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ON STATISTICALLY CONVERGENT SERIES

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ABSTRACT: In this article the notion of convergence of series is extended and statistically convergent series is introduced. Some properties are studied and interesting results are established. The deviations are established providing suitable examples.

KEY WARDS: statistical convergence, bounded series, conditional convergence, Cauchy criteria.

AMS Classification no. : 40A05, 46A45.

1. INTRODUCTION

The main object of this article is to introduce statistically convergent series and some definitions. The idea is similar to statistical convergence of sequences. The idea of statistical convergence was introduced by Fast [3], Buck [1] and Schoenberg [9] independently. Later on it was studied and linked with summability by Fridy ([4], [5]), Šalăt [8], Rath and Tripathy [7], Tripathy [10], Conner [2], Maddox [6] and many others.

Definition

A series $\sum_k x_k$ is said to be statistically convergent, if it's sequence of partial sums (s_n), where $s_n = x_1 + x_2 + x_3 + \ldots + x_n$ is statistically convergent. Throughout sums without limit means it is from $k = 1$ to ∞.

A series $\sum_k x_k$ is said to be bounded if it's sequence of partial sums is bounded.

The idea depends on the density of a certain subset A of the set N of natural numbers. A subset A of N is said to have density $\delta(A)$ if

$\lim_{n\to\infty} \frac{|A(n)|}{n} = \delta(A)$, where $A(n) = \{k \leq n{:}k \in A\}$ and $|A|$ denotes the cardinality of A. Clearly finite sets have zero density, $\delta(A^c) = \delta(N\text{-}A) = 1\text{-}\delta(A)$, whenever both sides exist. Throughout A^c is the complement of the set A in N. A sequence (x_n) is said to be statistically convergent to α, written as stat-lim $x_n = \alpha$, if for every $\epsilon > 0$,

$$\delta[\{k \in N{:}|x_k - \alpha| \geq \epsilon\}] = 0$$

A subset $K = \{k_j;\ j \in N\}$ of N is said to be *thin* if $\delta(K) = 0$, it is *nonthin* if either $\delta(K) \neq 0$ or K fails to have natural density. A series $\sum b_n$ is said to be a rearrangement of $\sum a_n$ if each $b_n = a_k$ for some $k \in N$.

Definition

A series is said to be *statistically conditionally convergent* if it is statistically convergent but not absolutely convergent.

Definition

A series $\sum a_n$ is said to be *statistically non-negative* term series if $\delta[\{k \in N{:}a_k < 0\}] = 0$.

A sequence (x_n) is said to be *statistically Cauchy* if for every $\epsilon > 0$, there exists m = m(ϵ) such that $\delta[\{k \in N : |x_n - x_m| \geq \epsilon\}] = 0$. A sequence (x_n) is said to be statistically bounded, if there exists a A > 0, such that $\delta[\{k \in N : |x_k| > A\}] = 0$, see Tripathy [10]. Clearly a statistically convergent series is statistically bounded, but not conversely.

2. PROPERTIES

The following lemmas will help us in establishing the results.

Lemma 1 (Lemma 1.1, Šalăt [8])

A sequence $x = (x_n)$ is statistically convergent to α if a only if there exists such a set $K = \{k_1 < k_2 < k_3 < \ldots.\} \subset N$ that $\delta(K) = 1$ and $\lim_{n\to\infty} x_{k_n} = \alpha$.

Lemma 2 (Theorem 1, Fridy [4])

A number sequence $x = (x_n)$ is statistically convergent if and only if it is statistically Cauchy.

A statistically convergent series may be unbounded. For this consider the example.

Example 1

Let $\sum a_n$ be defined by

$$a_n = \begin{cases} (-1)^n k, & n = k^2, \\ (-1)^n k, & n-1 = k^2,\ k \in N, \\ n^{-2}, & \textit{otherwise}. \end{cases}$$

Remark 1

All bounded series are not statistically convergent. This is clear on considering the series $\sum (-1)^n$.

Remark 2

A bounded statistically convergent series may or may not be convergent, which is clear from example 2.

Example 2

Consider the series $\sum a_n$, where

$$a_n = \begin{cases} (-1)^n, & n = k^2, \\ (-1)^n, & n-1 = k^2,\ k \in N, \\ n^{-2}, & \textit{otherwise}. \end{cases}$$

It is clear from definition and example 2 that "every convergent series is statistically convergent but not conversely". Using Weierstrass completeness principle we have

Proposition 1

A non-negative term series is statistically convergent if and only it is convergent.

Remark 3

If a series is statistically convergent then any rearrangement of it may or may not be statistically convergent. For this consider example 2 and it's following rearrangement.

$$\sum b_n = 1 - 1 + 4^{-1} + 1 - 1 + 9^{-1} + 1 - 1 + (16)^{-1} + 1 - 1 + (25)^{-1} + \ldots.$$

From the definition and above examples we have

Proposition 2

If $\sum a_n$ converges statistically, then stat-lim $a_n = 0$, but not conversaly.

Proposition 3

If $\sum a_n$ and $\sum b_n$ are two statistically convergent series, then for complex numbers $\alpha, \beta; \sum (\alpha a_n + \beta b_n)$ converges statistically to the sum $\alpha \sum a_n + \beta b_n$.

3. MAIN RESULTS

Theorem 1

If a series $\sum a_n$ is statistically convergent then there exists such a subset $K = \{k_1 < k_2 < k_3 < \ldots\}$ of N that $\delta(K) = 1$ *and* $\sum_i a_{k_i}$ *is convergent.*

Proof

Let $\sum a_n$ be statistically convergent, then by Lemma 1, there exists such a set $P = \{p_1 < p_2 < p_3 < \ldots\} \subset N$ with $\delta(P) = 1$ that (S_{p_i}) is convergent. Let us construct a subset $M = \{m_1, m_2, m_3, \ldots\}$ of N as $m_{2i} = q_i + 1$, $m_{2i-1} = q_i$, where $q_i \in P^c$, if there is repetition that is $m_i = m_{i+1}$ for some i, then count m_i and reject m_{i+1}. Then $\delta(M) = 0 \Rightarrow \delta(M^c) = 1$. Taking $K = M^c$, we have $\sum_{k \in K} a_k$ is convergent.

Remark 4

The converse of the above result fails even if $\sum a_n$ is bounded, which is clear from example 3.

Example 3

Let $\sum a_n$ be defined by $a_n = (-1)^n$, $n = k^2$, $k \in N$ and $a_n = 0$, otherwise.

Theorem 2

A statistically convergent series $\sum a_n$ has a thin subseries divergent to ∞ if and only if it has a thin subseries divergent to $-\infty$

Proof

Let $\sum a_n$ be statistically convergent and has a thin subseries divergent to ∞. Then by Proposition 1, it follows that it has positive terms as well as negative terms. By Theorem 1, there exists such a set $K \subset N$ that $\delta(K) = 1$ such that (s_n) converges on K. Let M be a thin subset of N such that $\sum_{k \in M} a_k = \infty$. Then from definition of statistical convergence of series it follows that $\sum_{k \in B} a_k = -\infty$, where $B = K^c - M \cap K^c$. Clearly $\delta(B) = 0$. Similarly the converse part follows.

We formulate the following results, the proofs are obvious.

Proposition 4

A series $\sum a_n$ of complex terms is statistically convergent if and only if the series of real and imaginary parts are statistically convergent.

Proposition 5

A number series $\sum a_n$ is statistically convergent if and only if for every $\epsilon > o$ there exists $m = m(\epsilon)$ such that

$$\lim_{n\to\infty} n^{-1}\left|\left\{k\in N: \left|\sum_{k=m}^{n} a_k\right| \geq \epsilon\right\}\right| = 0$$

Proposition 6

If $\sum a_n$ converges statistically then the remainder tends to 0 statistically and conversely.

Proposition 7

The following are equivalent:

i) $\sum a_n$ *is statistically convergent.*

ii) *There exists a convergent series $\sum b_n$ such that* $\delta[\{k \in N: a_k \neq b_k\}] = 0$.

iii) *There exists such a subset $K \subset N$ that $\delta(K) = 1$ and $\sum_{k\in K} a_k$ is convergent.*

(iv) *There exists series $\sum x_n$ and $\sum y_n$ such that $a_n = x_n + y_n$ for all $n \in N$, where $\sum x_n$ is convergent and $\sum y_n$ is statistically null.*

Remark 5

In case of statistically convergent sequences, the limit remains same for decomposition theorem, but for series the limit may be different.

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ON GENERALIZED ABSOLUTE σ-CONVERGENT SEQUENCE SPACES

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ABSTRACT: In this work, we investigate some topological properties of the sequences spaces defined by invariant convergence and related inclusions.

1. INTRODUCTION

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m_{(n)} \neq n$ for all positive integers n and m where

$$\sigma^m_{(n)} = \left(\sigma^{m-1)}_{(n)}\right), \ m = 1,2,...$$

A continuous linear functional ϕ on ℓ_∞ is called σ-mean if it has the properties:

(i) ϕ (x) $\geq$ 0 when the sequence x = (x_n) has $x_n \geq 0$ for all n,

(ii) ϕ (e) = 1 where e = (1,1,...) and

(iii) $\phi(x_{(n)}) = \phi(x)$, for all $\in \ell_\infty$

when σ(n) - n + 1, a σ-mean is often called a Banach limit and V_σ the set of bounded sequences all of whose invariant means are equal is the set of almost convergent sequences [C.C. Lorentz (1948)].

If x = (x_n), we write Tx = (Tx) = $(x_{\sigma(n)})$. The space V_σ can be characterized either (i) as the set of all bounded sequences x for which there is an L so that $\lim_n t_{mn}(x) = L$ uniformly in n where

$$t_{mn}(x) = \frac{1}{m+1}\sum_{i=0}^{m} x_{\sigma^1_{(n)}}, \qquad (n = 1,2,...)$$

or (ii) as the set of all bounded sequences x for which $\lim_{m} \frac{1}{m+1} \sum_{i=0}^{m} x_{\sigma^{1}_{(n)}}$ is of the L, where L = σ-lim x.

2. TOPOLOGICAL RESULTS

A paranormed space (X,h) is a topological linear space with the topology given by the paranorm h. It may be recalled that a paranorm h is a real subadditive function on X such that g(0) = 0, g(x) = g(-x) and such that multiplication is continuous, i.e. $\lambda_n \to \lambda$, $x_n \to x$ imply that $\lambda_n x_n \to \lambda x$ where λ_n, λ scalars and x_n, $x \in X$.

Let $p = (p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k and $\sup p_k = H < \infty$. This assumption is made throughout the rest of this paper.

Let

$$\left|\overline{S_\sigma}\right|_{(p_k)} = \left\{x I \sum_{k=1}^{\infty} k^{p_k-1} \left|\psi_{kn}(x) - \psi_{k\text{-}kn}(x)\right|^{p_k} \text{ congerges uniformly in } n\right\}$$

$$\left|\overline{\overline{S_\sigma}}\right|_{(p_k)} = \left\{x \sup_{n} \sum_{k=1}^{\infty} k^{p_k-1} \left|\psi_{kn}(x) - \psi_{k\text{-}kn}(x)\right|^{p_k} < \infty\right\},$$

where $\psi_{kn}(x) = \frac{1}{k+1} \sum_{m=0}^{k} t_{mn}(x), \ \psi_{-1,n} = t_{-1.n} = x_{n-1}$

$$[\omega_\sigma]_p = \left\{x : \lim_{n} \frac{1}{k+1} \sum_{m=0}^{k} |t_{mn}(x-L)| = 0 \text{ uniformly in } n \text{ for some } L\right\}$$

$$[\hat{S}_f]_{(p_k)} = \left\{x : \sum_{k=1}^{\infty} k^{p_k-1} |d_{kn}(x) - d_{k-1,n}(x)|^{p_k} \text{ converges uniformly in } n\right\}$$

$$[\hat{S}_f]_{(p_k)} = \left\{x : \sup_{n} \sum_{k=1}^{\infty} k^{p_k-1} |d_{kn}(x) - d_{k-1,n}(x)|^{p_k} < \infty\right\}$$

where $d_{kn} = d_{kn}(x) = \frac{1}{k+1} \sum_{m=0}^{k} t_{mn}(x)$

If $p_k = p$ (constant) for all k, then we write $\left[\bar{S}_\sigma\right]_p$ and $\left[\bar{\bar{S}}_\sigma\right]_p$ for $\left[\bar{S}_\sigma\right]_{(p_k)}$ and $\left[\bar{\bar{S}}_\sigma\right]_{(p_k)}$, respectively, if $p_k = 1$ we write $\left[\bar{S}_\sigma\right]$ and $\left[\bar{\bar{S}}_\sigma\right]$ for $\left[\bar{S}_\sigma\right]_{(p_k)}$ and $\left[\bar{S}_\sigma\right]_{(p_k)}$, respectively. Moreover $\sigma(n) = n+1$ we write for $\left[\hat{S}_f\right]_{(p_k)}$ and $\left[\hat{S}_f\right]_{(p_k)}$ [3], respectively.

Theorem 1

Let $p = (p_k)$ be bounded away from 0. Then $\left[\bar{S}_\sigma\right]_{(p_k)}$ is a complete linear topological space paranormed by

$$h(x) = \sup_n \left(\sum_{k=1}^{\infty} k^{p_k - 1} \left| \psi_{kn}(x) - \psi_{k-1.n}(x) \right|^{p_k} \right)^{1/M} \tag{1.1}$$

where $M = \max(1, \sup p_k)$. The space $\left[\bar{\bar{S}}_\sigma\right]_{(p_k)}$ is paranormed by (1.1).

Proof

We have

$$|x_k + y_k|^{p_k} \leq K\left(|X_k|^{p_k} + |y_k|^{p_k}\right), \tag{1.2}$$

Where $K = \max(1, 2^{H-1})$. Since,

$$|\lambda_k|^{p_k} \leq \max\left(1, |\lambda|^H\right), \tag{1.3}$$

we have $$\sum_{k=1}^{\infty} k^{p_k-1} \left| \psi_{kn}(\lambda x + \mu y) - \psi_{k-1.n}(\lambda x + \mu y) \right|^{p_k}$$

$$\leq K.K_1 \sum_{k=1}^{\infty} k^{p_k-1} \left| \psi_{kn}(x) - \psi_{k-1.n}(x) \right|^{p_k}$$

$$+K.K_2\sum_{k=1}^{\infty} k^{p_k-1}\left|\psi_{kn}(y)-\psi_{k-1.n}(y)\right|^{p_k}, \tag{1.4}$$

$$\left|\psi_{1n}(x^s-x^t)-\psi_{(0,n)}(x^s-x^t)\right| = \frac{1}{2}\left|X_n^s-X_n^t\right| \quad \rightarrow 0$$

as, s,t $\rightarrow \infty$ for each fixed n. Hence, (x^s) is a Cauchy sequence in **C**. Since **C** is complete, there exists $x \in$ **C** such that $x^s \rightarrow x$ coordinatewise as s $\rightarrow \infty$. It follows from (1.5) that given $\epsilon > 0$, there exists s_0 such that

$$\left(\sum_{k=1}^{\infty} k^{p_k-1}\left|\psi_{kn}(x^s-x^t)-\psi_{k-1.n}(x^s-x^t)\right|^{p_k}\right)^{1/M} < \epsilon \tag{1.6}$$

for s,t $> s_0$. Now making $t\rightarrow\infty$ and then taking supremum with respect to n in (1.6) we obtain

$$h(x^s - x) \leq \epsilon,$$

for S $>$ S_0. This proves that $x^s \rightarrow x$ and $x\in\left[\bar{S}_\sigma\right]_{(p_k)}$. Hence, $x\in\left[\bar{S}_\sigma\right]_{(p_k)}$ is complete.

By taking $\sigma(n)$ = n+1 in Theorem 1, we obtain the following result which is valid for almost everywhere convergence corollary. The spaces $\left[\hat{S}_1\right]_{(p_k)}$ and $\left[\hat{S}_1\right]_{(p_k)}$ are complete linear topological spaces relative to the paranormal defined by the function.

$$h(x) = \sup_n \left(\sum_{k=1}^{\infty} k^{p_k-1}\left|d_{kn}(x)-d_{k-1.n}(x)\right|^{p_k}\right)^{1/M}$$

Theorem 2

If p is a constant and $p \geq 1$, then

$$\left[\bar{S}_\sigma\right]_p \subset \left[\bar{\bar{S}}_\sigma\right]_p^{\bullet}$$

Proof

Suppose that $x \in [\bar{S}_\sigma]_p$. Hence there exists an integer M > 0 such that

$$\sum_{k \geq M}^{\infty} k^{p-1} |\psi_{kn}(x) - \psi_{k-1.n}(x)|^p \leq 1 \tag{2.1}$$

for each n it is enough to show that,

$$\sum_{k=1}^{M-1} k^{p-1} |\psi_{kn}(x) - \psi_{k-1.n}(x)|^p = O(1), \ \forall n. \tag{2.2}$$

(2.1) implies that, for k ≥ M

$$|\psi_{kn}(x) - \psi_{k-1.n}(x)|^p \leq \frac{1}{k^{p-1}} \leq 1$$

and so this implies

$$|\psi_{kn}(x) - \psi_{k-1.n}(x)| \leq 1, (k \geq M, \forall n) \tag{2.3}$$

since $$x_{\sigma^k_{(n)}} = (k+1)(\psi_{kn} - \psi_{k-1.n}) - (k-1)(\psi_{k-1.n} - \psi_{k-2.n}) \tag{2.4}$$

it follows from (2.3) that, for any fixed $k \geq M, |x_{\sigma^k_{(n)}}| = O(1) \forall n,$ and this implies that $|x_i|$ is bounded. Hence,

$$|\psi_{kn}(x) - \psi_{k-1.n}(x)| \leq \frac{1}{k(k+1)} \sum_{i=1}^{k} |x_{\sigma^1_{(n)}}|$$

$$= O(1) \frac{1}{k(k+1)} \sum_{i=1}^{n} x_{\sigma^k_{(n)}} = O(1), \forall k \quad and \ n.$$

It follows from this that (2.2) holds and this completes the proof.

Lemma

$[\omega_\sigma]_p - \lim x = L$ if and only if

(i) $\omega_\sigma - \lim x = L$

and (ii) $$\frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x-L) - \psi_{kn}(x-L)|^p = 0(1)$$

as m → ∞, uniformly in n.

Proof

Clearly $[\omega_\sigma]_p - \lim x = L$ for p = 1, Holder's inequality of p > 1

$$\left|\frac{1}{m}\sum_{k=1}^{m} t_{kn}(x-L)\right| \leq \frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x-L)|$$

$$\leq \frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x-L)|^p \; m^{1-1/p}$$

that ω_σ - lim x = L for p > 1. This proves (i). Again for p ≥ 1

$$\left(\frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x-L) - \psi_{kn}(x-L)|^p\right)^{1/p}$$

$$\leq \left(\frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x-L)|^p\right)^{1/p} + \left(\frac{1}{m}\sum_{k=1}^{m} |\psi_{kn}(x-L)|^p\right)^{1/p}$$

$$= \sum\nolimits_1 + \sum\nolimits_2 \quad \text{(say)}$$

Now, by hypothesis $\Sigma_1 = 0(1)$ as m → ∞, uniformly in n. Also, (i) gives us $|\Psi_{kn}(x-L)| \to 0$ as k → ∞, uniformly in n, and hence $\Sigma_2 = 0(1)$ as m → ∞ uniformly in n.

This proves (ii) since

$$\left(\frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x-L)|^p\right)^{1/p} \leq \left(\frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x-L) - \psi_{kn}(x-L)|^p\right)^{1/p}$$

$$+ \left(\frac{1}{m}\sum_{k=1}^{m} |\psi_{kn}(x-L)|^p\right)^{1/p}$$

the converse part follows immediately.

Theorem 3

$$[\bar{S}_\sigma]_p \subset [\omega_\sigma]_p \quad \text{and if } x \in \; [\omega_\sigma]_p \text{ , then}$$

$$[\omega_\sigma]_p - \lim x = \omega_\sigma - \lim x = L$$

Proof

Suppose that $x \in [\bar{S}_n]_p$. Hence, if we write

$$V_{m,n}(x) = \sum_{k=m}^{\infty} k^{p-1} |\psi_{kn}(x) - V_{k-1,n}(x)|^p$$

then $V_{m,n}$ (x) is finite for each m $\geq$ 1 and $V_{m,n}$ (x) $\rightarrow$ 0 as m $\rightarrow \infty$ uniformly in n.
Since every absolutely σ-convergent sequence is σ-convergent, we have ω_σ - lim x = L, (say).

Hence, to prove the theorem, it is enough to show, (by use of the Lemma) that

$$\frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x) - \psi_{kn}(x)|^p \rightarrow 0 \text{ as } m \rightarrow \infty$$

uniformly in n. Since for k $\geq$ 1

$$(t_{kn}(x) - \psi_{kn}(x)) = k(\psi_{kn}(x) - \psi_{k-1.n}(x))$$

We have

$$\frac{1}{m}\sum_{k=1}^{m} |t_{kn}(x) - \psi_{kn}(x)|^p = \frac{1}{m}\sum_{k=1}^{m} k^p |\psi_{kn}(x) - \psi_{k-1.n}(x)|^p$$

$$= \frac{1}{m}\sum_{k=1}^{m} k^p (V_{kn}(x) - V_{k+1.n}(x)) \; k^{1/p}$$

$$= \frac{1}{m}\sum_{k=1}^{m} k\left(V_{kn}(x) - V_{k+1.n}(x)\right)$$

$$= \frac{V_{1,n}(x)}{m} - V_{m+1,n}(x) + \frac{1}{m}\sum_{k=1}^{m}\left(V_{k+1,n}(x)\right) = 0(1)$$

as m → ∞ uniformly in n. This completes the proof.

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CONVERGENCE THEOREMS FOR SOME VARIANTS OF NEWTON'S METHOD OF ORDER GREATER THAN TWO

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ABSTRACT: Using the majorant method we find sufficient conditions for the convergence of a Chebysheff-Halley-type method in a banach space. Two different approaches are used. The first one utilizes divided differences of order one, whereas the second employs Fréchet-derivatives of order one and two. Our results improve all our previous results as well as those of others.

AMS (MOS) Subject Classification: 47H17, 65H10, 65J15, 49D15.

Key Words and Phrases: Banach space, Chebysheff-Halley method, majorant method, Fréchet-derivative, divided difference.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x* of the nonlinear equation.

$$F(x) = 0 \tag{1}$$

in a Banach space E, where F is a nonlinear operator defined on some convex subset D of E with values in E.

We recently introduced the Chebysheff-Halley-type method given by

$$y_n = x_n - [x_n, x_n]^{-1} F(x_n) \tag{2}$$

$$L_n = -[x_n, x_n]^{-1}([x_n, y_n] - [x_n, x_n]),\ M_n = (I - L_n)^{-1} \tag{3}$$

$$x_{n+1} = y_n - [x_n, x_n]^{-1} M_n([x_n, y_n] - [x_n, x_n])(y_n - x_n),\ n \geq 0, x_0 \in D \tag{4}$$

to find a solution x^* of equation (1) [5], [6], [8].

Here $[x, y]$ denotes a divided difference of order one which is an operator in $L(E,E)$ satisfying

$$[x,y]\,(x-y) = F(x) - F(y) \qquad \text{and} \qquad F'(x) = [x,x] \ \text{ for all } x \in D, \tag{5}$$

[5], [9], [17], where F' denotes the Fréchet-derivative of F.

Using the majorant method and the standard Newton-Kantorovich-type hypotheses we showed that the iteration $\{x_n\}$ $(n \geq 0)$ converges with order eventually three [5]. These results constitute major improvements over all previous ones.

In this study we improve on these results even further by assuming that the following Zabrejko-Nguen-type conditions are satisfied

$$\|([x+h_1,y+h_2]-[x,y])\| \leq A_1(t_1+\|h_1\|,t_1)+A_2(t_2+\|h_2\|,t_2) \tag{6}$$

for all $x \in U(x_0, t_1) = \{x \in E_1 \mid \|x - x_0\| \leq t_1\}$, $y \in U(x_0, t_2)$, $\|h_1\| \leq R - t_1$, $\|h_2\| \leq R - t_2$, $t_1, t_2 \geq 0$ for some fixed $R > 0$ such that $U(x_0, R) \subseteq D$ [1]. The functions A_1, A_2 are continuous in both variables, and such that if one of the variables is fixed then A_1 and A_2 are increasing functions of the other on $[0, R]$ with $A_1(0,0) = A_2(0,0)$.

Here we provide an error analysis as well as error bounds on the distances $\|x_{n+1} - x_n\|$ and $\|x_n - x^*\|$ for all $n \geq 0$. We also show how to choose the functions A_1 and A_2. Special choices of A_1 and A_2 will led to all the previous results [5], [6], [9], [14], [15], [17], [20].

In Section 3 of our study we present a different approach, and method (Chebysheff-Halley-Werner) for solving equation (1) that uses first and second Fréchet-derivatives only instead of first Fréchet-derivatives and divided differences of order one.

The computational cost slightly increases this way but many researchers find this approach more useful because it avoids mixing Fréchet derivatives and divided differences. Another reason why we present this second method is because we can compare our results (favorably) with earlier ones [5], [6], [12], [14], [15], [17], [18], [20], [21].

2. CONVERGENCE ANALYSIS FOR THE CHEBYSHEFF-HALLEY METHOD

We will need to introduce the constants

$$t_0 = 0,\ s_0 \geq \|y_0 - x_0\|,\ \beta \geq \|F'(x_0)^{-1}\| \tag{7}$$

for some fixed $x_0 \in D$,

$$a = 1 - \beta(2A_1(R_1,0) + A_2(R_1,0) + A_2(R_1, R_1)), \tag{8}$$

$$a_1 = 1 - \beta (A_1 (R_1,0) + A_2 (R,0) \tag{9}$$

for some fixed R_1 and R with $0 \leq R_1 \leq R$, the sequences for all $n \geq 0$

$$a_n = 1 - \beta (A_1 + A_2)(t_n, 0) \tag{10}$$

$$b_{n+1} = \beta \int_0^1 (A_1 + A_2)((1-t)\|x^* - x_0\| + t\|x_{n+1} - x_0\|, 0)\, dt, \tag{11}$$

$$c_{n+1} = \beta (1 - b_{n+1})^{-1}, \tag{12}$$

$$d_n = \beta \int_0^1 (A_1 + A_2)(\|x_n - x_0\| + t\|x^* - x_n\|, \|x_n - x_0\|)\|x^* - x_n\|\, dt, \tag{13}$$

$$e_n = 1 - \beta (A_1 + A_2)(\|x_n - x_0\|,\ 0), \tag{14}$$

$$h_{h+1} = \int_{s_n}^{t_{n+1}} (A_1 + A_2)(t, s_n)\, dt + (A_1 + A_2)(s_n, t_n)(t_{n+1} - s_n)$$

$$+ p_n \int_{t_n}^{s_n} (A_1 + A_2)(t, t_n)\, dt$$

$$+ q_n \left(\int_{t_n}^{s_n} A_1(t, t_n)\, dt + \int_{s_n}^{s_n + (s_n - t_n)} A_2(t, s_n)\, dt \right), \tag{15}$$

$$p_n = \frac{\beta (A_1(t_n, 0) + A_2(s_n, s_n))}{1 - \beta [2A_1(t_n, 0) + A_2(s_n, s_n) + A_2(t_n, 0)]},$$

$$q_n = \frac{1 - \beta (A_1(t_n, 0) + A_2(t_n, 0))}{1 - \beta [2A_1(t_n, 0) + A_2(s_n, s_n) + A_2(t_n, 0)]}, \tag{16}$$

$$\bar{p}_n = \frac{\beta (A_1(\|x_n - x_0\|, 0) + A_2(\|x_n - x_0\| + \|x_n - y_n\|, \|y_n - x_0\|))}{1 - \beta [2A_1(\|x_n - x_0\|, 0) + A_2(\|x_n - x_0\| + \|x_n - y_n\|, \|y_n - x_0\|) + A_2(\|x_n - x_0\|, 0)]}, \tag{17}$$

$$\bar{q}_n = \frac{\beta}{1 - \beta [2A_1(\|x_n - x_0\|, 0) + A_2(\|x_n - x_0\| + \|x_n - y_n\|, \|y_n - x_0\|) + A_2(\|x_n - x_0\|, 0)]}, \tag{18}$$

$$r_n = \frac{1}{1 - \beta [2A_1(t_n, 0) + A_2(s_n, s_n) + A_2(t_n, 0)]}, \tag{19}$$

$$s_{n+1} = t_{n+1} + \frac{\beta}{a_{n+1}} h_{n+1}, \tag{20}$$

$$t_{n+1} = s_n + \frac{\beta}{r_n} \left(A_1(t_n,0) + A_2(s_n,t_n)\right)(s_n - t_n), \tag{21}$$

and the functions

$$T(r) = s_0 + \frac{\beta}{a(r)} \{ \int_{s_0}^{r} (A_1+A_2)(t,r)\,dt + (A_1+A_2)(r,r)\,r + p(r)$$

$$\int_0^r (A_1+A_2)(t,r)\,dt + q(r)\left(\int_0^r A_1(t,r)\,dt + \int_{s_0}^r A_2(t,r)\,dt\right)$$

$$+ q(r)\left(A_1(r,0) + A_2(r,r)\right) r \}, \tag{22}$$

$$a(r) = 1 - \beta\left(A_1(r,0) + A_2(r,0)\right), \tag{23}$$

$$p(r) = \frac{\beta\left(A_1(r,0) + A_2(r,r)\right)}{1 - \beta\left(2A_1(r,0) + A_2(r,r) + A_2(r,0)\right)} \tag{24}$$

and $$q(r) = \frac{1}{1 - \beta\left(2A_1(r,0) + A_2(r,r) + A_2(r,0)\right)} \quad \text{on } [0,R] \tag{25}$$

We can now state and prove the main result:

Theorem 1

Let $F : D \subseteq E \to E$ be a nonlinear operator whose divided difference [.,.] satisfies condition (6) on D. More assume:

(i) *there exists a minimum non-negative number* R_1 *such that*

$$T(R_1) \le R_1; \tag{26}$$

(ii) *The numbers,* R, R_1 *with* $R_1 \le R$ *are such that the constants* a, a_0 *given by* (8) *and* (9) *respectively are positive, and*

$$U(x_0, R) \subseteq D \quad \text{with } R_1 \le R. \tag{27}$$

Then (a) *The scalar sequence* $\{t_n\}$ $(n \geq 0)$ *generated by* (15)-(16) *is monotonically increasing and bounded above by its limit, which is number* R_1;

(b) *the sequence* $\{x_n\}$ $(n \geq 0)$ *generated by* (2)-(3) *is well defined, remains in* $U(x_0, R_1)$ *for all* $n \geq 0$, *and converges to a solution* x^* *of the equation* $F(x) = 0$, *which is unique in* $U(x_0, R)$.

Moreover the following estimates are true for all $n \geq 0$

$$\|y_n - x_n\| \leq s_n - t_n, \tag{28}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{29}$$

$$\|x^* - x_n\| \leq R_1 - t_n, \tag{30}$$

$$\|x^* - s_n\| \leq R_1 - s_n, \tag{31}$$

$$\|F(s_{n+1})\| \leq \bar{h}_{n+1} \leq h_{n+1}, \tag{32}$$

$$\|x_{n+1} - x^*\| \leq c_{n+1}\bar{h}_{n+1} \leq R_1 - t_{n+1} \tag{33}$$

and

$$\|y_n - x_n\| \leq \|x^* - x_n\| + d_n(1 - a_n)^{-1}, \tag{34}$$

where

$$\bar{h}_{n+1} = \int_0^1 (A_1 + A_2)(\|y_n - x_0\| + t\|x_{n+1} - y_n\|, \|y_n - x_0\|)\|x_{n+1} - y_n\| dt$$

$$+ (A_1 + A_2)(\|x_n - x_0\| + \|y_n - x_n\|, \|x_n - x_0\|)\|x_{n+1} - y_n\|$$

$$+ \bar{p}_n \int_0^1 (A_1 + A_2)(\|x_n - x_0\| + t\|y_n - x_n\|, \|x_n - x_0\|)\|y_n - x_n\| dt$$

$$+ \frac{1}{\beta}\left[1 - \beta(A_1 + A_2)(\|x_n - x_0\|, 0\right]\bar{q}_n \int_0^1 [A_1(\|x_n - x_0\|) + t\|y_n - x_n\|, \|x_n - x_0\|\big)$$

$$+ A_2(\|y_n - x_0\| + (1 - t)\|x_n - y_n\|, \|y_n - x_0\|)]\|x_n - y_n\| dt.$$

Proof

(a) By (7), (15),(20),(21), and the monotonicity of the functions A_1 and A_2, we deduce that the sequence $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and nonnegative. Using (7), (15), (20) and (21) (for $n = 0$) we can get $t_0 \leq s_0 \leq t_1 \leq s_1 \leq R_1$. Let us assume that $t_k \leq s_k \leq t_{k+1} \leq s_{k+1} \leq R_1$ for $k = 0,1,2,\ldots,n$. Then by (20) and (21) we can have in turn

$$t_{k+2} \leq t_{k+1} + \frac{\beta}{a(R_1)}\left\{\int_{s_k}^{t_{k+1}}(A_1+A_2)(t,s_k)dt + (A_1+A_2)(s_k,t_k)(t_{k+1}-s_k)\right\}$$

$$\left\{+q_k\left(\int_{t_k}^{s_k}A_1(t,t_k)dt + \int_{s_k}^{s_k+(s_k-t_k)}A_2(t,s_k)dt\right) + p_k\int_{t_k}^{s_k}(A_1+A_2)(t,t_k)dt\right\}$$

$$+\frac{\beta r_k}{a(R_1)}(A_1(t_{k+1},0)+A_2(s_{k+1},t_{k+1}))(s_{k+1}-t_{k+1})$$

$$\leq \ldots \leq s_0 + \frac{\beta}{a(R_1)}\left\{\int_{s_0}^{t_{k+1}}(A_1+A_2)(t,R_1)dt + (A_1+A_2)(R_1,R_1)(t_{k+1}-s_0)\right\}$$

$$\left\{+q(R_1)\left(\int_0^{s_k}A_1(t,R_1)dt + \int_{s_0}^{s_k}A_2(t,R_1)dt\right) + p(R_1)\int_0^{s_k}(A_1+A_2)(t,R_1)dt\right\}$$

$$+\frac{\beta q(R_1)}{a(R_1)}(A_1(R_1,0)+A_2(R_1,R_1))(s_{k+1}-t_0) \leq T(R_1) \leq R_1 \quad \text{by (26).}$$

Hence, the scalar sequence $\{t_n\}$ $(n \geq 0)$ is bounded above by R_1. By hypothesis (26) R_1 is the minimum nonnegative zero of the equation $T(r) - r = 0$ in $[0,R_1]$ and from the above $R_1 = \lim_{n\to\infty} t_n$.

(b) Using condition (7), (20) and (21) we get $x_1, y_0 \in U(x_0,R_1)$ and that estimates (28) and (29) are true for $n = 0$. We first show that $[x_n, x_n]$ is invertible for all $n \geq 1$. Let us assume that (28) and (29) are true for $k = 0,1,2,\ldots,n-1$. Then we will obtain in turn

$$\|x_{k+1}-x_0\| \leq \|y_{k+1}-y_0\| + \|y_0-x_0\| \leq \|x_{k+1}-y_k\| + \|y_k-y_0\| - \|y_0-x_0\|$$

$$\leq \ldots \leq (t_{k+1}-s_k) + (s_k-s_0) + s_0 \leq t_{k+1} \leq R_1$$

and

$$\|y_{k+1}-x_0\| \leq \|y_{k+1}-y_0\| + \|y_0-x_0\| \leq \|y_{k+1}-x_{k+1}\| + \|x_{k+1}-y_k\| + \|y_k-y_0\| + \|y_0-x_0\|$$

$$\leq \ldots \leq (s_{k+1}-t_{k+1}) + (t_{k+1}-s_k) + (s_k-s_0) + s_0 \leq s_{k+1} \leq R_1.$$

That is, x_n, $y_n \in U(x_0, R_1)$ for all $n \geq 0$.

Using hypothesis (6) we can obtain

$$\|F'(x_0)^{-1}\|\|F'(x_k)-F'(x_0)\| \leq \beta\|[x_k,x_k]-[x_0,x_0]\|$$

$$\leq \beta(A_1(\|x_k-x_0\|,0)+A_2(\|x_k-x_0\|,0)) \leq \beta(A_1+A_2)(t_k,0)$$

$$\leq \beta(A_1+A_2)(R_1,0)<1 \quad \text{by condition } a>0$$

It now follows the Banach lemma on invertible operators [9], [13] that $[x_n, x_n]$ is invertible, and

$$\|[x_n,x_n]^{-1}\| \leq \frac{\beta}{a_n} \quad \text{for all } n \geq 0. \tag{35}$$

Using (2), (3) and (4) we can easily obtain the approximation

$$F(x_{n+1}) = \int_0^1 [F'(y_n+t(x_{n+1}-y_n)) - F(y_n)](x_{n+1}-y_n)dt$$

$$+(F'(y_n)-F'(x_n))(x_{n+1}-y_n)+M_n\int_0^1 (F'(x_n+t(y_n-x_n))-[x_n,y_n])(y_n-x_n)dt$$

$$+M_nF'(x_n)^{-1}([x_n,y_n]-[x_n,x_n]) \int_0^1 [F'(x_n+t(y_n-x_n))-[x_n,y_n]](y_n-x_n)dt.$$

Then, by using the estimates

$$\|M_n\|\cdot\|F'(x_n)^{-1}\|\cdot\|[x_n,y_n]-[x_n,x_n]\| \leq \bar{q}_n,$$

$$\|M_n\| \leq \bar{p}_n,$$

conditions (6), (35) and (28), we can obtain through the triangle inequality

$$\|F(x_{n+1})\| \leq \int_0^1 [A_1\|y_n-x_0\|+t\|x_{n+1}-y_n\|,\|y_n-x_0\|$$

$$+ A_2(\|y_n-x_0\|+t\|x_{n+1}-y_n\|,\|y_n-x_0\|]dt$$

$$+ [A_1(\|x_n-x_0\|+\|y_n-x_n\|,\|x_n-x_0\|)$$

$$+A_2(\|x_n-x_0\|+\|y_n-x_n\|,\|x_n-x_0\|)]\|x_{n+1}-y_n\|$$

$$+\bar{p}_n\int_0^1 (A_1+A_2)(\|x_n-x_0\|+t\|y_n-x_n\|,\|x_n-x_0\|)\|y_n-x_n\|dt$$

$$+\bar{q}_n\int_0^1[A_1(\|x_n-x_0\|+t\|y_n-x_n\|,\|x_n-x_0\|)$$

$$+A_2(\|y_n-x_0\|+(1-t)\|x_n-y_n\|,\|y_n-x_0\|]\|y_n-x_n\|dt$$

$$=\bar{h}_{n+1}\le\int_0^1 (A_1+A_2)(s_n+t(t_{n+1}-s_n),s_n),(t_{n+1}-s_n)dt$$

$$+(A_1+A_2)(s_n,t_n)(t_{n+1}-s_n)+p_n\int_{t_n}^{s_n}(A_1+A_2)(t,t_n)dt$$

$$+q_n\left[\int_0^1\left[A_1(t_n+t(s_n-t_n),t_n)+A_2(s_n+(1-t)(s_n-t_n),s_n)(s_n-t_n)\right]\right]dt$$

$$=h_{n+1}$$

From approximations (2) and (36) we get

$$\|y_{n+1}-x_{n+1}\|\le\|F'(x_{n+1})^{-1}\|\cdot\|F(x_{n+1})\|$$

$$\le\frac{h_{n+1}}{a_{n+1}}=s_{n+1}-t_{n+1},$$

which shows (28) for all $n\ge 0$.

Using hypotheses (6) and (35) we can obtain from (4)

$$\|x_{n+1}-y_n\|\le\|F'(x_n)^{-1}\|\cdot\|M_n\|\|([x_n,x_n]-[x_n,x_n])\|\cdot\|y_n-x_n\|$$

$$\le\bar{q}_n[A_1(\|x_n-x_0\|,0)+A_2(\|x_n-x_0\|+\|y_n-x_n\|,\|x_n-x_0\|)]\|y_n-x_n\|$$

$$\le\frac{\beta}{r_n}[A_1(t_n,0)+A_2(s_n,t_n)](s_n-t_n)=t_{n+1}-s_n,$$

which shows (29) for all $n\ge 0$.

It now follows from estimates (28) and (29) that the sequence $\{x_n\}$ ($n \geq 0$) is Cauchy in a Banach space E and as such it converges to some $x^* \in U(x_0, R_1)$ with $F(x^*) = 0$ (by (2)).

To show uniqueness, we assume that there exists another solution y^* of equation (1) in $U(x_0, R)$. Then from hypothesis (6), we get

$$\|F'(x_0)^{-1}\|\|([x^*,y^*]-[x_0,x_0])\| \leq \beta(A_1(\|x^*-x_0\|,0)+A_2(\|y^*-x_0\|,0))$$

$$\leq \beta(A_1(R_1,0)+A_2(R,0))<1, \qquad \text{(since } a_0 > 0\text{)}$$

from which it follows that the linear operator $[x^*, y^*]$ is invertible. From this fact, (4) and the approximation

$$F(x^*)-F(y^*)=[x^*,y^*](x^*-y^*)$$

we obtain $x^* = y^*$.

Estimates (30) and (31) follow easily from estimates (28) and (29), respectively.

Using the triangle inequality, and the approximations

$$x_{n+1}-x^*=(B_{n+1}^{-1})(F(x_{n+1}))$$

$$B_{n+1}=\int_0^1[x^*+t(x_{n+1}-x^*),x^*+t(x_{n+1}-x^*)]dt,$$

$$y_n-x_n=x^*-x_n+[x_n,x_n]^{-1}$$

$$\left(\int_0^1[x_n+t(x^*-x_n),x_n+t(x^*-x_n)]-[x_n,x_n]\right)(x^*-x_n)dt,$$

$$\|B_{n+1}^{-1}\| \leq \int_0^1[A_1((1-t)\|x^*-x_0\|+t\|x_{n+1}-x_0\|,0)$$

$$+A_2((1-t)\|x^*-x_0\|+t\|x_{n+1}-x_0\|,0)]dt$$

$$\leq \int_0^1[A_1((1-t)R_1+tR_1,0)+A_2((1-t)R_1+tR_1,0)]dt$$

$$=A_1(R_1,0)+A_2(R_1,0)<1 \qquad \text{(by hypothesis } a > 0\text{)},$$

we can immediately obtain estimates (33) and (34).

That completes the proof of the theorem.

Remark 1

(a) Let us denote the right-hand side of (6) by

$$A_3(t_1+\|h_1\|,t_1,t_2+\|h_2\|,t_2)$$

Then we can choose

$$A_3(t_1+\|h_1\|,t_1,t_2+\|h_2\|,t_2) = \sup_{\substack{x\in U(x_0,t_1),y\in U(x_0,t_2)\\ \|h_1\|\le R-t_1,\ \|h_2\|\le R-t_2}} \|[x+h_1,y+h_2]-[x,y]\|$$

Estimate (6) will now hold for the above choice of the function A_3.

(b) Let us assume that instead of condition (6) the following is true:

$$\|[x_0,x_0]^{-1}([x,y]-[z,z])\| \le q_1(r)\|x-z\|+q_2(r)\|y-z\|$$

for all x, $y \in U(x_0, R)$, and q_1, q_2 be two nondecreasing functions on $[0,R]$ with $q_1(0) = q_2(0) = 0$. For example we can choose.

$$q_1(r) = q_2(r) = \sup_{x,y,z\in U(x_0,R)} \frac{\|[x,y]-[z,z]\|}{\|x-z\|+\|y-z\|}$$

For some applications of these results to the solution of nonlinear integral equations we refer the reader to [9] and the references there.

Theorem 2

Let f: $U(x_0, R) \to E$ *be a nonlinear operator satisfying*

$$\|f(x,y)-f(z,z)\| \le k_1(r)\|x-z\|+k_2(r)\|y-z\|, \tag{37}$$

for all x, y, $z \in U(x_0, r)$, $r \le R$, *and for some nondecreasing function* k_1 *and* k_2 *on* $[0,R]$. *Then*

$$\|f(x+h_1,y+h_2)-f(x,y)\| \le v_1(t_1+\|h_1\|)-v_1(t_1)+v_2(t_2+\|h_2\|)-v_2(t_2), \tag{38}$$

for all $x \in U(x_0, t_1)$, $y \in U(x_0, t_2)$, $\|h_1\| \leq R - t_1$, $\|h_2\| \leq R - t_2$,

$v_1(r) = \int_0^r k_1(t)dt$ and $v_2(r) = \int_0^r k_2(t)dt$.

Proof

Let $x \in U(x_0, t_1)$, $y \in U(x_0, t_2)$, $\|h_1\| \leq R - t_1$ and $\|h_2\| \leq R_2$. Using (37) for $m \in N$, we obtain

$$\|f(x+h_1, y+h_2) - f(x,y)\| \leq \sum_{j=1}^{m} \|f(x+m^{-1}jh_1, y+m^{-1}jh_2)$$

$$-f(x+m^{-1}(j-1)h_1, y+m^{-1}(j-1)h_2)\|$$

$$\leq \sum_{j=1}^{m} k_1(t_1+m^{-1}j\|h_1\|)m^{-1}\|h_1\| + \sum_{j=1}^{m} k_2(t_2+m^{-1}j\|h_2\|)m^{-1}\|h_2\|$$

$$\leq v_1(t_1+\|h_1\|) - v_1(t_1) + v_2(t_2+\|h_2\|) - v_2(t_2) \quad \text{as } m \to \infty,$$

by the monotonicity of k_1, k_2 and the definition of the Riemann integral.

Therefore another choice for the functions A_1 and A_2 is given by

$$A_1(t_1+\|h_1\|, t_1) = \int_{t_1}^{t_1+\|h_1\|} q_1(t)dt, \quad A_2(t_2+\|h_2\|, t_2) = \int_{t_2}^{t_2+\|h_2\|} q2(t)dt, \quad (39)$$

Moreover if we let $q_1(r) = q_2(r) = q$ for some $q > 0$ and for all $r \in [0,R]$. Then our results can be reduced to the ones in [5], which have improved the ones in [5]-[8]. Furthermore, we can have

$$A_1(t_1+\|h_1\|, t_1) \leq q\|h_1\| \quad \text{and} \quad A_2(t_2+\|h_2\|, t_2) \leq q\|h_2\|,$$

which means that our estimates on the distances $\|y_n - x_n\|$ and $\|x_n - x^*\|$ can easily be proved to be better than the ones in [14], [15], [17], [20] (and references there) for all $n > 0$. These ideas can also be used for Steffensen's method [9].

Remark 2

(c) Estimates (33) and (34) can sometimes be solved explicitly for $\|x^* - x_n\|$ for all $n \geq 0$. For example, choose $q_1(r) = q_2(r) = q$ and A_1 and A_2 as in (39).

3. CONVERGENCE ANALYSIS FOR THE CHEBYSHEFF-HALLEY-WERNER METHOD

Suppose that the nonlinear operator F defined on some convex subset D to E_1 containing $U(x_0,R)$, with values in E_2, is twice Fréchet-differentiable at every interior point of $U(x_0,R)$ and satisfies the conditions

$$\|F'(x+h) - F'(x)\| \leq A(r, \|h\|), \tag{40}$$

$$\|F''(x)\| \leq M, \tag{41}$$

and $$\|F''(x+h) - F''(x)\| \leq B(r, \|h\|) \quad \text{for all } x \in U(x_0, R),$$
$$0 \leq r \leq R, 0 \leq \|h\| \leq R - r. \tag{40}$$

Here A, B are nonnegative and continuous functions of two variables such that if one of the variables is fixed, then they are nondecreasing functions of the other on the interval $[0,R]$. Moreover we assume that $\frac{\partial A(0,t)}{\partial t}$ is positive, continuous and nondecreasing on $[0,R - r]$, with $A(0,0) = 0$.

Note that by setting for all r, $\|h\|$, $A(r, \|h\|) = c\|h\|$ for some $c > 0$, we obtain the usual Lipschitz condition on F' (see [4], [9]), whereas for $A(r, \|h\|) = e(r)\|h\|$ we obtain some generalized conditions considered also in [9], but for Newton's method. Conditions of the form (1) we also considered in [22], for Newton's method.

We denote by $F'(x_n)$ and $F''(x_n)$ the first and second Fréchet-derivatives of F evaluated at $x = x_n$. Note that $F'(x_n) \in L(E_1, E_2)$ is a linear operator, whereas $F''(x_n) \in L(E_1, E_2))$ is a bilinear operator for all $n \geq 0$, [2], [3].

Let $x_0 \in E_1$ be arbitrary and define the Chebysheff-Halley-Werner method on E_1 for all $n \geq 0$ by

$$y_n = x_n - F'(x_n)^{-1} F(x_n), \tag{43}$$

$$H(x_n, y_n) = -F'(x_n)^{-1} F''(x_n)(y_n - x_n) \tag{44}$$

and $$x_{n+1} = y_n - \frac{1}{2} F'(x_n)^{-1}\left[I - \frac{a}{2} H(x_n, y_n)\right]^{-1} F''(x_n)(y_n - x_n)^2 \tag{45}$$

Halley's method has a very long history. One can refer to [5], [6], [9], [12], [14], [15], [17], [20], [21] and the references there for some background.

In this study we are concerned with the problem of approximating a locally unique zero x^* of the equation

$$F(x) = 0.$$

Using the majorant theory, we will show that under certain Newton-Kantorovich assumptions on the part (F, x_0) the Halley-Werner method converges to a locally unique zero x^* of equation (7). We also provide upper bounds on the distances $\|x_n - x^*\|$ and $\|y_n - x^*\|$ for all $n \geq 0$.

Finally, we show that our results improve earlier ones [11]-[22].

It is convenient to introduce the constants

$$\eta \geq \|y_0 - x_0\|, \beta \geq \|F'(x_0)^{-1}\|, \; t_0 = 0, \tag{46}$$

$$s_0 \geq \eta, \quad t_1 \geq s_0^* = \frac{\beta M \eta^2}{2 - \beta M \eta}, \tag{47}$$

the scalar iterations for all $n \geq 0$

$$s_{n+i} = t_{n+1} + D(t_{n+1}) P(t_n, s_n) \tag{48}$$

$$t_{n+2} = s_{n+1} + \frac{1}{2} D(t_{n+1}) C(t_{n+1}) M (s_{n+1} - t_{n+1})^2, \tag{49}$$

where $$D(t_n) = \frac{\beta}{1 - \beta A(0, t_n)}, \; C(t_n) = \frac{1}{1 - \frac{M|a|}{2} D(t_n)(s_n - t_n)} \tag{50}$$

and $$P(t_n, s_n) = \int_{s_n}^{t_{n+1}} A(s_n, t)\,dt + A(t_n, s_n - t_n)(t_{n+1} - s_n)$$

$$+\ \frac{1}{2}C(t_n)D(t_n)M(s_n-t_n)\ =\ \int_{t_n}^{s_n}A(t_n,t)dt$$

$$+\ \frac{|a|}{2}C(t_n)\ \int_{t_n}^{s_n}B(t_n,t)dt(s_n-t_n)^2. \tag{52}$$

Furthermore, we define the function T on $[0,R]$ by

$$T(r)=t_1+D(r)\left[\int_0^r A(r,t)dt+A(r,r)r+\frac{1}{2}C(r)D(r)Mr\int_0^r A(r,t)dt\right]$$

$$+\frac{r^2}{2}C(r)\int_0^r B(r,t)\,dt+\frac{|a|r^2}{2}C(r)M, \tag{52}$$

where $$C(r)\ =\ \frac{1}{1-\frac{M|a|}{2}D(r)r} \tag{53}$$

We can now prove the main result of this section:

Theorem 3

Let F: $D \subset E_1 \rightarrow E_2$ be a nonlinear operator defined on some convex subset D of a Banach space E_1 with values in a Banach space E_2. Assume:

a) *F is twice Fréchet-differentiable on $U(x_0,R) \subseteq D$ for some $x_0 \in d$, $R \geq 0$, and satisfies conditions* (40)-(42);

(b) *the inverse of the linear operator $F'(x_0)$ exists;*

(c) *there exists a minimum nonnegative number R_1, with*

$$T(R_1)\leq R_1, \tag{54}$$

$$R_1\leq R; \tag{55}$$

(d) *the following estimates are also true:*

$$\beta A(0,R_1)<1, \tag{56}$$

$$\frac{M|a|}{2}\ \frac{\beta R_1}{1-\beta A(0,R_1)}\ <\ 1 \tag{57}$$

and $\dfrac{\beta}{R-R_1}\int_{R_1}^{R} A(0,t)\,dt < 1$

if $R \neq R_1$ *or* $\beta A(0,R_1) < 1$ *if* $R = R_1$ (58)

Then

(i) *the saclar sequence* $\{t_n\}$ $(n \geq 0)$ *defined by* (48)-(49) *is monotonically increasing and bounded above by its limit* R_1 *for all* $n \geq 0$;

(ii) *the Chebysheff-Halley-Werner method* $\{x_n\}$ $(n \geq 0)$ *generated by* (43)-(45) *is well defined, remains in* $U(x_0, R_1)$ *for all* $n \geq 0$ *and converges to a unique zero* x^* *of equation* $F(x) = 0$ *in* $U(x_0, R)$.

Moreover the following estimates are true:

$$\|x_n - x^*\| \leq R_1 - t_n \tag{59}$$

and

$$\|y_n - x^*\| \leq R_1 - s_n \quad \textit{for all } n \geq 0. \tag{60}$$

Proof

(i) We will show that sequence $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and bounded above by R_1 and as such it converges to R_1 (by (c) and (54). From (46)-(49) and (55) $t_0 \leq s_0 \leq s_1 \leq t_2$. By assuming $t_k \leq s_k \leq t_{k+1}$, $k = 0,1,2,\ldots,n$ we obtain $t_{k+1} \leq s_{k+1} \leq t_{k+2}$ from (48), (49) and the hypotheses on A and B. Hence, $\{t_n\}$ $(n \geq 0)$ is monotonically increasing. From (46) and (55) $t_0 \leq t_1 \leq R_1$, and from (49) for $n = 0$, $t_2 \leq T(R_1 \leq R_1$. Let us assume that $t_k \leq R_1$ for $k = 0,1,2,\ldots,n+1$. Then from (47)-(49) we get in turn

$$t_{n+2} = t_{n+1} + D(t_{n+1})\left[P(t_n,s_n) + \frac{1}{2}C(t_{n+1})M(s_{n+1}-t_{n+1})^2\right]$$

$$\leq t_{n+1} + D(R_1)\left[P(t_n,s_n) + \frac{1}{2}C(t_{n+1})M(s_{n+1}-t_{n+1})^2\right]$$

$$\leq \ldots \leq t_1 + D(R_1)\left[\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} A(s_i,t)\,dt + A(R_1,R_1)\right]$$

$$\left[\left(\sum_{i=0}^{n}(t_{i+1}-s_i)\right) + \frac{1}{2}C(R_1)D(R_1)MR_1 + \frac{1}{2}D(R_1)R_1^2\right]$$

$$\left[\left(\sum_{i=0}^{n}\int_{t_i}^{s_i}B(t_i,t)dt\right)+\frac{|a|}{2}C(R_1)M\left(\sum_{i=0}^{n}(s_{i+1}-t_{i+1})^2\right)\right]\leq T(R_1)\leq R_1,\quad \text{(by (54))}.$$

Hence, $\{t_n\}$ $(n \geq 0)$ is bounded above by R_1. Moreover $t_k \leq s_k \leq t_{k+1} \leq R_1$ for all $k \geq 0$.

That completes the proof of part (i).

(ii) We will show that if

$$\|y_n - x_n\| \leq s_n - t_n \quad (n \geq 0), \tag{61}$$

$$\|F(x_n)\| \leq P(t_{n-1}, s_{n-1}) \quad (n \geq 1), \tag{62}$$

$$\|F'(x_{n+1})^{-1}\| \leq D(t_{n+1}) \quad (n \geq -1), \tag{63}$$

and
$$\frac{|a|}{2}\|H(x_n, y_n)\| \leq \frac{|a|}{2} M D(t_n)(s_n - t_n) < 1, \tag{64}$$

then
$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{65}$$

$$\|F(x_{n+1})\| \leq P(t_n, s_n) \tag{66}$$

and
$$\|y_{n+1} - x_{n+1}\| \leq s_{n+1} - t_{n+1} \quad \text{for all } n \geq 0.$$

From (41), (45), (62) and (63) we obtain

$$\|x_{n+1} - y_n\| \leq \frac{1}{2}\|F'(x_n)^{-1}\|\left\|\left(1 - \frac{1}{2}H(x_n, y_n)\right)^{-1}\right\|\|F''(x_n)\|\cdot\|F(x_n)\|$$

$$\leq \frac{1}{2}\frac{\beta M}{1 - \beta A(0, t_n)}C(t_n)P(t_{n-1}, s_{n-1}) = t_{n+1} - s_n$$

Hence, (60) is true.

From (40), (45), (46), (51), (61)-(63) and the approximation

$$F(x_{n+1})=\int_0^1\left[F'(y_n+t(x_{n+1}-y_n))-F'(y_n)\right](x_{n+1}-y_n)dt$$

$$+\left(F'(y_n)-F'(x_n)\right)(x_{n+1}-y_n)+\frac{a}{2}\left[I-H(x_n,y_n)\right]^{-1}F'(x_n)$$

$$F''(x_n)(y_n-x_n)\int_0^1\left[F'(x_n+t(y_n-x_n))-F'(x_n)\right](y_n-x_n)dt$$

$$+\left[I-\frac{a}{2}H(x_n,y_n)\right]^{-1}\int_0^1\left[F''(x_n+t(y_n-x_n))-F''(x_n)\right](1-t)\,dt(y_n-x_n)^2, \tag{68}$$

we obtain by using the triangle inequality in turn

$$\|F(x_{n+1})\|\le\int_{s_n}^{t_{n+1}}A(s_n,t)dt+A(t_n,s_n-t_n)(t_{n+1}-s_n)$$

$$+\frac{|a|}{2}C(t_n)D(t_n)M(s_n-t_n)\int_{t_n}^{s_n}A(t_n,t)dt$$

$$+\frac{1}{2}C(t_n)\int_{t_n}^{s_n}B(t_n,t)dt(s_n-t_n)^2 = P(t_n,s_n)$$

We have also used the estimates

$$\begin{aligned}\|x_{n+1}-x_0\|&\le\|x_{n+1}-y_0\|+\|y_0-x_0\|\\&\le\|x_{n+1}-y_n\|+\|y_n-y_0\|+\|y_0-x_0\|\\&\le\ldots\le(t_{n+1}-s_n)+(s_n-s_0)+s_0=t_{n+1}\le R_1,\end{aligned} \tag{69}$$

and

$$\begin{aligned}\|y_{n+1}-x_0\|&\le\|y_{n+1}-y_0\|+\|y_0-x_0\|\\&\le\|y_{n+1}-x_{n+1}\|+\|x_{n+1}-y_n\|+\|y_n-y_0\|+\|y_0-x_0\|\\&\le\ldots\le(s_{n+1}-t_{n+1})+(t_{n+1}-s_n)+(s_n-s_0)\le s_{n+1}\le R_1.\end{aligned} \tag{70}$$

Hence, (66) is true.

From (43), (61) and (66)

$$\|y_{n+1}-x_{n+1}\| \le \|F'((x_{n+1})^{-1}\| \cdot \|F(x_{n+1})\| \le D(t_{n+1})P(t_n,s_n)=s_{n+1}-t_{n+1}$$

Hence, (67) is also true.

Moreover, from (40), (46), (56), (69), and the estimate

$$\|F'(x_0)^{-1}\|.\|F'(x_n)-F'(x_0)\| \le \beta A(0,t_n) \le \beta A(0,R_1) < 1,$$

is follows from the Banach lemma on invertible operators [13] that $F'(x_n)^{-1}$ exists and

$$\|F'(x_n)^{-1}\| \le \frac{\|F'(x_0)^{-1}\|}{1-\|F'(x_0)^{-1}\|.\|F'(x_n)-F'(x_0)\|} \le D(t_n)$$

for all $n \ge 1$. Furthermore, from (41), (57), (61), (63), and the estimate

$$\frac{|a|}{2}\|H(x_n,y_n)\| \le \frac{|a|}{2}\|F'(x_n)^{-1}\|.\|F''(x_n)\|.\|x_y-y_n\|$$

$$\le \frac{M|a|}{2}D(t_n)(s_n-t_n) \le \frac{M|a|}{2}D(R_1)R_1 < 1,$$

It follows that $I-\frac{a}{2}H(x_n,y_n)$ is invertible, and

$$\left\|\left(I-\frac{a}{2}H(x_n,y_n)^{-1}\right)\right\| \le C(t_n) \quad \text{for all } n \ge 0.$$

Hence, the iterates generated by (43)-(45) are well defined for all $n \ge 0$. Also, by (65), (67) and (61)

$$\|x_{n+1}-x_n\| \le t_{n+1}-t_n \text{ and } \|y_{n+1}-y_n\| \le s_{n+1}-s_n \text{ for all } n \ge 0 \quad (71)$$

It now follows from (71) and (i) that the sequence $\{x_n\}$ ($n \ge 0$) is Cauchy in a Banach space, and as such it converges to some $x^* \in U(x_0, R_1)$, which by taking the limit as $n \to \infty$ in (43) becomes a zero of F, since $F(x^*) = 0$. Moreover, by (69) and (70) x_n, $y_n \in U(x_0, R_1)$ for all $n \ge 0$. The estimates (59) and (60) now follow from (71).

Finally to show uniqueness, we assume there exists another zero y^* of equation (1) in $U(x_0,R)$. Then from (40) and (63), we obtain

$$\|F'(x_0)^{-1}\|.\int_0^1 \|F'(y^*+t(x^*-y^*))-F'(x_0)\|\,dt$$

$$\leq \int_0^1 A(0,(1-t)\|x_0 - y^*\| + t\|x_0 - x^*\|)dt < 1, \quad \text{by (19)}.$$

It now follows from the above inequality that the linear operator $\int_0^1 F'(y^* + t(x^* - y^*))dt$ is invertible. From this fact, and the approximation

$$F(x^*) - F(y^*) - \int_0^1 F'(y^* + t(x^* - y^*))(x^* - y^*)dt$$

it follows that $x^* = y^*$. □

That completes the proof of the theorem.

Remark 3

(a) From the estimates

$$\|x_n - y_0\| \leq \|x_n - y_n\| + \|y_n - y_0\| \leq (t_n - s_n) + (s_n - s_0) \leq t_n - \eta \leq R_1 - \eta$$

and
$$\begin{aligned} \|y_{n+1} - y_0\| &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - y_0\| \\ &\leq (s_{n+1} - t_{n+1}) + (t_{n+1} - s_n)(s_n - s_0) \\ &\leq s_{n+1} - \eta \leq R_1 - \eta \end{aligned}$$

it follows that $x_n, y_n \in U(y_0, R_1 - \eta)$ for all $n \leq 0$. Note also that R_1 is the unique nonnegative zero of $T(r) - r = 0$ in $[0,R_1]$ (by (54)).

(b) We can use the Chebysheff-Halley-Werner method to approximate nonlinear equations with nondifferentiable operators. Indeed, consider the equation

$$F_1(x) = 0, \tag{72}$$

where $F_1(x) = F(x) + Q(x)$,

with F as before and Q satisfying an estimate of the form

$$\|Q(x+h) - Q(x)\| \leq E(r, \|h\|), x \in U(x_0,R), 0 \leq r \leq R, \leq \|h\| \leq R - r$$

where E is a nonnegative and continuous function of two variables such that if one of the variables is fixed then E is a non-decreasing function of the other on the interval $[0,R]$. Note that the differentiability of Q is not assumed here. Replace F in (41) by F_1 and leave the Fréchet-derivatives as they are. Define the sequences $\{\bar{t}_n\}$ and $\{\bar{s}_n\}$ $(n \geq 0)$ as the corresponding $\{t_n\}$ and $\{s_n\}$ $(n \geq 0)$ given (48) and (49) respectively. The change will be an extra term of the form $E(t_n, s_n - t_n)$ added in the definition of $P(t_n, s_n)$. Define T_1 by T in (52) the insert inside the bracket the term $E(r,r)$. Then following the proof of the above theorem step by step we can show a similar theorem with identical hypotheses and conclusions, but holding for equation (72). (See, also [4], [9], [22].)

(c) Following the proof of the theorem, we can show the result (see also [9]):

Theorem 4

Let $F: D \subset E_1 \rightarrow E_2$, E_1, E_2 be real Banach spaces, and D be an open convex domain. Assume that F has second order continuous Fréchet-derivatives on D and that the following conditions are satisfied:

$$\|F'(x) - F'(y)\| \leq \alpha \|x-y\|, \; \|F''(x)\| \leq M, \; \|F''(x) - F''(y)\| \leq N \|x-y\|, \quad \text{for all } x,y \in D$$

$$\|F'(x_0)^{-1}\| \leq \beta, \; \|y_0 - x_0\| \leq \eta,$$

$$\left[\frac{(2+a)}{(2-a)}M^2 + \frac{2N}{3(2-a)\beta}\right]^{1/2} \leq K,$$

$$h = K\beta\eta \leq \begin{cases} .485 & \text{if } 0 \leq a \leq 1, \\ .5 & \text{if } 1 < a \leq 2 \end{cases}$$

and

$$U(y_0, r_1 - \eta) \subset D$$

Moreover, we define

$$g(t) = \frac{1}{2}Kt^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}, \quad r_1 = \frac{1-\sqrt{1-2h}}{h}\eta, \text{ and } \theta = \frac{1-\sqrt{1-2h}}{1+\sqrt{1-2h}},$$

where r_1 is the smallest zero of the equation $g(t) = 0$. Then the Chebysheff-Halley-Werner method (43)-(45) is convergent. Also x_n, $y_n \in U(y_0, r_1 - \eta)$, for all $n \in N_0$. The limit x^* is the unique zero of the equation $F(x) = 0$ in $U(x_0, r_2^*)$, $r_1 \leq r_2^* < r_2$ if $\alpha = K$, (or $M = K$) and $r_2^* = r_2$ if $\alpha < K$ (or $M < K$).

Moreover, we have the following error estimates and optimal error constants:

$$\|x_n - x^*\| \leq r_1 - t_n^1, \ \|y_n - x^*\| \leq r_n^1 - s_1,$$

and $$r_1 - t_n^1 = \frac{(1-\theta^2)\eta}{1-\theta^{3n}}\theta^{3n} - 1, \quad \textit{for all } n \geq 0,$$

where $$s_n^1 = t_n^1 - \frac{g(t_n^1)}{g'(t_n^1)}, \quad t_0^1 = 0, \ h(t_n^1, s_n^1) = -g'(t_n^1) g''(t_n^1)(s_n^1 - t_n^1)$$

and $$t_{n+1}^1 = s_n^1 - \frac{1}{2}(s_n^1 - t_n^1)^2 \frac{g'(t_n^1)^{-1} g''(t_n^1)}{g'(t_n^1) - \frac{a}{2} h(t_n^1, s_n^1)} \quad \textit{for all } n \geq 0.$$

(d) Several sufficient conditions can be given to show for example that under the hypoteses of Theorems 3 and 4

$$s_n - t_n \leq s_n^1 - t_n^1 \quad \text{for all } n \geq 0.$$

One such condition can be

$$D(r)\left[\int_0^r A(r,t)dt + A(r,r)r + \frac{|a|}{2} C(r) D(r) Mr \int_0^r A(r,t)dt\right] \leq s_1^1 - t_1^1,$$

or $$\leq -\frac{g(r)}{g'(r)}$$

for all $r \in [0, \min\{r_1, R_1\}]$.

The details are left to the motivated reader.

(e) By Theorems 3 and 4, we conclude that under the order of convergence for the Chebysheff-Halley-Werner method is three, whereas for Newton's method it is only two [4], [13].

(f) Similar theorems can be proved if $\|h\|$ in (40) and (42) is replaced by a Holder condition of the form $\|h\|^p$ for some $p \in [0,1]$, [9].

(g) The function A can be chosen as

$$A(r, \|h\|) = \sup_{\substack{x,y \in U(x_0, r) \\ \|h\| \leq R - r}} \|F'(x+h) - F'(x)\|,$$

or $$A(r, \|h\|) = \int_r^{r+\|h\|} q(t)dt$$

where q is a nondecreasing function on the interval $[0,R]$ satisfying

$$\|F'(x) - F'(y)\| \leq q(r) \|x - y)\|$$

for all $x, y \in U(x_0, r)$.

Similarly, the function B can be given by

$$B(r,\|h\|) = \sup_{\substack{x,y \in U(x_0,r) \\ \|h\| \leq R-r}} \|F''(x+h) - F''(x)\|.$$

Other choices are to be equal to the usual Lipschitz or Ptak-like conditions usually imposed on F (see, e.g. [4], [9], [22]). Other choices are also possible.

One can refer to [9] for some possible applications of these ideas to the solution of integral equations.

(h) Finally, if the right-hand sides of conditions (40) and (42) change to $A(r,r+\|h\|)$, and $B(r,r+\|h\|)$ a new theorem similar to Theorem 3 can then follow immediately. Remarks similar to (a)-(g) above for the new condition can then follow also.

(i) Using the estimate

$$\|F''(x)\| \leq \|F''(x) - F''(x_0)\| + \|F''(x_0)\| \leq B(R_1,0) + \|F''(x_0)\| = M^*,$$

we see that hypotheses (41) can be replaced by the weaker one, given by

$$\|F''(x)\| \leq M^*.$$

(j) The Lipschitz condition (42) can be dropped, but the order of convergence will be slower (see, also [5], [9]).

4. APPLICATIONS

In this section we will give an example for Theorem 4 when $a = 1$ (similarly we can work for Theorem 3). We first note that by eliminating y_n $(n \geq 0)$ from approximations (43)-(45) we can obtain the method of tangent hyperbolas (or Chebysheff-Halley) which has been extensively studied in [1], [5], [6], [9], [12], [14], [15], [17], [18], [20], [21]. In all but our references it is assumed that $N > 0$, which means that their results cannot apply to solve quadratic operator equations of the form

$$P(x) = B(x,x) + L(x) + z, \tag{73}$$

where B, L are bounded quadratic and linear operators respectively with z fixed in E_1. We then have that $P'(x) = 2B(x) + L$ and $P''(x) = 2Q$. Hence we get $M = 2\ \|B\|$ and $N = 0$. Integral equations that can be formulated in the form $P(x) = 0$ have very important applications in radiative transfer [2], [3], [9], [10].

As a specific example, let us consider the solution of quadratic integral equations of the form

$$x(s) = y(s) + \lambda s(s) \int_0^1 q(s,t)x(t)dt \tag{74}$$

in the space $E_1 = C[0,1]$ of all functions continuous on the interval $[0,1]$, with norm

$$\|x\| = \max_{0 \le s \le 1} |x(s)|.$$

Here we assume that λ is a real number called the "albedo" for scattering and the kernel $q(s,t)$ is a continuous function of two variables s, t with $0 < s,t < 1$ and satisfying

(i) $0 \le q(s, t) \le 1,\ 0 < s,\ t \le 1,\ q(0,0) = 1;$
(ii) $q(s, t) + q(t, s) = 1,\ 0 < s,\ t \le 1.$

The function $y(s)$ is a given continuous function defined on $[0,1]$, and finally $x(s)$ is the unknown function sought in $[0,1]$.

Equations of this type are closely related with the work of S. Chandrasekhar [10], (Nobel prize of physics 1983), and arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gases, [2], [3], [9], [10].

There exists an extensive literature on equations like (74) under various assumptions on the kernel $q(s,t)$ and λ is a real or complex number. One can refer to the recent work of [2], [3], [9] and the references there. Here we demonstrate that the theorem via the iterative procedure (43)-(45) provides existence results for (74).

For simplicity (without loss of generality) we will assume that

$$q(s,t) = \frac{s}{s+t} \quad \text{for all } 0 < s,t \le 1,\ q(0,0) = 1$$

Note that $q(s,t)$ so defined satisfies (i) and (ii) above.

Let us now choose $\lambda = .25$, $y(s) = 1$ for all $s \in [0,1]$; and define the operator P on E_1 by

$$P(x) = \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)dt - x(s) + 1.$$

Note that every zero of the equation $P(x) = 0$ satisfies the equation (74).

Set $x_0(s) = 1$, use the definition of the first and second Fréchet-derivatives of the operator P to obtain using and the theorem,

$$N=0,\ \alpha = M = 2|\lambda| \max_{0\le s\le 1}\left|\int_0^1 \frac{s}{s+t} dt\right| = 2|\lambda|\ln 2 = .34657359$$

$$K = M\sqrt{3} = .600283066,$$

$$\beta = \| P'(1)^{(-1)} \| = 1.53039421,$$

$$\eta \ge \|P'(1)^{(-1)} P(1)\| \ge \beta\lambda \ln 2 = .265197107,$$

$$h = .243628554 < .5$$

$$r_1 = .3090766, \qquad r_2 = 1.867984353$$

and $\theta = .165459951.$

(For detailed computations, see also [2], [8] and [10].)

Therefore according to Theorem 4 equation (74) has a solution x and the two-point method (43)-(45) converges to x^*. Note that the results obtained in [1], [12], [14], [15], [17], [18], [20], [21] cannot apply here, since $N = 0$. For Theorem 3 we can take $A(r,t) = \alpha t$ and $B(r,t) = 0$ for all $r \in [0,R]$. The computational details for this case are left to the motivated reader.

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BAYESIAN ANALYSIS OF MATCHED BINARY RESPONSE SAMPLES

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ABSTRACT: In this paper, a Bayesian assessment of McNemar's test using Cox (1958) linear logistic model is provided in this paper.

KEYWORDS: McNemar's test, Logistic model, Conditional test, a Prior distributions, Saturated with parameters.

1. INTRODUCTION

Matched samples usually arise in situations where responses on individuals are available before and after treatment. It may also arise in situations when two individuals are matched on the basis of some characteristic. If responses are binary both X_{1i} before treatment and X_{2i} after treatment take value 0 and 1. Let $P(X_{1i} = 1) = P_{1i}$ *and* $P(X_{2i} = 1) = P_{2i}$, $(i = 1,2,3,\ldots,n)$. McNemar (1947) provides a test of the hypothesis H_0: $P_{1i} = P_{2i}$. McNemar's conditional test of H_0 has been justified by Cox (1958) using a linear logistic model for P_{1i} and P_{2i}. We discuss the implications of Cox's model in a Bayesian context and propose a large sample Bayesian test of H_0.

2. THE LOGISTIC MODEL

Cox (1958) has proposed the following reparametrization

$$\log\frac{p_{1i}}{1-p_{1i}} = \alpha_i, \quad \text{and} \quad \log\frac{p_{2i}}{1-p_{2i}} = \alpha_i + \beta \tag{2.1}$$

The reparametrization reduces the number of parameters from $2n$ to $n+1$. The hypothesis to be tested becomes $H_0 : \beta = 0$.

Four different types of responses are possible for (X_{1i}, X_{2i}). These are in obvious notation (0,0), (0,1), (1,0) and (1,1). Let the observed frequencies for these responses be a, b, c, & d respectively. The distinct models of analysis are available to eliminate nuisance parameters P_j. Cox (1970) for instance argues that pairs leading to $X_{1i} + X_{2i} = 0$ or 2 give no information about β. One thus considers only those pairs for which $X_{1i} + X_{2i} = 1$. Conditional on this, the distribution of b is given by

$$P(b) = \binom{b+c}{b} \left(\frac{e\beta}{1+e^{\beta}} \right)^{b} \left(\frac{1}{1+e^{\beta}} \right)^{c} \tag{2.2}$$

which is independent of a_i. Use of (2.2) leads to McNemar's test of H_0. However, Cox (1970) remarks that the requirement that the inference be true for all set of α_i is a serious one. We do not find the argument, leading to the neglect of pairs with $X_{1i} + X_{2i} = 1$ or 2, as sufficiently convincing. In particular (2.2) is true in cases where all α_i's are equal (a case corresponding to independence of X_{1i} and X_{2i}). However, in this case McNemar's test bears no resemblance to the usual test of H_0.

3. BAYSIAN ANALYSIS FOR A GENERAL PRIOR

In (2.1), α_i may be thought of as describing the property of the ith individual and β measures the difference on the logistic scale between before and after treatment responses. In many situations, it will be found both convenient and realistic to assume that α_i are independent and have the same distribution and that β is independent of α_i. This leads to the following representation of the prior distribution.

$$\pi(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta) = \pi(\beta) \prod_{i=1}^{n} \pi(\alpha_i) \tag{3.1}$$

He also observed that there are altogether 4 different types of responses. This permits as to introduce two parameters μ and σ^2 (say) in the description of $\pi(\alpha_i)$, so that together with β the total number of parameters will become 3. Thus 3 parameters will describe the whole model. Such a scheme will make the model sufficiently flexible and in the words of Cox (1970) saturated with parameters.

With this set up we have

$$P(X_{2i} = 0 \mid \mu, \sigma^2, \beta) = E\left(\frac{1}{1+e^{\alpha_i+\beta}} \mid \mu, \sigma^2, \beta \right) = \phi_{\beta}(\mu, \sigma^2)$$

$$P(X_{1i} = 0 \mid \mu, \sigma^2, \beta) = E\left(\frac{1}{1+e^{\alpha_i}} \mid \mu, \sigma^2, \beta \right) = \phi_{\beta}(\mu, \sigma^2) \tag{3.2}$$

Henceforth we shall write ϕ_β and ϕ_0 for $\phi_\beta(\mu,\sigma^2)$ and $\phi_0(\mu,\sigma^2)$ respectively. The probability for the four different types of responses are

$$(0,0) : \left(e^{\beta}\phi_\beta - \phi_0\right)/\left(e^{\beta-1}\right) = \theta_1$$

$$(0,1) : e^{\beta}\left(\phi_0 - \phi_\beta\right)/\left(e^{\beta-1}\right) = \theta_2$$

$$(1,0) : \left(\phi_0 - \phi_\beta\right)/\left(e^{\beta-1}\right) = \theta_3 \tag{3.3}$$

$$(1,1) : \left(1 - \phi_0\right) - \left(1 - \phi_\beta\right)/\left(e^{\beta-1}\right) = \theta_4$$

A serious but very sensible implication of the model is that $\theta_1\ \theta_2 > \theta_2\ \theta_3$ (or $\theta_1 > \theta_3$) which simply states that X_{1i} and X_{2i} are positively correlated. The likelihood given observations a, b, c and d is

$$L\left(\alpha\,\theta_1^a\ \theta_2^b\ \theta_3^c\ \theta_4^d\right) \tag{3.4}$$

When a, b, c and d are large, a Bayesian test of H_0 will depend on the maximum likelihood estimates of β and its posterior variance. We, therefore, discuss the maximum likelihood estimates of β (and ϕ_0, ϕ_β or μ and σ^2)

$$\left.\begin{aligned}
\frac{\partial \log L}{\partial \mu} &= \frac{1}{e^{\beta-1}}\left[\frac{d\phi_\beta}{d\mu}\left\{\frac{ae^{\beta}}{\theta_1} - \frac{(b+c)(1+e^{\beta})}{\theta_1+\theta_3}\,\frac{d_4}{\theta_4}\right\} - \frac{d\theta_0}{d\mu}\left\{\frac{a}{\theta_1} - \frac{(b+c)(1+e^{\beta})}{\theta_2+\theta_3} + \frac{e^{\beta}d_4}{\theta_4}\right\}\right] = 0 \\
\frac{\partial \log L}{\partial \sigma^2} &= \frac{1}{e^{\beta-1}}\left[\frac{d\phi_\beta}{d\sigma^2}\left\{\frac{ae^{\beta}}{\theta_1} - \frac{(b+c)(1+e^{\beta})}{\theta_2+\theta_3}\,\frac{d_4}{\theta_4}\right\} - \frac{d\theta_4}{d\sigma^2}\left\{\frac{a}{\theta_1} - \frac{(b+c)(1+e^{\beta})}{\theta_2+\theta_3} + \frac{e^{\beta}d_4}{\theta_4}\right\}\right] = 0
\end{aligned}\right\} \tag{3.5}$$

Solving these two, we either have $\dfrac{a}{\theta_1} = \dfrac{b+c}{\theta_2+\theta_3} = \dfrac{d}{\theta_4}$ or $\dfrac{d\phi_\beta}{d\mu}\Big/\dfrac{d\phi_0}{d\mu} = \dfrac{d\phi_\beta}{d\sigma^2}\Big/\dfrac{d\phi_0}{d\sigma^2}$

The latter will imply a relationship between μ, β and σ^2 regardless of a, b, c and d. Hence we have

$$\frac{a}{\theta_1} = \frac{b+c}{\theta_2+\theta_3} = \frac{d}{\theta_4} \tag{3.6}$$

When the values for θ_1, θ_2, θ_3 and θ_4 are put in log likelihood function we get

$$\log L = k + b\beta - (b+c)\ \log\left(1+\theta^{\beta}\right) \tag{3.7}$$

If there is no restriction on β, the maximum likelihood estimate of β is log b/σ. However, since $\theta_1 > \phi_0\phi_\beta$ permissible values of β have to satisfy this condition which gives

$$ad > \frac{(e+c)^2 e^{\beta}}{(1+e^{\beta})^2} \tag{3.8}$$

Thus in cases where $ad < bc$ the usual estimate for β i.e. logb/σ will violate (3.8). There are two points to be noted carefully. First even with the very general prior distribution for α_i, restriction (3.8) is to be taken care of McNemar's test would be questionable in cases where $ad < bc$. Secondly precise from for $\pi(\alpha_i)$ such as a normal distribution with mean μ and variance σ^2, which we discuss in Section 4, will generally impose restriction even more severe than (3.8).

4. NORMAL PRIOR DISTRIBUTIONS

Consider $\mu(\alpha_i)$ to be a normal distribution with mean μ and variance σ^2. It may be checked that now

$$\phi_0\ (-\mu,\sigma^2) = 1-\phi_0\ (\mu,\sigma^2) \tag{4.1}$$

In a slightly different context McCullagh (1977) has suggested a beta distribution for $\left(e^{\alpha_i}/1+e^{\alpha_i}\right)$. We remark that for small σ^2 there is virtually no difference between the normal prior for α_i or a beta prior for $\left(e^{\alpha_i}/1+e^{\alpha_i}\right)$. However, we use the following approximation for ϕ_β:

$$\phi_\beta = \frac{\left(e^{-(\mu+\beta)}+1\right)+\frac{1}{2}\sigma^2}{\left(e^{-(\mu+\beta)}+1\right)\left(e^{\mu+\beta}+1\right)+\sigma^2} \tag{4.2}$$

This approximation works extremely well for σ as large as 1 or perhaps even 1.5 for selected values of μ and σ^2. Table 1 gives the exact value of ϕ_β (evaluated numerically) in the top row followed by the approximate value obtained from (4.2) in the bottom row.

Table 1. Exact value and approximate values of ϕ_β for different values of μ and σ

$\sigma\downarrow$ $\mu\rightarrow$	.1	.5	1.0	2.0	5.0
.1	.4751 .4750	.3778 .3778	.2694 .2694	.1196 .1196	.0068 .0068
.5	.4765 .4764	.3843 .3840	.2798 .2794	.1289 .1290	.0075 .0076
1	.4800 .4794	.4008 .3980	.3069 .3033	.1554 .1555	.0100 .0102
2	.4875 .4848	.4369 .4248	.3707 .3523	.2318 .2248	.0195 .0323

It is clear from Table-1 that the approximation in (4.2) works provided σ is less than about 1.5. For very large σ, θ_2 and θ_3 become very small. This means that unless b and c are extremely small as compared to a and d, the approximation in (4.2) can be safely used.

Note that ϕ_0 and ϕ_β in (4.2) satisfy (4.1) and the condition that $\theta_1\theta_4 > \theta_2\theta_3$. The maximum likelihood estimate of ϕ_0 and ϕ_β remain as in (3.6). The condition that $\sigma^2 > 0$ means that

$$\frac{\left(\frac{1}{2}-\phi_0\right)e^{\mu}}{\left(e^{\mu}+1\right)\left[\left(e^{\mu}+1\right)\phi_0-1\right]} > 0 \tag{4.3}$$

Also we have

$$e^{2\mu} = \frac{\frac{2a-d}{N}-2\phi_0\phi_\beta}{e^{\beta}\left[\frac{a}{N}-2\phi_0\phi_\beta\right]} > 0 \tag{4.4}$$

where ϕ_0 and ϕ_β as in (4.2).

The maximum likelihood estimate of β is a value which maximizes $b\beta$-$(b+c)$ log $(1 + e^{\beta})$ and satisfies (4.3) and (4.4).

5. AN APPROXIMATE BAYESIAN TEST OF H_0

A Bayesian analysis would require a precise description of $\pi(\alpha_i)$ followed by a prior distributions of μ, σ^2 and β. Often, however, an approximate method is both useful and indicative of the nature of final analysis. We seek to provide an approximate Bayesian test of H_0. We recall that the most important implication of assuming any prior distribution for $\pi(\theta_i)$ is that $\theta_1 \theta_4 > \theta_2 \theta_3$. Any Bayesian analysis would have to take care of this restriction.

Instead of providing a prior distribution for μ, σ^2 and β in the general case we try a conjugate prior on θ_1, θ_2, θ_3, θ_4 which are functions of μ, σ^2 and β subject to the condition $\theta_1\theta_4 > \theta_2\theta_3$. Precise description of $\pi(\alpha_i)$ is available. This would require a check up if the conjugate prior for θ_1, θ_2, θ_3, θ_4, together with $\theta_1\theta_4 > \theta_2\theta_3$ leads to sensible distribution for μ, σ^2 and β. The prior distribution is

$$\alpha \Pi \theta_i^{c_i - 1}, c_i > 0 \tag{5.1}$$

such that $\theta_1 \theta_4 > \theta_2 \theta_3$. The posterior distribution θ_i's is thus

$$\alpha \Pi \theta_1^{c_1+a} \theta_2^{c_2+b} \theta_3^{c_3+c} \theta_4^{c_4+d} \tag{5.2}$$

subject to $\theta_1 \theta_4 > \theta_2 \theta_3$. Let $Z = \log \frac{\theta_1\theta_4}{\theta_2\theta_3}$ and $W = \log\theta_2 - \log\theta_3$, where H_0 becomes $H_0 : W = 0$. We, therefore, need the posterior distribution of W given $Z > 0$. There is no condition on values of c_i near 0 and $\log a_i$'s. The posterior distribution of Z and W is a bivariate normal with means μ_W, μ_Z, variances σ_W^2, σ_Z^2 and correlation coefficient ρ where

$$\left.\begin{aligned}
&\sigma_W^2 = 1 \\
&\mu_Z = \log\left(\frac{ad}{bc}\right) \\
&\sigma_W^2 = \frac{1}{b} + \frac{1}{c} \\
&\sigma_Z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}, \rho = \left(-\frac{1}{b} + \frac{1}{c}\right) / \sigma_Z \sigma_W
\end{aligned}\right\} \tag{5.3}$$

(see for example Lindley-1964)

Let $Y = W - \frac{\rho \sigma_W}{\sigma_Z} Z$, then L is independent of Z so that the mean and variance of L is unaffected by conditions $Z > 0$. Now

$$W = Y + \rho \frac{\sigma_W}{\sigma_Z} Z \tag{5.4}$$

From (5.4), the posterior mean and variance of W given $Z > 0$ is

$$\left.\begin{aligned} E(W|Z>0) &= E(Y) + \rho \frac{\sigma_W}{\sigma_Z} E(Z|Z>0) \\ V(W|Z>0) &= V(Y) + \left(\rho \frac{\sigma_W}{\sigma_Z}\right)^2 V(Z|Z>0) \end{aligned}\right\} \tag{5.5}$$

These work out to be

$$E(W|Z>0) = \mu_w + \rho \frac{\sigma_W}{\sigma\sqrt{2}\pi} \frac{e^{-b^2/2}}{1-\Phi(-b)}, \tag{5.6}$$

$$V(W|Z>0) = \sigma_W^2 - \rho^2 \frac{\sigma_W^2}{\sqrt{2\pi}} \left[\frac{be^{-b^2/2}}{1-\Phi(-b)} + \frac{\frac{1}{\sqrt{2\pi}} e^{-b^2}}{[1-\Phi(-b)]^2} \right] \tag{5.7}$$

where $b = \frac{\mu_Z}{\sigma_Z}$

As an approximation we may consider the conditional distribution of W to be normal with mean and variances given by (5.6) and (5.7) Test of H_0 would depend on the normal deviate

$$T = \frac{E(W|Z>0)}{\sqrt{V(W|Z>0}} \tag{5.8}$$

It is interesting to note that T is always greater than

$$T_0 = \frac{\mu W}{\sigma_W} \tag{5.9}$$

which would be normal deviate provided by McNemar's test. Also T uses all the observations a, b, c and d whereas T_0 uses only b and c. T is nearly equal to T_0 when b is close to 0 (when $\rho \to 0$) or when μ_Z/σ_Z is large (because in this case the condition $Z > 0$ becomes redundant).

6. NUMERICAL COMPARISONS

We have analyzed the following cases where different values of a, b, c and d are provided in the same order.

Case I: (10, 10, 18, 20)
Case II: (10, 4, 12, 20)
Case III: (13, 8, 20, 13)

First a comparison of maximum likelihood estimates of β is provided where

$\hat{\beta}_1$ = Maximum likelihood estimates of β is satisfying (3.8)

$\hat{\beta}_2$ = maximum likelihood estimates of β is satisfying (4.3), (4.4)

$\hat{\beta}_3$ = $E(W|Z > 0)$

$\hat{\beta}_4$ = $\log(b/c)$

The results are given in the Table 2.

Table 2: Values of $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\beta}_3$, $\hat{\beta}_4$, T and T_0 for different cases

Cases	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	T	T_0
Case-I	-0.60	-0.60	-0.60	-0.60	-1.65	-1.49
Case-II	-1.10	-1.10	-1.11	-1.10	-2.33	-1.90
Case-III	-0.92	-0.92	-1.03	-0.92	-2.67	-2.19

In all three cases $ad > bc$. Different estimates of β are nearly the same for cases I and II. However, for cases III these show slight differences. Then $ad > bc$ (which could be rare) these estimates are likely to differ substantially. Comparing T and T_0, we find that T is always bigger than T_0 in magnitude. Also these values are different in all three cases. Sometimes this can be crucial as in case II where T_0 will not reject H_0 at 5% level but T will reject H_0. These seems to be an advantage in using T in place of T_0 in that it utilizes all observations and also takes care of the restriction imposed by the model.

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A DECOMPOSITION OF SUPER-CONTINUITY

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ABSTRACT: The purpose of this paper is to give a decomposition of super-continuity. We prove that a mapping $f: (X, \tau) \to (Y, \sigma)$ is super-continuous if and only if f is α-continuous and D-continuous.

KEYWORDS AND PHRASES: δ-semi-continuous, D-continuous, α-continuous and super-continuous mappings.

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1. INTRODUCTION

The concept of δ-open set was first introduced by Veličko [12]. Super-continuity was studied in [1,5,10]. In this paper, we introduce the notions of D-set and *D-continuous mapping, and prove that a mapping is super-continuous if and only if it is both α-continuous and* D-continuous.

2. PRELIMINARIES

Throughout this paper spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. A subset A of a space (X,τ) is called regular open (resp. regular closed) if Int(Cl(A)) = A (resp. Cl(int(Cl(A)) = A), where Cl(A) and Int(A)) denotes the closure (resp. interior) of A. The δ-interior [12] of a subset A is the union of all regular open sets of X contained in A and is denoted by $\text{Int}_\delta(A)$. The subset A is called δ-open [12] if $A = \text{Int}\delta(A)$, i.e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set A is called δ-closed [12] if $A = \text{Cl}_\delta(A)$, where $\text{Cl}_\delta(A) = \{x \in X: U \cap A \neq \phi$, U is regular open $x \in U\}$. The family of all δ-open sets form a topology τ_δ on X such that $\tau_\delta \subset \tau$. Since the intersection of two regular open sets is regular open, the collection of all regular open sets forms a base for a coarser topology τ_δ called semi-regularization of τ, than the original one τ. It is well-known that $\tau_s = \tau_\delta$.

Definition 1

A subset A of a space X is called:

(a) semiopen [4] if $A \subset \mathrm{Cl}(\mathrm{Int}(A)$, equivalently there exists an open set U such that $U \subset A \subset \mathrm{Cl}(U)$

(b) α-open [11] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,

(c) δ-semiopen [8] if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A)$, equivalently there exists a δ-open set U such that $U \subset A \subset \mathrm{Cl}(U)$.

Definition 2

A mapping $f: X \rightarrow Y$ is called:

(a) Semi-continuous [4] if f^1 (V) is semiopen in X for every open set V of Y,

(b) α-continuous [11] if f^1 (V) is α-open in X for every open set V of Y,

(c) δ-semi-continuous [9] f^1 (V) is δ-semiopen in X for every open set V of Y,

(d) super-continuous [5] if f^1 (V) is δ-open in X for every open set V of Y,

3. A DECOMPOSITION OF SUPER-CONTINUITY

Definition 3

A subset A of a space X is called a D-set if $= U \backslash W$, where U is a δ-open set and W is a regular open set.

It is easily seen that a subset A is a Dset if and only if $A = U \cap F$, where U is a δ-open set and F is a regular closed set. A δ-open U set is D-set since $U = U \backslash \phi$.

Theorem 1: *Every D-set is δ-semiopen*

Proof

Let $A = U \cap F$ be a D-set, where U is δ-open and $F = \mathrm{Cl}(\mathrm{Int}(F))$. Since $A = U \cap F$ and F is closed, by Lemma 2 in [12] we have $\mathrm{Int}_\delta(A) \supset U \cap \mathrm{Int}\delta(F) = U \cap \mathrm{Int}(F)$. Since $\mathrm{Int}_\delta(A) \subset A \subset F$, $\mathrm{Int}_\delta(A) = \mathrm{Int}_\delta(A)) \subset \mathrm{Int}(F)$. On the other hand, $\mathrm{Int}_\delta(A) \subset A \subset U$ and thus $\mathrm{Int}_\delta(A) \subset U \cap \mathrm{Int}(F)$. Hence $\mathrm{Int}_\delta(A) = U \cap \mathrm{Int}(F)$. Now we prove that $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A)$. Let $x \in A$ and V be any open set containing x. Then $U \cap V$ is also open set containing x. Since $x \in F =$

Cl(Int(F)), there exists a point $z \in$ Int(F) such that $z \neq x$ and $z \in U \cap V$, which implies $z \in U \cap$ Int(F) $=$ Int$_\delta$(A). Hence $x \in$ Cl(Int$_\delta$(A)) and thus $A \subset$ Cl(Int$_\delta$(A).

Theorem 2

Let A be a subset of a space X. Then A is δ-open set if and only if it is both α-open set and D-set.

Proof

Since the necessity is clear, we prove only the sufficiency. Let $A = U \cap F$ be a D-set, where U is δ-open and *F is regular closed. Now* A being α-open, we have

$$U \cap F \subset \text{Int}(\text{Cl}(\text{Int}(U \cap F))) = \text{Int}(\text{Cl}(U \cap \text{Int}(F)))$$
$$\subset \text{Int}(\text{Cl}(U) \cap \text{Cl}(\text{Int}(F))) = \text{Int}(\text{Cl}(U) \cap F)$$
$$= \text{Int}(\text{Cl}(U)) \cap \text{Int}(F).$$

Since $U \cap$ Int(Cl(U)), $U \cap F = (U \cap F) \cap U \subset$ (Int(Cl(U)) $\cap F) \cap U = U \cap$ Int(F). But $U \cap F \supset U \cap$ Int(F). *Therefore,* $U \cap F = U \cap$ Int(F). By Lemma 2 in [12], F being closed, Int(F) $=$ Int$_\delta$(F) and hence $A = U \cap$ Int$_\delta$(F) is δ-open

From the discussion so far, we see that the relationships among D set, δ-semiopen set and some well-known open-like sets can be indicated in following diagram:

δ-open set   open set

D-set α-open set

δ-semiopen set   semiopen set

Example 1

(a) Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}\}$. Then $\{e,d\}$ is δ-semiopen because Cl(Int$_\delta$($\{c,d\}$)) $=$ Cl($\{c\}$) $= \{c,d\}$. $\{c,d\}$ is also a D-set since $\{c,d\}$ is regular closed and $\{c,d\} = X \cap \{c,d\}$. But $\{c,d\}$ is not α-open because Int(Cl(Int($\{c,d\}$))) $= \{c\} \not\supset \{c,d\}$.

(b) Let $X = \{a,b,c,d\}$ with topology $\tau = \{X, \phi, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. Then $\{c,d\}$ is δ-semiopen because Cl(Int$_\delta$($\{c,d\}$)) $= \{b,c,d\} \supset \{c,d\}$.

But $\{c,d\}$ is not D-set since $\{c,d\}$ is not δ-open, $\{c,d\} \neq X\backslash\{a\}$ and $\{c,d\} \neq X\backslash\{c\}$. Also $\{c,d\}$ is not α-open because $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\{c,d\}))) = \{c\} \not\supset \{c,d\}$.

(c) Let $X = \{a,b,c,d\}$ with topology $\tau = \{X,\phi,\{a\}$. Then $\{a,b\}$ is α-open because $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\{a,b\}))) = X \supset \{a,b\}$. But $\{a,b\}$ is not δ-semiopen because $\mathrm{Cl}(\mathrm{Int}_\delta(\{a,b\})) = \phi \not\supset \{a,b\}$.

Definition 4

A mapping f: $X \to Y$ is called D-continuous if f^1 (V) is D set in X for every open set V of Y.

By Theorem 1, we have an immediate result.

Theorem 3

Super-continuity ⇒ D-continuity ⇒ δ-semi-continuity ⇒ semi-continuity.
In above theorem, none of the implications is reversible.

Example 2

(a) Let $X = \{a,b,c,d\}$ with topology $\tau = \{X,\phi,\{a\}1, \{c\}, \{a,b\},\{a,c\}, \{a,b,c\}, \{a,c,d\}\}$, $Y = \{a,b,c,d\}$ with topology $\sigma = \{X,\phi,\{c,d\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be identity. Then f is D-continuous and δ-continuous but it is not super-continuous.

(b) Let $X = \{a,b,c,d\}$ with topology $\tau = \{X,\phi,\{a\},\{c\},\{a,c\},\{a,b,c\}, \{a,c,d\}\}$. $Y = \{a,b,c,d\}$ with topology $\sigma = \{X,\phi,\{c,d\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity. Then f is δ-semi-continuous and neither D-continuous nor α-continuous.

(c) Let $X = \{a,b,c\}$ with topology $\tau = \{X,\phi,\{a\}\}$, $Y = \{a,b,c\}$ with topology $\sigma = \{X,\phi,\{a,b\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity. Then f is α-continuous and hence it is semi-continuous. But f is not δ-semi-continuous.

By Theorem 2, we have the following decomposition of super-continuity:

Theorem 4

A mapping f: X → Y is super-continuous if and only if it is both α-continuous and D-continuous.

Although it is well known that super-continuity implies δ-continuity, in the above theorem we cannot be replaced α-continuity (or, D-continuity) by δ-continuity because δ-continuity is independent of both α-continuity and D-continuity (see Example 2 and following example).

Example 3

δ-continuity and D-continuity are independent of each other.

(a) Let $X = \{a,b,c\}$ with topology $\tau = \{X,\phi,\{a\},\{c\},\{a,b\},\{a,c\}, \{a,b,c\}, \{a,c,d\}\}$, $Y = \{a,b,c,d\}$ with topology $\sigma = \{Y,\phi,\{a,b\}, \{c,d\}\}$. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be identity. Then f is D-continuous but not δ-continuous.

(b) Let $X = \{a,b,c,d\}$ with topology $\tau = \{X,\phi,\{a\},\{c\},\{a,b\},\{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. $Y = \{a,b,c,d\}$ with topology $\sigma = \{Y,\phi,\{a,b\}, \{c,d\}\}$. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be identity. Then f is D-continuous but not δ-continuous.

Neither δ-continuity and α-continuity nor δ-continuity and D-continuity implies super-continuity (see Example 3(a) and 2(a)). To show that δ-continuity and continuity does not imply super-continuity, we have the following example.

Example 4

$X = \{a,b,c,d\}$ with topology $\tau = \{X,\phi,\{a\},\{c\},\{a,c\},\{a,b,c\}, \{a,c,d\}\}$, $Y = \{a,b,c,d\}$ with topology $\sigma = \{Y,\phi,\{a,b,c\}\}$. Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be identity. Then f is δ-continuous and continuous. But it is not D-continuous and hence not super-continuous.

It is well known that the inverse image of a T_i space ($i = 0,1,2$) under a continuous injection is also a T_i space ($i = 0,1,2$). However, in case of D-continuity, the assertion is true for $i = 0$.

Theorem 5

If $f : X \rightarrow Y$ us a D-continuous injective mapping and Y is a T_0 space, then X is also a T_0 space.

Proof

Let $x,y \in X$ with $x \neq y$. Then $f(x) \neq f(y)$. Since Y is T_0 at least one of $f(x)$ and $f(y)$, say $f(x)$, has an open neighborhood V such that $f(x) \in V$ and $f(y) \notin V$. Since f is D-continuous, $f^1(V)$ is D-set and $x \in f^1(V)$, $y \notin f^1(V)$. Then $f^1(V) = U \cap F$, where U is δ-open and F is regular closed, and there are two possible cases for $y \notin f^1(V)$: (i) $y \notin U$ and (ii) $y \notin F$. In case (i), $x \in U$ and $y \notin U$; in case (ii), $y \in X \setminus F$ and $x \in F$, $X \setminus F$ is open and $x \notin X \setminus F$. Hence X is a T_0 space.

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CELLULAR FOLDING

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ABSTRACT: In this paper we introduce the notion of cellular and neat cellular foldings on a category of complexes equipped with cellular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell. Then we obtain the necessary and sufficient conditions for a cellular map to be a cellular folding and a neat cellular folding respectively.

1. INTRODUCTION

Let K and L be directed complexes and f: $|K| \rightarrow |L|$ be continuous function. Then f: K $\rightarrow$ L is a cellular function if

(1) for each directed cell $\sigma \in$ K, f(σ) = $\pm$ τ where τ is a directed cell in L,

(2) dim (f(σ)) $\leq$ dim (σ), [4]

Let K and L be complexes of the same diensin n and K be equipped with finite cellular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell.

A cellular map Φ: $K \rightarrow L$ is *a cellular folding* if Φ satisfies the following:

(i) For each i-cell $e^i \epsilon K$, $\Phi(e^i)$ is an i-cell in L. i.e. Φ maps i-cells to i-cells.

(ii) If $\bar{e}$ contains n vertices. then $\overline{\Phi(e)}$ must contain n distinct vertices.

In the case of directed complexes it is also required that Φ maps directed i-cells of K to i-cells of L but of the same orientation.

A cellular folding $\Phi : K \rightarrow L$ is *a neat cellular folding* if $L^n - L^{n-1}$ consists of a single n-cell. Int L.

The set of complexes together with the neat cellular foldings form a category which is a subcategory of the category of cellular foldings and we denote it by $N(K,L)$. Thus if $N(K,L) \neq \phi$. then dim $L \geq$ dim K.

Throughout this paper we use the term complex to mean a complex equipped with cellular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell.

2. CHAIN MAPS AND CELLULAR FOLDING

The next theorem gives the necessary and sufficient condition for a cellular map to be a cellular folding.

Theorem (1)

Let K and L be complexes of the same dimension n and $\Phi : K \to L$ be a cellular map such that $\Phi(K) = L \neq K$. Then Φ is a cellular folding if the map $\Phi_p : C_p(K) \to C_p(L)$ between chain complexes $(C_p(K), \partial_p), (C_p(L), \partial'_p)$ is a chain map.

Proof

Let Φ be a cellular folding, ten it is a cellular map and we can define a homomorphism $\Phi_p : C_p(K) \to C_p(L)$ by

$$\Phi_p(\sigma) = \begin{cases} \Phi(\sigma) & \text{if } \Phi(\sigma) \text{ is a p-cell in L,} \\ \phi & \text{if } \dim(\Phi(\sigma)) < p, \end{cases} \qquad [4]$$

and since a cellular folding maps p-cells to p-cells, $\Phi_p(\sigma_\lambda)$ is a p-cell in L for all λ.

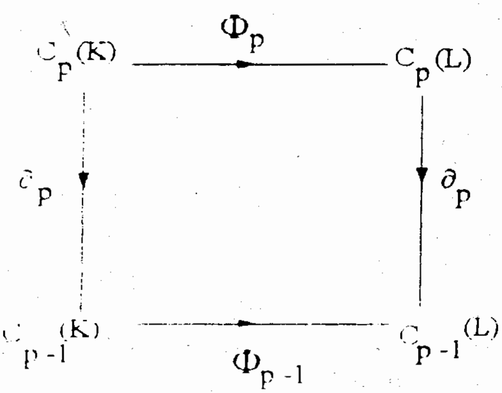
Thus for a p-chain $C = a_1 \sigma_1^p + a_2 \sigma_2^p + a_m \sigma_m^p \in C_p(K)$ where a_i's $\in Z$ and σ_i's are p-cells in K.

$$\Phi_p(C) = \Phi_p\left(a_1 \sigma_1^p + a_2 \sigma_2^p + \ldots + a_m, \sigma_m^p\right)$$

$$\Phi_p(C) = \Phi_p\left(a_1 \sigma_1^p\right) + \Phi_p\left(a_2 \sigma_2^p\right) + \ldots + \Phi_p\left(a_m, \sigma_m^p\right)$$

$$= a_1 \Phi_p\left(\sigma_1^p\right) + a_2 \Phi_p\left(\sigma_2^p\right) + \ldots + a_m \Phi_p\left(\sigma_m^p\right) \in C_p(L).$$

Now since the closure of both σ_λ^p and $\Phi(\sigma_\lambda^p)$ has the same number of distinct vertices then $\Phi_{p-1} 0 \partial_p = \partial'_p \, 0 \Phi_p$, where $\partial_p : C_p(K) \to C_{p-1}(K)$ and $\partial'_p : C_p(L) \to C_{p-1}(L)$ are the boundary operators, that is to say the following diagram commutes



Fig

and hence Φ is a chain map.
Conversely, suppose Φ is not a cellular folding, then there exists a j-cell σ in K such that $\Phi(\sigma)$ is a m-cell in L, where m $\neq$ j. Since Φ_p is a homomorphism from the p-th chain of K to the p-th chain of L, then

$$\Phi_j\left(\sum_{1=1}^{n-1} \lambda_1 \sigma_1^{(j)} + \lambda_n \sigma\right) = \sum_{1=1}^{n-1} \lambda_1 \Phi\left(\sigma_1^{(j)}\right) + \lambda_n \Phi(\sigma)$$

but $\Phi(\sigma)$ is not a j-cell, then Φ_j cannot be a j-chain map and hence our assumption is false and we have the result.

2.1 Examples

1. Let K be a complex such that $|K|$ is the infinite strip $\{(x,y): -\infty < x < \infty, 0 \leq y \leq 2\}$ equipped with an infinite number of 2-cells such that the closure of each 2-cell consists of four 0-cells and four 1-cells, see Fig.1.

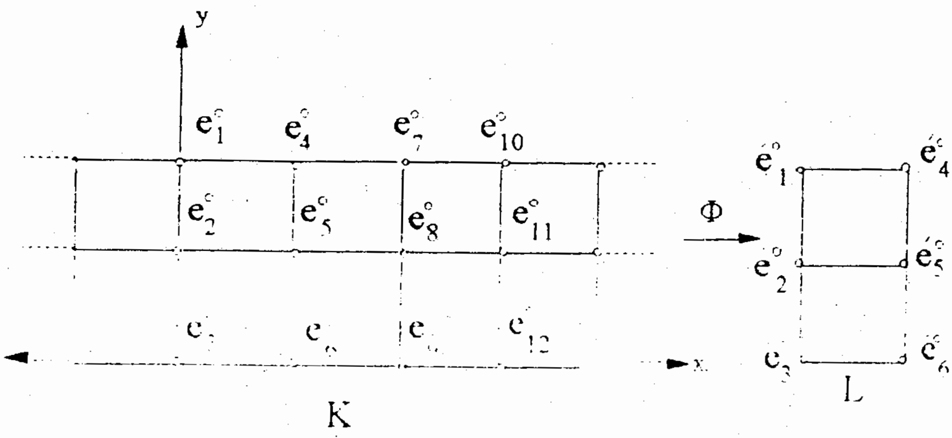


Fig.1

Let L be a complex with six 0-cells, seven 1-cells and two 2-cells. The cellular map $\Phi : K \to L$ defined by

$$\Phi\left(e_n^{\circ}\right) = e_m^{\prime\circ} \quad \left\{ \begin{array}{ll} where & m = 1,2,\ldots,6 \;\; and \\ 0 & n-m \;\; is \; a \; multiple \; of \; 6 \end{array} \right\}$$

$$\Phi\left(e_i^{2}\right) = \left\{ \begin{array}{ll} e_1^{\prime 2} & if \; i \; is \;\; odd \\ e_2^{\prime 2} & if \; i \; is \;\; even \end{array} \right\}$$

This map is a cellular folding.

2. Consider a complex K such that $|K| = S^2$ with cellular subdivision consisting of two 0-cells, four 1-cells and four 2-cells. Let $\Phi : K \to K$ be a cellular map defined by

$$\Phi\left(e_1^0, e_2^0\right) = \left(e_1^0, e_2^0\right)$$

$$\Phi\left(e_2^1, e_4^1\right) = \left(e_1^1, e_2^1\right)$$

$$\Phi\left(e_n^2\right) = e_1^2, \quad n = 1,2,3,4.$$

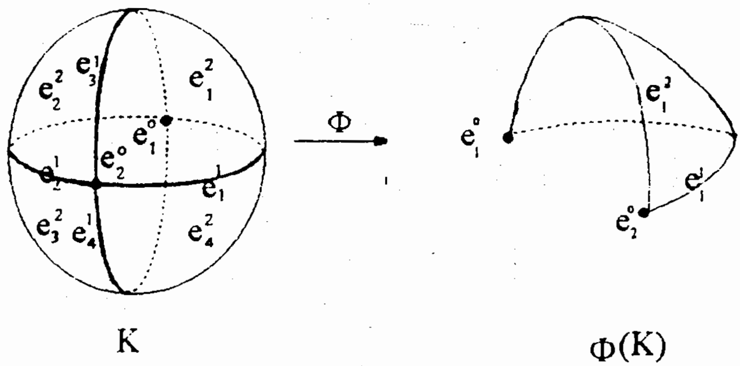


Fig.2

This map is a cellular folding with image consisting of two 0-cells, two 1-cells and a single 2-cell, see Fig.2.

3. Consider a complex K such that $|K|$ is a torus with cellular subdivision consisting of three 0-cells, six 1-0cells and three 2-cells.

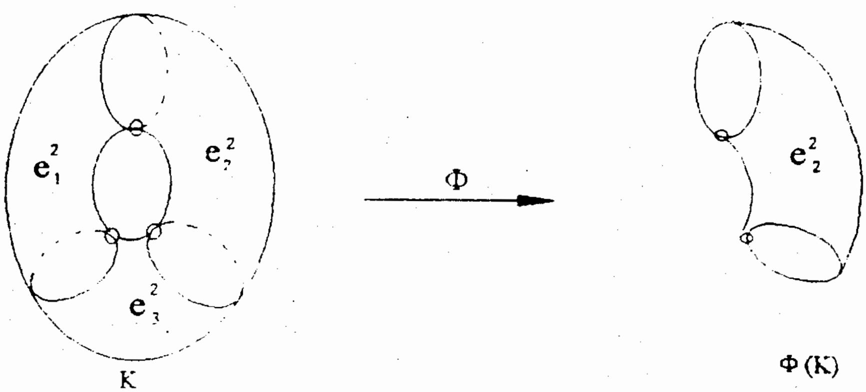


Fig.3

Any cellular map $\Phi : K \to K$ which has two vertices in the image is not a cellular folding since Φ_1 is not a chain map in this case.

4. Consider a complex K such that $|K|$ is a torus with cellular subdivision consisting of four 0-cells, eight 1-cells and four 2-cells.

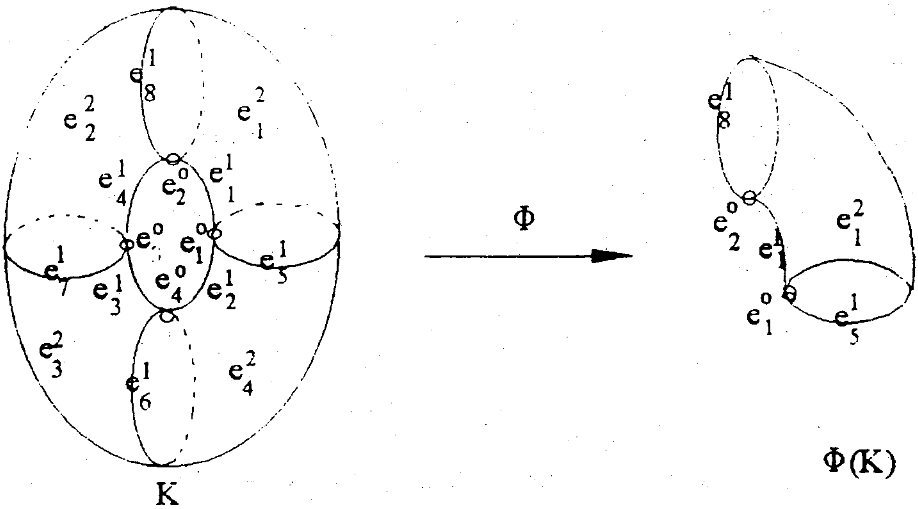


Fig.4.

A cellular map $\Phi : K \to K$ defined by

$$\Phi\left(e_1^0, e_2^0, e_3^0, e_4^0\right) = \left(e_1^0, e_2^0, e_3^0, e_4^0\right)$$

$$\Phi\left(e_1^1, e_2^1, \ldots, e_8^1\right) = \left(e_1^1, e_1^1, e_1^1, e_1^1, e_5^1, e_8^1, e_5^1, e_8^1\right)$$

$$\Phi\left(e_n^2\right) = e_1^2, \quad n = 1,2,3,4.$$

This map is a cellular folding with image consisting of two 0-cells, three 1-cells and a single 2-cell.

5. Consider a cell-complex K such that $|K| = S^2$ with cellular subdivision consisting of four 0-cells, six 1-cells and four 2-cells, see Fig.5.

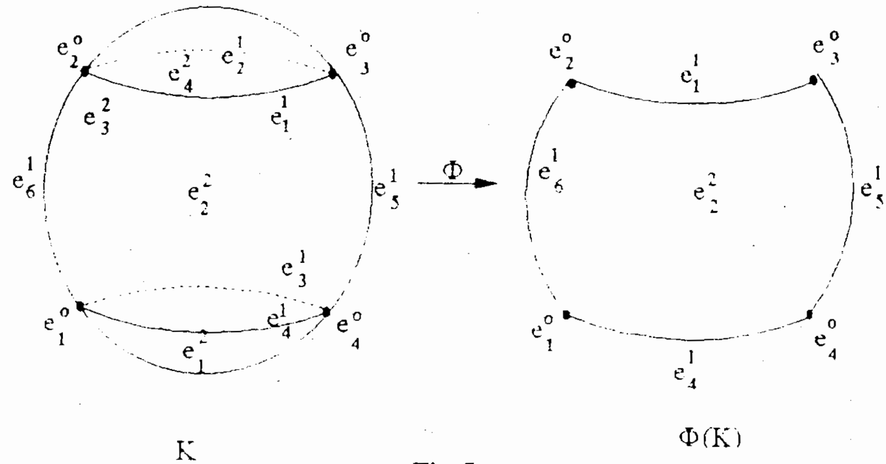


Fig.5

Let $\Phi : K \rightarrow K$ be a cellular map defined by

$$\Phi\left(e_1^0, e_2^0, e_3^0, e_4^0\right) = \left(e_1^0, e_2^0, e_3^0, e_4^0\right)$$

$$\Phi\left(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1\right) = \left(e_1^1, e_1^1, e_4^1, e_4^1, e_5^1, e_6^1\right)$$

$$\Phi\left(e_n^2\right) = e_2^2, \quad n = 1,2,3,4.$$

This map is not cellular folding since $\overline{e_1^2}, \overline{\Phi\left(e_1^2\right)}$ does not contain the same number of vertices.

3. NEAT CELLULAR FOLDING

The following theorem gives the necessary and sufficient condition for a cellular map to be a neat cellular folding.

Theorem (2)

If $\Phi \in N(K,L)$ such that $\Phi(K) = L \neq K$, then Φ is a neat cellular folding iff the map $\Phi_p : C_p \rightarrow C_p(L)$ between the chain complexes $\left(C_p(K), \partial_p\right), \left(C_p(L), \partial_p'\right)$ is a chain map and $H_p(K) \simeq \ker \Phi_p^*$, where $\Phi_p^* : H_p(K) \rightarrow H_p(L)$, $p \geq 1$ is the induced homomorphism.

Proof

Assuming that Φ is a neat cellular folding, then it is a cellular folding and hence the map Φ_p : $H_p(K) \rightarrow H_p(L)$ between the chain complexes $(C_p(K),\hat{\partial}_p),(C_p(L),\hat{\partial}_p)$ is a chain map. Now consider the induced homorphism Φ_p^* : $H_p(K) \rightarrow H_p(L)$, there is a short exact sequence:

$$0 \rightarrow \ker \Phi_p^* \rightarrow H_p(K) \rightarrow \text{Im } \Phi_p^*$$

where i^* is the induced homomorphism by the inclusion. since Φ is surjective. we have $Im\,\Phi_p^* \simeq H_p(L)=0$, but H_p (L) = 0 for neat cellular folding, hence the above sequence will take the form:

$$0 \rightarrow ker\,\Phi_p^* \rightarrow H_p(K) \rightarrow 0.$$

The exactness of this sequence implies that

$$H_p(K) \simeq ker\,\Phi_p^*.$$

Conversely, suppose Φ_p is a chain map between chain complexes and $H_p(K) \simeq ker\,\Phi_p^*$ but Φ is not neat then $L^n - L^{n-1}$ consists of more than one n-cells. Thus

$$H_o(L) \simeq Z^j, \quad H_p(L)=0, \quad \text{for p} = 1,2,\ldots,n$$

and $$H_p(K) \simeq H_p(L) \oplus ker\,\Phi_p^* \neq ker\,\Phi_p^* \quad \text{for p} = 0$$

hence the assumption is false and Φ is neat.
It should be noted that examples (2) and (4) are neat cellular foldings.

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ON A LOCAL FORM OF GENERALIZED LOBATCHEWSKI'S FUNCTIONAL EQUATION

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Synopsis

The paper deals with real functions defined on a real open interval (A,B) ($-\infty \leq A < B \leq +\infty$). A triple (f,g,h) of functions is said to be locally g-Lobatchewski if for each $x \in (A,B)$ there exists $\delta(x) > 0$ such that $f(x)^2 = g(x+k)h(x-K)$ holds for each k, $|k| < \delta(x)$. In the paper a full description of the family of all locally g-Lobatchewski triples is given.

1. INTRODUCTION AND PRELIMINARIES

In the general theory of functional equations, Lobatchewski's functional equation

$$f\left(\frac{x+y}{2}\right)^2 = f(x)f(y) \quad (\text{or } f(x)^2 = f(x+k)f(x-k)) \tag{L}$$

is well known. In the present paper we shall deal with the following generalized form of Lobatchewski's functional equation

$$f\left(\frac{x+y}{2}\right)^2 = g(x)h(y) \tag{GL}$$

with three unknown functions f,g,h. Especially we shall deal with a local form of these functional equations. With this aim the functional equation (GL) will be investigated in the form

$$f(x)^2 = g(x+k)h(x-k)) \tag{GL$_0$}$$

Note, that analogous generalizations of Cauchy equations have been investigated by Pexider [1].

Definition 1

A triple (f,g,h) $(f:(A,B) \to R,\ g:(A,B) \to R,\ h:(A,B) \to R,\ R-$ the real line, $(A,B) \subset R)$ is said to be locally g-Lobatchewski at $x \in (A,B)$ if there exists $\delta(x) > 0$ such that (GL_0) holds for each k, $|K| < \delta(x)$. We say that (f,g,h) is locally g-Lobatchewski if it is locally g-Lobatchewski at x for each $x \in (A,B)$. Let LGL stand for the set of all locally g-Lobatchewski triples.

In the following, using solutions of (GL), a full description of the families LGL will be given. Local form of functional equation (L) in paper [3] have been investigated. Recall some definitions and results of [3] that will be used in the following text.

Definition 2

([3]) A set $M \subset (A,B)$ is said to be an s-set if

(i) M is closed and countable;

(ii) for each $x \in M$ there exists $\delta_x > 0$ such that for each k, $0 < k < \delta_x$, $x+k \in M$ if and only if $x-k \in M$.

Definition 3

([3]) A set $N \subset (A,B)$ is said to be a semisymmetric $(ss-)$ set if

(i) N is closed;

(ii) for each $x \in N$ there exists $\delta_x > 0$ such that for each k, $0 < k < \delta_x$, $x+k \in N$ or $x-k \in N$.

Theorem A

([3]) Let $f: (A,B) \to R$. Then the following statements are equivalent:

(a) f is locally Lobatchewski at each $x \in (A,B)$, i.e. for each $x \in (A,B)$ there exists $\delta(x) > 0$ such that (L) holds for each k, $|K| < \delta(x)$;

(b) there exists an $ss-$set N such that $N = \{x \in (A,B): f(x) = 0\}$; for each interval (a,b) contiguous to N there exists a function $g: (a,b) \to R$ which fulfil (L), an $s-$set $M \subset (a,b)$ with the collection $\{J_n\}$ of contiguous intervals of M in (a,b) and a real sequence $\{a_n\}$, such that $f|J_n = a_n g|J_n$ holds for each n, and f is locally Lobatchewski at each $x \in M$.

2. RESULTS AND PROOFS

We begin with the following Theorem 1 which must be known, but we are not able to give any references. It is stated in more general form than we shall use in the proof of Theorem 2. In the following R^n stands for n-dimensional euclidean space and $K(x,r) \subset R^n$ denotes the open ball centered at x, with the radius $r > 0$.

Theorem 1

Let $D \subset R^n$ be an open and convex set.

(a) *Let functions $f:D \to R$, $g:D \to R$ and $h: D \to R$ fulfil (GL). Then*

I. *there exists an additive function $\varphi: R^n \to R$ and real constants a and b with $ab > 0$, such that*

$$f(x) = \pm\sqrt{ab}\, e^{\varphi(x)},\ g(x) = a e^{\varphi(x)} \ \ and \ \ h(x) = b e^{\varphi(x)} \quad (1)$$

holds for each $x \epsilon D$, or

II. *one of the following possibilities occurs*
$f = 0,\ g = 0,$ *h is arbitrary, or*
$f = 0,\ h = 0,$ *g is arbitrary.* (2)

(b) *Each triple (f,g,h) of real functions defined on D of the form (1) or (2), where $\varphi: R^n \to R$ is an additive function and a and b any real constants with $ab > 0$ fulfil (GL).*

In the proof of Theorem 1 we shall use the following Lemma. Let $D \subset R^n$ be open and convex set, $0 \epsilon D$. Let r be such a positive number that $K(0,2r) \subset D$. if a function $\varphi: D \to R$ fulfil the equation

$$\psi(x+y) = \psi(x)\ \psi(y) \quad (3)$$

for each $x, y \in K(0,r)$ then there exists a unique function $\tau: R^n \to R$ fulfilling (3) such that $\tau|D = \psi$. If $\psi \neq 0$, then $\psi(x) = e^{(\varphi(x)}$ holds for each $x \in D$, where $\varphi:R^2 \to R$ is an additive functions.

Proof of the Lemma

Let ψ be a solution of (3). By induction we can verify, that

$$\psi(x) = \psi(2^{-m}x)^{2^m} \quad (4)$$

holds for each $x \epsilon D$ and $m = 1,2,\ldots$.

Suppose that there exists $x_0 \epsilon D$ such that $\psi(x_0) = 0$. Choose arbitrary $x \epsilon D$. Then there exists a positive integer m such that $x' = 2^{-m} x \epsilon K(0,r/2)$ and $x_0' = 2^{-m} x_0 \epsilon K(0,r/2)$. Consequently $x' - x^{0'} \epsilon K(0,r)$. According to (4) we have

$$\psi(x) = \psi(x')^{2^m} = \psi((x' - x_0') + x_0')^{2^m} = \psi(x' - x_0')^{2^m}\,\psi(x_0) = 0$$

i.e. $\psi = 0$. Then obviously $\tau = 0$ ($\tau: R^n \to R$).

Suppose that $\psi(x) \neq 0$ holds for each $x \in D$. According to (4) $\psi(x) \geq 0$, hence $\psi(x) > 0$ holds for each $x \in D$. We can define the function $\psi_0: D \to R$ by

$$\psi_0(x) = \log\psi(x) \tag{5}$$

From (3) it follows that $\psi_0(x+y) = \psi_0(x) + \psi_0(y)$ holds for each $x,y \in K(0,r)$. The last equation is the Cauchy equation on a restricted domain and there exists a uniquely determined additive function $\psi : R^n \to R$ such that $\psi|D = \psi_0$. It follows from (5) that $\tau(x) = e^{\Psi(x)}$ holds for each $x \in R^n$.

Proof of Theorem 1

(a) The following two cases will be considered: (I) there exists $x_0 \in D$ such that $g(x_0) \neq 0$ and $h(x_0) \neq 0$; (II) $g(x) = 0$ or $h(x) = 0$ holds for each $x \in D$.

(I) Suppose $x_0 = 0 \in D$, $a = g(0) \neq 0$. Obviously $ab > 0$. Put $f_0(x) = (ab)^{-1} f(x/2)^2$ for every $x \in D$. Setting in (GL) $y = 0$ ($x = 0$) we obtain

$$g(x) = a f_0(x) \quad (h(x) = bf_0(x)). \tag{6}$$

Choose $r > 0$ such that $K(0,2r) \subset D$. If $x,y \in K(0,r)$, then $x + y \in D$ and a $bf_0(x+y) = f((x+y)/2)^2 = g(x)h(y) = a\,bf_0(x)\,f_0(y)$. The function f_0 fulfil the assumptions of the Lemma, hence $f_0(x) = e^{\psi(x)}$ holds for each $x \in D$, where $\psi : R^n \to R$ is an additive function. The expression (1) is an easy consequence of the definition of f_0 and (6).

Suppose now that $0 \notin D$, or $g(0) = 0$, or $h(0) = 0$. Then $0 \in D_0 = D - x_0$. It is easy to check that function $F : D_0 \to R$, $G : D_0 \to R$ and $H : D_0 \to R$ defined by

$$F(x) = f(x+x_0),\ G(x) = g(x+x_0),\ H(x) = h(x+x_0) \tag{7}$$

fulfil the equation (GL) on D_0, $G(0) \neq 0$, $H(0) \neq 0$. By the method used int he proof of Theorem 1 in [2] we can verify that expression (1) holds.

(ii) If $g(x) = 0$, or $h(x) = 0$ holds for each $x \in D$, then (GL) implies $f(x) = 0$ for every $x \in D$. We will prove that the expression (2) is satisfied. On the contrary suppose that there are $x \in D$ and $y \in D$ such that $g(x) \neq 0$ and $h(y) \neq 0$. Then $0 = f((x+y)/2)^2 = g(x)h(y) \neq 0$ a contradiction.

(b): Straightforward verification.

Theorem 2

Let f: (A,B) → R, g: (A,B) → R, h: (A,B) → R. Then the following statements are equivalent:

(a) (f,g,h)ϵ LGL;

(b) *there exists an ss-set* $N = \{x \epsilon (A,B) : f(x) = 0\}$ *and for each* $x \epsilon N$ *either* $g(x) = 0$*, or* $h(x) = 0$*; for each interval (a,b) contiguous to N there exists an additive function* ψ : R → R, an s-set M ⊂ (a,b) with the collection $\{J_n\}$ of contiguous intervals of M in (a,b) and real sequences $\{a_n\}$, $\{b_n\}$ with $a_n b_n > 0$, such that

$$f|J_n = \pm\sqrt{a_n b_n}\, e^{\varphi|J_n},\ g|J_n = a_n e^{\varphi|J_n},\ h|J_n = b_n e^{\varphi|J_n} \quad (8)$$

holds for each n, and (f,g,h) is locally g-Lobatchewski at each xϵM.

Proof

(a) implies (b): Let (f,g,h) ϵ LGL. First we show that f: (A,B) → R is locally Lobatchewski at each x ϵ(A,B). Let $\delta(x)$ have the meaning of Definition 1. Choose k, $|K| < \delta(x)$. Analogously to the proof of Theorem 2 in [2] we can check that $f(x+k)f(x-k) = f(x)^2$, hence f is locally Lobatchewski at each xϵ(A,B) and its form is described by Theorem A,(b). If x ϵ N, obviously either $g(x) = 0$, or $h(x) = 0$. Each non zero function Ψ:(a,b) → R ((a,b) ⊂ R), which fulfil (L) has the form $\psi(x) = ce^{\psi(x)}$, where ψ: R → R is an additive function and $c \neq 0$ is a real constant (Theorem 1, f = g = h). Let (a,b) be an interval contiguous to N. Then for (a,b) there exists an additive function ψ: R → R, an s-set M ⊂ (a,b) with the collection $\{J_n\}$ of contiguous intervals in (a,b) and a real sequence $\{c_n\}$ such that $f|J_n = c_n e^{\varphi|J_n}$ holds for each n, and f is locally Lobatchewski at each x ϵ M. The expression (8) is an consequence of Theorem 1.

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A NOTE ON SOLUTIONS OF THE FUNCTIONAL EQUATION f (x+h(x)) = f(x) + f(h(x))

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Synopsis

We consider the functional equation $f(x+h(x)) = f(x) + f(h(x))$ where h is a given function. We show that under some conditions imposed on h every solution f of this equation which is differentiable at zero and (in Theorem 2) continuous at the points of discontinuity of h is actually a linear function. This is improvement of former results by Forti from [2].

1. INTRODUCTION AND PRELIMINARIES

The present note is connected with a paper of Forti [2]. Using a slightly different method of proof we are able to improve some results obtained there. In order to keep the terminology used in [2] let us recall first the following.

Definition

Let X, Y be groups and A be a class of functions from X into Y. Let $Z \subset X \times X$. If every function f belonging to A and satisfying the Cauchy equation.

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

for all $(x,y) \in Z$ satisfies this equation for all $(x,y) \in X * X$ then we say that the condition (Z,X,Y) is redundant for the class A.

In the sequel we shall deal with conditions (Z_h, R, R) where for a given function $h: R \to R$ we denote $Z_h = \{(x,h(x)): x \in R\}$. In [3] Zdun proved that if h is strictly increasing, continuous and $h(0) = 0$ then (Z_h, R, R) is redundant for the class of functions differentiable at zero. Some generalizations of this result are due to Dhombres [1]. Here we make the same assumptions on h as Forti but we prove that (Z_h, R,R) is redundant for larger classes of functions.

2. MAIN RESULTS

In this section our main results are given. We shall prove first the following:

Theorem 1

Let h: R → R be continuous, h(0) = 0 and

(i) $xh(x) > 0$ *for all* $x \neq 0$; *or*

(ii) $x(h(x)+x) < 0$ *for all* $x \neq 0$.

Then (Z_h, R, R) *is redundant for the class of functions differentiable at 0.*

Proof

Let f: R → R be a function differentiable at 0, f(0) = 0 and define g: R → R by

$$g(x) = \begin{cases} f(x)/x, & x \neq 0, \\ f'(0) & x = 0 \end{cases}$$

Then g is continuous at 0 and moreover f solves the equation

$$f(x+h(x)) = f(x) + f(h(x)) \tag{2.1}$$

if and only if g solves

$$g(x+h(x)) = \frac{x}{x+h(x)}g(x) + \frac{h(x)}{x+h(x)}g(h(x)) \tag{2.2}$$

for $x \neq 0$.

Now it is sufficient to show that g is a constant function. We may also assume without loss of generality that g(0) = 0. Let us fix an $\epsilon > 0$. Continuity of g at 0 implies that there exists a $t > 0$ such that

$$|g|_{[-t,t]}| < \epsilon. \tag{2.3}$$

Put $T := \sup\{t > 0 : (2.3) \text{ holds for } t\}$. Then $|g|_{(-T,T)}| < \epsilon$. Suppose that $T < +\infty$ and consider both cases assumed in our theorem.

(i) Denote $B_T := \{t \in (0,T) : h(t) \geq T\}$ and $C_T := \{t \in (-T,0) : h(t) \leq -T\}$ and put

$$b = \begin{cases} \inf B_T, & B_T \neq 0, \\ T, & B_T = 0; \end{cases} \qquad c = \begin{cases} \sup \; C_T, & C_T \neq 0, \\ -T, & C_T = 0. \end{cases}$$

We have $0 < b \leq T$, $-T \leq c < 0$ because $h(0) = 0$ and h is continuous. Hence, $b+h(b) > T$ and $c+h(c) < -T$ as h is continuous and $xh(x) > 0$ for $x \neq 0$.

Take a $y \in (0,b+h(b))$. There exists an $x \in (0,b)$ such that $y = x+h(x)$. From (2.2) we derive (taking into account that $h(x) < T$.

$$|g(y)| = |g(x+h(x))| \leq \frac{x}{x+h(x)} |g(x)| + \frac{h(x)}{x+h(x)} |g(h(x))| < \varepsilon$$

Similarly if $y \in (c+h(c), o)$ then $|g(y)| < \epsilon$. Thus $|g(y)| < \epsilon$ for all $y \in (c+h(c), b+h(b)) >_F (-T,T)$ which contradicts our supposition and proves that $|g(y)| < \epsilon$ for all $y \epsilon R$. ϵ being arbitrary we conclude that $g = 0$ in the case (i).

(ii) Equation (2.2) may be equivalently transformed into

$$g(h(x)) = \frac{x+h(x)}{h(x)} g(x+h(x)) + \frac{-x}{h(x)} g(x). \tag{2.4}$$

Observe that now for any $x \neq 0$ we have $-x/h(x) \in (0,1)$ whence $x+h(x)/h(x) = 1-(-x/h(x)) \in (0,1)$. Denote by u the function given by $u(x) = x+h(x)$ for $x \in R$. It is easy to see that $h(x) < u(x) < o$ for $x > 0$ and $h(x) > u(x) > 0$ for $x < 0$. Denote by R_T and S_T the sets $\{t \in (0,t) : u(t) \leq -T\}$ and $\{t \in (-T,0) : u(t) \geq T\}$ respectively and put

$$r = \begin{cases} \inf R_T, & R_T \neq 0, \\ T, & R_T = 0 \end{cases}$$

$$s = \begin{cases} \sup \; S_T, & S_T \neq 0 \\ -T, & S_T = 0 \end{cases}$$

Continuity of u and $u(0) = 0$ imply that $0 < r < T$ and $-T \leq s < 0$. Moreover, $h(r) = u(r)-r < -T$ and $h(s) = u(s) - s > T$.

Fix a $y \in (h(r),0)$. By the continuity of h there exists an $x \in (0,r)$ such that $y = h(x)$ and (2.4) with the fact that $u(x) \in (-T,0)$ gives

$$|g(y)| = |g(h(x))| \le \frac{x+h(x)}{h(x)} |g(u(x))| + \frac{-x}{h(x)} |g(x)| < \epsilon.$$

In the same way we obtain for y $\epsilon(0,h(s))$ that $|g(y)| < \epsilon$. As in the case (i) we infer that g = 0 in the present case, too. This completes the proof of the theorem.

Let us remark that Examples 4 and 5 from [2] show that neither continuity of h at 0 can be dropped nor can h vanish for some x $\neq$ 0. Let us also remark that in general continuity of h cannot be replaced by a weaker condition, e.g. by requiring that h transforms intervals with 0 as an endpoint on to intervals. This is seen by the following.

Example: Define $h_1 : (0, +\infty) \to R$ by

$$h_1(x) = \begin{cases} x, & x \in [0,1/2), \\ (1/2)(1-x), & x \in [1/2, 1), \\ 1/2, & x \in [1,+\infty), \end{cases}$$

and h : R $\to$ R by

$$h(x) = \begin{cases} h_1(x) & x \le 0 \\ -h_1(-x), & x < 0. \end{cases}$$

Then h(0) = 0, xh(x) > 0 and the image under h of any interval with 0 as an endpoint is an interval while f : R $\to$ R given by

$$f(x) = \begin{cases} x-1, & x \le -1, \\ x, & |x| < 1, \\ x+1, & x \ge 1 \end{cases}$$

is a nonlinear function, differentiable at 0, continuous in a neighborhood of 0, and is a solution of (2.1).

Finally let us prove

Theorem 2

If h:R $\to$ R is such that h(0) = 0 and (iii) $-x^2 < xh(x) < 0$ for x $\neq$ 0 then (Z_h, R, R) is redundant for the class of functions differentiable at 0 and continuous at the points of discontinuity of h.

Proof

As in the proof of Theorem 1 fix an $\epsilon > 0$ and denote by T the supremum of all positive t's for which (2.3) holds. We have $|g|_{(-T,T)}| < \epsilon$.

Suppose that $T < +\infty$. Condition (iii) implies that $0<x+h(x)<x$ for $x > 0$ and $0 > h(x)+x>x$ for $x < 0$. Thus in particular $-h(x)/x\epsilon(0,1)$ whence $(x+h(x)/x) \epsilon (0,1)$. Moreover

$$0 < T+h(T) < T \quad \text{or} \quad -T<h(T) < 0. \tag{2.5}$$

An equivalent form of (2.2) yields

$$|g(T)| = \left|\frac{T+h(T)}{T} g(T+h(T)) + \frac{-h(T)}{T} g(h(T))\right|$$

$$\leq \frac{T+h(T)}{T} |g(T+h(T))| + \frac{-h(T)}{T} |g(h(T))| < \epsilon.$$

The last inequality follows from (2.5). By hypothesis g and h cannot simultaneously discontinuous at T. Now, if h is continuous at T then because of (2.5) there exists a $T_2 > T$ such that $x+h(x) \epsilon (0,T)$ and $h(x) \epsilon (-T,o)$ for $x \epsilon (0,T_2)$. As before we get $|g(x)| < \epsilon$ for $x \epsilon (0,T_2)$. Therefore in both cases there exists $T_0> T$ such that $|g(x)| < \epsilon$ for $x \epsilon[0,T_0)$. Analogously we can show that $|g(x)| < \epsilon$ for $x \epsilon(-T_0,0]$ where $T_0' > T$, which implies that $|g(x)| < \epsilon$ for $x \epsilon (-T_0', T_0) \supset_F (-T,T)$ and contradicts our supposition. As in Theorem 1 it follows that $g = 0$ which ends the proof of Theorem 2.

Example 8 in [2] shows that in general h and f cannot have common points of discontinuity.

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BOUNDARY VALUE PROBLEMS FOR THE CAUCHY-RIEMANN SYSTEM

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ABSTRACT: For the Cauchy-Riemann system the boundary value problems, with point conditions, are presented and the existence and uniqueness for their solutions is proved.

1. INTRODUCTION

In may problems of computational fluid dynamics there appear to be advantageous in working with the corresponding first-order elliptic systems. The simplest elliptic system of the first order is the Cauchy-Riemann system $\psi_y = -\phi_x$, $\phi_y = \psi_x$ and every pair of sufficiently smooth functions ϕ and ψ satisfying this system also satisfies the Laplace equation. The inhomogeneous Cauchy-Riemann system which, in planar cartesians, appears as below

$$\begin{aligned} \text{div } (\phi, \psi) &= f_1 \\ \text{curl } (\phi, \psi) &= f_2, \end{aligned} \tag{1}$$

has been of interest for the researchers in the last two decades, see for example [Borzi et al. 1997], [Chang & Gunzburger, 1990], [Ghil & Bagovind, 1979], Hafez & Phillips, 1985], [Lomax & Martin, 1974], [Neittaanmäki & Saranen, 1981], [Nicolides, 1992], Rose, [1981] and [Vanmaele et al, 1974].

Collectively the Cauchy-Riemann system (1) is elliptic while individually both the partial differential equations are hyperbolic. If ϕ and ψ are twice continuously and $f_1 = f_2 = 0$ then ϕ and ψ are harmonic. For the ellipticity of the system (1) we refer to [Wendland, 1979].

The following basic boundary value problems for the Cauchy-Riemann system are known. For a square domain $\Omega = (0,1)\times(0,1)$ with boundary Γ, Vanmaele et al. [1994] consider the Cauchy-Riemann system (1) with boundary conditions

$$\begin{aligned} &\phi \in H^{½}(\Gamma_1) \text{ is known on } \Gamma_1 = \{0\} \times (0,1), \\ &\psi \in H^{½}(\Gamma_1) \text{ is known on } \Gamma_2 = \Gamma\backslash\Gamma_1, \end{aligned} \tag{2}$$

and prove the existence and uniqueness of a solution $(\phi,\psi) \in H^1(\Omega) \times H^1(\Omega)$. The result is valid for any rectangular domain. Chang & Gunzburger [1990], [Neittaanmäki & Saranen, 1981b] and Vanmaele et al [1997] discuss the div-curl system (2,1) with the boundary conditions

$$(\phi,\psi) \times \boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{3}$$

where $\boldsymbol{n}$ is the outer unit normal and $(\phi,\psi) \times \boldsymbol{n} = \phi n_2 - \psi n_1$. The well-posedness of (1), (3) is proved in $H^1(\Omega) \times H^1(\Omega)$ subject to he compatibility conditions

$$\int_\Omega f_2 \, d\Omega = 0. \tag{4}$$

Subject to the compatibility condition

$$\int_\Omega f_1 \, d\Omega = 0, \tag{5}$$

The problem

$$(\phi,\psi) \,.\, \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{6}$$

considered with the system (1) is well-posed in $H^1(\Omega) \times H^1(\Omega)$, see [Vanmaele et al., 1994], [Chang & Gunzburger, 1990] and [Wendland, 1979].

In a rectangular domain Ω, for example $(0,2\pi) \times (0,\pi)$ for $g_i \in H^{1/2}(\Gamma_i)$, i = 1,2; the following boundary conditions for the div-curl system (1) are also important:

$$\begin{aligned} \psi &= g_1(x) \quad \text{on} \quad \Gamma_1 = (0,2\pi) \times \{0\} \\ \psi &= g_2(x) \quad \text{on} \quad \Gamma_2 = (0,2\pi) \times \{\pi\} \end{aligned} \tag{7}$$

and that both ϕ and ψ are periodic in the x-direction, that is,

$$\begin{aligned} \phi(0,y) &= \phi(2\pi,y), \\ \psi(0,y) &= \psi(2\pi,y). \end{aligned} \tag{8}$$

Equations (8) are sometimes referred to as periodic boundary conditions. Subject to the compatibility conditions

$$\int_0^{2\pi}\int_0^{\pi} f_1 \, dx \, dy = \int_0^{2\pi} \{g_2(x) - g_1(x)\} \, dx, \tag{9}$$

the boundary conditions (7), (8) together with (1) determine $\psi \in H^1(\Omega)$ completely and $\phi \in H^1(\Omega)$ up to an additive constant [Ghil & Balgovind, 1979].

In this paper we shall discuss the boundary value problems with point conditions for the Cauchy-Riemann system.

2. CAUCHY-RIEMANN BOUNDARY VALUE PROBLEM -I

Let Γ denote the boundary of a simply-connected domain $\Omega \subset \Re^2$ and $G(\overline{\Omega})$ be the subspace of the space $G(\overline{\Omega})$ generated by the functions which are harmonic in Ω and continuous in $\overline{\Omega} = \Omega \cup \Gamma$. We consider the Cauchy-Riemann system

$$\begin{cases} \psi_y + \psi_x = 0 \\ \\ \psi_x - \phi_y = 0 \end{cases} \quad in \ \Omega \tag{10}$$

with the boundary condition

$$\psi = f \ \text{ on } \ \Gamma, \tag{11a}$$

and $$\phi = \phi^P \text{ at a single point } P \in \overline{\Omega}. \tag{11b}$$

Theorem 2.1
For $f \in C(\Gamma)$, the problem (10)-(11) possesses a unique solution

$$(\phi, \psi) \in G(\overline{\Omega}) \times G(\overline{\Omega}).$$

Proof
Step I. It is well-known, see for example [Mikhlin, 1970], that the Dirichlet problem for the Laplace equation

$$\begin{cases} \Delta \psi = 0 \ \ in \ \Omega, \\ \ \psi = f \ \ on \ \Gamma, \end{cases} \tag{12}$$

is a well-posed problem in the pair of spaces $(G(\overline{\Omega}), C(\Gamma))$. ψ is determined uniquely everywhere is $G(\overline{\Omega})$. Let this unique solution be ψ^*.

Step II. We will prove now that for the unique $\psi^* \in G(\overline{\Omega})$ if there exists satisfying ϕ^* (10)-(11), then that ϕ^* will be unique. Let us assume that ϕ_1, ϕ_2 with $\phi_1 \neq \phi_2$ are two such solutions for the same ψ^* and that δ be defined as $\delta = \phi_1 - \phi_2$. From (10) is follows that

$$\delta_x = 0,$$
$$\delta_y = 0,$$

which imply that δ = constant. Also (11b) implies that $\delta = 0$ at P which shows further that $\delta \equiv 0$ and uniqueness of ϕ^* follows.

Step III. To prove the existence of ϕ^* we proceed as follows. Let $R(x,y)$ be an arbitrary point in $\overline{\Omega}$. We can always choose as path from P to R consisting of horizontal and vertical straight lines. For example, in Figure 1a, we show one such path, while in Figure 1b we show another. It is sufficient to consider Figure 1a. We choose a line PQ within $\overline{\Omega}$ such that $y = y_p$ = fixed along PQ. To construct ϕ^* along PQ we integrate the equation (10a) along PQ and obtain

$$\phi^*(x,y_P) = \phi^P - \int_{x_P}^{x} \psi_y^*(\lambda, y_P)\, d\lambda \tag{13}$$

To construct ϕ^* along QR, we integrate the equation (10b) along QR and obtain

$$\phi^*(x,y) = \phi^*(x,y_P) + \int_{y_P}^{y} \psi_x^*(x,\lambda)\, d\lambda \tag{14}$$

Using (13) in (14) we get

$$\phi^*(x,y) = \phi^P - \int_{x_P}^{x} \psi_y^*(\lambda, y_P)\, d\lambda + \int_{y_P}^{y} \psi_x^*(x,\lambda)\, d\lambda \tag{15}$$

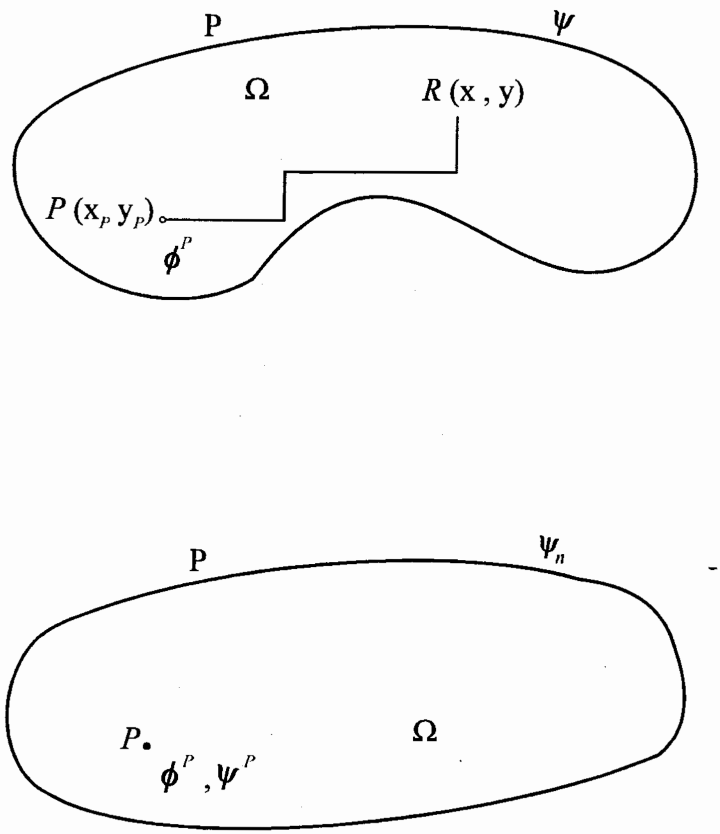

Figure 1. C-R Boundary Value Problem-I

It is obvious that all the terms on the right hand side of (15) are known which shows that ϕ^* is known at an arbitrary point R and hence everywhere in $\overline{\Omega}$.

Step IV. Now we show that $\phi^*(x,y)$ satisfies the original C-R system (10). Differentiating (15) with respect to x we obtain

$$\phi_x^*(x,y) = -\psi_y^*(x,y_P) + \int_{y_P}^{y} \psi_{xx}^*(x,\lambda)\,d\lambda$$

which, since ψ^* is harmonic, can further be simplified as

$$\phi_x^*(x,y) = -\psi_y^*(x,y_P) - \int_{y_P}^{y} \psi_{yy}^*(x,\lambda)\,d\lambda = -\psi_y^*(x,y). \tag{16}$$

Similarly differentiating (15) with respect to y we readily get

$$\phi_y^*(x,y) = \psi_x^*(x,y). \tag{17}$$

Step V. We recall from the *step I* that $\psi^* \in G(\overline{\Omega})$. It is well-known, see for example [Mikhlin, 1970], that a function which is harmonic is some domain has derivatives of all orders in that domain. Therefore is follows immediately that $\psi_{xy}^*, \psi_{yx}^* \in C(\Omega)$. Consequently it can easily be shown from (16), (17) that ϕ^* is harmonic in Ω and hence from *step III* is follows that $\phi^* \in G(\overline{\Omega})$. Thus there exists a unique $(\phi^*, \psi^*) \in G(\overline{\Omega}) \times G(\overline{\Omega})$.

Remarks 2.1 (Well-posedness and Ill-posedness)

- In the pair of spaces $(H^{1/2}(\Omega), H^{1/2}(\Gamma))$ the Dirichlet problem for the Laplace equation is well posed [Girault & Raviart, 1986]. It is easy to prove that the Cauchy-Riemann boundary value problem (10)-(11) is well-posed in the pair of spaces $(H^1(\Omega) \times H^1(\Omega), H^{1/2}(\Gamma))$.
- The Dirichlet problem for the Laplace equation is ill-posed in the pair of spaces $(H^1(\Omega), C(\Gamma))$. see for example [Mikhlin, 1970]. The Dirichlet problem (12) for ψ being ill-posed in $(H^1(\Omega), C(\Gamma))$ thus implies that the Cauchy-Riemann boundary value problem (10)-(11) is ill-posed in the pair of spaces $(H^1(\Omega) \times H^1(\Omega), C(\Gamma))$.

For the properties of the Sobolev spaces on the domain and regularity of the boundary, we refer to [Girault & Raviart, 1986].

Remarks 2.2

If, in a simple-connected domain $\Omega \subset \Re^2$ with boundary Γ, we consider the Cauchy-Riemann system (1) for $f_1, f_2 \in H^1(\Omega)$ with the boundary condition

$\psi = f \in H^{½}(\Gamma)$ on Γ.

$\phi = \phi^P$ at the single point $P \in \overline{\Omega}$

then there exits a unique solution $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$.

The proof can be established on the same lines as in the previous proof. For the Poisson equation $\Delta\psi = F$ in Ω (where $F = \partial_x f_2 + \partial_y f_1$) considered with the Dirichlet conditions $\psi|_\Gamma = f$, there exists a unique solution $\psi = \psi^*$ in $H^1(\Omega)$ [Girault & Raviart, 1986]. It is simple to show the uniqueness of $\phi = \phi^*$ in Ω. The existence of ϕ^* can be shown by construction, firstly along $y = y_p$:

$$\phi^*(x,y_P) = \phi^P + \int_{x_P}^{x} \left[f_1(\lambda,y_P) - \psi_y^*(\lambda,y_P)\right] d\lambda \tag{18}$$

and then $\forall\ (x,y) \in \overline{\Omega}$,

$$\phi^*(x,y) = \phi^P + \int_{x_P}^{x} \left[f_1(\lambda,y_P) - \psi_y^*(\lambda,y_P)\right] d\lambda + \int_{y_P}^{y} \left[\psi_x^*(x,\lambda) - f_2(x,\lambda)\right] d\lambda \tag{19}$$

Finally, it is to show that ϕ^* satisfies (1). From (19) it follows that

$$\phi_x^*(x,y) = f_1(x,y_P) - \psi_y^*(x,y_P) + \int_{y_P}^{y} \left[\psi_{xx}^*(x,\lambda) - \partial_x f_2(x,\lambda)\right] d\lambda$$

$$= f_1(x,y_P) - \psi_y^*(x,y_P) + \partial_y \int_{y_P}^{y} \left[f_1(x,\lambda) - \psi_y^*(x,\lambda)\right] d\lambda$$

$$= f_1(x,y) - \psi_y^*(x,y).$$

Similarly from (19) it can be shown that

$$\phi_y^*(x,y) = \psi_x^*(x,y) - f_2(x,y),$$

and the proof is complete.

3. CAUCHY-RIEMANN BOUNDARY VALUE PROBLEM - II

Below we consider the Neumann boundary condition in conjunction with two point conditions for the Cauchy-Riemann system.

Theorem 3.1

Let Γ denote the boundary of a simply-connected domain $\Omega \subset \Re^2$. Given $f_1, f_2 \in H^1(\overline{\Omega})$, $f \in H^{\frac{1}{2}}(\Gamma)$; the Cauchy-Riemann system (1) in Ω, with the boundary conditions (as depicted in Figure 2)

$$\psi_n = f \text{ on } \Gamma \tag{20}$$

and
$$\left.\begin{aligned} \psi &= \psi^P \\ \phi &= \phi^P \end{aligned}\right\} \text{ at a single point } P \in \overline{\Omega} \tag{21}$$

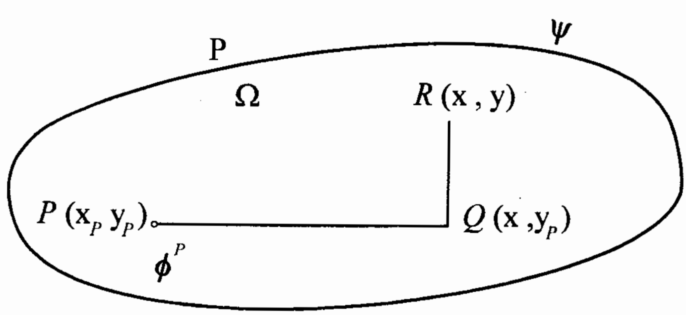

Figure 2. C-R Boundary Value Problem-II

possesses a unique solution $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$ subject to the compatibility condition

$$\int_\Gamma \{(f_2, f_1) \cdot n - f\} \, ds = 0. \tag{22}$$

Proof

Let us consider

$$\Delta\psi = F \text{ in } \Omega \tag{23}$$

where $F = \text{div}\,(f_2, f_1)$. It is well-known, see for example [Girault & Raviat, 1986], that the Neumann problem (20) for the Poisson equation (23) possesses a solution $\psi \in H^1(\Omega)$ which is unique up to an additive constant subject to the compatibility condition

$$\int_\Omega F \, dx \, dy = \int_\Gamma f \, ds,$$

which on substituting $F = \text{div}\,(f_2, f_1)$. and using divergence theorem, can further be written as the required compatibility condition (22). Using the point condition (21a) the indeterminacy is eliminated to determine ψ uniquely in Ω.

To use Remarks 2.2. we need to show that if $\psi \in H^1(\Omega)$ then there exists a unique continuous extension of ψ onto the boundary Γ, at least for certain types of boundary. We use a special case of Theorem 1.5.1.2 in [Grisvard, 1985]; let $\Omega \subset \Re^2$ have a $C^{0,1}$ boundary Γ (i.e. Lipschitz continuous boundary) then the mapping $u \to \gamma u$ which is defined for $u \in C^{0,1}(\overline{\Omega})$ has a unique continuous extension as an operator from $H^1(\Omega)$ onto $H^1(\Gamma)$.

Now for the Cauchy-Riemann system (1) with ψ known on $H^{1/2}(\Gamma)$ in conjunction with point condition (21b), the existence of a unique solution $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega)$ follows from Remarks 2.2.

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SECRET SHARING SCHEME AND DISHONEST PARTICIPANTS

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ABSTRACT: Secret Sharing Schemes are used in many institutions. For sensitive data these schemes constitute a fundamental protection tool forcing the adversary to attack multiple locations in order to learn or destroy the information. In this paper we generalize the problem to one in which the secret is some data which is distributed among all the dishonest participants.

1. INTRODUCTION

Secret Sharing Schemes are used in many commercial and distribution environments. Secret sharing schemes were introduced by Blakley [2] and Shamir [4]. A secret sharing scheme is a method of sharing a secret among a set of participants in such a way that certain subsets of participants are qualified to compute the secret by combining their shares. A secret sharing scheme is called perfect if in addition any non-qualified subset of participants has absolutely no information on the secret. One of the best examples is military where the key and password is shared between s individuals of ranks $r_1, \ldots r_w$ (where w is a positive integer), so that if a person of rank r_i is incapacitated, then a person of rank $r_j \geq r_i$ or at least two participants of rank less that r_i may replace the lost data. Brickle [3], Simmons [5], [6] and Beutelspacher [1] have adapted the basic schemes and constructed multilevel systems. In this paper we will show how secret sharing scheme can be modelled on matrix and demonstrate how these can be adapted to realize multilevel schemes when the participants are dishonest. To begin with, we must formally define the secret sharing schemes.

2. SHAMIR'S SECRET SHARING SCHEME [4]

In this scheme n parties P_i, $i = 1, \ldots .n$, k-share a secret s, $1 < k < n$, if and only if the following conditions re satisfied:

(1) Each P_i some information a_i known to the parties P_j, $j \neq i$.
(2) The secret s can be easily computed from any k of the a_i's.

(3) The knowledge of any k-1 of the a_i's no matter which ones they are, leaves s undetermined.

A set $\{a_i, \ldots\ldots a_n\}$ satisfying (2)-(3) is referred to as (k,n) threshold scheme for s. Let S be the finite set of secrets, and input is a secret taken from S. In addition to the parties, there is dealer in the system, who has a secret input s. A scheme is a probabilistic mapping which the dealer applies to the input and generates n pieces of information. The piece of information (what a participant must remember) is called share. For every i the dealer gives the i-th share to P_i. The dealer is only active in this initial stage. After the initial stage, the parties can communicate, according to some pre-defined, possible randomized, protocol. The parties are hones, that is they follow their protocols. However, they are curious and after the protocol has ended some of them can collude and try to gain some partial information on the secret.

2.1 Participants and Communication Model

Each participants in P is connected to a communication channel, C with the property that messages sent on C instantly reach every party connected to it. We assume that the system synchronized i.e. the participants can access at the same time.

2.2 Time Periods and Update Phases

Time is divided into time periods (e.g., a day, a week, etc.) which are determined by the common global clock. At the beginning of each time period the participants engage in an interactive update protocol (update phase). At the end of an update phase the participants hold me shares of secret s, i.e., share are changed periodically.

2.3 The Mobile Adversary Model

Corrupting a participant means any combination of learning the secret information (share) of he participant modifying its data, changing the intended behavior of the participants, discommoding the participants, and so on. For the sake of simplicity, we do not differentiate between malicious faults and "normal" participants failures (e.g., crashes, power failures etc.).

3. THE PROPOSED SCHEME

Our model of secret sharing scheme based on matrix is as follows. A matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

of order $m \times n$ with entries randomly chosen from a set $N = \{0,1\ldots9\}$ is taken to be the secret key. The subscript i and j of the elements a_{ij} of the matrix indicated respectively of row and column in which a_{ij} is located. The a_{ij} lies at the intersection of ith row and jth column of A. In this scheme there is a dealer who distributes the shares among the st of participants who are ranked and placed is levels $r_1,,,,r_w$ where W is a positive integer.

$(i\ j\ a_{ij})$, $1 \le i \le m$, $1 \le j \le n$ are shares of rank r_1.

If p is the number of shares then

$(i\ j\ a_{ij}\ a_{ij+1} \ldots a_{ij+p-1})$, $1 \le i \le m, j = 1, p+1, 2p+1, 3p+1, \ldots < n$ are shares of rank r_2.

If q is the number of shares then

$(i\ j\ a_{ij}\ a_{ij+1} \ldots a_{ij+q-1})$, $p < q$, $1 \le i \le m, j = 1, q+1+2q+1+3q+1, \ldots < n$ are shares of rank r_3.

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$(i\ j\ a_{ij}\ a_{ij+1} \ldots a_{in})$, $1 \le i \le m, j = 1$ are the share of rank r_w.

The order of the matrix is made public but the matrix is kept secret and taken to be the key. The shares are distributed privately to the participants. When a group of participants whose shares constitute the matrix A come together they can construct the secret key. We assume that there are l_i participants of rank r_i. However an incapacitated participant of rank r_i so that the secret can be recovered from the shares of all participants of rank $r_j \ge r_i$, or at least two participants of rank less than r_i. Also a person may change his share in embarrassment and confusion, then the error can be corrected by majority vote. Consider the case of the military. Each row of the matrix is assigned a different regiment. However if all the rows come together, they can reconstruct the matrix and hence the secret key. In the case of cheating the dealer can change only the cheated entries of the matrix without changing the whole matrix/data.

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EVERY IDEAL IS A SUBSET OF A PROPER P-IDEAL IN BCI-ALGEBRA

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ABSTRACT: In [2] We introduced he concept of p-ideal and posed the following problem: characterize p-semisimple BC-algebras. Now we show that every ideal is a subset of a p-ideal in BC-algebra.

Definition 1([2])

A nonempty subset A in a BCI-algebra X is called p-ideal of X if it satisfies

(1) $0 \in A$,
(2) $(x*y)*(y*z) \in A$ and $y \in A$ imply $x \in A$.

Lemma 2([2])

An ideal A of a BCI-algebra X is a P-Ideal of X if and only if $0*(0*x) \in A$ implies $X \in A$, whereas $x \in X$.

Lemma 3([2])

A BCI-algebra X is p-semisimple if and only if every ideal of X is a p-ideal.

Lemma 4([2])

Let A be an ideal of a BCI-algebra X. Then X/A is a p-semisimple BCI-algebra if and only of A is a p-ideal of X.

Let X be a BCI-algebra and A an ideal of X, denote

$$Z(A) = \{x \in X \mid 0*(0*x) \in A\}.$$

Obviously, $A \subseteq Z(A) \supseteq B(X)$(BCK-part of X).

Remark

Z(a) is not L(A) in [1].

Theorem 5

Let A be an Ideal in a BCI-algebra X, then $Z(A) \supseteq A$ is a p-ideal of X.

Proof

Let $x*y \in Z(A)$ and $y \in Z(A)$, where $x,y \in X$. Obviously $0 \in Z(A)$. By the definition of Z(A), we have $0*(0*y) \in A$, $0*(0*(x*y)) \in A$. Then

$$0*(0*x))*(0*(0*y)) = 0*(0*(x*y)) \in A.$$

Since A is an ideal, so $0*(0*x) \in A$. This means that $x \in Z(A)$. Hence Z(A) is an ideal of X. For any $x \in A$, since $(0*(0*x)*x = 0 \in A$, so $0*(0*x) \in A$, it is clean that $x \in Z(A)$. Thus $A \subseteq Z(A)$.

Suppose $0*(0*X) \in Z(A)$, then $0*(0*(0*(0*X)) \in A$. Since

$$0*(0*X) = 0*(0*(0*(0*X))),$$

so $0*(0*X) \in A$. This means $x \in Z(A)$. By Lemma 2, Z(A) is a p-ideal of X.

Combining Theorem 5 and Lemma 4 we have.

Theorem 6

Let A be an ideal in a BCI-algebra X, then X/Z(A) is a p-semisimple BCI-algebra.

In [3], it is shown that strong ideal and closed p-ideal coincide in BCI-algebra.

Theorem 7

Let A be an ideal in a BCI-algebra X. Then Z(A) is strong ideal if and only if A is closed.

Proof

Suppose that A is closed ideal of X. If x $\in$ Z(A), then 0*(0*X) $\in$ A. Since is closed, so 0*(0*(0*X) $\in$ A, which implies 0*x $\in$ Z(A). Thus Z(A) is closed ideal. By Theorem 5, Z(A) is closed p-ideal, By Theorem 3 in [3], Z(A) is strong ideal.

Conversely, suppose Z(A) is strong ideal. Then Z(A) is closed. Let x $\in$ A, by definition of ideal we have 0*(0x) $\in$ A. This shows that x $\in$ Z(A). Since Z(A) is closed, so 0*x $\in$ Z(A), i.e. 0*(0*(0*x)) $\in$ A. Thus

$$0*x = (0*(0*(0*(0*X)$$

This says that A is closed.

The following is easily verified by Lemma 2.

Theorem 8

Let A be an ideal of BCI-algebra X. Then Z(A)=A if and only if A is a p-ideal.

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YIELD STRESS OF SEMI-RIGID AND FLEXIBLE POLYMERS

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ABSTRACT: In order to measure the yield stress of aqueous solutions of Polyacrylamide and Xanthan gum, Zero-shear viscosity of these polymers have been determined. In steady shear, plateau regions of viscosity for both the polymers were found. Oscillatory dynamic testing is also carried out in the linear regime. It was found that both the polymers do not exhibit a true yield stress because G decrease with decrease of frequency in oscillatory dynamic mode.

1. INTRODUCTION

There is the growing importance of bio-polymers like Xanthan gum and synthetic polymers like polyacrylamide in such widely differing applications as the food industries and oil extraction. In the former, Xanthan gum is widely employed as a thickening or gelling agent [Cf.1], while in the latter both Xanthan gum and polyacrylamide are of potential use in Enhanced Oil Recovery [Cf.2].

The present experiments have involved in Xanthan gum Keltrol F (Supplied by Kelco) and the polyacrylamide Magnaflox E-10 (Supplied by Colloids). Walters *et al.* [Cf.3] have investigated the rheological properties of these two polymers in different concentrations and found that near room temperature 3%, 2% and 0.9% aqueous solutions of Xanthan gum have a similar shear viscosity response to 2%, 1.5% and 0.75% aqueous solutions of polyacrylamide, respectively. They found that in steady shear flow Xanthan gum solutions are highly shear thinning with low values of N_1 while polyacrylamide solutions are shear-thinning with high values on N_1. On the other hand in oscillatory shear flow, Xanthan gum have high values of G' while polyacrylamide solutions have low values of G'.

The conformation of the Xanthan gum molecule has been well covered by Rocherfort and Middleman [1] and Lim *et al.* [4]. It is usually considered to be 'semi rigid'. In contrast, polyacrylamide is generally regarded as being 'very flexible' and it is largely this difference which gives rise to the substantially different rheological response.

We remark that the concentrations employed in the present series of experiments would according to Wissbrun and Whitcomb [5,6] place the Xanthan gum solutions in the liquid-crystalline category.

Whitcomb and Macosko [6] have reported that concentrated Xanthan gum solutions (1% by weight and higher) exhibit an apparent yield stress. However, later work by Lim *et al.* [4], using dynamic oscillatory measurements and optical birefringence studies, have indicated that Xanthan gum does not exhibit a true yield stress because G' decreases with decrease of frequency, and that the effect may be attributed to the formation of a liquid crystal structure. They also measured the zero-shear viscosity of Xanthan gum solutions and found that the flow curve flattens at low shear rates.

Due to its semi-rigid molecular structure, Xanthan gum is highly resistant to molecular degradation, with no loss of viscosity being reported after prolonged steady shearing at $\gamma = 46{,}000\ s^{-1}$ [7].

However, the polymer is known to be susceptible to both biological and chemical attack particularly at elevated temperatures and in the presence of oxygen [8].

Walters *et al.* [3] have been shown that unlike polyacrylamide solutions, Xanthan gum solutions suffer from the effect of 'pre-shearing' in an oscillatory mode of deformation. An aqueous solution of 3% Xanthan gum was exposed to a steady shear flow and it was sheared for 60 seconds before oscillation. It was observed that both dynamic viscosity *n* and dynamic rigidity G drop significantly as the pre-shear rate was increased.

Luyten *et al.* [9] studied Xanthan gum solutions in steady shear flow at concentrations 0.025% to 0.2% (w/v %) in 0.1% Nacl. They found two plateau regions for these solutions with viscosities ranging from 0.75×10^{-2} to 4 (Pa.s).

Bewersdroff and Singh [10] measured the zero-shear viscosity of these solutions with concentrations ranging from 0.0025% to 0.075% (by weight). They found that there is a plateau region at shear rate $\gamma = 0.001$ (l/s) with a viscosity of 0.15 (Pa.s) for these concentrations.

2. BASIC RHEOMETRY

Consider a steady simple shear flow represented by cartesian velocity components:

$$V_x = \gamma y, \quad V_y = V_z = 0, \tag{1}$$

where γ is the constant shear rate. The corresponding stress distribution for a non-Newtonian elastic liquid can be written in the form [11,12].

$$\begin{aligned} \sigma_{xy} &= \sigma = \gamma\, \eta(\gamma), \\ \sigma_{xx} - \sigma_{yy} &= N_1(\gamma), \quad \sigma_{xx} - \sigma_{zz} = N_2(\gamma) \end{aligned} \tag{2}$$

where σ_{ik} is the stress tensor, η the apparent (or shear) viscosity and N_1 and N_2 are the first and second normal-stress differences, respectively. In conventional rheometry, it is customary to limit attention to η and N_1.

In an Oscillation, shear flow is given by

$$V_x = \alpha \omega y \cos \omega t, \; V_y = V_z = 0, \tag{3}$$

where α is a small amplitude, the relevant stress σ_{xy} is given by [4,5]

$$\sigma_{xy} = \alpha \omega \left[\acute{\eta} \cos \omega t + \frac{\acute{G}}{\omega} \sin \omega t \right] \tag{4}$$

where $\acute{\eta}$ is the dynamic viscosity and $\acute{G}$ the dynamic rigidity.

For non-linear materials such as polymer solutions, the response will only the linear if the amplitude is small; otherwise non-linear effects must be expected and have to be accommodated (see § 4).

In the following sections, we shall refer to all rheometrical functions defined above, except for the second normal stress difference N_2. The experiments were performed on Weissenberg Rheogoniometers and Controlled Stress Rheometer (both Manufactured by Carrimed U.K).

3. ZERO-SHEAR VISCOSITY OF XANTHAN GUM AND POLYACRYLAMIDE SOLUTIONS

In order to determine the zero-shear rate viscosity of Xanthan gum and polyacrylamide and to see whether there is yield stress in both the polymers or not, a controlled Stress Rheometer was employed. In this instrument a force or shear stress τ acts on the upper moving plate and this induces a movement v or shear rate γ on this surface relative to the lower fixed surface. In this manner it is capable of measuring viscosity over a shear rate range as low as 10^{-6}(l/s). The flow behaviour of the samples were investigated manually. The tests were performed in a cone-and-plate configuration with cone having an angle of 1° with plate radius 2.5 cm. The measuring temperature was 20 C° throughout the experiments.

For 3% Xanthan gum a plot of shear viscosity versus shear rate is given in Fig.(1). It is clear from the figure that the sufficiently low shear rates, the viscosity reaches a Newtonian plateau and as the shear rates are increased it falls monotonically. There is some scatter of data at low shear rates but there is no doubt that the viscosity shear rate curve flattens at sufficiently low shear rates.

From these results it is clear that the polymer flows at even low stresses and the viscosity, although large, is always finite and there is no yield stress in this polymer.

For the 2% polyacrylamide solution, steady shear viscosity was obtained with Controlled Stress Rheometer. The relevant data is given in Fig.2. At lower range of shear rates there is some scatter of data but the viscosity is finite for this polymer as well.

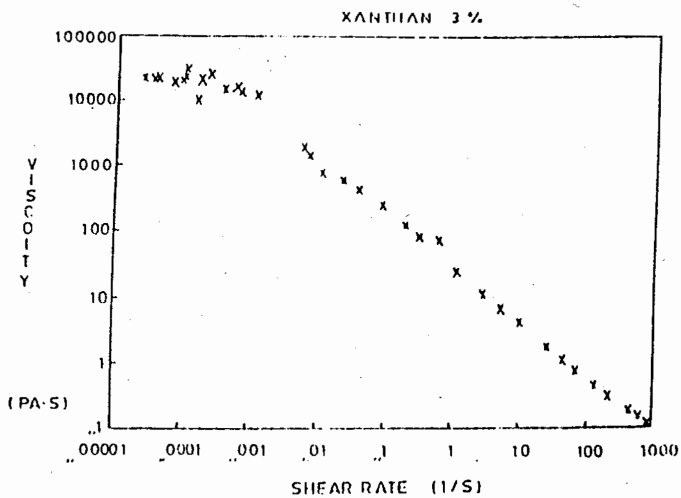

Fig.1 Steady Shear Data for 3% Xanthan gum with Controlled Stress Rheometer.

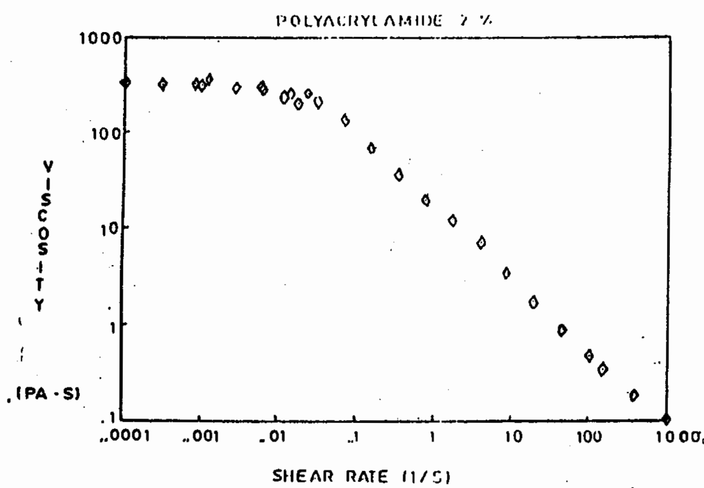

Fig.2 Steady Shear Data for 2% Polyacrylamide Solution with Controlled Stress Rheometer.

4. NON-LINEAR BEHAVIOUR IN OSCILLATORY SHEAR

It is of interest to consider large strain oscillatory shear flow for two reasons. First, it is important to confirm that the conventional dynamic testing is carried out in the linear regime. Secondly, the non-linear effects themselves are able to tell us something about the rheology of the solutions. In order to ensure that the oscillatory data is in the linear region, strain-stress experiments were performed. We employed the Weissenberg Rheometer (model R-16) in a cone-and-plate combination with cone 1° 32′ for the present series of experiments. Figs.(3) and (4) contain representative dynamic data for frequency $\omega = 5.0$ rad/sec for 2% polyacrylamide and 3% Xanthan gum solutions. In these figures dynamic viscosity $\acute{\eta}$ and rigidity $\acute{G}$ are normalised by dividing the corresponding values at very small amplitudes. The behaviour of these polymers is typical of that expected for elastic liquids, with a linear region followed by decreasing values of both $\acute{\eta}$ and $\acute{G}$ as the strain is increased, the fall in $\acute{G}$ being greater than the fall in $\acute{\eta}$ [Cf.11].

In Figs.(3) and (4), the fall in both $\acute{\eta}$ and $\acute{G}$ is more pronounced for the Xanthan gum solutions.

5. OSCILLATORY SHEAR FLOW

(i) Effect of Pre-shearing

It is well known that, unlike polyacrylamide solutions, Xanthan gum solutions suffer from the effect of pre-shearing. In order to see the effect on 3% Xanthan gum solution, we sheared the test sample in Controlled Stress Rheometer at a constant rate for a period of 60 seconds before being tested in oscillatory mode. The procedure was repeated at different pre-shearing rates of q = 3, 30, 90 and 150 (l/s). The relevant data is given in Figs.(5) and (6). These figures show that if the sample is pre-sheared at low shear rates, the dynamic viscosity and rigidity are increased. Even if the pre-shearing time is increased up to five minutes, the dynamic moduli are still found to be increased. If the same sample is pre-sheared at high shear rates, the dynamic moduli are decreased. There is a further decrease in the dynamic moduli if the pre-shearing rate is further increased. So the degree of elasticity in Xanthan gum solution in oscillatory shear depends on the rate of pre-shearing and consequently on the state of its structure.

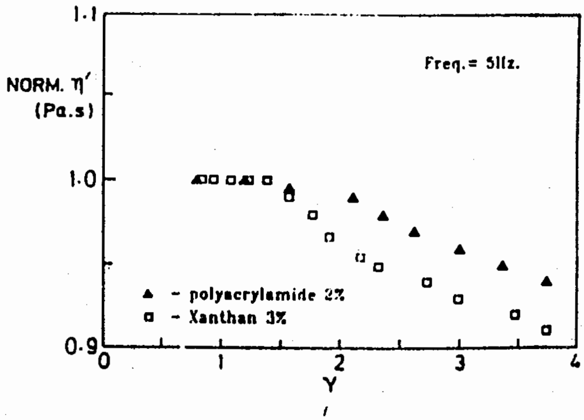

Fig.3 Dynamic viscosity data normalized with respect to its vanishingly small strain value for ω = 5 Hz. γ is strain.

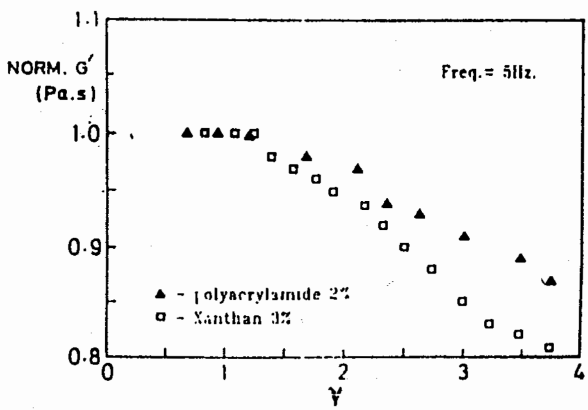

Fig.4 Dynamic rigidity data normalized with respect to its vanishingly small strain value for ω = 5 Hz. γ is strain.

It can be concluded from the above results that Xanthan gum solutions suffer from the effect of pre-shearing. These results are constant with the conclusion of Walter's *et al.* [Cf.3] who found that the dynamic moduli are decreased with increase of pre-shearing rate.

If the test sample is left after pre-shearing for a period which can be as long as one day it will recover, and the dynamic moduli will return to their 'unpre-sheared' values [Cf.3]. However, gentle shearing for a period of 60 seconds has the same effect. In order to investigate this behaviour a sample was pre-sheared at q = 800 (l/s) and subsequently pre-sheared at q = 3 (l/s) for 60 seconds. It was found that the dynamic moduli are once again increased and are higher than the data taken with the fresh sample even if the pre-shearing continues for upto five minutes. This is shown in the Figs.(5-6). So it appears that there is a structure in Xanthan gum solutions which is broken down by high shearing but is recovered by subsequent slight shearing. It is possible that this slight shearing re-aligns the molecules and builds up the structure.

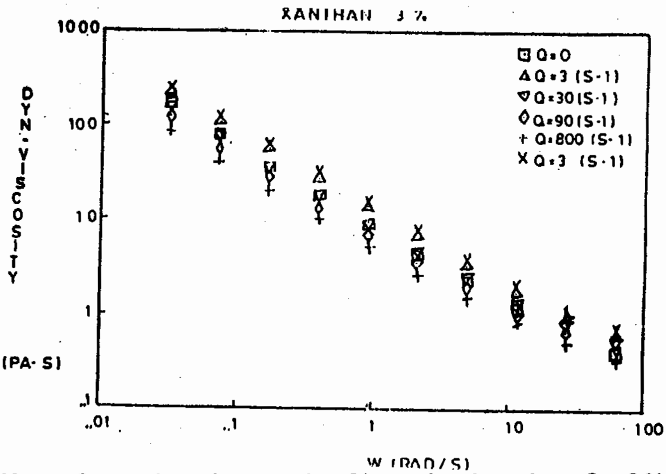

Fig.5 The effect of pre-shearing on the dynamic viscosity of a 3% Xanthan gum, q is the pre-shearing rate. $q^* = 3$ (l/s) * Following Shear rate at q = 800 (l/s).

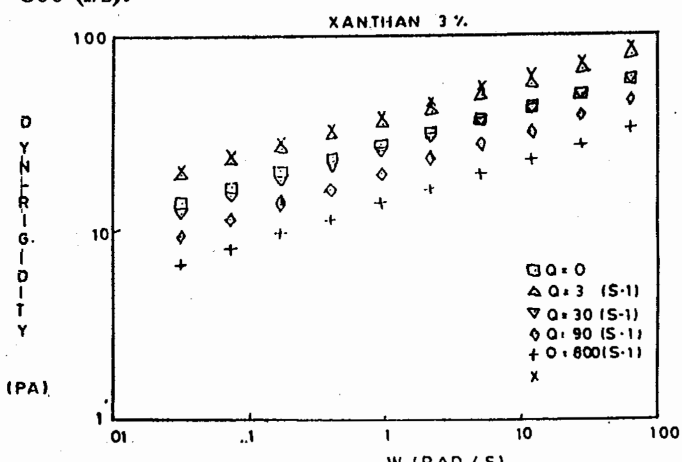

Fig.6 The effect of pre-shearing on the dynamic rigidity of a 3% Xanthan gum, q is the pre-shearing rate. $q^* = 3$ (l/s) * Following Shear rate at q = 800 (l/s).

(ii) Oscillatory Testing

The Controlled Stress Rheometer, old/Weissenberg Rheometer (model R-16) as well as the new model were used to obtain dynamic data for these polymers. As our main concern was to confirm the dynamic rigidity of both the polymers, we compared the dynamic experimental results of the Weissenberg Rheogoniometers with the Controlled Stress Rheometer data. Relevant comparison of dynamic data for 2% polyacrylamide and 3% Xanthan gum are given in Figs.(7) and (8).

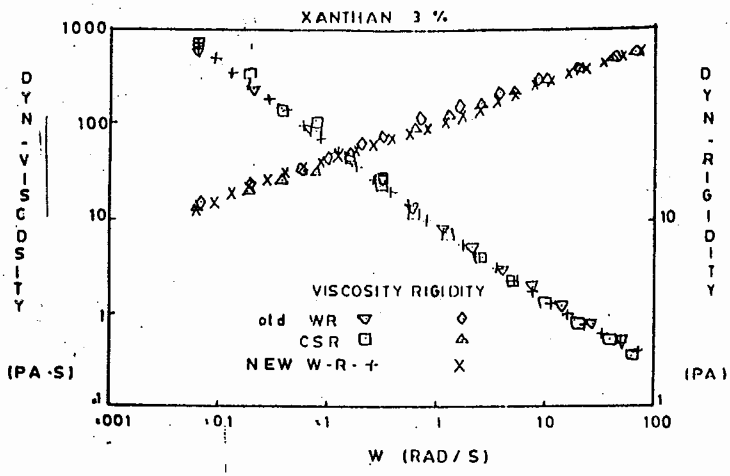

Fig.7 Comparison of Oscillatory shear data for 3% Xanthan gum with Weissenberg Rheogoniometer and Controlled Stress Rheometer.

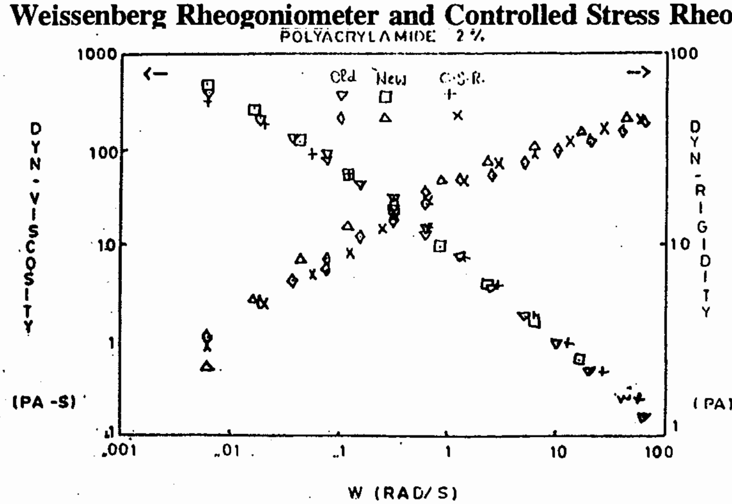

Fig.8 Comparison of Oscillatory shear data for 2% Polyacrylamide with Weissenberg Rheogoniometer and Controlled Stress Rheometer.

These figures show that there is a good agreement of most of the dynamic viscosity $\acute{\eta}$ data for both the polymers. The dynamic rigidity $\acute{G}$ data taken from both the instruments is also satisfactory. This gives us confidence in the values of the moduli employed in the interpretation of the experimental results. For the very low shear rates we expect the Controlled Stress Rheometer to give more reliable results.

6. TESTING FOR YIELD STRESS

We know that yield stress can be measured in oscillatory dynamic tests in which amplitude as well as frequency are varied [13,14]. Below the yield stress the material behaves as a viscoelsastic solid and the response in phase would be sinusoidal if the material is linear. As the amplitude is increased, the stress wave would begin to develop a flat top and at high amplitudes a sharp spike may be obtained [15]. The material is then above the yield. The yield stress can be measured from the point where the stress wave ceases to be sinusoidal [13].

Once again we employed the Controlled Stress Rheometer because it is popular instrument for measuring the yield stress. The instrument was used in cone-and-plate configuration with cone having an angle of 2° with plate radius 2.5 cm.

The representative dynamic data for 3% Xanthan gum and 2% polyacrylamide aqueous solutions at frequency $f = 10^{-3}$ Hz. and displacement amplitude 0.004 (rads) are given in Figs.(9-10). These plots have the same frequency but different phase differences. It is clear that the applied and measured waves for both the polymers are sinusoidal. The representative wave forms for 2% Xanthan gum and 1.5% polyacrylamide are given in Figs.(11-12). Both waves are once again sinusoidal.

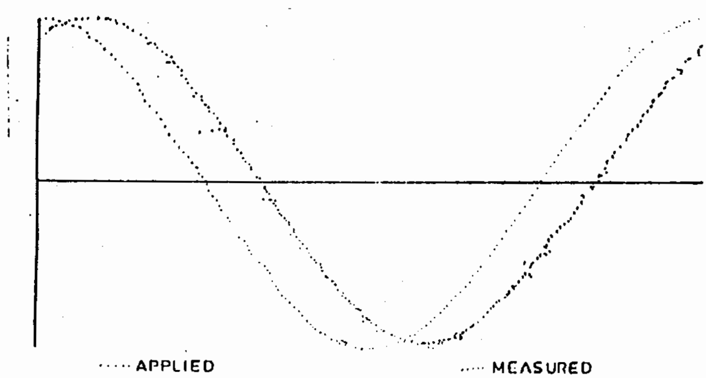

Fig.9 Waveform for 3% Xanthan gum solution at f = 0.001 Hz.

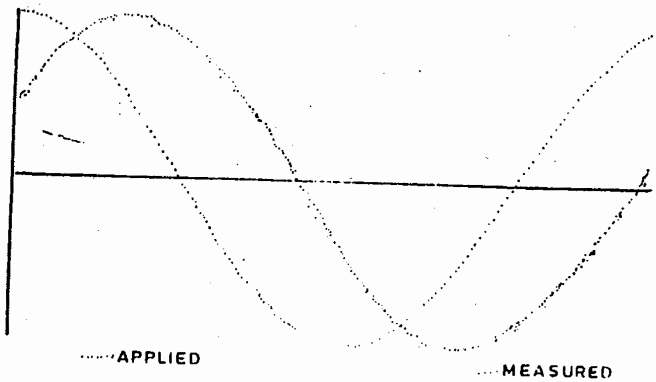

Fig.10 Waveform for 2% Polyacrylamide solution at f = 0.001 Hz.

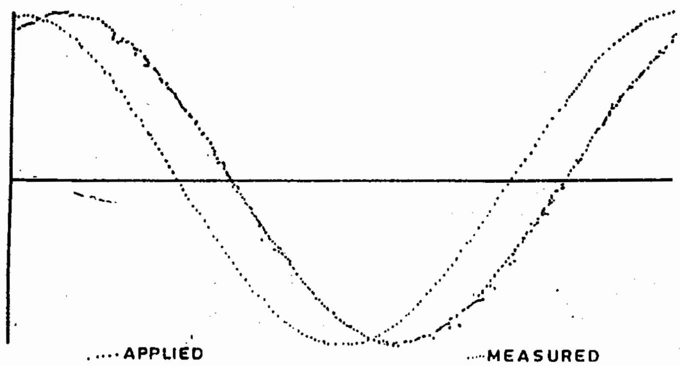

Fig.11 Waveform for 2% Xanthan gum solution at f = 3.33×10^{-3} Hz.

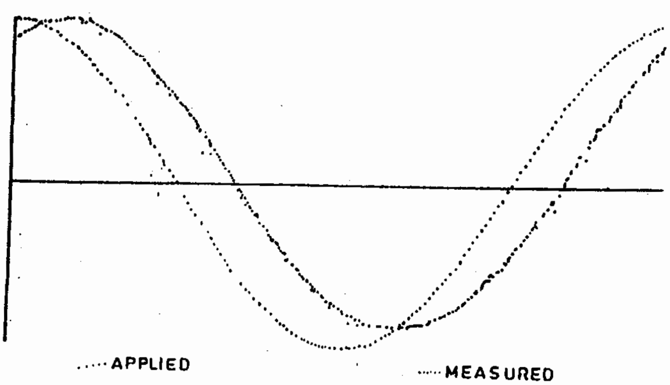

Fig.12 Waveform for 1.5% Polyacrylamide solution at f = 0.001 Hz.

For the lower concentration of 0.9% Xanthan gum and 0.75% polyacrylamide solutions there is some scatter of data at f = 0.0011 Hz., but still the waves are sinusoidal. The relevant data are given in Figs. (13-14) at amplitude 0.004 (rads.)

The output waveforms at high displacement amplitudes were found to be distorted (i.e. not sinusoidal) due to non-linear region but no flat top at high amplitudes was found. On the basis of these results (see also § 3), we are in a position to confirm that there appears to be no yield stress in 3% Xanthan gum and 2% polyacrylamide solution.

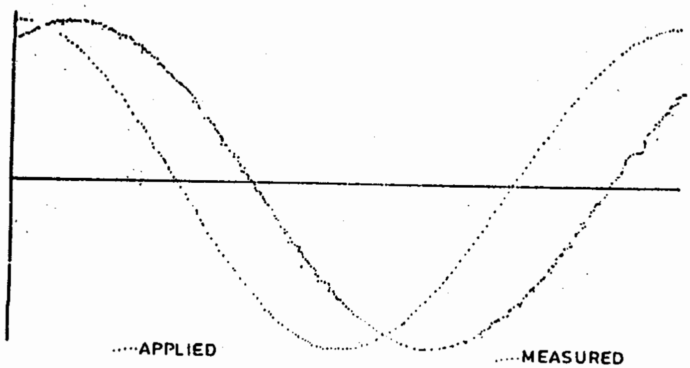

Fig.13 Waveform for 0.9% Xanthan gum at f = 0.001 Hz.

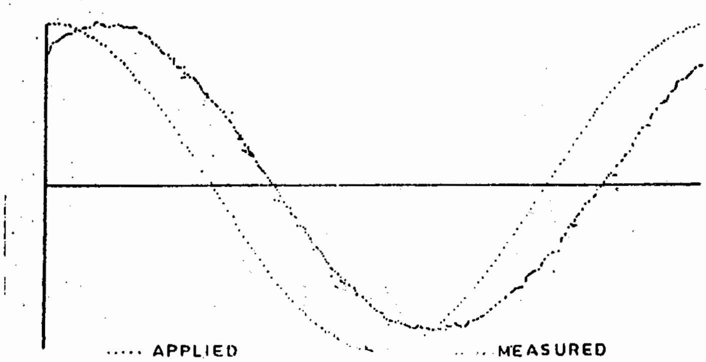

Fig.14 Waveform for 0.75% Polyacrylamide solution at f = 0.001 Hz.

7. CONCLUSIONS

The following conclusion can be drawn from the present study of polyacrylamide and Xanthan gum aqueous solutions:

1. The dynamic properties of Xanthan gum solutions can be significantly affected by pre-shearing.
2. The dynamic rigidity for un-presheared Xanthan gum solutions is higher than that of the corresponding polyacrylamide solutions, this is a consequence of the gel-like structure which is easily broken down by shear (See § 5). There is an increase in the dynamic rigidity following a slow rate of pre-shearing, and a marked decrease following a high rate of pre-shearing. A surprising observation was the subsequent increase of the dynamic moduli when a slow rate of pre-shearing immediately followed by a high one.
3. In Oscillatory dynamic mode, the dynamic rigidity G' for 3% Xanthan gum decreased rapidly as strain amplitude was increased which indicates the break up of solution structure when larger strains are imposed.
4. Xanthan gum and polyacrylamide solutions do not exhibit a true yield stress because G' decreases with decrease of frequency and the effect may attributed to the formation of a liquid crystal structure.
5. There is no yield stress in both the polymers as the zero-shear viscosity of Xanthan gum and polyacrylamide solutions show that the flow curve flattens at low shear rates.

On the basis of the above results, we are in a position to confirm that there appears to be no yield stress in Xanthan gum and polyacrylamide solutions.

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SECOND NORMAL STRESS DIFFERENCE OF BOGER FLUIDS M_1 AND D_4

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1. INTRODUCTION

A new class of polymer solutions, the so-called "Boger fluids" was introduced by Boger [1] in 1977. These fluids exhibit a nearly constant viscosity in steady shear while also demonstrating second order-behaviour of the first normal stress difference over a finite range of strain rates. A Boger fluid's combined properties of constant viscosity and high elasticity, together with optical clarity, make this type of fluid ideal for studying some elastic effects. An obvious instance for the use of such a test fluid would be in an experimental study where a difference in response between a Newtonian liquid and elastic liquid of the same viscosity could be immediately identified with "fluid elasticity" without the complication of shear-thinning effects. The first Boger fluid used to study non-viscometric flows were aqueous solutions consisting of a small amount of polymer dissolved in corn syrup [2,3]. Consequently, an organic based Boger fluid was introduced consisting of a small amount of polyisobutylene (PIB) dissolved in polybutene [4].

For several years, an item of considerable rheological concern has been the sign and magnitude of Second Normal Stress Difference N_2 relative to the First Normal Stress Difference N_1 in a simple shear flow. Although most workers agree that N_2 for polymeric systems is small, there has been disagreement on magnitude and even on sign. The understanding of N_2 is essential for the construction and evaluation of rheological theories and is also important in many practical fluid flows where hydrogyamics and secondary pattern are concerned, for instance it decides whether rectilinear flow is possible in flow through pipes of non-circular cross-section or not.

Boger fluids have been used extensively to test techniques in different laboratories and, also, their rheometrical properties have been studied for their own interest. The current one of interest was the Boger fluid M_1 and D_4.

Samples of polyisobutyelene solution labelled "M_1" were supplied by Dr. T. Sridhar of Monash University, U.S.A. It is a stable, highly elastic and constant viscosity liquid, which consists of a 0.244% polyisobutylene in a mixed solvent consisting of a 7% kerosine and 93% polybutene. The solution and melt viscosity measurements yield an average molecular weight of about 3.8 million. The solvent is a fluid in which N_1 behaves quadratically with shear rate and has a relaxation time of 66 microns which is approximately three orders of magnitude lower than the relaxation time of solution. The solvent viscosity is constant upto a shear rate of 5000 (l/s).

The Boger fluid designated "D_4" has a viscosity of 0.19 Pa.s. In this sample low molecular weight polyisobutene was used as a solvent for high molecular weight (4×10^6) polyisobutylene. This sample was prepared by professor A.S. Lodge of Monash University U.S.A. We have measured second normal stress difference of the Boger fluids M_1 and D_4 using the Weissenberg Rheogoniometer manufactured by Carrimed U.K.

2. MEASUREMENT OF SECOND NORMAL STRESS DIFFERENCE N_2

Various methods exist for the measurement of second normal stress difference N_2. We have used total normal force in cone-and-plate and plate-and-plate to measure N_2.

We undertook a series of experiments on Boger fluids designated M_1 and D_4 (both constant viscosity) to measure the second normal stress difference N_2.

The relevant expressions in terms of total normal force in the parallel-plate and cone-and-plate geometries are given by

$$\left(N_1 - N_{2_{\gamma_a}}\right) = \frac{2F}{\pi a^2}\left(1 + \frac{1}{2}\frac{d \ln F}{d \ln_{\gamma_a}}\right) \tag{1}$$

$$N_1(\gamma) = \frac{2F}{\pi a^2} \tag{2}$$

respectively, where F is the total force, a is the radius of the plate and Γ_a is rim shear rate. Substraction of equation (1) from equation (2) at a particular shear rate gives the second normal stress difference N_2. Hence this method requires the use of two separate experimental geometries and thus two completely different experiments on a given fluid to evaluate N_2. This leads to significant experimental error unless extremely careful and refined experiments are performed, especially so since N_2 is usually very small.

The first normal stress difference N_1 was obtained on a Weissenberg Rheogoniometer using the cone-and-plate geometry, with plate radius 3.75 cm. and cone angle 1.5°. The relevant steady shear data for M_1 and D_4 are given in Figs.(1) and (2). It is clear from these figures that viscosity for both the fluids is constant and N_1 behaves quadratically over the shear rates studied. The same apparatus was used to determine $N_1 - N_2$, this time using the parallel-plate geometry with plate radius 3.75 cms. Different gap sets ranging from 500 to 250 microns were used. Error bars are inevitable in experiments and it may also be associated with the method of calculation.

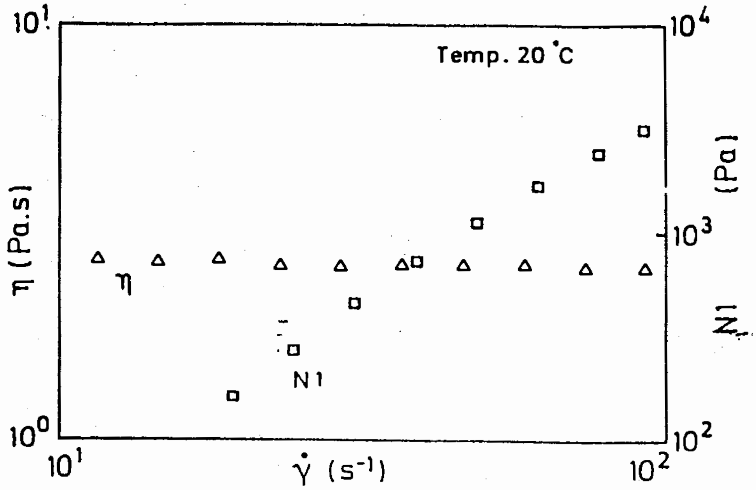

Fig.1 Steady shear data for Boger fluid M_1.

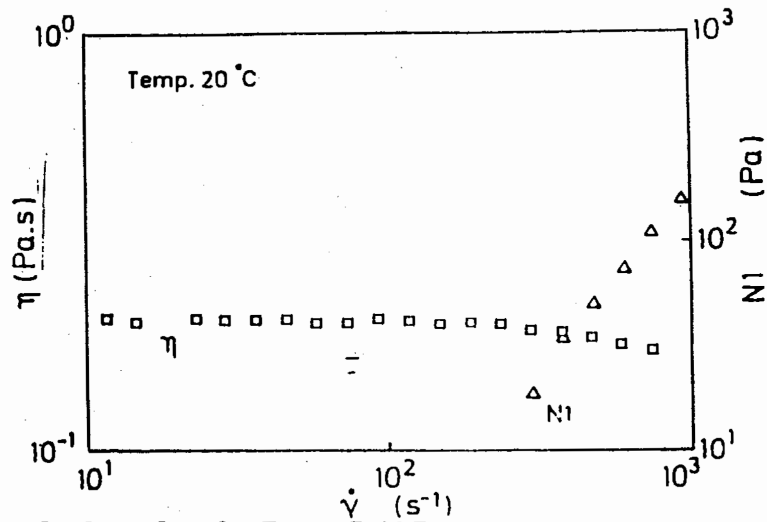

Fig.2 Steady shear data for Boger fluid D_4.

Considerable difficulty was obtained in determining the second normal stress difference N_2 for the fluids M_1 and D_4. In order to make any sense of the data, we found it essential to use shear stress as independent variable rather than the customary shear rate. In view of the reasonable expectation that $N_1 - N_2$ against the shear stress σ would be independent of temperature [5] such a procedure at least removed one troublesome influence on the raw data, namely the effect of even very small changes of temperature between the cone-and-plate and parallel-plate experiments which were needed to determine N_2.

The most trustworthy results of many repeated experiments on N_2 for the Boger fluid M_1 are given in Fig.3. In the figure the ratio of $-N_2/N_1$ is plotted giving equal weight to each gap set. Although there is considerable data scatter, most of the N_2 data are significantly large and all negative. So the ratio of N_2/N_1 is found to be negative and $|N_2/N_1| < 0.2$.

In Fig.4 the normal stress ratio, N_2/N_1 for the Boger fluid D_4 is given. It is clear that there is good agreement between the data of both runs. For this fluid, we found the values of N_2 to be very small, opposite in sign to N_1, and $N_2/N_1 < 0.2$.

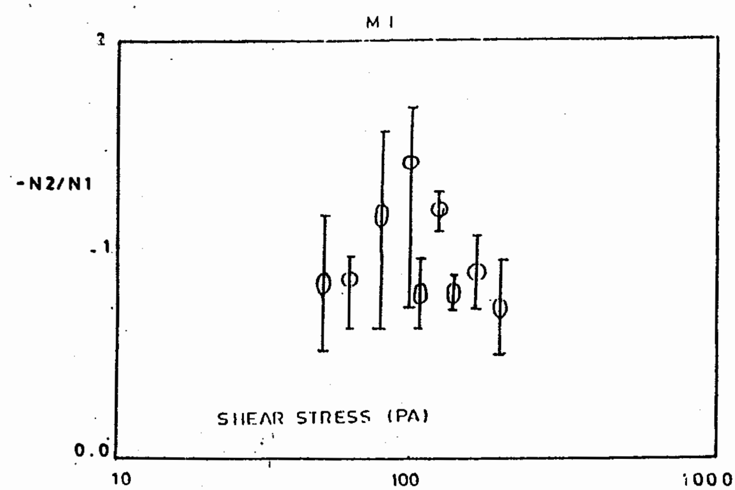

Fig.3 The normal stress ratio $-N_2/N_1$ as a function shear stress for Boger fluid M_1.

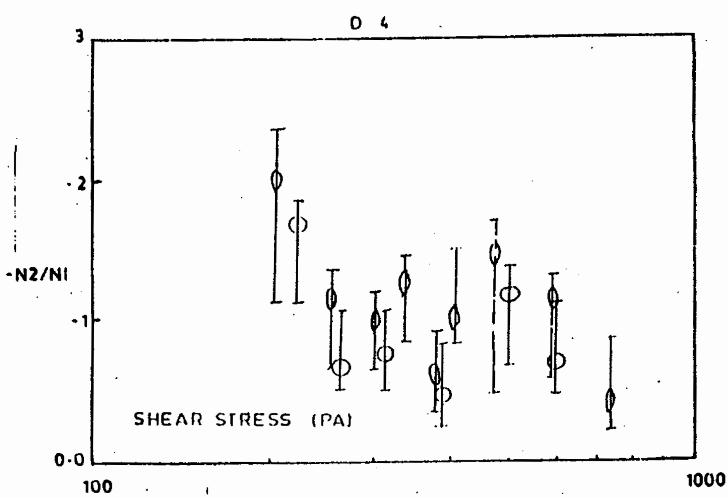

Fig.4 The normal stress ratio $-N_2/N_1$ as a function shear stress for Boger fluid D_4.

Although there is considerable scatter of data for both the fluids, the ratio of N_2/N_1 is significantly large and all negative. Thus on the basis of our data for both the fluids M_1 and D_4, the Weissenberg hypothesis $N_2 = 0$ is, at best, a rough approximation.

CONCLUSIONS

From the present measurement of the second normal stress difference N_2 of Boger fluids M_1 and D_4, the following conclusions can be drawn:

1) The second normal stress difference N_2 of Boger fluids, M_1 and D_4 is generally negative and much smaller in magnitude than N_1.
2) Total normal force measurements made in the cone-and-plate and parallel-plate geometries do in themselves offer reasonably accurate methods of determining N_2 if they are taken with painstaking care.
3) The main problem in this method is that N_2 is obtained by taking the small difference between two large experimentally determined quantities. This causes the N_2 values to be seriously affected even by small experimental errors and the determination of derivatives of experimental curves. Therefore, it is, difficult to avoid scatter in the second normal stress difference results.

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ON RATIONALITY OF ZASSENHAUS GROUPS

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ABSTRACT: An important problem in *Q-group* theory is to classify particular classes of those groups which are not *Q-groups*. In this paper we classify a number of Zassenhaus groups which are not *Q-groups*.

KEY WORDS AND PHRASES: Frobenius Groups, Zassenhaus Groups, *Q-groups*, *Q'-groups*, $L_2(n)$

AMS Subject Classification (1995): 20C 15

NOTATIONS: Notations will be those of [1] except where mentioned otherwise.

1. INTRODUCTION

A *Q-group* is a finite group all of whose ordinary complex representations have rational valued characters, otherwise it is called a *Q'-group*.

The interplay between the structure of a finite group and its representation has had, and continues to have, deep consequence for both theories. By imposing certain conditions on the group, such as being abelian or nilpotent, one is able to draw conclusions about its representations. Conversely, restrictions on the representations can lead to specific structures. It is in this context that we approach the study of *Q-groups*. It is quite interesting to note that the order of a non-trivial *Q-group* must be divisible by 2 thus ensuring the existence of an involution.

A transitive permutation group G of degree $n>2$ which has minimal degree $n-1$ i.e. no non-identity element fixes more than one letter and a subgroup H of G fixing a letter is non trivial, is called a **Frobenius Group**. A subgroup K of G fixing no letter is called Frobenius Kernel and H is termed as Frobenius Subgroup (also called complement). It is known that G is the semi-direct product of H and K.

Among the transitive groups, the class of doubly transitive groups in which only identity fixes more than two letters is very important and interesting one. We call such groups **Zassenhaus Groups.** If *G* is a Zessenhaus Group and *N* a subgroup of *G* fixing a letter, then *N* is a Frobenius subgroup of *G* with Frobenius Kernal *K* and complement *H*. With these notations we call *G* a Zessenhaus Group of type (H,K). This is an extremely important class of groups for it includes two of the families of simple groups, the group $L_2(q)$, $q > 3$. $\{L_2(q)$ is in fact image of special linear group $SL(2,q)$ in projective linear group $PGL(2,q)$, called projective special linear group. It is usually denoted by $PSL(2,q)$, but here, for simplicity of notation, we shall denote this group by $L_2(q)\}$ and the Suzuki Groups. Since the order of a Zassenhaus group is even, it thus bears directly on classification of *Q-group*.

2. DEFINITIONS AND SOME KNOWN RESULTS

Definition 2.1 [2]

Let *m* be an integer greater than *I*. The set $\{[n]\}$ of residue classes module m, where $1 \leq n < m$ and *n* is coprime to *m*, is a group and is denoted by G_m. A routine proof, using the Euclidean Algorithm, shows that G_m is an abelian group, under the usual multiplication of residue classes. More over $|Gm| \geq 2$ for $m > 2$.

Results regarding automorphism group of a cyclic group are summarized in the following proposition.

Proposition 2.1 [2]

Let *G* be a cyclic group of order *m*, then we have:

i) If *G* contains an element of order greater than 2 then $|Aut(G)| \geq 2$.
ii) If *m* is an odd prime, say *p*, then *Aut(G)* is cyclic of order $p - 1$.
iii) $Aut(G) \cong G_m$
iv) If $m = p^n$ for some prime *p* then *Aut(G)* is cyclic group of order $p^{n-1}(p - 1)$.

Proposition 2.2 [6]

Let *G* be a finite group. Then *G* is a *Q-group* if and only if, for every cyclic subgroup *H* of *G*, we have

$$N_G(H)/C_G(H) \cong Aut(H)$$

From proposition n2.1 it follows that a finite group G is a *Q'-group* if and only if there exists a cyclic subgroup H if G such that

$$N_G(H)/C_G(H) \ncong Aut(H).$$

Proposition 2.3 [6]

Let G be a Zassenhaus group of degree $n + 1$ and type (H,K). Also let N be the subgroup of G fixing a letter. Then:

i) N is a Frobenius group with kernel K of order n and complement H.
ii) K is disjoint from its conjugates and $N = N_G(K)$. Moreover $C_G(y) \subseteq K$ and $C^*_G(y) \subseteq N$ for all y in K^* where $K^* = K - \{e\}$.
iii) $|N_G(H):H| = 2$.
iv) $|G| = hn(n+1)$, where $h = |H|$ and h divides $n - 1$.

Proposition 2.4 [1]

Let G be a simple Zassenhaus group of degree $n + 1$ and type (H,K) in which $|H|$ is odd. Then H is cyclic and $N_G(H)$ is a Frobenius group.

Proposition 2.5 [1]

Let G be a simple Zassenhaus group of degree $n + 1$ and type (H,K) in which H is cyclic, H is inverted by an involution of G, and H has order $n - 1$ if n is even and order $\frac{1}{2}(n - 1)$ if n is odd. Then $n = p^r > 3$ for some prime p and G is isomorphic to $L_2(n)$.

3. SOME ZASSENHAUS Q'-GROUP

Theorem 3.1

Let G be a Zassenhaus group of degree $n + 1$, $n > 4$, and type (H,K). if $n - 1$ is a prime, then G is a *Q'-Group*.

Proof

By proposition 2.3 part (iv) we have $h = |H|$ and h divides $n - 1$ which is prime, say p. As $|H| \neq 1$, $H =$ p so that H is a cyclic subgroup of G. Therefore $Aut(H)$ is a cyclic group of order $p - 1$ {by proposition 2.1 part (ii)}. Since $n > 4$, we have $|Aut(H)| > 3$. By proposition 2.3 part (iii) we have $|N_G(H): H| = 2$, which forces $|N_G(H)| = 2p$. Also $H \subseteq C_G(H) \subseteq N_G(H) \Rightarrow 2, |C_G(H)| = p$ or $2p|$.

If $|C_G(H)| = p$ then $|N_G(H):C_G(H)| = 2$. But $|Aut(H)| = p - 1 > 3$, which shows that $Aut(H)$ is not isomorphic to $N_G(H)/C_G(H)$.

If $|C_G(H)| = 2p$ then $N_G(H)/C_G(H) = E$ and again $N_G(H)/C_G(H) \not\cong Aut(H)$. Thus, in both the cases, G is a Q'-group.'

4. SOME SIMPLE ZASSENHAUS Q'-GROUP

Theorem 4.1

Let G be a simple Zassenhaus group of degree $n + 1$ and type (H,K) in which H is cycle, H is inverted by an involution of G and H has order $n - 1$ if n is even and order $\frac{1}{2}(n - 1)$ if n is odd. Then G is a *Q'-Group*.

Proof

By proposition 2.5 such Zassenhaus groups are isomorphic to $L_2(n)$, where $n = p^r > 3$ for some prime p. We discuss the rationality of these groups by considering following two cases.

Case-I: n is odd

Since $n > 3$ and n is odd, we have $n \geq 5$. For $n = 5$, $G \cong L_2(5)$. By [4] and [5] we have $L_2(5) \cong A_5$. But by [3] A_5 is not a *Q-group*. Thus G is a *Q'-Group* for $n = 5$.

For $n = 7$ we have $|K| = 7$ {by proposition 2.3 part (i)}, so that K is cyclic. Also for $n = 7$, $|H| = \frac{1}{2}(7-1) = 3$ (by assumption). By proposition 2.3 part (ii) we have $N_G(K) = N$ and N, being Frobenius group, is semi-direct product of H and K, so that $|N_G(H)| = |N| = |H|\,|K| = 21$.

Now $C_G(y) \subseteq K$ for all $y \in K^{*} \Rightarrow C_G(K) \subseteq K$. But K is cyclic, therefore $K \subseteq C_G(K)$, so $C_G(K) = K$ and $|C_G(K)| = 7$. Thus $|N_G(K):C_G(H)| = 3$. But $|Aut(K)| = 6$ {by proposition 2.1 part (ii)}. Therefore $N_G(K)/C_G(K) \not\cong Aut(K)$ and G is a *Q'-group* for $n = 7$.

For $n = 9$ $G \not\cong L_2(9)$ and by [4] we have $L_2(9) \not\cong A_6$. But by [3] A_6 is not a Q-group. So G is a *Q'-group* for $n = 9$.

Next we take $n \geq 11$. In this case $|H| = \frac{1}{2}(n - 1) \geq 5$, so that $|Aut(H)| > 2$ {by proposition 2.1 part (iii)}. Since $|H|$ is cyclic, therefore, $H \subseteq C_G(H) \subseteq N_G(H)$. By proposition 2.3 part (iii) and by proportion 2.3 part (iii), $|N_G(H):H| = 2$, therefore $|N_G(H):C_G(H)| = 1$ or 2. But $|Aut(H)| > 2$. Hence $N_G(H)/C_G(H) \not\cong Aut(H)$. So G is a *Q'-group* for $n \geq 11$. Thus G is a *Q'-group* for odd n.

Case-II: *n* is even

Since $n > 3$, we first consider $n = 4$. In this case $G \cong L_2(4)$ and, by [4], $L_2(4) \cong A_5$. But A_5 is not a *Q-group* by [3]. Therefore G is a *Q'-group* for $n = 4$. Next we consider $n \geq 6$. In this case $|H| = n - 1 \geq 5$ and $Aut(H) > 2$. {by proposition 2.1}.

Since H is cyclic, therefore, $H \subseteq C_G(H) \subseteq N_G(H)$. By proposition 2.3 part (iii) $|N_G(H):H| = 2$, therefore $|N_G(H):C_G(H)| = 1$ or 2. But $|Aut(H)| > 2$. Hence $N_G(H)/C_G(H) \ncong Aut(H)$. Thus G is a *Q'-group*. Hence (because $|H| > 3$ and H is cyclic of odd order). Hence $N_G(H)/C_G(H) \ncong Aut(H)$. So G is a *Q'-group* for $n \geq 6$. That is G is a *Q'-group* for all n.

Theorem 4.2

Let G be a simple Zassenhaus group of degree $n + 1$ and type (H,K) in which $|H| = h > 3$ is odd. Then G is a *Q'-group*.

Proof

Since $|H|$ is odd, by proposition 2.4, H is cyclic. Thus $H \subseteq C_G(H) \subseteq N_G(H)$ and by proportion 2.3 part (iii), $|N_G(H):H| = 2$, so that $|N_G(H):C_G(H)| = 1$ or 2. But $|Aut(H)| > 2$ (because $|H| > 3$ and H is cyclic of odd order). Hence $N_G(H)/C_G(H) \ncong Aut(H)$. Thus G is a *Q'-group*.

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MAGNETIC SYMMETRY AND PHYSICAL PROPERTIES

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ABSTRACT: The number of independent non-vanishing constants required to describe the six known physical properties, involving a polar vector, has been determined by the character method of Bhagavantam and Hamermesh for each one of the 58 bicoloured magnetic point groups.

1. INTRODUCTION

Group theory has been successfully applied by [1,2] and then by [3] to enumerate the constants needed to describe various physical or magnetic properties in crystals. More recently a physical significance was given by [4] to the number of constants, relating to magnetic properties, appearing against the alternating representations of the 32 single coloured crystallographic point groups. Following this, the number of second-, third- and the fourth-order elastic coefficients for each of the 58 bicoloured magnetic point groups has been enlisted by [4]. In this paper, the number of constants needed to describe some of the known physical properties is enumerated for the 58 magnetic variants of the 32 srystallographic point groups on the basis of physical significance [4]. The results given here are new and were not considered by [5,6].

2. PHYSICAL PROPERTIES OF THE MAGNETIC POINT GROUPS

Only those physical properties with serial numbers 2, 3, 5, 8, 11 and 13 in Table 7a of [1] are considered here, and those properties which have no known physical significance are omitted. It is known that ferromagnetism is possible only in those crystal classes in which pyromagnetism is also possible; there are 31 ferromagnetic classes. These 31 classes consist of 12 conventional point groups and 19 bicoloured magnetic point groups. That is to say, a magnetic property such as pyromagnetism is exhibited by 12 conventional crystal classes. Whether or not the converse is true and whether the 58 bicoloured crystal classes exhibit non-magnetic properties are investigated in this paper for the six physical properties. The numbers obtained in respect of these properties by means of the character method of [1] are given in Table 1 for the 58 bicoloured crystal classes.

[1] implies that a symmetry operation and its complement will have the same effect on physical properties such as elasticity, photoelasticity, etc. and gives the impression that there is no distinction between a point group and its magnetic variants so far as the physical properties are concerned: for instance, according to [1] and [6], in the case of photoelasticity the crystal class 2mm and its variants $2\underline{mm}$ and $\underline{2}m\underline{m}$ requires the same 12 constants, where as, in the light of this paper, the above three classes require 12, 8 & 8 constants respectively (Table 1), which are different in nature.

TABLE 1. THE PHYSICAL PROPERTY NUMBERS OF THE 58 BICOLOURED CRYSTAL CLASSES

S.No.	Magnetic Point Groups	Physical Properties						S.No.	Magnetic Point Groups	Physical Properties					
		2	3	5	8	11	13			2	3	5	8	11	13
1.	$\underline{\bar{1}}$	3	-	18	-	63	-	30.	$\underline{\bar{3}}$	1	-	6	-	21	-
2.	$\underline{m}$	1	2	8	16	29	58	31.	$3\underline{m}$	-	-	2	4	8	16
3.	$\underline{2}$	2	2	10	16	34	58	32.	$3\underline{2}$	1	-	4	4	13	16
4.	$\underline{2}/m$	2	-	10	-	34	-	33.	$\bar{3}\underline{m}$	-	-	-	4	-	16
5.	$2/\underline{m}$	1	-	8	-	29	-	34.	$\underline{\bar{3}}m$	1	-	4	-	13	-
6.	$\underline{2}/\underline{m}$	-	2	-	16	-	58	35.	$\underline{\bar{3}}/\underline{m}$	-	-	2	-	8	-
7.	$\underline{2}m\underline{m}$	1	1	5	8	17	29	36.	$\underline{\bar{6}}$	1	-	4	4	11	18
8.	$2\underline{m}\,\underline{m}$	-	1	3	8	12	29	37.	**6**	-	-	2	4	10	18
9.	$\underline{2}\,\underline{2}2$	1	1	5	8	17	29	38.	$6/\underline{m}$	1	-	4	-	11	-
10.	$\underline{m}mm$	1	-	5	-	17	-	39.	$\underline{6}/m$	-	-	2	-	10	-
11.	$\underline{m}\,\underline{m}\,\underline{m}$	-	-	3	-	12	-	40.	$\underline{6}/\underline{m}$	-	-	-	4	-	18
12.	$m\underline{m}\,\underline{m}$	-	1	-	8	-	29	41.	$6/\underline{m}\,\underline{2}$	-	-	1	2	5	7
13.	$\underline{4}$	-	2	4	10	14	34	42.	$6\,m\,\underline{2}$	1	-	3	2	8	9
14.	$\underline{\bar{4}}$	1	2	4	10	15	34	43.	$\underline{\bar{6}}\,\underline{m}\,2$	-	-	1	2	3	9
15.	$\underline{4}/\underline{m}$	-	-	4	-	14	-	44.	$6\,\underline{m}\,\underline{m}$	-	-	1	2	3	7
16.	$4/\underline{m}$	1	-	4	-	15	-	45.	$\underline{6}\,\underline{m}\,m$	-	-	1	2	5	9
17.	$\underline{4}/m$	-	2	-	10	-	34	46.	$\underline{6}\,\underline{2}\,\underline{2}$	1	-	3	2	8	7
18.	$4\underline{m}\,\underline{m}$	-	-	1	3	5	12	47.	$\underline{6}\,\underline{2}\,\underline{2}$	-	-	1	2	5	9

S.No.	Magnetic Point Groups	Physical Properties						S.No.	Magnetic Point Groups	Physical Properties					
		2	3	5	8	11	13			2	3	5	8	11	13
19.	$\underline{4}m\underline{m}$	-	1	2	5	7	17	48.	$6/\underline{m}mm$	1	-	3	-	8	-
20.	$\bar{\underline{4}}2\underline{m}$	-	1	1	5	5	17	49.	$6/m\,\underline{m}\,\underline{m}$	-	-	-	2	-	7
21.	$\bar{\underline{4}}\,\underline{2}m$	1	1	3	5	10	17	50.	$6/\underline{m}\,\underline{m}\,\underline{m}$	-	-	1	-	3	-
22.	$\bar{4}\underline{2}\,\underline{m}$	-	-	2	3	7	12	51.	$6/m\,m\,\underline{m}$	-	-	1	-	5	-
23.	$4\underline{2}\,\underline{2}$	1	-	3	3	10	12	52.	$\underline{6}/\underline{m}\,m\,\underline{m}$	-	-	-	2	-	9
24.	$\bar{4}2\underline{2}$	-	1	2	5	7	17	53.	$\underline{m}3$	-	-	1	-	4	-
25.	$\underline{4}/\underline{m}m\underline{m}$	-	-	2	-	7	-	54.	$\bar{\underline{4}}\,3\,\underline{m}$	-	-	-	1	1	4
26.	$4/\underline{m}mm$	1	-	3	-	10	-	55.	$\underline{4}\,3\,\underline{2}$	-	-	1	1	3	4
27.	$4/m\underline{m}\,\underline{m}$	-	-	-	3	-	12	56.	$\underline{m}\,3\,\underline{m}$	-	-	-	-	1	-
28.	$4/\underline{m}\,\underline{m}\,\underline{m}$	-	-	1	-	5	-	57.	$m\,3\,\underline{m}$	-	-	-	1	-	4
29.	$\underline{4}/mm\underline{m}$	-	1	-	5	-	17	58.	$\underline{m}\,3\,m$	-	-	1	-	3	-

These physical properties are in general exhibited by crystals in the magnetic state. However, in the magnetic state the number of constants required differs from that required in the ordinary state. This work suggests that experimental determination of the physical constants of crystals in the magnetic state is advisable.

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TABLES OF THE KOSTKA FOULKES POLYNOMIALS

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ABSTRACT: In thispaper we obtain the transition matrices K(t) for n = 7,8,9 and x(t) for n = 8 which are useful to calculate the character tables for Symmetric groups and general linear groups respectively.

1. INTRODUCTION

Littlewood [3] has given an elegant expression for the Hall-Littlewood functions $Q_\lambda(x;t)$ in terms of the raising operator R_{ij} acting on the S-function $S_\lambda(x;t)$. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of n we have

$$Q_\lambda(x;t) = \prod_{1 \le i < j \le m} \left(1 + t\, R_{ij} + t^2 R_{ij}^2 + \ldots\right)$$

$$S\left(\lambda_1, \lambda_2, \ldots \lambda_n\right) (x;t) \tag{1.1}$$

where R_{ij} is the raising operator defined as

$$R_{ij} S_{\left(S_{\lambda_1, \ldots, \lambda_n}\right)}(x;t) = S_{\left(\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots \lambda_n\right)}(x;t)$$

This expression may be used to calculate the Kostka-Foulkes-polynomials $K_{\lambda\mu}(t)$ (see [4]). Morries in [5] has used this expression as a basis for a recursive method to calculate the Kostka-Foulkes-polynomials.

Our intension in this paper is to give a similar recursvie formula by using the horizontal strips as in [6] for the calculation of Kostka-Folkes-polynomial. We shall produce Kostka Foulkes matrices for n = 7,8,9. These matrices have been given for $n \le 6$ in [4].

Moreover the transition matrix X(t) between the power sum product function and the Hall-Littlewood symmetric function lead to the so-called Green's polynomial defined in [2]. In this last section we shall produce x(t) for n = 8.

We shall first briefly review some of the basic definitions. For this purpose we closely follow Macdonald [4].

2. BASIC DEFINITIONS

A partition λ of an integer n is a sequence of positive integers $\lambda_1 \geq \lambda_2, \ldots \geq \lambda_m$ such that $\lambda_1 + \lambda_2 + \ldots . \lambda_m = n$. The number of parts m in λ is called length of the partition and is denoted by $i(\lambda)$. If λ is a partition of n, then we shall use the notation $\lambda \vdash n$. The (Young) diagram of a partition λ is given by

$$[\lambda] = \{(i,j) \in Z : 1 \leq j \leq \lambda_i, 1 \leq i \leq (\lambda)\}$$

Let λ and μ be two partitions such that $\mu \subset \lambda$, that is $\mu_i \leq \lambda_i$ $(i = 1,2,\ldots)$ then the skew diagram λ/μ is the set theoretic difference $\theta = \lambda - \mu$. A horizontal r-strip is a skew diagram with r-squares which has atmost one square in each column.

Let $x_1, x_2, \ldots .$ be independent variables then a polynomial in these independent set of variable is said to be symmetric polynomial if it is invariant under the action of the symmetric group.

Let $\Lambda = \underset{k \geq o}{+} \Lambda^k$ be the ring of symmetric polynomials in the variables x_1, $x_2, \ldots$ where Λ^k is the ring of homogeneous symmetric polynomials of degree k.

Then the rth power sum symmetric function is defined as $p_r = \sum_{i=1}^{\infty} x_i^r$ and $\{\lambda : \lambda \vdash n\}$ form a Q-basis of Λ^k, where Q is the field of rational numbers. Furthemore the Schur function e_λ are defined by

$$e_\lambda(x) = \frac{\det\left(x_i^{\lambda_{i+m-j}}\right)}{\det\left(x_i^{m-j}\right)} \quad 1 \leq i, j \leq m$$

form a Z-hasis for Λ^k. Note that in contrast to Macdonald [4], we used e_λ rather than S_λ to denote Schur functions for a reason which will become apparent later.

Let t be an indeterminate independent of the $x_1, x_2 \ldots .$ and $P_\lambda(x;t) = P_\lambda(x_1, x_2 \ldots \ldots ;t)$ and $Q_\lambda(x;t) = Q_\lambda(x_1, x_2, \ldots ,; t)$ are Hall-Littlewood P- and Q-functions as defined in [2]. Now if $b_\lambda(t) = \prod_{i \geq 1} \phi_{m_{i(\lambda)}}(t)$ where $\phi_r(t) = (1-t)(1-t^2)\ldots .(1-t^5)$ and $m_i(\lambda)$ is the multiplicity of i in λ. Then

$$Q_\lambda(x;t) = b_\lambda(t)\, P_\lambda(x;t), \text{ and}$$

for $t = 0$.

$P_\lambda(x;0) = Q_\lambda(x;0) = e_\lambda(x)$, the Schur function in the variables $x_1, x_2, \ldots$. Then we have that $\{P_\lambda(x;t) : \lambda \vdash n\}$ form Z[t] basis for $\wedge(t)$. The transition matrix which connect these Z[t]-bases is the matrix of Kostka-folkes polynomials.

$$K(t) = \left(K_{\lambda\mu}(t)\right). \quad \text{That is}$$

$$e_\lambda(x) = \sum_\mu K_{\lambda\mu}(t)\, P_\mu(x;t),$$

where $\quad K_{\lambda\mu}(t) \in Z[t]$.

Now if $Q_r(x;t) = q_r(x;t)$. Then we define another symmetric function $S_\lambda(x;t) = \det\left(q_{\lambda_i - i + j}(x;t)\right)$, and we have that $\{Q_\lambda(x;t) : \lambda \vdash n\}$ and $\{S_\lambda(x;t) : \lambda \vdash n\}$ are Q(t)-bases of $\wedge$ Q(t).

3. RECURSIVE METHOD TO CALCULATE THE KOSTKA POLYNOMIALS

Butler in [1] has shown that for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in Z^m$ we may find a partition such that

$$S_\alpha(x;t) = \pm S_\mu(x;t)$$

We define the raising operator R_{ij} $(i<j)$ acting on the $S_\lambda(x;t)$ as:

$$R_{ij} S_{(\alpha_1,\ldots,\alpha_j,\ldots)}(x;t) = S_{(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_j-1,\ldots)}(x;t)$$

The expressionn (1.1) may be used to calculate $K_{\lambda\mu}(t)$. Furthermore we shall use this expression as a basis for a recursive method to calculate the Kostka Foulkes polynomials as Morris has done in [5].

Theorem

Let μ is a partition of n, and $\mu_o \geq \mu_1$ then

$$Q_{(\mu_o,\mu_1,\ldots\mu_m)}(x;t) = \sum_{r=0}^{\infty} t^r \sum_{\mu \vdash n} K_{\lambda,\mu}(t) \sum_{\substack{\mu/w \\ \text{hori-}r\text{-strip}}} S_{(\mu_o+r,w)}(x;t). \qquad (3.1)$$

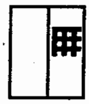
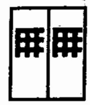
Proof See [6]

Example

For convenience we will write Q_λ and S_λ for Q_λ (x;t) S_λ (x;t) respectively.

$$Q_{(2)} = S_{(2)}$$

when $\mu = (2)$, for ɪ = 0,1,2 we have

μ/w: =  , i.e. w = (2) μ/w: =  , w = (1), μ/w: = , w = (0)

therefore $Q_{(32)} = S_{(32)} + t\, S_{(41)} + t^2\, S_{(5)}$

similarly

$$\begin{aligned} Q_{(3^2 2)} \quad = \quad & S_{(3^2 2)} + t\left\{S_{(431)} + S_{(42^2)}\right\} + t^2\left\{S_{(53)} + S_{(521)}\right\} + t^3 S_{(62)} + \\ & t\left\{S_{(34)} + t\left(S_{(4^2)} + S_{(431)}\right) + t^2\left(S_{(53)} + S_{(521)}\right) + t^3\left(S_{(62)} + S_{(61^2)} + t^4 S_{(71)}\right)\right\} + \\ & t^2\left\{S_{(35)} + tS_{(4^2)} + t^2 S_{(53)} + t^3 S_{(62)} + t^4 S_{(71)} + t^5 S_{(8)}\right\}. \end{aligned}$$

Now using properties of S-functions we have

$$\begin{aligned} Q_{(3^2 2)} \quad = \quad & S_{(3^2 2)} + \left(t + t^2\right) S_{(431)} + t^3 S_{(4^2)} + tS_{(42^2)} + \left(t^2 + t^3 + t^4\right) S_{(53)} + \\ & \left(t^2 + t^3\right) S_{(521)} + \left(t^3 + t^4 + t^5\right) S_{(62)} + t^4 S_{(61^2)} + \left(t^5 + t^6\right) S_{(71)} + t^7 S_{(8)}. \end{aligned}$$

Now by using the recursive relation given by equation (3.1) we will give the expansion of HL-Q_λ function in terms of S-functions, for $\lambda_1 + \lambda_2 +,\ldots\ldots + \lambda_m = 5$ and $\lambda_1 + \ldots\ldots + \lambda_m = 6$. These relations are then use to calculate the matrices $K(t) = [K_{\lambda\mu}]$.

The relation for $\lambda_1 + \ldots\ldots + \lambda_m \leq 4$ have already be given in [5]. For sake of simplicity we are not including those results here.

$$Q_{(\lambda_o,5)} = S_{(\lambda_o,5)}+tS_{(\lambda_o+1,4)}+t^2S_{(\lambda_o+2,3)}+$$
$$t^3S_{(\lambda_o,3,2)}+t^4S_{(\lambda_o+4,1)}+t^5S_{(\lambda_o+5)}.$$

$$Q_{(\lambda_o,41)} = S_{(\lambda_o,41)}+tS_{(\lambda_o,5)}+(t^2+t^2)S_{(\lambda_o+1,4)}+tS_{(\lambda_o+1,31)}+$$
$$(t^2+t^3)S_{(\lambda_o+2,3)}+t^2S_{(\lambda_o+2,21)}+(t^3+t^4)S_{(\lambda_o+3,2)}+t^3S_{(\lambda_o+3,1^2)}+$$
$$(t^4+t^5)S_{(\lambda_o+4,1)}+t^6S_{(\lambda_o+5)}$$

$$Q_{(\lambda_o,32)} = S_{(\lambda_o,32)}+tS_{(\lambda_o,41)}+t^2S_{(\lambda_o,5)}+tS_{(\lambda_o+1,2^2)}+$$
$$(t+t^2)S_{(\lambda_o+1,31)}+(t^2+t^3)S_{(\lambda_o+1,4)}+(t^2+t^3+t^4)S_{(\lambda_o+2,3)}+$$
$$(t^3+t^2)S_{(\lambda_o+1,21)}+(t^3+t^4+t^5)S_{(\lambda_o+3,2)}+t^4S_{(\lambda_o+3,1^2)}+$$
$$(t^5+t^6)S_{(\lambda_o+4,1)}+t^7S_{(\lambda_o+5)}.$$

$$Q_{(\lambda_o,31^2)} = S_{(\lambda_o,31^2)}+tS_{(\lambda_o,32)}+(t+t^2)S_{(\lambda_o,41)}+t^3S_{(\lambda_o+5)}+$$
$$tS_{(\lambda_o+1,21^2)}+t^2S_{(\lambda_o+1,2^2)}+(t+2t^2+t^3)S_{(\lambda_o+1,31)}+$$
$$(t^2+t^3+t^4)S_{(\lambda_o+1,4)}+t^2S_{(\lambda_o+2,1^3)}+(t^2+2t^3+t^4)S_{(\lambda_o+2,21)}+$$
$$(2t^3+t^4+t^5)S_{(\lambda_o+2,3)}+(t^3+t^4+t^5)S_{(\lambda_o+3,1^2)}+(2t^4+t^5+t^6)S_{(\lambda_o+3,2)}+$$
$$(t^5+t^6+t^7)S_{(\lambda_o+4,1)}+t^8S_{(\lambda_o+5)}.$$

$$Q_{(\lambda_o,2^21)} = S_{(\lambda_o,2^21)}+tS_{(\lambda_o,31^2)}+(t+t^2)S_{(\lambda_o,32)}+(t^2+t^3)S_{(\lambda_o,41)}+$$
$$t^4S_{(\lambda_o,5)}+(t+t^2)S_{(\lambda_o+1,21^2)}+(t+t^2+t^3)S_{(\lambda_o+1,2^2)}+$$
$$(2t^2+2t^3+t^4)S_{(\lambda_o,1,31)}+(t^3+t^4+t^5)S_{(\lambda_o+1,4)}+t^3S_{(\lambda_o+2,1^3)}+$$

$$(t^2+2t^3+2t^4+5^5)S_{(\lambda_o+2,21)}+(t^3+t^4+t^5+t^6)S_{(\lambda_o+2,3)}+$$

$$(t^4+t^5+t^6+5^5)S_{(\lambda_o+3,1^2)}+(t^4+2t^5+t^6+t^7)S_{(\lambda_o+3,2)}+$$

$$(t^6+t^7+t^8)S_{(\lambda_o+4,1)}+t^9S_{(\lambda_o+5)}.$$

$$Q_{(\lambda_o,21^3)} = S_{(\lambda_o,21^3)}+(t+t^2)S_{(\lambda_o,2^21)}+(t+t^2+t^3)S_{(\lambda_o,31^2)}+$$

$$(t^2+t^3+t^4)S_{(\lambda_o,32)}+(t^3+t^4+t^5)S_{(\lambda_o+41)}+t^6S_{(\lambda_o+1,1^4)}+$$

$$(t+2t^2+2t^3+t^4)S_{(\lambda_o+1,21^2)}+(t^2+2t^3+t^4+t^5)S_{(\lambda_o+1,2^2)}+$$

$$(t^2+2t^3+3t^4+2t^5+t^6)S_{(\lambda_o+1,31)}+(t^4+t^5+t^6+t^7)S_{(\lambda_o+1,4)}+$$

$$(t^2+t^3+t^4+5^5)S_{(\lambda_o+2,1^3)}+(2t^3+3t^4+3t^5+2t^6+t^7)S_{(\lambda_o+2,21)}+$$

$$(t^4+2t^5+2t^6+t^7+t^8)S_{(\lambda_o+2,3)}+(t^4+t^5+2t^6+t^7+t8)S_{(\lambda_o+3,1^2)}+$$

$$(t^5+2t^6+2t^7+t^8+t^9)S_{(\lambda_o+3,2)}+(t^7+t^8+t^9+t^{10})S_{(\lambda_o+4,1)}+t^{11}S_{(\lambda_o+5)}.$$

$$Q_{(\lambda_o,1^5)} = S_{(\lambda_o,1^5)}+(t+t^2+t^3+t^4)S_{(\lambda_o,21^3)}+(t^2+t^3+t^4+t^5+t^6)S_{(\lambda_o,2^21)}+$$

$$(t^3+t^4+2t^5+t^6+t^7)S_{(\lambda_o,31^2)}+(t^4+t^5+t^6+t^7+t^8)S_{(\lambda_o,32)}+$$

$$(t^6+t^7+t^8+t^9)S_{(\lambda_o,41)}+t^{10}S_{(\lambda_o,5)}+(t+t^2+t^3+t^4+t^5)S_{(\lambda_o+1,1^4)}+$$

$$(t^2+2t^3+3t^4+3t^5+3t^6+2t^7+t^8)S_{(\lambda_o+1,21^2)}+$$

$$(t^3+t^4+2t^5+2t^6+2t^7+t^8+t^9)S_{(\lambda_o+1,2^2)}+$$

$$(t^4+2t^5+3t^6+3t^7+3t^8+2t^9+t^{10})S_{(\lambda_o+1,31)}+$$

$$(t^7+t^8+t^9+t^{10}+t^{11})S_{(\lambda_o+1,4)}+$$

$$(t^3+t^4+2t^5+2t^6+2t^7+t^8+t^9)S_{(\lambda_o+2,1^3)}+$$

$$(t^4+2t^5+3t^6+4t^7+4t^8+3t^9+t^{11})S_{(\lambda_o+2,21)}+$$

$$(t^6+t^7+2t^8+2t^9+2t^{10}+t^{11}+t^{12})S_{(\lambda_o+2,3)}+$$

$$(t^6+t^7+2t^8+2t^9+2t^{10}+t^{11}+t^{12})S_{(\lambda_o+3,1^2)}+$$

$$(t^7+t^8+2t^9+2t^{10}+2t^{11}+t^{12}+t^{13})S_{(\lambda_o+3,2)}$$

$$(t^{10}+t^{11}+t^{12}+t^{13}+t^{14})S_{(\lambda_o+4,1)}+t^{15}S_{(\lambda_o+5)}$$

$\lambda_1+\lambda_2+\ldots+\lambda_m = 6$

$$Q_{(\lambda_o,6)} = S_{(\lambda_o,6)}+tS_{(\lambda_o+1,5)}+t^2S_{(\lambda_o+2,4)}+t^3S_{(\lambda_o+3,3)}+$$

$$t^4S_{(\lambda_o+4,2)}+t^5S_{(\lambda_o+5,1)}+t^6S_{(\lambda_o+6)}$$

$$Q_{(\lambda_o,51)} = S_{(\lambda_o,51)}+tS_{(\lambda_o,6)}+tS_{(\lambda_o+1,41)}+(t+t^2)S_{(\lambda_o+1,5)}+$$

$$t^2S_{(\lambda_o+2,31)}+(t^2+t^3)S_{(\lambda_o+2,4)}+t^3S_{(\lambda_o+3,21)}+$$

$$(t^3+t^4)S_{(\lambda_o+3,3)}+t^4S_{(\lambda_o+4,1^2)}+$$

$$t^4S_{(\lambda_o+4,2)}+(t^5+t^6)S_{(\lambda_o+5,1)}+t^7S_{(\lambda_o+6)}$$

$$Q_{(\lambda_o,42)} = S_{(\lambda_o,42)}+tS_{(\lambda_o,51)}+t^2S_{(\lambda_o,6)}+tS_{(\lambda_o+1,32)}+$$

$$(t+t^2)S_{(\lambda_o+1,41)}+(t^2+t^3)S_{(\lambda_o+1,5)}+t^2S_{(\lambda_o+2,2^2)}+$$

$$(t^2+t^3)S_{(\lambda_o+2,31)}+(t^2+t^3+t^4)S_{(\lambda_o+2,4)}+$$

$$(t^3+t^4)S_{(\lambda_o+3,21)}+(t^3+t^4+t^5)S_{(\lambda_o+3,3)}+t^5S_{(\lambda_o+4,1^2)}+$$

$$(t^4+t^5+t^6)S_{(\lambda_o+4,2)}+(t^6+t^7)S_{(\lambda_o+5,1)}+t^8S_{(\lambda_o+6)}.$$

$$\begin{aligned}
Q_{(\lambda_o,41^2)} \quad = \quad & S_{(\lambda_o,41^2)}+tS_{(\lambda_o,42)}+(t+t^2)S_{(\lambda_o,51)}+t^3S_{(\lambda_o,6)}+ \\
& +tS_{(\lambda_o+1,31^2)}+t^2S_{(\lambda_o+1,32)}+(t+2t^2+t^3)S_{(\lambda_o+1,41)}+ \\
& (t^2+t^3+t^4)S_{(\lambda_o+1,5)}+t^2S_{(\lambda_o+2,21^2)}+t^3S_{(\lambda_o+2,2^2)} \\
& (t^2+2t^3+t^4)S_{(\lambda_o+3,21)}+(2t^3+t^4+t^5)S_{(\lambda_o+2,4)}+t^3S_{(\lambda_o+3,1^3)}+ \\
& (t^3+2t^4+t^5)S_{(\lambda_o+3,21)}+(2t^4+t^5+t^6)S_{(\lambda_o+3,3)}+ \\
& (t^4+t^5+t^6)S_{(\lambda_o+4,1^2)}+(2t^5+t^6+t^7)S_{(\lambda_o+4,2)}+ \\
& (t^6+t^7+t^8)S_{(\lambda_o+5,1)}+t^9S_{(\lambda_o+6)}.
\end{aligned}$$

$$\begin{aligned}
Q_{(\lambda_o,3^2)} \quad = \quad & S_{(\lambda_o,3^2)}+tS_{(\lambda_o,42)}+t^2S_{(\lambda_o,51)}+t^3S_{(\lambda_o,6)}+ \\
& (t+t^2)S_{(\lambda_o+1,32)}+(t^2+t^3)S_{(\lambda_o+1,41)}+(t^3+t^4)S_{(\lambda_o+1,5)}+ \\
& t^3S_{(\lambda_o+2,2^2)}+(t^2+t^3+t^4)S_{(\lambda_o+2,31)}+(t^3+t^4+t^5)S_{(\lambda_o+2,4)} \\
& (t^4+t^5)S_{(\lambda_o+3,21)}+(t^3+t^4+t^5+t^6)S_{(\lambda_o+3,3)}+t^6S_{(\lambda_o+4,1^2)}+ \\
& (t^5+t^6+t^7)S_{(\lambda_o+4,2)}+(t^7+t^8)S_{(\lambda_o+5,1)}+t^9S_{(\lambda_o+6)}.
\end{aligned}$$

$$\begin{aligned}
Q_{(\lambda_o,321)} \quad = \quad & S_{(\lambda_o,321)}+tS_{(\lambda_o,3^2)}+(t+t^2)S_{(\lambda_o,42)}+(t^2+t^3)S_{(\lambda_o,51)}+ \\
& t^2S_{(\lambda_o,42)}+t^4S_{(\lambda_o,6)}+t^3S_{(\lambda_o+1,2^21)}+
\end{aligned}$$

$$(t+t^2)S_{(\lambda_o+1,31^2)}+(t+t^2+t^3)S_{(\lambda_o+1,32)}$$

$$(2t^2+2t^3+t^4)S_{(\lambda_o+1,41)}+(t^3+t^4+t^5)S_{(\lambda_o+1,5)}+(t^2+t^3)S_{(\lambda_o+2,21^2)}+$$

$$(t^2+t^3+t^4)S_{(\lambda_o+2,2^2)}+(t^2+t^3+t^4+t^5)S_{(\lambda_o+2,31)}+$$

$$(t^3+2t^4+t^5+t^6)S_{(\lambda_o+2,4)}+t^4S_{(\lambda_o+3,1^3)}+$$

$$(t^3+t^4+t^5+t^6)S_{(\lambda_o+3,21)}+(2t^4+2t^5+t^6+t^7)S_{(\lambda_o+3,3)}+$$

$$(t^5+t^6+t^7)S_{(\lambda_o+4,1^2)}+(t^5+2t^6+t^7+t^8)S_{(\lambda_o+4,2)}+$$

$$(t^7+t^8+t^9)S_{(\lambda_o+5,1)}+t^{10}S_{(\lambda_o+6)}$$

$$Q_{(\lambda_o,31^3)} = S_{(\lambda_o,31^3)}+(t+t^2)S_{(\lambda_o,321)}+t^3S_{(\lambda_o,3^2)}+(t+t^2+t^3)S_{(\lambda_o,41^2)}+$$

$$(t^2+t^3+t^4)S_{(\lambda_o,42)}+(t^3+t^4+t^5)S_{(\lambda_o,51)}+t^6S_{(\lambda_o,6)}+$$

$$tS_{(\lambda_o+1,21^3)}+(t^2+t^3)S_{(\lambda_o+1,2^21)}$$

$$(t+2t^2+2t^3+t^4)S_{(\lambda_o+1,31^2)}+(t^2+2t^3+2t^4+t5)S_{(\lambda_o+1,32)}+$$

$$(t^2+2t^3+3t^4+2t^5+t^6)S_{(\lambda_o+1,41)}+(t^4+t^5+t^6+t^7)S_{(\lambda_o+1,5)}+$$

$$t^2S_{(\lambda_o+2,1^4)}(t^2+2t^3+2t^4+t^5)S_{(\lambda_o+2,21^2)}+$$

$$(t^4+2t^5+2t^6+t^7+t^8)S_{(\lambda_o+2,4)}+(t^3+t^4+t^5+t^6)S_{(\lambda_o+3,1^3)}+$$

$$(2t^4+3t^5+3t^6+2t^7+t^8)S_{(\lambda_o+3,21)}+(t^5+3t^6+2t^7+t^8+t^9)S_{(\lambda_o+3,3)}+$$

$$(t^5+t^6+2t^7+t^8+t^9)S_{(\lambda_o+4,1^2)}+(t^6+2t^7+2t^8+t9+t^{10})S_{(\lambda_o+4,2)}+$$

$$(t^8+t^9+t^{10}+t^{11})S_{(\lambda_o+5,1)}+t^{12}S_{(\lambda_o+6)}.$$

$$
\begin{aligned}
Q_{(\lambda_o,2^3)} = {} & S_{(\lambda_o,2^3)}+(t+t^2)S_{(\lambda_o,321)}+t^3S_{(\lambda_o,3^2)}+t^3S_{(\lambda_o,41^2)}+ \\
& (t^2+t^3+t^4)S_{(\lambda_o,42)}+(t^4+t^5)S_{(\lambda_o,51)}+t^6S_{(\lambda_o,6)}+ \\
& (t+t^2+t^3)S_{(\lambda_o+1,2^21)}+(t^2+t^3+t^4)S_{(\lambda_o+1,31^2)}+ \\
& (t^2+2t^3+2t^4+t^5)S_{(\lambda_o+1,32)}+(t^3+2t^4+2t^5+t^6)S_{(\lambda_o+1,41)}+ \\
& (t^5+t^6+t^7)S_{(\lambda_o+1,5)}+(t^3+t^4+t^5)S_{(\lambda_o+2,21^2)}+ \\
& (t^2+t^3+2t^4+t^5+t^6)S_{(\lambda_o+2,2^2)}(t^3+2t^4+3t^5+2t^6)S_{(\lambda_o+2,31)}+ \\
& (t^4+t^5+2t^6+t^7+t^8)S_{(\lambda_o+2,4)}+t^6S_{(\lambda_o+3,1^3)}+ \\
& (t^4+2t^5+2t^6+2t^7+t^8)S_{(\lambda_o+3,21)}+(t^5+2t^6+2t^7+t^8+t^9)S_{(\lambda_o+3,3)}+ \\
& (t^7+t^8+t^9)S_{(\lambda_o+4,1^2)}+(t^6+t^7+2t^8+t9+t^{10})S_{(\lambda_o+4,2)}+ \\
& (t^9+t^{10}+t^{11})S_{(\lambda_o+5,1)}+t^{12}S_{(\lambda_o+6)}.
\end{aligned}
$$

$$
\begin{aligned}
Q_{(\lambda_o,2^21^2)} = {} & S_{(\lambda_o,2^21^2)}+tS_{(\lambda_o,2^3)}+tS_{(\lambda_o,31^3)}+(t+2t^2+t^3)S_{(\lambda_o,321)}+ \\
& (t^2+t^4)S_{(\lambda_o,3^2)}+(t^2+t^3+t^4)S_{(\lambda_o,41^2)}+(2t^3+t^4+t^5)S_{(\lambda_o,42)}+ \\
& (t^4+t^5+t^6)S_{(\lambda_o,51)}+t^7S_{(\lambda_o,6)}+(t+t^2)S_{(\lambda_o+1,21^2)}+ \\
& (t+2t^2+2t^3+t^4)S_{(\lambda_o+1,2^21)}+(2t^2+3t^3+2t^4+t^5)S_{(\lambda_o+1,31^2)}+ \\
& (t^2+3t^3+3t^4+2t^5+t^6)S_{(\lambda_o+1,32)}+(t^3+3t^4+3t^5+2t^6+t^7)S_{(\lambda_o+1,41)}+ \\
& (t^5+t^6+2t^7+t^8)S_{(\lambda_o+1,5)}+t^3S_{(\lambda_o+2,1^4)}+ \\
& (t^2+2t^3+3t^4+2t^5+t^6)S_{(\lambda_o+2,21^2)}+(2t^3+2t^4+3t^5+t^6+t^7)S_{(\lambda_o+2,2^2)}+
\end{aligned}
$$

$$(t^3+4t^4+4t^5+4t^6+2t^7+t^8+t^9)S_{(\lambda_o+2,31)}+(2t^5+2t^6+2t^7+t^8+t^9)S_{(\lambda_o+2,4)}+$$

$$(t^4+t^5+t^6+t^7)S_{(\lambda_o+3,1^3)}+(t^4+3t^5+4t^6+3t^7+2t^7+2t^8+t^9)S_{(\lambda_o+3,21)}+$$

$$(t^5+2t^6+3t^7+2t^8+t^9+t^{10})S_{(\lambda_o+3,3)}+(t^6+t^7+2t^8+t^9+t^{10})S_{(\lambda_o+4,1^2)}\cdot$$

$$(2t^7+2t^8+2t^9+t^{10}+t^{11})S_{(\lambda_o+4,2)}+(t^9+t^{10}+t^{11}+t^{12})S_{(\lambda_o+5,1)}+$$

$$t^{13}S_{(\lambda_o+6)}$$

$$Q_{(\lambda_o,21^4)} = S_{(\lambda_o,21^4)}+(t+t^2+t^3)S_{(\lambda_o,2^11^2)}+(t^2+t^4)S_{(\lambda_o,2^3)}+$$

$$(t+t^2+t^3+t^4)S_{(\lambda_o,31^3)}+(t^2+2t^3+2t^4+2t^5+t^6)S_{(\lambda_o,321)}+$$

$$(t^4+t^5+t^7)S_{(\lambda_o,3^2)}+(t^3+t^4+2t^5+t^6+t^7)S_{(\lambda_o,41^2)}+$$

$$(t^4+t^5+2t^6+t^7+t^8)S_{(\lambda_o,42)}+(t^6+t^7+t^8+t^9)S_{(\lambda_o,51)}+$$

$$t^{10}S_{(\lambda_o,6)}+tS_{(\lambda_o+1,1^5)}+(t+2t^2+2t^3+2t^4+t^5)S_{(\lambda_o+1,21^3)}+$$

$$(t^2+3t^3+3t^4+3t^5+2t^6+t^7)S_{(\lambda_o+1,2^21)}+$$

$$(t^2+2t^3+4t^4+4t^5+4t^6+2t^7+t^8)S_{(\lambda_o+1,31^2)}+$$

$$(t^2+3t^3+3t^4+3t^5+2t^6+t^7)S_{(\lambda_o+1,2^21)}+$$

$$(t^3+2t^4+4t^5+4t^6+3t^7+2t^8+t^9)S_{(\lambda_o+1,32)}+$$

$$(t^4+2t^5+3t^6+4t^7+3t^8+2t^9+t^{10})S_{(\lambda_o+1,41)}+$$

$$(t^7+t^8+t^9+t^{10}+t^{11})S_{(\lambda_o+1,5)}+(t^2+t^3+t^4+t^5+t^6)S_{(\lambda_o+2,1^4)}+$$

$$(2t^3+3t^4+5t^5+4t^6+4t^7+2t^8+t^9)S_{(\lambda_o+2,21^2)}+$$

$$(2t^4+2t^5+4t^6+3t^7+3t^8+t^9+t^{10})S_{(\lambda_o+2,2^2)}+$$

$$(t^4+3t^5+5t^6+6t^7+5t^8+4t^9+2t^{10}+t^{11})S_{(\lambda_o+2,31)}+$$

$$(t^6+t^7+3t^8+2t^9+2t^{10}+t^{11}+t^{12})S_{(\lambda_o+2,4)}+$$

$$(t^4+t^5+2t^6+2t^7+2t^8+t^9+t^{10})S_{(\lambda_o+3,1^3)}+$$

$$(t^5+3t^6+4t^7+3t^8+3t^9+3t^{10}+2t^{11}+t^{12})S_{(\lambda_o+3,21)}+$$

$$(2t^7+2t^8+3t^9+3t^{10}+2t^{11}+t^{12}+t^{13})S_{(\lambda_o+3,3)}+$$

$$(t^7+t^8+2t^9+2t^{10}+2t^{11}+t^{12}+t^{13})S_{(\lambda_o+4,1^2)}+$$

$$(t^8+t^9+3t^{10}+2t^{11}+2t^{12}+t^{13}+t^{14})S_{(\lambda_o+4,2)}+$$

$$(t^{11}+t^{12}+t^{13}+t^{14}+t^{15})S_{(\lambda_o+5,1)}+t^{16}S_{(\lambda_o+6)}.$$

$$Q_{(\lambda_o,1^6)} = S_{(\lambda_o,1^6)}+(t+t^2+t^3+t^4+t^5)S_{(\lambda_o,21^4)}+$$

$$(t^2+t^3+2t^4+t^5+2t^6+t^7+t^8)S_{(\lambda_o,2^21^2)}+$$

$$(t^3+t^5+t^6+t^7+t^9)S_{(\lambda_o,2^3)}+(t^3+t^4+2t^5+2t^6+2t^7+t^8+t9)S_{(\lambda_o,31^3)}+$$

$$(t^4+2t^5+2t^6+3t^7+3t^8+2t8+2t^9+2t^{10}+t^{11})S_{(\lambda_o,321)}+$$

$$(t^6+t^8+t^9+t^{10}+t^{12})^{10}S_{(\lambda_o,3^2)}+(t^6+t^7+2t^8+2t^9+2t^{10}+t^{11}+t^{12})S_{(\lambda_o+41^2)}+$$

$$(t^7+t^8+2t^9+t^{10}+2t^{11}+t^{12}+t^{13})S_{(\lambda_o+42)}+$$

$$(t^{10}+t^{11}+t^{12}+t^{13}+t^{14})S_{(\lambda_o+51)}+t^{15}S_{(\lambda_o,6)}$$

$$(t+t^2+t^3+t^4+t^5+t^6)S_{(\lambda_o+1,1^5)}+$$

$$(t^2+2t^3+3t^4+4t^5+4t^6+4t^7+3t^8+2t^9+t^{10})S_{(\lambda_o+1,21^3)}+$$

$$(t^3+2t^4+3t^5+4t^6+5t^7+5t^8+4t^9+3t^{10}+2t^{11}+t^{12})S_{(\lambda_o+1,2^21)}+$$

$$(t^4+2t^5+4t^6+5t^7+6t^8+6t^9+5t^{10}+4t^{11}+2t^{12}+t^{13})S_{(\lambda_o+1,31^2)}+$$

$$(t^5+2t^6+3t^7+4t^8+5t^9+5t^{10}+4t^{11}+3t^{12}+2t^{13}+t^{14})S_{(\lambda_o+1,32)}+$$

$$(t^7+2t^8+3t^9+4t^{10}+4t^{11}+4t^{12}+3t^{13}+2t^{14}+t^{15})S_{(\lambda_o+1,41)}+$$

$$(t^{11}+t^{12}+t^{13}+t^{14}+t^{15}+t^{16})S_{(\lambda_o+1,5)}+$$

$$(t^3+t^4+2t^5+2t^6+3t^7+2t^8+2t^9+t^{10}+t^{11})S_{(\lambda_o+2,1^4)}+$$

$$(t^4+2t^5+4t^6+5t^7+7t^8+7t^9+7t^{10}+5t^{11}+4t^{12}+2t^{13}+t^{14})S_{(\lambda_o+2,21^2)}+$$

$$(t^5+t^6+3t^7+3t^8+5t^9+4t^{10}+5t^{11}+3^{12}+3t^{13}+t^{14}+t^{15})S_{(\lambda_o+2,2^2)}+$$

$$(t^6+2t^7+4t^8+5t^9+7t^{10}+7t^{11}+7t^{12}+5t^{13}+4t^{14}+2t^{15}+t^{16})S_{(\lambda_o+2,31)}+$$

$$(t^9+t^{10}+2t^{11}+2t^{12}+3t^{13}+2t^{14}+2t^{15}+t^{16}+t^{17})S_{(\lambda_o+2,4)}+$$

$$(t^6+t^7+2t^8+3t^9+3t^{10}+3t^{11}+3t^{12}+3t^{13}+t^{14}+t^{15})S_{(\lambda_o+3,1^3)}+$$

$$(t^7+2t^8+3t^9+5t^{10}+6t^{11}+6t^{12}+6t^{13}+5t^{14}+3t^{15}+2t^{16}+t^{17})S_{(\lambda_o+3,21)}+$$

$$(t^{11}+t^{12}+2t^{13}+2t^{14}+3t^{15}+2t^{16}+2t^{17}+t^{18}+t^{19})S_{(\lambda_o+4,2)}+$$

$$(t^{15}+t^{16}+t^{17}+t^{18}+t^{19}+t^{20})S_{(\lambda_o+5,1)}+t^{21}S_{(\lambda_o,6)}.$$

The above relations give us the transition matrices K(t) for n = 7,8,9.

4. THE TRANSITION MATRIX X(t)

The transition matrix which connecting $P_\lambda(x;t)$ or $Q_\lambda(x;t)$ with power-sum bases

$$\left\{p_\rho(x) = p_1^{\rho_1} p_2^{\rho_2} \ldots . p_n^{\rho_n}, \rho = \left(1^{\rho_1} \ldots . n^{\rho_n}\right)\right\}$$

is given by $p_\rho(x) = \sum_\lambda X_\rho^\lambda(t)\ p_\lambda(x;t)$

and $$Q_\lambda(x;t) = \sum_\rho Z_\rho(t)\ X_\rho^\lambda(t)\ p_\rho(x)$$

where $$Z_\rho(t) = \prod_{i\geq 1}\left(1-t^{\rho_i}\right)^{-1} i^{\rho_i} \rho_i 1 \text{ and } X\rho^\lambda(t) \in Z[t].$$

The $X_\rho^\lambda(t)$ are the Green polynomials which one of fundamental interst in the complex character theory of general linear group G-L(n,q). In that case $t = q^{-1}$. We have the polynomials

$$Q_\rho^\lambda(q) = q^{n(\lambda)}\ X_\rho^\lambda(q^{-1}).$$

Green [2] has calculated a complete set of polynomials $Q_\rho^\lambda(q)$ for n = 2,3,4,5 and table for n = 6,7 are in [5]. We note the following errors in the table for n = 7.

$$Q_{3,1^4}^{3,1^4}(q) = (1+q)\left(1+q+q^2+q^a\right)\left(1+q+q^2+q^3+q^4-5q^5+3q^6\right)$$

$$Q_{421}^{321^4}(q) = (1-q)\left(1+q+q^2-q^6\right)$$

$$Q_{32^2}^{31^4}(q) = (1-q)\left(1+q^2-2q^5-q^7+q^9\right)$$

$$Q_{2^31}^{321^2}(q) = (1-q)\left(1+q+3q^2+2q^3+4q^4-q^6\right)$$

$$Q_{43}^{32^2}(q) = (1-q)\left(1-q^2+q^3+q^4-q^5\right)$$

we have also constructed the matrix X(t) for n = 8,9. We include the table only for n = 8.

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Table of X(t) for n=8

	1^8	21^6	$2^2,1^4$	$2^3,1^2$
8	1	1	1	1
71	$t+7$	$t+5$	$t+3$	$t+1$
62	$t^2+7t+20$	$t^2+5t+10$	t^2+t+2	t^2+3t+4
61^2	$t^3+7t^2+27t+21$	$t^3+5t^2+15t+9$	t^3+3t^2+7t+1	t^3+t^2+3t-3
53	$t^3+7t^2+20t+28$	$t^3+5t^2+10t+10$	t^3+3t^2+4t+4	t^3+t^2+2t+2
521	$t^4+7t^3+27t^2+69t+64$	$t^4+5t^3+15t^2+29t+16$	$t^4+3t^3+7t^2+9t$	$t^4+t^3+3t^2+t$
51^3	$t^6+7t^5+27t^4+76t^3+105t^2+85t+35$	$t^6+5t^5+15t^4+34t^3+35t^2+25t+5$	$t^6+3t^5+7t^4+12t^3+5t^2+t-5$	$t^6+t^5+3t^4+2t^3-t^2-3t-3$
4^2	$t^4+7t^3+20t^2+28t+14$	$t^4+5t^3+10t^2+10t+4$	$t^4+3t^3+4t^2+4t+2$	$t^4+t^3+2t^2+2t$
431	$t^5+7t^4+27t^3+69t^2+106t+70$	$t^5+5t^4+15t^3+29t^2+30t+10$	$t^5+3t^4+7t^3+9t^2+6t+2$	$t^5+t^4+3t^3+t^2+2t-2$
42^2	$t^6+7t^5+27t^4+69t^3+126t^2+134t+56$	$t^6+5t^5+15t^4+29t^3+40t^2+26t+4$	$t^6+3t^5+7t^4+9t^3+10t^2+2t$	$t^6+t^5+3t^4+t^3+4t^2-2t+4$
421^2	$t^7+7t^6+27t^5+76t^4+167t^3+247t^2+225t+90$	$t^7+5t^6+15t^5+34t^4+59t^3+61t^2+35t$	$t^7+3t^6+7t^5+12t^4+15t^3+7t^2-3t-6$	$t^7+t^6+3t^5+2t^4+3t^3-3t^2-t$
41^4	$t^{10}+7t^9+27t^8+76t^7+174t^6+288t^5+358t^4+344t^3+245t^2+125t+35$	$t^{10}+5t^9+15t^8+34t^7+64t^6+80t^5+80t^4+56t^3+25t^2+5t-5$	$t^{10}+3t^9+7t^8+12t^7+18t^6+12t^5+6t^4-8t^3-11t^2-11t-5$	$t^{10}+t^9+3t^8+2t^7+4t^6-4t^5-8t^3+t^2-3t+3$

321^3	$t^{11}-t^{10}+3t^9-4t^8+6t^7-8t^6+6t^5-8t^4+5t^3-3t^2+3t$	$t^{11}+4t^{10}+9t^9+16t^8+24t^7+27t^6+15t^5-2t^4-15t^3-19t^2-4t+4$	$t^{11}+2t^{10}+3t^9+4t^8+4t^7+t^6-3t^5-4t^4-5t^3-t^2+2t+2$	$t^{11}+t^9-t^6-t^5-2+t^4+t^3+t^2$	$t^{11}+t^{10}+t^8-3t^6+t^4-3t^3+2t^2+2t-2$	$t^{11}-t^{10}+t^8-2t^7+t^6-t^4+t^3+2t^2-4t+2$
31^5	$t^{15}-t^{14}+3t^{13}-4t^{12}+6t^{11}-9t^{10}+7t^9-11t^8+9t^7-9t^6+11t^5-6t^4+8t^3-5t^2+3t-3$	$t^{15}+4t^{14}+9t^{13}+16t^{12}+24t^{11}+31t^{10}+26t^9+14t^8+4t^7-9t^6-14t^5-6t^4+t^3+4t^2+9t+6$	$t^{15}+2t^{14}+3t^{13}+4t^{12}+4t^{11}+3t^{10}-2t9-4t^8-6t^7-7t^6-4t^5+t^3+2t^2+3t$	$t^{15}+t^{13}-t^{10}-2t^9-2t^8-t^6+2t^5+2t^4+t^3+t-2$	$t^{15}+t^{14}-t^{12}-2t^{10}-t^9-t^8-2t^7+t^5+t^3+t^2$	$t^{15}-t^{14}+t^{12}-2t^{11}+t^9-t^8-2t^6-t^5+t^3+t^2$
2^4	$t^{12}-t^{11}+3t^{10}-4t^9+7t^8-9t^7+13t^6-12t^5+16t^4-8t^3+14t^2-2t+6$	$t^{12}+t^{11}+9t^{10}+16t^9+20t^8+16t^7+4t^6-11t^5-20t^4-20t^3-13t^2-5t-1$	$t^{12}+2t^{11}+3t^{10}+4t^9+2t^8-2t^6-3t^5-2t^4-2t^3-t^2-t-1$	$t^{12}+t^{10}+t^5-t^2-t-1$	$t^{12}+t^{11}+t^9-t^8-2t^7+t^6-2t^5-2t^4+t^3-t^2+t+2$	$t^{12}-t^{11}+t^9-t^8+t^6-2t^4+t^3-t^2-t+2$
$2^9\ 1^2$	$t^{19}-t^{12}+3t^{11}-4t^{10}+6t^9-8t^8+10t^7-12t^6+13t^5-11t^4+11t^8-8t^2+4t-4$	$t^{13}+4t^{12}+9t^{11}+16t^{10}+24t^9-27t^8+20t^7+3t^6-20t^5-35t^4-32t^3-16t^2-2t+1$	$t^{13}+2t^{12}+3t^{11}+4t^{10}+4t^9+t^8-2t^7-5t^6-6t^5-3t^4+2t^2-1$	$t^{13}+t^{11}-t^8-t^6+t^4-2t+1$	$t^{13}+t^{12}+t^{10}-3t^8-t^7-2t^5+t^4+t^3-t^2+t+1$	$t^{13}-t^{12}+t^{10}-2t^9+t^8+t^7-2t^6+3t^4-3t^3-t^2+3t-1$
2^21^4	$t^{16}-t^{15}+3t^{14}-4t^{13}+6t^{12}-9t^{11}+11t^{10}-15t^9+5t^8-21t^7+23t^6-22t^5+20t^4-17t^3+11t^2+89t+4$	$t^{16}+4t^{15}+9t^{14}+16t^{13}+24t^{12}+31t^{11}+31t^{10}+19t^9+t^8-24t^7-44t^6-45t^5-29t^4+11t^3+4t^2-5t+5$	$t^{16}+2t^{15}+3t^{14}+4t^{13}+4t^{12}+3t^{11}-t^{10}-5t^9-9t^8-8t^7-6t^6+3t^4+3t^3+4t^2+17t-1$	$t^{16}+t^{14}-t^{11}-t^{10}-t^9+t^8+2t^5-t^4+t^3-17t+1$	$t^{16}+t^{15}+t^{13}-2t^{11}-2t^{10}-2t^9-5t^8+t^6+2t^5+t^4+t^3+t^2+t+1$	$t^{16}-t^{15}+t^{13}-2t^{12}+2t^{10}-3t^8+4t^7-3t^6+3t^4-3t^3+t^2-31t-1$
21^6	$t^{21}-t^{20}+3t^{19}-4t^{18}+6t^{17}-9t^{16}+10t^{15}-14t^{14}+14t^{13}-17t^{12}+17t^{11}-17t^{10}+17t^9-14t^8+14t^7-10t^6+9t^5-6t^4+4t^3-3t^2+t-1$	$t^{21}+4t^{20}+9t^{19}+16t^{18}+24t^{17}+31t^{16}+35t^{15}+30t^{14}+15t^{13}-5t^{12}-28t^{11}-47t^{10}-55t^9-50t^8-35t^7-15^6+4t^5+16t^4+194t^3-16t^2+11t+4$	$t^{21}+2t^{20}+3t^{19}+4t^{18}+4t^{17}+3t^{16}+t^{15}-4t^{14}-7t^{13}-9t^{12}-10t^{11}-7t^{10}-3t^9+2t^8+5t^7+7t^6+6t^5+4t^4+t^3-t-2$	$t^{21}+t^{19}-t^{16}-t^{15}-2t^{14}-t^{13}-t^{12}+t^{10}+t^9+2t^8+t^7+t^6-t^3-t$	$t^{21}+t^{20}+t^{18}-2t^{16}-t^{15}-3t^{14}-3t^{13}+t^{12}-t^{11}+t^{10}+5^{t9}+t^8+t^7+3t^6-2t^5-2t^4+t^3-2t^2-t+1$	$t^{21}-t^{20}+t^{18}-2t^{17}+t^{15}-t^{14}-t^{13}+3t^{12}-t^{11}-t^{10}+3^{t9}-t^8-t^7+t^6-2t^4+t^3-t+1$

	2^4	$3,1^5$	321^8	$32^2,1$	$3^2,1_2$	$3^2,2$
8	1	1	1	1	1	1
71	$t-1$	$t+4$	$t+2$	t	$t+1$	$t-1$
62	t^2-t+4	t^2+4t+5	t^2+2t+1	t^2+1	t^2+t-1	t^2-t+1
61^2	t^3-t^2+3t-3	t^3-4t^2+9t+6	t^3+2t^2+3t	t^3+t-2	t^3+t^2	t^3-t^2
53	t^3-t^2+4t-4	t^3+4t^2+5t+1	t^3+2t^2+t+1	t^3+t+1	t^3+t^2-t+1	t^3-t^2+t+1
521	$t^4-t^3+3t^2-3t$	$t^4+4t^3+9t^2+12t+4$	$t^4+2t^3+3t^2+2t-2$	t^4+t^2	t^4+t^3-2	t^4-t^3+2t-2
51^3	$t^6-t^5+3t^4-4t^3+t^2-3t+3$	$t^6+4t^5+9t^4+16t^3+15t^2+10t+5$	$t^6+2t^5+3t^4+4t^3-t^2-2t-1$	$t^6+t^4-t^2-2t+1$	$t^6+t^5+t^3-3t^2-2t+2$	$t^6-t^5+t^3-t^2-2t+2$
4^2	$t^4-t^3+4t^2-4t+6$	$t^4+4t^3+5t^2+t-1$	$t^4+2t^3+t^2+t+1$	t^4+t^2+t-1	$t^4+t^3-t^2+t+2$	$t^4-t^3+t^2+t-2$
431	$t^5-t^4+3t^3-3t^2+2t-2$	$t^5+4t^4+9t^3+12t^2+4t-5$	$t^5+2t^4+3t^3+2t^2+1$	t^5+t^3-1	t^5+t^4+t+1	$t^5-t^4+2t^2-3t+1$
42^2	$t^6-t^5+3t^4-3t^3+6t^2-2t+8$	$t^6+4t^5+9t^4+12t^3+9t^2-t-4$	$t^6+2t^5+3t^4+2t^3+t^2-t-2$	$t^6+t^4+t^2-t$	t^6+t^5-t-1	$t^6-t^5+2t^3-2t^2-t+1$
421^2	$t^7-t^6+3t^5-4t^4+7t^3-9t^2+9t-6$	$t^7+4t^5+9t^4+16t^3+20t^2+10t$	$t^7+2t^6+3t^5+4t^4+2t^3-2t^2-4t$	$t^7+t^5-2t^2$	$t^7+t^6+t^4-t^3-2t^2$	$t^7-t^6+t^4-t^3-2t^2+2t$
41^4	$t^{10}-t^9+3t^8-4t^7+6t^6-8t^5+6t^4-8t^3+5t^2-3t+3$	$t^{10}+4t^9+9t^8+16t^7+16t^7+24t^6+21t^5+16t^4+14t^3+5t^2+5t+5$	$t^{10}+2t^9+3t^8+4t^7+4t^6-t^5-4t^4-4t^3-5t^2-t+1$	$t^{10}+t^8-3t^5-2t^3+t^2+t+1$	$t^{10}+t^8+t^7-3t^5-2t^4-t^3-t^2+2t+2$	$t^{10}-t^9+t^7-2t^6-t^5+2t^4-t^3+t^2+2t-2$
3^2 2	$t^7+t^6+3t^5+3t^4+6t^3-6t^2+6t-6$	$t^7+4t^6+9t^5+12t^4-9t^3-9t-6$	$t^7+2t^6+3t^5+2t^4+t^3-1$	$t^7+t^5+t^3-t+2$	t^7+t^6	$t^7-t^6+2t^4-2t^3+2t$
3^2 1^2	$t^8-t^7+3t^6-4t^5+7t^4-13t^3+13t^2-14t+8$	$t^8+4t^7+9t^6+16t^5+20t^4+11t^3-6t^2-11t-4$	$t^8+2t^7+3t^6+4t^5+2t^4-t^3-2t^2+t+2$	$t^8+t^6-t^3-2t^2+t$	$t^8+t^7+t^5-t^4-t^3+3t^2+t-1$	$t^8-t^7+t^5-t^4-t^3+t^2+t-1$
32^2 1	$t^9-t^8+3t^7-4t^6+7t^5-9t^4+9t^3-8t^2+4t-2$	$t^9+4t^8+9t^7+16t^6+20t^5+16t^4-t^3-16t^2-14t-5$	$t^9+2t^8+3t^7+4t^6+2t^5-3t^3-2t^2-1$	$t^9+t^7-t^3+2t-1$	$t^9+t^8+t^6-t^5-2t^4+2t^3-t^2-2t+1$	$t^9-t^8+t^6-t^5+t^2-1$

2^21^4	$t^{16}+7t^{15}+27t^{14}+76t^{13}+174t^{12}+343t^{11}+595t^{10}+913t^9+1213t^8+1491t^7+1567t^6+1418t^5+1084t^4+679t^3+331t^2+337t+20$	$t^{16}+5t^{15}+15t^{14}+34t^{13}+64t^{12}+105t^{11}+149t^{10}+181t^9+177t^8+151t^7+81t^6-60t^4-81t^3-65t^2+29t-10$	$t^{16}+3t^{15}+7t^{14}+12t^{13}+18t^{12}+23t^{11}+23t^{10}+17t^9+t^8-9t^7-21t^6-22t^5-16t^4-5t^3+3t^2+37t+4$	$t^{16}+t^{15}+3t^{14}+2t^{13}+4t^{12}+t^{11}+t^{10}-3t^9-3t^8-5t^7-3t^6+3t^3-t^2+t-2$
21^6	$t^{21}+7t^{20}+27t^{19}+76t^{18}+174t^{17}+343t^{16}+602t^{15}+954t^{14}+1374t^{13}+1807t^{12}+2177t^{11}+2407t^{10}+2441t^9+2266t^8+1918t^7+1470t^6+1009t^5+610t^4+316t^3+133t^2+41t+7$	$t^{21}+5t^{20}+15t^{19}+34t^{18}+64t^{17}+105t^{16}+154t^{15}+200t^{14}+230t^{13}+231t^{12}+197t^{11}+131t^{10}+45t^9-40t^8-106t^7-140t^6-141t^5-116t^4-80t^3-45t^2-19t-5$	$t^{21}+3t^{20}+7t^{19}+12t^{18}+18t^{17}+23t^{16}+26t^{15}+22t^{14}+14t^{13}-t^{12}-15t^{11}-29t^{10}-35t^9-34t^8-26t^7-14t^6-3t^5+6t^4+8t^3+9t^2+5t+3$	$t^{21}+t^{20}+3t^{19}+2t^{18}+4t^{17}+t^{16}+2t^{15}-4t^{14}-2t^{13}-9t^{12}-3t^{11}-9t^{10}+t^9-4t^8+6t^7+7t^5+4t^3-t^2+t-1$
1^3	$t^{28}+7t^{27}+27t^{26}+76t^{25}+174t^{24}+343t^{23}+602t^{22}+961t^{21}+1415t^{20}+1940t^{19}+2493t^{18}+3017t^{17}+3450t^{16}+3736t^{15}+3836t^{14}+3736t^{13}+3450t^{12}+3017t^{11}+2493t^{10}+1940t^9+1415t^8+961t^7+602t^6+343t^5+174t^4+76t^3+27t^2+7t+1$	$t^{28}+5t^{27}+15t^{26}+34t^{25}+64t^{24}+105t^{23}+154t^{22}+205t^{21}+249t^{20}+276t^{19}+277t^{18}+247t^{17}+186t^{16}+100t^{15}-100t^{13}-186t^{12}-247t^{11}-277t^{10}+276t^9-249t^8-205t^7-154t^6-105t^5-64t^4-34t^3-15t^2-5t-1$	$t^{28}+3t^{27}+7t^{26}+12t^{25}+18t^{24}+23t^{23}+26t^{22}+25t^{21}+19t^{20}+8t^{19}-7t^{18}-23t^{17}-38t^{16}-48t^{15}-52t^{14}-48t^{13}-38t^{12}-23t^{11}-7t^{10}+8t^9+19t^8+25t^7+26t^6+23t^5+18t^4+12t^3+7t^2+3t+1$	$t^{28}+t^{27}+3t^{26}+2t^{25}+4t^{24}+t^{23}+2t^{22}-3t^{21}-3t^{20}-8t^{19}-7t^{18}-9t^{17}-6t^{16}-4t^{15}+4t^{13}+6t^{12}+9t^{11}+7t^{10}+8t^9+3t^8+3t^7-2t^6-t^5-4t^4-2t^3-3t^2-t-1$

41^4	421^2	42^2	431	4^2	51^3
1	1	1	1	1	1
$t+3$	$t+1$	$t-1$	t	$t-1$	$t+2$
t^2+3t+2	t^2+t	t^2-t+2	t^2-1	t^2-t	t^2+2t
t^3+3t^2+5t+3	t^3+t^2+t-1	t^3-t^2+t-1	t^3-t	t^3-t^2-t+1	t^3+2t^2+2t+1
t^3+3t^2+2t-2	t^3+t^2	t^3-t^2+2t-2	t^3-t+1	t^3-t^2	t^3+2t^2-2
$t^4+3t^3+5t^2+3t$	$t^4+t^3+t^2-t$	$t^4-t^3+t^2-t$	t^4-t^2	$t^4-t^3-t^2+t$	$t^4+2t^3+2t^2-t-1$
$t^6+3t^5+5t^4+6t^3+5t^2+3t+1$	$t^6+t^5+t^4-t^2-t-1$	$t^6-t^5+t^4-2t^3+t^2-t+1$	$t^6-t^4-t^2+1$	$t^6-t^5-t^4+t^2+t-1$	$t^6+2t^5+2t^4+t^3$
$t^4+3t^3+2t^2-2t-2$	t^4+t^3	$t^4-t^3+2t^2-2t+2$	t^4-t^2+t+1	t^4-t^3+2	t^4+2t^3-2t-1
$t^5+3t^4+5t^3+3t^2-4t-4$	$t^5+t^4+t^3-t^2$	$t^5-t^4+t^3-t^2$	t^5-t^3+2t-1	$t^5-t^4-t^3+t^2+2t-2$	$t^5+2t^4+2t^3-t^2-4t$
$t^6-3t^5+5t^4-3t^3-2t^2-4t$	$t^6+t^5+t^4-t^3$	$t^6-t^5+t^4-t^3+2t^2$	$t^6-t^4+t^2-t$	$t^6-t^5-t^4+t^3+2t^2-2t$	$t^6+2t^5+2t^4-t^3-4t^2-t+1$
$t^7+3t^6+5t^5+6t^4+3t^3-3t^2-3t$	$t^7+t^6+t^5-t^3-t^2-t+2$	$t^7-t^6+t^5-2t^4+3t^3-3t^2+t$	t^7-t^5	$t^7-t^6-t^5+3t^3-t^2-3t+2$	$t^7+2t^6+2t^5+t^4-3t^3-3t^2$
$t^{10}+3t^9+5t^8+6t^7+6t^6+2t^5+t^2+t-1$	$t^{10}+t^9+t^8-2t^5-2t^4+t^2+t-1$	$t^{10}-t^9+t^8-2t^7+2t^6-2t^5+t^2+t-1$	$t^{10}-t^8-t^5+t^2+t-1$	$t^{10}-t^9-t^8+2t^6-2t^4+t^2+t-1$	$t^{10}+2t^9+2t^8+t^7-t^6-2t^5-2t^4+t^3$
$t^7+3t^6+5t^5+3t^4-2t^3-6t^2-4t$	$t^7+t^6+t^5-t^4-2$	$t^7-t^6+t^5-t^4+2t^3-2t^2$	$t^7-t^5+t^3-t$	$t^7-t^6-t^5+t^4+2t^3-2t^2-2t+2$	$t^7+2t^6+2t^5-t^4-4t^3-3t^2+t+2$
$t^8+3t^7+5t^6+6t^5+3t^4-5t^3-9t^2-4t$	$t^8+t^7+t^6-t^4-t^3-t^2$	$t^8-t^7+t^6-2t^5+3t^4-5t^3+3t^2$	$t^8-t^6+t^3-t$	$t^8-t^7-t^6+3t^4-t^3-3t^2+2t$	$t^8+2t^7+2t^6+t^5-3t^4-5t^3-t^2+2t+1$
$t^9+3t^8+5t^7+6t^6+3t^5-3t^4-11t^3-8t^2+4$	$t^9+t^8+t^7-t^5-t^4-t^3$	$t^9-t^8+t^7-2t^6+3t^5-3t^4+t^3$	$t^9-t^7+t^3-2t^2+1$	$t^9-t^8-t^7+3t^5-t^4-3t^3+4t-2$	$t^9+2t^8+2t^7+t^6-3t^5-5t^4-4t^3+2t^2+4t$

1^8	$t^{28}-t^{27}+3t^{26}-4t^{25}+6t^{24}-9t^{23}+10t^{22}-15t^{21}+15t^{20}-20t^{19}+21t^{18}-23t^{17}+26t^{16}-24t^{15}+28t^{14}-24t^{13}+26t^{12}-23t^{11}+21t^{10}-20t^9+15t^8-15t^7+10t^6-9t^5+6t^4-4t^3+3t^2-t+1$	$t^{28}+4t^{27}+9t^{26}+16t^{25}+24t^{24}+31t^{23}+35t^{22}+34t^{21}+26t^{20}+11t^{19}-9t^{18}-31t^{17}-51t^{16}-65t^{15}-70t^{14}-65t^{13}-51t^{12}-31t^{11}-9t^{10}+11t^9+26t^8+34t^7+35t^6+31^5+24t^4+16t^3+9t^2+4t+1$	$t^{28}+2t^{27}+3t^{26}+4t^{25}+4t^{24}+3t^{23}+t^{22}-2t^{21}-6t^{20}-9t^{19}-11t^{18}-11t^{17}-9t^{16}-5t^{15}+5t^{13}+9t^{13}+11t^{11}+11t^{10}+9t^9+6t^8+2t^7-t^6-3t^5-4t^4-t^3-3t^2-2t-1$	$t^{28}+t^{26}-t^{23}-t^{22}-2t^{21}-2t^{20}-t^{19}-t^{18}+t^{17}+t^{16}+3t^{15}+2t^{14}+3t^{13}+t^{12}+t^{11}-t^{10}-t^9-2t^8-2t^7-t^6-t^5+t^2+1$	$t^{28}+t^{27}+t^{25}-2t^{23}-t^{22}-2t^{21}-4t^{20}-t^{19}-t^{17}+3t^{16}+4t^{15}+2t^{14}+4t^{13}+3t^{12}-t^{11}-t^9-4t^8-2t^7-t^6-2t^5+t^3+t+1$	$t^{28}-t^{27}+t^{25}-2t^{24}+t^{22}-2t^{21}+3t^{19}-2t^{18}+t^{17}+3t^{16}-2t^{15}+2t^{13}-3t^{12}-t^{11}+2t^{10}-3t^9+2t^7-t^6+2t^4-t^3+t-1$

521	53	6;1²	62	71^-	8^-
1	1	1	1	1	1
t	$t-1$	$t+1$	$t-1$	t	$t-1$
t^2	t^2-t	t^2+t-1	t^2-t+1	t^2-1	t^2-t
t^3-1	t^3-t^2-t+1	t^3+t^2	t^3-t^2	t^3-t	t^3-t^2-t+1
t^3	t^3-t^2+1	t^3+t^2-t-1	t^3-t^2+t-1	t^3-t	t^3-t^2
t^4-t+1	$t^4-t^3-t^2+2t-1$	t^4+t^3-2t	t^4-t^3	t^4-t^2-t+1	$t^4-t^3+t^2+t$
t^6-t^3	$t^6-t^5-t^4+t^3$	$t^6+t^5-t^3-t^2$	$t^6-t^5-t^3+t^2$	$t^6-t^4-t^3+t$	$t^6-t^5-t^4+t^2+t-1$
t^4-1	$t^4-t^3-t^2-t$	$t^4+t^3-t^2-t$	$t^4-t^3+t^2-t$	t^4-t^2	t^4-t^3
t^5-t^2	$t^5-t^4-t^3+2t^2-t$	$t^5+t^4-2t^2-t+1$	t^5-t^4-t+1	$t^5-t^3-t^2+t$	$t^5-t^4-t^3+t^2$
t^6-t^3+t-1	$t^6-t^5-t^4+2t^3-t^2-t+1$	$t^6+t^5-2t^3-2t^2+t+1$	t^6-t^5+t-1	$t^6-t^4-t^3+t$	$t^6-t^5-t^4+t^3$
$t^7-t^4-t^3+t^2$	$t^7-t^6-t^5+t^4$	$t^7+t^6-t^4-3t^3+2t$	$t^7-t^6-t^4+t^3$	$t^7-t^5-t^4-t^3+2t^2+t-1$	$t^7-t^6-t^5+t^3+t^2-t$
$t^{10}-t^7-t^6+t^3$	$t^{10}-t^9-t^8+t^7-t^6+t^5+t^4-t^3$	$t^{10}+t^9-t^7-2t^6-t^5+t^3+t^2$	$t^{10}-t^9-t^7+t^5+t^3-t^2$	$t^{10}-t^8-t^7-t^6+t^5+t^4+t^3-t$	$t^{10}-t^9-t^8+2t^5-t^2-t+1$
$t^7-t^4+t^2-t$	$t^7-t^6-t^5+2t^4-t^3+t-1$	$t^7+t^6-2t^4-2t^3+2t$	t^7-t^6	$t^7-t^5-t^4+t^2$	$t^7-t^6-t^5+1$
$t^8-t^5-t^4+t^3-t^2+1$	$t^8-t^7-t^6+t^5+t^3-t^2-t+1$	$t^8+t^7-t^5-3t^4-t^3+3t^2+t-1$	$t^8-t^7-t^5+t^4-t^3+t^2+t-1$	$t^8-t^6-t^5-t^4+2t^3+t^2-t$	$t^8-t^7-t^6+t^4+t^3-t^2$
$t^9-t^6-t^5+t^4$	$t^9-t^8-t^7+t^6+t^4-t^3-t^2+t$	$t^9+t^8-t^6-3t^5-$ $2t^4+2t^3+3t^2-1$	$t^9-t^8-t^6+t^5+t^2-2t+1$	$t^9-t^7-t^6-t^5+t^4+2t^3-t$	$t^9-t^8-t^7+t^5+t^4-t^3$
$t^{11}-t^8-t^3+t$	$t^{11}-t^{10}-t^9+t^8-t^7+2t^6-$ $2t^4+t^2+t-1$	$t^{11}+t^{10}-t^8-2t^7-$ $3t^6+3t^4+3t^3-2t$	$t^{11}-t^{10}-t^8+t^6+t^4-t^3$	$t^{11}-t^9-t^8-t^7+2t^5+2t^4-2t^2-$ $t+1$	$t^{11}-t^{10}-t^9+2t^6-t^3-t^2+t$
$t^{15}-t^{12}-t^{11}-t^9+t^7+$ $t^6+2t^5-t^3-t^2-t+1$	$t^{15}-t^{14}-t^{13}+t^{12}-t^{11}+$ $t^{10}+t^9-t^8-t^7+t^6+t^5-$ $t^4+t^3-t^2-t+1$	$t^{15}+t^{14}-t^{12}-2t^{11}-2t^{10}-$ $t^9+t^8+2t^7+2t^6+t^5-t^3-t^2$	$t^{15}-t^{14}-t^{12}+t^9+t^8-t^5-$ t^3+t^2	$t^{15}-t^{13}-t^{12}-t^{11}+t^9+$ $2t^8+t^7-t^5-t^4-t^3+t$	$t^{15}-t^{14}-t^{13}+t^{10}+t^9+t^8-t^7-$ $t^6-t^5+t^2+t-1$

$3^2 2$	$t^7+7t^6+27t^5+69t^4+126t^3+162t^2+126t+42$	$t^7+5t^6+15t^5+29t^4+40t^3+36t^2+14t$	$t^7+3t^6+7t^5+9t^4+10t^3+6t^2+2t+2$	$t^7+t^6+3t^5+t^4+4t^3+2t$
$3^2 1^2$	$t^8+7t^7+27t^6+76t^5+167t^4+275t^3+309t^2+202t+56$	$t^8+5t^7+15t^6+34t^5+59t^4+71t^3+49t^2+10t-4$	$t^8+3t^7+7t^6+12t^5+15t^4+11t^3+t^2-2t$	$t^8+t^7+3t^6+2t^5+3t^4-t^3-3t^2-2t-4$
$3 2^2 1$	$t^9+7t^8+27t^7+76t^6+167t^5+295t^4+401t^3+392t^2+244t+70$	$t^9+5t^8+15t^7+34t^6+59t^5+81t^4+75t^3+40t^2-10$	$t^9+3t^8+7t^7+12t^6+15t^5+15t^4+5t^3-4t+2$	$t^9+t^8+3t^7+2t^6+3t^5+t^4-t^3+2$
$3 2 1^3$	$t^{11}+7t^{10}+27t^9+76t^8+174t^7+336t^6+534t^5+688t^4+693t^3+509t^2+251t+64$	$t^{11}+5t^{10}+15t^9+34t^8+64t^7+100t^6+120t^5+110t^4+61t^3+5t^2-19t-16$	$t^{11}+3t^{10}+7t^9+12t^8+18t^7+20t^6+14t^5+4t^4-11t^3-11t^2-9t$	$t^{11}+t^{10}+3t^9+2t^8+4t^7-6t^4-3t^3-3t^2+t$
$3 1^5$	$t^{15}+7t^{14}+27t^{13}+76t^{12}+174t^{11}+343t^{10}+575t^9+821t^8+1009t^7+1071t^6+979t^5+762t^4+496t^3+259t^2+99t+21$	$t^{15}+5t^{14}+15t^{13}+34t^{12}+64t^{11}+105t^{10}+139t^9+155t^8+141t^7+101t^6+47t^5-26t^3-31t^2-21t-9$	$t^{15}+3t^{14}+7t^{13}+12t^{12}+18t^{11}+23t^{10}+19t^9+13t^8-3t^7-13t^6-25t^5-22t^4-20t^3-9t^2-5t+1$	$t^{15}+t^{14}+3t^{13}+2t^{12}+4t^{11}+t^{10}-t^9-5t^8-7t^7-7t^6-5t^5+2t^3+5t^2+3t+3$
2^4	$t^{12}+7t^{11}+27t^{10}+76t^9+167t^8+295t^7+421t^6+484t^5+448t^4+328t^3+182t^2+70t+14$	$t^{12}+5t^{11}+15t^{10}+34t^9+59t^8+81t^7+85t^6+66t^5+34t^4+4t^3-10t^2-10t-4$	$t^{12}+3t^{11}+7t^{10}+12t^9+15t^8+15t^7+9t^6+4t^5+2t^2+2t+2$	$t^{12}+t^{11}+3t^{10}+2t^9+3t^8+t^7+t^6+2t^5+2t^4+4t^3+2t^2+2t$
$2^3 1^2$	$t^{13}+7t^{12}+27t^{11}+76t^{10}+174t^9+336t^8+554t^7+780t^6+925t^5+901t^4+691t^3+392t^2+148t+28$	$t^{13}+5t^{12}+15t^{11}+34t^{10}+64t^9+100t^8+130t^7+136t^6+105t^5+45t^4-15t^3-40t^2-30t-10$	$t^{13}+3t^{12}+7t^{11}+12t^{10}+18t^9+20t^8+18t^7+8t^6-3t^5-11t^4-9t^3+4t+4$	$t^{13}+t^{12}+3t^{11}+2t^{10}+4t^9+2t^7-4t^6+t^5-3t^4+5t^3+2t-2$

$t^{11}+3t^{10}+5t^9+6t^8+6t^7+2t^6-8t^5-12t^4-7t^3+t^2+3t$	$t^{11}+t^{10}+t^9-2t^6-2t^5-2t^4+t^3+t^2+t$	$t^{11}-t^{10}+t^9-2t^8+2t^7-2t^6+t^3+t^2-t$	$t^{11}-t^9-t^6+t^5-t^3+t^2$	$t^{11}-t^{10}-t^9+2t^7-2t^5+t^3+t^2-t$	$t^{11}+2t^{10}+2t^9+t^8-t^7-4t^6-6t^5-2t^4+3t^3+4t^2+t-1$
$t^{15}+3t^{14}+5t^{13}-6t^{12}+6t^{11}+5t^{10}-3t^9-9t^8-7t^7-5t^6-3t^5+4t^3+3t^2-t-3$	$t^{15}+t^{14}+t^{13}-t^{10}-3t^9-3t^8-t^7+t^6+t^5+2t^4+2t^3+t^2-t-1$	$t^{15}-t^{14}+t^{13}-2t^{12}+2t^{11}-3t^{10}+t^9-t^8+3t^7-t^6+t^5-t^2-t+1$	$t^{15}-t^{13}-t^{10}+t^6+t^3-t$	$t^{15}-t^{14}-t^{13}+2t^{11}-t^{10}-t^9+t^8+t^7-t^6-t^5+2t^4-t^2-t+1$	$t^{15}+2t^{14}+2t^{13}+t^{12}-t^{11}-2t^{10}-5t^9-4t^8-t^7+t^6+4t^5+2t^4+t^3-t^2-t+1$
$t^{12}+3t^{11}+5t^{10}+6t^9+3t^8-3t^7-9t^6-10t^5-6t^4+4t^2+4t+2$	$t^{12}+t^{11}+t^{10}-t^8-t^7-t^6$	$t^{12}-t^{11}+t^{10}-2t^9+3t^8-3t^7+3t^6-2t^5+2t^4-2$	$t^{12}-t^{10}-t^5+t^2+t-1$	$t^{12}-t^{11}-t^{10}+3t^8-t^7-3t^6+4t^4-2t^2-2t+2$	$t^{12}+2t^{11}+2t^{10}+t^9-3t^8-5t^7-4t^6-t^5+3t^4+3t^3+2t^2-1$
$t^{13}+3t^{12}+5t^{11}+6t^{10}+6t^9+2t^8-6t^7-14t^6-15t^5-7t^4+3t^3+8t^2+6t+2$	$t^{13}+t^{12}+t^{11}-2t^8-2t^7-2t^6+t^5+t^4+t^3$	$t^{13}-t^{12}+t^{11}-2t^{10}+2t^9-2t^8+2t^7-2t^6+t^5+t^4-t^3-2t+2$	$t^{13}-t^{11}-t^8+t^6-t^4+2t^2-1$	$t^{13}-t^{12}-t^{11}+2t^9-2t^7+t^5+t^4-t^3$	$t^{13}+2t^{12}+2t^{11}+t^{10}-t^9-4t^8-6t^7-5t^6+6t^4+6t^3+2t^2-2t-2$
$t^{16}+3t^{15}+5t^{14}+6t^{13}+6t^{12}+5t^{11}-t^{10}-11t^9-13t^8-15t^7-9t^6+10t^4+13t^3+7t^2-33t-2$	$t^{16}+t^{15}+t^{14}-t^{11}-3t^{10}-3t^9-t^8+t^7+t^6+2t^5+2t^4+t^3-t^2-t$	$t^{16}-t^{15}+t^{14}-2t^{13}+2t^{12}-3t^{11}+3t^{10}-3t^9-t^8-3t^7+3t^6-2t^4+t^3-t^2+35t-2$	$t^{16}-t^{14}-t^{11}-t^{10}+t^9-t^8+t^4+t^3-2t^2+15t+1$	$t^{16}-t^{15}-t^{14}+2t^{12}-t^{11}-t^{10}+t^9-3t^8-t^7-t^6+2t^5-t^3-t^2+33t$	$t^{16}+2t^{15}+2t^{14}+t^{13}-t^{12}-2t^{11}-5t^{10}-7t^9-2t^8+t^7+7t^6+8t^5+4t^4-t^3-4t^2-18t$
$t^{21}+3t^{20}+5t^{19}+6t^{18}+6t^{17}+5t^{16}+2t^{15}-6t^{14}-14t^{13}-17t^{12}-15t^{11}-11t^{10}-5t^9+6t^8+14t^7-14t^6+9t^5+6t^4+2t^3-3t^2-5t-3$	$t^{21}+t^{20}+t^{19}-t^{16}-2t^{15}-4t^{14}-2t^{13}-t^{12}+t^{11}+t^{10}+3t^9+4t^8+2t^7-t^5-2t^3-t^2-t+1$	$t^{21}-t^{20}+t^{19}-2t^{18}+2t^{17}-3t^{16}+2t^{15}-2t^{14}+2t^{13}-t^{12}+t^{11}+t^{10}-t^9+2t^8-2t^7+2t^6-3t^5+2t^4-2t^3+t^2-t+1$	$t^{21}-t^{19}-t^{16}-t^{15}+t^{13}+t^{12}+t^{10}+t^9-t^7-t^6-t^3+t$	$t^{21}-t^{20}-t^{19}+2t^{17}-t^{16}-2t^{15}+2t^{14}+2t^{13}-t^{12}-3t^{11}+3t^{10}+t^9-2t^8-2t^7+2t^6+t^5-2t^4+t^2+t-1$	$t^{21}+2t^{20}+2t^{19}+t^{18}-t^{17}-2t^{16}-3t^{15}-6t^{14}-6t^{13}-3t^{12}+2t^{11}+7t^{10}+6t^9+6t^8+3t^7-t^5-5t^4-4t^3-2t^2+t+2$
$t^{28}+3t^{27}+5t^{26}+6t^{25}+6t^{24}+5t^{23}+2t^{22}-3t^{21}-9t^{20}-14t^{19}-17t^{18}-17t^{17}-14t^{16}-8t^{15}+8t^{13}+14t^{12}+17t^{11}+17t^{10}+14t^9+9t^8+3t^7-2t^6-5t^5-6t^4-6t^3-5t^2-3t-1$	$t^{28}+t^{27}+t^{26}-t^{23}-2t^{22}-3t^{21}-3t^{20}-2t^{19}-t^{18}+t^{17}+2t^{16}+4t^{15}+4t^{14}+4t^{13}+2t^{12}+t^{11}-t^{10}-2t^9-3t^8-3t^7-2t^6-t^5+t^2+t+1$	$t^{28}-t^{27}+t^{26}-2t^{25}+2t^{24}-3t^{23}+2t^{22}-3t^{21}+3t^{20}-2t^{19}+3t^{18}-t^{17}+2t^{16}-2t^{12}+t^{11}-3t^{10}+2t^9-3t^8+3t^7-2t^6+3t^5-2t^4+2t^3-t^2+t-1$	$t^{28}-t^{26}-t^{23}-t^{22}+t^{19}+t^{18}+t^{17}+t^{16}+t^{15}-t^{13}-t^{12}-t^{11}-t^{10}-t^9+t^6+t^5+t^2-1$	$t^{28}-t^{27}-t^{26}+2t^{24}-t^{23}-2t^{22}+2t^{21}+3t^{20}-3t^{18}+t^{17}+2t^{16}-4t^{14}+2t^{12}+t^{11}-3t^{10}+3t^8+t^7-2t^6-t^5+2t^4-t^2-t+1$	$t^{28}+2t^{27}+2t^{26}+t^{25}-t^{24}-2t^{23}-3t^{22}-4t^{21}-5t^{20}-5t^{19}-2t^{18}+2t^{17}+5t^{16}+6t^{15}+6t^{14}+6t^{13}+5t^{12}+2t^{11}-2t^{10}-5t^9-5t^8-4t^7-3t^6-2t^5-t^4+t^3+2t^2+2t+1$

$t^{12}-t^9-t^8+t^7+t^5-t^4-t^3+1$	$t^{12}-t^{11}-t^{10}+t^9+t^7-t^6-t^5+2t^2-1$	$t^{12}+t^{11}-t^9-3t^8-2t^7+t^6+2t^5+2t^4+t^3-t^2-t$	$t^{12}-t^{11}-t^9+t^8+t^6-2t^4+t^3-t^2+t$	$t^{12}-t^{10}-t^9-t^8+t^7+t^6+t^5-t^3$	$t^{12}-t^{11}-t^{10}+t^8+t^7-t^6$
$t^{13}-t^{10}-t^9+t^6$	$t^{13}-t^{12}-t^{11}+t^{10}-t^9+2t^8-2t^6+3t^3-t^2-2t+1$	$t^{13}+t^{12}-t^{10}-2t^9-3t^8-t^7+2t^6+4t^5+3t^4-t^3-t^2-t+1$	$t^{13}-t^{12}-t^{10}+t^8+t^7-2t^5+t^4-t^3+t^2+t-1$	$t^{13}-t^{11}-t^{10}-t^9+t^7+3t^6+t^5-2t^4-2t^3+t$	$t^{13}-t^{12}-t^{11}+2t^8-t^5-t^4+t^3$
$t^{16}-t^{13}-t^{12}-t^{10}+t^9+2t^8+t^7+t^6-t^3-16t$	$t^{16}-t^{15}-t^{14}+t^{13}-t^{12}+t^{11}+t^{10}-t^9+t^8+t^7+t^6-t^5+t^4-t^3-t^2-15t$	$t^{16}+t^{15}-t^{13}-2t^{12}-2t^{11}-2t^{10}+3t^8+4t^7+3t^6-3t^4-3t^3-t^2+t+1$	$t^{16}-t^{15}-t^{13}+2t^{10}-t^8-t^6+2t^5-t^4+t^3-t^2-t+1$	$t^{16}-t^{14}-t^{13}-t^{12}+3t^9+2t^8-t^6-3t^5-t^4+2t^2+t-1$	$t^{16}-t^{15}-t^{14}+t^{11}+t^{10}+t^9-t^8-t^7-t^6+t^3+t^2-t$
$t^{21}-t^{18}-t^{17}-t^{15}+t^{12}+2t^{11}+t^{10}-t^7-t^5-t^4+t$	$t^{21}-t^{20}-t^{19}+t^{18}-t^{17}+t^{16}+2t^{11}-2t^{10}-t^5+t^4-t^3+t^2+t-1$	$t^{21}+t^{20}-t^{18}-2t^{17}-2t^{16}-t^{15}-t^{14}+t^{13}+3t^{12}+3t^{11}+3t^{10}+t^9-t^8-3t^7-3t^6-2t^5+t^3+2t^2+t-1$	$t^{21}-t^{20}-t^{18}+t^{15}+t^{14}-t^{13}+t^{12}-t^{11}+t^{10}-t^9+t^8-t^7-t^6+t^3+t-1$	$t^{21}-t^{19}-t^{18}-t^{17}+2t^{14}+2t^{13}+t^{12}-t^{10}-2t^9-2t^8+t^5+t^4+t^3-t$	$t^{21}-t^{20}-t^{19}+t^{16}+2t^{14}-t^{12}-t^{11}-t^{10}-t^9+2t^7+t^5-t^2-t+1$
$t^{28}-t^{25}-t^{24}-t^{22}-t^{20}+t^{19}+2t^{18}+2t^{17}+t^{16}-t^{12}-2t^{11}-2t^{10}+t^9+t^8+t^6+t^4+t^3-1$	$t^{28}-t^{27}-t^{26}+t^{25}-t^{24}+t^{23}-t^{21}+t^{20}+t^{19}+t^{18}-t^{17}-t^{16}-t^{12}-t^{11}+t^{10}+t^9+t^8-t^7+t^5-t^4+t^3-t^2-t+1$	$t^{28}+t^{27}-t^{25}-2t^{24}-2t^{23}-t^{22}+t^{19}+2t^{18}+3t^{17}+3t^{16}+2t^{15}-2t^{13}-3t^{12}-3t^{11}-2t^{10}-t^9+t^6+2t^5+2t^4+t^3-t-1$	$t^{28}-t^{27}-t^{25}+t^{22}+t^{19}+t^{17}-t^{16}-2t^{14}-t^{12}+t^{11}+t^9+t^6-t^3-t+1$	$t^{28}-t^{26}-t^{25}-t^{24}+2t^{21}+t^{20}+t^{19}+t^{18}-t^{16}-2t^{15}-2t^{13}-t^{12}+t^{10}+t^9+t^8+2t^7-t^4-t^3-t^2+1$	$t^{28}-t^{27}-t^{26}+t^{23}+t^{21}+t^{20}-t^{18}-t^{17}-2t^{16}+2t^{12}+t^{11}+t^{10}-t^8-t^7-t^5+t^2+t+1$

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DEFINING NEW CONSISTENCY RELATIONS FOR SPLINE SOLUTIONS OF TENTH ORDER BOUNDARY-VALUE PROBLEMS

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Dedicated to the memory of Dr. M. Rafique

ABSTRACT: Linear, tenth-order boundary-value problems (special case) are solved, using polynomial splines of degree ten. The spline function values at the midknots of the interpolation interval, and the corresponding values of the even-order derivatives are related through consistency relations. The algorithm developed approximates the solutions, and their higher-order derivatives, of differential equations.
Two numerical illustrations are given to show the practical usefulness of the algorithm developed. It is observed that this algorithm is second-order convergent.

KEYWORDS: Tenth-order,Two-point boundary-value problems; finite-difference methods; Ten-degree splines.

1 INTRODUCTION

When an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability is as ordinary convection the ordinary differential equation is sixth order; when the instability sets in as overstability, it is modelled by an eighth-order ordinary differential equation.

Suppose, now, that a uniform magnetic field is also applied across the fluid in the same direction as gravity. When instability sets in now as ordinary convection, it is modelled by a tenth-order boundary-value problem; when instability sets in as overstability, it is modelled by a twelfth-order boundary-value problem (for details, see Chandrasekhar [4]). Finite difference methods of solution for such problems were developed by Boutayeb and Twizell [1,2,3], Djidjeli *et al.* [5], Twizell [11],Twizell and Boutayeb [12], and Twizell *et al.* [13]. Siddiqi and Twizell developed spline solutions for sixth, eighth, tenth and twelfth order problems in [7,8,9,10], respectively.

Usmani [14], solved fourth-order boundary-value problem using quartic splines.

In the present paper tenth-order boundary-value problems are solved using tenth-degree splines, introducing some new consistency relations.

These problems have the form

$$\left.\begin{array}{l} y^{(x)} + \phi(x)y \;=\; \psi(x)\,, \qquad -\infty < a \le x \le b < \infty\,, \\ y(a) = A_0\,,\; y^{(ii)}(a) = A_2\,,\; y^{(iv)}(a) = A_4\,,\; y^{(vi)}(a) = A_6\,, \\ y^{(viii)}(a) = A_8\,,\; y(b) = B_0\,,\; y^{(ii)}(b) = B_2\,,\; y^{(iv)}(b) = B_4\,, \\ y^{(vi)}(b) = B_6\,,\; y^{(viii)}(b) = B_8\,, \end{array}\right\} \qquad (1.1)$$

where $y = y(x)$, and $\phi(x)$ and $\psi(x)$ are continuous functions defined in the interval $x \in [a,\ b]$. A_i and B_i, $i = 0, 2, 4, 6, 8$, are finite real constants.

2 THE TENTH-DEGREE SPLINES

2.1 Consistency Relations

The interval $[a,\ b]$ is divided into $n \geq 18$ equal parts, thus introducing $n+1$ grid points x_i so that

$$x_i \;=\; a + ih\,, \qquad i = 0, 1, 2, \ldots, n,$$

$$x_0 \;=\; a, \quad x_n \;=\; b \quad \text{and} \quad h \;=\; \frac{b-a}{n}.$$

The exact solution of the problem (1.1) at $x = x_i$ is $y(x_i)$. Let s_i be the approximation to y at x_i determined by the tenth-degree spline defined on the sub-interval $[x_i,\ x_{i+1}]$ by

$$\begin{aligned} Q_i(x) \;=\;& a_i(x-x_i)^{10} + b_i(x-x_i)^9 + c_i(x-x_i)^8 + d_i(x-x_i)^7 \\ &+ e_i(x-x_i)^6 + f_i(x-x_i)^5 + g_i(x-x_i)^4 \end{aligned}$$

$$+ u_i(x-x_i)^3 + v_i(x-x_i)^2 + w_i(x-x_i) + z_i,$$
$$i = 0, 1, \ldots, n-1. \qquad (2.1)$$

The tenth-degree spline $s(x) \in C^9[a,\ b]$ can, thus, be defined as

$$s(x) \ = \ Q_i(x), \quad x \in [x_i,\ x_{i+1}], \quad i = 0, 1, \ldots, n-1\ . \qquad (2.2)$$

The coefficients of (2.1) are determined, (see [9]), as

$$a_i = \frac{1}{3628800} s_{i+1/2}^{(x)}\ , \qquad (2.3)$$

$$b_i = \frac{1}{362880} s_i^{(ix)}\ , \qquad (2.4)$$

$$c_i = \frac{1}{40320} s_{i+1/2}^{(viii)} - \frac{1}{80640} h s_i^{(ix)} - \frac{1}{322560} h^2 s_{i+1/2}^{(x)}\ , \qquad (2.5)$$

$$d_i = \frac{1}{5040} s_i^{(vii)}\ , \qquad (2.6)$$

$$e_i = \frac{1}{720} s_{i+1/2}^{(vi)} - \frac{1}{1440} h s_i^{(vii)} - \frac{1}{5760} h^2 s_{i+1/2}^{(viii)} + \frac{1}{17280} h^3 s_i^{(ix)} + \frac{1}{55296} h^4 s_{i+1/2}^{(x)}\ , \qquad (2.7)$$

$$f_i = \frac{1}{120} s_i^{(v)}\ , \qquad (2.8)$$

$$g_i = \frac{1}{24} s_{i+1/2}^{(iv)} - \frac{1}{48} h s_i^{(v)} - \frac{1}{192} h^2 s_{i+1/2}^{(vi)} + \frac{1}{576} h^3 s_i^{(vii)} + \frac{5}{9216} h^4 s_{i+1/2}^{(viii)} - \frac{1}{5760} h^5 s_i^{(ix)} - \frac{61}{1105920} h^6 s_{i+1/2}^{(x)}\ , \qquad (2.9)$$

$$u_i = \frac{1}{6} s_i'''\ , \qquad (2.10)$$

$$v_i = \frac{1}{2} s_{i+1/2}'' - \frac{1}{4} h s_i''' - \frac{1}{16} h^2 s_{i+1/2}^{(iv)} + \frac{1}{48} h^3 s_i^{(v)} + \frac{5}{768} h^4 s_{i+1/2}^{(vi)} - \frac{1}{480} h^5 s_i^{(vii)} - \frac{61}{92160} h^6 s_{i+1/2}^{(viii)} + \frac{17}{80640} h^7 s_i^{(ix)} + \frac{277}{4128768} h^8 s_{i+1/2}^{(x)}\ , \qquad (2.11)$$

$$w_i = s_i' \qquad (2.12)$$

and

$$z_i = s_{i+1/2} - \frac{1}{2} h s_i' - \frac{1}{8} h^2 s_{i+1/2}'' + \frac{1}{24} h^3 s_i''' + \frac{5}{384} h^4 s_{i+1/2}^{(iv)} - \frac{1}{240} h^5 s_i^{(v)} - \frac{61}{46080} h^6 s_{i+1/2}^{(vi)} + \frac{17}{40320} h^7 s_i^{(vii)} + \frac{277}{2064384} h^8 s_{i+1/2}^{(viii)} - \frac{31}{725760} h^9 s_i^{(ix)}$$

$$- \frac{50521}{3715891200} h^{10} s^{(x)}_{i+1/2} \qquad (2.13)$$

The odd-order derivatives of the splines are calcutated, (see [9]), as

$$hs_i^{(ix)} = (s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) - \tfrac{1}{8}h^2(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) , \qquad (2.14)$$

$$\begin{aligned} hs_i^{(vii)} = & (s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2}) - \tfrac{1}{24}h^2(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) \\ & + \tfrac{1}{384}h^4(s^{(x)}_{i+1/2} - s^{(x)}_{i+1/2}) , \end{aligned} \qquad (2.15)$$

$$\begin{aligned} hs_i^{(v)} = & (s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) - \tfrac{1}{24}h^2(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2}) \\ & + \tfrac{7}{5760}h^4(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) - \tfrac{1}{15360}h^6(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) , \end{aligned} \qquad (2.16)$$

$$\begin{aligned} hs_i''' = & (s''_{i+1/2} - s''_{i-1/2}) - \tfrac{1}{24}h^2(s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) \\ & + \tfrac{7}{5760}h^4(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2}) - \tfrac{31}{967680}h^6(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) \\ & + \tfrac{17}{10321920}h^8(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) \end{aligned} \qquad (2.17)$$

and

$$\begin{aligned} hs_i' = & (s_{i+1/2} - s_{i-1/2}) - \tfrac{1}{24}h^2(s''_{i+1/2} - s''_{i-1/2}) \\ & + \tfrac{7}{5760}h^4(s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) - \tfrac{31}{967680}h^6(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2}) \\ & + \tfrac{127}{154828800}h^8(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) - \tfrac{31}{743178240}h^{10}(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) . \end{aligned} \qquad (2.18)$$

The even-order derivatives of the splines are defined as, see [9]

$$\begin{aligned} h^8 s^{(viii)}_{i-1/2} = & (s_{i-9/2} - 8s_{i-7/2} + 28s_{i-5/2} - 56s_{i-3/2} + 70s_{i-1/2} - 56s_{i+1/2} \\ & + 28s_{i+3/2} - 8s_{i+5/2} + s_{i+7/2}) - \frac{1}{3715891200}h^{10}\Big(s^{(x)}_{i-9/2} \\ & + 59040s^{(x)}_{i-7/2} + 9234220s^{(x)}_{i-5/2} + 196710304s^{(x)}_{i-3/2} \\ & + 826623270s^{(x)}_{i-1/2} + 196710304s^{(x)}_{i+1/2} + 9234220s^{(x)}_{i+3/2} \\ & + 59040s^{(x)}_{i+5/2} + s^{(x)}_{i+7/2}\Big) , \end{aligned} \qquad (2.19)$$

$$i = 5,\ 6, \ldots\ldots, n-4 ,$$

$$\begin{aligned} & (s_{i-11/2} - 2s_{i-9/2} - 19s_{i-7/2} + 104s_{i-5/2} - 238s_{i-3/2} + 308s_{i-1/2} \\ & - 238s_{i+1/2} + 104s_{i+3/2} - 19s_{i+5/2} - 2s_{i+7/2} + s_{i+9/2}) \end{aligned}$$

$$= \frac{1}{464486400}h^8 \Big(s^{(viii)}_{i-11/2} + 59038s^{(viii)}_{i-9/2} + 9116141s^{(viii)}_{i-7/2}$$
$$+ 178300904s^{(viii)}_{i-5/2} + 906923282s^{(viii)}_{i-3/2} + 1527092468s^{(viii)}_{i-1/2}$$
$$+ 906923282s^{(viii)}_{i+1/2} + 178300904s^{(viii)}_{i+3/2} + 9116141s^{(viii)}_{i+5/2}$$
$$+ 59038s^{(viii)}_{i+7/2} + s^{(viii)}_{i+9/2}\Big) \ , \qquad (2.\ 20)$$

$$h^6 s^{(vi)}_{i-1/2} = \frac{1}{4}\big(-s_{i-9/2} + 12s_{i-7/2} - 52s_{i-5/2} + 116s_{i-3/2}$$
$$- 150s_{i-1/2} + 116s_{i+1/2} - 52s_{i+3/2} + 12s_{i+5/2} - s_{i+7/2}\big)$$
$$+ \frac{1}{1486356800}h^{10}\Big(s^{(x)}_{i-9/2} + 59036s^{(x)}_{i-7/2} + 8998052s^{(x)}_{i-5/2}$$
$$+ 159301092s^{(x)}_{i-3/2} + 468393398s^{(x)}_{i-1/2} + 159301092s^{(x)}_{i+1/2}$$
$$+ 8998052s^{(x)}_{i+3/2} + 59036s^{(x)}_{i+5/2} + s^{(x)}_{i+7/2}\Big) \ , \qquad (2.\ 21)$$

$$i = 5,\ 6, \ldots\ldots, n-4 \ ,$$

$$\big(s_{i-11/2} + 70s_{i-9/2} - 211s_{i-7/2} - 184s_{i-5/2} + 1490s_{i-3/2}$$
$$- 2332s_{i-1/2} + 1490s_{i+1/2} - 184s_{i+3/2} - 211s_{i+5/2} + 70s_{i+7/2}$$
$$+ s_{i+9/2}\big)$$
$$= \frac{1}{9676800}h^6\Big(s^{(vi)}_{i-11/2} + 59038s^{(vi)}_{i-9/2} + 9116141s^{(vi)}_{i-7/2}$$
$$+ 178300904s^{(vi)}_{i-5/2} + 906923282s^{(vi)}_{i-3/2} + 1527092468s^{(vi)}_{i-1/2}$$
$$+ 906923282s^{(vi)}_{i+1/2} + 178300904s^{(vi)}_{i+3/2} + 9116141s^{(vi)}_{i+5/2}$$
$$+ 59038s^{(vi)}_{i+7/2} + s^{(vi)}_{i+9/2}\Big) \ , \qquad (2.\ 22)$$

$$h^4 s^{(iv)}_{i-1/2} = \frac{1}{240}\big(7s_{i-9/2} - 96s_{i-7/2} + 676s_{i-5/2} - 1952s_{i-3/2}$$
$$+ 2730s_{i-1/2} - 1952s_{i+1/2} + 676s_{i+3/2} - 96s_{i+5/2}$$
$$+ 7s_{i+7/2}\big) - \frac{1}{891813888000}h^{10}\Big(7s^{(x)}_{i-9/2} + 413240s^{(x)}_{i-7/2}$$
$$+ 62278100s^{(x)}_{i-5/2} + 1017050568s^{(x)}_{i-3/2} + 2677072970s^{(x)}_{i-1/2}$$
$$+ 1017050568s^{(x)}_{i+1/2} + 62278100s^{(x)}_{i+3/2}$$
$$+ 413240s^{(x)}_{i+5/2} + 7s^{(x)}_{i+7/2}\Big) \ , \qquad (2.\ 23)$$

$$i = 5,\ 6, \ldots\ldots, n-4\ ,$$

$$\begin{aligned}
&\big(s_{i-11/2} + 718s_{i-9/2} + 7661s_{i-7/2} - 14296s_{i-5/2} - 23278s_{i-3/2} \\
&+ 58388s_{i-1/2} - 23278s_{i+1/2} - 14296s_{i+3/2} + 7661s_{i+5/2} \\
&+ 718s_{i+7/2} + s_{i+9/2}\big) \\
= \ &\frac{1}{80640}h^4 \left(s^{(iv)}_{i-11/2} + 59038s^{(iv)}_{i-9/2} + 9116141s^{(iv)}_{i-7/2}\right. \\
&+ 178300904s^{(iv)}_{i-5/2} + 906923282s^{(iv)}_{i-3/2} + 1527092468s^{(iv)}_{i-1/2} \\
&+ 906923282s^{(iv)}_{i+1/2} + 178300904s^{(iv)}_{i+3/2} + 9116141s^{(iv)}_{i+5/2} \\
&\left.+ 59038s^{(iv)}_{i+7/2} + s^{(iv)}_{i+9/2}\right)\ , \qquad (2.\ 24)
\end{aligned}$$

$$\begin{aligned}
h^2 s''_{i-1/2} \ = \ &\frac{1}{5040}\big(-9s_{i-9/2} + 128s_{i-7/2} - 1008s_{i-5/2} + 8064s_{i-3/2} \\
&- 14350s_{i-1/2} + 8064s_{i+1/2} - 1008s_{i+3/2} + 128s_{i+5/2} \\
&- 9s_{i+7/2}\big) + \frac{1}{18728091648000}h^{10}\left(9s^{(x)}_{i-9/2} + 531304s^{(x)}_{i-7/2}\right. \\
&+ 79802048s^{(x)}_{i-5/2} + 1271457208s^{(x)}_{i-3/2} + 3241844782s^{(x)}_{i-1/2} \\
&+ 1271457208s^{(x)}_{i+1/2} + 79802048s^{(x)}_{i+3/2} \\
&\left.+ 531304s^{(x)}_{i+5/2} + 9s^{(x)}_{i+7/2}\right)\ , \qquad (2.\ 25)
\end{aligned}$$

$$i = 5,\ 6, \ldots\ldots, n-4\ ,$$

$$\begin{aligned}
&\big(s_{i-11/2} + 6550s_{i-9/2} + 318509s_{i-7/2} + 1828616s_{i-5/2} + 36050s_{i-3/2} \\
&- 4379452s_{i-1/2} + 36050s_{i+1/2} + 1828616s_{i+3/2} + 318509s_{i+5/2} \\
&+ 6550s_{i+7/2} + s_{i+9/2}\big) \\
= \ &\frac{1}{360}h^2 \left(s''_{i-11/2} + 59038s''_{i-9/2} + 9116141s''_{i-7/2}\right. \\
&+ 178300904s''_{i-5/2} + 906923282s''_{i-3/2} + 1527092468s''_{i-1/2} \\
&+ 906923282s''_{i+1/2} + 178300904s''_{i+3/2} + 9116141s''_{i+5/2} \\
&\left.+ 59038s''_{i+7/2} + s''_{i+9/2}\right) \qquad (2.\ 26)
\end{aligned}$$

and

$$\big(s_{i-11/2} - 10s_{i-9/2} + 45s_{i-7/2} - 120s_{i-5/2} + 210s_{i-3/2}$$

$$
\begin{aligned}
&- 252s_{i-1/2} + 210s_{i+1/2} - 120s_{i+3/2} + 45s_{i+5/2} \\
&- 10s_{i+7/2} + s_{i+9/2}) \\
= \quad & \frac{1}{3715891200}h^{10}\left(s^{(x)}_{i-11/2} + 59038s^{(x)}_{i-9/2} + 9116141s^{(x)}_{i-7/2}\right. \\
&+ 178300904s^{(x)}_{i-5/2} + 906923282s^{(x)}_{i-3/2} + 1527092468s^{(x)}_{i-1/2} \\
&+ 906923282s^{(x)}_{i+1/2} + 178300904s^{(x)}_{i+3/2} + 9116141s^{(x)}_{i+5/2} \\
&\left.+ 59038s^{(x)}_{i+7/2} + s^{(x)}_{i+9/2}\right) , \qquad (2.27)
\end{aligned}
$$

$$i = 6,\ 7, \ldots\ldots, n-5 .$$

Following Siddiqi and Twizell [9], the new consistency relations are defined, to determine the even-order derivatives of the splines.

Corresponding to equation (2.21), the following new consistency relation is defined as

$$
\begin{aligned}
h^6 s^{(vi)}_{i-1/2} \quad = \quad & \frac{1}{836}(577s_{i-9/2} - 3780s_{i-7/2} + 11140s_{i-5/2} - 19772s_{i-3/2} \\
&+ 23670s_{i-1/2} - 19772s_{i+1/2} + 11140s_{i+3/2} - 3780s_{i+5/2} \\
&+ 577s_{i+7/2}) - \frac{1}{388310630400}h^8\left(577s^{(viii)}_{i-9/2} + 34062300s^{(viii)}_{i-7/2}\right. \\
&+ 5104996420s^{(viii)}_{i-5/2} + 79935388708s^{(viii)}_{i-3/2} \\
&+ 194937414390s^{(viii)}_{i-1/2} + 79935388708s^{(viii)}_{i+1/2} \\
&\left.+ 5104996420s^{(viii)}_{i+3/2} + 34062300s^{(viii)}_{i+5/2} + 577s^{(viii)}_{i+7/2}\right) , \qquad (2.28)
\end{aligned}
$$

$$i = 5,\ 6, \ldots\ldots, n-4 ,$$

Corresponding to equation (2.23), the following new consistency relations are defined as

$$
\begin{aligned}
h^4 s^{(iv)}_{i-1/2} \quad = \quad & \frac{1}{40128}(-3353s_{i-9/2} + 20136s_{i-7/2} - 13628s_{i-5/2} - 73064s_{i-3/2} \\
&+ 139818s_{i-1/2} - 73064s_{i+1/2} - 13628s_{i+3/2} + 20136s_{i+5/2} \\
&- 3353s_{i+7/2}) + \frac{1}{18638910259200}h^8\left(3353s^{(viii)}_{i-9/2}\right. \\
&+ 197940984s^{(viii)}_{i-7/2} + 29773590908s^{(viii)}_{i-5/2} + 478392670664s^{(viii)}_{i-3/2} \\
&+ 1084329369942s^{(viii)}_{i-1/2} + 478392670664s^{(viii)}_{i+1/2} \\
&\left.+ 29773590908s^{(viii)}_{i+3/2} + 197940984s^{(viii)}_{i+5/2} + 3353s^{(viii)}_{i+7/2}\right) \qquad (2.29)
\end{aligned}
$$

and

$$h^4 s^{(iv)}_{i-1/2} = \frac{1}{22909056}\left(-118501 s_{i-9/2} - 7550472 s_{i-7/2} + 70581908 s_{i-5/2}\right.$$
$$- 212477368 s_{i-3/2} + 299128866 s_{i-1/2} - 212477368 s_{i+1/2}$$
$$\left. + 70581908 s_{i+3/2} - 7550472 s_{i+5/2} - 118501 s_{i+7/2}\right)$$
$$+ \frac{1}{221686353100800} h^6 \left(118501 s^{(vi)}_{i-9/2} + 6995317440 s^{(vi)}_{i-7/2}\right.$$
$$+ 1036318791580 s^{(vi)}_{i-5/2} + 14726918959744 s^{(vi)}_{i-3/2}$$
$$+ 13749899372670 s^{(vi)}_{i-1/2} + 14726918959744 s^{(vi)}_{i+1/2}$$
$$\left. + 1036318791580 s^{(vi)}_{i+3/2} + 6995317440 s^{(vi)}_{i+5/2} + 118501 s^{(vi)}_{i+7/2}\right) \quad (2.30)$$

Corresponding to equation (2.25), the following new consistency relations are defined as

$$h^2 s''_{i-1/2} = \frac{1}{19261440}\left(97957 s_{i-9/2} - 569640 s_{i-7/2} - 146420 s_{i-5/2}\right.$$
$$+ 23406568 s_{i-3/2} - 45576930 s_{i-1/2} + 23406568 s_{i+1/2}$$
$$\left. - 146420 s_{i+3/2} - 569640 s_{i+5/2} + 97957 s_{i+7/2}\right)$$
$$- \frac{1}{8946676924416000} h^8 \left(97957 s^{(viii)}_{i-9/2} + 5782811640 s^{(viii)}_{i-7/2}\right.$$
$$+ 870926755660 s^{(viii)}_{i-5/2} + 14115998901448 s^{(viii)}_{i-3/2}$$
$$+ 31490485944990 s^{(viii)}_{i-1/2} + 14115998901448 s^{(viii)}_{i+1/2}$$
$$\left. + 870926755660 s^{(viii)}_{i+3/2} + 5782811640 s^{(viii)}_{i+5/2} + 97957 s^{(viii)}_{i+7/2}\right), \quad (2.31)$$

$$h^2 s''_{i-1/2} = \frac{1}{11546164224}\left(4532977 s_{i-9/2} + 288910680 s_{i-7/2}\right.$$
$$- 2786303972 s_{i-5/2} + 2001865630 s_{i-3/2} - 35051592090 s_{i-1/2}$$
$$+ 2001865630 s_{i+1/2} - 2786303972 s_{i+3/2} + 288910680 s_{i+5/2}$$
$$\left. + 4532977 s_{i+7/2}\right) - \frac{1}{1117299219628032000} h^6 \left(4532977 s^{(vi)}_{i-9/2}\right.$$
$$+ 267589498416 s^{(vi)}_{i-7/2} + 39646871472652 s^{(vi)}_{i-5/2}$$
$$+ 558687812102416 s^{(vi)}_{i-3/2} + 708000430315398 s^{(vi)}_{i-1/2}$$

$$+ 558687812102416 s^{(vi)}_{i+1/2} + 39646871472652 s^{(vi)}_{i+3/2}$$
$$+ 267589498416 s^{(vi)}_{i+5/2} + 4532977 s^{(vi)}_{i+7/2}\Big) \qquad (2.32)$$

and

$$h^2 s''_{i-1/2} = \frac{1}{778761910379520}\big(5505416771 s_{i-9/2} + 3817691026680 s_{i-7/2}$$
$$- 51899872442380 s_{i-5/2} + 951914094240584 s_{i-3/2}$$
$$- 1807674836483310 s_{i-1/2} + 951914094240584 s_{i+1/2}$$
$$- 51899872442380 s_{i+3/2} + 3817691026680 s_{i+5/2}$$
$$+ 5505416771 s_{i+7/2}\big) - \frac{1}{627993604530044928 00}$$
$$h^4 \Big(5505416771 s^{(iv)}_{i-9/2} + 324893597111400 s^{(iv)}_{i-7/2}$$
$$+ 42209318785024340 s^{(iv)}_{i-5/2} + 741043040230402 96 s^{(iv)}_{i-3/2}$$
$$- 2967245258784946770 s^{(iv)}_{i-1/2} + 741043040230402 96 s^{(iv)}_{i+1/2}$$
$$+ 42209318785024340 s^{(iv)}_{i+3/2} + 324893597111400 s^{(iv)}_{i+5/2}$$
$$+ 5505416771 s^{(iv)}_{i+7/2}\Big) \qquad (2.33)$$

Sine the system of equations (2.27) provides $n - 10$ equations in n unknowns ($s_{i-1/2}$, $i = 1, 2, 3, /ldots, n$), ten more equations are needed. These are defined in the next susection in the form of end conditions, see [9].

2.2 End Conditions

$$\big(252 s_0 - 462 s_{1/2} + 330 s_{3/2} - 165 s_{5/2} + 55 s_{7/2} - 11 s_{9/2} + s_{11/2}\big)$$
$$= \frac{77}{2} h^2 s''_0 - \frac{583}{96} h^4 s^{(iv)}_0 + \frac{14597}{11520} h^6 s^{(vi)}_0 - \frac{1094863}{2580480} h^8 s^{(viii)}_0$$
$$+ \frac{h^{10}}{3715891200}\Big(620169186 s^{(x)}_{1/2} + 728622378 s^{(x)}_{3/2} + 169184763 s^{(x)}_{5/2}$$
$$+ 9057103 s^{(x)}_{7/2} + 59037 s^{(x)}_{9/2} + s^{(x)}_{11/2}\Big), \qquad (2.34)$$

$$\big(-168 s_0 + 330 s_{1/2} - 297 s_{3/2} + 220 s_{5/2} - 121 s_{7/2} + 45 s_{9/2}$$
$$- 10 s_{11/2} + s_{13/2}\big)$$

$$
\begin{aligned}
= & -21h^2 s_0'' + \frac{27}{16}h^4 s_0^{(iv)} + \frac{1479}{5760}h^6 s_0^{(vi)} - \frac{538419}{1290240}h^8 s_0^{(viii)} \\
& + \frac{h^{10}}{3715891200}\Big(728622378 s_{1/2}^{(x)} + 1517976327 s_{3/2}^{(x)} + 906864244 s_{5/2}^{(x)} \\
& + 178300903 s_{7/2}^{(x)} + 9116141 s_{9/2}^{(x)} + 59038 s_{11/2}^{(x)} + s_{13/2}^{(x)}\Big) , \qquad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
& \big(72s_0 - 165s_{1/2} + 220s_{3/2} - 253s_{5/2} + 210s_{7/2} - 120s_{9/2} \\
& + 45s_{11/2} - 10s_{13/2} + s_{15/2}\big) \\
= & \ 5h^2 s_0'' + \frac{35}{48}h^4 s_0^{(iv)} - \frac{419}{1152}h^6 s_0^{(vi)} - \frac{16253}{258048}h^8 s_0^{(viii)} \\
& + \frac{h^{10}}{3715891200}\Big(169184763 s_{1/2}^{(x)} + 906864244 s_{3/2}^{(x)} \\
& + 1527092467 s_{5/2}^{(x)} + 906923282 s_{7/2}^{(x)} + 178300904 s_{9/2}^{(x)} + 9116141 s_{11/2}^{(x)} \\
& + 59038 s_{13/2}^{(x)} + s_{15/2}^{(x)}\Big) , \qquad (2.36)
\end{aligned}
$$

$$
\begin{aligned}
& \big(-18s_0 + 55s_{1/2} - 121s_{3/2} + 210s_{5/2} - 252s_{7/2} + 210s_{9/2} \\
& - 120s_{11/2} + 45s_{13/2} - 10s_{15/2} + s_{17/2}\big) \\
= & \ \frac{1}{4}h^2 s_0'' - \frac{71}{192}h^4 s_0^{(iv)} - \frac{719}{23040}h^6 s_0^{(vi)} - \frac{6551}{5160960}h^8 s_0^{(viii)} \\
& + \frac{h^{10}}{3715891200}\Big(9057103 s_{1/2}^{(x)} + 178300903 s_{3/2}^{(x)} \\
& + 906923282 s_{5/2}^{(x)} + 1527092468 s_{7/2}^{(x)} + 906923282 s_{9/2}^{(x)} + 178300904 s_{11/2}^{(x)} \\
& + 9116141 s_{13/2}^{(x)} + 59038 s_{15/2}^{(x)} + s_{17/2}^{(x)}\Big) \qquad (2.37)
\end{aligned}
$$

and

$$
\begin{aligned}
& \big(2s_0 - 11s_{1/2} + 45s_{3/2} - 120s_{5/2} + 210s_{7/2} - 252s_{9/2} \\
& + 210s_{11/2} - 120s_{13/2} + 45s_{15/2} - 10s_{17/2} + s_{19/2}\big) \\
= & -\frac{1}{4}h^2 s_0'' - \frac{1}{192}h^4 s_0^{(iv)} - \frac{1}{23040}h^6 s_0^{(vi)} - \frac{1}{5160960}h^8 s_0^{(viii)} \\
& + \frac{h^{10}}{3715891200}\Big(59037 s_{1/2}^{(x)} + 9116141 s_{3/2}^{(x)} + 178300904 s_{5/2}^{(x)} \\
& + 906923282 s_{7/2}^{(x)} + 1527092468 s_{9/2}^{(x)} + 906923282 s_{11/2}^{(x)} \\
& + 178300904 s_{13/2}^{(x)} + 9116141 s_{15/2}^{(x)} + 59038 s_{17/2}^{(x)} + s_{19/2}^{(x)}\Big) \\
& \qquad (2.38)
\end{aligned}
$$

The remaining last five end conditions can be determined similarly.

3 THE SPLINE SOLUTION

For the spline solution, the following system of equations can be written, see [9]

$$\left.\begin{array}{ll} (i) & \mathbf{MY} = \mathbf{C} + \mathbf{T}\ , \\ (ii) & \mathbf{MS} = \mathbf{C}\ , \\ (iii) & \mathbf{ME} = \mathbf{T}\ . \end{array}\right\} \tag{3.1}$$

where $\mathbf{Y} = \left(y_{i-1/2}\right)$, $\mathbf{T} = (t_i)$, $\mathbf{E} = \left(\hat{e}_{i-1/2}\right)$, $i = 1, 2\ldots, n$,

$$\mathrm{M} = \mathrm{M}_0 + \frac{1}{3715891200} h^{10}\mathrm{BF}\ , \tag{3.2}$$

$$\mathbf{S} = \left(s_{i-1/2}\right),\quad i = 1, 2, \ldots\ldots, n \tag{3.3}$$

and

$$\mathbf{C} = (\hat{c}_i),\quad i = 1, 2, \ldots\ldots, n\ . \tag{3.4}$$

Also, M_0 and B are eleven-band symmetric matrices, with

$$\mathrm{M}_0 = [\mathrm{M}_1\ \ \mathrm{M}_2]\ , \tag{3.5}$$

where

$$\mathrm{M}_1 = \begin{bmatrix} -462 & 330 & -165 & 55 & -11 & 1 \\ 330 & -297 & 220 & -121 & 45 & -10 \\ -165 & 220 & -253 & 210 & -120 & 45 \\ 55 & -121 & 210 & -252 & 210 & -120 \\ -11 & 45 & -120 & 210 & -252 & 210 \\ 1 & -10 & 45 & -120 & 210 & -252 \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & 1 & -10 & 45 & -120 & 210 \\ & & 1 & -10 & 45 & -120 \\ & & & 1 & -10 & 45 \\ & & & & 1 & -10 \\ & & & & & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$
M_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & & & & & \\
-10 & 1 & & & & \\
45 & -10 & 1 & & & \\
-120 & 45 & -10 & 1 & & \\
210 & -120 & 45 & -10 & 1 & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-252 & 210 & -120 & 45 & -10 & 1 \\
210 & -252 & 210 & -120 & 45 & -11 \\
-120 & 210 & -252 & 210 & -121 & 55 \\
45 & -120 & 210 & -253 & 220 & -165 \\
-10 & 45 & -121 & 220 & -297 & 330 \\
1 & -11 & 55 & -165 & 330 & -462
\end{bmatrix}
$$

while

$$
B = [B_1 \quad B_2 \quad B_3] \;, \tag{3.6}
$$

where

$$
B_1 = \begin{bmatrix}
620169186 & 728622378 & 169184763 & 9057103 \\
728622378 & 1517976327 & 906864244 & 178300903 \\
169184763 & 906864244 & 1527092467 & 906923282 \\
9057103 & 178300903 & 906923282 & 1527092468 \\
59037 & 9116141 & 178300904 & 906923282 \\
1 & 59038 & 9116141 & 178300904 \\
 & \ddots & \ddots & \ddots \\
 & 1 & 59038 & 9116141 \\
 & & 1 & 59038 \\
 & & & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$B_2 = \begin{bmatrix} 59037 & 1 & 0 & 0 \\ 9116141 & 59038 & 1 & 0 \\ 178300904 & 9116141 & 59038 & 1 \\ 906923282 & 178300904 & 9116141 & 59038 \\ 1527092468 & 906923282 & 178300904 & 9116141 \\ 906923282 & 1527092468 & 906923282 & 178300904 \\ & & & \\ \ddots & \ddots & \ddots & \ddots \\ & & & \\ 178300904 & 906923282 & 1527092468 & 906923282 \\ 9116141 & 178300904 & 906923282 & 1527092468 \\ 59038 & 9116141 & 178300904 & 906923282 \\ 1 & 59038 & 9116141 & 178300904 \\ 0 & 1 & 59038 & 9116141 \\ 0 & 0 & 1 & 59037 \end{bmatrix}$$

and

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & & & \\ 59038 & 1 & & \\ 9116141 & 59038 & 1 & \\ & & & \\ \ddots & \ddots & \ddots & \\ & & & \\ 178300904 & 9116141 & 59038 & 1 \\ 906923282 & 178300904 & 9116141 & 59037 \\ 1527092468 & 906923282 & 178300903 & 9057103 \\ 906923282 & 1527092467 & 906864244 & 169184763 \\ 178300903 & 906864244 & 1517976327 & 728622378 \\ 9057103 & 169184763 & 728622378 & 620169186 \end{bmatrix}$$

The matrix F is defined as

$$F = \operatorname{diag}\left(\phi_{i-1/2}\right), \quad i = 1, 2, \ldots, n, \tag{3.7}$$

and the vector $\mathbf{C} = (\hat{c}_i)$, $i = 1, 2, \ldots, n$ **can be defined as**

$$\hat{c}_1 = -252A_0 + \frac{77}{2}h^2A_2 - \frac{583}{96}h^4A_4 + \frac{14597}{11520}h^6A_6 - \frac{1094863}{2580480}h^8A_8$$

$$+\frac{h^{10}}{3715891200}\left(620169186\psi_{1/2}+728622378\psi_{3/2}\right.$$
$$\left.+169184763\psi_{5/2}+9057103\psi_{7/2}+59037\psi_{9/2}+\psi_{11/2}\right), \quad (3.8)$$

$$\hat{c}_2 = 168A_0-21h^2A_2+\frac{27}{16}h^4A_4+\frac{1479}{5760}h^6A_6-\frac{538419}{1290240}h^8A_8$$
$$+\frac{h^{10}}{3715891200}\left(728622378\psi_{1/2}+1517976327\psi_{3/2}\right.$$
$$+906864244\psi_{5/2}+178300903\psi_{7/2}+9116141\psi_{9/2}$$
$$\left.+59038\psi_{11/2}+\psi_{13/2}\right), \quad (3.9)$$

$$\hat{c}_3 = -72A_0+5h^2A_2+\frac{35}{48}h^4A_4-\frac{419}{1152}h^6A_6-\frac{16253}{258048}h^8A_8$$
$$+\frac{h^{10}}{3715891200}\left(169184763\psi_{1/2}+906864244\psi_{3/2}\right.$$
$$+1527092467\psi_{5/2}+906923282\psi_{7/2}+178300904\psi_{9/2}$$
$$\left.+9116141\psi_{11/2}+59038\psi_{13/2}+\psi_{15/2}\right), \quad (3.10)$$

$$\hat{c}_4 = 18A_0+\frac{1}{4}h^2A_2-\frac{71}{192}h^4A_4-\frac{719}{23040}h^6A_6-\frac{6551}{5160960}h^8A_8$$
$$+\frac{h^{10}}{3715891200}\left(9057103\psi_{1/2}+178300903\psi_{3/2}\right.$$
$$+906923282\psi_{5/2}+1527092468\psi_{7/2}+906923282\psi_{9/2}$$
$$\left.+178300904\psi_{11/2}+9116141\psi_{13/2}+59038\psi_{15/2}+\psi_{17/2}\right), \quad (3.11)$$

$$\hat{c}_5 = -2A_0-\frac{1}{4}h^2A_2-\frac{1}{192}h^4A_4-\frac{1}{23040}h^6A_6-\frac{1}{5160960}h^8A_8$$
$$+\frac{h^{10}}{3715891200}\left(59037\psi_{1/2}+9116141\psi_{3/2}\right.$$
$$+178300904\psi_{5/2}+906923282\psi_{7/2}+1527092468\psi_{9/2}$$
$$+906923282\psi_{11/2}+178300904\psi_{13/2}+9116141\psi_{15/2}$$
$$\left.+59038\psi_{17/2}+\psi_{19/2}\right) \quad (3.12)$$

and

$$\hat{c}_i = +\frac{h^{10}}{3715891200}\left(\psi_{i-11/2}+59038\psi_{i-9/2}\right.$$
$$+9116141\psi_{i-7/2}+178300904\psi_{i-5/2}+906923282\psi_{i-3/2}$$
$$+1527092468\psi_{i-1/2}+906923282\psi_{i+1/2}+178300904\psi_{i+3/2}$$
$$\left.+9116141\psi_{i+5/2}+59038\psi_{i+7/2}+\psi_{i+9/2}\right), \quad (3.13)$$
$$i=6,7,\ldots\ldots,n-5\,.$$

$\hat{c}_{n-4}$, $\hat{c}_{n-3}$, $\hat{c}_{n-2}$, $\hat{c}_{n-1}$ and $\hat{c}_n$ are defined similar to $\hat{c}_5$, $\hat{c}_4$, $\hat{c}_3$, $\hat{c}_2$ and $\hat{c}_1$, respectively, except that the boundary values B_0, B_2, B_4, B_6, and B_8 will replace A_0, A_2, A_4, A_6, and A_8 at the other end.

After determining $s_{i-1/2}$, $i = 1, 2, \ldots, n$, s_0 and s_n can be determined using the differential equation (1.1).

Also, $s^{(x)}_{i-1/2}$, $i = 1, 2, \ldots, n$, $s^{(x)}_0$ and $s^{(x)}_n$ can be determined using (1.1). The derivatives $s^{(viii)}_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using (2.19) and (2.20). The derivatives $s^{(vi)}_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using ((2.21 or (2.28)) and (2.22). The derivatives $s^{(iv)}_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using ((2.23) or (2.29) or (2.30)) and (2.24) and $s''_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using ((2.25) or (2.31) or (2.32) or (2.33)) and (2.26).

Now it is possible to determine the odd-order derivatives of the spline.

s'_i, s'''_i, $s^{(v)}_i$, $s^{(vii)}_i$ and $s^{(ix)}_i$, for $i = 1, 2, \ldots, n-1$ are determined using (2.18), (2.17), ... ,(2.14) respectively, while s'_0, s'_n, s'''_0, s'''_n, $s^{(v)}_0$, $s^{(v)}_n$, $s^{(vii)}_0$, $s^{(vii)}_n$, $s^{(ix)}_0$ and $s^{(ix)}_n$ are determined through the following relations which were obtained while determining (2.14)—(2.18)

$$h(s'_i - s'_{i-1}) = h^2 s''_{i-1/2} + \frac{1}{24} h^4 s^{(iv)}_{i-1/2} + \frac{1}{1920} h^6 s^{(vi)}_{i-1/2} + \frac{1}{322560} h^8 s^{(viii)}_{i-1/2} + \frac{1}{92897280} h^{10} s^{(x)}_{i-1/2} \qquad (3.14)$$

$$h(s'''_i - s'''_{i-1}) = h^2 s^{(iv)}_{i-1/2} + \frac{1}{24} h^4 s^{(vi)}_{i-1/2} + \frac{1}{1920} h^6 s^{(viii)}_{i-1/2} + \frac{1}{322560} h^8 s^{(x)}_{i-1/2} \,. \qquad (3.15)$$

$$h(s^{(v)}_i - s^{(v)}_{i-1}) = h^2 s^{(vi)}_{i-1/2} + \frac{1}{24} h^4 s^{(viii)}_{i-1/2} + \frac{1}{1920} h^6 s^{(x)}_{i-1/2} \,. \qquad (3.16)$$

$$h(s^{(vii)}_i - s^{(vii)}_{i-1}) = h^2 s^{(viii)}_{i-1/2} + \frac{1}{24} h^4 s^{(x)}_{i-1/2} \,. \qquad (3.17)$$

$$h(s^{(ix)}_i - s^{(ix)}_{i-1}) = h^2 s^{(x)}_{i-1/2} \,. \qquad (3.18)$$

4 NUMERICAL RESULTS

In this section, two problems are discussed to compare the maximum absolute errors with the analytical solutions, see [9]. Numerical results relating to the solution of tenth-order BVPs are rare in the literature. The value of n used in Tables 1 and 2 is that which gives the smallest maximum error moduli for Problems 4.1 and 4.2 . Some unexpected results for the higher derivatives were obtained near the boundaries of the given interval. These results were due to equations (2.20), (2.22), (2.24) and (2.26). The absolute errors in the function values were, however, very small. The absolute errors in the function values and all derivatives were seen to be small at points remote from the boundaries, as observed in [9].

Problem 4.1

Consider

$$\left.\begin{array}{l} y^{(x)} - xy = -(89 + 21x + x^2 - x^3)e^x , \quad -1 \leq x \leq 1 , \\ y(-1) = 0 = y(1) , \; y''(-1) = \frac{2}{e} , \; y''(1) = -6e , \\ y^{(iv)}(-1) = -\frac{4}{e} , \; y^{(iv)}(1) = -20e , \\ y^{(vi)}(-1) = -\frac{18}{e} , \; y^{(vi)}(1) = -42e , \\ y^{(viii)}(-1) = -\frac{40}{e} , \; y^{(viii)}(1) = -72e \end{array}\right\} \quad (4.1)$$

The analytical solution of the above differential system is

$$y(x) = (1 - x^2)e^x \quad (4.2)$$

The maximum errors (in absolute value) in $y_i^{(k)}$, $k = 0, 1, 2, \ldots, 9$, are shown in the Table 1 .

Problem 4.2

Consider

$$
\left.
\begin{aligned}
y^{(x)} + y &= -10[2x\sin(x) - 9\cos(x)] \ , \qquad -1 \le x \le 1 \ , \\
y(-1) &= 0 = y(1) \ , \\
y''(-1) &= 4\sin(-1) + 2\cos(-1) \ , \\
y''(1) &= -4\sin(1) + 2\cos(1) \ , \\
y^{(iv)}(-1) &= -8\sin(-1) - 12\cos(-1) \ , \\
y^{(iv)}(1) &= 8\sin(1) - 12\cos(1) \ , \\
y^{(vi)}(-1) &= 12\sin(-1) + 30\cos(-1) \ , \\
y^{(vi)}(1) &= -12\sin(1) + 30\cos(1) \ , \\
y^{(viii)}(-1) &= -16\sin(-1) - 56\cos(-1) \ , \\
y^{(viii)}(1) &= 16\sin(1) - 56\cos(1) \ .
\end{aligned}
\right\} \qquad (4.3)
$$

The analytical solution of the above differential system is

$$ y(x) = (x^2 - 1)\cos(x) \qquad (4.4) $$

The maximum errors (in absolute value) in $y_i^{(k)}$, $k = 0, 1, 2, \ldots, 9$, **are shown in the Table 2 .**

Table 1: Maximum absolute errors for Problem 4.1 with $n = 18$ **.**

$y_i^{(k)}$	$x \in [x_5, x_{n-5}]$	$x \notin [x_5, x_{n-5}]$
$k = 0$	0.2069×10^{-2}	0.1056×10^{14}
$k = 1$	0.2090×10^{-2}	0.1552×10^{16}
$k = 2$	0.5104×10^{-2}	0.2406×10^{17}
$k = 3$	0.5163×10^{-2}	0.1806×10^{18}
$k = 4$	0.1257×10^{-1}	0.7300×10^{18}
$k = 5$	0.1276×10^{-1}	0.4380×10^{19}
$k = 6$	0.3082×10^{-1}	0.3835×10^{15}
$k = 7$	0.3108×10^{-1}	0.1523×10^{17}
$k = 8$	0.7805×10^{-1}	0.3171×10^{18}
$k = 9$	0.1936×10	0.1902×10^{19}

Table 2: Maximum absolute errors for Problem 4.2 with $n = 32$.

$y_i^{(k)}$	$x \in [x_5,\ x_{n-5}]$	$x \notin [x_5,\ x_{n-5}]$
$k = 0$	0.2655×10^{-3}	0.4159×10^{14}
$k = 1$	0.3578×10^{-3}	0.1007×10^{16}
$k = 2$	0.6552×10^{-3}	0.2485×10^{17}
$k = 3$	0.9090×10^{-3}	0.2471×10^{18}
$k = 4$	0.1618×10^{-2}	0.5749×10^{18}
$k = 5$	0.2254×10^{-2}	0.6132×10^{19}
$k = 6$	0.4048×10^{-2}	0.1654×10^{17}
$k = 7$	0.5719×10^{-2}	0.1183×10^{19}
$k = 8$	0.1107×10^{-1}	0.3204×10^{20}
$k = 9$	0.1200×10^{-1}	0.3418×10^{21}

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THE STOKES-BITSADZE SYSTEM

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ABSTRACT: The stream function/Airy stress function formulation of the classical Stokes equations in the plane is shown to be mathematically equivalent to the Bitsadze's canonical system of equations and is identified as Stokes-Bitsadze System (SBS). The new formulations, the div-curl formulations, of the Stokes-Bitsadze System are presented.

1. INTRODUCTION

The classical Stokes problem has played a fundamental role in the computer solution of incompressible viscous flows for over three decades. Not only do the Stokes equations govern completely the slow viscous flow of incompressible fluids, but also their solution is a key feature in algorithms for stationary and time-dependent flows governed by the nonlinear Navier-Stokes equations.

2. DIFFERENT FORMULATIONS OF STOKES EQUATIONS

In this paper we are concerned with the two-dimensional Stokes problem as an elliptic boundary value problem in the plane. There are numerous formulations of the Stokes equations in two dimensions, each deriving from the equations governing creeping incompressible flows

$$\text{div}\ \sigma = \boldsymbol{0} \quad \text{(conservation of momentum)}, \tag{1.1}$$

$$\text{div}\ \boldsymbol{u} = 0 \quad \text{(incompressibility condition)}. \tag{1.2}$$

We have assumed that there are no body forces present; $\boldsymbol{u} = (u, v)$ is the velocity of the fluid, while σ denotes the Cauchy stress tensor. It is possible to split the stress σ into an isotropic part and an anisotropic part

$$\sigma = -pI + \boldsymbol{T}, \tag{1.3}$$

where, after assuming a scaling with respect to density, p is the kinematic pressure and T is the extra-stress tensor. Both σ and T are symmetric (conservation of angular momentum). For a Newtonian fluid, T is related to the velocity gradient and is given by

$$T = \eta\{\nabla u + (\nabla u)^t\}, \tag{1.4}$$

where η is the (constant) kinematic viscosity. The first formulation of the Stokes equations is therefore the first-order system in the variables (u,p,T):

$$\begin{aligned} -\nabla p + \operatorname{div} T &= 0, \\ T - \eta\,\{\nabla u + (\nabla u)^T &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \tag{1.5}$$

In cartesians, equations (1.5) constitute an elliptic system of six equations in the six unknown variables $(u, v, p, T^{xx}, T^{xy}, T^{yy})$.

The most common formulation of the Stokes equations is in terms of the primitive variables (u,p):

$$\begin{aligned} -\nabla p + \eta\Delta u &= 0, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.6}$$

where (1.6a) is obtained from the substitution of (1.5b) into (1.5a). In cartesians, equations (1.6) constitute an elliptic system of three equations in the three unknowns (u, v, p), equations (1.6a) being second-order in the velocity.

Introducing the stream-function ψ such that

$$u = \psi_y, \quad v = \psi_x \tag{1.7}$$

the incompressibility condition (1.2) is automatically satisfied given continuity of the second-order derivatives of ψ. Moreover the pressure may be eliminated from the two equations (1.6a) to obtain

$$\Delta^2 \psi = 0, \tag{1.8}$$

where Δ^2 is the biharmonic operator. This is a single fourth-order equation in the single variable ψ. Alternatively, in terms of the vorticity

$$\omega = \operatorname{curl} \mathbf{u} \equiv v_x - u_y, \tag{1.9}$$

the equations (1.6) may be written in stream-function vorticity formulation:

$$\begin{aligned} \Delta\omega &= 0, \\ \Delta\psi &= -\omega \end{aligned} \tag{1.10}$$

There is also the velocity-vorticity-pressure formulation:

$$\begin{aligned} \eta\ \mathbf{curl}\ \omega - \nabla p &= \mathbf{0}, \\ \text{curl}\ \boldsymbol{u} - \omega &= 0, \\ \text{div}\ \boldsymbol{u} &= 0, \end{aligned} \tag{1.11}$$

with $\mathbf{curl}\ \omega = (-\omega_y, \omega_x)$.

The researchers have also been interested in the stream function/stress function formulation, see for example [Coleman, 1981], [Davies & Devlin, 1993], [Owens & Phillips, 1994], [Cassidy, 1996] and [Thatcher, 1997].

3. THE STREAM FUNCTION/STRESS FUNCTION FORMULATION [Coleman, 1981]

Let the components of the velocity be given in terms of stream function $\psi(x,y)$ by (1.7) and the components of extra stress tensor $\boldsymbol{T}$ be given, in terms of the Airy stress function[1] $\phi(x,y)$ and pressure p by,

$$\begin{cases} \sigma^{xx} = -p+T^{xx} = \phi_{yy}, \\ \sigma^{xy} = T^{xy} = -\phi\text{-}xy, \\ \sigma^{xy} = -p+T^{yy} = \phi_{xx}, \end{cases} \tag{1.12}$$

where upper indices denote stress components while the lower indices denote the second derivatives. Then the momentum and mass balance equations (1.1) and (1.2) are satisfied by continuity. The tensor $\boldsymbol{T}$ can thus be expressed as

$$\boldsymbol{T} = \begin{pmatrix} p+\phi_{yy} & -\phi_{xy} \\ -\phi_{xy} & p+\phi_{xx} \end{pmatrix} \tag{1.13}$$

Using the equation (1.13) the following equations are obtained from (1.4) and (1.7)

$$\begin{aligned} p + \phi_{yy} &= 2\,\eta\psi_{xy}, \\ -\phi_{xy} &= \eta(\psi_{yy} - \psi_{xx}), \\ p + \phi_{xx} &= -2\eta\psi_{xy}, \end{aligned} \tag{1.14}$$

[1] The Airy stress function was introduced by [Airy, 1863].

The pressure p can then be eliminated between (1.14a) and (1.14c) giving the following second order elliptic system in ϕ and ψ

$$\begin{aligned} \phi_{xx} - \phi_{yy} &= -4\,\eta\psi_{xy}, \\ -\,\phi_{xy} &= \eta(\psi_{yy} - \psi_{xx}), \end{aligned} \tag{1.15}$$

or the stream function/stress function formulation of the Stokes system.

4. THE STOKES-BITSADZE SYSTEM

Let us re-scale the dependent variables in (1.15) as follows; $\eta\psi \to \omega$, $\phi \to \phi$. The equations (1.15) are then reduced to

$$\begin{aligned} \phi_{xx} - \phi_{yy} + 2\,\psi_{xy} &= 0, \\ \psi_{xx} - \psi_{yy} + 2\,\phi_{xy} &= 0, \end{aligned} \tag{1.16}$$

The Stokes system (1.16) in ψ, ϕ formulation is of special interest to us. Indeed it is the famous second order elliptic system known as the Bitsadze system [Nakhushev, 1988]. Henceforth we will call the system (1.16) as the Stokes-Bitsadze system (SBS). The ellipticity of SBS, in the sense of Petrovskii [1946], is proved by Thatcher [1997].

4.1 Related Background

By introducing the notations; $w = \phi + i\psi$, $z = x + iy$ and $\bar{z} = x - iy$, the SBS (1.16) may be written in the form

$$w_{\bar{z}\bar{z}} = 0, \tag{1.17}$$

where $2\partial_{\bar{z}} = \partial_x + i\partial_y$. From (1.17) the regular solution of SBS (1.16) can be represented in the form

$$w_{\bar{z}\bar{z}} = 0, \tag{1.18}$$

where $f(z)$ and $g(z)$ are arbitrary analytic functions of the complex variable z. On the grounds of (1.18) Bitsadze [1964] shows that in the circular domain $|z - z_0| < \varepsilon$ the homogeneous Dirichlet problem for the SBS (1.16) has the infinite set of linearly independent solutions given by

$$w = \left\{\varepsilon^2 - |z - z_0|^2\right\} g(z), \tag{1.19}$$

where $g(z)$ is a function which is arbitrary and analytic in the domain $|z - z_0| < \varepsilon$. Bitsadze [1964] concludes that the Dirichlet problem for the SBS (1.16) is neither Fredholmian nor Noetherian[2]. For the Fredholm and Noether theory we refer to [Bitsadze, 1968 & 1988] and [Mikhlin, 1970]. Bitsadze [1988] shows that the Fredholmian character of the Neumann problem is also violated for the SBS[3]. Wendland [1979] considers the Dirichlet problem for the SBS (1.16) and proves the violation of Lopatinski condition to show the problem to be non-Fredholm. For the details on Lopatinski condition we refer to [Wendland, 1979].

5. THE DIV-CURL FORMULATIONS OF THE STOKES-BITSADZE SYSTEM

Below we present the Stokes-Bitsadze system in two different div-curl formulations.

5.1 Formulation-I

We can write the Stokes-Bitsadze System (1.16) as

$$\begin{aligned} \partial_x(\psi_y + \phi_x) + \partial_y(\psi_x - \phi_y) &= 0, \\ \partial_x(\psi_x + \phi_y) - \partial_y(\psi_y + \phi_x) &= 0. \end{aligned} \tag{1.20}$$

Let us introduce $\Phi(x,y)$ and $\Psi(x,y)$ which are defined as

$$\begin{aligned} \Phi(x,y) \text{ div } &\equiv (\phi,\psi) = \psi_y + \phi_x, \\ \Psi(x,y) \text{ curl} &\equiv (\phi,\psi) = \psi_x + \phi_y. \end{aligned} \tag{1.21}$$

2 The situation contrasts greatly with a system of a single elliptic equation, see for details [Kuz'min, 1967] and [Bitsadze, 1968].

3 Similar facts can also be observed when the number of independent variables is more than two. For some multidimensional analogs of Bitsadze systems we refer to [Yanushauskas, 1995], [Treneva, 1985] and [Kuz'min, 1967].

It follows immediately that the SBS (1.16) has the double div-curl formulation

$$\begin{aligned} \operatorname{div}\ (\Phi,\Psi) &= 0, \\ \operatorname{curl}\ (\Phi,\Psi) &= 0, \end{aligned} \tag{1.22}$$

where

$$\begin{aligned} \Phi(x,y) &= \operatorname{div}\ (\phi,\psi) \\ \Psi(x,y) &= \operatorname{curl}\ (\phi,\psi) \end{aligned} \tag{1.23}$$

5.2 Remark

It is easy to see that SBS (1.16) remains unchanged either (ϕ,ψ) is replaced by $(-\phi,\psi)$ or (Φ,Ψ) is replaced by $(-\Psi,\Phi)$.

5.3 Formulation-II

Let ϕ and ψ be sufficiently continuously differentiable functions. The equation (1.16a) of the SBS can be written as

$$\Delta\phi = -2\psi_{xy} + 2\phi_{yy} \tag{1.24}$$

which on differentiating with respect to x gives

$$\partial_x(\Delta\phi) = -\partial_y(2\psi_{xx} - 2\phi_{xy}). \tag{1.25}$$

From equations (1.16b) and (1.25) we immediately obtain

$$\partial_x(\Delta\phi) = -\partial_y(\Delta\psi). \tag{1.26}$$

Similarly we can easily show that

$$\partial_x(\Delta\psi) = \partial_y(\Delta\phi). \tag{1.27}$$

Hence we have the following div-curl formulation

$$\begin{aligned} \operatorname{div}\ (\Delta\phi,\ \Delta\psi) &= 0, \\ \operatorname{curl}\ (\Delta\phi,\ \Delta\psi) &= 0, \end{aligned} \tag{1.28}$$

5.4 Remarks

The formulation (1.28) is the vorticity-pressure formulation for the Stokes equations. It is one order higher than Formulation-I.

6. CONCLUSION

The stream function/stress function formulation of the Stokes equations is shown to be mathematically equivalent to the well-known Bitsadze system and is identified as Stokes-Bitsadze System (SBS). New formulations 'the div-curl formulations' of the Stokes-Bitsadze System are presented.

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DR. LAL MUHAMMAD CHAWLA (1917-1998)

Muhammad Amin

Dr. Lal Muhammad Chawla, an exceptionally brilliant mathematician of international repute and an eminent scholar of the Holy Quran, died in Lahore on November 5, 1998. He was gentle, kind-hearted, considerate and a valuable friend and colleague.

Dr. Lal Muhammad Chawla was born on November 1, 1917 in village Mahatpur (district Jallundar, India). He received his early education in his home district and after passing his high school examination, he joined the Islamia College, Lahore from where he graduated with Honours in 1937. He did his M.A. in mathematics from University of the Punjab, Lahore in 1939. He obtained his D.Phil in Pure Mathematics from the Oxford University in 1955.

Dr. Chawla's career as a college and University teacher is spread over a period of 47 years from 1939 to 1986: Lecturer Islamia College, Lahore, 1939 to 1947; Lecturer/Senior Lecturer Government College, Lahore, 1947 to 1957; Senior Professor and Head of Mathematics Department, Government College, Lahore, 1957 to 1969; Principal Central Training College Lahore, April to September 1969; Chairman, Board of Intermediate & Secondary Education, Sargodha; 1969 to 1970. He was visiting professor, University of Illinois, Urbana 1965 to 1966 and University of Florida, Gainsville, 1966 to 1967; Tenured Professor of Mathematics, Kansas State University, Manhattan, 1970 to 1982; Professor of Mathematics, King Abdul Aziz University, Jeddah, 1982 to 1986.

Dr. Chawla was a keen research worker and published more than forty research papers in the fields of Algebra and Number Theory. In 1968, he discovered an arithmetic function which won international fame and was named in his honour as Chawla Arithmetic Function. Quite a few persons further worked on the new function and were able to earn their Ph.D.'s for their work based on Chawla Arithmetic Function.

In order to disseminate latest scientific and mathematical research, Dr. Chawla and two other colleagues at Govt. College Lahore founded the *Journal of Natural Sciences and Mathematics* in 1961. The journal proved its merit and soon got international recognition and reputation. Dr. Chawla remained editor till 1986. Dr. Chawla was also one of the pioneers of introducing modern mathematical disciplines at Postgraduate level at the Punjab University. His services in the cause of higher education in mathematics in the country and abroad have made him a man of beloved memory.

Dr. Chawla was a devout and enlightened Muslim. In the later years of his life he spent most of his time and energy on the study of Al-Quran-ul-Karim. He started writing the book titled, *A Study of Al-Quran-ul-Karim* in 1975 and it was completed in 1994 in four volumes. Seven more books on Al-Quran-ul-Karim, Hadith and Prophet Muhammad (Allah's blessing and peace be upon him) followed by the time he breathed his last. All of his works have been published. These books have been well received by religious scholars in Pakistan and abroad.

In the end, I pray to Allah to grant him eternal peace and Janat-ul-Firdous (Aameen).

(Muhammad Amin)
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CONTENTS

ON STATISTICALLY CONVERGENT SERIES *B.C. Tripathy* 1

ON GENERALIZED ABSOLUTE σ-CONVERGENT SEQUENCE SPACES *H. Kara* . . 9

CONVERGENCE THEOREMS FOR SOME VARIANTS OF NEWTON'S METHOD OF ORDER GREATER THAN TWO *Ioannis K. Argyros* 17

BAYESIAN ANALYSIS OF MATCHED BINARY RESPONSE SAMPLES *Abdul Kadir A. Kattan* 43

A DECOMPOSITION OF SUPER-CONTINUITY *Jin Han Park* 53

CELLULAR FOLDING *E.M. Elkholy* 59

ON A LOCAL FORM OF GENERALIZED LOBATCHEWSKI'S FUNCTIONAL EQUATION *Hasan M. Ymeri* 67

A NOTE ON SOLUTIONS OF THE FUNCTIONAL EQUATION $f(x+h(x)) = f(x) + f(h(x))$ *Hasan M. Ymeri* 73

BOUNDARY VALUE PROBLEMS FOR THE CAUCHY-RIEMANN SYSTEM *Muhammad Tahir* 79

SECRET SHARING SCHEME AND DISHONEST PARTICIPANTS *Farast Tahir* . . . 89

EVERY IDEAL IS A SUBSET OF A PROPER P-IDEAL IN BCI-ALGEBRA *Xiaohong Zhang and S.A. Bhatti* 93

YIELD STRESS OF SEMI-RIGID AND FLEXIBLE POLYMERS *Abdul Qayyum Bhatti* 97

SECOND NORMAL STRESS DIFFERENCE OF BOGER FLUIDS M_1 AND D_4 *Abdul Qayyum Bhatti* 111

ON RATIONALITY OF ZASSENHAUS GROUPS *Abdul Hameed and S.M. Husnine* 117

MAGNETIC SYMMETRY AND PHYSICAL PROPERTIES *M. Shafique Baig* 123

TABLES OF THE KOSTKA FOULKES POLYNOMIALS *N. Sultana* 127

DEFINING NEW CONSISTENCY RELATIONS FOR SPLINE SOLUTIONS OF TENTH ORDER BOUNDARY-VALUE PROBLEMS *Shahid S. Siddiqi* 153

THE STOKES-BITSADZE SYSTEM *Mohammad Tahir* 173

DR. LAL MUHAMMAD CHAWLA *Muhammad Amin* 181