=0$$

and conditions (i)$-$(vii) of the theorem reduce to the corresponding Routh-Hurwitz criterion.

Remark 2

$\varphi(\dddot{x})=a$, $\psi(\ddot{x},\dddot{x}) = f_2(\dddot{x}),h(\dot{x},\ddot{x})=c\ddot{x}$, then the conditions of the theorem reduce to those of Chukwu [5].

Notation

In what follows the capitals D,D_0,D_1, ...denote finite positive constants whose magnitudes depend only on the functions φ,ψ,h,g,f and p as well as on the constants $a,b,c,d,e,\epsilon_0,\epsilon,\epsilon_1,\epsilon_2,\epsilon_3$, and A; but are independent of solutions of the differential equation under consideration. The D's are not necessarily the same each time they occur, but each D_i, i = 1,2,3,... retains its identity throughout.

2. THE FUNCTION V(x,y,z,w,u)

The actual proof of the theorem will rest mainly on the certain properties of the continuous function $V = V(x,y,z,w,u)$ defined by

$$V = V_1 + V_2 + V_3,$$

where
$$2V_1 = u^2 + 2awu + \frac{2d(ab-c)}{ad-e}zu + 2\int_0^w \psi(z,s)ds$$

$$+\left[a^2 - \frac{d(ab-c)}{ad-e}\right]w^2 + 2\left[c + \frac{ad(ab-c)}{ad-e} - \delta\right]zw$$

$$+2\delta yu + 2a\delta yw + 2wf(x) + 2wg(y) + 2a\int_0^z h(y,s)\,ds$$

$$+\left[\frac{bd(ad-c)}{ad-e} - d - a\delta\right]z^2 + 2b\delta yz + 2azg(y)$$

$$-2eyz + 2azf(x) + (\delta c - ae)y^2 + \frac{2d(ab-c)}{ad-e}\int_0^y g(s)ds$$

$$+\frac{2d(ab-c)}{ad-e}yf(x) + 2\delta\int_0^s f(s)ds \tag{2.1}$$

$$\delta = \frac{e(ab-c)}{ad-e} + e, \tag{2.2}$$

$$V_2 = \begin{cases} (A+2)(\Phi+u)\ \mathrm{sgn}\ x, & \text{if } |x| \geq |\Phi+u|, \\ (A+2)x\ \mathrm{sgn}\ (\Phi+u), & \text{if } |x| \leq |\Phi+u|, \end{cases} \tag{2.3}$$

$$\Phi(w) = \int_0^w \phi(s)\,ds,$$

$$V_3 = \begin{cases} -(A+1)w\ \mathrm{sgn}\ x, & \text{if } |u| \geq |w|, \\ -(A+2)u\ \mathrm{sgn}\ w, & \text{if } |u| \leq |w|, \end{cases} \tag{2.4}$$

The properties of the function V, which are required for the proof of the theorem are summarized in Lemma 1 and Lemma 2.

Lemma 1

Subject to the assumptions (i)-(vii) of the theorem, there is a constant D_1 such that

$$V(x,y,z,w,u) \geq -D_1 \text{ for all } x,y,z,w \text{ and } u, \tag{2.5}$$

$$V(x,yy,z,w,u) \to \infty \text{ as } x^2+y^2+z^2+w^2+u^2 \to \infty \tag{2.6}$$

Proof

The function $V_1(x,y,z,w,u)$ can be rearranged as follows:

$$2V_1 = \left[u+aw+\frac{d(ab-c)}{ad-e}z+\delta y\right]^2 + \frac{d\Delta}{(ad-e)^2}\left[z+\frac{e}{d}y\right]^2 + \Delta_2[w+az]^2$$

$$+\frac{d(ad-e)}{d'(ab-c)}\left[\left(\frac{ab-c}{ad-e}\right)f(x)+\left(\frac{ab-c}{ad-e}\right)d'y+\left(\frac{ad'}{d}\right)z+\left(\frac{d'}{d}\right)w\right]^2+\left(\frac{\varepsilon}{a}\right)+\sum_{i=1}^{5} W_1, \tag{2.7}$$

where $W_1 = 2\delta\int_0^x f(s)ds - \dfrac{d(ab-c)}{d'(ad-e)}f^2(x)$,

$$W_2 = \frac{d(ab-c)}{d'(ad-e)}\left[2\int_0^y g(s)ds - yg(y)\right] + \left[\delta c - ae - \frac{e^2\Delta}{d(ad-e)^2} - \delta^2\right]y^2,$$

$$W_3 = 2a\int_0^z h(y,s)ds - (ac)z^2,$$

$$W_4 = \int_0^w \psi(z,s)ds - bw^2,$$

$$W_5 = 2e\left[\frac{cd-be}{ad-e}\right]yz,$$

Δ and Δ_2 are defined by (1.3) and (1.5), respectively.

The function W_2 can be estimated as in [5]. In fact the estimates there show tnat

$$W_2 \geq \left[\frac{e\,\Delta}{4d(ad-e)}\right] y^2$$

From (vii), (ii), (vi) and (2.2) we obtain

$$W_1 \geq 2\epsilon \int_0^x f(s)ds + \frac{2(ab-c)}{(ad-e)} \int_0^x f(s)[e - f'(s)] - \frac{(ab-c)}{(ad-e)} f^2(0)$$

$$\geq 2\epsilon \int_0^x f(s)ds + \frac{2(ab-c)}{(ad-e)} f^2(0) \ .$$

Because $f^2(0)$ is not necessarily zero, set $D_0 = \dfrac{(ab-c)}{(ad-e)} f^2(0)$, then we

have the following estimate

$$W_1 \geq 2\epsilon \int_0^x f(s)\,ds - D_0$$

We get from (v) for $z \neq 0$

$$W_4 = 2a \int_0^z \left[\frac{h(y,s)}{s} - c\right] s ds \geq 0.$$

but $W_3 = 0$ when $z = 0$, hence we have $W_3 \geq 0$ for all y and z.

We find from (iv) for all z and $w \neq 0$,

$$W_3 = \int_0^w \left[\frac{\psi(z,s)}{s} - b\right] s ds \geq 0.$$

but $W_4 = 0$ when $w = 0$, there we obtain $W_4 \geq 0$ for all z and w.

Combining the estimates for W_1, W_2, W_3 and W_4 with (2.7) we deduce that

$$2V_1 \geq \left[u+aw+\frac{d(ab-c)}{ad-e}z+\delta y\right]^2 + \frac{d\Delta}{(ad-e)^2}\left[z+\frac{e}{d}y\right]^2$$

$$+\Delta_2[w+az]^2+\left[\frac{e\Delta}{4d(ad-e)}\right]y^2$$

$$+\left(\frac{\epsilon}{a}\right)w^2+2\epsilon\int_0^x f(s)ds+2\epsilon\left[\frac{cd-be}{ad-e}\right]yz-D_0.$$

Then we have

$$2V_1 \geq D_2\int_0^x f(s)ds+2D_3y^2+2D_4z^2+2D_5w^2+2D_6u^2+2\epsilon\left[\frac{cd-be}{ad-e}\right]yz-D_0.$$

Let $\qquad W_6 = D_3y^2+2\epsilon\left[\dfrac{cd-be}{ad-e}\right]yz+D_4z^2 \quad .$ (2.8)

Since $\qquad |yz| \leq \left(\dfrac{1}{2}\right)(y^2+z^2) \quad ,$

then we obtain

$$W_6 \geq D_3y^2-\epsilon\left[\frac{cd-be}{ad-e}\right](y^2+z^2)+D_4z^2 \geq D_7(y^2+z^2)$$

for some D_7, $D_7 = \left(\dfrac{1}{2}\right)\min\{D_3,D_4\}$, , if

$$\epsilon \leq \left[\frac{ad-e}{2(cd-be)}\right]\min\{D_3,D_4\}. \tag{2.9}$$

Consequently

$$2V_1 \geq D_2\int_0^x f(s)ds + (D_3+D_7)y^2 + (D_4+D_7)z^2 + 2D_5w^2 + 2D_6u^2 - D_0. \quad (2.10)$$

From (2.3) and (v) we obtain

$$|V_2| \leq (A+2)[\varphi(\theta w)|w| + |u|] \leq (A+2))[(a+\epsilon_0)|w| + |u|],$$

since $\Phi(0)$ implies that $\Phi(w) = w\varphi(\theta w)$, $0 \leq \theta \leq 1$.

Also, by (2.4)

$$|V_3| \leq (A+1)|w|.$$

Then it follows

$$2V_2 + 2V_3 \geq -D_8(|w| + |u|). \quad (2.11)$$

Summing up the above discussion yields

$$2V \geq D_2\int_0^x f(s)ds + (D_3+D_7)y^2 + (D_4+D_7)z^2 + 2D_5w^2 + 2D_6u^2 - D_8(|w|+|u|) - D_0$$

From the result obtained, it is evident that (2.5) and (2.6) can be easily verified.

Lemma 2

Let $(x(t), y(t), z(t), w(t), u(t))$ be any solution of (1.2). Then the limit

$$\dot{V}^+(t) = \lim_{h\to 0^+} \sup \left[\frac{V(x(t+h),y(t+h),z(t+h),w(t+h),u(t+h)) - V(x(t),y(t),z(t),w(t),u(t))}{h}\right]$$

exists and there is a constant D_9 such that

$$\dot{V}^+ \leq -1 \text{ provided } x^2(t)+y^2(t)+z^2(t)+w^2(t)+u^2(t) \geq D_9 \quad (2.12)$$

whenever $0<\dot{\epsilon}_0=\max[\epsilon_0,\epsilon,\epsilon_1,\epsilon_2,\epsilon_3]\le\epsilon_4$ is sufficiently small.

Proof

The existence of $\dot{V}^+$ is quite immediate, since V_1 has continuous first partial derivatives, and V_2 and V_3 are easily shown to be locally Lipschitizian in x, w and u, w and u, respectively. Therefore the composite $V = V_1 + V_2 + V_3$ is at least locally Lipschitzian in x,y,z,w and u.

An easy calculation from (1.2) and (2.1) shows that

$$\frac{d}{dt}V_1(x,y,z,w,u)=-[\varphi(w)-a]u^2-\left[a\frac{\psi(z,w)}{w}-c+\delta-\frac{ad(ab-c)}{ad-e}\right]w^2$$

$$-z^2\left[\frac{d(ab-c)}{2}\frac{h(y,z)}{z}-\{\delta b+(ag'(y)-e)\}\right]-\left[\delta yg(y)-\frac{d(ab-c)}{ad-e}f'(x)y^2\right]$$

$$-a[\varphi(w)-a]wu-\left[\frac{h(y,z)}{z}-c\right]zu-\frac{d(ab-c)}{ad-e}[\varphi(w)-a]zu-\delta[\varphi(w)-a]yu$$

$$+[g'(y)-d]zw+[f'(x)-e]yw-\delta\left[\frac{h(y,z)}{z}-c\right]yz-a[e-f'(x)]yz$$

$$-\frac{d(ab-c)}{ad-c}\psi(z,w)z-\delta y\psi(z,w)+\frac{bd(ab-c)}{ad-e}zw+b\delta yw+w\int_0^w\psi_z(z,s)ds$$

$$+az\int_0^z h_y(y,s)+\left[u+aw+\frac{a(ab-c)}{ad-e}z+\delta y\right]p(t,x,y,z,w,u) \qquad (2.13)$$

From $\frac{\psi(z,w)}{w}\ge b$ and $\delta=\frac{e(ab-c)}{ad-e}+\epsilon$ we find

$$\left[a\frac{\psi(z,w)}{w}-c+\delta-\frac{ad(ab-c)}{ad-e}\right]w^2\geq\epsilon w^2 \tag{2.14}$$

By using $\frac{h(y,z)}{z}\geq c$, $\left[ab-c+\delta - \frac{(ab-c)}{ad-e}\right]=\epsilon$ and (1.4) we have

$$z^2\left[\frac{d(ab-c)}{ad-e}\frac{h(y,z)}{z} - \{\delta b+(ag'(y)-e)\}\right]\geq$$

$$z^2\left[\frac{(cd-be)(ab-c)}{ad-e}-\{ag'(y)-e\} - \epsilon b\right]\geq\left(\frac{1}{2}\right)$$

$$z^2\left[\frac{(cd-be)(ab-c)}{ad-e} - \{ag'(y)-e\}\right]=\left(\frac{\Delta_1}{2}\right)z^2 \tag{2.15}$$

Since $\frac{g(y)}{y}\geq d$ and $f'(x)\leq e$, it is clear that

$$-\left[\delta yg(y) - \frac{d(ab-c)}{ad-e}f'(x)y^2\right]\leq(\epsilon d)y^2 \tag{2.16}$$

Because of (v) and (iv), it follows that

$$z\int_0^z h_y(y,s)\,ds\leq 0 \quad \text{and} \quad w\int_0^w \psi_z(z,s)\,ds\leq 0 \tag{2.17}$$

On gathering the estimates (2.14)-(2.17) into (2.13) we obtain

$$\dot{V}_1\leq\frac{\epsilon d}{8}y^2-\frac{\Delta_1}{8}z^2-\frac{\epsilon}{4}w^2+\left[u+aw+\frac{d(ab-c}{ad-e}z+\delta y\right]$$

$$p(t,x,y,z,w,u) - \sum_{i=8}^{15}W_i \tag{2.18}$$

where $$W_8 = \left(\frac{1}{4}\right)[\varphi(w)-a]u^2 + \left[\frac{h(y,z)}{z} - c\right]zu + \left(\frac{\Delta_1}{16}\right)z^2,$$

$$W_9 = \left(\frac{1}{4}\right)[\varphi(w)-a]u^2 + \frac{d(ab-c}{ad-e}[\varphi(w)-a]zu + \left(\frac{\Delta_1}{16}\right)z^2,$$

$$W_{10} = \left(\frac{1}{4}\right)[\varphi(w)-a]u^2 + a[\varphi(w)-a]wu + \left(\frac{\epsilon}{4}\right)w^2,$$

$$W_{11} = \left(\frac{1}{4}\right)[\varphi(w)-a]u^2 + \delta[\varphi(w)-a]yu + \left(\frac{\epsilon d}{4}\right)y^2,$$

$$W_{12} = \left(\frac{\epsilon}{4}\right)w^2 - [g'(y)-d]zw + \frac{d(ab-c)}{(ad-e)}\left[\frac{\psi(z,w)}{w} - b\right]zw + \left(\frac{\Delta_1}{16}\right)z^2,$$

$$W_{13} = \left(\frac{\epsilon}{4}\right)w^2 + [e-f'(x)]yw + \delta\left[\frac{\psi(z,w)}{w} - b\right]zw + \left(\frac{\epsilon d}{4}\right)y^2,$$

$$W_{14} = \left(\frac{\Delta_1}{16}\right)z^2 + \delta\left[\frac{h(y,z)}{z} - c\right]yz + \left(\frac{\epsilon d}{4}\right)y^2,$$

$$W_{15} = \left(\frac{\Delta_1}{16}\right)z^2 + a[e-f'(x)]yz + \left(\frac{\epsilon d}{8}\right)y^2,$$

From (1.4) and (viii) we have

$$\dot{V}_1 \le -\frac{\epsilon d}{8}y^2 - \frac{\epsilon b}{4}z^2 - \frac{\epsilon}{4}w^2$$

$$+ \left[|u| + a|w| + \frac{d(ab-c)}{ad-e}|z| + \delta|y|\right]A - \sum_{i=8}^{15} W_1 \tag{2.19}$$

It can be seen from the similar estimates arising in the course of [23] that

$$W_8 \geq 0,\ W_9 \geq 0,\ W_{10} \geq 0,\ W_{11} \geq 0,\ W_{12} \geq 0,\ W_{13} \geq 0,\ W_{14} \geq 0 \text{ and } W_{15} \geq 0 \tag{2.20}$$

Thus, in view of (2.19) and (2.20), there are constants D_{10}, D_{11} and D_{12} such that $\dot{V}_1$ satisfies

$$\dot{V}_1 \leq \left(D_{10}y^2 + D_{11}z2 + D_{12}w^2\right) + A\left[|u| + a|w| + \frac{d(ab-c)}{ad-e}|z| + \delta|y|\right] \tag{2.21}$$

From (2.3),(2.4) and (1.2) we have

$$\dot{V}_2^+ = \begin{cases} -(A+2)[\psi(z,w)+h(y,z)+f(x)-p(t,x,y,z,w,u)] \text{ sign } x, \\ \text{if } |x| \geq |\Phi+u|, \\ (A+2)\, y \text{ sgn } (\Phi+u), \text{ if } |x| \leq |\Phi+u| \end{cases} \tag{2.22}$$

$$\dot{V}_3^+ = \begin{cases} (A+1)\, u \text{ sgn } u, \text{ if } |u| \geq |w| \\ -(A+1)[\varphi(w)u+\psi(z,w)+h(y,z)+g(y)+f(x) \\ -p(t,x,y,z,w,u)] \text{ sign } w, \text{ if } |u| \leq |w|, \end{cases} \tag{2.23}$$

Thus it will be clear from (ii), (iv), (v), (vi) and (viii) that $\dot{V}_2^+$ and $\dot{V}_3^+$ at the least satisfy

$$\dot{V}_2^+ \leq \begin{cases} -(A+2)f(x) \text{ sign } x + D(|y|+|z|+|w|+1), \\ \text{if } |x| \geq |\Phi+u|, \\ (A+2)\,|y|, \text{ if } |x| \leq |\Phi+u| \end{cases} \tag{2.24}$$

$$\dot{V}_3^+ \leq \begin{cases} -(A+1)|u|, \text{ if } |u| \geq |w|, \\ (A+1)|f(w)| + D(|y|+|z|+|w|+|u|+1), \\ \text{if } |u| \leq |w|, \end{cases} \tag{2.25}$$

From (2.21), (2.24) and (2.25) it can be shown that $\dot{V}^+ = \dot{V}_1 + \dot{V}_2^+ + \dot{V}_3^+$ necessarily satisfies:

$$\dot{V}^+ \leq -D_{13}(y^2+z^2+w^2)+A\left[|u|+a|w|+\frac{d(ab-c)}{ad-e}|z|+\delta|y|\right]$$

$$-(A+2)f(x) \ \text{sgn} \ x+(A+1)|f(x)+D(|y|+|z|+|w|+|u|+1) \tag{2.26}$$

according to $|x| \geq |\Phi+u|$ if $|w| \geq |u|$ or

$$\dot{V}^+ \leq -D_{13}(y^2+z^2+w^2)+A\left[|u|+a|w|+\frac{d(ab-c)}{ad-e}|z|+\delta|y|\right]$$

$$+(A+2)|y|+(A+1)|f(x)|+D(|y|+|z|+|w|+|u|+1) \tag{2.27}$$

according to $|x| \leq |\Phi+u|$ if $|w| \geq |u|$, and

$$\dot{V}^+ \leq -D_{13}(y^2+z^2+w^2)+A\left[|u|+a|w|+\frac{d(ab-c)}{ad-e}|z|+\delta|y|\right]$$

$$-(A+2)f(x) \ \text{sgn} \ x-(A+1)|u|+D(|y|+|z|+|w|+1) \tag{2.28}$$

according to $|x| \geq |\Phi+u|$ if $|w| \leq |u|$ or

$$\dot{V}^+ \leq -D_{13}(y^2+z^2+w^2)+A\left[|u|+a|w|+\frac{d(ab-c)}{ad-e}|z|+\delta|y|\right]$$

$$-(A+1)|u|+(A+2)|y| \tag{2.29}$$

according to $|x| \leq |\Phi+u|$ if $|w| \leq |u|$.

The remainder of the claim can be proved by using the techniques similar those used by Chukwu [5] and hence is omitted.

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Punjab University
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ON SEMI NILPOTENT ELEMENTS OF A RING

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In this note we introduce a new notion in associative rings called the semi-nilpotent elements. An element x of R is called a semi-nilpotent element of R if x^n-x is a nilpotent element for some n and if $x^n - x = 0$. We say the element is trivially semi-nilpotent. We obtain some interesting properties about these elements. For more about rings please refer [1].

Definition 1

An element x of a associative ring R is called semi-nilpotent if $x^n - x$ is a nilpotent element. If $x^n - x = 0$ we say x is a trivial semi nilpotent element of R.

Theorem 2

If x is a nilpotent element of a ring R, then x is a semi-nilpotent element of R.

Proof

Given $x \in R$ with $x^n = 0$ clearly $x^n - x = -x$ so $(-x)^n = 0$; hence our clain.

Example 1

Let $Z_2 = (0,1)$ be the field of characteristic two and $G = \langle g \,|\, g^2 = 1 \rangle$ The group $Z_2\, G = \{0,1,g,1+g\}$ has be a semi-nilpotent element $g^2 - g$

$= 1+g$ (as $g^2 = 1$ and $-1 = +1$ in Z_2) and $(1+g)^2 = 0$ which is nilpotent element of z_2G.

Proposition 3

Even an element which is a unit in R can be a semi nilpotent element.

Proof

Obvious; from example 1 as $g \in Z_2G$ with $g^2 = 1$. Hence our claim.

Proposition 4

Every idempotent element of the ring R is a trival semi nilpotent element of R.

Proposition 5

A semi-nilpotent element need not always be a nilpotent element of the ring R.

Proof

Evident from example 1 $g \in Z_2G$ is semi nilpotent but clearly g is not a nilpotent element of $Z_2 G$.

Proposition 6

Let $Z_2 = (0,1)$ be the field of characteristic two and G be a group having elements of even order. Then elements of G are semi-nilpotent elements of the group ring $Z_2 G$.

Proof

Let $g \in G$ with $g^{2n} = 1$ then $g^{2n} - g = 1+g$ so $(1+g)^{2n} = 0$. Hence our clain.

Theorem 7

Let k be a field of characteristic zero and G a torsion free abelian group. The group ring KG has no semi-nilpotents.

Proof

Obvious as KG has no zero divisors KG has no nontrivial semi-nilpotents.

Problems

Can KG have no trivial semi-nilpotents if G is a torsion free non abelian group?

Clearly this problem is equivalent to the zero divisor conjecture for group rings. Hence unless that conjecture is settled nothing could be said about this problem.

Example 2

Let $G = <g \mid g^2 = 1>$ and $Z_3 = (0,1,2)$ be the prime field of characteristic three. The group ring $Z_2G = (0,1,g,2g,2,\ 1+g,\ 2+2g,\ 2+g,\ 2g+1\}$. The group ring Z_2G has no nontrival semi nilpotents. In view of this example we put forth the following problem.

Problem

If Z_p be the prime field of characteristic p, p a prime and $G = <g \mid g^q = 1>$ be a cyclic group of order q.

(1) If $(p,q) = 1$ can Z_pG the group ring have nontrival semi nilpotents?

(2) If p/q, can Z_pG, the group ring have nontrival semi nilpotents?

Example 3

Let $G = <g|g^3 = 1>$ and $Z_3 = (0,1,2\}$ be the prime field of characteristic 3. The group ring $Z_3G = \{0, 1, 2, g, g^2, 2g, 2g^2, 1+g, 1+2g, 2+g, 2+2g, 1+g^2, 1+2g^2, g^2+2, 2+2g^2, g+g^2, 2g+g^2, 2g^2+g, 2g+2g^2, 1+g+g^2, 2+g+g^2, 2g+g^2+1, 2g^2+g+1, 2+2g+g^2, 2+2g^2+g, 1+2g+2g^2, 2+2g+2g^2\}$. In view of the above example Z_3G has nontrivial semi nilpotents. For $(1+g)^3 = (1+g) = 1+1+2+2g = 1+2g$; clearly $(1+2g)^3 = 0$, hence Z_3G has nontrivial semi-nilpotents.

Theorem 8

Let $Z_p = \{0,1,\ldots,p-1\}$ be the prime field of characteristic p and $G = <g|g^p = 1>$; then the group ring Z_pG has nontrivial semi-nilpotents.

Proof

Consider the element $1+(p-2)g \in Z_pG$, $\{1+(p-2)g\}^p - \{1+(p-2)g\} = 1+(p-2) + p-1+2g = (p-2)+2g$. Clearly $((p-1)+2g)^p\, p-2+2 = 0$. Hence the theorem.

Theorem 9

Let $Z_p = \{0,1,\ldots,p-1\}$ be the prime field of characteristic p and $G = <g|g^p = 1>$. Every element α in Z_pG such that sum of its coefficients is $p-1$ is a non trivial semi-nilpotent element of Z_pG.

Proof

Clearly if $\alpha \in Z_pG$ where $\alpha = \Sigma\alpha_i g^i$ with $\Sigma\alpha_i = p-1$ then; we have $\alpha^p - \alpha$ is such that the sum of its coefficients is p hence $(\alpha^p - \alpha)^p = 0$. Hence our claim. In view of the above theorem we prove the following problems.

Problem

Does there exists any other nontrivial semi nilpotents in Z_pG where $G = \langle g \mid g^p = 1 \rangle$?

Theorem 10

Let Z_p be a prime field of characteristic p and S_n the symmetric group of degree n, $n>p$. then the group ring Z_pS_n has nontrivial semi nilpotents.

Proof

Since S_n is the symmetric group of degree n, $n>p$ if we take the permutation s_p which permutes only p elements in S_n then that permutation s_p generates a cyclic group of degree p. Hence by Theorem 9 we have nontrivial semi nilpotents in Z_pS_n.

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Punjab University
Journal of Mathematics
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A FORMAL ALGORITHM OF A SAMPLING PROCEDURE FOR A 3-WAY STRATIFIED POPULATION

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SUMMARY: A formal algorithm of a sampling procedure for a 3-way stratified population described by Chaudhary and Kumar (1988) is presented.

Keywords: Assigned probabilities, controlled selection, fixed allocations, preferred samples, random allocations.

1. INTRODUCTION

Stratified random sampling is one of the most widely used sampling techniques. In practice in many surveys it is possible to stratify the population with respect to a number of stratifying variables. The need for stratification by more than one variable might arise in part because various kinds of information may be required from the same survey or because a sample of first stage units may be needed to serve over a period of time in a great number of diverse surveys. But at same time the sampler faces the problem that the number of non-empty strata that can be formed from the combination of several stratifying variables may be very large and possibly even greater than the permitted sample size.

In order to achieve extra control over the selection of units, some of the constraints of stratified random sampling have to be relaxed. For example, sampling within one stratum may not be independent of sampling in other strata; the selection of a unit in one stratum may deny the selection of units from other strata. Further empty strata are permissible and samples containing zero elements from one or more non-empty strata are allowed. Consequently the theory of simple stratified random sampling may not generalize in this case. However, controlled selection techniques may be used for sample selection and estimation purposes.

Goodman and Kish (1950) described a method called controlled selection for drawing samples for two or more way stratified population. The notion of controlled selection is that it increases the probability of selection of a preferred sample beyond that which is possible with stratified random sampling, while maintaining the assigned probabilities of selection for each population element, thus preserving the property of a probability sample. At the same time the probabilities of selection of non-proffered samples are reduced, generally to zero. paterson (1954), Bryant (1960), Hess, Riedel and Fitzpatric (1961) Jessen (1969,70,75) and Waterton (1980) also presented multiple stratification designs. Ghazali (1994) has given the formal algorithm for the Bryant et al method and suggested alternative simple method for improving the design. Ernst (1981) has proved that a solution will always exist in the two-dimensional case but Hess, Riedel and Fitzpatric (1975) have given a counter example for the three-dimensional case.

Waterton (1988) has described the method given by Hess et al (1961). Ghazali (1996) has shown that the procedure could fail even in simple cases and suggested some modifications to remedy the problems. Ghazali (1996) has shown that the Method-3 given by Jessen (1970) fails to attain its objectives and yields solution only under specific conditions. Some modifications are suggested to improve the design.

Chaudhary and Kumar (1988) have extended the Bryant et al (1960) 2-way stratification design to the 3-way stratification case. They have given only a verbal description but we present hare a formal algorithm for implementing their procedure.

2. NOTATION

Suppose that a population of N units is stratified by three stratifying variables (factors) with I, J, and K categories. Thus there are $1 \times j \times k$ strata cells. Suppose that N_{ijk} is the number of population units in the ijk-th stratum cell such that

$$N_{i..} = \sum_{j=1}^{J} \sum_{k=1}^{1} N_{ijk}, \qquad N_{.j.} = \sum_{i=1}^{I} \sum_{k=1}^{K} N_{ijk},$$

$$N_{..k} = \sum_{i=1}^{I} \sum_{j=1}^{J} N_{ijk}, \qquad N = \sum_{i=1}^{I} N_{i..} = \sum_{j=1}^{J} N_{.j.} = \sum_{k=1}^{K} N_{..k},$$

and $W_{ijk} = N_{ijk}/N$ is the proportion of population units in the ijk-th cell. Further given the sample size n, let $E_{ijk} = nW_{ijk}$ be the number of sample units to be drawn from the ijk-th stratum cell under proportional stratification. E_{ijk} may be written as

$$E_{ijk} = n^*_{ijk} + p_{ijk} \tag{2.1}$$

where n^*_{ijk} is an integer and $0 \le p_{ijk} < 1$. Similarly,

$$E_{i..} = \sum_{j=1}^{J} \sum_{k=1}^{K} E_{ijk} = n^*_{i..} + p_{i..}, \tag{2.2}$$

$$E_{.j.} = \sum_{i=1}^{I} \sum_{k=1}^{K} E_{ijk} = n^*_{.j.} + p_{.j.}, \tag{2.3}$$

and

$$E_{..k} = \sum_{i=1}^{I} \sum_{j=1}^{J} E_{ijk} = n^*_{..k} + p_{..k}, \tag{2.4}$$

where $n^*_{i..}$, $n^*_{.j.}$ and $n^*_{..k}$ are integers and $0 \le p_{i..}, p_{.j.}, p_{..k} < 1$.

Also suppose that n_{ijk}, $n_{i..}$, $n_{.j.}$ and $n_{..k}$ are the number of sample units allocated to the ijk-th cell, i-th level of the first factor, j-th level of the second factor, and the k-th level of the third factor respectively in a sample drawn by some design.

3. 3-WAY STRATIFICATION DESIGN:

For the population and the sample, defined in Section 2, assume that $P_{i..}$, $P_{.j.}$ and $p_{..k}$, defined in 2.2-4 respectively, are equal to zero for all i, j, and k and set $n_{i..} = n^*_{i..}$, $n_{.j.} = n^*_{.j.}$ and $n_{..k} = n^*_{..k}$. Also assume that $n_{i..} \geq 1$, $n_{.j.} \geq 1$, and $n_{..k} \geq 1$. In the case when some of the $n_{i..}$, $n_{.j.}$, or $n_{..k}$ are equal to zero, they may be forced upward to be greater or equal to unity by regrouping the adjacent categories.

In order to draw the sample we shall use the alternative scheme proposed by Ghazali (1994). Chaudhary and Kumar (1988) have also proposed a similar modification. However, they do not say why the modification is done. As the implication of the scheme for the biased estimator they write "It would not be improper to mention that the bias was likely to increase with the use of modified sampling scheme and in terms of the mean square error, the estimator would become less efficient." In fact, the consequences of this scheme are opposite to the above quotation. As mentioned by Ghazali (1994), the main objective of the modification is to reduce the discrepancies between the number of units to be drawn from a stratum cell under proportional stratification and the expected number of units under the Bryant design. This in turn reduces the bias and hence the mean square error of the biased estimator. It also reduces the variance of the unbiased estimator. The alternative scheme is given below:

n_{ijk}, as defined in 2.1, is taken as the fixed allocation. Thus in total

$u = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} n^*_{ijk}$ units are allocated to the cells in advance as fixed allocation.

Let $b_{i..} = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{ijk'}$ $\quad b_{.j.} = \sum_{i=1}^{I} \sum_{k=1}^{K} p_{ijk'}$

$$b_{..k} = \sum_{i=1}^{I} \sum_{j=1}^{J} p_{ijk} \quad \text{and}$$

$$b = \sum_{i=1}^{I} b_{i..} = \sum_{j=1}^{J} b_{.j.} = \sum_{k=1}^{K} b_{..k} = n-u,$$

where $b_{i..}$, $b_{.j.}$, $b_{..k}$, for all i, j, and k, and b are assumed to be integers. In order to draw a sample of size b by the Bryant design a cube of size b×b×b is constructed. The sub cells of this cube are identified by $\theta(p,q,r)$, p,q,r = 1, ..., b. A sub cell is selected at random from b^3 sub cells and 1 is placed in the selected sub cell. The values of p, q, and r corresponding to the selected sub cell are eliminated and from the remaining $(b-1)^3$ sub cells again a sub cell is selected at random and 1 is placed in the selected sub cell. This process is repeated a further b-2 times. By adding $n_{i..}$ adjacent rows, $n_{.j.}$ adjacent columns and $n_{..k}$ adjacent layers of the cube an ixjxk cell. Now we present the formal algorithm.

4. THE ALGORITHM

Let U_{pqr}, V_p^1, V_p^2, and V_r^3 be 0-1 indicator variables. Then the procedure is given as below:

Step 1

Set $U_{pqr} = 0 \qquad p,q,r = 1, \ldots, b,$

$V_p^1 = 0$, $V_q^2 = 0$, and $V_r^3 = 0$.

Step 2

Define $A_1 = \{p: V_p^1 = 0, 1 \leq p \leq b\},$

$A_2 = \{q: V_q^2 = 0, 1 \leq q \leq b\},$

$$A_3 = \{r: V_r^3 = 0,\ 1 \le r \le b\}.$$

Select a value of p, q, and r at random such that $p \in A_1$, $q \in A_2$ and $r \in A_3$. For these chosen values of p, q and r, set $V_p^1 = 1$, $V_p^2 = 1$, $V_p^3 = 1$ and $U_{pqr} = 1$.

Repeat step 2 b-1 times.

Step 3

Let $b_{R0} = b_{C0} = b_{F0} = 0$,

$$b_{Ri} = \sum_{f=1}^{i} b_{f..} \qquad i = 1,......,I,$$

$$b_{Cj} = \sum_{f=1}^{j} b_{.f.} \qquad j = 1,......,J,$$

$$b_{Fk} = \sum_{f=1}^{k} b_{..f} \qquad k = 1,......,K,$$

Define

$$X_{iqr} = \sum_{p=b_{R(i-1)}+1}^{b_{Ri}} U_{pqr} \qquad i=1,......,I,$$

$$Z_{ijr} = \sum_{q=b_{C(j-1)}+1}^{b_{Cj}} X_{iqr} \qquad j=1,......,J,$$

$$b_{ijk} = \sum_{r=b_{F(k-1)}+1}^{b_{Fk}} Z_{ijr} \qquad k=1,......,K,$$

Then b_{ijk} is the number of sample units to be drawn from ijk-th cell under the Bryant design. The final sample constitutes the fixed allocations plus the random allocation obtained by the Bryant design, i.e. $n_{ijk}^* + b_{ijk}$.

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Punjab University
Journal of Mathematics
Vol. xxx (1997) pp.157 - 160

PROF. DR. MOHAMMAD RAFIQUE (1940 - 1996)

Prof. Dr. Mohammad Rafique breathed his last on June 17, 1996 Saudi Arabia while serving there as Professor of Applied Mathematics at King Fahd University of Petroleum and Minerals, Dhahran. Prof. Rafique, who was a man of many parts, will be long remembered by his friends, colleagues and students for his qualities of head and heart.

Dr. M. Rafique was born in 1940 at Jallunder and migrated along with his family to Pakistan on the eve of Partition. His school and college studies were carried out at Sahiwal whereas for his Master's degree in Mathematics he moved to Lahore and got admission in Punjab University. He had a distinguished academic record with first position in B.A. and second position in M.A. examinations. After earning his M.A. degree in Mathematics from the Punjab University in 1962 he joined the Department of Mathematics as a Lecturer in September, 1962. His appointment at the Department of Mathematics was a recognition of his academic excellence by the then Head (Chairman) of the Department, Prof. Dr. Manzur Hussain, who was always eager to bring good people under his paternal fold. Since M. Rafique was keen to enhance his knowledge and make himself more useful to the department, his efforts to earn a scholarship to pursue higher studies abroad proved fruitful and he was selected by the Ministry of Education, Government of Pakistan for a scholarship under the Colombo Plan.

The period September, 1964 to July 1967 was spent by M. Rafique at Bangor College of the University of Wales. Within a period of three years he completed M.Sc. and Ph.D. degrees from that University and rejoined his parent department on July 11, 1967.

Dr. Rafique was a very enthusiastic research worker and despite very incongenial environment and totally inadequate facilities he published a large number of research papers, mostly in collaboration with Prof. Dr. Muhammad Saleem of Physics Department. In fact Dr. Rafique and Dr. Saleem are the duo of research scholars who did pioneering work in establishing culture of research in the fields of applied mathematics and mathematical physics, at Punjab University.

When the internationally known Pakistani theoretical physicist Prof. Abdus Salam who was then Director of International Centre (ICTP) Trieste became familiar with the work of Dr. Rafique, he asked him to apply for a research fellowship at his Centre.

After having been offered a research fellowship Dr. Rafique availed it and spent the period September 1971 to September 1972 at ICTP.

He continued his teaching and research at the Mathematics Department, Punjab University till September 1977, when he left for Libya to join the Department of Mathematics at Al-Fateh University Tripoli on a teaching assignment. He spent five years at Tripoli and returned to the Punjab University on September 1, 1982.

He was appointed as Professor of Mathematics on April 14, 1983.

Dr. Rafique was not only a very enthusiastic research worker but a very devoted and dedicated teacher. He was always liked and respected by his students. He took pains in preparing his lectures and tried his best in communicating his material to the students to their satisfaction.

His command over the subject, his deep interest in research, his remarkable sharp critical approach and his zealousness in academic pursuits always earned him high praise from all concerned quarters.

He wrote more than 50 research papers almost all of which were either published in prestigious foreign journals or contributed to international conferences. By any standard, this is a commendable feat and ensures him a high place among the eminent mathematicians of our country.

He had a remarkable command over the English language and could express himself in good English.

It is significant to remark that he was the first Pakistani mathematician whose book on Special Relativity written in collaboration with Dr. Mohammad Saleem was published by a renowned publisher of the Great Britain. This book has been appreciated in academic circles throughout the world for its lucid presentation and for its due emphasis on relativistic collisions and group theoretical concepts on which a comprehensive treatment is not available in any other book on this subject.

When the computers were introduced in our University for scientific research for the first time in mid-eighties, he gained, in a short time, such a comprehensive knowledge of the subject that even the experts in the field were pleasantly surprised and every body envied his capabilities. No doubt, without his extreme interest, the role of computers in scientific research in our University would not have been significant even today. He learned the various software packages and made use of them in his work. He also helped others in learning computational skills. He delivered a series of lectures on FORTRAN programming. These lectures, after some modification, were published in the form of a book by the University.

Dr. Rafique was a deeply religious person. He took performance of his duties as a matter of religious obligation and always tried to help others - in particular he was generous in extending financial help to the needy. He had memorised most of the Quranic text and was a voracious reader of Islamic literature.

A most distinguished feature of Dr. Rafique's personality was his modesty. He never boasted or displayed an iota of arrogance. He was always humble and helpful. Temperamentally, he was a cool and calm person who will rarely exhibit his passions but at heart he was very affectionate and always enjoyed helping others. He was a man of extremely pleasant manners and would never talk ill of any one. His friends and colleagues had immense respect for him and will always remember his charming and graceful personality.

May God bless his soul!
(Khalid Latif Mir)

Note: I gratefully acknowledge considerable help from Prof. Dr. Mohammad Saleem, former Dean Faculty of Science, Punjab University Lahore and a life-long friend of the late Dr. Rafique, in preparing this obituary. (KLM)

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CONTENTS