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THE COMMUTATIVE NEUTRIX PRODUCT OF $\Gamma^{(r)}(x)$ AND $\delta^{(s)}(x)$

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ABSTRACT: The Gamma function $\Gamma(x)$ is defined as a distribution and the commutative neutrix product of $\Gamma^{(r)}(x)$ and $\delta^{(s)}(x)$ is evaluated for $r, s = 0,1,2,\ldots$.

KEY WORDS AND PHRASES: Distribution, delta-function, Gamma function, neutrix, neutrix limit, neutrix product.

AMS Subject Classification (1991): 46F10.

In the following, we let N be the neutrix, see van der Corput [1], having domain $N' = \{1,2,\ldots,n,\ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda}\ln^{r-1} n,\ \ln^r n: \quad \lambda > 0,\ r = 1,2,\ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0,$

(iii) $\rho(x) = \rho(-x),$

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1.$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1,2,\ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D. Then if f is an arbitrary distribution in D', we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

for $n = 1,2,\ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see [2].

DEFINITION 1

Let f and g be distributions in D' for which on the interval (a,b), f is the k-th derivative of a locally summable function F in $L^p\,(a,b)$ and $g^{(k)}$ is a locally summable function in $L^q\,(a,b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a,b) by

$$fg = \sum_{i=0}^{K} \binom{k}{i} (-1)^i \left(F g^{(i)}\right)^{(k-i)}$$

The following definition for the commutative neutrix product of two distributions was given in [3] and generalizes Definition 1.

DEFINITION 2

Let f g be distributions in D' and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the neutrix product $f \square g$ of f and g exists and is equal to the distribution h on the interval (a,b) if

$$\operatorname*{N-lim}_{n\to\infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle$$

for all function ϕ in D with support contained in the interval (a,b). Note that if

$$\lim_{n\to\infty} \langle f_n g_n : \phi \rangle = \langle h, \phi \rangle,$$

we simply say that the product f.g exists and equals h, see [2].

This definition of the neutrix product is clearly commutative. A non-commutative neutrix product, denoted by f $\circ$ g, was considered in [5].

It is obvious that if the product $f.g$ exists then the neutrix product $f \square g$ exist and $f.g = f \square g$. Further, it was proved in [4] that if the product fg exists by Definition 1 then the product $f.g$ exists by Definition 2 and $fg = f.g$.

The following two theorems hold in [8] and [9] respectively.

THEOREM 1

Let f and g be distributions in D' and suppose that the neutrix products $f \square g^{(i)}$ (or $f^{(i)} \square g$) exist on the interval (a,b) for $i = 0,1,2,\ldots,r$. Then the neutrix products $f^{(k)} \square g$ (or $f \square g^{(k)}$) exist on the interval (a,b) for $k = 0,1,2,\ldots,r$ and

$$f^{(k)} \square g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (f \square g^{(i)})^{(k-i)}$$

or
$$f\Box g^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \, (f^{(i)}\Box g)^{(k-i)}$$

on the interval (a,b) for $k = 1,2,\ldots, r$.

THEOREM 2

The neutrix product $x^{-r} \Box\delta^{(s)}(x)$ exists and

$$x^{-r} \Box\delta^{(s)}(x) = c_{rs}\, \delta^{(r+s)}(x), \tag{1}$$

where

$$c_{rs} = \frac{(-1)^{s-1}}{(r-1)!\,(r+s)!} \int_{-1}^{1} v^{r+s} \rho^{(s)}(v) \int_{-1}^{1} \ln|v-u|\, \rho^{(r)}(u)\,du\,dv,$$

for $r = 1,2,\ldots$ and $s = 0, 1,2,\ldots$. In particular

$$x^{-r}\delta^{(r-1)}(x) = \frac{(-1)^r r!}{(2r)!}\, \delta^{(2r-1)}(x), \tag{2}$$

for $r = 1,2,\ldots$ Further,

$$\frac{(-1)^s}{(s-1)!} x^{-r}\Box\delta^{(s-1)}(x) + \frac{(-1)^r}{(r-1)!} x^{-s}\Box\delta^{(r-1)}(x) = \frac{(-1)^{r+s}}{(r+s-1)!}\, \delta^{(r+s-1)}(x), \tag{3}$$

for $r,s = 1,2,\ldots$

Now let us consider the Gamma function $\Gamma(x)$. This function is defined for $x > 0$ by

$$\Gamma(X) = \int_0^\infty t^{x-1} e^{-t} dt$$

and it follows that $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$. $\Gamma(x)$ is then defined by

$$\Gamma(x) = x^{-1}\,\Gamma(x+1)$$

for $-1 < x < 0$. The function

$$\begin{aligned} f(x) &= \Gamma(x) - x^{-1} \\ &= x^{-1}\, s\, \Gamma(x+1) - x^{-1} \\ &= \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!} x^{i-1} \end{aligned}$$

is an infinitely differentiable function on the interval $(-1,1)$ and so we define the distribution $\Gamma(x)$ on this interval by

$$\begin{aligned} \Gamma(x) &= x^{-1} + f(x) \\ &= x^{-1} + \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!} x^{i-1}, \end{aligned}$$

where x^{-1} is interpreted in the distributional sense. More generally $\Gamma(x)$ is defined by

$$\Gamma(x) = [x(x+1)\ldots(x+r)]^{-1}\,\Gamma(x+r+1)$$

for $-r-1 < x < -r$ and $r = 1,2,\ldots$. The function

$$\begin{aligned} f_r(x) &= \Gamma(x) - (-1)^r (x+r)\, r! \\ &= \frac{r!\,\Gamma(x+r+1) - (-)^r x(x+1)\ldots(x+r-1)}{x(x+1)\ldots(x+r)\,r!} \end{aligned}$$

is infinitely differentiable on the open interval $(-r-1, -r+1)$ and we define the distribution Γ on this interval by

$$\Gamma(x) \quad = \quad (-1)^r\,[(x + r)\,r!]^{-1} + f^{(r)}(x)$$

for $r = 0,1,2,\ldots$ where $(x + r)^{-1}$ is interpréted in the distributional sense, see [6]. The distribution $\Gamma(x)$ is of course an ordinary summable function for $x > 0$.

We now prove the following theorem.

THEOREM 3

The neutrix product $\Gamma^{(r)}(x) \,\Box\, \delta^{(s)}(x)$ exists and

$$\Gamma^{(r)}(x)\Box\delta^{(s)}(x) = (-1)^r r!\, C_{r+1,s}\,\delta^{(r+s+1)}(x) + \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^i\Gamma^{(r+i+1)}(1)}{r+i+1}\,\delta^{(s-i)}(x), \tag{4}$$

for $r,s = 0,1,2,\ldots$ In particular

$$\Gamma^{(r-1)}(x)\,.\,\delta^{(r-1)}(x) = -\frac{r!(r-1)!}{(2r)!}\,\delta^{(2r-1)}(x) + \sum_{i=0}^{r-1}\binom{r-1}{i}\frac{(-1)^i\Gamma^{(r+i)}(1)}{r+i}\,\delta^{(r-i-1)}(x), \tag{5}$$

for $r = 1,2,\ldots$ Further,

$$\Gamma^{(r-1)}(x)\Box\delta^{(s-1)}(x) + \Gamma^{(s-1)}(x)\Box\delta^{(r-1)}(x) = -\frac{(s-1)!(r-1)!}{(r+s-1)!}\,\delta^{(s+r-1)}(x) + $$

$$-\sum_{i=0}^{s-1}\binom{s-1}{i}\frac{(-1)^i\Gamma^{(r+i)}(1)}{(r+i)}\,\delta^{(s-i-1)}(x) +$$

$$-\sum_{i=0}^{r-1}\binom{r-1}{i}\frac{(-1)^i\Gamma^{(s+i)}(1)}{(s+i)}\,\delta^{(r-i-1)}(x), \tag{6}$$

for $r,s = 1,2,\ldots.$

Proof

We first of all note that

$$\Gamma^{(r)}(x)\Box\delta^{(s)}(x) = 0$$

on every interval (a,b) not containing the origin. On the open interval $(-1,1)$ we have

$$\Gamma^{(r)}(x) = (-1)^r r!\,x^{-r-1} + f^{(r)}(x)$$

$$= (-1)^r r!\,x^{-r-1} + \sum_{i=0}^{\infty} \frac{\Gamma^{(r+i+1)}(1)}{(r+i+1)\,i!} x^i.$$

From equation (1)

$$z^{-r-1}\Box\delta^{(s)}(x) = c_{r+1,s}\,\delta^{(r+s+1)}(x), \tag{7}$$

for $r,s = 0,1,2,\ldots$ and since $f^{(r)}(x)$ is infinitely differentiable the product of $f^{(r)}(x)$ and $\delta^{(s)}(x)$ exists by Definition 1. We have

$$f^{(r)}(x)\,\delta^{(s)}(x) = \sum_{i=0}^{s} \frac{\Gamma^{(r+i+1)}(1)}{(r+i+1)i!}\, x^i \delta^{(s)}(x)$$

$$= \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i \Gamma^{(r+i+1)}(1)}{r+i+1}\,\delta^{(s-i)}(x), \tag{8}$$

since $$x^i \delta^{(s)}(x) = \frac{(-1)^i s!}{(s-i)!}\,\delta^{(s-i)}(x),$$

for $i = 0,1,2,\ldots s$ and is zero for $i = s+1,\ s+2,\ldots.$

The neutrix distribution product is clearly distributive and so

$$\Gamma^{(r)}(x)\square\delta^{(s)}(x) = (-1)^r\, r!x^{-r-1}\square\delta^{(s)}(x) + f^{(r)}(x)\square\delta^{(s)}(x).$$

Equation (4) then follows on using equations (7) and (8), equation (5) follows from equations (2) and (8) and equation (6) follows from equations (3) and (8).

The following theorem was also proved in [8].

THEOREM 4

The neutrix products $x_+^{-r}\square\delta^{(s)}(x)$ *exists and*

$$x_+^{-r}\square\delta^{(s)}(x) = \frac{1}{2}\, c_{rs}\,\delta^{(r+s)}(x) \tag{9}$$

for $r = 1,2,\ldots$ *and* $s = 0,1,2,\ldots$. *In particular*

$$x_+^{-r}\square\delta^{(r-1)}(x) = \frac{(-1)^r r!}{2(2r)!}\,\delta^{(2r-1)}(x), \tag{10}$$

for $r = 1,2,\ldots$. *Further,*

$$\frac{(-1)^s}{(s-1)!}x_+^{-r}\square\delta^{(s-1)}(x) + \frac{(-1)^r}{(r-1)!}x_+^{-s}\square\delta^{(r-1)}(x) = \frac{(-1)^{r+s}}{2(r+s-1)!}\,\delta^{(r+s-1)}(x), \tag{11}$$

for $r,s = 1,2,\ldots$.

We next define the distribution $\Gamma(x_+)$ by

$$\begin{aligned}\Gamma(x_+) &= x_+^{-1} + f(x_+)\\ &= x_+^{(-1)} + \sum_{i=1}^{\infty}\frac{\Gamma^{(i)}(1)}{i!}x_+^{i-1},\end{aligned}$$

where x_+^{-1} is interpreted in the distribtuional sense.

We now prove

THEOREM 5

The neutrix product $\Gamma^{(r)}(x_+) \,\Box\, \delta^{(s)}(x)$ exists and

$$\Gamma^{(r)}(x_+)\Box\delta^{(s)}(x) = (-1)^r r!\frac{1}{2}c_{r+1,s}\,\delta^{(r+s+1)}(x) + \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^i\Gamma^{(r+i+1)}(1)}{2(r+i+1)}\,\delta^{(s-i)}(x), \tag{12}$$

for $r,s = 0,1,2,\ldots$. *In particular*

$$\Gamma^{(r-1)}(x_+)\Box\delta^{(r-1)}(x) = -1\frac{r!(r-1)!}{2(2r)!}\,\delta^{(2r-1)}(x) + \sum_{i=0}^{r-1}\binom{r-1}{i}\frac{(-1)^i\Gamma^{(r+i)}(1)}{2(r+i)}\,\delta^{(r-i-1)}(x), \tag{13}$$

for $r = 1,2,\ldots$. *Further,*

$$\Gamma^{(r-1)}(x_+)\Box\delta^{(s-1)}(x) + \Gamma^{(s-1)}(x_+)\Box\delta^{(r-1)}(x) = -\frac{(s-1)!(r-1)!}{2(r+s-1)!}\,\delta^{(s+r-1)}(x) + $$
$$-\sum_{i=0}^{s-1}\binom{s-1}{i}\frac{(-1)^i\Gamma^{(r+i)}(1)}{2(r+i)}\,\delta^{(s-i-1)}(x) + $$
$$-\sum_{i=0}^{r-1}\binom{r-1}{i}\frac{(-1)^i\Gamma^{(s+i)}(1)}{2(s+i)}\,\delta^{(r-i-1)}(x), \tag{14}$$

for $r,s = 1,2,\ldots$.

Proof

On the open interval $(-1,1)$ we have

$$\Gamma^{(r)}(x_+) = (-1)^r r!\,x_+^{-r-1} + f^{(r)}(x_+)$$

$$= (-1)^r r!\,x_+^{-r-1} + f_1^{(r)}(x_+) + f_2^{(r)}(x_+),$$

where

$$f_1^{(r)}(x_+) = \sum_{i=0}^{s} \frac{\Gamma^{(r+i+1)}(1)}{(r+i+1)i!} x_+^i, \qquad f_2^{(r)}(x_+) = \sum_{i=s+1}^{s} \frac{\Gamma^{(r+i+1)}(1)}{(r+i+1)i!} x_+^i.$$

From equation (9)

$$x_+^{-r-1} \Box \delta^{(s)}(x) = \frac{1}{2} c_{r+1,s}\, \delta^{(r+s+1)}(x), \tag{15}$$

for $r,s = 0,1,2,\ldots,$

$$f_1^{(r)}(x_+) \Box \delta^{(s)}(x) = \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i \Gamma^{(r+i+1)}(1)}{2(r+i+1)} \delta^{(s-i)}(x), \tag{16}$$

since $\qquad x_+^i \Box \delta^{(s)} = \dfrac{(-1)^i s!}{2(s-i)!} \delta^{(s-i)}(x),$

for $i = 0,1,2,\ldots s$, see [4] and

$$f_2^{(r)}(x_+)\delta^{(s)}(x) = 0, \tag{17}$$

since $f_2^{(r)}(x_+)$ is s times continuously differentiable and $f_2^{(r+i)}(0) = 0$ for $i = 0,1,\ldots, s$.

Equation (12) then follows on using equations (15),(16) and (17), equation (13) follows from equation (10), (16) and (17) and equation (14) follows from equations (11), (16) and (17).

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KOEBE DOMAIN FOR CONVEX FUNCTIONS OF COMPLEX ORDER

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ABSTRACT: In this note we shall give the Koebe domain for convex functions of complex order.

KEY WORDS AND PHRASES: Univalent functions, starlike functions of complex order, convex functions of complex order, Koebe domain.

1. INTRODUCTION

Let A denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + \dots$ which are analytic in $D = \{z \mid |z| < 1\}$. A function $f(z)$ in A is said to be a convex function of complex order $b(b \neq 0$, complex), that is $f(z) \in C(b)$ if and only if $f(z) \neq 0$ in D and

$$Re\left(1 + \frac{1}{b} z \frac{f''(z)}{f'(z)}\right) > 0 \quad z \in D \tag{1.1}$$

The class $C(b)$ was introduced by P. Wiatrowski [4].

It should be noticed that by giving specific values to b we obtain the following important subclasses:

i) $C(1) = C$ is well known class of convex functions

ii) $C(1 - \beta)$, $0 \leq \beta < 1$ is the class of convex functions of order β introduced by M.S. Robertson [2].

iii) $C(e^{-1\lambda}\cos\lambda), |\lambda|<\frac{\pi}{2}$ is a class of functions for which $zf'(z)$ is λ- Spiralike.
This class was introduced by M.S. Robertson [2].

iv) $C[(1-\beta)e^{-1\lambda}\cos\lambda], 0\leq\beta<1, |\lambda|<\frac{\pi}{2}$ is the class of functions for which $zf'(z)$ is λ- Spirallike of order β

Definition

Let $S(1-b)$ ($b \neq 0$, complex), denotes the class of functions $f(z) = z + a_2z^2 + a_3 z^3 + \ldots$ in $D = \{z \mid |z| < 1\}$. Which satisfy $(f(z)/z) \neq 0$ for $z = re^{i\theta} \in D$, and

$$Re\left[1+\frac{1}{b}\left(z\,\frac{f'(z)}{f(z)}-1\right)\right] > 0 \quad z \in D$$

Then $f(z)$ is said to be starlike functions of complex order.

Definition

Let F be a set of functions $f(z)$, each regular in D. The Koebe domain for a set F is denoted by K(F) and is the collection of points w such that w is in $f(D)$ for every function $f(z)$ in F. In symbols

$$K(F) = \bigcap_{f\in F} f(D)$$

Supposing that the set F is invariant under the rotation, so $e^{-ia}f(d^{-1a}z)$ is in whenever $f(z)$ is in F. Then the Koebe domain will be either the single point $\omega = 0$ or an open disc $|\omega| < R$. In the second case R is often easy to find. Indeed, supposing that we have a sharp lover bound $M(r)$ for $f(re^{i\theta})$ for all functions in F, and F contains only univalent functions. Then

$$R = \lim_{-1-} M(r)$$

gives the disc $|\omega| < R$ as the koebe domain for the set F.

KOEBE DOMAIN FOR THE CLASS C(b)

In this section we shall give the koebe domain for the class of convex functions of complex order under the condition $|b|+4\ |b|\ |1-b|<1$.

Lemma 2.1

A sufficient condition for the univelence of $f(z)$ in C(b) is

$$|b| + 4\ |b|\ |1 - b| < 1$$

Proof

Since $f(z) \in$ C(b) we can write.

$$1 + \frac{1}{b} = \frac{f''(z)}{f'(z)} = P(z) \tag{2.1}$$

where P(z) is analytic in D and satisfies the condition P(0) = 1. Re (P(z)) > 0.

From the equality (2.1) we find

$$\frac{f''(z)}{f'(z)} = \frac{b\{P(z) - 1\}}{z} \tag{2.2}$$

If we calculate the derivative of (2.2) we obtain that

$$\left(\frac{f''(z)}{f'(z)}\right)' = \frac{bzP'(z) - bP(z) + b}{z^2} \tag{2.3}$$

Simple calculation from (2.3) shows that

$$\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{b^2[P(z)]^2 - 2b^2P(z) + b}{2z^2} \tag{2.4}$$

From (2.3) and (2.4) we get

$$\{f(z),z\} = \frac{b^2}{2z^2}\{2zP'(z) - [P(z)]^2 + 1\} + \frac{b(1-b)}{2z^2}[P(z) - 1]^2 \tag{2.5}$$

If we take the absolute value of the both sides in (2.5) we obtain

$$|\{f(z),z\}| \leq \frac{|b|}{2|z^2|}|2zP'(z) - [P(z)]^2 + 1| + \frac{|b|\,|1-b|}{2z^2}|p(z) - 1| \tag{2.6}$$

On the other hand, M.S. Robertson [3] proved the inequalities given below if P(z) is analytic in D and satisfies the condition P(0) = 1, Re (P(z)) > 0, then

$$|2zp'(z) - [p(z)]^2 + 1| \leq \frac{4|z^2|}{(|-|z|)^2} \tag{2.7}$$

$$|p(z) - 1| \leq \frac{2|z|}{(|-|z|)} \tag{2.8}$$

Now considering the relations (2.6), (2.7) and (2.8) all together, we can conclude that

$$|f(z),z| \leq \frac{2}{(1-|z^2|)^2} \quad |b| + |b|\,|1-b| \tag{2.9}$$

Hence, from the Nehari test [7], we get

$$|f(z),z| \leq \frac{2}{(1-|z^2|)^2} \quad |b| + 4|b|\,|1-b| \leq \frac{2}{(1-|z^2|)^2} \tag{2.10}$$

The inequality shows that the theorem is true.

Corollary 2.1

Since $C(b) = C$ for $b = 1$, the inequality $|b| + 4|b||1-b| \leq 1$ reduces $1 = 1$. This shows that all convex functions are univalent in the unit disc.

Corollary 2.2

Let $b = e^{-i\lambda} \cos\lambda$, then the inequality $|b| + 4|b||1-b| \leq 1$ becomes

$$|\cos\lambda| + |1 - 4\sin\lambda| \leq 1$$

This inequality was found by M.S. Robertson [3].

Lemma 2.2

Let f(z) be regular in unit circle and normalized so that $f(0) = f'(0) - 1 = 0$. A necessary and sufficient condition for $f(z) \in C(b)$, is that for each member s(z), $s(z) = z + b_2z^2 + b_3z^3 + \dots$ of $s(1-b)$, the equation

$$s(z) = z\left(\frac{f(z) - f(n)}{z - n}\right)^2 \quad z, n \in D,\ z \neq n \tag{2.11}$$

must be satisfied.

Proof

Let f(z) convex function of complex order in D, then this function is analytic, regular and continuous in the unit disc. Therefore the equation (2.11) can be written in the form.

$$s(z) = z(f'(z))^2 \tag{2.12}$$

If we take the logarithmic derivative from (2.12) and simple calculations shows that

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$$s(z) = z(f'(z))^2 \tag{2.12}$$

If we take the logarithmic derivative from (2.12) and simple calculations shows that

$$Re\left[\frac{1}{2b}\left(z\frac{S'(z)}{S(z)}-1\right)+1\right] = Re\left[1+\frac{1}{b}\,z\,\frac{f''(z)}{f'(z)}\right] \qquad (2.13)$$

Considering the relation (2.13) and the definitions of convex functions of complex order, the definition of starlike function of complex order together we obtain that the function s(z) is starlike functions of complex order.

Conversely, let s(z) is starlike functions of complex order in D, then simple calculations from (2.11) we obtain that

$$\frac{1}{b}\left(z\frac{S'(z)}{S(z)}-1\right)+1 = \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(n)} - \frac{z+n}{z-n}\right] + \frac{b-1}{b} \qquad (2.14)$$

If we take $F(z,n) = \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(n)} - \frac{z+n}{z-n}\right] + \frac{b-1}{b}$

the relation (2.14) can be written in the form

$$F(z,n) = \frac{1}{b}\left[z\frac{S'(z)}{S(z)}-1\right]+1 \qquad (2.15)$$

Considering the relation (2.15) and the definition of starlike functions of complex order together we obtain

$$Re\ F(z,n) > 0 \qquad (2.16)$$

$$F(z,n) = 1 + \frac{1}{b}\left(\frac{2}{f(n)} - \frac{2}{n}\right) z + \dots \qquad (2.17)$$

$$\lim_{n\to z} F(z,n) = 1 + \frac{1}{b}\,z\,\frac{f''(z)}{f'(z)} \qquad (2.18)$$

Therefore by using continuity the claim is proved. Hence it follows that f(z) is convex function of complex order.

Corollary

By the relation (2.15), (2.16), (2.17), (2.18) the function

$$f(z,n) = \frac{1}{2}\left[\frac{2zf'(z)}{f(z)-f(n)} - \frac{z+n}{z-n}\right] + \frac{b-1}{b}$$

has a positive real part and is analytic in D, using the caratheodory inequality for this function we obtain

$$\left|\frac{1}{b}\left(\frac{2}{f(n)} - \frac{2}{n}\right)\right| \leq 2 \tag{2.19}$$

simple calculations from (2.19) we arrive that

$$Re\, \frac{f(z)}{z} > \frac{1}{1+|b|} \tag{2.20}$$

Corollary

If we take $\eta = 0$ in $f(z,n)$ we obtain

$$F(z,0) = \frac{1}{b}\left(2z\,\frac{f'(z)}{f(z)} - 1\right) + 1 - \frac{1}{b} \tag{2.21}$$

from the inequality (2.21) we arrive that

$$Re\left(1+\frac{1}{b}\left(z\,\frac{f'(z)}{f(z)} - 1\right)\right) > \frac{1}{2},\ z \in D \tag{2.22}$$

The inequality (2.22) shows that all convex functions of complex order are starlike functions of complex order of order $\frac{1}{2}$

Theorem 2.1

The Koebe domain for the class *C (b)* is

$$R = \lim_{r \to 1^-} m(r) = \lim_{r \to 1^-} \frac{2r}{(1+|b|)(1+r)} = \frac{1}{1+|b|}$$

This result is sharp because the external function is

$$f_r(z) = \frac{2z}{(1+|b|)(1+z)}$$

Proof

Let $f(z) \in C(b)$, the inequality (2.20) can be written in the form

$$Re\left(\frac{1+|b|}{2}\frac{f(z)}{z}\right) \geq \frac{1}{2},\ z \in D \tag{2.24}$$

Therefore the function $\left(\frac{1+|b|}{2}\frac{f(z)}{z}\right)$ is subordinate to the function $\left(\frac{1}{1+z}\right)$. Using the subordination principle then we have

$$\frac{2r}{(1+|b|)(1+r)} \leq |f(z)| \leq \frac{2r}{(1+|b|)(1-r)} \tag{2.25}$$

The inequality (2.25) shows that the proof of this theorem is complete.

Corollary 2.3

If $b = 1 - \beta$, $0 \le \beta \le 1$, then the Koebe domain for the class of the convex function of order β is

$$R_{ki} = \frac{1}{2-\beta} \qquad (2.26)$$

It should be noticed that this result is sharper than the M.S. Robertson's result [2]. M.S. Robertson has shown the Koebe domain for the convex function of order β to be [1].

$$R_{k2} = \begin{cases} Log_2 & if\ \beta = \dfrac{1}{2} \\ \dfrac{2^{2\beta-1}-1}{2\beta-1} & if\ \beta \neq \dfrac{1}{2} \end{cases}$$

If we compare the result of (2.26) and (2.27) we can clearly see the numerical difference between them

β	R_{k1}	R_{k2}
1/2	0.666	0.301
1/3	0.600	0.618
1/4	0.517	0.586
1/5	0.555	0.567
1/6	0.545	0.555
1/7	0.538	0.546
1/8	0.533	0.540
1/9	0.529	0.536
1/10	0.5226	0.532
1/11	0.532	0.529
1/12	0.521	0.526

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β	R_{k1}	R_{k2}
1/13	0.520	0.524
1/14	0.518	0.522
1/15	0.517	0.521
1/16	0.516	0.519
1/17	0.515	0.518
1/18	0.514	0.517
1/19	0.513	0.516
1/20	0.512	0.515

Corollary 2.4

Let $b = 1$ then we obtain the Koebe domain for convex functions

$$R = \frac{1}{2}$$

This result is well known.

Corollary 2.5

Let $b = e^{-i\lambda}\cos\lambda$, then

$$R = \frac{1}{1+\cos\lambda}$$

This is Koebe domain of the class $C(e^{-i\lambda}\cos\lambda)$

Corollary 2.6

The Koebe domain of the class $C(1-\beta)e^{-i\lambda}\cos\lambda)$ is

$$R = \frac{1}{1+(1-\beta)\cos\lambda}$$

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ON THE REVERSAL OF A RANDOM EVOLUTION

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ABSTRACT: In this note, we construct reversed random evolutions on finite state space, continuous-time markov chains. The proofs use the sample path approach of random evolutions. These results completely settle the issue concerning the proper order of the operators in the random evolution structures, and further explains the connections between forward and backward random evolutions.

KEY WORDS AND PHRASES: Random evolution, markov chain, semigroup of bounded linear operators

Research on the random evolution of a family of semigroups $\{T_i(t), t \geq 0, i = 1, \ldots, N\}$ with switching among semigroups controlled by a finite state, stationary markov chain v was begun by Griego-Hersh [3] to study equation of the form

$$\frac{\partial \tilde{u}}{\partial t} = \tilde{A}\tilde{u} + Q\tilde{u}, \; \tilde{u}(0) = \tilde{f}$$

where Q is the infinitesimal matrix of v, $\tilde{A}$ is the infinitesimal generator of a semigroup $\tilde{R}(t)$ on a Banach space $\tilde{B}$, and $\tilde{u}(t) = \tilde{R}(t)\, \tilde{f}$, where $\tilde{f} \in \tilde{B}$

In [6 and 7], the author showed that the random evolutions of Griego and Hersh were backward random evolutions and constructed forward random evolutions. Forward random evolutions provide a probabilistic approach to the study of a different but analogous class of Cauchy problems.

The purpose of this note is to directly obtain reversed random evolutions from the markov chain. This gives additional insight as to how the random evolution structure relates to the mechanics of the Markov chain. Here we repeat some of the arguments contained in [8] for completeness.

Surveys of the literature on random evolutions are given in the papers of Cohen [1], Hersh [4] and in the books of Ethier and Kurtz [2], Iordache [5], and Korolyuk and Swishchuk [9].

2. REVERSED RANDOM EVOLUTIONS

Suppose $\nu = \{\nu(t), t \geq 0\}$ is a right-continuous Markov chain with state space $\{1,\ldots,N\}$, stationary transition probabilities $p_{ik}(t)$, and infinitesimal matric $Q = < q_{ik} > = <p'_{ik}(0)>$. P_i is the probability measure defined on sample paths $\omega(t)$ for ν under the condition $\omega(0) = i$. E_i denotes integration with respect to P_i. For a sample path $\omega \in \Omega$ of ν, $\tau_j(\omega)$ is the time of the *jth* jump, and $N(t,\omega)$ is the number of jumps up to time t.

Let $\{T_i(t), t \geq 0, i = 1, \ldots, N\}$ be a family of strongly continuous semigroups of bounded linear operations on a fixed Banach space B. A_i is the infinitesimal generator of T_i. Let D_i be the domain of A_i. $\tilde{B}$ is the N-fold Cartesian product of B with itself. A generic element of $\tilde{B}$ is denoted by $\tilde{f} = \langle f_i \rangle$ where $f_i \in B$, $i = 1,\ldots,N$. We equip $\tilde{B}$ with any appropriate norm so that $\|\tilde{f}\| \to 0$ as $\|f_i\| \to 0$ for each i.

Definition 2.1

A backward random evolution $\{R(t,\omega), t \geq 0\}$ is defined by the product

$$R(t) = T_{\nu(0)}(\tau_1)T_{\nu(\tau_1)}(\tau_1 - \tau_2)\dots T_{\nu(\tau_{N(T)})}(t - \tau_{N(T)})$$

Definition 2.2

For $t \geq 0$ define the matrix operator $\tilde{R}(t)$ on $\tilde{B}$ specified componentwise by

$$(\tilde{R}(t)\tilde{f})_i = E_i[R(t)f_{\nu(t)}].$$

The following results now follow from Griego and Hersh [3].

Theorem 2.3

$\{\tilde{R}(t), t \geq 0\}$ is a strongly continuous semigroup of bounded linear operations on $\tilde{B}$.

Theorem 2.4

The infinitesimal generator $\tilde{A}$ of $\tilde{R}(t)$ is given by $\tilde{A} = diag(A_1,\dots,A_N)$ + Q in matrix form, or considering $\tilde{A}$ as acting on column vectors we get

$$(\tilde{A}\tilde{f})_i = A_i f_i + \sum_j q_{ij} f_j$$

Corollary 2.5

The Cauchy problem for an unknown vector $\tilde{u}(t)$, $t > 0$,

$$\frac{\partial u_i}{\partial t} = A_i u_i + \sum_j q_{ij} u_j, \; \tilde{u}(0+) = \tilde{f} \tag{2.1}$$

is solved by $\tilde{u}(t) = \tilde{R}(t)\tilde{f}$, *for* $\tilde{f} \in \tilde{D} = D_1 \times D_2 \times \ldots \times D_N$.

Definition 2.6

A forward random evolution is $\{S(t,\omega),\ t \geq 0\}$ defined by the product

$$S(t) = T_{v(\tau_n)}(t-\tau_n)T_{v(\tau_n-1)}(\tau_n-\tau_{n-1})\ldots T_{v(\tau_1)}(\tau_2-\tau_1)T_{v(0)}(\tau_1)$$

where $N(t) = n$.

Here, a forward random evolution is obtained by reversing the order of the operators in a backward random evolution.

Definition 2.7

For $t \geq 0$ define the expectation semigroup $\tilde{U}(t)$ on $\tilde{B}$ (componentwise) by

$$\left(\tilde{U}(t)\tilde{f}\right)_k = \sum_i E_i[S(t)f_i;\ v(t) = k],$$

where $E_i[S(t)f_i;\ v(t) = k] = E_i[S(t)f_i I_{(v(t)=k)}]$.

The following result is proved in [6 and 8].

Theorem 2.8

The Cauchy problem for an unknown vector $\tilde{u}(t)$, $t > 0$, *is solved by*

$$\frac{\partial u_k}{\partial t} = A_k u_k + \sum_j q_{ik}\ u_i,\ \tilde{u}(0+) = \tilde{f} \qquad (2.2)$$

is solved by $\tilde{u}(t) = \tilde{U}(t)\tilde{f}$, for $\tilde{f} \in \tilde{D}$.

The system (2.1) is reminiscent of the backward Kolmogrov system. For this reason we call (2.1) the backward system for the random evolution. $\tilde{R}(t)$ is called a backward random evolution semigroup. The generator of $\tilde{U}(t)$ is the matrix transpose of $\tilde{A}$, the generator of $\tilde{R}(t)$. The system of equation (2.2) taken with the system of equations (2.1) form a formally adjoint system. It is on this account that we call the semigroup $\tilde{U}(t)$ the transpose of the semigroup $\tilde{R}(t)$. The system (2.2) is the analogue of the forward Kolmogorov system. For this reason, we call (2.2) the forward system for the random evolution. $\tilde{U}(t)$ is called a forward random evolution semigroup.

Having constructed the forward random evolution semigroup $\tilde{U}$, we will now use the operator set-up in definition 2.6 to obtain "reversed" random evolutions (see[8]). Let use rename *S(t)*, *M(t)*, and use it to obtain a backward Kolmogorov system. In so doing, we show the backward Kolmogorov system depends essentially on the fact that almost all sample functions of *v(t)* have a first discontinuity, which is a jump. similarly, the forward Kolmogorov system depends essentially on the existence of a last discontinuity, which is a jump, in the interval $[0,t]$. Thus, the forward Kolmogorov system for random evolutions may also be obtained by using *R(t)* from definition 2.1 in place of *S(t)* in Definition 2.7.

Definition 2.9

A reversed random evolution $\{M(t,\omega),\ t \geq 0\}$ is defined by the product

$$M(t) = T_{v(\tau_n)}(t-\tau_n)T_{v(\tau_n-1)}(\tau_n-\tau_{n-1})\cdots T_{v(0)}(\tau_1)$$

where $N(t) = n$.

The proof of the following lemma is essentially the same as that of Lemma 2 in [3] and is thus omitted.

Lemma 2.10

If $g:\Omega \to B$ is Bochner P_i -integrable for a fixed $i = 1,\ldots,n$, then for each $t \geq 0$ the function $\omega \to M(t,\omega)g(\omega)$ is Bochner P_i-integrable and

$$E_i[M(t)g \mid F_t](\omega) = M(t,\omega)E_i[g \mid F_t](\omega),$$

for almost all ω with respect to P_i, where F_t is the σ-algebra generated by the random variables $v(u)$, $0 \leq u \leq t$, that is, F_t is the past up to time t for the Markov chain.

$$\tilde{B}_d = \{\tilde{f} \in \tilde{B}: f_i - f,\ 1 \leq i \leq N\}$$

Definition 2.11

For $t \geq 0$ define the (matrix) operator $\tilde{T}(t)$ on $\tilde{B}_d$ specified componentwise by

$$(\tilde{T}(t)\tilde{f})_i = E_i[M(t)f_{v(t)}], \qquad \text{where } \tilde{f} = \langle f \rangle.$$

As in Lemma 2.10 we see that $M(t)f_{v(t)}$ is Bochner integrable. Below we will show that $\tilde{T}(t)$ defines a semigroup on $\tilde{B}$. We call $\tilde{T}(t)$ the "expectation semigroup" associated with the random evolution $M(t)$.

Theorem 2.12

$\{\tilde{T}(t),\ t \geq 0\}$ is a strongly continuous semigroup of bounded linear operators on $\tilde{B}_d$.

Proof

That $\tilde{T}(t)$ is strongly continuous in t and is a bounded linear operator followed from [3] or [8]. Thus we need only check the semigroup property. It suffices to show that for each i,

$$\left(\tilde{T}(t+s)\tilde{f}\right)_i = \left(\tilde{T}(t)\tilde{T}(s)\tilde{f}\right)_i .$$

Let $\theta_t\omega$ be the shifted path defined by the requirement that $v(u,\theta_t\omega) = v(u+t,\omega)$ for every $u \geq 0$. Define $go\theta_t$ by $(g\circ\theta_t)(\omega) = g(\theta_t\omega)$. Then the Markov property of $v(t)$ is expressed by the formula $E_i[g\circ\theta_t \mid F_t](\omega) = E_{v(t,\omega)}[g]$ for almost all $\omega[P_i]$. We can omit ω and write simply $E_i[g\circ\theta_t \mid F_t](\omega) = E_{v(t)}[g]$. It is easy to check that $M(t)$ satisfies the multiplicative formula

$$M(t + s) = M(t) \text{ o } \theta_s M(s).$$

As a result for fixed i, we have

$$\left(\tilde{T}(t+s)\tilde{f}\right)_i = E_i[M(t+s)f]$$

$$= E_i\left[E_i[M(t+s)f \mid F_s]\right]$$

$$= E_i\left[E_i[M(t) \circ \theta_s\ M(s)\ f_{v(t+s)} \mid F_s]\right]$$

$$= E_i\left[M(t) \circ \theta_s\ E_i[M(s)\ f_{v(t+s)} \mid F_s] \mid F_s\right]$$

(by Lemma 2.10 and the fact that $M(t)$ o θ_s depends on F_u, $u \geq s$)

$$= E_{v(s)}\left[M(t) E_i[M(s)f]\right]$$

(by the Markov property of ν)

$$= \left(\tilde{T}(t)\,\tilde{T}(s)\tilde{f}\right)_i \qquad Q.E.D.$$

Theorem 2.13

The Cauchy problem for an unknown vector $\tilde{u}(t)$, $t > 0$,

$$\frac{\partial \tilde{u}_i}{\partial t} = A_i \tilde{u}_i + \sum_{j=1}^{N} q_{ij}\, \tilde{u}_j,\ \tilde{u}(0) = \tilde{f} \tag{2.3}$$

is solved by $\tilde{u}(t) = \tilde{T}(t)\tilde{f}$,

where $\tilde{f} \in \tilde{D} = \left\{\tilde{f} \in \tilde{B}_d : f_j = f,\ 1 \le j \le N,\ f \in \bigcap D_j\right\}.$

Proof

We need to prove that the infinitesimal generator of $\tilde{T}(t)$ is $\tilde{A}$ = diag $(A_1,\ldots,A_n)$ + Q with domain $\tilde{D}$. If τ is the first jump of ν then

$$\left(\tilde{T}(t)\tilde{f}\right)_i = E_i[M(t)f;\tau > t] + E_i\left[M(t)f;\tau\, E_i \ge t\right]$$

$$= T_i(t) f_i P_i(\tau > t) + \int_0^t E_i[M(t)f \mid \tau = r] P_i(\tau \in dr)$$

$$= e^{-q_i t} T_i(t) f_i + \int_0^t E_i\left[E_{\nu(r)}[M(t-r)]T_i(r) f \mid \tau = r\right] P_i(\tau \in dr)$$

$$= e^{-q_i t} T_i(t) f_i + \int_0^t \sum_{j \neq i} E_i[E_j[M(t-r)] T_i(r) f] \left(\frac{q_{ij}}{-q_{ii}} \right) (-q_{ii} p_{ii}(r)) \, dr)$$

$$= e^{-q_i t} T_i(t) f_i + \int_0^t \sum_{j \neq i} \left(\tilde{T}(t-r) T_i(r) \tilde{f} \right)_j q_{ij} p_{ii}(r) \, dr$$

Letting s-lim denote limit in the norm of $\tilde{B}$ we have, for $f_i \in D_i$,

$$(\tilde{A}\tilde{f})_i = s - \lim_{t \searrow 0} \frac{1}{t} \left[\tilde{T}(t) \tilde{f})_i - f_i \right]$$

$$= s \lim_{t \searrow 0} \frac{1}{t} \left[e^{-q_i t} T_i(t) f_i - f_i \right]$$

$$ts - \lim_{t \searrow 0} \frac{1}{t} \int_0^t \sum_{j \neq i} \left(\tilde{T}(t-r) T_i(r) \tilde{f} \right)_j q_{ij} p_{ii}(r) \, dr$$

$$= A_i f_i - q_i f_i + \sum_{j \neq 1} q_{ij}, f_j$$

$$= A_i f_i + \sum_{j = i} q_{ij} f_j \; .$$

By standard semigroup theory we obtain the results of the theorem.

REMARK

The construction of reversed random evolutions settles an issue about the order of the operators in a random evolution (see [3 and 4]).

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APPLICATION OF THE LAPLACE METHOD TO THE ASYMPTOTIC BEHAVIOUR OF THE TAIL PROBABILITIES

Hasan M. Ymeri

ABSTRACT: The object of this paper is to formulate some assertions about the probability tail behaviour of various distributions under large deviations. Here we review some results from Laplace's theory and give its consequences for the theory of large deviations. At the end of the paper we shall discuss an approach to quantil approximation problem via large deviation theory.

KEY WORDS: Large deviation theory, Laplace Theory, quantil approximation.

Procedures using saddlepoint approximations, conjugate distributions or Edgeworth expansions are widely applied both to large deviation theory of probability and to mathematical statistics, [1-4].

Concerning the Laplace method for determining the asymptotic behaviour of so called laplace integrals it seems that its application to the mentioned fields has received considerably smaller attention up to now. Therefore in the present paper we review some results from Lapalce's theory and give its consequences for the theory of large deviations.

To be more precisely, we shall formulate some assertions about the probability tail behaviour of various distributions under large deviations.

Throughout this paper we shall consider laplace integrals

$$F(\lambda) = \int_a^b f(x)\, e^{\lambda S(x)}\, dx,$$

where f and S satisfy the conditions

i) $f, S \in \mathbb{C}([a, b])$;

ii) $\max\{S(x) \mid x \in [a, b]\}$ is attained only at the point $x = a$;

iii) $f, S \in \mathbb{C}^\infty([a, a + \in])$ for some $\in > 0$, and $S'(a) \neq 0$;

iv) $\int_a^b |f(x)|\, e^{\lambda_o S(x)}\, dx < \infty$, for some $\lambda_o > 0$.

Theorem 1

The following asymptotic expansion formula and representation formula are true.

$$F(\lambda) \approx e^{\lambda S(a)} \sum_{k=0}^{\infty} C_k\, \lambda^{-k-1} \quad \text{as} \to \infty \tag{1}$$

and $$F(\lambda) \sim -\frac{f(a)\, e^{\lambda S(a)}}{\lambda\, S'(a)} \quad \text{as} \to \infty \tag{2}$$

where $$c_k = (-1)^{k+1} M^k \left(\frac{f(x)}{S^i(x)}\right)\Big|_{x=a}$$

Remarks

1. Relation (1) means that for arbitrary n

$$\lim_{\lambda \to \infty} \lambda^{n+1} \left[F(\lambda)/e^{\lambda S(a)} - \sum_{k=0}^{n} c_k\, \lambda^{-k-1} \right] = 0$$

2. Relation (2) means that

$$\lim_{\lambda \to \infty} F(\lambda) \frac{\lambda\, S'(a)}{f(a)\; e^{S(a)\lambda}} = -1$$

3. M^K denotes the k-th power of the differential operator

$$M = \frac{1}{S'(x)} \frac{d}{dx}$$

Now, let (Ω, α, P) be a probability space and $\underline{x}$ a random variable defined on it. If $\underline{x}$ is generalized Gamma distributed with parameters $\alpha > 0$, $b > 0$ and $p > 0$ then the asymptotic $(\lambda \to \infty)$ behaviour of the tail probability

$$P(\underline{x} \geq \lambda) = \int_{\lambda}^{\infty} \frac{\alpha\; b^{p/\alpha}}{\Gamma(p/\alpha)}\, x^{p-1}\, e^{-bx^{\alpha}}\, dx, \qquad \lambda > 0 \tag{3}$$

can be determined by applying Theorem 1 with functions

$$f(x) = x^{p-1}$$

and

$$S(x) = -bx^{\alpha}$$

The followings Theorem summarizes the respective conclusions.

Theorem 2

The asymptotic behaviour of the tail probabilities (3) is described by the following asymptotic representation formula

$$P(\underline{x} \geq \lambda) \sim \frac{b^{p/\alpha - 1}}{\Gamma(p/\alpha)}\, \lambda^{p-\alpha}\, e^{-b\lambda^{\alpha}} \qquad \lambda \to \infty \tag{4}$$

and the asymptotic expansion formula

$$P(\underline{x} \geq \lambda) \approx \frac{\alpha\, b^{p/\alpha}}{\Gamma(p/\alpha)} \lambda^{p} e^{-b\lambda^{\alpha}} \sum_{k=0}^{\infty} c_k \lambda^{-(k+1)\alpha} \qquad \text{as } \lambda \to \infty \tag{5}$$

where one has to take

$$c_o = (\alpha\, b)^{-1},$$

$$c_k = (\alpha\, b)^{-k-1} \prod_{i=1}^{k} (p - i\,\alpha),\ k = 1,2,...$$

Remarks

4. If p/α is an integer then the series in (5) will be only a finite sum taking into account terms up to the summation index $k = p/\alpha - 1$.

5. Although the Gaussian distribution is not really a special case of the generalized Gamma distribution it is possible to apply Theorem 2 to the study of the asymptotic tail behaviour of the Gaussian distribution too as will be shown in formulas (6) and (7).

Proof

Replacing x by λx in (3) we get

$$P(\underline{x} \geq \lambda) = \frac{\alpha\, b^{p/\alpha}}{\Gamma(p/\alpha)} \lambda^{p} F(\lambda^{\alpha})$$

where $\quad F(\lambda^{\alpha}) = \int_{1}^{\infty} x^{p-1} e^{-b\lambda^{\alpha}x^{\alpha}} dx$

is a Laplace integral with parameter λ^{α} instead of λ and the same functions f and S as above.

The maximum of the function $S(x) = -bx^\alpha$ for $x \in [1,\infty)$ is attained only at the point $x = 1$.

Obviously, conditions (i) and (iii) are fulfilled, too. Choosing $\lambda_o = 1$, one gets:

$$\int_1^\infty x^{p-1} e^{-b\lambda_o^\alpha x^\alpha} dx \int_0^\infty x^{p-1} e^{-bx^\alpha} dx = b^{-p/\alpha} 1/\alpha \, \Gamma(p/\alpha) < \infty$$

Hence, all assumptions of Theorem 1 are fulfilled and we have the asymptotic representation formula

$$P(x \geq \lambda) \sim \frac{b^{p/\alpha-1}}{\Gamma(p/\alpha)} \lambda^{p-\alpha} e^{-b\lambda^\alpha} \qquad \text{as} \qquad \lambda \to \infty,$$

In order to determine the coefficients c_k of the asymptotic expansion formula (5) we have to consider

$$(-1)^{k+1} \left(\frac{1}{-\alpha b x^{\alpha-1}} \frac{d}{dx}\right)^k \left(\frac{x^{p-1}}{-\alpha b x^{\alpha-1}}\right) \Big|_{x=1}$$

$$= (\alpha b)^{-k-1} \left(\frac{1}{x^{\alpha-1}} \frac{d}{dx}\right)^k (x^{p-\alpha}) \Big|_{x=1}$$

It is easy to deduce by mathematical induction that

$$c_o = (\alpha b)^{-1},$$

$$c_k = (\alpha b)^{-k-1} \prod_{i=1}^{k} (p - i\alpha), \; k = 1,2,...$$

Taking into account the factor $\lambda^p \, \alpha b^{p/\alpha}/\Gamma(p/\alpha)$ we get the asymptotic expansion formula (5).

We shall formulate now some special conclusions from Theorem 2.

Let $\underline{x}$ be a standard Gaussian random variable. Then the asymptotic behaviour of the tail probabilities

$$P(\underline{x} \geq \lambda) = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{\lambda}^{\infty} e^{-x^2/2}\, dx$$

can be obtained applying Theorem 2 with parameters $\alpha = 2$, $b = 1/2$ and $p = 1$.

We get the wellknown formulas

$$P(\underline{x} \geq \lambda) \sim \frac{1}{\sqrt{2\pi}\,\lambda}\, e^{-\lambda^2/2} \qquad \text{as} \quad \lambda \to \infty \tag{6}$$

and

$$P(\underline{x} \geq \lambda) \approx \frac{1}{\sqrt{2\pi}\,\lambda}\, e^{-\lambda^2/2}\, x$$

$$x \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!}{2^{k-1}\,(k-1)!}\, \lambda^{-2k}, \qquad \lambda \to \infty \tag{7}$$

Putting $\alpha = 1$, we have a Gamma distributed random variable with the parameters $b > 0$ and $p > 0$. We get the asymptotic representation formula

$$P(\underline{x} \geq \lambda) \sim \frac{1}{\Gamma(p)}\, (b\lambda)^{p-1}\, e^{-b\lambda}, \qquad \lambda \to \infty \tag{8}$$

We determine the coefficients c_k now. In the case of p being not integer we make use of the relation

$$\Gamma(x+1) = x\Gamma(x),\ x \in \mathbb{R},\ x \neq -1, -2, \ldots$$

Then the coefficients can be written in the form

$$c_k = b^{-k-1} \frac{\Gamma(p)}{\Gamma(p-k)}$$

If p is an integer we get the same result putting

$$\frac{\Gamma(p)}{\Gamma(p-k)} = 0 \quad \text{for} \quad k > p-1$$

Therefore we have the asymptotic expansion formula

$$P(\underline{x} \geq \lambda) \approx (b\lambda)^{p-1} e^{-b\lambda} \sum_{k=0}^{\infty} \left[(b\lambda)^k \Gamma(p-k)\right]^{-1}, \quad \lambda \to \infty \tag{9}$$

where in the case of an integer p the series is defined to be only finite sum taking into account terms up to the summation index $k = p^{-1}$.

For another application of the laplace method to large deviation theory let $\underline{x}$ be a Weibull distributed random variable with parameters $b > 0$ and $p > 0$. Then $(\alpha = P)$

$$P(x \geq \lambda) \sim e^{-b\lambda p}, \qquad \lambda \to \infty \tag{10}$$

and in the asymptotic expansion formula (5) the coefficients c_k are equal to zero for $k \in \{1, 2, ...\}$.

Let us return now to the case of a standard Gaussian random variable $x.z_{1-\alpha}$ denotes the 1-α quantil of the distribution function of $\underline{x}$, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_1-\alpha} e^{-x^2/2}\, dx = 1 - \alpha$$

Obviously, The quantil $z_{1-\alpha}$ tends to infinity whenever α tends to zero. Hence, for sufficiently small $\alpha > 0$ the solution $\lambda = z^*_{1-\alpha}$ of the equation

$$\alpha = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2}$$

can be used as an approximation to $z_{1-\alpha}$. Let $\lambda \geq 1$. Then

$$\alpha \leq (2\pi)^{-1/2} \exp\{-\lambda^2/2\}$$

and $$\lambda \leq (-2 \ln \alpha - \ln (2\pi))^{1/2}$$

This yields

$$\alpha \geq [(2\pi)(-2 \ln \alpha - \ln (2\pi))]^{-1/2} \exp\{-\lambda^2/2\}$$

and

$$\lambda \geq \{-2 \ln \alpha - \ln (2\pi) - \ln (-2 \ln \alpha)\}^{1/2}$$

so that

$$z^*_{1-\alpha} \sim \{-2 \ln \alpha - \ln (2\pi)\}^{1/2} \qquad \text{as } \alpha \to 0$$

However, $\{-2 \ln \alpha - \ln (2\pi)\}^{1/2}$ is still a quite crude approximation to $z_{1-\alpha}$ in statistical relevant cases of α. A numerical study shows that

$$\{-2 \ln \alpha - \ln (2\pi) - \ln (-2 \ln \alpha)\}^{1/2}$$

is a better approximation to $z_{1-\alpha}$ and differs from it not more than 0.12 if $\alpha \leq 0.05$ and not more than 0.022 if $\alpha \leq 0.0001$.

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SOME NEW ASPECT IN THE THEORY OF DISTRIBUTIONS

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ABSTRACT: In this paper, the problem of an alternative definition of distribution x_x^{-r} using the locally summable function $\ln x_+$ is discussed. Although the distribution $x_+^{-r} \ln^r x_+$ is considered as a single entity and not as a product of the distribution x_+^{-r} and the locally summable function $\ln^r x_+$, our results show us that differentiation of $x_+^{-r} \ln^r x_+$ acts as if it were such a product.

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KEY WORDS: Alternative definition of distribution, Locally summable function.

For $\lambda > -1$, the distribution x_+^λ is the locally summable function defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x>0, \\ 0, & x<0, \end{cases}$$

When $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$, the distribution x_+^λ is defined inductively by the equation

$$\left(x_+^{\lambda+1}\right)' = (\lambda + 1)\, x_+^\lambda$$

It follows that if $-r-1 < \lambda < -r$, then

$$\langle x_+^\lambda , \Phi(x) \rangle = \int_0^\infty x^\lambda \left[\Phi(x) - \sum_{i=0}^{r-1} \frac{\Phi^{(i)}(0)}{i!} x^i \right] dx$$

$$= \int_0^\infty x^\lambda \left[\Phi(x) - \sum_{i=0}^{r-2} \frac{\Phi^{(i)}(0)}{i!} x^i - \frac{\Phi^{r-1}(0)}{(r-1)!} x\, H(1-x)\, x^{r-1} \right] dx$$

$$+ \frac{\Phi^{(r-1)}(0)}{(r-1)!\,(\lambda+r)^1}$$

for arbitrary test function Φ in the space D of infinitely differentiable functions with compact support, where H denotes Heaviside's function.

Note that if $r = 1$, then $\sum_{i=0}^{-1}$ is understood to mean an empty sum.

Gelfand and Shilov [2] define the distribution $F_{-r}(x_+, \lambda)$, when $-r-1 < \lambda < -r$, by the equation

$$< F_{-r}(x_+, \lambda),\ \Phi(x) >$$

$$= \int_0^\infty x^\lambda \left[\Phi(x) - \sum_{i=0}^{r-2} \frac{\Phi^i(0)}{i!} x^i - \frac{\Phi^{r-1}(0)}{(r-1)!} H(1-x)\, x^{r-1} \right] dx,$$

for arbitrary Φ in D.

They then define the distribution x_+^{-r} by

$$x_+^{-r} = F_{-r}(x_+, -r) \tag{2}$$

for $r = 1,2,\ldots$. We will denote $F_{-r}(x_+, -r)$ simply by $F(x_+, -r)$ and it follows easily that

$$\frac{d}{dx}F(x_+,-r) = -rF(x_+,-r-1)+\frac{(-1)^r}{r!}\delta^{(r)}(x)$$

Thus with x_+^{-r} defined by equation (2), equation (1) is not satisfied with $r = -2,-3,\ldots$.

This seems to be rather unfortunate and so an alternative definition of x_+^{-r} was given in [1] by letting $\ln x_+$ be the locally summable function defined by

$$\ln x_+ = \begin{Bmatrix} \ln x, & x>0 \\ 0, & x<0 \end{Bmatrix}$$

then defining x_+^{-1} by the equation

$$(\ln x_+)' = x_+^{-1} \tag{3}$$

and more generally defining x_+^{-r} inductively by the equation

$$\left(x_+^{-r+1}\right)' = -(r-1)\, x_+^{-r} \tag{4}$$

for $r = 2,3,\ldots$. With this definition of x_+^{-r}, equation (1) is then satisfied for all values of λ.

It can be proved easily that

$$x_+^{-1} = F(x_+,\ -1)$$

and it then follows by induction that

$$x_+^{-r} = F(x_+,-r)+\frac{(-1)^r}{(r-1)!}\Psi(r-1)\,\delta^{(r-1)}(x) \tag{5}$$

for $r = 1,2,\ldots$, where

$$\Psi(r) = \begin{cases} 0, & r=0 \\ \sum_{i=1}^{r} 1/i, & r \geq 1 \end{cases}$$

The distribution $x_+^{\lambda} \ln^s x_+$ is defined by

$$\frac{\partial^s}{\partial \lambda^s} x_+^{\lambda} = x_+^{\lambda} \ln^s x_+$$

for $\lambda \neq -1, -2, \ldots$ and $s = 1, 2, \ldots$. Then $x_+^{\lambda} \ln^s x_+$ is locally summable function for $\lambda > -1$ and

$$\left\langle x_+^{\lambda} \ln^s x_+, \Phi \right\rangle = \int_0^{\infty} x^{\lambda} \ln^s x \left[\Phi(x) - \sum_{i=0}^{r-1} \frac{\Phi^{(i)}(O)}{i!} x^i \right] dx$$

$$= \int_0^{\infty} x^{\lambda} \ln^s x \left[\Phi(x) - \sum_{i=0}^{r-2} \frac{\Phi^{(i)}(O)}{i!} x^i - \frac{\Phi^{r-1}(O)}{(r-1)!} \times H(1-x)\, x^{r-1} \right] dx$$

$$+ \frac{(-1)^s\, s!\, \Phi^{(r-1)}(O)}{(r-1)!\,(\lambda + r)^{s+1}}$$

for $-r-1 < \lambda < -r$, $s = 1, 2, \ldots$. and arbitrary Φ in D.

It follows easily from the definition that

$$\left(x_+^{\lambda} \ln^s x_+\right)' = \lambda x_+^{\lambda-1} \ln^s x_+ + s x_+^{\lambda-1} \ln^{s-1} x_+ \qquad (6)$$

for $\lambda \neq -1, -2, \ldots$. and $s = 0, 1, 2, \ldots$. Although the distribution $x_+^{\lambda} \ln^s x_+$ is considered as a single entity and not as a product of the distribution x_+^{λ} and the locally summable function $\ln^s x_+$, equation (6) shows us that differentiation of $x_+^{\lambda} \ln^s x_+$ acts as if it were such a product.

Now we consider the problem of defining $x_+^{-r} \ln^s x_+$ so that equation (6) is satisfied for all λ and $s = 0,1,2,\ldots$. Gelfand and Shilov [2] define $x_+^{-r} \ln^s x_+$ by equation

$$\frac{\partial^s}{\partial \lambda^s} F_{-r}(x_+, \lambda)\,|_{\lambda=-r} = x_+^{-r} \ln^s x_+$$

for $r,s = 1,2,\ldots$. From now on, we will denote this distribution by

$$F(x_+, -r) \ln^s x_+$$

so that $\langle F(X_+, -r) \ln^s x_+, \Phi(x) \rangle$

$$= \int_0^\infty x^{-r} \ln^s x \left[\Phi(x) - \sum_{i=0}^{r-2} \frac{\Phi^{(i)}(O)}{i!} x^i - \frac{\Phi^{(r-1)}(O)}{(r-1)!} H(1-x)\, x^{r-1} \right] dx$$

for arbitrary Φ in D.

Theorem 1

$$[F(x_+, -r) \ln^s x_+]' = -rF(x_+, -r-1) \ln^s x_+ + sF(x_+, -r-1) \ln^{s-1} x_+$$

for $r,s = 1,2,\ldots$.

Proof

For arbitrary Φ in D we have

$$\langle [F(x_+, -r) \ln^s x_+]', \Phi(x) \rangle = -\langle F(x_+, -r) \ln^s x_+, \Phi'(x) \rangle$$

$$= \int_0^\infty x^{-r} \ln^s x \left[\Phi^1(x) - \sum_{i=0}^{r-2} \frac{\Phi^{(i+1)}(O)}{i!} x^i - \frac{\Phi^r(O)}{(r-1)!} \times H(1-x) x^{r-1} \right] dx$$

$$= \int_0^\infty x^{-r-1} \ln^{s-1} x(-r \ln x+s)\left[\Phi(x) - \sum_{i=0}^{r-1} \frac{\Phi^{(i)}(O)}{i!}x^i - \frac{\Phi^{(r)}(O)}{r!}H(1-x)x^r\right] dx$$

$$= \left\langle -rF(x_+,-r-1) \ln^s x_+ + sF(x_+,-r-1) \ln^{s-1} x_+,\ \Phi(x)\right\rangle$$

for $r,s = 1,2,\ldots$. The result of the theorem follows.

It follows from the theorem that with Gelfand and Shilov's definition of the distribution $x_+^{-r} \ln^s x_+$, equation (6) is satisfied for all λ and $s = 1,2,\ldots$, even though it is not satisfied for $r = -1,-2,\ldots$ when $s = 0$.

In order to define $x_+^{-r} \ln^s x_+$ so that equation (6) is satisfied for all λ and $s = 0,1,2,\ldots$, we first of all define $x_+^{-r} \ln^s x_+$, by the equation

$$\left(\ln^{s+1} x_+\right)^1 = (s+1)\ x_r^{-1} \ln^s x_+$$

for $s = 0,1,2,\ldots$, so that equation (6) is satisfied with $\lambda = 0$ and $s = 1,2,\ldots$.

Theorem 2

$$x_+^{-1} \ln^s x_+ = F(x_+, -1) \ln^s x_+$$

for $s = 0,1,2,\ldots$.

Proof

We have

$$(s+1)\left\langle x_r^{-1} \ln^s x_r, \Phi(x)\right\rangle = -\left\langle \ln^{s+1} x_+, \Phi^1(x)\right\rangle$$

$$= -\int_0^1 \ln^s xd\,[\Phi(x) - \Phi(O)] - \int_1^\infty \ln^{s+1} xd\ \Phi(x)$$

$$= (s+1) \int_0^\infty x^{-1} \ln^s x[\Phi(x) - \Phi(O)H(1-x)]\, dx$$

$$= (s+1) \left\langle F(x_+,-1) \ln^s x_+,\ \Phi(x)\right\rangle$$

for $s = 0,1,2,\ldots$ and arbitrary Φ in D. The result of the theorem follows.

More generally we now define $x_+^{-r} \ln^s x_+$ by the equation

$$x_r^{-r} \ln^s x_+ = F(x_+,-r) \ln^s x_+ + \frac{(-1)}{(r-1)!} \Psi_o(r-1)\, \delta^{r-1}(x)$$

for $r,s = 1,2\ldots$, where

$$\Psi_s(r) = \begin{Bmatrix} 0, & r = 0 \\ s\sum_{i=1}^{r} \dfrac{\Psi_{s-1}(i)}{i}, & r \geq 1 \end{Bmatrix}$$

for $s = 1,2,\ldots$, with particular case $\Psi_o(r)$ being equal to $\Psi(r)$ defined above. Note that in the particular case $r = 1$, $x_+^{-1} \ln^s x_+$ is in agreement with Theorem 2.

Theorem 3

$$\left(x_+^{-r} \ln^s x_+\right)^1 = -rx_+^{-r-1} \ln^s x_+ + sx_+^{-r-1} \ln^{s-1} x_+$$

for $r,s = 1,2,\ldots$.

Proof

Using the definition of $x_+^{-r} \ln^s x_+$ and Theorem 1 we have

$$\left(x_+^{-r} \ln^s x_+\right)^1 = -rF\left(x_+, -r-1\right) \ln^s x_+ + sF\left(x_+, -r-1\right) \times$$

$$\times \ln^{s-1} x + \frac{(-1)^r}{(r-1)!} \Psi_s(r-1)\, \delta^{(r)}(x) =$$

$$= -r\left[F\left(x_+, -r-1\right) \ln^s x_+ + \frac{(-1)^{r+1}}{r!} \Psi_s(r)\,\delta^{(s)}(s)\right] +$$

$$+ s\left[F\left(x_+, -r-1\right) \ln^{s-1} x_+ + \frac{(-1)^{r+1}}{r!} \times \Psi_{s-1}(r)\right] \delta^{(r)}(x) =$$

$$= -rx_+^{-r-1} \ln^s x_+ + sx_+^{-r-1} \ln^{s-1} x_+.$$

for $r,s = 1,2,\ldots$

It follows that with this definition of $x_+^{-r} \ln^s x_+$, equation (6) is satisfied for all λ and $s = 0,1,2,\ldots$.

The distribution $x_-^{\lambda} \ln^s x_-$ is defined by replacing x by $-x$ in the distribution $x_+^{\lambda} \ln^s x_+$ for $\lambda \neq -1, -2, \ldots$ and $s = 0,1,2,\ldots$ and the distribution $F(x_-, -r) \ln^s x_-$ is defined by replacing x by $-x$ in the distribution $F(x_+, -r) \ln^s x_+$ for $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$. We therefore define the distribution $x_-^{-r} \ln^s x_-$ by replacing x by $-x$ in the distribution $x_+^{-r} \ln^s x_-$ for $r = 1,2,$ and $s = 0,1,2,\ldots$. It follows that

$$x_-^{-r} \ln^s x_- = F\left(x_-, -r\right) \ln^s x_- - \frac{1}{(r-1)!} \Psi_s(r-1)\,\delta^{r-1}(x)$$

for $r = 1,2, \ldots$ and $s = 0,1,2,\ldots$. and that

$$(x_-^{\lambda} \ln^s x_-^{1} = -\lambda x_-^{\lambda-1} \ln^s x_- - sx_-^{-\lambda-1} \ln^{s-1} x_-$$

for all λ and $s = 0,1,2,\ldots$.

We finally define the distribution $x^{-r} \ln^s |x|$ by

$$x^{-r} \ln^s |x| = x_+^{-r} \ln^s x_+ + (-1)^r x_-^{-r} \ln^s x_-$$

for $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$. It follows that

$$x^{-s} \ln^s |x| = F(x_+, -r) \ln^s x_+ + (-1)^r F(x_-, -r) \ln^s x_-$$

so that this definition of $x^{-r} \ln^s |x|$ is in agreement with Gelfand and Shilov's definition. We then of course have

$$(x^{-r} \ln^s |x|)' = -rx^{-r-1} \ln^s |x| + sx^{-r-1} \ln^{s-1} |x|$$

for $r = 1,2,\ldots$ and $s = 0,1,2,\ldots$.

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ON DIVERGENT SERIES WITH NON-INTEGER PARTIAL SUMS

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1. INTRODUCTION

It is a well known fact that the nth partial sum of the harmonic series never equals an integer when $n > 1$. This is a surprising property considering that these partial sums grow unboundedly with increasing n. In view of such a result one may naturally question whether there exist other such positive-rational-termed divergent series having non-integer partial sums. In this paper we shall answer in the affirmative, by considering a family of series whose terms are formed from the integer multiples of the reciprocals of generalized Fibonacci sequences given by.

$$U_n = PU_{n-1} - QU_{n-2},$$

where $(P,Q) = 1$ with $U_0 = 0$, $U_1 = 1$. For our purposes the choice of such a class of sequences is entirely a reasonable one, as the resulting family of series contains as a special case the harmonic series (i.e. when $(P,Q) = (2,1)$). In order to achieve the main goal of this paper it will first be necessary, in Section 2, to extend the above result concerning the partial sums of the harmonic series, so as to include the case of integer multiples of this series. In particular, we shall demonstrate that for any sequence of integers $\{b_n\}$ in which for every prime $(b_p,p) = 1$, the corresponding partial sums $\sum_{r=1}^{n} \frac{b_r}{r}$, are non-integer for $n > 1$. By modifying the proof of this result, it will then be possible to establish a similar conclusion (see Theorem 3.1) for the partial sums of these series

$$\sum_{n=1}^{\infty} \frac{b_n}{U_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where U_n is generated with respect to a relatively prime pair (P,Q) which satisfy either $|P| > Q>0$ or $P \neq 0$, $Q < 0$ and the sequence $\{b_n\}$ is such that $(b_p, U_p) = 1$ for every prime p. The required divergent series can be obtained from this family by imposing an additional assumption upon the sequence $\{b_n\}$. As we shall see, one of the main difficulties with establishing Theorem 3.1 lies in finding the values that P and Q may assume in order that $U_n \neq 0$ for $n > 1$. By employing some well known results on generalized Fibonacci sequences, we will show that the only relatively prime pairs (P,Q), in which $U_n = 0$ for some $n \geq 1$, are $(-1,1)$ and $(1,1)$. Consequently, the above restrictions on P and Q have been chosen so as to avoid such problem values. In addition to these results, we shall further demonstrate in Section 3 that the partial sums of (1) remain non-integer, when the terms of the series are "thinned out", by summing over index values which are either odd or a multiple of a fixed integer.

2. HARMONIC CASE

The argument upon which we shall establish the non-integer status of the partial sums (denoted S_n) in this and the following section, will be based on the simple criterion of divisibility. As S_n can be reduced to an expression involving a single fraction, with denominator equal to either $n!$ or $U_1U_2 \ldots U_n$, our task will be to produce a factor of these denominators which fails to be a divisor of the corresponding numerators. Once such a factor has been found, we can then conclude that S_n is non-integer. To this end, we require the following technical lemma.

Lemma 2.1

Suppose p_1 and p_2 are consecutive primes. If the integer $m \neq p_1$ is such that $1 \leq m \leq p_2$, then $(p_1, m) = 1$.

Proof

We need only consider those integer $m \in (p_1, p_2)$. The result holds trivially if p_1 and p_2 are twin primes, as consecutive integers are relatively prime. Suppose $p_1 + 1 < p_2$ and assume m is an arbitrary composite number satisfying $p_1 < m < p_2$ with $(p_1, m) \neq 1$. Clearly then $m \geq 2p_1$, however by Chebychev's theorem there exists a prime $q \in (p_1, 2p_1)$. Hence there must exist another prime q such that $p_1 < q < p_2$, but this is a contradiction as p_1 and p_2 are consecutive primes. Consequently $(p_1,m) = 1$ for all integers between p_1 and p_2.

We are now in a position to establish the non-integer status of the partial sums of those series formed from the integer multiples of the Harmonic series. It should be noted, that an alternate proof exists for the case $b_n = 1$ of Theorem 2.1, which is based on an argument involving the occurrence of the highest power of 2 in the sequence 1, 2,...,n (see [3, p.176]). This argument however cannot readily be modified to prove the following more general result.

Theorem 2.1

Suppose $\{b_n\}$ is a sequence of integers with the properly that $(p,b_p) = 1$ for every prime p, then the n-th partial sum of the series $\sum_{n=1}^{\infty} \frac{b_n}{n}$ is never an integer for $n > 1$.

Proof

If $n = 2$ then the result holds trivially since by assumption, 2 does not divide b_2 and so $S_2 = b_1/1 + b_2/2$ cannot be an integer. Assume now $n > 2$ and consider

$$S_n = \sum_{r=1}^{n} \frac{b_r}{r} = \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n!} = \frac{M_n}{n!} \qquad \ldots\ldots (2)$$

where $a_m = n!/m$ for $m = 1,2\ldots,n$. Let p be the largest prime strictly less than n. Clearly $p \mid a_m$ for all $m \neq p$, since p is present as a factor in each such terms a_m. Now, by definition $p < n \leq p'$ where p' is the next consecutive prime to p, so by Lemma 2.1 $(p,m) = 1$ for all $m \in \{1,2,\ldots,n\}\backslash\{p\}$. This in turn implies via hypothesis that p cannot divide the integer $a_p b_p$. Thus we have produced a factor of $n!$ which fails to divide into the numerator M_n of (2). Consequently $M_n/n!$ cannot be an integer.

3. GENERALIZED FIBONACCI CASE

In this section we shall extend Theorem 2.1 by establishing a similar result for those series of the form in (1) where U_n is a generalized Fibonacci sequence. From this class of series, it then will be possible to construct the desired divergent series. The key to generalizing Theorem 2.1 lies with employing a well-known result of generalized Fibonacci sequences (see Theorem VI of [1]), which states that the greatest common divisor of any two terms U_m, U_n in a given sequence is equal to the term in the sequence having index (m,n), that is $(U_m, U_n) = U_{(m,n)}$. By combining this property with Lemma 2.1, we will show that U_p (where p is the largest prime less than n) is the required factor of the denominator in the fractional expression for S_n which allows us as before, to deduce the non-integer status of S_n. However, before reaching this stage we must first ensure that the partial sums of these series are well-defined, by choosing P and Q so that $U_n \neq 0$ for $n > 1$. To achieve this end let us momentarily digress, by introducing a well-known algebraic formula for U_n and its companion sequence V_n, the so-called generalized Lucas numbers.

Remark 3.1

As U_n and V_n are defined by the following second-order linear recurrence relations

$$U_n = PU_{n-1} - QU_{n-2} \text{ and } V_n = PV_{n-1} - QV_{n-2},$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = P$, *it can easily be shown that*

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where α, β are the roots of $x^2 - Px + Q = 0$ when the relatively prime pair (P,Q) are such that the discriminant $\Delta = P^2 - 4Q \neq 0$. Clearly (α,β) form either a complex conjugate or a real distinct pair of roots for the above quadratic. It should however be noted, that the only time when $\Delta = 0$ for a $(P,Q) = 1$ occurs when $(P,Q) = (2,1)$ or $(P,Q) = (-2,1)$, which correspond to the cases $U_n = n$ and $U_n = (-1)^{n+1} n$ respectively.

With the above expression for U_n in hand, it is clear that $U_m = 0$ for some $m > 1$ if and only if $\alpha^m = \beta^m$. Via this equality, we shall argue that if a sequence $\{U_n\}$ contains a zero for $n > 1$, then necessarily the ordered pair (P,Q) can only assume either $(-1,1)$ or $(1,1)$. To facilitate our argument, it will be necessary to make use of the following well-known result of algebra concerning symmetric functions, a proof of which is not included-however, interested readers can refer to Theorem 1.9 o f[2].

Lemma 3.1

Let R be a ring. Then every symmetric polynomial in $R[x_1,\ldots,x_n]$ is expressible as polynomial in the variables $\sigma_1,\ldots,\sigma_n$ over R, where σ_i is the coefficient of x^{n-i} in the monic polynomial

$$\phi(x) = (x - x_1) \ldots (x - x_n).$$

The proof of the following result was partly motivated from the discussion following Theorem 1 of [1].

Proposition 3.1

If the generalized Fibonacci sequence $\{U_n\}$ is generated with respect to the relatively prime pair $(P,Q) \neq (1,1), (-1,1)$ then $U_n \neq 0$ for all $n \geq 1$.

Proof

We establish the contrapositive. Suppose there exists an $m > 1$ such that $U_m = 0$, then by Remark 3.1 $\alpha^m = \beta^m$, where α, β are the roots of $x^2 - Px + Q = 0$. Consider now $V_m = 2\alpha^m$ and $Q^m = \alpha^{2m}$, we claim that $(V_m, Q^m) = 1$, this will be demonstrated later. Note that as both $2\alpha^m$ and α^{2m} are integer, we must have $\alpha^m \in \mathbf{Z}\backslash\{0\}$. Consequently if $(2\alpha^m, \alpha^{2m}) = 1$ then $\alpha^m = \beta^m = \pm 1$, that is, both α and β are then m-th roots of unity. We consider now the following two possible cases.

Case 1: $\Delta = P^2 - 4Q < 0$

In this instance (α,β) are a complex conjugate pair given by

$$\alpha = \frac{P + i\sqrt{4Q - P^2}}{2} \quad \text{and} \quad \beta = \frac{P - i\sqrt{4Q - P^2}}{2}$$

Since α,β are the m-th roots of unity they must necessarily have their complex moduli equal to one, thus $Q = |\alpha|^2 = |\beta|^2 = 1$. But as $P^2 < 4Q = 4$, it is immediately apparent that P can only assume the values 1 or -1.

Case 2: $\Delta = P^2 - 4Q > 0$

If α,β are distinct reals which are the m-th roots of unity, then we can only have $(\alpha,\beta) = (-1,1)$ or $(1,-1)$. In any event $P = \alpha + \beta = 0$ and $Q = \alpha\beta = -1$.

Hence from these cases we conclude, as $P \neq 0$, that if a zero occurs in a given generalized Fibonacci sequence $\{U_n\}$, then the ordered pair (P, Q) can only assume the values $(-1,1)$ or $(1,1)$. To complete the argument let us now validate the claim made earlier. It will suffice to prove that $(V_n, Q) = 1$ for every $n \geq 1$. To this end consider the following binomial expansion

$$(\alpha + \beta)^n = \alpha^n + \beta^n + \alpha\beta I_n(\alpha,\beta) = V_n + \alpha\beta I_n(\alpha,\beta). \quad \ldots (3)$$

Clearly $I_n(\alpha,\beta)$ is a symmetric polynomial in $Z[\alpha,\beta]$. Consequently, by Lemma 3.1, $I_n(\alpha,\beta)$ can be expressed as a polynomial in $\sigma_1 = P$ and $\sigma_2 = Q$ over the ring $R = Z$. As a result $I_n(\alpha,\beta)$ must be an integer for all $n \geq 1$. Hence from (3) we conclude as P is relatively prime to Q, that $(V_n, Q) = 1$ for all $n \geq 1$.

With the above result established, we now have only one more obstacle to overcome, before reaching the main theorem of this section. In modifying the proof of Theorem 2.1, our argument will become invalid if $U_p = \pm 1$ for any prime p, as this factor will divide into the numerator, of the fractional expression for S_n. One approach to avoiding this problem, is to find P and Q which give rise to sequences $\{U_n\}$ whose absolute terms are monotonically increasing, since then $|U_{n+1}| > |U_n| > 1$ for $n \geq 1$. Fortunately, there is an abundance of such P and Q, to see this we first must consider the following result.

Lemma 3.2

Consider the sequence $\{U_n\}$ generated with respect to the relatively prime pair (P,Q). If $\tilde{U}_n$ is another generalized Fibonacci sequence with $(\tilde{P},\tilde{Q}) = (-P,Q)$ then $\tilde{U}_n = (-1)^{n+1} U_n$.

Proof

We argue using induction. Clearly $\tilde{U}_1 = 1 = (-1)^2 U_1$, thus suppose the result holds for all integers $n \leq m$. Then

$$\begin{aligned} \tilde{U}_{m+1} &= -P(-1)^{m+4} U_m - Q(-1)^m U_{m-1} \\ &= P(-1)^{m+2} U_m - Q(-1)^{m+2} U_{m+1} \\ &= (-1)^{m+2} \{PU_m - QU_{m-1}\} \end{aligned}$$

Hence $\tilde{U}_{m+1} = (-1)^{m+2} U_{m+1}$ and so the result holds for $n = m + 1$.

Now if $P > 0$ and $Q < 0$ then clearly $U_{n+1} > U_n > 1$ for $n > 1$, while if $\tilde{P} = -P$ and $Q = \tilde{Q}$ then by Lemma 3.2 $|\tilde{U}_n| = U_n$ and so $|\tilde{U}_{n+1}| > |\tilde{U}_n| > 1$ when $n > 1$. For the case $P > Q > 0$ we first prove by induction that $\{U_n\}$ is monotone increasing. Clearly $U_1 - U_0 = 1 > 0$, assuming $U_m - U_{m-1} > 0$ for $m \geq 1$ then

$$\begin{aligned} U_{m+1} - U_m &= (P-1)\,U_m - QU_{m-1} \\ &\geq QU_m - QU_{m-1} \\ &= Q\{U_m - U_{m-1}\} > 0. \end{aligned}$$

Consequently by a similar application of Lemma 3.2 as above, we deduce that $|U_{n+1}| > |U_n| > 1$ when $n > 1$, for those sequences generated with respect to a (P,Q) satisfying $|P| > Q > 0$.

Hence if either $P \neq 0$, $Q < 0$ or $|P| > Q > 0$ then $U_p \neq \pm$ for all prime p.

Remark 3.2

Note that in the case $|Q| > P > 0$ it generally cannot be concluded that $U_p \neq \pm 1$ for every prime p, since there are infinitely many such (P,Q) where $U_3 = 1$. Indeed, since $U_3 = P^2 - Q$, one may set $Q = P^2 - 1$ for any $P \in \mathbf{Z}\backslash\{0,1\}$, with $(P,Q) = 1$ by construction.

As the above restrictions on P and Q exclude the case $(P,Q) = (1,1)$ or $(-1,1)$ and so the possibility of $U_n = 0$ for any $n > 1$, we can now argue in a similar manner as before, to prove the following generalization of Theorem 2.1.

Theorem 3.1

Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to a relatively prime pair (P,Q) which satisfy either $|P| > Q > 0$ or $P \neq 0$, $Q < 0$. If $\{b_n\}$ is a sequence of integers with the property that

$(b_p\,,\ U_p) = 1$ for every prime p, then the n-th partial sum of the series $\sum_{n=1}^{\infty} b_n/U_n$ is never an integer for $n > 1$.

Proof

If $n = 2$ then the result holds trivially since by assumption U_2 cannot divide b_2 and so $S_2 = b_1/U_1 + b_2/U_2$ cannot be an integer. Assume now $n > 2$ and consider

$$S_n = \sum_{r=1}^{n} \frac{b_r}{U_r} = \frac{a_1 b_1 + a_2 b_2 + \cdots a_n b_n}{U_1 U_2 \ldots U_n} = \frac{M_n}{U_1 U_2 \ldots U_n}, \quad \ldots . \quad (4)$$

where $a_m = (U_1 U_2 \ldots U_n)/U_m$ for $m = 1,2,\ldots,n$. Let p be the largest prime strictly less then n. Clearly $U_p | a_m$ for all $m \neq p$, as U_p is present as a factor in each such term a_m. Now be definition $p < n \leq p'$ where p' is the next consecutive prime to p, so by Lemma 2.1.

$$(U_p,\ U_m) = U_{(p,m)} = U_1 = 1,$$

for all $m \in \{1,2,\ldots n\}\backslash\{p\}$. This in turn implies, via the hypothesis that U_p cannot divide $a_p b_p$. Thus we have produced a factor of $U_1\ U_2\ \ldots\ U_n$ which fails to divide into the numerator M_n of (4). Consequently $M_n/(U_1\ U_2\ \ldots\ U_n)$ cannot be an integer.

It is now a simple matter to extract the positive-rational-termed divergent series, having non-integer partial sums. To induce divergence of the series in Theorem 3.1, we shall need to impose an additional assumption on the terms of the sequence $\{b_n\}$ as follows.

Corollary 3.1

Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to a relatively prime pair (P,Q) satisfying $P>Q>0$ or $P>0$, $Q<0$. Let N be any fixed positive integer and $\{b_n\}$ a sequence of positive integers with the property that $(b_p, U_p) = 1$ and $b_p N > U_p$ for every prime p. Then *the n-th partial sum of the divergent series* $\sum_{n=1}^{\infty} \frac{b_n}{U_n}$ *is never an integer for* $n > 1$.

Proof

The terms of the series in question are positive and by Theorem 3.1 have partial sums S_n, that are non-integer for $n > 1$. In addition we clearly must have

$$S_n > \sum_{p \leq n} \frac{b_p}{U_p}$$

However $\frac{b_p}{U_p} > \frac{1}{N} > 0$ and so $b_p/U_p \neq o(1)$ as $p \to \infty$, consequently the sum on the right in the above inequality must grow unboundedly with increasing n. Thus $S_n \to \infty$ as $n \to \infty$.

Remark: 3.3

If we set $N = 1$ *in above, then one could choose* $b_n = U_{n+1}$, *since then* $(b_p\ , U_p) = (U_{p+1}\ , U_p) = U_1 = 1$ *and* $U_{p+1} > U_p$. *Hence, one such divergent series would have partial sums given by*

$$S_n = \frac{U_2}{U_1} + \frac{U_3}{U_2} + \cdots + \frac{U_{n+1}}{U_n},$$

which are non-integer for $n > 1$.

To conclude we now demonstrate that the series in Theorem 3.1 continue to have non-integer partial sums, when summed over index values that are either odd or a multiple of a fixed positive integer.

Corollary 3.2

Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to a relatively prime pair (P,Q) satisfying $|P|>Q>0$ or $P>0$, $Q<0$. If $\{b_n\}$ is a sequence of integers with the property that $(b_p\ ,\ U_p) = 1$ for every prime p. Then the n-th partial sum of the series $\sum_{n=1}^{\infty} b_{2n-1}/U_{2n-1}$ is never an integer for $n > 1$.

Proof

Without loss of generality assume $\{b_n\}$ is sequence with $b_{2m} = U_{2m}$. Then Theorem 3.1 the partial sums

$$S_{2m} = \sum_{r=1}^{m} \frac{b_{2r-1}}{U_{2r-1}} + m \qquad \text{or} \qquad S_{2m-1} = \sum_{r=1}^{m} \frac{b_{2r-1}}{U_{2r-1}} + m - 1,$$

are non-integer for $m > 1$. Thus in either case, the partial sums over odd index values must be non-integer.

To establish a similar conclusion for those summands taken over indexes of the form nr, where r is a fixed integer, it will be necessary to impose an alternate assumption on the sequence $\{b_n\}$ as follows.

Corollary 3.3

Suppose $\{U_n\}$ is as above and let r be a fixed positive integer. Let $\{b_n\}$ be a sequence of integers with the property that $(b_p\ ,\ U_{pr}/U_r) = 1$ for every prime p. Then the n-th partial sum of the series $\sum_{n=1}^{\infty} b_{nr}/U_{nr}$ is never an integer for $n > 1$.

Proof

Set $\tilde{U}_n = U_{nr}/U_r$ noting here that $U_r \neq 0$ as $r \geq 1$. We first show that $\tilde{U}_n$ is a generalized Fibonacci sequence generated with respect to $(\tilde{P},\tilde{Q}) = (v_r, Q^r)$. The following well-known identity

$$U_{nr} = V_r\, U_{(n-1)r} - Q^r\, U_{(n-2)r} \qquad (5)$$

can be verified directly by substituting the expressions for U_n and V_n of Remark 3.1. Dividing both sides of (5) by U_r yields

$$\tilde{U}_n = V_r \tilde{U}_{n-1} - Q^r \tilde{U}_{n-2},$$

in addition we clearly have $\tilde{U}_0 = 0$ and $\tilde{U}_1 = 1$. Consequently $\tilde{U}_n$ is an integer and so $U_r \mid U_{nr}$ for all $n \geq 1$. Furthermore as P and Q are such that the sequence $\{ \mid U_n \mid \}$ is monotone increasing it follows that

$$\left|\frac{U_{(n+1)r}}{U_r}\right| > \left|\frac{U_{nr}}{U_r}\right| > \left|\frac{U_r}{U_r}\right| = 1,$$

for $n > 1$. Hence by assumption and Theorem 3.1 the partial sums

$$S_n = \sum_{m=1}^{n} \frac{b_m}{\tilde{U}_m} = U_r \left(\frac{b_1}{U_{1r}} + \frac{b_2}{U_{2r}} + \cdots + \frac{b_n}{U_{nr}} \right)$$

are non-integer for $n > 1$. The result now readily follows as U_r is an integer.

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BOUNDARY VALUE PROBLEMS FOR RETARDED IMPLICIT DIFFERENTIAL EQUATIONS

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Questions of existence and uniqueness of solutions for systems of retarded implicit differential equations with nonlinear boundary conditions are the subject of our study. Given are quite general sufficient conditions for existence of unique solution of such problems. The solution is given as the limit of the method of successive approximations. It is shown that the Seidel-type methods are convergent to this solution under the same assumptions as the method of successive approximations, but the error estimates are better than the corresponding ones obtained for the method of successive approximations.

1. INTRODUCTION

A problem of a numerical solution of boundary value problems for retarded differential equation was considered by many authors(see, for example [1],[2],[3]). The corresponding results are established under the assumption that the original problem has a unique solution. Existence and uniqueness of solutions of boundary value problems for implicit ordinary differential equations were discussed in [4]. The results stated in this paper are an extension of paper [4].

We consider the following retarded implicit problem

$$F\big(t,x(t),x(\alpha_1(t)),\ldots,x(\alpha_r(t)),\dot{x}(t)\big) = \Theta,\ t \in I = [0,b], \qquad (1a)$$

$$g(x(0),x(b)) = \Theta, \tag{1b}$$

where $F:I \times (R^p)^{r+2} \to R^p$, $g:R^p \times R^p \to R^p$, $\alpha_i:I \to I$ are given (Θ is zero vector in R^p). Our objective is to obtain sufficient conditions under which problem (1) has a solution. Due to this fact it is convenient to transform problem (1) into an integral one.

Put $s = x(0)$, $y(t) = x'(t)$, $t \in I$. Then problem (1) takes the following form

$$F\left(t,s+\int_0^t y(\tau)d\tau,s+\int_0^{a_1(t)} y(\tau)d\tau,\ldots,s+\int_0^{a_r(t)} y(\tau)d\tau,y(t)\right) = \Theta,\ t \in I \tag{2a}$$

$$g\left(s,s+\int_0^b y(\tau)d\tau\right) = \Theta \tag{2b}$$

Notice that a solution of (2) consists of two elements: $s \in R^p$ and $y \in C(I,R^p)$ ($C(I,R^p)$ denotes the collection of all continuous functions $x: I \to R^p$).

A solution of (2) may be now found by the method of successive approximations constructed by

$$F\left(t,s_n+\int_0^t y_n(\tau)d\tau,s_n+\int_0^{a_1(t)} y_n(\tau)d\tau,\ldots,s_n+\int_0^{a_r(t)} y_n(\tau)d\tau,y_{n+1}(t)\right)$$
$$= \Theta,\ t\in I$$

$$s_{n+1} = s_n - B^{-1}\cdot g\left(s_n,s_n+\int_0^b y_n(\tau)d\tau\right) \tag{3}$$

for $n = 0,1,\ldots$, where s_0 and y_0 are given and B is some $p \times p$ non-singular matrix connected with g. To solve problem (2) also the Seidel-type methods may be applied:

$$F\left(t,\tilde{s}_n+\int_0^t\tilde{y}_n(\tau)d\tau,\tilde{s}_n+\int_0^{\alpha_1(t)}\tilde{y}_n(\tau)d\tau,\ldots,\tilde{s}_n+\int_0^{\alpha_r(t)}\tilde{y}_n(\tau)d\tau,\tilde{y}_{n+1}(t)\right)=\Theta,\ t\in I, \quad (4)$$

$$\tilde{s}_{n+1} = \tilde{s}_n - B^{-1}.g\left(\tilde{s}_n,\tilde{s}_n+\int_0^b\tilde{y}_{n+1}(\tau)d\tau\right),\ \tilde{s}_o = s_0,\quad \tilde{y}_0 = y_0$$

$$\bar{s}_{n+1} = \bar{s}_n - B^{-1}.g\left(\bar{s}_n,\bar{s}_n+\int_0^b\bar{y}_n(\tau)d\tau\right),\ \bar{s}_o = s_0,\quad \bar{y}_0 = y_0, \quad (5)$$

$$F\left(t,\bar{s}_{n+1}+\int_0^t\bar{y}_n(\tau)d\tau,\bar{s}_{n+1}+\int_0^{\alpha_1(t)}\bar{y}_n(\tau)d\tau,\ldots,\bar{s}_{n+1}+\int_0^{\alpha_r(t)}\bar{y}_n(\tau)d\tau,\bar{y}_{n+1}(t)\right) = \Theta,$$

for $n = 0,1\ldots$. It is shown that the sequences (3)-(5) are convergent to the solution of problem (2) under the same conditions, but the error estimates for the Seidel-type methods (4) and (5) are better than for method (3).

2. ASSUMPTIONS AND LEMMAS

Let us now introduce some assumptions.

A_1 There exist constants $K_i > 0$ and $K > 0$ such that the relations:

$$\|F(t,z_0,z_1,\ldots,z_r,z) - F(t,\bar{z}_0,\bar{z}_1,\ldots,\bar{z}_r,z)\| \le \sum_{i=0} K_i\|z_i - \bar{z}_i\|,$$

$$\|F(t,z_0,z_2,\ldots,z_r,z) - F(t,z_0,z_1,\ldots,z_r,\bar{z})\| \le K\|z-\bar{z}\|$$

hold for $z_0, z_1, \ldots, z_r, \bar{z}_0, \bar{z}_1, \ldots, \bar{z}_r, z, \bar{z} \in R^p$.

A_2 There exist a nonsingular matrix B of order p and constants a_0, $b_0 \geq 0$, $b_0 < 1$ such that the condition

$$\|s_1 - s_2 - B^{-1}[g(s_1, s_1 + z_1) - g(s_2, s_2 + z_2)]\| \leq b_0\|s_1 - s_2\| + a_0\|z_1 - z_2\|$$

holds for s_1, s_2, z_1, $z_2 \in R^p$.

A_3 For any $t \in I$ and y_0, $y_1, \ldots, y_r \in R^p$ there exists $\tilde{y} \in R^p$ such that $F(t, y_0, y_1, \ldots, y_r, \tilde{y}) = \Theta$.

A_4 There exists a solution $u^* \in C(I, R_+)$, $R_+ = [0, \infty)$, of the equation

$$u = Lu + v, \tag{6}$$

where $$(Lu)(t) = \frac{\bar{K}a_0}{1 - b_0}\int_0^b u(\tau)d\tau + \frac{K_0}{K}\int_0^t u(\tau)d\tau$$

$$+ \sum_{i=1}^{r} \frac{K_i}{K}\int_0^{a_i(t)} u(\tau)d\tau, \quad \bar{K} = \frac{\sum_{i=0}^{r} K_i}{K},$$

$$v(t) = \sup_{0 \leq \tau \leq r}\left(\frac{1}{K}\right)\left\|F\left(\tau, s_0 + \int_0^r y_0(\tau)d\tau, s_0 + \int_0^{a_1(\tau)} y_0(\tau)d\tau, \ldots, s_0 + \int_0^{a_r(\tau)} y_0(\tau)d\tau\, y_0(\tau)\right)\right\| +$$

$$+ \frac{\bar{K}}{1 - b_0}\left\|B^{-1} \cdot g\left(s_0, s_0 + \int_0^b y_0(\tau)d\tau\right)\right\|$$

A_5 In the class of measurable functions u: $I \to R_+$, $u \leq u^*$, function $u = 0$ is the unique solution the equation

$$Lu = u. \tag{7}$$

Lemma A

The sequences $\{u_n(t)\}$, $\{\omega_n\}$ defined by the following formulas:

$$u_0(t) = u^*(t),\ t \in I$$

$$u_{n+1}(t) = \bar{K}\omega_n + \frac{K_0}{K}\int_0^t u_n(\tau)d\tau + \sum_{i=1}^{r}\frac{K_i}{K}\int_0^{\alpha_i(t)} u_n(\tau)d\tau,\ n = 0,1,..., \tag{8}$$

$$\omega_0 = \frac{1}{1-b_0}\left[a_0\int_0^b u^*(\tau)d\tau + \left\|B^{-1}g\left(s_0, s_0 + \int_0^b y_0(\tau)d\tau\right)\right\|\right]$$

$$\omega_{n+1} = b_0\omega_n + a_0\int_0^b u_n(\tau)d\tau,\quad n = 0,1,... \tag{9}$$

are convergent to zero.

Proof

It is clearly that $u_n(t) \geq 0$ and $\omega_n \geq 0$ for any $n = 0,1...$. By induction, we can prove $u_{n+1}(t) \leq u_n(t)$ and $\omega_{n+1} \leq \omega_n$. Hence, the proof is complete.

Lemma B

The sequences $\{\tilde{u}_n(t)\}, \{\tilde{\omega}_n\}$ and $\{\bar{u}_n(t)\}, \{\bar{\omega}_n\}$ defined by the following formulas:

$$\tilde{u}_0(t) = u^*(t),\ t \in I$$

$$\tilde{u}_{n+1}(t) = \bar{K}\tilde{\omega}_n + \frac{K_0}{K}\int_0^t \tilde{u}_n(\tau)d\tau + \sum_{i=1}^{r}\frac{K_i}{K}\int_0^{\alpha_i(t)} \tilde{u}_n(\tau)d\tau,\ n = 0,1,..., \tag{10}$$

$$\tilde{\omega}_0 = \omega_0,\ \tilde{\omega}_{n+1} = b_0\tilde{\omega}_n + a_0\int_0^b \tilde{u}_{n+1}(\tau)d\tau \tag{11}$$

and

$$\bar{u}_0(t) = u^*(t),\ t \in I$$

$$\bar{u}_{n+1}(t) = \bar{K}\bar{\omega}_{n+1} + \frac{K_0}{K}\int_0^t \bar{u}_n(\tau)d\tau + \sum_{i=1}^{r}\frac{K_i}{K}\int_0^{\alpha_i(t)} \bar{u}_n(\tau)d\tau,\ n = 0,1,..., \tag{12}$$

$$\bar{\omega}_0 = \omega_0 \quad \bar{\omega}_{n+1} = b_0\bar{\omega}_n + a_0\int_0^b \bar{u}_n(\tau)d\tau,\ \ n = 0,1 \tag{13}$$

are convergent to zero. Moreover, the estimates

$$\tilde{u}_n(t) \le u_n(t),\ \bar{u}_n(t) \le (t),\ \ t \in I, \tilde{\omega}_n \le \omega_n,\ \ \bar{\omega}_n \le \omega_n,$$

are satisfied for $n = 0,1....$

Proof

We can do the proof by induction.

3. EXISTENCE RESULTS

Our objective in this section is to give sufficient conditions under which problem (2) has a solution.

Theorem A

Let assumptions A_1 - A_5 hold.

Then problem (2) has a solution $(\bar{y},\bar{s})$ being the limit of the sequences $\{y_n\}$ and $\{s_n\}$ defined by (3).

Estimates

$$\|\bar{y}(t)-y_n(t)\| \le u_n(t),\ t\in I,\quad \|\bar{s}-s_n\| \le \omega_n$$

are true. Moreover, $(\bar{y},\bar{s})$ is a unique solution of (2) in the class of functions satisfied the relations

$$\|\bar{y}(t)-y_0(t)\| \le u^*(t),\ t\in I\quad \|\bar{s}-s_0\| \le \omega_0$$

Proof

Using assumptions A_1 - A_2 to equalities:

$$F\left(t,s_n+\int_0^t y_n(\tau)d\tau, s_n+\int_0^{a_1(t)} y_n(\tau)d\tau,\dots,s_n+\int_0^{a_r(t)} y_n(\tau)d\tau, y_{n+1}(t)\right)-$$

$$-F\left(t,s_n+\int_0^t y_n(\tau)d\tau, s_n+\int_0^{a_1(t)} y_n(\tau)d\tau,\dots,s_n+\int_0^{a_r(t)} y_n(\tau)d\tau, y_{n+1}(t)\right)=$$

$$F\left(t,s_0+\int_0^t y_0(\tau)d\tau, s_0+\int_0^{a_1(t)} y_0(\tau)d\tau,\dots,s_0+\int_0^{a_r(t)} y_0(\tau)d\tau, y_0(t)\right)-$$

$$-F\left(t,s_n+\int_0^t y_n(\tau)\,d\tau,s_n+\int_0^{a_1(t)} y_n(\tau)d\tau,\dots,s_n+\int_0^{a_r(t)} y_n(\tau)d\tau,y_0(t)\right)-$$

$$-F\left(t,s_0+\int_0^t y_0(\tau)\,d\tau,s_0+\int_0^{a_1(t)} y_n(\tau)d\tau,\dots,s_0+\int_0^{a_r(t)} y_0(\tau)d\tau,y_0(t)\right),\ t\in I$$

$$s_{n+1}-s_0=s_n-s_0-B^{-1}\left[g\left(s_n,s_n+\int_0^b y_n(\tau)d\tau\right)-g\left(s_0,s_0+\int_0^b y_0(\tau)\,d\tau\right)\right]-$$

$$-\ B^{-1}g\left(s_0,s_0+\int_0^b y_0(\tau)d\tau\right),$$

we obtain the following inequalities:

$$z_{n+1}(t)\le\bar{K}.v_n+\frac{K_0}{K}\int_0^t z_n(\tau)d\tau+\sum_{i=1}^{r}\frac{K_i}{K}\int_0^{\alpha_1(t)} z_n(\tau)d\tau+\xi(t),\ t\in I, \tag{14}$$

$$v_{n+1}\le b_0v_n+a_0\int_0^b z_n(\tau)d\tau+\eta, \tag{15}$$

where $z_n(t)\ =\ \|y_n(t)-y_0(t)\|,\quad v_n\ =\ \|s_n-s_0\|,$

$$\xi(t)\ =\ \underset{0\le\tau\le t}{supp}\left(\frac{1}{K}\right)\left\|F\left(\tau,s_0+\int_0^\tau y_0(\xi)d\xi,s_0\right)\left(\int_0^{\alpha_1(t)} y_0(\xi)d\xi+\int_0^{\alpha_r(t)} y_0(\xi)d\xi,y_0(\tau)\right)\right\|,$$

$$\eta = \left\| B^{-1} g\left(s_0, s_0 + \int_0^b y_0(\tau)d\tau\right)\right\|, \quad \bar{K} = \frac{\sum_{i=0}^{r} K_i}{K}$$

Note that

$$z_0(t) = \|y_0(t) - y_0(t)\| = 0 \le u^*(t) = u_0(t), \; v_0 = \|s_0 - s_0\| = 0 \le \omega_0$$

Furthermore, Using assumptions A_4 - A_5, by induction we can prove that the following inequalities:

$$\|y_n(t) - y_0(t)\| \le u^*(t) \equiv u_0(t), \; t \in I, \; \|s_n - s_0\| \le \omega_0 \tag{16}$$

hold for $n = 0,1....$ Moreover, from the relations:

$$F\left(t, s_{n+m} + \int_0^t y_{n+m}(\tau)d\tau, s_{n+m} + \int_0^{a_1(t)} y_{n+m}(\tau)d\tau, \ldots, s_{n+m} + \right.$$

$$\left. + \int_0^{a_r(t)} y_{n+m}(\tau)d\tau, y_{n+m+1}(t)\right) -$$

$$-F\left(t, s_{n+m} + \int_0^t y_{n+m}(\tau)d\tau, s_{n+m} + \int_0^{a_1(t)} y_{n+m}(\tau)d\tau, \ldots, s_{n+m} + \right.$$

$$\left. + \int_0^{a_r(t)} y_{n+m}(\tau)d\tau, y_{m+1}(t)\right) =$$

$$F\left(t, s_m + \int_0^t y_m(\tau)d\tau, s_m + \int_0^{a_1(t)} y_m(\tau)d\tau, \ldots, s_m + \int_0^{a_r(t)} y_m(\tau)d\tau, y_{m+1}(t)\right)$$

$$-F\left(t, s_{n+m} + \int_{\sigma}^{t} y_{n+m}(\tau)\,d\tau, s_{n+m} + \int_{0}^{\alpha_1(t)} y_{n+m}(\tau)d\tau, \ldots, s_n + \right.$$

$$\left. + \int_{0}^{\alpha_r(t)} y_{n+m}(\tau)d\tau, y_{m+1}(t)\right),\ t \in I$$

$$s_{n+m+1} - s_{m+1} = s_{n+m} - s_m - B^{-1}\left[g\left(s_{n+m}, s_{n+m} + \int_0^b y_{n+m}(\tau)d\tau\right) - g\left(s_m, s_m + \int_0^b y_m(\tau)\,d\tau\right)\right]$$

and assumption A_1 - A_2 result the following inequalities

$$\|y_{n+m+1}(t) - y_{m+1}(t)\| \le \bar{K}\|s_{s+m} - s_m\| + \frac{K_0}{K}\int_0^t \|z_{n+m}(t) - y_m(\tau)\|d\tau$$

$$+\sum_{i=1}^{r} \frac{K_i}{K} \cdot \int_0^{\alpha_i(t)} \|y_{n+m}(\tau) - y_m(\tau)\|,\ t \in I,$$

$$\|s_{n+m+1} - s_{m+1}\| \le b_0\|s_{n+m} - s_m\| + a_0 \int_0^b \|y_{n+m}(\tau) - y_m(\tau)\|\ d\tau$$

for $n,m = 0,1,\ldots.$

Hence, Using estimate (16), by induction for m, we can prove that the following estimates:

$$\|y_{n+m+1}(t) - y_{m+1}(t)\| \le u_{m+1}(t), \quad \|S_{n+m+1} - S_{m+1}\| \le \omega_{m+1} \tag{17}$$

are true for $n,m = 0,1,\ldots.$

The sequences $\{y_n(t)\}$, $\{S_n\}$ are convergent and have limits because both sequences $\{u_n\}$, $\{\omega_n\}$ are convergent to zero by Lemma A.

Let $\lim\limits_{n\to\infty} y_n(t) = \bar{y}(t),\ t \in I,\ \lim\limits_{n\to\infty} s_n = \bar{s}.$

Then from equality (3), it is clearly that $(\bar{y}, \bar{s})$ is a solution of problem (2).

From estimates (17) we can get

$$\|\bar{y}(t) - y_n(t)\| \le u_n(t)\ ,\ t \in I\ ,\ \|\bar{s} - s_n\| \le \omega_n\ .$$

The proof is complete.

The next theorem shows that the sequences (3)-(5) are convergent to the solution of problem (2) under same conditions.

Theorem B

Under the assumptions of Theorem A, we have

$$\lim_{n\to\infty} \tilde{y}_n(t) = \lim_{n\to\infty} \bar{y}_n(t) = \bar{y}(t)\ ,\ t \in I, \tag{18}$$

$$\lim_{n\to\infty} \tilde{s}_n(t) = \lim_{n\to\infty} \bar{s}_n = \bar{s}, \tag{19}$$

where $(\bar{y}, \bar{s})$ is the solution of problem (2).

Moreover, the error estimates

$$\|\tilde{y}_n(t) - \bar{y}(t)\| \le \tilde{u}_n(t)\ ,\ t \in I\ ,\quad \|\tilde{s}_n - \bar{s}\| \le \tilde{\omega}_n\ , \tag{20}$$

$$\|\bar{y}_n(t) - \bar{y}(t)\| \le \bar{u}_n(t)\ ,\ t \in I\ ,\quad \|\bar{s}_n - \bar{s}\| \le \bar{\omega}_n\ , \tag{21}$$

are satisfied for $n = 0,1,\ldots$.

Proof

Notice that, By Theorem A, problem (2) has the solution $(\bar{y}, \bar{s})$. Using assumptions A_1 and A_2 to the following equalities:

$$F\left(t,\bar{s}+\int_0^t \bar{y}(\tau)d\tau,\bar{s}+\int_0^{a_1(t)}\bar{y}(\tau)d\tau,\ldots,\bar{s}+\int_0^{a_r(t)}\bar{y}(\tau)d\tau,\bar{y}(t)\right)-$$

$$F\left(t,\bar{s}+\int_0^t \bar{y}(\tau)d\tau,\bar{s}+\int_0^{a_1(t)}\bar{y}(\tau)d\tau,\ldots,\bar{s}+\int_0^{a_{r2}(t)}\bar{y}(\tau)d\tau,\tilde{y}_{n+1}(t)\right)=$$

$$F\left(t,\tilde{s}_n+\int_0^t \tilde{y}_n(\tau)d\tau,\tilde{s}_n+\int_0^{a_1(t)}\tilde{y}_n(\tau)d\tau,\ldots,\tilde{s}_n+\int_0^{a_r(t)}\tilde{y}_n(\tau)d\tau,\tilde{y}_{n+1}(t)\right)-$$

$$-F\left(t,\bar{s}+\int_0^t \bar{y}(\tau)d\tau,\bar{s}+\int_0^{a_1(t)}\bar{y}(\tau)d\tau,\ldots,\bar{s}+\int_0^{a_r(t)}\bar{y}(\tau)d\tau,\tilde{y}_{n+1}(t)\right),$$

$$\tilde{s}_{n+1}-\bar{s}=\tilde{s}_n-\bar{s}-B^{-1}\left[g\left(\tilde{s}_n,\tilde{s}_n+\int_0^b \tilde{y}_{n+1}(\tau)d\tau\right)-g\left(\bar{s},\bar{s}+\int_0^b \bar{y}(\tau)d\tau\right)\right],$$

we obtain

$$\|\tilde{y}_{n+1}(t)-\bar{y}(t)\|\le\bar{K}\|\tilde{s}_n-\bar{s}\|+\frac{K_0}{K}\int_0^t\|\tilde{y}_n(\tau)-\bar{y}(t)\|d\tau$$

$$+\sum_{i=1}^{r}\frac{K_i}{K}\int_0^{a_i(t)}\|\tilde{y}_n(\tau)-\bar{y}(\tau)\|\,d\tau,$$

$$\|\tilde{s}_{n+1}-\bar{s}\|\le b_0\,\|s_n-\bar{s}\|+a_0\int_0^b\|\tilde{y}_{n+1}(t)-\bar{y}(\tau)\|\,d\tau.$$

Furthermore, By induction with respect to n, we can show that estimate (20) is satisfied. By a similar argument, we can verify (21).

From estimates (20)-(21) and Lemma B we obtain equalities (18)-(19). The proof is complete. Notice that the sequences (3)-(5) are convergent under the same conditions but the error estimates for sequences (4) and (5) are better than for sequence (3).

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NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEM FOR RETARDED DIFFERENTIAL EQUATIONS

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In this paper we study the numerical solution of boundary value problems for retarded implicit differential equations. For the boundary value problems the backward Euler method is combined with iterative ones. The conditions for this convergence are obtained.

I. INTRODUCTION

We consider the retarded implicit differential equation of the form

$$F(t,y(t),y(\alpha(t)),y'(t)) = \Theta \in R^q,\ t \in [a,b],\ a < b, \tag{1}$$

with the nonlinear boundary condition

$$g(y(a),\ y(b)) = \Theta, \tag{2}$$

where F and g are given continuous vectors in R^q (Θ is zero element in R^q). Here α is a retarded function. Indeed, α is given. We assume that $\alpha(a) = a$, $a \leq \alpha(t) \leq t$ for $t \in I$ and α is continuous on I. Let $\varphi \in C^1(I,R^q)$ be a unique solution of problem (1-2) If F does not depend on the third variable then we have an implicit differential equation of the form

$$F(t,y(t),\ y'(t)) = \Theta$$

and problems of numerical solutions for such types of equations were discussed in [2]. The results stated in this paper are an extension of [2]. Some results for numerical solutions of equations of the form

$$y'(t) = f(t,y(t)), \qquad t \in [a,b]$$

can be found in papers [1], [3].

We consider the simplest possible procedure based on the backward Euler method combined with iterative ones. We construct an implicit method defined by the nonlinear equation

$$\Phi\ (t_i,y(t_i),\ Z(t_i),\ (y(t_{i+1})\ -\ y(t_i))/h,h)\ =\ \Theta,\ i\ =\ 0,1,\ldots,N\ -\ 1,\ (3)$$

where Φ is an approximation of F. We need to solve equation (3) for $y(t_{i+1})$. Indeed, $y(t)$ is an approximation of the exact solution $\varphi(t)$ of problem (1-2) at the point t while $Z(t)$ is an approximation of $\varphi(a(t))$. We choose a positive integer number N and select mesh points t_i (i = $0,1,\ldots,N$) by the formula $t_i = a + i.h$ with the step $h = (b - a)/N$.

To solve problems (1-2) numerically we construct the implicit method defined by the following nonlinear equations

$$\Phi\left(t_i,y_i^{[j]},Z_i^{[j]},\left(y_{i+1}^{[j]}-y_i^{[j]}\right)/h,h\right),\ =\ \Phi,\quad i\ =\ \overline{0,N-1},\ j\ =\ 0,1\ldots, \qquad (4)$$

$$Z_i^{[j]}\ =\ y_{k_i}^{[j]}+h.\Phi_1\left(t_{k_i},y_{k_i}^{[j]},h,e(i)\right),\quad i\ =\ \overline{0,N-1},\quad j\ =\ 0,1,\ldots, \qquad (5)$$

$$y_0^{[j+1]}\ =\ y_0^{[j]}-B^{-1}.g\left(y_0^{[j]},y_N^{[j]}\right),\quad j\ =\ 0,1,\ldots, \qquad (6)$$

with $y_0^{[0]}$ is an arbitrary vector in R^q.

Note that in the above notations

(1) y_i, Z_i are the approximations of the exact solution $\varphi(t)$ of problem (1-2) at the point t_l and $\alpha(t_l)$, respectively

(2) k_i is the integer part of $(\alpha(t_i) - a)/h$.

Indeed, $\alpha(t_i) = t_{k_i} + h.e(i)$ for $e(i) = (\alpha(t_i) - a)/h - k_i$.

The conditions for Φ, Φ_1 will be defined later. Of course, we may take $\Phi = F$ and $\Phi_1 = 0$.

(3) B is a nonsingular $q \times q$ matrix and j is a number of approximations.

The procedure (4-6) works as follows:

(1) Take $j = O$ and for the initial given value $y_0^{[0]}$ find $y_i^{[0]}$ for $i = 1,\ldots,N$ solving the equations (4) and (5),

(2) determine the value $y_0^{[1]}$ from (6),

(3) apply again (4) and (5) for finding $y_i^{[1]}$ $(i = 1,2,\ldots,N)$, and so on to obtain the sequences of the values $y_i^{[j]}$ for $i = 0,1,\ldots,N$ and $j = 0,1,\ldots$.

II. ASSUMPTIONS AND DEFINITIONS

To obtain some results we need the following assumptions and definitions.

Assumption A1

Suppose that:

1^0 $\Phi: I \times R^q \times R^q \times R^q \times H \to R^q$, $H = [0,h_0]$, $h_0 > 0$,

2^0 there exists a function $\epsilon_1: I \times H \to R_+ = [0,\infty)$ and constants $L_1, L_2 \geq 0$ such that for $t \in I$ and $u_1, u_2, \bar{u}_1, \bar{u}_2, u_3 \in R^q$, $h \in H$ we have

$$\left\| \Phi(t,u_1,u_2,u_3,h) - \Phi(t,\overline{u_1},\overline{u_2},u_3,h) \right\|$$

$$\leq L_1 \|u_1 - \overline{u_1}\| + L_2 \|u_2 - \overline{u_2}\| + \epsilon_1(t,h)$$

and $\lim_{N \to \infty} h . \sum_{i=0}^{N-1} \epsilon_1(t_i, h) = 0$, where $\| . \|$ denotes a norm in R^q,

3^0 there exists a constant $L_3 > 0$ such that the inequality

$$\|\Phi(t,u_1,u_2,u_3,h) - \Phi(t,u_1,u_2,u_3,\overline{u_3})\| \geq L_3 \|u_3 - \overline{u_3}\|$$

holds for $t \in I$, u_1, u_2, u_3, $\bar{u}_3 \in R^q$ and $h \in H$.

4^0 $\Phi_1 : I \times R^q \times R^q \times H \times H_0 \to R^q$, $H_0 = [0,1)$ and there exists a constant $L_4 \geq 0$ such that the inequality

$$\|\Phi_1(t,u_1,u_2,h,x)\| \leq L_4$$

holds for any point (t,u_1,u_2,h,x) from the domain of Φ_1.

Assumption A2

Assume that:

1^0 there exists a nonsingular square matrix B of order q and constants $m_1 \geq 0$, $d > 0$ such that $\| B^{-1} \| \leq d$ and

$$\|B(u_1 - u_2) - g(u_1,u_3) + g(u_2,u_3)\| \leq m_1 \|u_1 - u_2\| \quad \text{for } u_1,u_2,u_3 \in R^q,$$

where the matrix norm is consistent with the vector norm,

2^0 there exists a constant $m_2 \geq 0$ such that

$$\|g(u_1,u_2) - g(u_1,\overline{u_2})\| \leq m_2 \|u_2 - \overline{u_2}\| \quad \text{for } u_1,u_2,\bar{u}_2 \in R^q.$$

Now we introduce the definitions of consistency and convergence.

Definition 1

The method (4-6) is said to be consistent with problem (1-2) on the solution φ if the following both conditions

$$\|\Phi(t,\varphi(t),\varphi(\alpha(t)),(\varphi(t+h)-\varphi(t))/h,h)\| \leq \epsilon_2(t,h),$$

$$\lim_{N\to\infty} h.\sum_{i=0}^{N-1} \epsilon_2(t_i,h) = 0$$

are satisfied, where $\epsilon_2 : I \times H \to R_+$.

Definition 2

The method (4-6) is said to be convergent to the solution φ of problem (1-2) if

$$\lim_{\substack{N\to\infty \\ j\to\infty}} \max_{i=0,1,\ldots,N} \left\| y_i^{[j]} - \varphi(t_i) \right\| = 0,$$

$$\lim_{\substack{N\to\infty \\ j\to\infty}} \max_{i=0,1,\ldots,N} \left\| Z_i^{[j]} - \varphi(\alpha(t_i)) \right\| = 0,$$

III. CONVERGENCE OF METHOD (4-6)

Now we can formulate the theorem on the convergence of method (4-6).

Theorem

If Assumptions A1 and A2 are satisfied and:

1^0 there exists the unique solution $\varphi \in C^1(I,R^q)$ of problem (1-2) and $\| \varphi'(t) \| \leq L_5$, where $L_5 \geq 0$,

2^0 the method (4-6) is consistent with problem (1-2) on the solution φ,

3^0 $m = d(m_1 + m_2 D) < 1$ with $D = \exp(L(b-a))$, then the method (4-6) is convergent to φ if $N, j \to \infty$. Moreover, we have the following estimations of errors

$$\max_{i=0,1,\ldots,N} \left\| y_i^{[j]} - \varphi(t_i) \right\| \le D.m^j \left\| y_0^{[j]} - \varphi_0 \right\|$$

$$+ B_N\left[1 + d.m_2.D.(1-m^j)/(1-m)\right],\ j = 0,1\ldots \tag{7}$$

$$\max_{i=0,1,\ldots,N} \left\| Z_i^{[j]} - \varphi(\alpha(t_i)) \right\| \le D.m^j \left\| y_0^{[j]} - \varphi_0 \right\|$$

$$+ B_N\left[1 + d.m_2.D.(1-m^j)/(1-m)\right] + h(L_4 + L_5),\ j = 0,1\ldots \tag{8}$$

where

$$B_N = h.\sum_{n=0}^{N-1} (1+hL)^{N-1-n}.\left[hL_0 + \left(\epsilon_1(t_n,h) + \epsilon_2(t_n,h)\right)/L_3\right],$$

$$L = (L_1 + L_2)/L_3, \qquad L_0 = L_2(L_4 + L_5)/L_3$$

Proof

We note that the equality

$$\Phi\left(t_i, y_i^{[j]}, \left(y_{[i+1]}^{[j]} - y_i^{[j]}\right)/h, h\right) - \Phi\left(t_i, y_i^{[j]}, Z_i^{[j]}(\varphi(t_{i+1}) - \varphi(t_i))/h, h\right)$$

$$= \Phi\left(t_i, \varphi(t_i), \varphi(\alpha(t_i)), (\varphi(t_{i+1}) - \varphi(t_i))/h, h\right)$$

$$- \Phi\left(t_i, \varphi(t_i), \varphi(\alpha(t_i)), (\varphi(t_{i+1}) - \varphi(t_i))/h, h\right)$$

$$- \Phi\left(t_i, y_i^{[j]}, Z_i^{[j]}, (\varphi(t_{i+1}) - \varphi(t_i))/h, h\right)$$

is satisfied. By Assumption A1 and definition of consistency, we obtain

$$\left(\frac{L_3}{h}\right).\left(\left\|y_{i+1}^{[j]}-\varphi\left(t_{i+1}\right)\right\|-\left\|y_i^{[j]}-\varphi\left(t_i\right)\right\|\right)$$

$$\leq L_1\left\|y_i^{[j]}-\varphi\left(t_i\right)\right\|+L_2\left\|Z_i^{[j]}-\varphi\left(\alpha\left(t_i\right)\right)\right\|+\epsilon_1\left(t_i,h\right)+\epsilon_2\left(t_i,h\right).$$

Hence,

$$\upsilon_{i+1}^{[j]} \leq \left(\frac{1+hL_1}{L_3}\right)\upsilon_i^{[j]}+\frac{hL_2}{L_3}.\omega_i^{[j]}+\frac{h}{L_3}\left[\epsilon_1\left(t_i,h\right)+\epsilon_2\left(t_i,h\right)\right], \tag{9}$$

where

$$\upsilon_i^{[j]} = \left\|y_i^{[j]}-\varphi\left(t_i\right)\right\|, \quad \omega_i^{[j]} = \left\|Z_i^{[j]}-\varphi\left(\alpha\left(t_i\right)\right)\right\|, \ (i = 0,1,\ldots,N).$$

Moreover,

$$\omega_i^{[j]} = \left\|y_{k_i}^{[j]}+h\,\Phi_1\left(t_{k_i},y_{k_i}^{[j]},Z_{k_i}^{[j]},h,e(i)\right)-\varphi\left(\alpha\left(t_i\right)\right)+\varphi\left(t_{k_i}\right)-\varphi\left(t_{k_i}\right)\right\|$$

$$\leq \upsilon_{k_i^{[j]}}+h.L_4+\int_{t_{k_i}}^{\alpha(t_i)}\left\|\varphi'(\tau)\right\|\,d\tau \leq V_i^{[j]}+h\left(L_4+L_5\right) \tag{10}$$

where $V_i^{[j]} = \max_{0\leq l\leq i} \upsilon_l^{[j]}$

From (9), we see that

$$\upsilon_{i+1}^{[j]} \leq \left(\frac{1+hL_1}{L_3}\right).V_i^{[j]}+\frac{hL_2}{L_3}.\left[V_i^{[j]}+h\left(L_4+L_5\right)\right]+\frac{h}{L_3}.\left[\epsilon_1\left(t_i,h\right)+\epsilon_2\left(t_i,h\right)\right]$$

$$= \left[1+h\frac{\left(L_1+L_2\right)}{L_3}\right].V_i^{[j]}+h^2L_2\frac{\left(L_4+L_5\right)}{L_3}.\left[\epsilon_1\left(t_i,h\right)+\epsilon_2\left(t_i,h\right)\right].$$

$$= (1+hL)\,V_i^{[j]} + h^2 L_0 + \frac{h}{L_3}\cdot\left[\epsilon_1(t_i,h) + \epsilon_2(t_i,h)\right]$$

Using the definition for $V_i^{[j]}$

$$V_{i+1}^{[j]} = \max_{0\le l\le i+1} = \max\left(V_i^{[j]}, \upsilon_{i+1}^{[j]}\right).$$

we have

$$V_{i+1}^{[j]} \le (1+hL)\,V_i^{[j]} + h^2 L_0 + \frac{h}{L_3}\left[\epsilon_1(t_i,h) + \epsilon_2(t_i,h)\right], i = \overline{0,N-1},$$

and hence

$$V_i^{[j]} \le (1+hL)^i\,V_0^{[j]} + \sum_{n=0}^{i-1} (1+hL)^{i-1-n}$$

$$\cdot\left[h^2 L_0 + \frac{h}{L_3}\left(\epsilon_1(t_n,h) + \epsilon_2(t_n,h)\right)\right], i = \overline{0,N}, \tag{11}$$

Moreover, by Assumption A2 and (11), we obtain

$$V_0^{[j+1]} = \|y_0^{[j+1]} - \varphi_0\| = y_0^{[j]} - B^{-1} g\left(y_0^{[j]}, y_N^{[j]}\right) - \varphi_0\,\|$$

$$= \left\|B^{-1}\cdot\left[B\left(y_0^{[j]} - \varphi_0\right) - g\left(y_0^{[j]}, y_N^{[j]}\right)\right]\right\|$$

$$= \left\|B^{-1}\left[B\left(y_0^{[j]} - \varphi_0\right) - g\left(y_0^{[j]}, \varphi_N\right) + g\left(y_0^{[j]}, \varphi_N\right) - g\left(y_0^{[j]}, y_N^{[j]}\right) + g(\varphi_0, \varphi_N)\right]\right\|$$

$$\le c\left(m_1\left\|y_0^{[j]} - \varphi_0\right\| + m_2\left\|y_N^{[j]} - \varphi_N\right\|\right) \le d.\left(m_1 . V_0^{[j]} + m_2 . V_N^{[j]}\right)$$

$$\leq d\{ \left[m_1 + m_2(1+hL)^N\right].V_0^{[j]} + m_2.\sum_{n=0}^{N-1}(1+hL)^{N-1-n}$$

$$.\left[h^2 L_0 + h/L_3\left(\epsilon_1\left(t_n,h\right)+\epsilon_2\left(t_n,h\right)\right)\right]\}$$

$$\leq d.\{ \left(m_1 + m_2.D\right).V_0^{[j]} + h.m_2.\sum_{n=0}^{N-1}(1+hL)^{N-1-n}$$

$$\left[h L_0 + \left(\epsilon_1\left(t_n,h\right)+\epsilon_2\left(t_n,h\right)\right)/L_3\right]\} \quad j = 0,1,...$$

Hence

$$V_0^{[j]} \leq m^j V_0^{[0]} + d..m_2.B_N\left(1-m^j\right)/(1-m) \tag{12}$$

with $B_N = h.\sum_{n=0}^{N-1}(1+hL)^{N-1-n}.\left[hL_0 + \left(\epsilon_1\left(t_n,h\right) + \epsilon_2\left(t_n,h\right)\right)/L_3\right].$

Combining (11) and (12) we arrive at the following estimation

$$\max_{i=0,1,...,N} \left\|y_i^{[j]} - \varphi\left(t_i\right)\right\| \leq D.m^j \left\|y_0^{[j]} - \varphi_0\right\|$$

$$+ B_N\left[1 + d.m_2.D.\left(1-m^j\right)/(1-m)\right] \tag{13}$$

Similarly, from (10) and (13) we have

$$\max_{i=0,1,...,N} \left\|Z_i^{[j]} - \varphi\left(\alpha\left(t_i\right)\right)\right\| \leq D.m^j \left\|y_0^{[j]} - \varphi_0\right\|$$

$$+ B_N\left[1 + d.m_2.D.\left(1-m^j\right)/(1-m)\right] + h\left(L_4 + L_5\right) \tag{14}$$

Summarizing (13) and (14) with conditions of Theorem, we can see that the sequence $\{y_i^{[j]}\}$ is convergent to the solution of problem (1-2) if j, $N \to \infty$.

IY. Numerical calculations

Now we consider the boundary problem

$$22y' + Sin(ty') + 2y - Sin\left(8y\left(\frac{1}{2}t\right)\right) - \pi\left(t^2 + 22t\right) = 0, \ t \in [0,2]$$

$$Cos(y(0)) + Sin(y(2)) + 5y(0) - 1 = 0 \tag{15}$$

We can check that the function $y(t) = \pi t^2/2$ is the unique solution of this problem. To solve problem (15) numerically we apply the nonlinear algorithm for $y_{i+1}^{[j]}$ defined by the equations

$$22.\frac{y_{i+1}^{[j]} - y_i^{[j]}}{h} + Sin\left(t.\frac{y_{i+1}^{[j]} - y_i^{[j]}}{h}\right) + 2y_i^{[j]} - Sin\left(8Z_i^{[j]}\right) - \pi\left(t_i^{[j]} + 22t_i\right) = 0, \tag{16}$$

$$i = 0,1,\ldots,N-1, \ j = 0,1,\ldots,$$

$$Z_i^{[j]} = y_{k(i)}^{[j]} \tag{17}$$

with $y_0^{[0]}(0) = y_0$,

y_0 is a fixed initial value

$$y_0^{[j+1]}(0) = y_0^{[j]}(0) - \frac{1}{6}\left[Cos\left(y_0^{[j]}(0)\right) + Sin\left(y_N^{[j]}\right) + 5y_0^{[j]}(0) - 1\right] \tag{18}$$

where $h = 1/N$, $k(i)$ is the integer part of $t_i/(2h)$

So it is easy to see that

$$\Phi\ (t,u_1,u_2,u_3,h) = 22u_3 + Sin(tu_3) + 2u_1 - Sin(8u_2) - \pi(t^2 + 22t)$$

and Assumptions A1 and A2 are satisfied with $L_1 = 2, L_2 = 8, L_3 = 20$ and $m_1 = 2, m_2 = 1, m_3 = \dfrac{1}{6}$.

The equation (16) has exactly one solution for $y_{i+1}^{[j]}$ because

$$\left|\frac{1}{22}.Cos\left(t.\frac{y_{i+1}^{[j]} - y_i^{[j]}}{h}\right).t\right| \le \frac{1}{11} < 1.$$

Moreover $\quad m = d.(m_1 + m_2.D) = \dfrac{1}{6}.\left(2 + 1.\exp\left(2.\dfrac{2+20}{20}\right)\right)$

$$= \frac{1}{6}(2+e) < \frac{1}{6}.4.8 \approx 0.8 < 1$$

All conditions of Theorem are satisfied, so the method (16-18) is convergent to the solution $\varphi(t)$ of (15). The numerical results are shown in the following table.

h	j	$y^{[j]}(0)$	$y^{[j]}(2)$	$\varphi(2)-y^{[j]}(2)$
0.1	1	0.04662	6.04459	0.23859
	5	0.04201	6.04691	0.23627
	10	0.03801	6.06323	0.21995
0.05	1	0.02851	6.16652	0.11666
	5	0.02412	6.16854	0.11464
	10	0.02412	6.17256	0.11062
0.01	1	0.00572	6.25983	0.02327
	5	0.00463	6.269970	0.01134
	10	0.00424	6.27905	0.00413

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ON OPEN MAPPING AND CLOSED GRAPH THEOREMS

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ABSTRACT: We shall prove open mapping and closed graph theorems independent of category argument for locally convex K-spaces. A version of closed graph theorem about a polish group acting transitively on a complete metric space is obtained.

INTRODUCTION

The application of Baire's category argument is an essential tool for the proofs of some fundamental theorems of functional analysis. Some authors have sought proofs of these results without using Baire's theorem (see Khan and Rowlands [2] and Swartz [7]. Here we continue with this theme; in particular following techniques of Pap and Swartz [6] we shall establish versions of open mapping and closed graph theorems for locally convex K-spaces. We shall also establish a closed graph theorem for homogeneous spaces by employing category dependent open mapping theorem due to Koshi and Takesaki [3]. Incidentally our work deals with the inter implication of these two important theorems as well.

TERMINOLOGY

Let X and Y be Hausdorff locally convex spaces. For any linear mapping T between X and Y, there is a natural linear map T′ (adjoint of T) from

$$D(T') = \{y' \in Y' : y'\,T \text{ is continuous on } X\}$$

to X′ and is defined as T′ Y′ = y′T.

Note that many of the properties of T are reflected through corresponding properties of T'.

A sequence $\{x_k\}$ in X is called K-convergent if every subsequence of $\{x_k\}$ has a subsequence $\{X_{n_k}\}$ such that the subserires $\sum_{k=1}^{(1)} K_{n_k}$ converges to an element of X. A K-convergent sequence obviously converges to 0 and if X is complete, then any sequence which converges to 0 is K-convergent. In general a sequence which converges to 0 is not K-convergent. For instance in real space C_{00} the sequence $\left\{\frac{e_i}{i}\right\}$, where $\{e_i\}$ is the sequence with 1 at ith coordinate and 0 elsewhere, converges to 0 but is not K-convergent. We shall call X to be a K-space if every sequence converging to 0 is K-convergent.

Let P be a family of seminorms on X and β be family of bounded subsets of X. The pair (β,1.1) induce a locally convex topology on x' via family P' of seminorms given by

$$P'(x') - \sup\{|x'(x)| : c \in A, A \in \beta\}.$$

Similarly if Y is a locally convex space with family Q of seminorms, then Q' will be induced family of seminorms defining a locally convex topology on Y'. also the pair (β,Q) will induce a locally convex topology on L (x,y) via the family of seminorms

$$|T|_{q,A} = \sup\{q(T(x)) : x \in A, A \in \beta\}.$$

(For details see Swartz [8]).

RESULTS

Proposition 1

Let $D(T) \subseteq X$ and $T:D(T) \rightarrow Y$ be linear. Then

(a) T' is closed

(b) If T is closed and Y is reflexive, then $D(T')$ is dense in Y'.

Proof:

(a) Let $\{y_\alpha\}$ be a net in $D(T')$ such that $y_\alpha' \to y'$ and $T' y_\alpha' \to x'$ for some $x' \in X'$ and $y' \in Y'$. Then for $x \in D(T)$, $y'_\alpha (Tx) \to y' (Tx)$ and $y_\alpha' (Tx) = T' y_\alpha'(x) \to x' (x)$. Thus $y' (Tx) = x' (x)$ for all $x \in D(T)$. Since x' is continuous so $y' \in D(T')$ and $T'y' = x'$. Thus T' is closed.

(b) It is sufficient to show that given $y_o \neq 0$, there exists $Y' \in D(T')$ such that $y'(Y_o) \neq 0$. Clearly $(o,y_o) \notin G(T)$, graph of T. As $G(T)$ is closed in X x Y so by Hahn-Banach theorem, there exists $Z' \in (X \text{ x } Y)'$ such that $z' (O, Y_o) \neq 0$ and $z' (x,Tx) = 0$ for all $(x,Tx) \in G(T)$. Define $y' \in Y'$ by $y' (y) = z' (0,y)$. Note that $y' (y_o) \neq 0$. Further define $x' \in X'$ by $x' (x) = z'(x,0)$. Then

$$0 = z'(x,Tx) = z'(x,0) + z'(0,Tx) = x'(x) + y' (T(x))$$

Thus $y'(Tx) = -x' (x)$, for all $x \in X$ and hence $Y' \in D(T')$.

Proposition 2

Let T be a linear map with domain $D(T) \subseteq X$ and range in Y where Y is complete. if T is bounded and closed, then D(T) is closed.

Proof

Let $x \in \overline{D(T)}$. Then there exists a net $\{x_\alpha\}$ in D(T) such that $x_\alpha \to x$. Then for each $q \in Q$ there exists some $M > 0$ such that $q (Tx_\alpha - T x_\lambda) \leq Mp (x_\alpha - x_\lambda)$, for all $\alpha, \lambda > \alpha'$.

So $\{Tx_\alpha\}$ is cauchy and hence converges to some $y \in Y$. Since T is closed, therefore $y = T(x)$ and $x \in D(T)$ as desired.

Theorem 1

If **X** is a locally convex K-space, then $T':D(T') \to X'$ is continuous.

Proof

Similar to the proof of theorem 3 of [6].

Theorem 2 (closed graph theorem).

Let X be a locally convex K-space and Y be a reflexive locally convex space. If $T:X \to Y$ is linear and closed, then T is continuous.

Proof

First we show that $D(T') = Y'$. By prop. 1(b), $D(T')$ is dense in Y'. Also by theorem 1 and prop. 1(a), T' is continuous and closed. Therefore by prop. 2, $D(T')$ is closed and so $D(T') = Y'$.

$$\begin{aligned} q(Tx) &= \sup \{|y'(T(x))| : q'(y') \le 1\}. \\ &= \sup \{|T'y'(x)| : q'(y') \le 1\}. \\ &\le p(x) \sup \{p'(T'y') : q'(y') \le 1\} \\ &\le p(x) \sup \{|T'|_{p',A'} : q'(y') \le 1\} \end{aligned}$$

where $p' \in P'$ and $y' \in A' = \{y' \in Y' : q'(y') \le 1\}$.

Hence $q(Tx) \le M.p(x)$ for all $x \in X$ implies that T is continuous. The above closed graph theorem enables us to give the following.

Theorem 3. (open mapping theorem)

Let X be a reflexive locally convex space and Y be a locally convex K-space. If $T:X \rightarrow Y$ is linear, onto and continuous, then T is open.

Proof

Put $A = X/N(T)$. Obviously quotient map ϕ from X to A is open. Let S be the induced map from A onto Y. The map $T = S \, 0 \, \phi$ is continuous so S is continuous. Also $S^{-1} : y \rightarrow A$ is closed. The space A is reflexive from X is so. By theorem 2, S^{-1} is continuous and hence S is open. This gives that T is open.

For any topological space X, G will denote the topological transformation group defined on X which is polish (separable complete metric) group acting transitively on X (that is for each $x,y \in X$ there exists an element $g \in G$ with $g.\, x = y$). The action $\Psi : G \times X \rightarrow X$ is defined as $\Psi(g,x) = g.x$ for $g \in G$ and $x \in X$.

Theorem 4. (cf. [4]).

Let G be the polish group acting transitively on a complete metric space X. Define for each $x \in X$, the map $F:G \rightarrow X$ by $F(g) = g.x$. If F is continuous and bijecitve, then F^{-1} is continuous.

Proof

By theorem B [3], F is open. Clearly F^{-1} exists and so F^{-1} is continuous.

The above theorem enables us to establish the following.

Theorem 5 (Closed graph theorem for homogeneous spaces).

Let G,X and F be as in theorem 4. The map F for each $x \in X$ is continuous if and only if its graph G(F), is closed.

Proof

Suppose that F is continuous and $(g,x) \in \overline{G(F)}$. Then there exists $g_n \in G$ such that $g_n \to g$ and $F(g_n) \to x$. But $F(g_n) \to F(g)$ and so $x = F(g)$. Now $(g,x) = (g,F(g)) \in G(F)$ and so G(F) is closed. Conversely let G(F) be closed and so it is complete is $(G \times X, d_{G\times X})$ where

$$d_{G \times X} [(g,x), (g', x')] = \{[d_G (g,g')]^2 + [d_x (x, x')]^2 \}^{1/2}$$

Consider $f{:}G(F) \to G$ given by

$$f(g,F(g)) = g$$

obviously f is bijective. The continuity of f follows from

$$d_G(f(g,F(g)), f(g, F(g_o)) = d_G (g, g_o)$$

$$\leq d_{G \times X}((g, F(g)), (g_o, F(g_o)))$$

Now $d_x (F(g), F(g_o)) \leq d_{G \times X}(, F(g)), (g_o, F(g_o)))$

$$= d_{G \times X} (f^{-1} (g), f^{-1} (g_o))$$

and the continuity of f^{-1} imply that F is continuous.

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A CLASSIFICATION OF THE ELEMENTS OF $Q^*(\sqrt{p})$ AND A PARTITION OF $Q^*(\sqrt{p})$ UNDER THE ACTION OF MODULAR GROUP PSL (2,Z)

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ABSTRACT: Let α denote a real quadratic irrational number $\frac{a+\sqrt{n}}{c}$, where n is a non-square positive integer and $a, \frac{a^2-n}{c}, c$ are relatively prime integers. Let G denote the modular group $<x,y: x^2 = y^3 = 1>$. In this paper we have obtained a classification of $Q^*(\sqrt{p})$ and a partition of $Q^*(\sqrt{p})$ under the action of the modular group.

INTRODUCTION

Every quadratic irrational number $\frac{a'+b'\sqrt{n'}}{c'}$, where n' is non-square can be uniquely represented as $\frac{a+\sqrt{n}}{c}$ where $\frac{a^2-n}{c}=b$ is an integer and $(a,b,c) = 1$ (see Q. Mushtaq [3]). We denote the set of all such numbers for a particular n by $Q^*(\sqrt{n})$. Q. Mushtaq [3] has shown that under the action of the Modular group PSL (2,Z), $Q^*(\sqrt{n})$ is invariant. Imrana Kousar, S.M. Husnine, A. Majeed in [4] have investigated the behaviour of ambiguous and totally positive or totally negative elements of $Q^*(\sqrt{n})$ under the action of the modular group. The same authors in [5] have discussed the action of the group $H = < t,y : t^3 = y^3 = 1>$ on the quadratic field. In section 1, we classify the elements of $Q^*(\sqrt{p})$ for any odd prime p with respect to the odd-even nature of a,b,c.

For number theoretic results we refer the readers to [1] and [2] and for the modular group action to [3].

SECTION I

We start with the following definition.

Definition 1

Let $\alpha = \dfrac{a+\sqrt{p}}{c}$, $b = \dfrac{a^2-p}{c}$ and $(a,b,c) = 1$, we say that α is of type $[u,v,w]$, where u,v,w denote the even or odd character of a,b,c respectively.

Theorem 1

Let $\alpha = \dfrac{a+\sqrt{p}}{c} \in Q^*(\sqrt{p})$, where p is an odd prime, $b = \dfrac{a^2-p}{c}$ and $(a,b,c) = 1$. If α is any of the type

(i)	[odd, even, odd]	and	$4\mid b$
(ii)	[odd, odd, even]	and	4 divides c

(iii)	[even, odd, odd]	and	4 divides exactly one of $(b-1)$ and $(c-1)$
(iv)	[odd, even, even]		

Then $p \equiv 1 \pmod 4$

However, if α is any of the type

(v)	[odd, even, odd]	and	$4 \mid b$
(vi)	[odd, even, odd]	and	$4 \mid c$
(vii)	[even, odd, odd]	and	4 divides either both of $(b-1)$ and $(c-1)$ or none of them.

Then $p \equiv 3 \pmod 4$.

Proof

(i) Let α be of type [odd, even, odd] and $4 \mid b$. Then

$$a = 2k_1 + 1, \quad b = 4k_2, \quad c = 2k_3 + 1$$

As $$b = \frac{a^2 - p}{c}$$

we have $$4k_2 = \frac{(2k_1 + 1)^2 - p}{2k_3 + 1}$$

i.e. $$p = 4k_4 + 1 \quad \text{where} \quad k_4 = k_1^2 + k_1 - 2k_2k_3 - k_2$$

Hence $$p \equiv 1 \pmod 4$$

(ii) Let α be of type [odd, odd, even] and $4 \mid c$. Then

$$a = 2k_1 + 1, \quad b = 2k_2 + 1, \quad c = 4k_3$$

so that $$b = \frac{a^2 - p}{c}$$

gives $$p = 4k_4 + 1$$

where $k_4 = k_1^2 + k_1 - 2k_2 k_3 - k_3$

Hence $p \equiv 1 \pmod 4$

(iii) Let α be an element of type [even, odd, odd] and 4 divides exactly one of $(b-1)$ and $(c-1)$.

Take $4 \mid (b-1)$ and $4 \nmid (c-1)$

Then $a = 2k_1, \quad b = 4k_2 + 1, \quad c = 2k_3 + 1$ (k_3 is odd)

and $b = \dfrac{a^2 - p}{c}$

give $p = 2(2k_1^2 - 4k_2k_3 - 2k_2 - k_3) - 1$

Since k_3 is odd,

$$2k_1^2 - 4k_2k_3 - 2k_2 - k_3$$

is also odd. So,

$$p = 2(2k_4 + 1) - 1 = 4k_4 + 1$$

Hence $p \equiv 1 \pmod 4$

(iv) Let α be of type [odd, even, even]. Then

$a = 2k_1 + 1, b = 2k_2, c = 2k_3$

As $b = \dfrac{a^2 - p}{c}$

$$p = 4(k_1^2 + k_1 - k_2k_3) + 1 = 4k_4 + 1$$

where $k_4 = k_1^2 + k_1 - k_2k_3$

Hence $p \equiv 1 \pmod 4$.

(v) Let α be of type [odd, even, odd] and $4|b$ then

$$a = 2k_1 + 1, \quad b = 2k_2 \ (k_2 \text{ is odd}), \qquad c = 2k_3 + 1$$

so that from $$b = \frac{a^2 - p}{c}$$

we have $$p = 2k_4 + 1,$$

where $$k_4 = 2k_1^2 + 2k_1 - 2k_2k_3 - k_2$$

Now k_4 is odd because k_2 is odd. And so

$$p = 2(2k_5 + 1) + 1 = 4k_5 + 3$$

Hence $p \equiv 3 \pmod 4$

(vi) Let α be of type [odd, odd, even] and $4|c$ then similarly as (v) from

$$a = 2k_1 + 1, \ b = 2k_2 + 1, \ c = 2k_3 \ (k_3 \text{ odd})$$

and $$b = \frac{a^2 = p}{c}$$

we obtain $$2k_2 + 1 = \frac{(2k_1 + 1)^2 - p}{2k_3}$$

$$p = 4k_5 + 3$$

i.e. $$p \equiv 3 \pmod 4$$

(vii) Let α be an element of type [even, odd, odd] and 4 divides both of $(b-1)$ and $(c-1)$ or none of them.

First we take $4|(b-1)$ and $4|(c-1)$. Then

$$a = 2k_1, \quad b = 4k_2 + 1, \; c = 4k_3 + 1$$

so

$$b = \frac{a^2-p}{c}$$

gives $\quad p = 4k_4 - 1,$

where $\quad k_4 = k_1^2 - 4k_2k_3 - k_2 - k_3$

Thus $\quad p \equiv 3 \pmod 4.$

Now we take $4 \nmid (b-1)$ and $4 \nmid (c-1)$. Then by substituting

$$a = 2k_1, \quad b = 2k_2 + 1, \; (k_2 \text{ odd}) \quad c = 2k_3 + 1 \quad (k_3 \text{ odd})$$

in

$$b = \frac{a^2-p}{c}$$

gives $\quad p = 2k_4 - 1$

where $\quad k_4 = 2k_1^2 - 2k_2k_3 - k_2 - k_3$

is even because k_2 and k_3 are odd and so

$$p = 2(2k_5) - 1$$

Proves $p \equiv 3 \pmod 4$.

Theorem 2

Let $\alpha \in Q^*(\sqrt{p})$, $b = \dfrac{a^2-p}{c}$ and $(a,b,c) = 1$. Then there exists no element of types:

(i) [odd, odd, odd]
(ii) [even, even, even]
(iii) [even, odd,even]

(iv) [even, even, odd] in $Q^*(\sqrt{p})$

Proof

(i) Let a and c be odd. Then a^2 is odd and $a^2 - p$ is even and $b = \frac{a^2-p}{c}$ = even/odd so, $b = \frac{a^2-p}{c}$ an integer forces that b cannot be odd.

(ii) Let a, b and c be even. Then $(a,b,c) \neq 1$ so, a = even, b = even, c = even is impossible for an element $\alpha \in Q^*(\sqrt{p})$.

(iii) Let a and b be even. Then a^2 is even and $a^2 - p$ is odd, so a^2 is even and $a^2 - p$ is odd.

so, $b = \frac{a^2-p}{c}$ = odd/even, not an integer, contrary to our assumption.

(iv) Let a = even and c = odd.

Then a^2 is even and $a^2 - p$ is odd. so, $b = \frac{a^2-p}{c}$ = odd/odd.

Hence $b = \frac{a^2-p}{c}$ an integer forces that b cannot be even.

(i) [odd, odd, odd]
(ii) [even, even, even]
(iii) [even, odd,even]

(iv) [even, even, odd] in $Q^*(\sqrt{p})$

Proof

(i) Let a and c be odd. Then a^2 is odd and $a^2 - p$ is even and

$b = \frac{a^2-p}{c}$ = even/odd so, $b = \frac{a^2-p}{c}$ an integer forces that b cannot be odd.

(ii) Let a, b and c be even. Then $(a,b,c) \neq 1$ so, a = even, b = even, c = even is impossible for an element $\alpha \in Q^*(\sqrt{p})$.

(iii) Let a and b be even. Then a^2 is even and $a^2 - p$ is odd, so

a^2 is even and $a^2 - p$ is odd.

so, $b = \frac{a^2-p}{c}$ = odd/even, not an integer, contrary to our assumption.

(iv) Let a = even and c = odd.

Then a^2 is even and $a^2 - p$ is odd. so, $b = \frac{a^2-p}{c}$ = odd/odd.

Hence $b = \frac{a^2-p}{c}$ an integer forces that b cannot be even.

Theorem 3

Let $\alpha \in Q^*(\sqrt{p}),\ b = \frac{a^2-p}{c},\ p \equiv 3 \pmod 4$

and a be odd and c be even then b cannot be even. In particular, if $p \equiv 3 \pmod 4$ then there exist no element of type [odd, even, even].

Proof

Let a be odd and c even and $p \equiv 3 \pmod 4$, then

$$a = 2k_1 + 1,\ c = 2k_2,\ p = 4k_3 + 3$$

As $$b = \frac{a^2-p}{c}$$

we have $$b = \frac{(2k_1+1)^2-(4k_3+3)}{2k_2}$$

$$= \frac{2k_4-1}{k_2} = \frac{odd}{odd} = odd \quad \text{(If } k_2 \text{ is odd)}$$

$$= \frac{odd}{even} \quad \text{(not an integer, if } k_2 \text{ is even)}$$

So, $b = \frac{a^2-p}{c}$ an integer forces that b cannot be even. Hence if $p \equiv 3 \pmod 4$ then there exist no element of type [odd, even, even].

Definition 2

Let $$\alpha = \frac{a+\sqrt{p}}{c},\quad b = \frac{a^2-p}{c}$$

and $$\alpha' = \frac{a'+\sqrt{p}}{c'}, \quad b' = \frac{a'^2-p}{c}$$

be the elements of $Q^*(\sqrt{p})$. We say that α and α' are of the same type (and write as $\alpha \sim \alpha'$) if, as pairs (a,a'), (b,b') and (c,c') are even or odd.

Theorem 4

The relation $\sim$ on pairs α,α' of the elements of $Q^*(\sqrt{p})$ is an equivalence relation.

Proof

The relation is reflexive as (a,a), (b,b), (c,c) as pairs are even or odd. Now, if $\alpha\sim\alpha'$ then (a,a'), (b,b'), (c,c') are even or odd and so clearly (a',a), (b',b), (c',c) are even or odd, proving $\sim$ is symmetric.

Let $\alpha\sim\alpha'$ and $\alpha'\sim\alpha''$

Then (a,a'), (b,b'),(c,c') and (a',a''),(b',b''), (c',c'') are even or odd. This shows that (a,a''), (b,b''), (c,c'') are even or odd. Therefore $\sim$ is transitive.

Hence the relation $\sim$ is an equivalence relation.

The equivalence relation defined above partitions $Q^*(\sqrt{p})$ into equivalence classes. These equivalence classes are.

[odd, even, odd]	and	$4\mid b$
[odd, odd, even]	and	$4\mid c$
[even, odd, odd]	and	4 divides exactly one of $(b-1)$ and $(c-1)$

[odd, even, even] for $p \equiv 1 \pmod 4$

However, if $p \equiv 3 \pmod 4$, then the equivalence classes are

[odd, even, odd]	and	$4 \nmid b$
[odd, odd, even]	and	$4 \nmid c$
[even, odd, odd]	and	4 divides either both of $(b-1)$ and $(c-1)$ or none of them.

SECTION II

Invariant subset of $Q^*(\sqrt{p})$

In case $p \equiv 1 \pmod 4$, we have obtained an important partition of $Q^*(\sqrt{p})$, under the action of modular group. The result is stated in the following theorem.

Theorem 5

(1) Let $\alpha = \dfrac{a+\sqrt{p}}{c} \in Q^*(\sqrt{p}), \quad b = \dfrac{a^2-p}{c}$

and α be of type [odd, even, even]. For any $g \in G$, let

$$g(\alpha) = \frac{a'+\sqrt{p}}{c'} \in Q^*(\sqrt{p}), \quad b' = \frac{a'^2-p}{c'}$$

Then α and α' are of the same type.

Proof

Since G is generated by

$$x(z) = \frac{-1}{z}, \quad y(z) = \frac{z-1}{z}$$

each $g \in G$ is a word in x,y or y^2. So, it is enough to show that $x(\alpha)$ and $y(\alpha)$ are of the same type.

$$x(\alpha) = \frac{-1}{\alpha} = \frac{-1}{(a+\sqrt{p})/c} = \frac{-c}{a+\sqrt{p}}$$

$$= \frac{-a_1+\sqrt{p}}{b} = \frac{a_1+\sqrt{p}}{c_1} \quad \text{(say)}$$

$$a_1 = -a, \quad b_1 = \frac{a_1^2-p}{c_1} = c, \quad c_1 = b$$

And $$y(\alpha) = \frac{\alpha-1}{\alpha} = \frac{(-a+b)+\sqrt{p}}{b} = \frac{a_2+\sqrt{p}}{c_2} \quad \text{(say)}$$

So, $$a_2 = -a+b, \quad b_2 = \frac{a_2^2-p}{c_2} = b-2a+c, \quad c_2 = b$$

Since a is odd and b, c are even $a_1 = a$ is odd, $b_1 = c$ is even, $c_1 = b$ is even $a_2 = -a + b$ is odd, $b_2 = b - 2a + c$ is even, $c_2 = b$ is even.

This shows that the set of elements of $Q^*(\sqrt{p})$ of type [odd, even, even] is invariant under the action of G.

Cor:

Let $$\alpha = \frac{a+\sqrt{p}}{c} \in Q^*(\sqrt{p})$$

of type [odd, even, even] with $p \equiv 1 \pmod 4$. Then the orbit α^G of α consists of the same type as α.

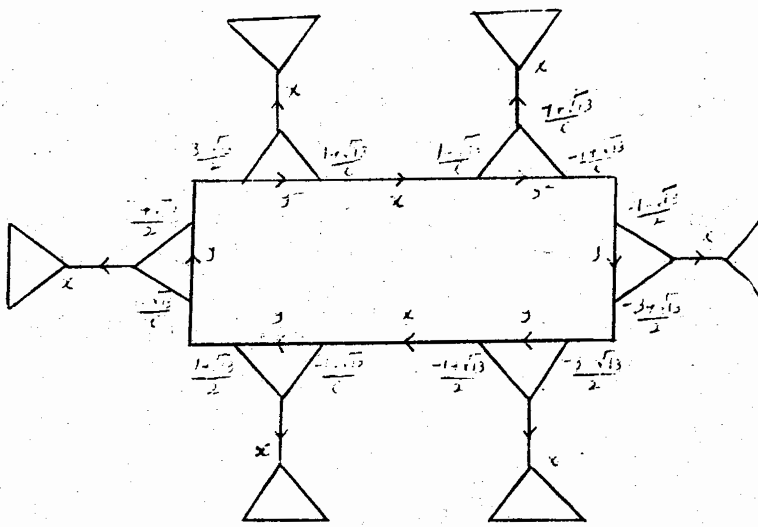

Figure 1

Theorem 6

Let $p \equiv 1 \pmod 4$ and

$$\alpha = \frac{a+\sqrt{p}}{c} \in Q^*(\sqrt{p})$$

be an element of types (i), (ii) and (iii) of theorem 1. Then the orbit α^G of α contains all types of elements other than the type (iv).

Proof

$$x(\alpha) = \frac{-1}{\alpha} = \frac{-1}{(a+\sqrt{p})/c} = \frac{-c}{a+\sqrt{p}/c}$$

$$= \frac{a_1+\sqrt{p}}{c_1} \quad \text{(say)}$$

$$a_1 = -a,\ b_1 = \frac{a_1^2-p}{c_1} = c,\ c_1 = b$$

and
$$y(\alpha) = \frac{\alpha-1}{\alpha} = \frac{(-a+b)+\sqrt{p}}{b} = \frac{a_2+\sqrt{p}}{c_2} \quad \text{(say)}$$

So,
$$a_2 = -a+b,\ b_2 = \frac{a_2^2-p}{c_2} = b-2a+c,\ c_2 = b$$

Let α be an element of type [odd, even, odd]. Then $a_1 = a$ is odd, $b_1 = c$ is odd, $c_1 = b$ is even and $a_2 = -a+b$ is odd, $b_2 = \text{v} - 2a+c$ is odd, $c_2 = b$ is even.

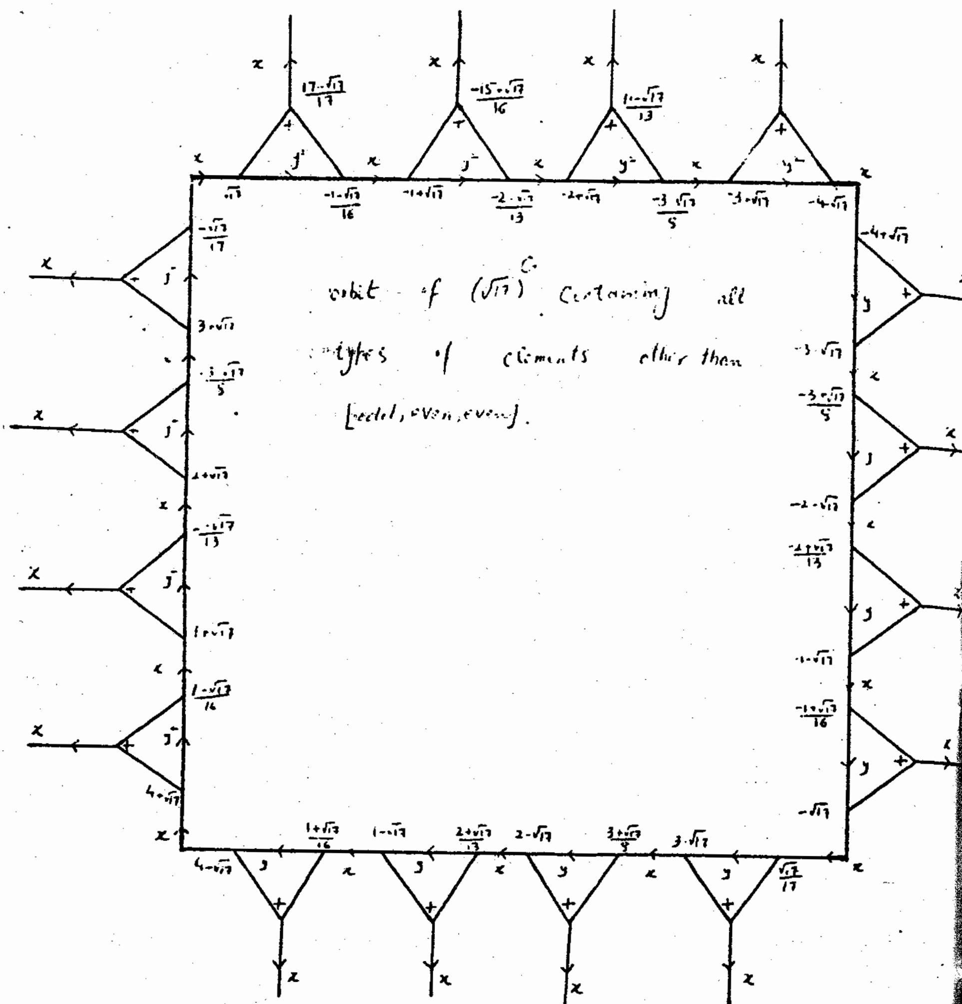

Figure 2 **Orbit of G on $Q^*(\sqrt{17})$, all types of elements other than [odd, even, even]**

Thus [odd, even, odd] $\underset{\to}{x}$ [odd, odd, even]

$\underset{\to}{y}$ [odd, odd, even]

Now let α be an element of type [odd, odd, even]. Then $a_1 = -a$ is odd, $b_1 = c$ is even, $c_1 = b$ is odd and $a_2 = -a + b$ is even $b_2 = b - 2a + c$ is odd, $c_2 = b$ is odd.

Thus [odd, odd, even] $\underset{\to}{x}$ [odd, even, odd]

$\underset{\to}{y}$ [even, odd, odd]

Next, let α be an element of type [even, odd, odd]

Then $a_1 = -a$ is even, $b_1 = c$ is odd, $c_1 = b$ is odd. And $a_2 = a + b$ is odd, $b_2 = b - 2a + c$ is even, $c_1 = b$ is odd.

Thus

[even, odd, odd] $\underset{\to}{x}$ [even, odd, odd]

$\underset{\to}{y}$ [odd, even, odd]

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COINCIDENCE POINTS OF R-WEAKLY COMMUTING MAPS

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ABSTRACT: In this paper two coincidence point theorems are proved which extend some results of Ciric [1] and Liu [2].

KEY WORDS AND PHRASES: Coincidence point, R-weakly commuting, maps metric space.

AMS (1991) Subject Classification. 54H25.

Throughout this paper, let (X, d) be a metric space, f, g and h be selfmappings of X. For x_0 in X, if there exists a sequence $\{x_n\}$ in X such that $hx_{2n+1} = fx_{2n}$, $hx_{2n+2} = gx_{2n+1}$ for $n \geq 0$, then $O(f,g,h,x_0) = [hx_n : n \geq 1]$ is called the orbit of (f,g,h) at x_0. (X,d) is called (f,g,h)-orbitally complete at x_0 if every Cauchy sequence in $O(f,g,h,x_0)$ converges in X. For T in {f,g,h}, T is said to be orbitally continuous at x_0 if it is continuous on $O(f,g,h,x_0)$. A point x in X is called a coincidence point of f and g if fx = gx. We recall that f and g is R-weakly commuting if there exists some positive real number R such that $d(fgx,gfx) \leq Rd(fx,gx)$ for x in X(cf.[3]).

Let M be a class of real valued functions Q with the properties:

a) $0 < Q(t) < t$ for $t > 0$ and $Q(0) = 0$;

b) $u(t) = t/(t-Q(t))$ is non-increasing on $(0,+\infty)$;

c) $\int_0^y u(t)dt < +\infty$ for $y > 0$; and

d) $Q(t)$ is non-decreasing on $(0, +\infty)$.

Liu [2] proved the existence of coincidence points for mappings f, g and h which satisfy $\min\{d(fx,gy), d(fx,hx), d(hy,gy)\} - \min\{d(hx,gy), d(fx,hy)\} \leq r d(hx,hy)$ for all x,y in X and some r in (0,1).

The purpose of this paper is to investigate the existence of coincidence points for mappings f, g and h which satisfy.

$$\min\{d(fx,gy), d(fx,hx), d(hy,gy)\} - \min\{d(hx,gy), d(fx,hy)\}$$

$$\leq Q(\max\{d(hx,hy), \min[d(fx,hx), d(gy,hy)\}]) \qquad (*)$$

for all x,y in X and some Q in M. We follow the techniques of Liu [2] to obtain the following results.

Theorem 1

Let f, g and h satisfy (*) and there exists a point x_0 in X such that (A) f is orbitally continuous at x_0, f and h are R-weakly commuting; (B) h is orbitally continuous at x_0 and (X,d) is (f,g,h)-orbitally complete at x_0. Then f and h or g and h have a coincidence point.

Proof

Suppose that $x_n = x_{n+1}$ for some $n \geq 0$. Then x_n is a coincidence point of f and h or g and h. Now suppose that $x_n \neq x_{n+1}$ for $n \geq 0$. Set $d_n = d(hx_n, hx_{n+1})$. By (*) for $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$\min\{d_{2n}, d_{2n+1}\} = \min\{d_{2n+1}, d_{2n}, d_{2n+1}\} - \min\{d(hx_{2n}, hx_{2n+2}), 0\}$$

$$\leq Q(\max\{d_{2n}, \min\{d_{2n}, d_{2n+1}\}\})$$

$$= Q(d_{2n})$$

We assert that $d_{2n+1} < Q(d_{2n})$. Otherwise $d_{2n} \leq Q(d_{2n}) < d_{2n}$, which is impossible. Hence $d_{2n+1} \leq Q(d_{2n})$. Similarly, we have $d_{2n+2} \leq Q(d_{2n+1})$. Thus $d_{n+1} \leq Q(d_n)$ for $n \geq 0$. Let $t_0 = d_0$ and $t_n = Q(t_{n-1})$ for $n \geq 1$. It is easy to see that $0 < t_n = Q(t_{n-1}) = Q^n(t_0) < t_{n-1}$. This implies that the sequence $\{t_n\}$ is convergent. Obviously we have

$$d(hx_n, hx_{n+p}) \leq \sum_{i=n}^{n+p-1} d_1 \leq \sum_{i=n}^{n+p-1} t_1 \leq \int_{t_{n+p}}^{t_n} u(t)dt$$

for all $n, p \geq 1$. Thus $\{hx_n\}_{n\geq 1}$ is Cauchy sequence. (B) implies that there exists v in X such that $hx_n \to v$ as $n \to \infty$. Since f and h are orbitally continuous at x_0 then $fhx_n \to fv$ and $h^2x_n \to hv$ as $n \to \infty$. It follows from R-weak commutativity of f and h that:

$$d(fv,hv) \leq d(fv,fhx_{2n}) + d(fhx_{2n}, hfx_{2n} + d(hfx_{2n}, hv)$$

$$\leq d(fv,fhx_{2n}) + Rd(hx_{2n}, fx_{2n}) + d(hhx_{2n+1}, hv)$$

Let n tend to infinity. The $d(fv, hv) \leq 0$. Consequently $fv = hv$. The completes the proof.

By using the method similar to the proof of Theorem 1 we have.

Theorem 2

Let f,g,h and x_0 be as in Theorem 1 and satisfy (C) g is orbitally continuous at x_0, g and h are R-weakly commuting.

Then f, g and h have a common coincidence point i.e., there exists t in X such that $ft = gt = ht$.

Remark 1

Theorem 1 of Ciric [1] and Theorem 3.1 of Liu [2] are special cases of the above theorem 1.

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DEFINING NEW CONSISTENCY RELATIONS FOR SPLINE SOLUTIONS OF SIXTH AND EIGHTH ORDER BOUNDARY-VALUE PROBLEMS

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Dedicated to the memory of Dr. M. Rafique

ABSTRACT: Linear, sixth-order boundary-value problems (special case) are solved, using polynomial splines of degree six.
The spline function values at the midknots of the interpolation interval, and the corresponding values of the even-order derivatives are related through consistency relations. The algorithm developed approximates the solutions, and their higher-order derivatives, of differential equations. Four numerical illustrations are given to show the practical usefulness of the algorithm developed. It is observed that this algorithm is second-order convergent.

KEYWORDS: Sixth-order, Eighth-order,Two-point boundary-value problems; finite-difference methods; Sextic splines,Octic Splines.

1 INTRODUCTION

When an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability is as ordinary convection the ordinary differential equation is sixth order; when the instability sets in as overstability, it is modelled by an eighth-order ordinary differential equation.

Suppose, now, that a uniform magnetic field is also applied across the fluid in the same direction as gravity. When instability sets in now as ordinary convection, it is modelled by a tenth-order boundary-value problem; when instability sets in as overstability, it is modelled by a twelfth-order boundary-value problem (for details, see Chandrasekhar [4]).

Usmani [10], solved fourth-order boundary-value problem using quartic splines.

In the present paper sixth-order and eighth-order boundary-value problems are solved using sextic and octic splines, respectively, introducing some new consistency relations.
These problems have the form

$$\left.\begin{array}{l} y^{(vi)} + \phi(x)y = \psi(x)\,, \qquad -\infty < a \le x \le b < \infty\,, \\ y(a) = A_0\,,\ y^{(ii)}(a) = A_2\,,\ y^{(iv)}(a) = A_4\,, \\ y(b) = B_0\,,\ y^{(ii)}(b) = B_2\,,\ y^{(iv)}(b) = B_4 \\ \text{for sixth} - \text{order BVPs} \\ \text{and} \\ \tilde{y}^{(viii)} + \tilde{\phi}(x)\tilde{y} = \tilde{\psi}(x)\,, \qquad -\infty < a \le x \le b < \infty\,, \\ \tilde{y}(a) = \tilde{A}_0\,,\ \tilde{y}^{(ii)}(a) = \tilde{A}_2\,,\ \tilde{y}^{(iv)}(a) = \tilde{A}_4\,,\ \tilde{y}^{(vi)}(a) = \tilde{A}_6\,, \\ \tilde{y}(b) = \tilde{B}_0\,,\ \tilde{y}^{(ii)}(b) = \tilde{B}_2\,,\ \tilde{y}^{(iv)}(b) = \tilde{B}_4\,,\ \tilde{y}^{(vi)}(b) = \tilde{B}_6 \\ \text{for eighth} - \text{order BVPs} \end{array}\right\} \tag{1.1}$$

where $y = y(x)$, and $\phi(x)$, $\psi(x)$ and $\tilde{\phi}(x)$, $\tilde{\psi}(x)$ are continuous functions defined in the interval $x \in [a,\ b]$, and A_i and B_i, $i = 0, 2, 4$, are finite real constants for sixth-order case and $\tilde{A}_i$ and $\tilde{B}_i$, $i = 0, 2, 4, 6$ are finite real constants, for eighth-order case.

2 THE SEXTIC AND OCTIC SPLINES

2.1 CONSISTENCY RELATIONS

The interval $[a,\ b]$ is divided into n equal parts, where $n \geq 10$ for sextic spline and $n \geq 14$ for octic spline, thus introducing $n+1$ grid points x_i so that

$$x_i = a + ih\ , \quad i = 0, 1, 2, \ldots, n,$$

$$x_0 = a, \quad x_n = b \quad \text{and} \quad h = \frac{b-a}{n}.$$

The exact solutions of the problems (1.1) at $x = x_i$ are $y(x_i)$ and $\tilde{y}(x_i)$, respectively. Let s_i be the approximation to y at x_i determined by the sextic spline $Q_i(x)$, defined on the sub-interval $[x_i,\ x_{i+1}]$ by

$$\begin{aligned} Q_i(x) = \ & a_i(x-x_i)^6 + b_i(x-x_i)^5 + c_i(x-x_i)^4 \\ & + d_i(x-x_i)^3 + e_i(x-x_i)^2 + f_i(x-x_i) + g_i\ , \\ & i = 0, 1, \ldots, n-1. \end{aligned} \tag{2.1}$$

The sextic spline $s(x) \in C^5[a,\ b]$ can, thus , be defined as

$$s(x) = Q_i(x), \quad x \in [x_i,\ x_{i+1}], \quad i = 0, 1, \ldots, n-1\ . \tag{2.2}$$

The octic spline is defined similarly (see [12]).
The coefficients of (2.1) are determined, (see [11]), as

$$a_i = \frac{1}{720} s^{(vi)}_{i+1/2}\ , \tag{2. 3}$$

$$b_i = \frac{1}{120} s^{(v)}_i\ , \tag{2. 4}$$

$$c_i = \frac{1}{24} s^{(iv)}_{i+1/2} - \frac{1}{48} h s^{(v)}_i - \frac{1}{192} h^2 s^{(vi)}_{i+1/2}\ , \tag{2. 5}$$

$$d_i = \frac{1}{6} s'''_i\ , \tag{2. 6}$$

$$e_i = \frac{1}{2} s''_{i+1/2} - \frac{1}{4} h s'''_i - \frac{1}{16} h^2 s^{(iv)}_{i+1/2} + \frac{1}{48} h^3 s^{(v)}_i$$

$$+\frac{5}{768}h^4 s^{(vi)}_{i+1/2}\,, \tag{2.7}$$

$$f_i = s'_i\,, \tag{2.8}$$

$$\begin{aligned} g_i = {} & s_{i+1/2} - \frac{1}{2}hs'_i - \frac{1}{8}h^2 s''_{i+1/2} + \frac{1}{24}h^3 s'''_i + \frac{5}{384}h^4 s^{(iv)}_{i+1/2} \\ & - \frac{1}{240}h^5 s^{(v)}_i - \frac{61}{46080}h^6 s^{(vi)}_{i+1/2}\,. \end{aligned} \tag{2.9}$$

The coefficients of octic spline are determined similarly, (see [12]). The odd-order derivatives of the sextic spline are defined as (see [11])

$$hs^{(v)}_i = (s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) - \tfrac{1}{8}h^2(s^{(vi)}_{i+1/2} + s^{(vi)}_{i-1/2})\,, \tag{2.10}$$

$$\begin{aligned} hs'''_i = {} & (s''_{i+1/2} - s''_{i-1/2}) - \tfrac{1}{24}h^2(s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) \\ & + \tfrac{1}{384}h^4(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2}) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} hs'_i = {} & (s_{i+1/2} - s_{i-1/2}) - \tfrac{1}{24}h^2(s''_{i+1/2} - s''_{i-1/2}) \\ & + \tfrac{7}{5760}h^4(s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) - \tfrac{1}{15360}h^6(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2})\,. \end{aligned} \tag{2.12}$$

The odd-order derivatives of the octic spline are defined similarly (see [12])

The following recurrence relations are obtained to define even-order derivatives of the sextic spline, (see [11])

$$\begin{aligned} h^4 s^{(iv)}_{i-1/2} = {} & (s_{i-5/2} - 4s_{i-3/2} + 6s_{i-1/2} - 4s_{i+1/2} + s_{i+3/2}) \\ & - \tfrac{1}{46080}h^6(s^{(vi)}_{i-5/2} + 724s^{(vi)}_{i-3/2} + 6230s^{(vi)}_{i-1/2} \\ & + 724s^{(vi)}_{i+1/2} + s^{(vi)}_{i+3/2})\,, \quad i = 3,\ 4, \ldots\ldots, n-2\,, \end{aligned} \tag{2.13}$$

$$\begin{aligned} & (s_{i-7/2} + 2s_{i-5/2} - 17s_{i-3/2} + 28s_{i-1/2} - 17s_{i+1/2} \\ & + 2s_{i+3/2} + s_{i+5/2}) \\ = {} & \tfrac{1}{5760}h^4\left(s^{(iv)}_{i-7/2} + 722s^{(iv)}_{i-5/2} + 10543s^{(iv)}_{i-3/2} + 23548s^{(iv)}_{i-1/2}\right. \\ & \left. + 10543s^{(iv)}_{i+1/2} + 722s^{(iv)}_{i+3/2} + s^{(iv)}_{i+5/2}\right)\,, \end{aligned} \tag{2.14}$$

$$h^2 s''_{i-1/2} = -\frac{1}{16}(s_{i-5/2} - 20s_{i-3/2} + 38s_{i-1/2} - 20s_{i+1/2} + s_{i+3/2}) + \frac{1}{92160}h^4(s^{(iv)}_{i-5/2} + 700s^{(iv)}_{i-3/2} - 3322s^{(iv)}_{i-1/2} + 700s^{(iv)}_{i+1/2} + s^{(iv)}_{i+3/2}), \quad (2.15)$$

$$i = 3, 4, \ldots\ldots, n-2,$$

$$\begin{aligned}&\left(s_{i-7/2} + 74s_{i-5/2} + 79s_{i-3/2} - 308s_{i-1/2} + 79s_{i+1/2} + 74s_{i+3/2} + s_{i+5/2}\right)\\ =\ &\tfrac{1}{120}h^2\left(s''_{i-7/2} + 722s''_{i-5/2} + 10543s''_{i-3/2} + 23548s''_{i-1/2} + 10543s''_{i+1/2} + 722s''_{i+3/2} + s''_{i+5/2}\right)\end{aligned} \quad (2.16)$$

and

$$\begin{aligned}&\left(s_{i-7/2} - 6s_{i-5/2} + 15s_{i-3/2} - 20s_{i-1/2} + 15s_{i+1/2} - 6s_{i+3/2} + s_{i+5/2}\right)\\ =\ &\tfrac{1}{46080}h^6\left(s^{(vi)}_{i-7/2} + 722s^{(vi)}_{i-5/2} + 10543s^{(vi)}_{i-3/2} + 23548s^{(vi)}_{i-1/2} + 10543s^{(vi)}_{i+1/2} + 722s^{(vi)}_{i+3/2} + s^{(vi)}_{i+5/2}\right),\end{aligned} \quad (2.17)$$

$$i = 4, 5, \ldots\ldots, n-3.$$

Following S.S.S. and E.H.T. [11], the following new consistency relation is determined, which can be considered to be the replacement for equation (2.15)

$$h^2 s''_{i-1/2} = -\frac{1}{12}(s_{i-5/2} - 16s_{i-3/2} + 30s_{i-1/2} - 16s_{i+1/2} + s_{i+3/2}) + \frac{1}{552960}h^6(s^{(vi)}_{i-5/2} + 712s^{(vi)}_{i-3/2} + 4718s^{(vi)}_{i-1/2} + 712s^{(vi)}_{i+1/2} + s^{(vi)}_{i+3/2}), \quad (2.18)$$

$$i = 3, 4, \ldots\ldots, n-2,$$

The consistency relations for octic spline are defined similarly (see [12])

Following S.S.S. and E.H.T. [12], some new consistency relations are determined to define even-order derivatives of the octic spline.
To replace the following equation (2.74) [12]

$$\begin{aligned} h^4 s^{(iv)}_{i-1/2} &= \frac{1}{8}\left(-19s_{i-7/2} + 122s_{i-5/2} - 317s_{i-3/2} + 428s_{i-1/2}\right. \\ &\quad \left. - 317s_{i+1/2} + 122s_{i+3/2} - 19s_{i+5/2}\right) + \frac{1}{10321920}h^6 \\ &\quad \left(19s^{(vi)}_{i-7/2} + 124366s^{(vi)}_{i-5/2} + 5501981s^{(vi)}_{i-3/2}\right. \\ &\quad + 11541508s^{(vi)}_{i-1/2} + 5501981s^{(vi)}_{i+1/2} + 124366s^{(vi)}_{i+3/2} \\ &\quad \left. + \ 19s^{(vi)}_{i+5/2}\right) \ , \ i = 5,\ 6, \ldots\ldots, n-4\ , \qquad (2.\ 19) \end{aligned}$$

the following consistency relation is determined

$$\begin{aligned} h^4 s^{(iv)}_{i-1/2} &= \frac{1}{6}\left(-s_{i-7/2} + 12s_{i-5/2} - 39s_{i-3/2} + 56s_{i-1/2}\right. \\ &\quad \left. - 39s_{i+1/2} + 12s_{i+3/2} - s_{i+5/2}\right) + \frac{1}{61931520}h^8 \\ &\quad \left(s^{(viii)}_{i-7/2} + 6548s^{(viii)}_{i-5/2} + 305383s^{(viii)}_{i-3/2}\right. \\ &\quad + 1182472s^{(viii)}_{i-1/2} + 305383s^{(viii)}_{i+1/2} + 6548s^{(viii)}_{i+3/2} \\ &\quad \left. + \ s^{(vi)}_{i+5/2}\right) \ . \qquad (2.\ 20) \end{aligned}$$

To replace the following equation (2.76) [12]

$$\begin{aligned} h^2 s''_{i-1/2} &= \frac{1}{6418560}\left(-7561s_{i-7/2} - 378746s_{i-5/2} + 8001593s_{i-3/2}\right. \\ &\quad -15230572s_{i-1/2} + 8001593s_{i+1/2} - 378746s_{i+3/2} \\ &\quad \left. - 7561s_{i+5/2}\right) + \frac{1}{172530892800}h^4\left(7561s^{(iv)}_{i-7/2}\right. \\ &\quad + 49374026s^{(iv)}_{i-5/2} + 1426444807s^{(iv)}_{i-3/2} \\ &\quad - 5929096628s^{(iv)}_{i-1/2} + 1426444807s^{(iv)}_{i+1/2} \\ &\quad \left. + 4937402s^{(iv)}_{i+3/2} + 7561s^{(iv)}_{i+5/2}\right) \ , \ i = 5,\ 6, \ldots, n-4\ , \\ &\qquad (2.\ 21) \end{aligned}$$

the following two relations are determined

$$\begin{aligned} h^2 s''_{i-1/2} &= \frac{1}{192}(35s_{i-7/2} - 226s_{i-5/2} + 781s_{i-3/2} \\ &\quad -1180s_{i-1/2} + 781s_{i+1/2} - 226s_{i+3/2} \\ &\quad + 35s_{i+5/2}) - \frac{1}{247726080}h^6 \Big(35s^{(vi)}_{i-7/2} \\ &\quad + 229094s^{(vi)}_{i-5/2} + 10127149s^{(vi)}_{i-3/2} + 21693332s^{(vi)}_{i-1/2} \\ &\quad + 10127149s^{(vi)}_{i+1/2} + 229094s^{(vi)}_{i+3/2} + 35s^{(vi)}_{i+5/2}\Big) \end{aligned} \quad (2.\ 22)$$

and

$$\begin{aligned} h^2 s''_{i-1/2} &= \frac{1}{180}(2s_{i-7/2} - 27s_{i-5/2} + 270s_{i-3/2} \\ &\quad -490s_{i-1/2} + 270s_{i+1/2} - 27s_{i+3/2} \\ &\quad + 2s_{i+5/2}) - \frac{1}{1857945600}h^8 \Big(2s^{(viii)}_{i-7/2} \\ &\quad + 13093s^{(viii)}_{i-5/2} + 591278s^{(viii)}_{i-3/2} + 2109014s^{(viii)}_{i-1/2} \\ &\quad + 591278s^{(viii)}_{i+1/2} + 13093s^{(viii)}_{i+3/2} + 2s^{(viii)}_{i+5/2}\Big) \ . \end{aligned} \quad (2.\ 23)$$

2.2 END CONDITIONS

The following end conditions for sextic spline are obtained (see [11])

$$\begin{aligned} &(20s_0 - 35s_{1/2} + 21s_{3/2} - 7s_{5/2} + s_{7/2}) \\ = \ &\tfrac{7}{2}h^2 s''_0 - \tfrac{77}{96}h^4 s^{(iv)}_0 + \tfrac{h^6}{46080}\Big(13005s^{(vi)}_{1/2} \\ + \ &9821s^{(vi)}_{3/2} + 721s^{(vi)}_{5/2} + s^{(vi)}_{7/2}\Big) \end{aligned} \quad (2.24)$$

$$\begin{aligned}
&\left(-10s_0 + 21s_{1/2} - 21s_{3/2} + 15s_{5/2} - 6s_{7/2} + s_{9/2}\right) \\
= \ & -\tfrac{3}{4}h^2 s_0'' - \tfrac{25}{64}h^4 s_0^{(iv)} + \tfrac{h^6}{46080}\Big(9821 s_{1/2}^{(vi)} \\
+ \ & 23547 s_{3/2}^{(vi)} + 10543 s_{5/2}^{(vi)} + 722 s_{7/2}^{(vi)} + s_{9/2}^{(vi)}\Big)
\end{aligned} \qquad (2.25)$$

and

$$\begin{aligned}
&\left(2s_0 - 7s_{1/2} + 15s_{3/2} - 20s_{5/2} + 15s_{7/2} - 6s_{9/2} + s_{11/2}\right) \\
= \ & -\tfrac{1}{4}h^2 s_0'' - \tfrac{1}{192}h^4 s_0^{(iv)} + \tfrac{h^6}{46080}\Big(721 s_{1/2}^{(vi)} + 10543 s_{3/2}^{(vi)} \\
& +23548 s_{5/2}^{(vi)} + 10543 s_{7/2}^{(vi)} + 722 s_{9/2}^{(vi)} + s_{11/2}^{(vi)}\Big) \ .
\end{aligned} \qquad (2.26)$$

The remaining last three end conditions may be written similarly . The end conditions for octic spline can be seen from [12].

3 THE SPLINE SOLUTION

For the solution of sixth-order BVPs the following system of equations can thus be written (see [11])

$$\begin{aligned}
(i) \quad & \mathbf{MY} = \mathbf{C} + \mathbf{T}\ , \\
(ii) \quad & \mathbf{MS} = \mathbf{C}\ , \\
(iii) \quad & \mathbf{ME} = \mathbf{T}\ .
\end{aligned} \qquad (3.1)$$

where $\mathbf{Y} = \left(y_{i-1/2}\right)$, $\mathbf{T} = (t_i)$, $\mathbf{E} = \left(\hat{e}_{i-1/2}\right)$, $i = 1, 2 \ldots, n$,

$$\mathrm{M} \ = \ \mathrm{M}_0 + \frac{1}{46080} h^6 \mathrm{BF}\ , \qquad (3.2)$$

$$\mathbf{S} \ = \ \left(s_{i-1/2}\right)\ , \quad i = 1, 2, \ldots\ldots, n \qquad (3.3)$$

and

$$\mathbf{C} \ = \ (\hat{c}_i), \quad i = 1, 2, \ldots\ldots, n\ . \qquad (3.4)$$

Also, M_0 and B are seven-band symmetric matrices, with

$$M_0 = \begin{bmatrix} -35 & 21 & -7 & 1 & & & & & & \\ 21 & -21 & 15 & -6 & 1 & & & & & \\ -7 & 15 & -20 & 15 & -6 & 1 & & & & \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 & & & \\ & & & & \ddots & \ddots & & & & \\ & & & 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ & & & & 1 & -6 & 15 & -20 & 15 & -7 \\ & & & & & 1 & -6 & 15 & -21 & 21 \\ & & & & & & 1 & -7 & 21 & -35 \end{bmatrix} \tag{3.5}$$

and

$$B = \begin{bmatrix} 13005 & 9821 & 721 & 1 & & & & & \\ 9821 & 23547 & 10543 & 722 & 1 & & & & \\ 721 & 10543 & 23548 & 10543 & 722 & 1 & & & \\ 1 & 722 & 10543 & 23548 & 10543 & 722 & 1 & & \\ & & & \ddots & \ddots & & & & \\ & & 1 & 722 & 10543 & 23548 & 10543 & 722 & 1 \\ & & & 1 & 722 & 10543 & 23548 & 10543 & 721 \\ & & & & 1 & 722 & 10543 & 23547 & 9821 \\ & & & & & 1 & 721 & 9821 & 13005 \end{bmatrix} \tag{3.6}$$

The matrix F is defined as

$$F = \operatorname{diag}\left(\phi_{i-1/2}\right), \quad i = 1, 2, \ldots, n \tag{3.7}$$

and the vector $C = (\hat{c}_i)$, $i = 1, 2, \ldots, n$ is defined as

$$\hat{c}_i = \begin{cases} -20A_0 + \frac{7}{2}h^2A_2 - \frac{77}{96}h^4A_4 + \frac{h^6}{46080}(13005\psi_{1/2} \\ +\ 9821\psi_{3/2} + 721\psi_{5/2} + \psi_{7/2})\ , & i = 1\ , \\ 10A_0 - \frac{3}{4}h^2A_2 - \frac{25}{64}h^4A_4 + \frac{h^6}{46080}(9821\psi_{1/2} \\ +\ 23547\psi_{3/2} + 10543\psi_{5/2} + 722\psi_{7/2} + \psi_{9/2})\ , & i = 2\ , \\ -2A_0 - \frac{1}{4}h^2A_2 - \frac{1}{192}h^4A_4 + \frac{h^6}{46080}(721\psi_{1/2} \\ +\ 10543\psi_{3/2} + 23548\psi_{5/2} + 10543\psi_{7/2} \\ +\ 722\psi_{9/2} + \psi_{11/2})\ , & i = 3\ , \\ \frac{h^6}{46080}(\psi_{i-7/2} + 722\psi_{i-5/2} + 10543\psi_{i-3/2} \\ +\ 23548\psi_{i-1/2} + 10543\psi_{i+1/2} + 722\psi_{i+3/2} \\ +\ \psi_{i+5/2})\ ,\ i = 4, 5, \ldots, n-3. \end{cases} \tag{3.8}$$

The definitions of $\hat{c}_{n-2}$, $\hat{c}_{n-1}$ and $\hat{c}_n$ are anologous to those of $\hat{c}_3$, $\hat{c}_2$ and $\hat{c}_1$ respectively, except that the boundary values B_0, B_2 and B_4 will replace A_0, A_2 and A_4 at the other end .

After determining $s_{i-1/2}$, $i = 1, 2, \ldots, n$, s_0 and s_n can be determined using the differential equation (1.1). Moreover, $s^{(vi)}_{i-1/2}$, $i = 1, 2, \ldots, n$, $s^{(vi)}_0$ and $s^{(vi)}_n$ can be determined using (1.1). The derivatives $s^{(iv)}_{i-1/2}$, $i = 1, 2, \ldots, n$, (that is, the fourth derivative of the spline at the midknots) can be determined using (2.13) and (2.14) and $s''_{i-1/2}$, $i = 1, 2, \ldots, n$, (that is, the second derivative of the spline at the midknots) can be determined using (2.15) and (2.16).

Now it is possible to determine the odd-order derivatives of the spline; s'_i , $i = 1, 2, \ldots, n-1$ are determined using (2.12); s'''_i , $i = 1, 2, \ldots, n-1$ are determined using (2.11); $s^{(v)}_i$, $i = 1, 2, \ldots, n-1$ are determined using (2.10), while s'_0, s'_n, s'''_0 , s'''_n , $s^{(v)}_0$ and $s^{(v)}_n$ are determined through the following relations, which are obtained while determining (2.10)—(2.12) .

$$h(s'_i - s'_{i-1}) = h^2 s''_{i-1/2} + \frac{1}{24}h^4 s^{(iv)}_{i-1/2} + \frac{1}{1920}h^6 s^{(vi)}_{i-1/2}\ , \tag{3.9}$$

$$h(s'''_i - s'''_{i-1}) = h^2 s^{(iv)}_{i-1/2} + \frac{1}{24}h^4 s^{(vi)}_{i-1/2}\ , \tag{3.10}$$

and

$$h(s_i^{(v)} - s_{i-1}^{(v)}) = h^2 s_{i-1/2}^{(vi)} . \tag{3.11}$$

For the solution of eighth-order case see([12])

4 NUNERICAL RESULTS

In this section, two problems are discussed to compare the maximum absolute errors for both old and new consistency relations alongwith the analytical solutions.

All calculations are computed in double-precision arithmetic.

PROBLEM 4.1

Consider the following sixth-order boundary-value problem

$$\begin{aligned} y^{(vi)} + xy &= -(24 + 11x + x^3)e^x , \quad 0 \leq x \leq 1 , \\ y(0) &= 0 = y(1) , \ y''(0) = 0 , \ y''(1) = -4e , \\ y^{(iv)}(0) &= -8 , \ y^{(iv)}(1) = -16e \end{aligned} \tag{4.1}$$

and the following eighth-order boundary-value problem

$$\begin{aligned} y^{(viii)} + xy &= -(48 + 15x + x^3)e^x , \quad 0 \leq x \leq 1 , \\ y(0) &= 0 = y(1) , \ y''(0) = 0 , \ y''(1) = -4e , \\ y^{(iv)}(0) &= -8 , \ y^{(iv)}(1) = -16e , \ y^{(vi)}(0) = -24 , \\ y^{(vi)}(1) &= -36e . \end{aligned} \tag{4.2}$$

The analytical solution of the above differential systems is

$$y(x) = x(1-x)e^x \tag{4.3}$$

The maximum errors (in absolute value) in $y_i^{(k)}$, $k = 0, 1, 2, \ldots, 5$, are shown in Table 1 for 6th-order case and $k = 0, 1, 2, \ldots, 7$, in Table 2 for 8th-order case.

PROBLEM 4.2

Consider the following sixth-order boundary-value problem

$$\begin{aligned}
y^{(vi)} + y &= 6[2x\cos(x) + 5\sin(x)]\,, \qquad -1 \le x \le 1\,,\\
y(-1) &= 0 = y(1)\,,\\
y''(-1) &= -4\cos(-1) + 2\sin(-1)\,,\\
y''(1) &= 4\cos(1) + 2\sin(1)\,,\\
y^{(iv)}(-1) &= 8\cos(-1) - 12\sin(-1)\,,\\
y^{(iv)}(1) &= -8\cos(1) - 12\sin(1)
\end{aligned} \tag{4.4}$$

and the following eighth-order boundary-value problem

$$\begin{aligned}
y^{(viii)} - y &= -8[2x\cos(x) + 7\sin(x)]\,, \qquad -1 \le x \le 1\,,\\
y(-1) &= 0 = y(1)\,,\\
y''(-1) &= -4\cos(-1) + 2\sin(-1)\,,\\
y''(1) &= 4\cos(1) + 2\sin(1)\,,\\
y^{(iv)}(-1) &= 8\cos(-1) - 12\sin(-1)\,,\\
y^{(iv)}(1) &= -8\cos(1) - 12\sin(1)\,,\\
y^{(vi)}(-1) &= -12\cos(-1) + 30\sin(-1)\,,\\
y^{(vi)}(1) &= 12\cos(1) + 30\sin(1)\,.
\end{aligned} \tag{4.5}$$

The analytical solution of the above differential systems is

$$y(x) = (x^2 - 1)\sin(x)\,. \tag{4.6}$$

The maximum errors (in absolute value) in $y_i^{(k)}$, $k = 0, 1, 2, \ldots, 5$, are shown in Table 1 for 6th-order case and $k = 0, 1, 2, \ldots, 7$, in Table 2 for 8th-order case.

Table 1: Maximum absolute errors for Problem 4.1 (6th-order) with $n = 100$.

$y_i^{(k)}$	Old for $[x_3, x_{n-3}]$	New for $[x_3, x_{n-3}]$	Old for otherwise	New for otherwise
$k = 0$	0.54×10^{-6}	0.56×10^{-6}	0.94×10^{-7}	0.35×10^{-7}
$k = 1$	0.17×10^{-5}	0.17×10^{-5}	0.82×10^{-5}	0.23×10^{-5}
$k = 2$	0.53×10^{-5}	0.55×10^{-5}	0.10×10^{-1}	0.72×10^{-3}
$k = 3$	0.17×10^{-4}	0.18×10^{-4}	0.42×10	0.29×10
$k = 4$	0.10×10^{-3}	0.69×10^{-4}	0.71×10^{3}	0.49×10^{2}
$k = 5$	0.36×10^{-1}	0.12×10^{-1}	0.47×10^{5}	0.33×10^{4}

Table 2: Maximum absolute errors for Problem 4.1 (8th-order) with $n = 32$.

$y_i^{(k)}$	Old for $[x_4, x_{n-4}]$	New for $[x_4, x_{n-4}]$	Old for otherwise	New for otherwise
$k = 0$	0.81×10^{-7}	0.91×10^{-6}	0.18×10^{4}	0.10×10^{4}
$k = 1$	0.24×10^{-6}	0.26×10^{-5}	0.43×10^{5}	0.43×10^{5}
$k = 2$	0.84×10^{-6}	0.90×10^{-5}	0.36×10^{7}	0.36×10^{7}
$k = 3$	0.27×10^{-5}	0.26×10^{-4}	0.38×10^{10}	0.38×10^{10}
$k = 4$	0.11×10^{-4}	0.88×10^{-4}	0.90×10^{12}	0.90×10^{12}
$k = 5$	0.48×10^{-4}	0.27×10^{-3}	0.88×10^{14}	0.88×10^{14}
$k = 6$	0.53×10^{-3}	0.88×10^{-3}	0.44×10^{16}	0.44×10^{16}
$k = 7$	0.11×10^{-1}	0.12×10^{-1}	0.94×10^{17}	0.94×10^{17}

Table 3: Maximum absolute errors for Problem 4.2 (6th-order) with $n = 100$.

$y_i^{(k)}$	Old for $[x_3, x_{n-3}]$	New for $[x_3, x_{n-3}]$	Old for otherwise	New for otherwise
$k = 0$	0.58×10^{-6}	0.73×10^{-6}	0.87×10^{-7}	0.92×10^{-7}
$k = 1$	0.17×10^{-5}	0.22×10^{-5}	0.52×10^{-5}	0.28×10^{-5}
$k = 2$	0.49×10^{-5}	0.72×10^{-5}	0.40×10^{-2}	0.55×10^{-3}
$k = 3$	0.16×10^{-4}	0.23×10^{-4}	0.10×10	0.11×10
$k = 4$	0.50×10^{-4}	0.71×10^{-4}	0.10×10	0.30×10^{1}
$k = 5$	0.52×10^{-2}	0.14×10^{-2}	0.47×10^{4}	0.30×10^{3}

Table 4: Maximum absolute errors for Problem 4.2 (8th-order) with $n = 64$.

$y_i^{(k)}$	Old for $[x_4, x_{n-4}]$	New for $[x_4, x_{n-4}]$	Old for otherwise	New for otherwise
$k = 0$	0.12×10^{-5}	0.46×10^{-5}	0.16×10^{4}	0.17×10^{4}
$k = 1$	0.29×10^{-5}	0.78×10^{-5}	0.38×10^{5}	0.38×10^{5}
$k = 2$	0.71×10^{-5}	0.12×10^{-4}	0.16×10^{8}	0.32×10^{7}
$k = 3$	0.20×10^{-4}	0.25×10^{-4}	0.34×10^{10}	0.34×10^{10}
$k = 4$	0.59×10^{-4}	0.46×10^{-4}	0.80×10^{12}	0.80×10^{12}
$k = 5$	0.18×10^{-3}	0.12×10^{-3}	0.78×10^{14}	0.78×10^{14}
$k = 6$	0.57×10^{-3}	0.29×10^{-3}	0.39×10^{16}	0.39×10^{16}
$k = 7$	0.47×10^{-2}	0.53×10^{-2}	0.83×10^{17}	0.83×10^{17}

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CONTENTS