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BEST APPROXIMATION IN CONVEX METRIC SPACES [1]

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Abstract

Some results regarding common fixed points and best approximation in convex metric spaces are obtained.

Let X be a metric space with metric d $I = [0,1]$. The X is called a convex metric space if there exists a mapping $W : X \times X \times I \rightarrow X$ satisfying

$$d(x, W(y, q, \lambda)) \leq \lambda d(x, y) + (1 - \lambda) d(x, q)$$

for all $x, y, q, \in X$ and all $\lambda \in I$. Every normed space X is a convex metric space with W defined by $W(x, q, \lambda) = \lambda x + (1 - \lambda)q$ for all $x, q \in X$ and all $\lambda \in I$. However, the converse is not true, in general (see, e.g., Takahashi [7]). A subset D of a convex metric space X is called (1) convex if $W(x, q, \lambda) \in D$ for all $x, q \in D$ and all $\lambda \in I$; (2) q-starshaped if there exists $q \in D$ such that $W(x, q, \lambda) \in D$ for all $x \in D$ and all $\lambda \in I$. A convex metric space X is said to satisfy the property (I) if

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y)$$

for all $x, y \in X$ and all $\lambda \in I$. We observe that the property (I) holds in every normed space X. Throughout this paper, X denotes a convex metric space satisfying the property (I). For more details, we refer to Beg and Shahzad [3] and Guay, Singh and Whitfield [4].

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Let D be a subset of X and $S, T : X \to X$ two mappings. Then T is called (3) S-nonexpansive on D if $d(Tx, \ Ty) \leq d(Sx, Sy)$ for every $x, y \in D$; (4) S-contraction on D if $d(Tx, Ty) \leq kd(Sx, Sy)$ for every $x, y \in D$ and some $k \in [0, 1)$. The mappings S and T commute on D if $STx = TSx$ for all $x \in D$ for all $x \in D$. We denote by $F(S)$ (resp. $F(T)$) the set of fixed points of S (resp. T). Let C be a subset of X and $\hat{x} \in X$. Then the set $P_C(\hat{x}) = \{x \in C : d(x, \hat{x}) = d(\hat{x}, C)\}$ is the set of best C-approximants to $\hat{x}$, where $d(\hat{x}, C) = \ inf\{d(\hat{x}, y) : y \in C\}$.

In 1992, Beg and Shahzad [3] obtained the following results in the setting of convex metric spaces.

Theorem 1:

Let D be a closed and q-starshaped subset of X and $T : D \to D$ a nonexpansive mapping. If $cl(T(D))$ is compact, then $F(T) \neq \phi$.

Theorem 2:

Let $T, S : X \to X$ be two mapings, C a subset of X such that $T(\partial C) \subset C$, and $\hat{x} \in F(T) \cap F(S)$, where ∂C denotes the boundary of C. Suppose S is continuous and affine on $P_C(\hat{x})$, S and T are commuting on $P_C(\hat{x})$, and T is S-nonexpansive on $P_C(\hat{x}) \cup \{\hat{x}\}$. If $P_C(\hat{x})$ is nonempty, compact, and q-starshaped with $q \in F(S)$, and if $S(P_C(\hat{x})) = P_C(\hat{x})$, then $P_C(\hat{x}) \cap F(T) \cap F(S) \neq \phi$

In this paper, we prove a common fixed point theorem for commuting mappings in convex metric spaces. We use this result to generalize Theorem 2 (above) and Theorem 3 of Sahab, Khan and Sessa [6].

We shall make use of the following result due to Al-Thagafi [1], which we state here as a lemma.

Lemma 3:

Let D be a closed subset of a metric space and $S, T : D \to D$ two mappings such that $T(D) \subset S(D)$. Suppose $cl(T(D))$ is complete, S is continuous, S and T are commuting and T is S-contraction. Then $F(S) \cap F(T)$ is singleton.

Theorem 4:

Let D be a closed subset of X and $S, T : D \to D$ two mappings such that $T(D) \subset S(D)$. Suppose D is q-starshaped with $q \in F(S)$, $cl(T(D))$ is compact, S is continuous and affine, S and T are commuting, and T are commuting, and T is S-nonexpansive. Then $F(S) \cap F(T) \neq \phi$.

Proof:

Let $\{\beta_n\}$ be a fixed sequence of positive numbers less than 1 and converging to 1. For each natural number n, define a sequence of maps T_n by

$$T_n x = W(Tx, q, \beta_n).$$

Clearly, each T_n maps D into itself because $T : D \to D$ and D is q-starshaped. Since $T(D) \subset S(D)$, D is q-starshaped and S is affine, it follows that $S(D)$ is q-starshaped and so $T_n(D) \subset S(D)$. Since S is affine and commutes with T, we have

$$\begin{aligned} T_n Sx &= W(TSx, q, \beta_n) \\ &= W(TSx, Sq, \beta_n) \\ &= W(STx, Sq, \beta_n) \\ &= S(W(Tx, q, \beta_n)) \\ &= ST_n x \end{aligned}$$

for all $x \in D$.

Furthermore,

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, q, \beta_n), W(Ty, q, \beta_n)) \\ &\leq \beta_n d(Tx, Ty) \\ &\leq \beta_n d(Sx, Sy) \end{aligned}$$

for all $x, y \in D$. Since $cl(T(D))$ is compact, each $cl(T_n(D))$ is compact. By Lemma 3, $F(S) \cap F(T_n) = \{x_n\}$ for some $x_n \in D$, that is, $x_n = Sx_n = T_n x_n =$

$W(Tx_n, q, \beta_n)$. Since $cl(T(D))$ is compact, $\{Tx_n\}$ has a subsequence $\{Tx_{n_m}\}$ (say) converging to y. It further implies that $\{x_{n_m}\} = \{W(Tx_{n_m}, q, \beta_{n_m})\}$ converges to y because $\beta_{n_m} \to 1$ as $m \to \infty$. So, from $y = \lim_{m\to\infty} x_{n_m} = \lim_{m\to\infty} Tx_{n_m} = \lim_{m\to\infty} Sx_{n_m}$ and the continuity of S, we have $y = Ty = Sy$. Hence $F(S) \cap F(T) \neq \phi$.

Remark 5:

Theorem 4 generalizes Theorem 2.2 of Al-Thagafi [1] and Theorem 4 of Habiniak [5]. It also contains Theorem 1 (above).

Theorem 6:

Let $S, T : X \to X$ be two mappings, C a subset of X such that $T(\partial C \cap C) \subset C$, and $\hat{x} \in F(S) \cap F(T)$. Suppose S is continuous and affine on $P_C(\hat{x})$, S and T are commuting on $P_C(\hat{x})$, and T is S-nonexpansive on $P_S(\hat{x}) \cup \{\hat{x}\}$. If $P_C(\hat{x})$ is nonempty, closed and q-starshaped with $q \in F(S)$, $cl(T(P_C(\hat{x})))$ is compact, and $S(P_C(\hat{x})) = P_C(\hat{x})$, then $P_C(\hat{x}) \cap F(T) \cap F(S) \neq \phi$.

Proof:

Let $y \in P_C(\hat{x})$. Then, as in Lemma 3.2 [2], $y \in \partial C \cap C$ and so $Ty \in C$ because $T(\partial C \cap C) \subset C$. Since T is S-nonexpansive on $P_C(\hat{x}) \cup \{\hat{x}\}$ and $S(P_C(\hat{x})) = P_C(\hat{x})$, it follows that

$$\begin{aligned} d(Ty, \hat{x}) &= d(Ty, T\hat{x}) \\ &\leq d(Sy, S\hat{x}) \\ &= d(Sy, \hat{x}) \\ &= d(\hat{x}, C) \end{aligned}$$

Thus $Ty \in P_C(\hat{x})$ and so $T(P_C(\hat{x})) \subset P_C(\hat{x}) = S(P_C(\hat{x}))$. Hence the conclusion follows from Theorem 4 with $D = P_C(\hat{x})$.

Remark:

Theorem 6 contains Theorem 2 (above) and Theorem 3 of Sahab, Khan and Sessa [6] as special cases.

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DIFFERENTIAL OPERATORS IN HALL-LITTLEWOOD SYMMETRIC FUNCTIONS

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Abstract

This paper is concerned with the differential operators which were first introduced by Hammond [3] and were extensively employed in the work of Macmahon [5]. Our aim is to generalised some results proved by Macdonald in [4]. By doing this we obtain a new proof of Marnaghan- Nakayama type formula.

1. Introduction

The operators $D(a_n), D(h_n)$ corresponding to elementary symmetric function 'a_n' and homogeneous symmetric function h_n were introduced by Hammond [3]. The $D(S_\lambda)$ corresponding to Schur function S_λ by Foulk [1], also see Foulk [2]. Moreover in [4] Macdonald considered the operator $D(P_\lambda)$ corresponding to the power sum symmetric function P_λ. Our aim is to generalised the Macdonald results by considering the differential operator over the ring symmetric polynomials $\wedge[t]$

1.1 Basic Definitions

This section contains the basic definitions and results which will be used later. We follow closely the notations used in [4].

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ be a partition of n i.e. $\lambda_1 + \lambda_2 \cdots \lambda_m = n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_m > 0$ we some time write partition in the form $\lambda = (1^{m1} \cdots 1^{m2} \cdots \gamma_r^m)$, where $m_i = m_i(\lambda)$ is the number of parts of λ which are equal to i. If $\lambda = (\lambda_1, \cdots, \lambda_m)$, then $\lambda \rightarrow n$ means λ is a partition of n.

Moreover
$|\lambda| = \lambda_1, \cdots, \lambda_m$ is called weight of the partition λ.

Let $x_1, x_2, \cdots$ be infinite set of indeterminates then $p_r = \sum_{k=1}^{n} x_k^r$, $a_r = \sum x_1, x_2 \cdots x_r$ summed over all selections of r suffixes, repetition of suffixes being allowed are powersum symmetric functions, elementry function and homogeneous symmetric functions respectively.

If χ_p^λ denote the value of irreducible linear character χ^λ of S_n (symmetric group) corresponding to the elements of S_n of type ρ. Then Schur function corresponding to the partition λ of n, is defined by

$$S_\lambda = \frac{1}{n!} \sum_\rho Z_\rho^{-1} \chi_p^\lambda P_\rho$$

where P_ρ is power symmetric function corresponding to the partition ρ, and

$$Z_p = \prod_i \; i^{m_i} \; m_i!$$

where m_i is the number of parts of ρ equal to i. If $\wedge$ denotes the ring of symmetric polynomials. Then Macdonald [4] have shown that S_λ form an orthonormal basis of $\wedge$. Therefore any symmetric function $f \varepsilon \wedge$ is uniquely determined by its scalar product with S_λ, namely

$$f = \sum_\lambda (f, S_\lambda) S_\lambda$$

Now for any partitions λ and μ we define skew S-functions $S_{\lambda/\mu}$ by the relations

$$< S_{\lambda/\mu}, S_v > = < S_\lambda, S_\mu \; S_v > \tag{1.1}$$

for all partitions v. Equivalently, if $g_{\mu v}^\lambda$ are the integers defined by

$$S_\mu S_v = \sum_\lambda g_{\mu v}^\lambda S_\lambda$$

Then we have

$$S_{\lambda/\mu} = \sum_\lambda g_{\mu v}^\lambda S_\lambda$$

In particular, it is clear that $S_{\lambda/0} = S_\lambda$ where 0 denotes the zero partition. Also $g^\lambda_{\mu v} = 0$ unless $|\lambda| = |\mu| + |v|$, so that $S_{\lambda/\mu}$ is homogeneous of degree $|\lambda| - |\mu|$ and is zero if $|\lambda| < |\mu|$.

2. Differential Operators Acting on Ring of Symmetric Function $\wedge$.

Differential operator $D(S_\lambda)$ acting on Schur function S_λ denoted by D_λ defined as

$$D_\lambda = \sum_\rho \frac{\chi^\lambda_\rho}{m_i\,!\,m_2\,!\cdots}\;\frac{\partial^{m_1+m_2+\cdots}}{\partial p_1^{m_1}\partial p_2^{m_2}\cdots}$$

Using equations (1.1) we have

$$< D_\mu S_\lambda, S_v > = < S_\lambda, S_\mu S_v >$$

It follows that

$$D_\mu S_\lambda = S_{\lambda/\mu}$$

Also Macdonald [4] was proved that

$$D(P_n)h_N = h_{N-n}$$

And therefore

$$D(P_n) = \sum_{r\geq 0} h_r \frac{\partial}{\partial h_{n+r}} \tag{2.1}$$

3. The Ring of Symmetric Polynomials $\wedge$ [t].

Let t be an independent indeterminates of the variables $x_1, x = (x_1, x_2, \cdots, x_n)$ and Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$. Also suppose that $P_\lambda = P_\lambda(x_1, \cdots, x_n; t)$ and $Q_\lambda = Q_\lambda(x_1, \cdots, x_n; t)$ be the Hall-Littlewood P and Q-functions respectively given by Macdonald [4]. Now if $b_\lambda(t) = \prod_{i\geq 1} \phi m_i(t)$, where

$$\phi_r(t) = (1-t) = (1-t)\cdots(1-t)^2\cdots(1-t^r)$$

Then

$$Q_\lambda(x;t) = b_\lambda(t)P_\lambda(x;t)$$

And for $r \geq 1$ we defined another function

$$q_r(x;t) = Q_r(x;t)$$

Then we have

$$q_r(x;t) = (1-t)P_r(x;t) = (1-t)$$

Moreover $\{P_\lambda(x;t)|\lambda \;\; n\}$ form Z [t]-basis of $\wedge$ [t] and the sets

$$\{Q_\lambda(x;t) : \lambda \rightarrow n\} \quad \text{and} \quad \{q_\lambda(x;t) : \lambda \rightarrow n\}$$

from Q [t] basis of $\wedge \oplus_z$ Q[t], where Q is ring of rational integers.

We may defined a scalar product on $\wedge[t]$ with values in Q[t] by requiring that bases $q_r(x;t)$ and $m_\lambda(x)$ be dual to each other by

$$< q_\lambda, \; m_\mu >= \delta_{\lambda\mu}$$

where $\delta_{\lambda\mu}$ is Kronecher delta.

3.1 Lemma

Let λ be a partition of n, define $Z_\lambda(t) = Z_\lambda \prod_{\lambda \geq 1}(1-t^{\lambda_i})^{-1}$, then $q_\lambda(x;t)$ $= \sum Z_\lambda(t)^{-1}P_\lambda(x)$ is power sum symmetric function.

Proof

See [6].

3.2 Skew HL-Functions

Let P_λ and Q_λ be HL P and Q-functions. Then for each pair of partitions λ and μ of n we define skew HL-function $Q_{\lambda/\mu}$ by

$$< Q_{\lambda/\mu}, P_v >=< Q_\lambda, P_\mu P_v >= f^\lambda_{\mu v}(t) \tag{3.2}$$

or equivalently

$$Q_{\lambda/\mu} = \sum_{v} f^{\lambda}_{\mu v}(t)$$

where $f^{\lambda}_{\mu v}(t)$ are polynomials defined by

$$P_\mu P_v = \sum_{\lambda} f^{\lambda}_{\mu v}(t) P_\lambda$$

It was proved by Macdonald [4] that

$$p_\lambda(x, z; t) = \sum_{v} \qquad P_{\lambda/\mu}(x; t) P_\mu(z; t) \tag{3.3}$$

where $Z = (z_1, z_2, \cdots, z_n)$ is another set of independent variables.

3. Differential Operator on $\wedge$ [t]

Suppose that $f \varepsilon \wedge$ [t], the ring of symmetric polynomial. Then we define $D(f) : \wedge[t] \to \wedge[t]$ be the adjoint of multiplication by f, that is $< D(f)$u, v $>=<$ u, fv $>$, for all u, v$\varepsilon \wedge [t]$. Then clearly $D : \wedge[t] \to$ End $\wedge$ [t] is a ring homomorphism.

We now investigate the operator for

$$(i) \quad f = Q_\mu(x; t), \qquad (ii) \quad f = q_\lambda(x; t) \quad (iii) \quad f = p_n$$

Case I:

When $f = Q_\mu(x; t) = Q_\mu$ for any partition μ, let D_μ denotes $D(Q_\mu)$. Then according to definition we have

$$\begin{aligned} < D_\mu P_\lambda, \; Q_v > &= < P_\lambda, Q_\mu Q_v > \\ &= < P_{\lambda/\mu}, Q_v > \end{aligned}$$

Therefore we have

$$D_\mu P_\lambda = P_{\lambda/\mu}.$$

Also from equation (3.3) for any $f \varepsilon \wedge [t]$ we have

$$F(x,z;t)=\sum D_\mu f(x;t)P_\mu(z;t).$$

Case II

Now consider $D(q_\lambda)$. From equation (3.1) we obtain

$$\begin{aligned}< D(q_\lambda)m_\mu,\ q_v > &= < m_\mu, q_\lambda q_v > \\ &= < m_\mu, q_{\lambda\cup v} >\end{aligned}$$

which is zero unless $\mu=\lambda\cup v$. Hence it follows that

$$D(q_\lambda)m_\mu=\left\{\begin{array}{ll}0, & unless\ \mu=\lambda U v\\ m_v, & if\ u=\lambda U v\end{array}\right\}$$

and in particular

$$D(q_\lambda)m_\mu=\left\{\begin{array}{ll}0 & if\ n\ is\ not\ a\ part\ of\ \mu\\ m_v & if\ n\ is\ a\ part\ of\ \mu\end{array}\right\}$$

where v is a partition obtained by removing a part n from μ. It follows that for every $f(x_0,x_1,\cdots,x_n)\ \varepsilon\ \wedge[t]$, $(D(q_n)f)\ (x_1,x_2,\cdots,x_m)$ is the coefficient of x_o^n in f.

Case III

Let λ be a partition of N. Consider $D(p_n)$ if $N\geq n$ we have $< D(p_n),\ q_N p_\lambda >=$ $< q_N, p_n p_\lambda >$

From Lemma (3.1) we have

$$q_n = \sum Z_\lambda(t)^{-1}p_\lambda$$

$\prod$

where

$$Z_\lambda(t) = Z_\lambda\prod_{i\geq 1}(1-t^{\lambda_i})^{-1},\quad Z_\lambda=i^{m_i}\ m_i!$$

it follows that

$$q_N q_M = \sum_{\lambda,\mu} \frac{1}{Z_\lambda(t) Z_\mu(t)} \quad q_{\lambda \cup v}$$

Now consider

$$< D(p_n),\ q_N q_M,\ p_v > = < q_N q_M, p_n p_v >$$

where $v \longrightarrow N + M - n$. We wish to prove that

$$D(p_n) q_N q_M = q_{N-n} q_M + q_N q_{M-n}$$

For this purpose we will prove that the coefficient of $p_n p_v$ in $q_N q_M$ and the coefficient of p_v in $q_{N-n} q_M + q_N q_{M-n}$ are the same. Now suppose that

$$\lambda = (1^{a_1}\ 2^{a_2}, \cdots, n^{a_n} \cdots)$$

and

$$\mu = (1^{b_1}\ 2^{b_2},\ \cdots, n^{b_n}, \cdots), \qquad \text{then}$$

$$\lambda U \mu = (1^{a_1+b_1}\ 2^{a_2+b_2} \cdots n^{a_n+b_n} \cdots)$$

and

$$\begin{aligned} Z_{\lambda U \mu}(t) &= \frac{1^{a_1+b_1}!2^{a_2+b_2}(a_2+b_2)! \cdots n^{a_n+b_n}(a_n+b_n)!}{(1-t)^{b_1+a_1}(1-t^2)^{b_2+a_2} \cdots (1-t^n)^{b_n+a_n}} \\ &= Z_\lambda(t) Z_\mu(t) \prod \frac{(a_i+b_i)!}{a_i! b_i!} \\ &= Z_\lambda(t) Z_\mu(t) \left[\prod_{i=1_{i\neq n}}^{N+M} \frac{(a_i+b_i)!}{a_i! b_i!} \right] \frac{(a_n+b_n)!}{a_n! b_n!} \end{aligned}$$

Therefore the coefficient of $p_v\ p_n$, where $v \rightarrow N + M - n$ is

$$\frac{1}{Z_{\lambda U v}(t)} \left[\prod_{i=1_{i\neq n}}^{N+M} \frac{(a_i+b_i)!}{a_i! b_i!} \right] \frac{(a_n+b_n)!}{a_n! b_n!}$$

Similarly the coefficient of p_v in $q_{N-n}\ q_M + q_N q_{M-n}$ is

$$\left[\prod_{i=1_{i\neq n}}^{N+M}\frac{(a_i+b_i)!}{a_i!b_i!}\right]\left\{\frac{(a_n-1+b_n)!}{(a_n-1)!b_n!}+\frac{(a_n+b_n-1)!}{a_n!(b_n-1)!}\right]$$

If we put $a_n+b_n=A_n$, then

$$\frac{(A_n-1)!}{(A_n-b_n-1)!b_n!}+\frac{(A_n-1)!}{(A_n-b_n)!(b_n-1)!}=\frac{A_n!}{(A_n-b_n)!b_n!}$$

Hence we have

$$D(p_n)q_Nq_M=q_{N-n}q_M+q_Nq_{M-n}$$

Now, by induction we can prove that

$$D(p_n)q_{\lambda_1\lambda_2\cdots\lambda_m}=\sum_{i=1}^{m}q_{\lambda_1\lambda_2\cdots\lambda_1-n\cdots\lambda_m}$$

That is, $D(p_n)=\sum q_r\frac{\partial}{\partial_{q_{n+r}}}$ acting on symmetric polynomials expressed as polynomials in q's. Further, since

$$<D(p_n)P_\lambda,p_\mu>=<P_\lambda,p_np_\mu>$$

We obtain

$$<D(p_n)P_\lambda,p_\mu>=\begin{cases}0, & if\ \lambda\neq\mu U(n)\\ Z_\lambda(t), & if\ \lambda=\mu U(n)\end{cases}$$

It follows that

$$<D(p_n)P_\lambda=\begin{cases}0, & if\ n\ is\ not\ a\ part\ of\ \lambda\\ Z_\lambda(t)Z_\mu(t)^{-1}, & if\ \mu\ is\ partition\ obtained\ from\\ & \lambda\ after\ removing\ one\ part\ n\end{cases}$$

We have

$$Z_\lambda(t)Z_\mu(t)^{-1}=nm_n(\lambda)(1-t^n)^{-1}$$

$$(1-t^n)Z_\lambda(t)Z_\mu(t)^{-1}=nm_n(\lambda)$$

Furthermore, we have

$$\begin{aligned} n\frac{\partial}{\partial p_n}P_\lambda &= n\frac{\partial}{\partial p_n}\left[p_1^{m_1}p_2^{m_2}\cdots p_n^{m_n}\cdots\right] \\ &= nm_n[p_1^{m_1}p_2^{m_2}\cdots p_n^{m_{n-1}}\cdots] \\ &= nm_np_\mu \end{aligned}$$

and so we have

$$(1-t^n)D(p_n)P_\lambda = n\frac{\partial}{\partial p_n}P_\lambda$$

or

$$(1-t^n)D(p_n) = n\frac{\partial}{\partial p_n}$$

In particular each $D(p_n)$ is a derivation of $\wedge[t]$. We have already proved that

$$D(p_n)q_N = q_{N-n}$$

Hence it follows that

$$n\frac{\partial}{\partial p_n}q_N = (1-t^n)q_{N-n}$$

which is Marnaghan Nakayama type formula.

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ON CHARACTERIZATION OF THE CHEVALLEY GROUP BY THE SMALLER CENTRALISER OF A CENTRAL INVOLUTION

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Abstract

In this paper we investigate a finite groups G having a subgroup isomorphic to the smaller centraliser of an involution in $F_4(2)$.

1. Introduction

Let $F_4(2)$ denote the Chevalley group of type F_4 over the field $T = \{0, 1\}$. $|F_4(2)| = 2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$.

The Center of a Sylow$_2$ - subgroup S of F_4 is a four group. The elements of order two in this subgroup of S lie in three distinct conjugacy classes in $F_4(2)$. Let t_1, t_2 and $t_3 = t_1t_2$ be these involutions in the center of S. Now in $F_4(2)$:

$$\begin{aligned} C(t_1) &\cong C(t_2) \quad \text{and} \\ C(t_1) \cap C(t_2) &\cong C(t_3) \end{aligned}$$

The Chevalley group $F_4(2^n)$ of type F_4 over the field of 2^n elements have been characterized by Guterman [3] in terms of the centralisers of 2-central involutions and this characterization is given by the following theorem.

Theorem 1

Let G be a finite group. Suppose the center of a Sylow$_2$ - subgroup of G contains elements y_1, y_2 and $y_3 = y_1 y_2$ of order two such that $C_G(y_i) \cong C(t_1) i = 1, 2, 3$ Then $G \cong F_4(2^n)$

In [5] Thomas has given an improved characterization of the Chevalley group $F_4(2^n)$ in terms of only the centraliser $C(t_1)$ for all $n > 2$. Later Husnine [4] treated the case for $n = 1$.

We propose a further improvement on the above result by assuming the smaller centralier. In fact, we make the following conjecture whch is more general.

Conjecture

Let G be a finite simple group with an involution y_3 lying in the center of a Sylow$_2$-subgroup. Suppose $C = C_G(y_3)$ is isomorphic to $C(t_3)$, the centraliser of t_3 in $F_4(2)$. Then G is isomorphic to $F_4(2)$.

We identify C with C_3. We shall refer tables 3 and 4 of [6].

For necessary details about the group $F_4(2)$, we refer the reader to [5].

It is easily observed that $Z(S) = S_{21} \dot{S}_{24}$.

SEC I: Description of $F_4(2)$

1.1

The root system $\sum$ of type (F_4) consists of 48 roots; $\pm\xi_i \pm \xi_i \pm \xi_j$, $\frac{1}{2}(\pm\xi_i \pm \xi_j \pm \xi_m \pm \xi_n)$, where $i, j, m, n = 1, 2, 3, 4$ and i, j, m, n are all distinct. We take $r_1 = \xi_4$, $r_2 = \xi_3 - \xi_4$, $r_5 = \xi_2 - \xi_3$, and $r_{10} = 1/2(\xi_1 - \xi_2 - \xi_3 - \xi_4)$ as a system of fundamental roots. If we denote the root $ar_1 + br_2 + cr_5 + dr_{10}$ by (abcd), then the positive roots are:

$$\begin{array}{llll} r_1 = 1000, & r_2 = 0100, & r_3 = 1100, & r_4 = 2100, \\ r_5 = 0100, & r_6 = 0110, & r_7 = 1110, & r_8 = 2110, \\ r_9 = 2210, & r_{10} = 0001, & r_{11} = 1001, & r_{12} = 1101, \\ r_{13} = 2101, & r_{14} = 1111, & r_{15} = 2111, & r_{16} = 2211, \\ r_{17} = 3211, & r_{18} = 2102, & r_{19} = 2112, & r_{20} = 2212, \\ r_{21} = 3212, & r_{32} = 4212, & r_{23} = 4312, & r_{24} = 4322. \end{array}$$

Let Δ be the additive group generated by $\sum$. We define an inner product $<,>$ on $V = R \otimes \Delta$, the vector space over the real numbers R, by $< \xi_i, \xi_j >= 0$ and $< \xi_i, \xi_j >= 1$ for $i, j = 1, 2, 3, 4;\quad i \neq j$. For $r, s \in \sum$ let $\lambda(r) =< r, r >$ and $s(r) = 2 < s, r > / < r, r >$. The values $\lambda(r_j)$ and $r_j(r_i)$ for $i = 1, 2, 5, 10$ and $1 \leq j \leq 24$ are given in table - 1.

For each $i, 1 \leq i \leq 24$ and each $s \in \sum$ let $\tilde{w}_i(s) = s - s(r_i)r_i$. Then $\tilde{w}_i$ is a permutation of $\sum$. The pemutation group $\tilde{W}$ generated by $\{\tilde{w}, | 1 \leq i \leq 24\}$ is the Weyl group of $\sum$.

1.2 $\tilde{W}$ is of order $2^7 3^2$ and is generated by $\tilde{w}_1, \tilde{w}_2, \tilde{w}_5$ and $\tilde{w}_{10}$. If $a_{ij} = |\tilde{w}_i \tilde{w}_j|$, then generators $\tilde{w}_1, \tilde{w}_2$ and $\tilde{w}_5$ and $\tilde{w}_{10}$ together with the relations

$(\tilde{w}_i \tilde{w}_j)^{a_{ij}} = 1, \{i, j\} \subseteq \{1, 2, 5, 10\}$, form a presentation of $\tilde{W}$.

It will be convenient to think of the elements $\tilde{w} \in \tilde{W}$ as permutations of $\{\pm i | 1 \leq i \leq 24\}$ defined as follows:

$$\tilde{w}(i) = \begin{cases} j \text{ if } \tilde{w}(r_i) = r_j; \quad \tilde{w}(-i) = -\tilde{w}(i) \\ -j \text{ if } \tilde{w}(r_i) = -r_j \end{cases}$$

The values $\tilde{w}_i(j)$ for $i = 1, 2, 5, 10$ and $1 \leq i \leq 24$ are also included in Table-1

Table - 1

i	$\tilde{w}_1(i)$	$\tilde{w}_2$ (i)	$\tilde{w}_5(i)$	$\tilde{w}_{10}(i)$	$r_i(r_1)$	$r_i(r_2)$	$r_i(r_5)$	$r_i(r_{10})$	$\lambda(r_i)$	$\bar{i}$
1	-1	3	1	11	2	- 1	0	-1	1	2
2	4	- 2	6	2	- 2	2	- 1	0	2	1
3	3	1	7	12	0	1	- 1	- 1	1	4
4	2	4	8	18	2	0	- 1	- 2	2	3
5	5	6	- 5	5	0	- 1	2	0	2	10
6	8	5	2	6	- 2	1	1	0	2	11
7	7	7	3	14	0	0	1	- 1	1	18
8	6	9	4	19	2	- 1	1	- 2	2	12
9	9	8	9	20	0	1	0	- 2	2	13
10	11	10	10	- 10	- 1	0	0	2	1	5
11	10	12	11	1	1	- 1	0	1	1	6
12	13	11	14	3	- 1	1	- 1	1	1	8
13	12	13	15	13	1	0	- 1	0	1	9
14	15	14	12	7	- 1	0	1	1	1	19
15	14	16	13	15	1	- 1	1	0	1	20
16	17	15	16	16	- 1	1	0	0	1	22
17	16	17	17	21	1	0	0	- 1	1	23
18	18	18	19	4	0	0	- 1	2	2	7
19	19	20	18	8	0	- 1	1	2	2	14
20	22	19	20	9	- 2	1	0	2	2	15
21	21	21	21	17	0	0	0	1	1	24
22	20	23	22	22	2	- 1	0	0	2	16
23	23	22	24	23	0	1	- 1	0	2	17
24	24	24	23	24	0	0	1	0	2	21

1.3 $\tilde{W}$ acts transitively on $\{r \in \sum |\lambda(r) = i\}, \quad i = 1, 2.$

Let Γ be a field with two elements and let F be the Chevalley group of type (F_4) over Γ. Then F has the following properties.

1.4 F is simple.

1.5 F has non trivial center.

1.6 For each $i, 1 \leq i \leq 24$, there exists a homomorphism $\phi : \ SL(2,2) \rightarrow F$.

For each $\alpha \in \Gamma$, we define.

$$x_i(\alpha) = \phi \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad x_{-1}(\alpha) = \phi_i \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad w_i = \phi_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that $w_i^2 = x_i(\alpha)^2 = (w_i x_i(1))^3 = 1$

1.7 For each $i, 1 \leq i \leq 24$, let $S_i = \{x_i(\alpha) | \alpha \in \Gamma\}$. Then each S_i is a group of order 2. The elements of S_i multiply according to the rule $x_i(\alpha)x_i(\beta) = x_i(\alpha + \beta), \alpha, \beta \in \Gamma$.

1.8 Let $S =< S_i | 1 \leq i \leq 24 >$. Then S is a Sylow_2- subgroup of F.

Any element $x \in S$ can be expressed uniquely in the form $x = \Pi_{i=1}^{24} x_i(\alpha_i)$ Which we shall abbreviate as $x = \Pi x_i(\alpha_i)$. Hence S has order 2^{24}. The product of any two elements of S may be obtained by use of the commutators $[x_i(1), x_j(1)], 1 \leq i \leq 24$. The nontrivial commutators are listed in Table-2.

2 In this section we collect the definitions and known results which are to be used in this sequel. We define the conjugate of y under x to be $x^{-1}yx$ for the elements x, y of the group G. We also write y^x for $x^{-1}yx$ and H^x for $x^{-1}Hx$ for the elements x and y and the subset H of G. The set of all conjugates of an element of y in G is denoted by $\text{ccl}_G(y)$.

2.1 [4]. Let $W_1 =< w_1, w_2, w_5 >$ and $C_1 = \{sws' | s \in S, s' \in S_w, w \in W_1\}$.

Then $C_1 = C_F(x_{21}(1))$ *(ii)* $\mathcal{Z}(C_1) = S_{21}$

2.2 [4]. Let $W_2 =< w_1, w_2, w_{10} >$ and $C_2 = \{sws' | s \in S, w \in W_2, s' \in S_w\}$

Table-2

Values $(i, j : m)$ for which $[x_i(1), x_j(1)] = x_m(1)$			
(1, 10 : 11)	(1, 12 : 13)	(1, 14 : 15)	(1, 16 : 17)
(1, 5 : 6)	(2, 8 : 9)	(2, 19 : 20)	(2, 22 : 23)
(3, 10 : 12)	(3, 11 : 13)	(3, 14 : 16)	(3, 15 : 17)
(4, 5 : 8)	(4, 6 : 9)	(4, 29 : 22)	4, 20 : 23)
(5, 18 : 19)	(5, 23 : 24)	(6, 18 : 20)	(6, 22 : 24)
(7, 10 : 14)	(7, 11 : 15)	(7, 12 : 16)	(7, 13 : 17)
(8, 18 : 22)	(8, 20 : 24)	9, 18 : 23)	(9, 19 : 24)
(10, 17 : 21)	(11, 16 : 21)	(12, 15 : 21)	(13, 14 : 21)
Values $(ij : m, n)$ for which $[x_i(1), x_j(1)] = x_m(1)x_n(1)$			
(1, 2 : 3, 4)	(1, 6 : 7, 8)	(1, 20 : 21, 22)	
(3, 5 : 7, 9)	(3, 19 : 21, 23)	(7, 18 : 21, 24)	
(2, 11 : 12, 18)	(2, 15 : 16, 24)	(4, 10 : 13, 18)	
(4, 14 : 17, 24)	(5, 12 : 14, 20)	(5, 13 : 15, 22)	
(6, 11 : 14, 19)	(6, 13 : 16, 23)	(8, 10 : 15, 19)	
(8, 12 : 17, 23)	(9, 10 : 16, 20)	(9, 11 : 17, 22)	

Then $C_2 = C_F(x_{24}(1))$; $Z(C_2) = S_{24}$

2.3 [4]. Let $W_3 =< w_1, w_2 >$ and $C_3 = \{sws'|s \in S, w \in W_3, s' \in S_w\}$

Then $C_3 = C_F(x_{21}(1)x_{24}(1))$.

From now onwards, since the only involution in any root subgroup S_i of $F_4(2)$ is $x_i(1)$, we will write x_i for $x_j(1)$ except where there is ambiguity.

2.4 ([3];.4.1). Let v be an automorphism of S. Then

$$\{v(S_{21}), v(S_{24})\} = \{S_{21}, S_{24}\}$$

2.5 For each $j = 1, 2, 5, 10$, let $D_j = \Pi S_i(i \neq j)$.

Then $D_{10} = C(x_{17})$ and $D_5 = C_s(x_{23})$. We write M for D_{10}. Then M and D_5 and subgroups S of order 2^{23} with centers of order 2^3. For our convenience we write D for D_5.

2.6 For any three elements x, y, z of a group G, we have,

$$[x, yz] = [x, y]^z [x, z]$$

2.7 Let x be an involution of S. Then x is conjugate in S to an involution of one of the forms $\Pi_{i \in 1} \; x_i$ or $\phi\left(\pi_{i \in 1} x_i\right)$ where I is one of the sets listed in Table 3.

The integers c(1) listed in Table 3 satisfy $\text{ccl}_s \phi \left(\Pi_{i \in 1} x_i\right) 2^{c(I)}$.

2.8 Let $x \in S$ be an involution. Then there is an involution y os one of the forms listed Table 4, such that x is conjugate in C_3, the centralizer of $x_{21}x_{24}$ in $F_4(2)$, to y or to $\phi(y)$, where ϕ is the graph automorphism of $F_4(2)$ defined in (2.14). The sets $F_3(y)$, listed in Table 4 are complete sets of respresentatives of distinct c conjugacy classes of involutions in S, which are fused in C_3 to give $C \cap$ $\text{cll}_{C_3}(y)$. The integers $d(y)$ i Table 4 satisfy $d(y) = |S \cap ccl_{c_3}(y)|$.

Table 3

I	$c(I)$	I	$c(I)$	I	$c(I)$	I	$C(I)$	I	$c(I)$
24	0	9,17	4	7,18	7	4,18,21	7	5,7,17	9
21,24	0	9,17,21	4	7,18,21	7	4,16	8	5,7,17,21	9
23	1	9,16	5	7,18,16	7	4,16,18	8	5,7,16	10
2,23	1	9,16,21	5	7,18,16,21	7	4,15	9	5,7,16,21	10
17,23	2	9,15	5	7,18,14	9	4,15,18	9	5,11	11
22	2	9,14	7	6	6	4,12	10	5,10	12
21,22	2	9,13	7	6,21	6	4,11	11	2	7
17,22	3	18	8	6,17	7	4,7	10	2,24	7
16,22	4	18,21	5	6,16	6	3,4	8	2,21	7
20	3	17,18	5	6,16,21	6	3,4,21	8	2,21,24	7
20,21	3	16,18	6	6,16,17	7	3,4,21,24	8	2,18	8
17,20	4	15,18	7	6,15	9	3,4,7	10	2,18,24	8
16,20	4	14,18	8	6,14	7	5	7	2,17	8
16,17,20	4	13,18	9	6,14,21	7	5,21	7	2,17,24	8
15,20	6	13,18,24	6	6,14,17	8	5,17	8	2,17,18	9
19	4	13,18,20	6	6,14,15	9	5,16	9	2,14	10
19,21	4	13,18,19	7	6,12	10	5,15	7	2,14,18	10
17,19	5	8	8	6,7	8	5,15,21	7	2,13	10
16,19	6	8,21	5	6,7,21	8	5,15,17	8	2,13,18	10
15,19	5	8,17	5	6,7,17	8	5,15,16	9	2,10	12
15,17,19	5	8,17,21	5	6,7,17,21	8	5,14	8	2,7	11
15,19,20	5	8,16	5	6,7,15	10	5,14,21	8	2,3	9
14,19	6	8,15	7	4	6	5,14,17	9	2,23,24	9
14,19,22	6	8,15,21	6	4,24	6	5,14,16	9	2,3,21	9
14,15,19,20	6	8,15,16	6	4,21	6	5,14,15,16	9	2,3,21,24	9
9	4	8,14	7	4,21,24	6	5,7	9	2,3,18	11
9,21	4	8,13	9	4,18	7	5,7,21	9	2,3,7	11

Table 4

y	F_3 (y)	d (y)
x_{21}	$\{x_{21}\}$	1
$x_{21}x_{24}$	$\{x_{21}, x_{24}\}$	1
x_{23}	$\{x_i \| i = 19, 20, 22, 23\}$	$2(2^4-1)$
$x_{21}x_{23}$	$\{x_{21}u \| u \in F_3(x_{23})\}$	$2(2^4-1)$
$x_{17}x_{23}$	$\{\Pi_i \in Ix_i \| I = \{17, 23\}$ $\{17, 22\}, \{16, 23\}, \{16, 20\}, \{15, 22\},$ $\{15, 19\}, \{14, 20\}, \{14, 19\}$	$2^2(2^2-1)(2^4-1)$
$x_{16}x_{22}$	$\{17, 19\}, \{16, 22\}, \{16, 19\}, \{16, 17, 20\},$ $\{15, 20\}, \{15, 23\}, \{15, 17, 19\}, \{15, 22, 23\},$ $\{15, 19, 20\}, \{14, 22\}, \{14, 23\}, \{14, 16, 19\},$ $\{14, 15, 20\}, \{14, 20, 23\}, \{14, 19, 22\},$ $\{14, 15, 19. 20\}$	$2^4(2^2-1)(2^4-1)$
x_9	$\{x \| I = 5, 6, 8, 9\}$	$2^4(2^4-1)$
x_9x_{21}	$\{ux_{21} \| u \in F_3(x_9)\}$	$2^4(2^4-1)$
$x9x17$	$\{\Pi_i \in Ix_i \| I = \{9, 17\}, \{9, 16\}$ $\{8, 17\}, \{8, 15\}, \{6, 16\}, \{6, 14\}, \{5, 15\},$ $\{5, 14\}$	$2^4(2^4-1)(2^4-1)$
$x_9x_{17}x_{21}$	$\{ux_{21} \| u \in F_3(x_9x_{17}\}$	
x_9x_{15}	$\{\Pi_i \in Ix_i \| I = \{9, 15\}, \{9, 14\}$ $\{8, 16\}, \{8, 14\}, \{8, 15, 16\}, \{6, 17\}, \{6, 15\},$ $\{6, 14, 17\}, \{6, 16, 17\}, \{6, 16, 17\}, \{6, 14, 15\}$ $\{5, 17\}$ $\{15, 16\}, \{5, 14, 17\}, \{5, 15, 16\}$ $\{5, 14, 15, 16\}. \{5, 15, 17\}$ $\{5, 14, 16\}$	$2^7(2^2-1)(2^4-1)$
x_9x_{13}	$\{\Pi_i \in Ix_i \| I = \{9, 13\}, \{9, 12\},$ $\{8, 11\}, \{8, 13\}, \{6, 10\}$ $\{6, 12\}, \{5, 10\}, \{5, 11\}$	$2^8(2^2-1)(2^4-1)$
x_{18}	$\{x_{18}\}$	2^5
$x_{18}x_{21}$	$\{x_{18}, x_{21}\}$	2^5
$x_{14}x_{18}$	$\{x_ix_{18}i\| = 14, 15, 16, 17\}$	$2^6(2^4-1)$
$x_{13}x_{18}$	$\{x_ix_{18}\|10, 11, 12, 13\}$	$2^6(2^4-1)$
$x_{13}x_{18}x_{24}$	$\{ux_{24}\|u \in F_3(x_{13}x_{18})\}$	$2^6(2^4-1)$
$x_{13}x_{20}x_{18}$	$\{(\Pi_i \in Ix_i)x_{18}\|I = \{13, 20\}$ $\{13, 19\}, \{12, 22\}, \{12, 19\}, \{12, 23, 24\},$ $\{11, 23\}, \{11, 20\}, \{11, 23, 24\}, \{11, 23, 24\}, \{10, 23\}$ $\{10, 22\}, \{10, 23, 24\}, \{10, 22, 24\},$	$2^7(2^2-1)(2^4-1)$
x_4	$\{x_i\|i = 2, 4\}$	$2^6(2^2-1)$

Table 4 (Continued)

y	$F_3(y)$	$d(y)$
x_4x_{21}	$\{ux_{21}\mid u \in F_3(x_4)\}$	$2^6(2^2-1)$
x_4x_{24}	$\{ux_{24}\mid u \in F_3(x_4)\}$	$2^6(2^2-1)$
$x_4x_{21}x_{24}$	$\{ux_{21}x_{24}\mid u \in F_3(x_4)\}$	$2^6(2^2-1)$
x_4x_{16}	$\{(\Pi_i \in Ix_i)x_{18}\mid I = \{4,16\},\{4,15\}$ $\{2,17\},\{2,14\},\{2,17,24\}\}$	$2^8(2^2-1)^2$
x_4x_7	$\{ux_7\mid u \in F_3(x_4)\}$	$2^{10}(2^2-1)$
x_4x_{18}	$\{ux_{18}\mid u \in F_3(x_4)\}$	$2^7(2^2-1)$
$x_4x_{18}x_{21}$	$\{ux_{18}x_{21}\mid u \in F_3(x_4)\}$	$2^7(2^2-1)$
$x_4x_{16}x_{18}$	$\{(\Pi_i \in Ix_i)x_{18}\mid I = \{4,16\},\{4,15\},\{2,17\},$ $\{2,14\}$	$2^8(2^2-1)^2$
x_4x_{12}	$\{(\Pi_i \in Ix_i)x_{18}\mid I = \{4,12\},\{4,11\},\{2,13\},$ $\{2,10\},\{2,13,18\}$	$2^{10}(2^2-1)^2$
x_3x_4	$\{(\Pi_i \in Ix_i)x_{18}\mid I = \{3,4\},\{3,2\},\{1,4\}$	$2^8(2-1)^2(1+2.2)$
$x_3x_4x_{24}$	$\{ux_{24}\mid u \in F_3(x_3x_4)\}$	$2^8(2-1)^2(1+2.2)$
$x_3x_4x_{21}x_{24}$	$\{ux_{21}x_{24}F_3(x_3x_4)\}$	$2^8(2-1)^2(1+2.2)$
$x_3x_4x_{18}$	$\{ux_{18}\mid u \in F_3(x_3x_4)\}$	$2^{10}(2-1)^2(1+2.2)$

2.9 Burnside's theorem: Let S be an Sylow-p-subgroup of G. If elements of $Z(S)$ are conjugate in G, then they are conjugate in $N_G(S)$.

We investigate the action of $N_G(D)$ on $Z_3(D) = S_{16}S_{20}Z_2(D)$ in the next section. For this we denote by $Z_1(D)$ the center of D and by $Z_i(D)$ the inverse imageof the center of $D/Z_{i-1}(D)$ in D for $i > 1$. It is easy to calculate $Z_i(D)$ from the table 2.

3. Action of $N_G(D)$ on $Z_3(D)$

In this section we prove the following theorem.

Theorem A:

There exists an involution $u\varepsilon N_G(D)$ which acts upon $Z_3(D) = S_{16}S_{20}Z_2(D)$

such that $x_{24}^u = x_{23}$ and u centralises $x_{16}, x_{17}, x_{20}, x_{21}, x_{22}$.

The proof is completed in the sequence of Lemmas.

3.1 Lemma:

$$\begin{aligned} Z_1(D) &= S_{23}S_{21}S_{24} \\ Z_2(D) &= S_{17}S_{22}Z_1(D) \\ Z_3(D) &= S_{16}S_{20}Z_2(D) \end{aligned}$$

Proof:

It is easy to verify from tbale 2.

3.2 Lemma

$$N_{C_3}(S) = S$$

It is directly verified from the structure of C_3 and from tables 1 & 2.

3.3 Lemma

$N_G(S) = S$. Thus no two elements of $Z(S)$ are conjugate in G.

Proof

Let $x \varepsilon N_G(S)$

Then x induces an inner automorphism of S. Thus by 2.4.

$$[S_{21}^x, S_{24}^x] = [S_{21}, S_{24}]$$

Hence $x_{21}x_{24}$ is centralised by x. So $x \in C_3$. Thus $x \in N_{c_3}(S)$.

Now, due to lemma 3.2, $x \in S$. Hence $N_G(S) = S$.

Finally, by Burnside's theorem, it is obvious that no two elements of $Z(S)$ are conjugate in G.

Lemma 3.4

$N_G(D) = S$, Hence $N_G(D) \cap C_3 = S$

Proof

From structure of C_3 and table 1, we find that normaliser of D in C_3 is S.

3.5 Lemma

S is not normal in $N_G(D)$.

Proof

If S is normal in $N_G(D)$, then according to [8, sec3], G has a normal complement to C_3 of odd order. We get a contratiction as G is a simple group according to hypothesis of theorem C.

3.6 Lemma

$$\frac{N_G(D)}{D} \equiv S_3$$

Proof

Due to Lemma 3.4, there is no element in $N_G(D) - S$ which belong to C_3 where C_3 is the centraliser of $x_{21}x_{24}$.

According to Lemma 3.3, no two elements of $Z(S)$ are conjugate in G. As $Z(D) - Z(S)$ has four involutions. So in $N_G(D), x_{21}x_{24}$ can be conjugate to itself and four involutions in $Z(D) - Z(S)$.

Hence $[N_G(D) : S]$ is less than or equal to 5.

If $5/N_G(D)$, then there will be an element of order 5 in $N_G(D)$. Thus $x_{21}x_{24}$ is conjugate in $N_G(D)$ to every involution in $Z(D) - Z(S)$.

Now $Z(S) \subset Z(D)$, $x_{21}x_{24} \in Z(S)$ and

$(x_{21}x_{24})^{N_G(D)} \cap Z(D) = \{x_{21}x_{24}\} \cup x_{23}Z(S)$. This according to { 9, Prop. 2} implies that $x_{21}x_{24}$ is centralized by $N_G(D)$. This is contradiction to Lemma

3.4.

Hence $[N_G(D):S]$ is not exactly 5.

Also $[N_G(D):S] \neq 2$ or 4. Since S is Sylow.2. Subgroup.

So, we are left with $[N_G(D):S] = 3$ or 1.

But $[N_G(D):S] = 1$ contradicts lemma 3.5.

So we are left with $[N_G(D):S] = 3$ As S is not normal in $N_G(D)$, therefore, $N_G(D)/D$ is non-abelain.

And, we know that a non-abelian group of order 6 is isomorphic to S_3

So, $N_G(D)/D = S_3$

Due to above theorem 3.6, we can choose $g_1 \in P_1$ and $g_2 \in P_2$, where $g_2 = g_1\ x_5 g_1$ and P_1, P_2 are distinct Sylow 2-subgroup of $N_G(D)$ other than S and g_1 is an involution in $P_1 - D$. Then both g_1 and g_2 normalize $Z_i(D)$ for all $i = 1, 2, 3, 4, 5, 6, 7$.

Where $Z_7(D) = D$. Also $g_2 g_1 g_2 \in S$ and $g_1 w_1 g_1, g_2 w_2 g_2 \in C_3$

3.7 Lemma

There is an involution u in $N_G(D)$ such $x_{24}^{u} = x_{23}$ and centralizes x_{17}, x_{21}, x_{22}. Hence we can assume $x_{24}^{g_1} = x_{23}$ and g_1 centralizes x_{17}, x_{21}, x_{22}.

Proof

First of all we examine the action of g_1 on $Z_G(D)$. Since $g_1 \notin C_3$, and we know that no two elements of $Z(S)$ are conjugate in G, therefore

$$\begin{aligned} x_{21}^{g1} &\in Z(D) - Z(S) \\ x_{24}^{g1} &\in Z(D) - Z(S) \end{aligned}$$

Now

$$Z(D) = S_{23} S_{21} S_{24}$$

$$\begin{aligned} &= \{x_{23}, x_{21}, x_{24}, x_{23}x_{21}, x_{23}x_{24}, x_{21}x_{24}, x_{23}x_{21}x_{24}1\} \\ Z(S) &= S_{21}S_{24} \\ &= \{x_{21}, x_{24}, x_{21}x_{24}, 1\} \end{aligned}$$

Due to Lemma 3.2, and commutator relations in S we are left with the following possibilities.

(i)

$x_{24}^{g1} = x_{23}x_{21}x_{24}$ this shows that $[x_{24}(x_{23}, x_{21}, x_{24})]$ is fixed by g_1.

$$x_{24}^{g_2} = x_{24}^{g_1 x_5 g_1} = (x_{23}x_{21}x_{24})^{x_5 g_1} = (x_{23}x_{21})^{g_1} = x_{23}x_{21}$$

$\Longrightarrow x_{24}^{g_2}x_{23}x_{21}$

$$x_{21}^{g_1} \in \{x_{23}, x_{24}\}; \qquad x_{21}^{g_2} \in \{x_{23}x_{24}, x_{24}\}$$

(ii)

$$x_{24}^{g_1} = x_{23} \quad ; \quad x_{24}^{g_2} = x_{24}^{g_1 x_5 g_1} = x_{23}^{x_5 g_1} = (x_{23}x_{24})^{g_1} = x_{23}x_{24}$$

Thus $x_{24}^{g_2} = x_{23}x_{24}$

$$\begin{aligned} x_{21}^{g_1} &\in \{x_{23}x_{21}x_{24}, x_{24}\} \\ x_{21}^{g_2} &\in \{x_{23}x_{21}, x_{24} \end{aligned}$$

(iii)

$$\begin{aligned} x_{24}^{g_1} &= x_{23}x_{21} \\ x_{24}^{g_1} &= x_{24}^{g_1 x_5 g_1} = (x_{23}x_{21})^{x_5 g_1} = (x_{23}x_{21}x_{24})^{g_1} = x_{23}x_{21}x_{24} \end{aligned}$$

therefore $x_{24}^{g_2} = x_{23}x_{21}x_{24}$

and

$$\begin{aligned} x_{21}^{g_1} &\in \{x_{23}x_{24}, x_{24}\} \\ x_{21}^{g_2} &\in \{x_{23}, x_{24}\} \end{aligned}$$

(iv)

$$
\begin{aligned}
x_{24}^{g_1} &= x_{23}x_{24} \qquad \Rightarrow \qquad x_{23}^{g_1} = x_{23} \\
x_{24}^{g_1} &= x_{24}^{g_1 x_5 g_1} \\
&= (x_{23}x_{24})^{x_5 g_1} \\
&= x_{23}^{g_1} = x_{23} \\
\text{therefore } x_{24}^{g_2} &= x_{23}
\end{aligned}
$$

and

$$
\begin{aligned}
x_{21}^{g_1} &\in \{x_{23}x_{21}, x_{24}\} \\
x_{21}^{g_2} &\in \{x_{23}x_{21}x_{24}, x_{24}\}
\end{aligned}
$$

since role of g_1 and g_2 can be interchanged.

$$
\begin{array}{lllllll}
(a) & x_{24}^{g_1} & = & x_{23}x_{21}x_{24} & ; & x_{21}^{g_1} = & x_{23} \\
(b) & x_{24}^{g_1} & = & x_{23}x_{24} & ; & x_{21}^{g_1} = & x_{23}x_{21} \\
(c) & x_{24}^{g_1} & = & x_{23}x_{21} & ; & x_{21}^{g_1} = & x_{21} \\
(d) & x_{24}^{g_1} & = & x_{23} & ; & x_{21}^{g_1} = & x_{21}
\end{array}
$$

Next, we examine the action of g_1 on $Z_2(D) = S_{17}S_{22}Z_1(D)$ since

$\bar{S}_{17}\bar{S}_{22} \subseteq Z(\bar{S})$ where $\bar{S} = S/Z(D)$ and $\bar{S}$ is self-normalizing in $N_G(D)/Z(D)$. we have due to Burnside's theorem that $\bar{S}_{17}$ and $\bar{S}_{22}$ are centralized by g_1.

$$\bar{S}_{17} = x_{17}S_{23}S_{21}S_{24}$$

Possibilities

$$
\begin{array}{lllllll}
(i) & x_{17}^{g_1} & = & x_{17} & & & \\
(ii) & x_{17}^{g_1} & = & x_{17}x_{23} & \Rightarrow & x_{23}^{g_1} & = x_{23} \\
(iii) & x_{17}^{g_1} & = & x_{17}x_{21} & \Rightarrow & x_{21}^{g_1} & = x_{21} \\
(iv) & x_{17}^{g_1} & = & x_{17}x_{24} & \Rightarrow & x_{24}^{g_1} & = x_{24} \\
(v) & x_{17}^{g_1} & = & x_{17}x_{21}x_{23} & \Rightarrow & (x_{21}x_{23})^{g_1} & = x_{21}x_{23} \\
(vi) & x_{17}^{g_1} & = & x_{17}x_{23}x_{24} & \Rightarrow & (x_{23}x_{24})^{g_1} & = x_{23}x_{24} \\
(vii) & x_{17}^{g_1} & = & x_{17}x_{21}x_{24} & \Rightarrow & (x_{21}x_{24})^{g_1} & = x_{21}x_{24} \\
(viii) & x_{17}^{g_1} & = & x_{17}x_{23}x_{21}x_{24} & \Rightarrow & (x_{21}x_{23}x_{24})^{g_1} & = x_{21}x_{23}x_{24}
\end{array}
$$

Now **Case a:** Contradicts with (ii)-(viii) and we are left with the possibility $x_{17}^{g_1} = x_{17}$.

but

$$x_{23} = x_{21}^{g_1} = [x_{10}, x_{17}]^{g_1}$$

from table no.2.

$$\begin{aligned} &\Rightarrow \quad [x_{10}^{g_1}, x_{17}^{g_1}]^{g_1} = x_{23} \\ &\Rightarrow \quad x_{10}^{g_1} x_{17} x_{10}^{g_1} = x_{23} x_{17} \\ &\Rightarrow \quad x_{17}^{x_{10}^{g_1}} = x_{23} x_{17} \end{aligned}$$

This shows that x_{17} is conjugate to $x_{23}x_{17}$ in S, which is contradiction to table no. 3. So all possibilities are eliminated by case a.

Case b:

$$(i) \qquad x_{17}^{g_1} = x_{17} x_{23}$$

and from table 2.

$$[x_{10}^{g_1}, x_{17}^{g_1}] = x_{21}^{g_1}$$

$$[x_{10}^{g_1}, x_{17} x_{23}] = x_{23} x_{21}$$

$$\Rightarrow (x_{17} x_{23})^{x_{10}^{g_1}} = x_{17} x_{21}$$

$$\Rightarrow x_{17} x_{23} \sim x_{17} x_{21}$$

Put

$$\begin{aligned} \phi(x_{17}) &= x_{23} \\ \phi(x_{23}) &= x_{17} \\ \phi(x_{21}) &= x_{24} \end{aligned}$$

We get $x_{17}x_{23} \sim x_{23}x_{24} \sim x_{23}$

$$\Rightarrow x_{17}x_{23} \sim x_{23}$$

but table 3 shows that $x_{17}x_{23}$ is not conjugate to x_{23}

This gives contradiction to the structure of S.

Next

$$x_{17}^{g_1} = x_{17}x_{21}x_{24}$$

$$[x_{10}^{g_1}, x_{17}^{g_1}] = x_{21}^{g_1} \quad \text{From table no 2}$$

$$[x_{10}^{g_1}, x_{17}x_{21}x_{24}] = x_{23}x_{21}$$

$$\Rightarrow x_{17}x_{21}x_{24} \sim x_{17}x_{23}x_{24}$$

$$\Rightarrow x_{17}x_{21}x_{24} \sim x_{17}x_{23}x_{24} \sim x_{17}x_{23}$$

$$\Rightarrow x_{17}x_{21}x_{24} \sim x_{17}x_{24} \sim x_{17}x_{23}$$

$$\Rightarrow x_{17}x_{24} \sim x_{17}x_{23}$$

But under graph automorphism ϕ. from table no. 1.

$$\phi(x_{17}) = x_{23}$$

$$\phi(x_{24}) = x_{21}$$

$$\phi(x_{23}) = x_{17}$$

Hence we get

$$x_{23}x_{21} \sim x_{23}$$

From table no. 4, we see that $x_{23}x_{21}$ and $x_{17}x_{23}$ belong to two different conjugacy classes, so in this case we get contradiction.

Case c

$$x_{24}^{g_1} = x_{23}x_{21}$$

and

$$x_{21}^{g_1} = x_{21}$$

gives us

$$(x_{21}x_{24})^{g_1} = x_{23}$$

Now,

$$[x_5, x_{23}]^{g_1^{-1}} = x_{24}^{g_1^{-1}}$$

$$[x_5^{g_1^{-1}}, x_{21}x_{24}] = x_{23}x_{21}$$

$$\Rightarrow (x_{21}x_{24})^{x_5^{g_1^{-1}}} = x_{23}x_{24}$$

Hence

$$x_{21}x_{24} \sim x_{23}x_{24} \sim x_{23}$$

$$\Rightarrow x_{21}x_{24} \sim x_{23}$$

This is contradiction from table 3.

Case d

$$x_{21}^{g_1} = x_{21}$$

and

$$x_{24}^{g_1} = x_{23}$$

leaves us with the possibility

$$x_{17}^{g_1} = x_{17}x_{21}$$

Thus

$$x_{17}^{g_1} = x_{17}$$

Next, by considering

$$\bar{S}_{22} = x_{22}S_{23}S_{21}S_{24}$$

Possibilities

$$\begin{array}{llllll}
(i) & x_{22}^{g_1} & = & x_{22} & & \\
(ii) & x_{22}^{g_1} & = & x_{22}x_{23} & \Rightarrow & x_{23}^{g_1} = x_{23} \\
(iii) & x_{22}^{g_1} & = & x_{22}x_{21} & \Rightarrow & x_{21}^{g_1} = x_{21} \\
(iv) & x_{22}^{g_1} & = & x_{22}x_{24} & \Rightarrow & x_{24}^{g_1} = x_{24} \\
(v) & x_{22}^{g_1} & = & x_{22}x_{23}x_{21} & \Rightarrow & (x_{21}x_{23})^{g_1} = x_{21}x_{23} \\
(vi) & x_{22}^{g_1} & = & x_{22}x_{23}x_{24} & \Rightarrow & (x_{23}x_{24})^{g_1} = x_{23}x_{24} \\
(vii) & x_{22}^{g_1} & = & x_{22}x_{21}x_{24} & \Rightarrow & (x_{21}x_{24})^{g_1} = x_{21}x_{24} \\
(viii) & x_{22}^{g_1} & = & x_{22}x_{23}x_{21}x_{24} & \Rightarrow & (x_{21}x_{23}x_{24})^{g_1} = x_{21}x_{23}x_{24}
\end{array}$$

Now **Case a:** Contradicts with (ii) - (viii) and we are left with the possibility.

$$x_{22}^{g_1} = x_{22}$$

But

$$[x_2^{g_1}, x_{22}^{g_1}] = x_{23}^{g_1}$$

$$[x_2^{g_1}, x_{22}] = x_{21}^{g_1}$$

$$\Rightarrow x_{22} \sim x_{22}x_{21}$$

This is contradiction in S as x_{22} and $x_{22}x_{21}$ belong to different conjugacy classes in S.

Cases (b), (c) and (d) $\Rightarrow$ g_1 carries x_{17} to x_{17} or $x_{17}x_{21}$and x_{22} to x_{22} or $x_{22}x_{23}x_{24}$

g_1, g_2 are involutions belonging to $N(D)$

Also

$$N(D)/D \cong SL(2,2)$$

Let

$$|g_1g_1| = 3.2^r \quad \text{for some } r$$

Writing $g = (g_1g_2)^{2r}$, we have $|g| = 3$

We have eliminated that part of g_1g_2 which belongs to D, that is which fixes x_{23}, x_{21}, x_{24}. So the action of g on D can be determined by that of g_1g_2.

Now,

$$|S_{17}S_{24}| = 4$$

$$S_{17}S_{21} = [x_{17}, x_{21}, x_{17}, x_{21}, I]$$

Since g_1, g_2 both centralize x_{21}, the remaining two elements $x_{17}, x_{17}x_{21}$ of $S_{17}S_{21}^{\#}$ must be centralized by g_1g_2 and g will move on all points of $S_{23}S_{24}$ except identity.

Without loss of generality, we can assume

$$x_{23}^g = x_{24}, x_{24}^g = x_{23}x_{24}, \quad (x_{23}x_{24})^g = x_{23}$$

$$\text{therefore} \quad g^3 = 1$$

$$\Rightarrow g^2 = g^{-1}$$

Let $u_1 = g^2 x_5 g = g^{-1} x_5 g = x_5^g$ then

$$x_{23}^{u_1} = x_{23}^{g^{-1}x_5 g} = (x_{23}x_{24})^{x_5 g} = x_{23}^{g} = x_{24}$$

$$x_{17}^{u_1} = x_{23}^{g^{-1}x_5 g} = x_{17}$$

$$x_{21}^{u_1} = x_{21}^{g^{-1}x_5 g} = x_{21}$$

Therefore u_1 takes x_{23} to x_{24} and centralizes x_{17} and x_{21}.

Let

$$x_{21}^{u_1} = x_{22}x_{23}x_{24}$$

Then

$$\begin{aligned} x_{22}^{u_2} &= x_{22}^{u_1 x_5 u_1} \\ &= (x_{22}x_{23}x_{24})^{x_5 u_1} \\ &= (x_{22}x_{23})^{u_1} \\ &= x_{22}x_{23} \end{aligned}$$

As before

Let

$$|u_1 u_2| = 3.2^s$$

and write

$$h = (u_1 u_2)^{2s}$$

so that $|h| = 3$

and h centralizes x_{17} and x_{21}.

without loss, we can again assume.

$$x_{23}^h = x_{24}, \quad x_{23}^h = x_{23}x_{24}, \quad (x_{23}x_{24})^h = x_{23}$$

$$x_{22}^h = x_{22}x_{23}, \quad (x_{22}x_{23})^h = x_{22}x_{23}x_{24}, \quad (x_{22}x_{23}x_{24})^h = x_{22}$$

Let $u = h^2x_5x_6h$

3.8 Lemma

Let u be an involution as in (1.1) in $N_G(D)$ which act upon $Z_3(D) = S_{16}S_{20}Z_2(D)$ and takes x_{24} to x_{23} and centralizes $x_{16}, x_{17}, x_{20}, x_{21}$.

Proof

Since $\bar{S}_{16}\bar{S}_{20} \subseteq Z(\bar{S})$ where $\bar{S} = S/Z_2(D)$, we have $S_{16}Z_2(D)$ and $S_{20}Z_2(D)$ are normal in $N_G(D)$ due to Burniside's theorem. Now the action of g_1 on $Z_2(D)$ as in Lemma (3.6) and structure of S yield $x_{16}^{g_1} = x_{16}z, \quad z \in Z_2(D)$.

Let x_{22} appear in $x_{16}^{g_1}$

$$[x_{16}^{g_1}, x_6] = [x_{22}, x_6] = x_{24}$$

$$\Rightarrow [x_{16}, x_6^{g_1}] = x_{24}^{g_1}$$

$$\Rightarrow [x_{16}, x_{16}^{g_1}] = x_{23}$$

$\Rightarrow x_{16}$ is conjugate to $x_{16}x_{23}$ under the action of $x_{21}^{g_1}$.

But under graph automorphism ϕ we see that from table no. 1.

Put

$$\phi(x_{16}) = x_{22}$$

$$\phi(x_{23}) = x_{17}$$

$$\Rightarrow x_{22} \sim x_{17}x_{22}$$

This is contradiction according to table no. 3.

Hence x_{22} cannot appear in $x_{16}^{g_1}$.

Let x_{23} appear in $x_{16}^{g_1}$

$$[x_{16}^{g_1}, x_5] = [x_{23}, x_5] = x_{24}$$

$$\Rightarrow [x_{16}, x_5^{g_1}] = x_{24}^{g_1}$$

$$\Rightarrow [x_{16}, x_5^{g_1}] = x_{23}$$

$$\Rightarrow x_{16} \sim x_{16}x_{23}$$

Under group automorphism ϕ.

$$\phi(x_{16}) = x_{22}$$

$$\phi(x_{23}) = x_{17}$$

$\Rightarrow x_{22}$ is conjugate to $x_{22}x_{17}$ in S but this gives contradiction to the structure of S.

Hence x_{23} cannot appear in $x_{16}^{g_1}$

now the possibilities are:

$$x_{16}^{g_1} = x_{16}x_{17},\ x_{16}x_{21},\ x_{16}x_{17}x_{21}$$

or

$$x_{16}^{g_1} = x_{16}$$

(i) if $x_{16}^{g_1} = x_{16}x_{17}$ we take $g = g_1x_1$

$$\begin{aligned} x_{16}^{g} &= x_{16}^{g_1x_1} \\ &= (x_{16}x_{17})^{x_1} \\ &= x_{16} \end{aligned}$$

(ii) $x_{16}^{g_1} = x_{16}x_{21}$ we take $g = g_1x_{11}$

i.e.

$$\begin{aligned} x_{16}^{g} &= x_{16}^{g_1x_{11}} \\ &= (x_{16}x_{21})^{x_{11}} \\ &= x_{16} \end{aligned}$$

(iii) If $x_{16}^{g_1} = x_{16}x_{17}x_{21}$ we take $g = g_1xx_{11}$

i.e.

$$\begin{aligned} x_{16}^{g_1x_1x_{11}} &= (x_{16}x_{17}x_{21})^{x_1x_{11}} \\ &= (x_{16}x_{17}x_{21})^{x_{11}} \\ &= x_{16} \end{aligned}$$

This g does not effect those elements which satisfy lemma 1.1. Hence we find that g takes x_{24} to x_{23} and centralizes $x_{16}, x_{17}x_{21}x_{22}$

next, by considering

$$x_{20}^{g_1} = x_{20}z, \qquad z \in Z_2(D)$$

let x_{17} appear in $x_{20}^{g_1}$.

$$\Rightarrow [x_{20}^{g_1}, x_{10}] = [x_{17}, x_{10}] = x_{21}$$

$$\Rightarrow [x_{20}^{g_1}, x_{10}] = x_{21}$$

$$\Rightarrow [x_{20}, x_{10}^{g_1}] = x_{21}$$

$$\Rightarrow x_{10}^{g_1}x_{20}(x_{10}^{g_1})^{-1} = x_{20}x_{21}$$

This implies that x_{20} is conjugate to $x_{20}x_{21}$ in S. But according to table (3), these two elements belong to different conjugacy classes in S.

Thus x_{17} cannot appear in $x_{20}^{g_1}$.

Hence the possibilities are

$$\begin{aligned} x_{20}^{g_1} &= x_{20}x_{22}x_{21} \\ x_{20}^{g_1} &= x_{20}x_{22}x_{21}x_{24} \\ x_{20}^{g_1} &= x_{20}x_{23}x_{24} \\ x_{20}^{g_1} &= x_{20} \end{aligned}$$

(i) If $x_{20}^{g_1} = x_{20}x_{22}x_{21}$ we take $g = g_1x_1$

i.e. $x_{20}^{g} = x_{20}^{g_1x_1}$

$$\begin{aligned} &= (x_{20}x_{22}x_{21})^{x_1} \\ &= x_{20} \end{aligned}$$

(ii) if $x_{20}^{g_1} = x_{20}x_{23}x_{24}$ we write $g = g_1x_5g_1x_5g_1x_5g_1$ that is

$$\begin{aligned} x_{20}^{g} &= x_{20}^{g_1x_5g_1x_5g_1x_5g_1} \\ &= (x_{20}x_{23}x_{24})^{x_5g_1x_5g_1x_5g_1} \\ &= (x_{20}x_{23})^{g_1x_5g_1x_5g_1} \\ &= (x_{20}x_{24})^{x_5g_1x_5g_1} \\ &= (x_{20}x_{24})^{g_1x_5g_1} \\ &= (x_{20}x_{24})^{x_5g_1} \\ &= (x_{20}x_{23}x_{24})^{g_1} \\ &= x_{20} \end{aligned}$$

and

$$\begin{aligned} x_{24}^{g_1x_5g_1x_5g_1x_5g_1} &= (x_{23})^{x_5g_1x_5g_1x_5g_1} \\ &= (x_{23}x_{24})^{g_1x_5g_1x_5g_1} \\ &= (x_{23}x_{24})^{x_5g_1x_5g} \\ &= (x_{23})^{g_1x_5g_1} = (x_{24})^{x_5g_1} = (x_{24})^{g_1} = x_{23} \end{aligned}$$

Hence we found that g takes x_{20} to x_{20}, x_{24} to x_{23} and acts on all elements of $Z_2(D)$ as g_1 in lemma 3.7.

Applying the technique as in lemma (3.7) we get an involution u in $N_G(D)$ which takes x_{23} to x_{24} and centralizes $x_{20}, x_{16}, x_{17}, x_{21}$. Hence lemma is proved. We take g to be the involution satisfying this lemma.

This completes the proof of lemma 3. and there by theorem A is established.

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BIFURCATING PERIODIC SOLUTIONS OF POLYNOMIAL SYSTEMS

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Abstract

Periodic solutions of certain one dimensional non-autonomous differential equations (1.6) are investigated, the independent variable is complex. The motivation which is explained in section one is the connection with certain polynomials two-dimensional systems. Several classes of coefficients are considered; in each case the aim is to estimate the maximum number of periodic solutions into which a given solution can bifurcate under pertubation of the coefficients.

1. Introduction

In this paper, we consider systems of the form

$$\dot{x} = \lambda x + y + p_n(x, y)$$
$$\dot{y} = -x + \lambda y + q_n(x, y) \qquad (1.1)$$

where p_n and q_n are homogeneous polynomials of degree n. In polar form, (1.1) is

$$\dot{r} = \lambda r + f(\theta)r^n, \quad \dot{\theta} = -1 + g(\theta)r^{n-1} \qquad (1.2)$$

where f and g are homogeneous polynomials of degree $n+1$ in $\cos\theta$ and $\sin\theta$. Now let

$$\rho = r^{n-1}(1 - r^{n-1}g(\theta))^{-1} \qquad (1.3)$$

Then ρ satisfies the first order-non-autonomous equation

$$\frac{d\rho}{d\theta} = \alpha(\theta)\rho^3 - \beta(\theta)\rho^2 - \lambda(n-1)\rho \qquad (1.4)$$

where

$$\alpha(\theta) = -(n-1)g(\theta)(f(\theta) + \lambda g(\theta))$$

and

$$\beta(\theta) = (n-1)f(\theta) - 2\lambda(n-1)g(\theta) - g'(\theta)$$

Thus $\alpha(\theta)$ and $\beta(\theta)$ are homogeneous polynomials in $\cos\theta$ and $\sin\theta$ of degree $2(n+1)$ and $(n+1)$ respectively. The transformation (1.3) is defined in

$$D = \{(r,\theta); r^{n-1}g(\theta) < 1\} \tag{1.5}$$

which is an open set containing the origin. It is clear that the limit cycles (an isolated closed orbits) of (1.1) correspond to positive 2π - periodic solutions of (1.4). Considerable interest has been shown in systems of the form (1.1), and much of it has been stimulated by this relationship (see[5],[6],[13]).

In the case $n = 2$, transformation (1.3) was introduced by Lins Neto [11] and [7]. The connection between (1.1) and (1.4) for $n > 2$ was explained in [12].

In order to be able to keep the track of the number of periodic solutions of almost any class of differential equations it is useful, if not essential, to work with the appropriate complexified form. This is because the number of zeros of the homomorphic function in a bounded region of the complex plane cannot be changed by small perturbations of the function.

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 + \gamma(t)z \tag{1.6}$$

where z is complex but t remains real, and the coefficients α, β, γ are real valued functions. We specify $\omega \varepsilon R$ and seek information about the number of solutions which satisfy the periodic boundary condition.

$$z(0) = z(\omega)$$

We say that such solutions are 'periodic' whether or not the coefficients in (1.6) are themselves periodic.

Quadratic systems have been investigated by means of equation (1.4) by Coll, Gasull & Llibre[6]. Also Lins Neto [11] has given examples which demonstrate that there is no upper bound for the number of periodic solutions of (1.6); unless, the coefficients are suitably restricted. The real questions therefore, is the maximum possible number of periodic solutions for various classes of coefficients.

In this paper, we concentrate on the number of periodic solutions for various classes of coefficients of equation (1.6) which bifurcate out of the origin - a problem which directly parallels the investigation of small amplitude limit cycles of (1.1). Following [1] and Lins Neto [11], we consider two types of coefficients: (i) polynomials in t, and (ii) polynomials in $\cos t$ and $\sin t$.

In doing so, we will answer some questions asked in [1]. The idea is to consider various classes C of equations of the form (1.6), and for each to claculate the maximum possible multiplicity of the origin, which we denote by $\mu_{\max}(C)$.

The multiplicity of $z = 0$ as a solution of (1.6) is the multiplicity of $z = 0$ as a zero of the displacement function $q : c \to z(\omega, 0, c) - c$, as usually defined in complex function theory. To compute the multiplicity (which we call μ), we write $z(t; 0, c) = \sum a_n(t)c^n$ where $0 \leq t \leq \omega$ and c in a neighbourhood of 0, and substitute directly into the equation. This gives a recursive set of linear differential equations for $a_n(t)$; the initial conditions are $a_1(0) = 1,\ a(0) = 0$ for $j > 1$. It can be seen that

$$\dot{a}_1(t) = a_1(t)\gamma(t)$$

where

$$a_1(t) = \exp\left[\int_0^w \gamma(s)ds\right]$$

Thus

$$\mu > 1 \text{ iff } \left[\int_0^w \gamma(s)ds = 0 \right. \tag{1.7}$$

Since we are interested in the case when $z = 0$ is a multiple solution, we shall assume that (1.7) holds. Under the transformation

$$\xi = z \exp\left[-\int_0^w \gamma(s)ds\right]$$

(1.6) becomes

$$\dot{\xi} = \hat{\alpha}(t)\xi^3 + \hat{\beta}(t)\xi^2 \tag{1.8}$$

where

$$\hat{\alpha}(t) = \alpha(t)\exp\left(2\int_0^w \gamma(s)ds\right)$$

and

$$\hat{\beta}(t) = \beta(t)\exp\left(\int_0^w \gamma(s)ds\right)$$

We note that, if the function α, β and γ are periodic, then so are $\hat{\alpha}$ and $\hat{\beta}$. Also if the multiplicity of $z = 0$ as a periodic solution of (1.6) is $\mu > 1$, then the multiplicity of $z = 0$ as a periodic solution of (1.8) is μ. We therefore suppose that $\gamma(t) \equiv 0$ that is $\lambda = 0$ in (1.6) and so we consider equations of the form

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 \tag{1.9}$$

For this $a_1(t) \equiv 1$ and for $n > 1$, the functions $a_n(t)$ are determined by the relation.

$$\dot{a}_n = \alpha \sum_{i+j+k=n_{i,j,k\geq 1}} a_1 a_j a_k + \beta \sum_{i+j=n_{i,j\geq 1}} a_i a_j \tag{1.10}$$

These equations can be solved recursively, but their calculation is tedious, involving integration by parts. Let $\eta_i = \dot{a}_i(\omega)$; then $\mu = k$ if $\eta_1 = 1$, $\eta_2 = \cdots = \eta_{k-1} = 0$ but $n_k \neq 0$. These η_i are called focal values. The formulae for $a_k(t)$ and η_k for $k \leq 8$ are given in [1]. Using the procedure we have calculated $a_9(t)$ and η_9 to answer the questions raised by Alwash in [1].

We will present $a_k(t)$ and η_9 for $k \leq 9$ in the next section. For the calculation of these focal values for each class we have used "REDUCE" a symbolic manipulating system. These focal values involve the definite integral of the type $\int \alpha(t)\bar{\beta}(t)dt$ where $\alpha(t)$ and $\beta(t)$ are polynomials in t with unknown coefficients and $\bar{\beta}(t) = \int \beta(t)dt$. To be able to calculate the maximum multiplicity of the origin, certain criteria are required which ensure that the origin is a centre, which we will present in section 3. In section 4, we consider equation (1.6) when the coefficients α and β are (i) polynomials in t and (ii) polynomials in cos t and sin t. We have proved that $\mu_{\max}$© is given by the table below, when the coefficients α, β are polynomial in t.

α/β	1	2	3	4	5
1	3	4	4	8	8
2			8		

and if we denote Ω_3 the class of equation (1.6) in which α is trignometric polynomial of degree 6 and β is of degree 3, then $\mu_{\max}(\Omega_3) = 7$.

2. Calculation of η_9 and method of perturbation

In the following theorem we give $a_n(t)$ for $n \leq 9$. We use a bar over a function to denote its indefinite integral, that is $\bar{\xi}(t) = \int_0^w \xi(s)ds$. To calculate η_9 we used the relation (1.10) with $n = 9$.

Theorem (2.1)

For equation (1.9), the functions $a_2; \cdots; a_9$ are given by the following formulae.

$$
\begin{aligned}
a_2 &= \bar\beta \\
a_3 &= \bar\beta^2 + \bar\alpha \\
a_4 &= \bar\beta^3 + 2\bar\beta\bar\alpha + \overline{\bar\beta\alpha} \\
a_5 &= \bar\beta^4 + 3\bar\beta^2\bar\alpha + \overline{\bar\beta^2\alpha} + 2\bar\beta\;\overline{\bar\beta\alpha} + \frac{3}{4}\bar\alpha^2 \\
a_6 &= \bar\beta^5 + 4\bar\beta^3\bar\alpha + \overline{\bar\beta^3\alpha} + 3\bar\beta^2\overline{\bar\beta\alpha} + 2\bar\beta\overline{\bar\beta^2\alpha} + \frac{9}{2}\bar\beta\bar\alpha^2 + 3\overline{\bar\beta\alpha}\bar\alpha - \frac{1}{2}\beta\bar\alpha^2 \\
a_7 &= \bar\beta^6 + 5\bar\beta^4\bar\alpha + \overline{\bar\beta^4\alpha} + 4\bar\beta^3\overline{\bar\beta\alpha} + 2\overline{\bar\beta^3\alpha}\bar\beta + \frac{17}{2}\bar\beta^2\bar\alpha^2 + 3\overline{\bar\beta^2\alpha}\bar\alpha \\
&\quad + 2(\overline{\bar\beta\alpha})^2 + 2\overline{\bar\beta^2\bar\alpha\alpha} + 8\overline{\bar\beta\alpha}\bar\beta\bar\alpha - \bar\beta\overline{\beta\bar\alpha^2} + \frac{5}{2}\bar\alpha^3 \\
a_8 &= \bar\beta^7 + 6\bar\beta^5\bar\alpha + \overline{\bar\beta^5\alpha} + 5\bar\beta^4\overline{\bar\beta\alpha} + 2\overline{\bar\beta^4\alpha}\bar\beta + 4\bar\beta^3\overline{\bar\beta^2\alpha} + 3\overline{\bar\beta^3\alpha}\bar\alpha \\
&\quad + 3\overline{\bar\beta^3\alpha}\bar\beta^2 + 3\overline{\bar\beta^3\bar\alpha\alpha} + \frac{27}{2}\bar\beta^3\bar\alpha 2 - 3/2\bar\beta^2\overline{\beta\bar\alpha 2} + 15\bar\beta^2\overline{\bar\beta\alpha}\bar\alpha \\
&\quad + 4\overline{\bar\beta^2\bar\alpha\alpha}\bar\beta + \overline{\bar\beta^2\alpha\overline{\bar\beta\alpha}} + 12\overline{\bar\beta^2\alpha}\;\;\overline{\bar\beta\alpha} + 8\overline{\bar\beta^2\alpha}\;\;\overline{\beta\bar\alpha} + 5\bar\beta(\overline{\beta\bar\alpha})^2 - \frac{1}{2}\overline{\beta\bar\alpha^3} \\
&\quad - 3/2\overline{\beta\bar\alpha^2}\bar\alpha + 10\bar\beta\bar\alpha^3
\end{aligned}
$$

$$
\begin{aligned}
a_9 \;=\;& \bar\beta^8 + 7\bar\beta^6\bar\alpha + \overline{\bar\beta^6\alpha} + 6\bar\beta^5\overline{\bar\beta\alpha} + 2\overline{\bar\beta^5\alpha}\bar\beta + 5\bar\beta^4\overline{\bar\beta^2\alpha} + 3\overline{\bar\beta^4\alpha}\bar\alpha \\
& +3\overline{\bar\beta^3\alpha}\bar\beta^2 + 5\overline{\bar\beta^4\bar\alpha\alpha} + 39/2\bar\beta^4\bar\alpha^2 - 2\bar\beta^3\overline{\beta\bar\alpha^2} + 24\bar\beta^3\overline{\bar\beta\alpha}\bar\alpha \\
& +6\overline{\bar\beta^3\bar\alpha\alpha}\bar\beta - 10\overline{\bar\beta^3\alpha\overline{\bar\beta\alpha}} + 12\bar\beta\bar\alpha\overline{\bar\beta^3\alpha} + 4\overline{\beta\bar\alpha\overline{\bar\beta^3\alpha}} + 4\bar\beta^3\overline{\bar\beta^3\alpha} \\
& +\frac{43}{6}\bar\alpha^3\bar\beta^2 + 4\overline{\beta\bar\beta\bar\alpha^3} + 4\bar\beta^2\bar\alpha\overline{\bar\beta^2\alpha} - 10\overline{\beta\bar\beta\bar\alpha\overline{\bar\beta^2\alpha}} + 15/2\bar\alpha^2\overline{\bar\beta^2\alpha} \\
& +2\bar\beta^2\overline{\bar\beta^2\alpha} - 2\bar\beta^4\bar\alpha + 8\overline{\bar\beta^3\bar\alpha\beta} + 2\overline{\bar\beta\bar\beta^2\alpha\overline{\bar\beta\alpha}} + 26\overline{\bar\beta\alpha} \quad \overline{\bar\beta^2\alpha}\bar\beta \\
& +6\overline{\bar\beta^2\alpha}\alpha\bar\alpha - 6\overline{\bar\beta^2\alpha}\alpha\bar\alpha + 12\bar\beta^2\overline{\bar\beta\bar\alpha\alpha} + 16\overline{\bar\beta^2\alpha}\beta\bar\alpha\bar\beta - 16\overline{\bar\beta^3\alpha\overline{\bar\beta\alpha}} \\
& +9\bar\beta^2\overline{(\bar\beta\alpha)^2} + 9\overline{(\bar\beta\alpha)^2}\bar\alpha - \overline{\beta\bar\alpha^3}\bar\beta + 35/8\bar\alpha^4 - 6\bar\alpha\bar\beta\overline{\beta\bar\alpha^2} + 6\overline{\beta\bar\alpha\beta\bar\alpha^2} \\
& +33\bar\alpha^2\bar\beta\overline{(\bar\beta\alpha)} - 24\overline{\bar\alpha^2\beta\overline{\bar\beta\alpha}} + 6\overline{\bar\beta^2\bar\alpha\alpha}\bar\alpha - 4\overline{\beta\bar\alpha\overline{\beta\bar\alpha^2}}
\end{aligned}
$$

From these we can now deduce the following result which enables us to calculate the multiplicity of the origin.

Theorem 2.2

The solution $z = 0$ of (1.9) has multiplicity k where $2 \le k \le 9$ if and only if $\eta_1 = 0$ for $2 \le i \le k-1$ and $\eta_k \neq 0$ where

$$
\begin{aligned}
\eta_2 &= \int_0^\omega \beta \\
\eta_3 &= \int_0^\omega \alpha \\
\eta_4 &= \int_0^\omega \alpha\bar\beta \\
\eta_5 &= \int_0^\omega \alpha\bar\beta^2 \\
\eta_6 &= \int_0^\omega (\alpha\bar\beta^3 - 1/2\bar\alpha^2\beta) \\
\eta_7 &= \int_0^\omega \alpha\bar\beta^4 + 2\alpha\bar\alpha\bar\beta^2 \\
\eta_8 &= \int_0^\omega \alpha\bar\beta^5 + 3\alpha\bar\alpha\bar\beta^3 + \alpha\bar\beta^2\overline{\bar\beta\alpha} - 1/2\bar\alpha^3\beta
\end{aligned}
$$

and

$$\eta_9 = \int_0^\omega \alpha\bar{\beta}^6 - 5\alpha\bar{\alpha}\bar{\beta}^4 - 2\bar{\beta}^3 \ \overline{\beta\bar{\alpha}} + 20\overline{\beta\bar{\alpha}^2} + 2\overline{\beta\bar{\alpha}} \ \ \beta\bar{\alpha}^2$$

Having determined the multiplicity μ, the aim will be to construct equations with the maximum possible number of distinct real periodic solutions. The idea is to make a sequence of perturbations of the coefficients, each of which causes one periodic solution to bifurcate out of the origin.

We start with an equation of the form (1.9) for which $\mu = k$ say. Let U' be a neighbourhood of the origin in the complex plane containing no periodic solutions other than $z = 0$. Then by theorem (2.4) in [1], the total number of periodic solutions with initial points in U is unchanged by sufficiently small perturbation of the coefficients. If possible, we perturb the coefficients α, β and γ so that $\eta_2 = \eta_3 = \cdots = n_{k-2} = 0$ but $n_{k-1} \neq 0$. There is then a non-trivial periodic solution $\psi(t)$, say, with $\psi(0) \in U'$, and the only periodic solutions in U' are ψ and the zero solution. Since complex solutions occur in conjugate pairs, it follows that ψ is real. Now let W_1 be a nieghbourhood of ψ and U_1, be a neighbourhood of 0 such that $U_1 \cup W_1 \subset U'$ and $U_1 \cap W_1 = \phi$. The number of periodic solutions with initial points in each of U_1 and W_1 is preserved under sufficiently small perturbations of the coefficients. We then seek to perturb the coefficients further such that $\eta_2 = \eta_3 = \cdots = \eta_{-3} = 0$, but $n_{k-2} \neq 0$. In this case $\mu = k - 2$. Now a second real non-trivial periodic solution has initial point in U_1; there remains a real periodic solution with initial point in W_1. Thus we have two non-trivial real periodic solutions and the zero solution is of multiplicity $k - 2$. Continuing in this way we end up with an equationof the form (1.9) with $\mu = 2$ and $k - 2$ distinct non-trivial real periodic solutions.

3. Conditions for a centre

To find maximum possible value of μ (the multiplicity of $z = 0$) for various classes of equations, we evaluate the quantities $\eta_k = a_k(\omega)$ which are given in section two, until a value K of k is found with the property that $\eta_k = 0$ for all k if $\eta_2 = \eta_3 = \cdots = \eta_{k+1} = 0$. Then $\mu_{\max}$ is the smallest such K. In association with the method which we have described for calculating the η_k, we need conditions which are sufficient for $z = 0$ to be a centre. Only then we know that we need to calculate no more of the η_k. Now we will state here the conditions for $z = 0$ to be a centre (see [1]), because we will need them in the next section.

Theorem (3.1)

Suppose that there is a differentiable function $\sigma(\omega) = \sigma(0)$ and continuous function f and g defined on $I = \sigma([0,\omega])$ such that

$$\alpha(t) = f(\sigma(t))\dot{\sigma}, \quad \beta(t) = g(\sigma(t)) = \dot{\sigma}$$

Then the origin is centre for the equation

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 \tag{3.1}$$

Corollary (3.2)

Suppose that in equation (3.1) α is a constant multiple of β and that $\int_0^\omega \beta(t)dt = 0$. Then the origin is a centre.

Corollary (3.3)

Suppose that α or β is identically zero and the other has mean value zero. Then the origin is a centre.

Corollary (3.4)

Suppose that α and β contain only odd powers of $\sin(t)$ or cost. Then the origin is a centre.

4. Polynomial Coefficients and Periodic Coefficients

In this section we consider equation (1.9) in which α and β are (i) polynomials in t (ii) trignometric function of t (polynomials in $\cos t$ and $\sin(t)$). Let (b_1, b_2) denote the class of equations of the form (1.9) in which α is degree b_1 and β of degree b_2. The question was raised in [11] whether an equation of the form (1.9) can have more than $b/2+3$ periodic solutions where $b = \max(b_1, b_2)$. We shall see later that there may indeed be more, even when $b_1 = 1$. We take $\omega = 1$ for convenience when coefficients are polynomials in t and $\omega = 2\pi$ when the coefficients are polynomials in $\cos t$ and $\sin t$. First we will consider the class $C_{1,k}$ in which α is of degree one and β is of degree k.

Theorem (4.1)

Let $C_{1,k}$ be the class of equations of the form $\dot{z} = \alpha(t)z^3 + \beta(t)z^2$ in which $\alpha(t)$ is of degree 1 and $\beta(t)$ is of degree k, $k = 1, 2, 3, 4$. Then we have the following results $\mu_{\max}(C_{1,k}) = 3, 4, 4, 8$ for $k = 1, 2, 3, 4$ respectively.

Proof

We need only consider the case in which degree $\alpha(t)$ is four; the cases in which degree of $\alpha(t)$ is less than four are then special cases.

(i) Let $\alpha(t) = a + bt$

$$\beta(t) = c + dt + et^2 + ft^3 + gt^4$$

Then by Theorem (3.1), we have

$$\eta_2 = c + \frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5}, \qquad \eta_3 = a + \frac{b}{2}$$

Thus multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu_2 = 3$ if $\eta_3 \neq 0$ but $\eta_2 = 0$. If $\eta_2 = \eta_3 = 0$, we take

$$a = -\frac{b}{2} \qquad \text{and} \qquad c = \left(\frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5}\right) \tag{4.1}$$

Then $\alpha(t)$ and $\beta(t)$ are of the form

$\alpha(t) = b(t - 1/2)$, $\beta(t) = d(t - 1/2) + e(t^2 - 1/3) + f(t^3 - 1/4) + g(t^4 - 1/5)$, and η_4 becomes

$$\eta_4 = \frac{-b(14e + 21f + 24g)}{5040}$$

Now $\eta_4 = 0$, then either $b = 0$ or

$$e = \frac{-(21f + 24g)}{14} \tag{4.2}$$

If $b = 0$ then $\alpha(t) = 0$ and $\eta_2 = 0$ gives that the meanvalue of $\beta(t)$ is zero, hence by Corollary (3.3) the origin is a centre, so we suppose that $b \neq 0$. Now if (4.2) holds then we have

$$\eta_5 = bg(-77d + 49f + 76g)/13582800$$

If now $\eta_5 = 0$ then either $g = 0$ or

$$d = \frac{(49f + 76g)}{77} \tag{4.3}$$

because $b \neq 0$. If $g = 0$, then $\alpha(t)$ and $\beta(t)$ are of the form

$$\begin{aligned} \alpha(t) &= b\left(t - \frac{1}{2}\right) \\ \beta(t) &= \left[d + f\left(t^2 - t - \frac{1}{2}\right)\right]\left(t - \frac{1}{2}\right) \end{aligned}$$

Let $\sigma(t) = t^2 - t$, then $\dot{\sigma}(t) = 2t - 1$

Also $\sigma(0) = \sigma(1)$ and we can write $\alpha(t)$ and $\beta(t)$ as

$$\begin{aligned} \alpha(t) &= \frac{1}{2} b\, \dot{\sigma} \\ \beta(t) &= \frac{1}{2}\left[d + f\left(\sigma - \frac{1}{2}\right)\right]\, \dot{\sigma} \end{aligned}$$

Then by Theorem (3.1), the origin is a centre with $f(\sigma) = \frac{1}{2}b_1$ and

$g(\sigma) = \frac{1}{2}\left[d + f\left(\sigma - \frac{1}{2}\right)\right]$. We therefore suppose that $g \neq 0$. With (4.3) holding we compute η_6, which we found is a constant multiple of ξ, where

$$\xi = bg(-2882679800b - 62475f^2 - 249900fg - 245544g^2)$$

If in addition $\eta_6 = 0$ then we have either $bg = 0$ or

$$2882679800b + 62475f^2 - 249900fg + 245544g^2 = 0 \tag{4.4}$$

We have already considered the possibility that $bg = 0$, so we suppose that (4.4) holds. Further computation gives

$$\eta_7 = bg(f+2g)(11613f^2 + 46452fg + 45968g^2)$$

Recalling that $bg = 0$ has been considered, now, if in addition $\eta_7 = 0$ then either $f = -2g$ or

$$11613f^2 + 46452fg + 45968g^2 = 0 \tag{4.5}$$

If $f = -2g$ then we find that

$$\eta_8 = \frac{-3583g^7}{1364980730520520539716584000000}$$

that is η_8 is a constant multiple of g^7. If $bg \neq 0$, $f + 2g \neq 0$ but (4.5) holds then

$$f = k_1 g, \quad i = 1,2$$

where

$k_1 = -1.7958495$, $k_2 = -2.2041505$, and in each case η_8 is constant multiple g^7. Then $\eta_8 = 0$ iff $g = 0$. But if $g = 0$, then the origin is a centre, hence $\mu_{\max}(C_{1.4}) = 8$.

(ii) If degree of $\beta(t) = 1$, then $e = f = g = 0$ in the above calculation and if $\eta_2 = \eta_3 = 0$ then

$$\alpha(t) = b\left(t - \frac{1}{2}\right)$$
$$\beta(t) = d\left(t - \frac{1}{2}\right),$$

that is $\alpha(t)$ is constant multiple of $\beta(t)$ and

$\int_0^1 \beta(t)dt = 0$. Hence by Corollary (3.2), the origin is a center.
Thus $\mu_{\max}(C_{1,1}) = 3$.

(iii) If degree of $\beta(t) = 2$. We have $f = g = 0$. In this case $\eta_2 = c + \frac{d}{2} + \frac{e}{3}$ and $\eta_3 = a + \frac{b}{2}$. If $\eta_2 = \eta_3 = 0$ then $\eta_4 = -14be/5040$. Now $\eta_4 = 0$ iff either $b = 0$ or $e = 0$. If $b = 0$ then the origin is a center as proved above. Thus we suppose $b \neq 0$. If $e = 0$ then we have class $C_{1,1}$ and we know by (ii), that $\mu_{\max}(C_{1.2}) = 4$.

(iv) If degree of $\beta(t) = 3$, we have $g = 0$ and

$$\begin{aligned} \alpha(t) &= a + bt \\ \beta(t) &= c + dt + et^2 + ft^3 \end{aligned}$$

Then if $\eta_2 = \eta_3 = 0$, we have

$$\eta_4 = b\frac{(14e + 121f)}{5040}$$

Now if $\eta_4 = 0$ then either $b = 0$ or $14e + 21f = 0$. If $b_1 = 0$ then the origin is centre by the same argument as given f or $C_{1,4}$ or if $14e + 21f = 0$ then

$$\alpha(t) = b\left(t - \frac{1}{2}\right), \quad \beta(t) = \left[d + f(t^2 - t\frac{1}{2}\right]\left(t - \frac{1}{2}\right)$$

Defining $\sigma(t) = t^2 - t$ we have $\dot{\sigma}(t) = 2t - 1$

Also $\sigma(0) = \sigma(1)$ and $\sigma(t) = \frac{1}{2}b\dot{\sigma}$

$\beta(t) = \frac{1}{2}[d + f(\sigma - 1/2]\dot{\sigma}$. Then the origin is a center by Theorem (3.1) with $f(\sigma) = b/2$, and $g(\sigma) = 1/2(d + f(\sigma - 1/2))$. Hence $\mu_{\max}(C_{1,3}) = 4$.

Now we use the technique described at the end of section 2 to construct a class of equation in $C_{1,4}$ with six non-trival real periodic solutions.

Theorem (4.2)

In the equation $\dot{z} = \alpha(t)z^3 + \beta(t)z^2$

Let $\alpha(t) = \{(-b/2 + \epsilon_5) + bt\}g^2$

$$\begin{aligned} \beta(t) = & \left\{\left(-\frac{1}{7} + \frac{3\varepsilon_1}{44} - \frac{\varepsilon_3}{2} - \frac{\varepsilon_4}{3} + \epsilon_6\right) + \left(-\frac{2}{7} - \frac{7\varepsilon_1}{11} + \epsilon_3\right)t \right. \\ & \left. + \left(\frac{9}{7} + \frac{3\varepsilon_1}{2} + \epsilon_4\right)t^2 + (-2 - \epsilon_1)t^3 + t^4\right\} g \end{aligned}$$

with $bg \neq 0$ and

$$-2882679800b_1 = (62475 \in_1 - 4356 + \in_2)g^2$$

If $\in_i$, $1 \leq i \leq 6$ are chosen to be non-zero and in that order such that each $\in_k$ is sufficiently small compared with $\in_{k-1}$, then (1.9) has six non-trivial real periodic solution.

Proof

The coefficients are chosen so that the multiplicity of the origin. μ is 8 if $\in_i = 0$ for $1 \leq i \leq 6$. Choose $\in_i = 0$ for $2 \leq i \leq 6$ then it can be checked that $\eta_2 = \eta_3 = \cdots = \eta_6 = 0$ but $\eta_7 = 0$; thus the multiplicity of the origin is reduced by one; hence $\mu = 7$. Therefore one real periodic solution bifurcates out of the origin. Next with $\in_2 \neq 0$ but $\in_3 = \in_4 = \cdots = \in_6 = 0$ we have that $\eta_2 = \eta_3 = \cdots = \eta_5 = 0$ but η_6 is a constant multiple of $\in_2$; so $\mu = 6$. If $\in_2$ is small enough there are two real non-trivial periodic solutions. Continuing in this way, we have six non-trivial real periodic solutions.

Corollary (4.2)

With $\alpha(t)$ and $\beta(t)$ as given in Theorem (4.1), the equation

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 + \gamma + \delta \tag{4.4}$$

has eight real periodic solutions if γ and δ are small enough.

Proof

If $\gamma = 0$ and $\delta = 0$, $\mu = 2$ then (4.4) has six real periodic solution. If γ is non-zero but small enough, then $\mu = 1$ and by the argument used in the above theorem there are seven distinct real periodic solution; $z = 0$ is another such solution.

In the next theorem we consider the class of equation $C_{1,5}$.

Theorem (4.3)

Let $C_{1,5}$ denote the class of equations of the form

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 \tag{4.5}$$

where

$$\begin{aligned}\alpha(t) &= a + bt\\ \beta(t) &= c + dt + et^2 + ft^3 + gt^4 + ht^5\end{aligned}$$

Then $\mu_{\max}(C_{1.5}) = 8$ when $f = 0$.

Proof

Initially we start with $f \neq 0$ then by using Theorem (2.2), we have

$$\begin{aligned}\eta_2 &= c + \frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5} + \frac{h}{6}\\ \eta_3 &= a + \frac{b}{2}\end{aligned}$$

The multiplicity $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 0$ if $\eta_2 = 3$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$ then we substitute for 'a' from $\eta_3 = 0$ and 'c' from $\eta_2 = 0$. Then

$$\eta_4 = \frac{b(-14e - 21f - 24g - 25h)}{5040}$$

Now $\eta_4 = 0$ if either $b = 0$ or

$$e = \frac{-(21f + 24g + 25h)}{14} \tag{4.6}$$

If $b = 0$ then $\eta_3 = 0$ gives $a = 0$ hence $\alpha(t) = 0$, $\eta_2 = 0$ then implies that the mean value of $\beta(t)$ is zero. Therefore by Corollary (3.3) the origin is a centre, thus we take $b \neq 0$. If (4.6) holds then we compute

$$\eta_5 = b(2g + 5h)(-1001d + 637f + 988g + 1175h)$$

Now if in addition $\eta_5 = 0$ then either

$$2g + 5h = 0 \qquad \text{or} \tag{4.7}$$

$$d = \frac{(637f + 988g + 1175h)}{1001} \tag{4.8}$$

because $b \neq 0$. If (4.7) holds then

$$
\begin{aligned}
\alpha(t) &= b\frac{(2t-1)}{2} \\
\beta(t) &= (2t-1)\frac{\{2d + f(2t^2 - 2t - 1) + h(2t^4 - 4t^3 - 2t^2 + 4t + 2)\}}{4}
\end{aligned}
$$

Define $\sigma(t) = t^2 - t$ then $\dot{\sigma}(t) = 2t - 1$.

$\sigma(0) = \sigma(1)$ and $\alpha(t)$. $\beta(t)$ can be written as

$$
\begin{aligned}
\alpha(t) &= \frac{b\dot{\sigma}(t)}{2} \\
\beta(t) &= \dot{\sigma}(t)\frac{\{2d + f(2\sigma - 1) + h(\sigma^2 - 4\sigma + 2)\}}{4}
\end{aligned}
$$

The the origin is a centre by Theorem (3.1) with

$$
f(\sigma) = \frac{b}{2} \quad \text{and} \quad g(\sigma) = \frac{\{2d + f(2\sigma - 1) + h(2\sigma^2 - 4\sigma + 2)\}}{4}
$$

Thus we suppose that $2g + 5h \neq 0$. If (4.8) holds then by Theorem (2.2) η_6 is constant multiple of $-b(2g + 5h)w$ where

$$
\begin{aligned}
w = {} & 712019106000b + 15431325f^2 + 61725300fg + 89376000fh \\
& + 60649368g^2 + 173372340gh + 122308350h^2
\end{aligned}
$$

Now $\eta_6 = 0$ only if $w = 0$ because we have already discussed the possibility of $b(2g + 5h) = 0$, in each case the origin is a centre. From $w = 0$ we substitute for b and obtain

$\eta_7 = -b(2g + 5h)$ (homogeneous cubic in f, g and h)

$\eta_8 = b(2g + 5h)$(homogeneous quartic inf, g and h)

We cannot draw any conclusion looking at the η_7 and η_8 therefore for simplification we take $f = 0$. Then η_7 and η_8 becomes.

$$\begin{aligned}\eta_7 &= -b(2g + 5h)\{8933880800g^3 + 3781037992g^2h \\ &\quad +53243188960gh^2 + 2494585560h^3\}\end{aligned}$$

and

$$\begin{aligned}\eta_8 &= b(2g + 5h)\{375325959174988223251 2g^4 + 2117802626346208005280g^3h \\ &\quad +4477014569382653595192 0g^2h^2 + 420265067480971125706 00gh^3 \\ &\quad +1478081150436369733750 0h^4\}\end{aligned}$$

To have the multiplicity of the origin greater than six, we need to prove that the cubic in η_7 and quartic in η_8 have no common zeros. To prove this we start by supposing that both have common zero. Then we we get a linear relation in g and h; say $g + kh = 0$ and for this value of g, η_8 is constant multiple of $bh^5 \neq 0$. Hence $\mu_{\max}(C_{1,5}) = 8$ with $f = 0$.

Theorem (4.4)

Let $g = \alpha$ be a real root of equation

$$8933880800g^3 + 3781037992g^2 + 53243188960g + 2494585560 = 0$$

Choose

$$
\begin{aligned}
g &= \alpha + \in_1 \\
d &= (988\alpha + 1175/1001 + \frac{988\varepsilon_1}{1001} + \in_3 \\
e &= -(24\alpha + 25)/14 - \frac{12\varepsilon_1}{7} + \in_4 \\
c &= -(1222\alpha + 1585)/1001 - \frac{\varepsilon_3}{2} - \frac{\varepsilon_4}{3} - \frac{1222}{10010} + \in_6 \\
a &= -\frac{b}{2} + \in_5 \\
b &= -\{60649368(\alpha + \in_1)^2 + 173372340(\alpha + \in_1) + 1223083507\} \\
&\quad /71202191600 + \in_2
\end{aligned}
$$

such that $| \in_6 | << | \in_5 | << | \in_4 | << | \in_3 | << | \in_2 | << | \in_1 |$

Then the equation (1.9) has six distinct non-trivial real periodic solutions where

$$
\begin{aligned}
\alpha(t) &= a + bt \\
\beta(t) &= c + dt + et^2 + ft^3 + gt^4 + ht^5
\end{aligned}
$$

with $f = 0, \quad h = 1$.

Theorem (4.5)

Let $C_{2,3}$ denote the class of equations of the form (1.9) in which α is of degree 2 and β of degree 3 respectively. Then $\mu_{\max}(C_{2,3}) = 8$

Proof

Let

$$
\alpha(t) = A_1 + A_2(2t - 1) + A_3(2t - 1)^2
$$

$$= (A_1 + A_2 + A_3) + 2A_2 - 4A_3)t + 4A_3t^2$$

$$\begin{aligned}\beta(t) &= B_1 + B_2(2t-1) + B_3(2t-1)^2 + B_4(2t-1)^3 \\ &= (B_1 - B_2 + B_3 - B_4) + (2B_2 - 4B_3 + 6B_4)t + (4B_3 - 12B_4)t^2 + 8B_4t^3\end{aligned}$$

Then by Theorem (2.2) $\eta_2 = (B_3 + 3B_1)/3$. Then $\mu = 2$ if $\eta_2 \neq 0$. If $\eta_2 = 0$, we take $B_1 = -B_3/3$ and $\eta_3 = (A_3 + 3A_1)/3$. The multiplicity of the origin, $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_3 = 0$ we substitute $A_1 = -A_3/3$. Then η_4 becomes $\eta_4 = (3B_4A_3 + 7B_2A_3 - 7B_3A_2)/315$.

To have origin of multiplicity greater than four we set $\eta_4 = 0$; supposing that $B_3 \neq 0$. Let $A_2 = \{A_3(7B_2 + 3B_4)\}/(7B_3)$. Then we compute

$$\eta_5 = 4B_4A_3(5B_4 + 11B_2)/72765$$

If $\eta_5 = 0$, we have either $B_4 = 0$ or $A_3 = 0$ or

$$5B_4 + 11B_2 = 0 \tag{4.9}$$

If $B_4 = 0$ then we are left with class $C_{2,2}$ for which $\mu_{\max}(C_{2,2}) = 4$. Thus we take $B_4 \neq 0$. If $A_3 = 0$, then we have class $(C_{1,3})$ for which $\mu_{\max} = 4$ again, therefore we take $A_3 \neq 0$. If (4.9) holds we take

$$B_2 = -5B_4/11 \tag{4.10}$$

Then

$$\eta_6 = -\{4A_3B_4(1694B_3^3 - 126B_3B_4^2 + 4719A_3B_4)\}/(2403646245B_3)$$

Now $\eta_6 = 0$ iff

$$1694B_3^3 - 126B_3B_4^2 + 4719A_3B_4 = 0 \tag{4.11}$$

because we have considered $A_3B_4 = 0$ before. From (4.11) we substitute for B_3^3 then η_7 is constant multiple of

$$-A_3B_4^3(3009391B_3^2A_3 - 2016B_3B_4^3 + 5204A_3B_4^2)/B_3$$

From $\eta_7 = 0$, we substituted

$$B_3B_4^3 = (3009391B_3^2A_3 + 5204A_3B_4^2)/2016$$

Then $\eta_8 = -2B_4A_3^3(850563329B_3^2 + 1410471B_4^2)/1124971341108615B_3^2$ which is non-zero. Hence $\mu_{\max}(C_{2,3}) = 8$.

(b) Trignometric Coefficients

In [1] Alwash studied the class Ω_3 of equation (1.9) in which α is of degree six and β of degree three. Let $\mu_{\max}(\Omega_3) = v_3$. Since he did not calculate η_9, the best estimate he could give was $v_3 \geq 7$. To check the multiplicity of the origin greater than 7, we computed till η_9. We will give here, the results obtained by using a computer program different from Alwash. This program is written in REDUCE. The $\alpha(t)$ and $\beta(t)$ were given by

$$\begin{aligned}
\alpha(t) &= (c+d)\cos^3 t \sin t + (e+d)\sin^3 t \cos t)(\cos^2 t + \sin^2 t) \\
\beta(t) &= (a\cos t + b\sin t)(\cos^2 t + \sin^2 t)
\end{aligned}$$

Here $w = 2\pi$ in Theorem (2.2). Then by Theorem (2.2), we have

$$\eta_2 = \eta_3 = \eta_4 = 0 \quad \text{but} \quad \eta_5 = -\frac{\pi}{4}ab(c + 2d + e)$$

μ is 5 if $\eta_5 \neq 0$. To have multiplicity at the origin greater than 5, we set $\eta_5 = 0$ either

$$\begin{aligned}
a &= 0 \quad \text{or} \quad b = 0 \quad \text{or} \\
e &= -(c + 2d) \qquad (4.12)
\end{aligned}$$

If $a \neq 0$, $b = 0$ then $\alpha(t)$ and $\beta(t)$ becomes

$$\begin{aligned}
\alpha(t) &= (c+d)\cos^3 t \sin t + (e+d)\cos t \sin^3 t \\
\beta(t) &= a\cos t
\end{aligned}$$

Let $\sigma(t) = a \sin t$, then $\sigma(0) = \sigma(2\pi)$ and also σ differentiable. Hence by Theorem (3.1) origin is a centre with $g(\sigma) = 1$ and $f(\sigma) = \frac{1}{a^4}[\sigma^2(c+d)\sigma + (e-c)\sigma^3)]$

If $b \neq 0$ but $a = 0$ then $\alpha(t)$ is the same but $\beta(t) = b \quad \sin t$. Let $\sigma(t) = -b\cos$ then $\sigma(0) = \sigma(2\pi) = -b$. Hence origin is a centre by Theorem (3.1) with $g(\sigma) = 1$ and $f(\sigma) = \frac{1}{b^4}[(e-c)\sigma^3 - b^2(e+d)\sigma]$. Now if $ab \neq 0$ but (4.12) holds then $\eta_6 = 0$ but $\eta_7 = \frac{\pi}{8}ab(a+b)(a-b)(c+d)$

$$\eta_8 = 0$$

$$\eta_9 = \frac{\pi}{32}ab(a+b)(a-b)(c+d)\{6(11b^2 + a^2)) + (c+d)\}$$

i.e.

$$\eta_9 = \frac{1}{4}\{6(11b^2 + a^2)' + (c+d))\}\eta_7$$

Thus $\eta_9 = 0$ whenever $\eta_7 = 0$. To prove that $\mu_{\max}(\Omega_3) = 7$ we need to prove that if $\eta_7 = 0$ then the origin is a center. Now $\eta_7 = 0$ if either

$$a + b = 0 \quad \text{or} \quad a - b = 0 \quad \text{or} \quad c + d = 0$$

If $a - b = 0$ but $a + b \neq 0, \quad c + d \neq 0$ then $\alpha(t)$ and $\beta(t)$ becomes

$$\begin{aligned} \alpha(t) &= (c+d)\cos^3 t \sin t - (c+d)\cos t \sin^3 t \\ \beta(t) &= a(\cos t + \sin t) \end{aligned}$$

Let $\sigma(t) = \sin t - \cos t$. Then $\sigma(0) = \sigma(2\pi)$. Also if we define

$f(\sigma) = (c+d)\left(\frac{\sigma^2-1}{2}\right)\sigma$ and $g(\sigma) = a$ then origin is a centre by Theorem (3.1). Thus $a - b \neq 0$. Now if $a + b = 0$ but $c + d \neq 0$ then $\alpha(t)$ is the same but $\beta(t) = a(\cos t + \sin t)$. Then origin is a center by Theorem (3.1) with $g(\sigma) = a$ and $f(\sigma) = (c+d)\left(\frac{1-\sigma^2}{2}\right)\sigma$. Hence $a + b \neq 0$. Now if $c + d = 0$ as $a + b \neq 0, \quad a - b \neq 0$. Then $\alpha(t) \equiv 0$ and meanvalue of $\beta(t)$ is zero, therefore by Corollary (3.3) the origin is a center. Thus we have the following result.

Theorem

Let $\mu_{\max}(\Omega_3) = v_3$. Then $v_3 = 7$.

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CONSTRAINED INTERPOLATION

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Abstract

The problem of the Constrained Interpolation is considered in this paper. The problem assumes data on the same side (above or below) of a function of independent variable and looks for an interpolant that also lies on the same side of the function. To tackle this problem, a rational cubic spline involving three free parameters in its description is used. Necessary and sufficient conditions have been determined so that when these conditions are satisfied, the rational cubic spline lies on the same side of a linear function.

keywords: Interpolation, Constrained interpolation, Rational cubic spline.

1. Introduction

The classical problem of interpolation - to determine a function that matches the given discrete values - is extened to Constrained Interpolation as below:

Given data points $\{(t_i, f_i) : i = 0, 1, \ldots, n\}$ such that

$$s.f_i \geq s.g(t_i) \quad \text{for all } i = 0, 1, \ldots, n \tag{1.1}$$

$$\text{where } s = \begin{cases} -1 \\ +1 \end{cases}$$

construct an interpolant $S(t)$ such that

$$S(t_i) = f_i \quad \text{for all } i = 0, 1, \ldots, n$$

$$\text{and } s.S(t) \geq s.g(t) \quad \text{for all} \quad t \in [t_0, t_n]$$

The interpolant $S(t)$ may also be required to have a certain degree of continuity. For visual purpose, $S(t) \in C^1[t_0, t_n]$ gives a reasonable pleasing effect. The function $g(t)$ may be any continuous function. However, this paper considers the case where $g(t)$ is a straight line, that is,

$$g(t) = mt + c$$

where m is the slope of the straight line and c is the y-intercept.

We use a rational cubic spline, Sarfraz(1994) to construct $S(t)$. This spline involves three free parameters in each interval. A particular case of this rational cubic spline when one parameter is taken equal to 2 is considered in Gregory et al(1994). To construct a constrained interpolant, we impose restrictions upon these parameters. When these parameters are assigned the restricted values, a desired interpolant is constructed.

The rest of the paper is organized as follows: In Section 2 of this paper, a rational cubic spline is described. In Section 3 we develop necessary and sufficient conditions which ensure the construction of a constrained interpolant. Numerical examples have been included in Section 4 for illustration.

2. Rational Cubic Spline

A piecewise rational cubic spline $S \in C^1[t_0, t_n]$, is defined for $t \in [t_i, t_{i+1}], i = 0, 1, 2, \ldots, n-1$, by

$$S(t) = S_i(t, \alpha_i, \beta_i, \gamma_i) = \frac{p_i(t)}{q_i(t)} \tag{2.1}$$

where

$$p_i(t) = \alpha_i f_i(1-\theta)^3 + V_i\theta(1-\theta)^2 + W_i\theta^2(1-\theta) + \beta_i f_{i+1}\theta^3$$

$$q_i(t) = \alpha_i(1-\theta)^2 + \gamma_i\theta(1-\theta) + \beta_i\theta^2$$

$$V_i = (\gamma_i + \alpha_i)f_i + \alpha_i h_i d_i \text{ and } W_i = (\gamma_i + \beta_i)f_{i+1} - \beta_i h_i d_{i+1}$$

where f_i and d_i are respectively, the data values and the first derivative values at the knots $t_i, i = 0, 1, 2, \ldots, n$ with $t_0 < t_1 < \ldots < t_n, h_i = t_{i+1} - t_i, \theta = \frac{(t-t_i)}{h_i}$ and $\alpha_i, \beta_i, \gamma_i$ are free parameters. The spline $S(t)$ has the Hermite interpolation properties, that is,

$$S(t_i) = f_i, \quad i = 0, 1, \ldots, n$$

$$S^{(1)}(t_i) = d_i, \quad i = 0, 1, \ldots, n$$

The derivative values d_i (if not given) may be estimated by using the standard three points difference formula. A comperhensive survey of estimating d_i's is given in Boehm et al(1984). Since the Hermite conditions are satisfied for the arbitrary values of α_i, β_i, and γ_i so $S(t)$ produces infinite number of interpolant (by varying the values α_i, β_i, and γ_i a new interpolant is constructed) through the same data. For example, when we take $\alpha_i = \beta_i = 1$ and $\gamma_i = 2$, for all $i = 0, 1, \ldots, n-1$ the rational cubic spline $S(t)$ reduces to the standard cubic Hermite spline and the restrictions

$$\alpha_i \geq 0, \quad \beta_i \geq 0, \quad \gamma_i > 0 \qquad (2.2)$$

ensure a positive denominator in (2.1). This encourges us to look for the values of α_i, β_i, and γ_i that also satisfy the conditions of constrained interpolation.

3. Constrained Interpolation

A particular case of constrained interpolation (1.1) may be defined as below: Given data points $\{(t_i, f_i) : i = 0, 1, \ldots, n\}$ above a straight line $L_1(t) = m_1 t + c_1$, that is,

$$f_i \geq m_1 t_i + c_1 \text{ for all } i = 0, 1, \ldots, n \qquad (3.1)$$

determine a function $S(t)$ such that

$$S(t_i) = f_i \ \forall i = 0, 1, \ldots, n \tag{3.2}$$

$$S(t) \geq L_1(t) \tag{3.3}$$

$$\text{and } S(t) \in C^1[t_0, t_n] \tag{3.4}$$

The rational cubic spline (2.1) satisfies the conditions (3.2) and (3.4). We, therefore, require (3.3) to be satisfied.

In each interval $[t_i, t_{i+1}]$, we have $q_i(t) > 0$ when α_i, β_i, and γ_i satisfy the conditions (2.2). Therefore, (3.3) is reduced to

$$U_1(t) = p_i(t) - L_1(t)q_i(t) \geq 0$$

Since, in each interval $[t_i, t_{i+1}]$, the Bezier form of the line $L_1(t)$ is given by:

$$L_1(t) = A_i(1-\theta) + B_i\theta$$

$$\text{where } A_i = L_1(t_i) \text{ and } B_i = L_1(t_{i+1})$$

Therefore, the cubic polynomial $U_1(t)$ can be expressed as:

$$U_1(t) = \alpha_i(f_i - A_i)(1-\theta)^3 + \phi_i\theta(1-\theta)^2 + \varphi_i\theta^2(1-\theta) + \beta_i(f_{i+1} - B_i)\theta^3 \tag{3.5}$$

$$\phi_i = V_i - \alpha_i B_i - \gamma_i A_i \text{ and } \varphi_i = W_i - \gamma_i B_i - \beta_i A_i$$

Since $U_1(t_i) = \alpha_i(f_i - A_i) \geq 0$ and $U_1(t_{i+1}) = \beta_i(f_{i+1} - B_i) \geq 0$ so we may look for $U_1(t)$ satisfying the following condition:

$$U_1(t) \geq 0 \ \forall t \in [t_i, t_{i+1}] \tag{3.6}$$

In case $U_1(t_i) = 0 = U_1(t_{i+1})$, we have

$$f_i = m_1 t_i + c_1$$

$$f_{i+1} = m_1 t_{i+1} + c_1$$

Then the assignment

$$d_i = m_1, \quad d_{i+1} = m_1$$

gives $S(t) = m_1t_i + c_1$ and so $U_1(t) = 0$ satisfies the condition (3.6). However, if $U_1(t_i) = 0$ but $U_1(t_{i+1}) \neq 0$, then $f_i = m_1t_i + c_1$ and the assignment $d_i = m_1$ guarantees

$$S(t) \geq m_1t + c_1$$

$$\text{and } U_1(t) = S(t) - m_1t_i + c_1 > 0$$

Similarly, if $U_1(t_{i+1}) = 0$ but $U_1(t_i) \neq 0$ then $d_{i+1} = m_1$ guarantees $U_1(t) \geq 0$.

For rest of the discussion we assume that both $U_1(t_i)$ and $U_1(t_{i+1})$, are non-zero that is, $U_1(t_i) \neq 0$ and $U_1(t_{i+1}) \neq 0$. Then,

$$f_i > m_1t_i + c_1$$

$$f_{i+1} > m_1t_{i+1} + c_1$$

In (3.5), $\alpha_i(f_i - A_i) > 0$ and $\beta_i(f_{i+1} - B_i) > 0$ therefore $U_1(t) > 0$ if $\phi_i \geq 0$ and $\varphi_i \geq 0$ that leads to the following sufficient conditions:

Theorem 3.1

Let $\{(t_i, f_i) : i = 0, 1, \ldots, n\}$ be given data satisfying condition (3.1). Then the rational cubic spline (2.1) also lies above the straight line $L_1(t)$ provided the following conditions are satisfied:

$$\begin{aligned} &\alpha_i \geq 0 \ \text{ if } \ f_i - (m_1t_{i+1} + c_1) + h_id_i \geq 0 \\ &\alpha_i \in [0, \frac{-\gamma_i\{f_i - (m_1t_i + c_1)\}}{f_i - (m_1t_{i+1} + c_1) + h_id_i}] \quad \text{otherwise} \end{aligned} \qquad (3.7)$$

$$\begin{aligned} &\beta_i \geq 0 \ \text{ if } \ f_{i+1} - (m_1t_i + c_1) - h_id_{i+1} \geq 0 \\ &\beta_i \in [0, \frac{-\gamma_i\{f_{i+1} - (m_1t_{i+1} + c_1)\}}{f_{i+1} - (m_1t_i + c_1) - h_id_{i+1}}] \quad \text{otherwise} \end{aligned} \qquad (3.8)$$

The above theorem only gives sufficient condition. However, using the necessary and sufficient conditions for cubic polynomials given in Schmidt and Hess(1988), the following necessary and suffficient conditions can easily be developed:

Theorem 3.2

The rational cubic spline (2.1) lies above a straight line if and only if the free parameters present in the description of the spline satisfy either the conditions (3.7), (3.8) or the condition given by:

$$36f_i f_{i+1}[\chi_1^2+\chi_2^2+\chi_1\chi_2-3\Delta_i(\chi_1+\chi_2)+3\Delta_i^2]+3(f_{i+1}\chi_1-f_i\chi_2)$$
$$(2h_i\chi_1\chi_2-3f_{i+1}\chi_1+3f_i\chi_2)+4h_i(f_{i+1}\chi_1^3-f_i\chi_2^3)-h_i^2\chi_1^2\chi_2^2\geq 0 \qquad (3.9)$$

where $\chi_1 = U_1'(t_i)$ and $\chi_2 = U_1'(t_{i+1})$

Similarly, if the data $\{(t_i, f_i) : i = 0, 1, \ldots, n\}$ lies below the straight line $L_2(t) = m_2 t + c_2$ then the following theorem gives the conditions which ensure that the rational cubic spline (2.1) also lies below the straight line:

Theorem 3.3

The rational cubic spline (2.1) lies below a straight line if and only if the free parameters present in the description of the spline satisfy either the conditions:

$$\alpha_i \geq 0 \ \text{ if } \ f_i-(m_2t_{i+1}+c_2)+h_id_i\leq 0$$
$$\alpha_i \in [0, \frac{-\gamma_i\{f_i-(m_2t_i+c_2)\}}{f_i-(m_2t_{i+1}+c_2)+h_id_i}] \quad \text{otherwise} \qquad (3.10)$$

$$\beta_i \geq 0 \ \text{ if } \ f_{i+1}-(m_2t_i+c_2)-h_id_{i+1}\leq 0$$
$$\beta_i \in [0, \frac{-\gamma_i\{f_{i+1}-(m_2t_{i+1}+c_2)\}}{f_{i+1}-(m_2t_i+c_2)-h_id_{i+1}}] \quad \text{otherwise} \qquad (3.11)$$

$$36f_i f_{i+1}[\chi_1^2+\chi_2^2+\chi_1\chi_2-3\Delta_i(\chi_1+\chi_2)+3\Delta_i^2]+3(f_{i+1}\chi_1-f_i\chi_2)$$
$$(2h_i\chi_1\chi_2-3f_{i+1}\chi_1+3f_i\chi_2)+4h_i(f_{i+1}\chi_1^3-f_i\chi_2^3)-h_i^2\chi_1^2\chi_2^2\leq 0 \qquad (3.12)$$

4. Numerical Example

In this section we consider a data that lies above the line $y = 1 + x/2$. We first use cubic osculatory method which produces a curve shown in Figure 1. A portion of this curve in the intervals $[3, 7]$ and $[9, 13]$ lies below the line and so is not in accordance with the shape of the data. We then use the rational cubic spline (2.1) and assign values to free parameters developed in Section 3. Two curve in Figure 2 and Figure 3 are generated for different values of free parameters. Both curves lie above the given straight line and preserve the inherant shape of the data.

Table

x	2	3	7	8	9	13	14
y	12	4.5	6.5	12	7.5	9.5	18

This example shows that the use of rational cubic spline (2.1) is two fold: it generates a desired curve and allows user to modify the shape of the curve simply by changing the values of free parameters.

5. Conclusions and Suggestions

This paper has given a simple method to construct curves through data that lie above/below a line. The method described here is local and allows user to refine the first draft of the constrained curve to a desired shape.

The future work will investigate for the generalization of these methods to 2D case where the requirement is to produce a surface through constrained data, that is, the data which lies above/below a plane. The work related to the situations where such a data is arranged over a rectangular grid is currently under preparation and will be presented in a subsequent paper.

The present study is only for the single valued curves. The problem can also be extended to parametric case where even stronger conditions could be imposed. For example, one may require an interpolant which simultaneouly lies above, below and between various lines.

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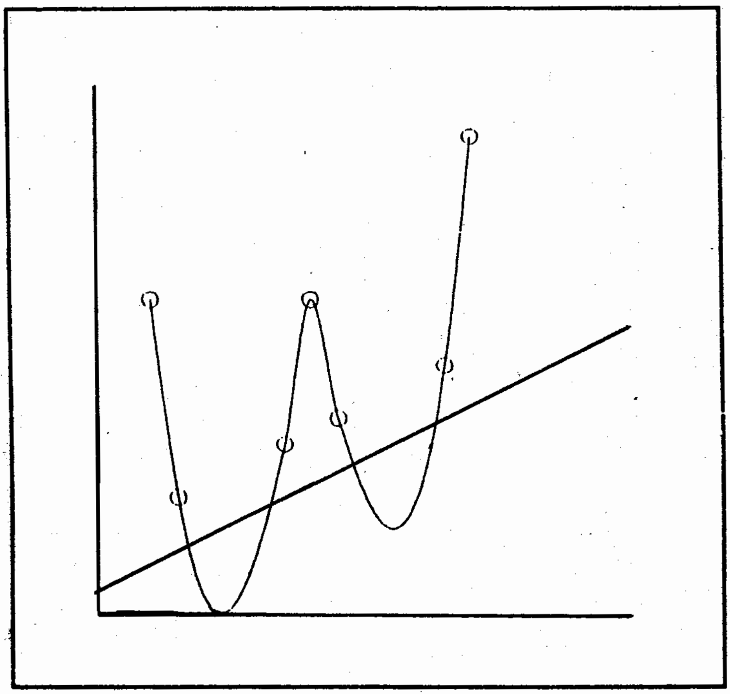

Figure 1

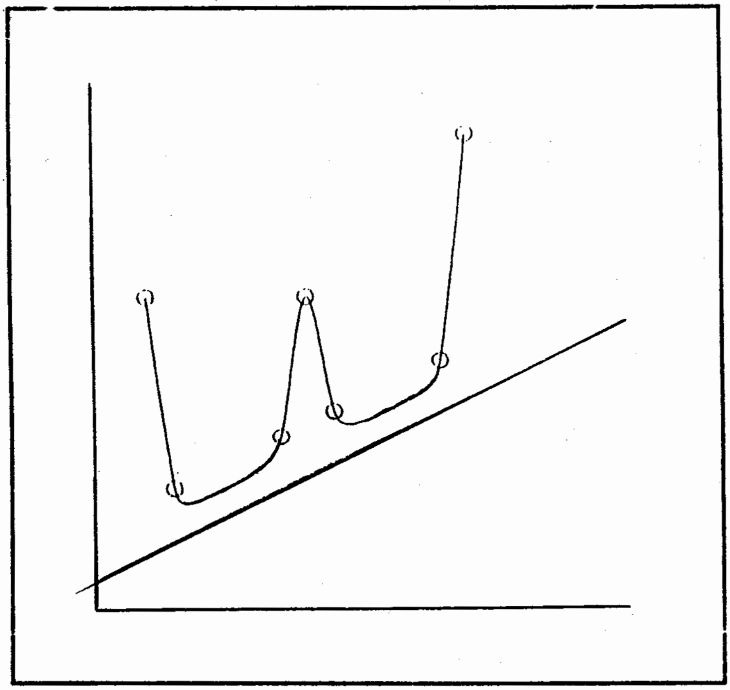

Figure 2

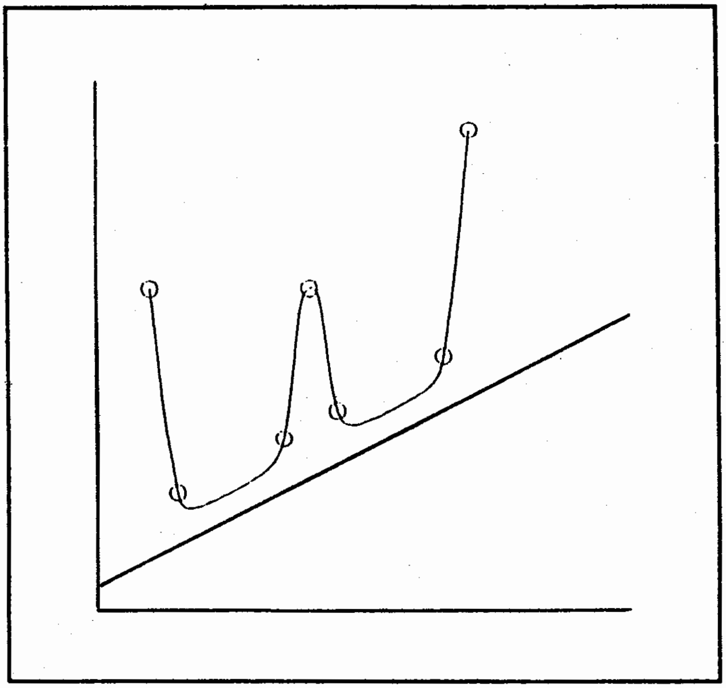

Figure 3

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DEFINING NEW CONSISTENCY RELATIONS FOR SPLINE SOLUTIONS OF TWELFTH ORDER BOUNDARY-VALUE PROBLEMS

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Dedicated to the memory of Dr. M. Rafique

Abstract

Linear, twelfth-order boundary-value problems (special case) are solved, using polynomial splines of degree twelve.

The spline function values at the midknots of the interpolation interval, and the corresponding values of the even-order derivatives are related through consistency relations. The algorithm developed approximates the solutions, and their higher-order derivatives, of differential equations.

Two numerical illustrations are given to show the practical usefulness of the algorithm developed. It is observed that this algorithm is second-order convergent.

Keywords:

Two-point boundary-value problems; finite-difference methods; twelfth-degree splines.

AMS Classification: 65L10

1. Introduction

When an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability is as ordinary convection the ordinary differential equation is sixth order; when the instability sets in as overstability, it is modelled by an eighth-order ordinary differential equation.

Suppose, now, that a uniform magnetic field is also applied across the fluid in the same direction as gravity. When instability sets in now as ordinary convection, it is modelled by a tenth-order boundary-value problem; when instability sets in as overstability, it is modelled by a twelfth-order boundary-value problem (for details, see Chandrasekhar [4]). Finite difference methods of solution for such problems were developed by Boutayeb and Twizell [1,2,3], Djidjeli et al. [5], Twizell [10], Twizell and Boutayeb [11], and Twizell et al. [12].

Usmani [13], solved fourth-order boundary-value problem using quartic splines.

In the present paper twelfth-order boundary-value problems are solved using twelfth-degree splines, introducing new consistancy relations.
These problems have the form

$$\left.\begin{array}{l} y^{(xii)} + \phi(x)y \;=\; \psi(x)\,, \qquad -\infty < a \leq x \leq b < \infty\,, \\ y^{(2k)}(a) = A_{2k}\,,\; y^{(2k)}(b) = B_{2k}\,,\; k \;=\; 0,\; 1,\; 2,\; \ldots\;,5 \end{array}\right\} \qquad (1.1)$$

where $y = y(x)$, and $\phi(x)$ and $\psi(x)$ are continuous functions defined in the interval $x \in [a,\; b]$. A_i and B_i, $i = 0, 2, 4, 6, 8, 10$, are finite real constants.

2. The Twelfth-degree spline

2.1 Consistency relations

The interval $[a,\; b]$ is divided into $n \geq 22$ equal parts, thus introducing $n+1$ grid points x_i so that

$$x_i \;=\; a + ih\,, \qquad i = 0, 1, 2, \ldots, n,$$

$$x_0 \;=\; a, \;\; x_n \;=\; b \quad \text{and} \quad h \;=\; \frac{b-a}{n}$$

The exact solution of the problem (1.1) at $x = x_i$ is $y(x_i)$. Let s_i be the approximation to y at x_i determined by the twelfth-degree spline defined on the sub-interval $[x_i,\; x_{i+1}]$ by

$$\begin{aligned} Q_i(x) &= a_i(x-x_i)^{12} + b_i(x-x_i)^{11} + c_i(x-x_i)^{10} + d_i(x-x_i)^9 \\ &+ e_i(x-x_i)^8 + f_i(x-x_i)^7 + g_i(x-x_i)^6 \\ &+ l_i(x-x_i)^5 + r_i(x-x_i)^4 + u_i(x-x_i)^3 \\ &+ v_i(x-x)i)^2 + w_i(x-x_i) + z_i, \\ &i = 0, 1, \ldots, n-1. \end{aligned} \tag{2. 1}$$

The twelfth-degree spline $s(x) \in C^{11}[a,\ b]$ can, thus, be defined as

$$s(x) \ = \ Q_i(x), \quad x \in [x_i,\ x_{i+1}], \quad i = 0, 1, \ldots, n-1 \tag{2.2}$$

The coefficients of (2.1) are determined, (see[9999]), as

$$a_i = \frac{1}{479001600} s_{i+1/2}^{(xii)} \tag{2. 3}$$

$$b_i = \frac{1}{39916800} s_i^{(xi)} \tag{2. 4}$$

$$c_i = \frac{1}{3628800} s_{i+1/2}^{(x)} - \frac{1}{7257600} h s_i^{(xi)} - \frac{1}{29030400} h^2 s_{i+1/2}^{(xii)} \tag{2. 5}$$

$$d_i = \frac{1}{362880} s_i^{(ix)} \tag{2. 6}$$

$$\begin{aligned} e_i &= \frac{1}{40320} s_{i+1/2}^{(viii)} - \frac{1}{80640} h s_i^{(ix)} - \frac{1}{322560} h^2 s_{i+1/2}^{(x)} + \frac{1}{967680} h^3 s_i^{(xi)} \\ &+ \frac{1}{3096576} h^4 s_{i+1/2}^{(xii)} \end{aligned} \tag{2. 7}$$

$$f_i = \frac{1}{5040} s_i^{(vii)} \tag{2. 8}$$

$$\begin{aligned} g_i &= \frac{1}{720} s_{i+1/2}^{(vi)} - \frac{1}{1440} h s_i^{(vii)} - \frac{1}{5760} h^2 s_{i+1/2}^{(viii)} + \frac{1}{17280} h^3 s_i^{(ix)} \\ &+ \frac{1}{55296} h^4 s_{i+1/2}^{(x)} - \frac{1}{172800} h^5 s_i^{(xi)} \\ &- \frac{61}{33177600} h^6 s_{i+1/2}^{(xii)}, \end{aligned} \tag{2. 9}$$

$$l_i = \frac{1}{120} s_i^{(v)} \tag{2. 10}$$

$$\begin{aligned} r_i &= \frac{1}{24} s_{i+1/2}^{(iv)} - \frac{1}{48} h s_i^{(v)} - \frac{1}{192} h^2 s_{i+1/2}^{(vi)} + \frac{1}{576} h^3 s_i^{(vii)} + \frac{5}{9216} h^4 s_{i+1/2}^{(viii)} \\ &- \frac{1}{5760} h^5 s_i^{(ix)} - \frac{61}{1105920} h^6 s_{i+1/2}^{(x)} + \frac{17}{967680} h^7 s_i^{(xi)} \end{aligned}$$

$$+\frac{277}{49545216}h^8 s^{(xii)}_{i+1/2} \quad (2.11)$$

$$u_i = \frac{1}{6}s'''_i \quad (2.12)$$

$$\begin{aligned} v_i = {} & \frac{1}{2}s''_{i+1/2} - \frac{1}{4}hs'''_i - \frac{1}{16}h^2 s^{(iv)}_{i+1/2} + \frac{1}{48}h^3 s^{(v)}_i + \frac{5}{768}h^4 s^{(vi)}_{i+1/2} \\ & - \frac{1}{480}h^5 s^{(vii)}_i - \frac{61}{92160}h^6 s^{(viii)}_{i+1/2} + \frac{17}{80640}h^7 s^{(ix)}_i \\ & + \frac{277}{4128768}h^8 s^{(x)}_{i+1/2} - \frac{31}{1451520}h^9 s^{(xi)}_i \\ & - \frac{50521}{7431782400}h^{10} s^{(xii)}_{i+1/2} \end{aligned} \quad (2.13)$$

$$w_i = s'_i \quad (2.14)$$

and

$$\begin{aligned} z_i = {} & s_{i+1/2} - \frac{1}{2}hs'_i - \frac{1}{8}h^2 s''_{i+1/2} + \frac{1}{24}h^3 s'''_i + \frac{5}{384}h^4 s^{(iv)}_{i+1/2} \\ & - \frac{1}{240}h^5 s^{(v)}_i - \frac{61}{46080}h^6 s^{(vi)}_{i+1/2} + \frac{17}{40320}h^7 s^{(vii)}_i \\ & + \frac{277}{2064384}h^8 s^{(viii)}_{i+1/2} - \frac{31}{725760}h^9 s^{(ix)}_i \\ & - \frac{50521}{3715891200}h^{10} s^{(x)}_{i+1/2} + \frac{691}{159667200}h^{11} s^{(xi)}_i \\ & + \frac{540553}{392398110720}h^{12} s^{(xii)}_{i+1/2} \end{aligned} \quad (2.15)$$

The odd-order derivatives of the splines are calculated, (see[9]), as .

$$hs^{(xi)}_i = (s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) - \frac{1}{8}h^2(s^{(xii)}_{i+1/2} - s^{(xii)}_{i-1/2}) , \quad (2.16)$$

$$\begin{aligned} hs^{(ix)}_i = {} & (s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) - \frac{1}{24}h^2(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) \\ & + \frac{1}{384}h^4(s^{(xii)}_{i+1/2} - s^{(xii)}_{i+1/2}) , \end{aligned} \quad (2.17)$$

$$\begin{aligned} hs^{(vii)}_i = {} & (s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2}) - \frac{1}{24}h^2(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) \\ & + \frac{7}{5760}h^4(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) - \frac{1}{15360}h^6(s^{(xii)}_{i+1/2} - s^{(xii)}_{i-1/2}) , \end{aligned} \quad (2.18)$$

$$hs^{(v)}_i = (s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) - \frac{1}{24}h^2(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2})$$

$$+ \frac{7}{5760}h^4(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) - \frac{31}{967680}h^6(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2})$$
$$+ \frac{17}{10321920}h^8(s^{(xii)}_{i+1/2} - s^{(xii)}_{i-1/2}) \qquad (2.\ 19)$$

$$hs_i''' = (s''_{i+1/2} - s''_{i-1/2}) - \frac{1}{24}h^2(s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2})$$
$$+ \frac{7}{5760}h^4(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2}) - \frac{31}{967680}h^6(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2})$$
$$+ \frac{127}{154828800}h^8(s^{(x)}_{i+1/2} - s^{(x)}_{i-1/2}) - \frac{31}{743178240}h^{10}(s^{(xii)}_{i+1/2}$$
$$- s^{(xii)}_{i-1/2}), \qquad (2.20) \qquad (2.\ 20)$$

and

$$hs_i' = (s_{i+1/2} - s_{i-1/2}) - \frac{1}{24}h^2(s''_{i+1/2} - s''_{i-1/2})$$
$$+ \frac{7}{5760}h^4(s^{(iv)}_{i+1/2} - s^{(iv)}_{i-1/2}) - \frac{31}{967680}h^6(s^{(vi)}_{i+1/2} - s^{(vi)}_{i-1/2})$$
$$+ \frac{127}{154828800}h^8(s^{(viii)}_{i+1/2} - s^{(viii)}_{i-1/2}) - \frac{73}{3503554560}h^{10}(s^{(x)}_{i+1/2}$$
$$- s^{(x)}_{i-1/2}) + \frac{691}{653996851200}h^{12}(s^{(xii)}_{i+1/2} - s^{(xii)}_{i-1/2}) \qquad (2.\ 21)$$

The even-order derivatives of the splines are defined, (see[9]), as

$$h^{10}s^{(x)}_{i-1/2} = \big(s_{i-11/2} - 10s_{i-9/2} + 45s_{i-7/2} - 120s_{i-5/2} + 210s_{i-3/2}$$
$$- 252s_{i-1/2} + 210s_{i+1/2} - 120s_{i+3/2} + 45s_{i+5/2} - 10s_{i+7/2}$$
$$+ s_{i+9/2}\big) - \frac{1}{1961990553600}h^{12}\Big(s^{(xii)}_{i-11/2} + 531430s^{(xii)}_{i-9/2}$$
$$+ 238294829s^{(xii)}_{i-7/2} + 11184969416s^{(xii)}_{i-5/2}$$
$$+ 143515424210s^{(xii)}_{i-3/2} + 507617624228s^{(xii)}_{i-1/2}$$
$$+ 143515424210s^{(xii)}_{i+1/2} + 11184969416s^{(xii)}_{i+3/2}$$
$$+ 238294829s^{(xii)}_{i+5/2} + 531430s^{(xii)}_{i+7/2} + s^{(xii)}_{i+9/2}\Big),$$
$$i = 6,\ 7, \ldots\ldots, n-5 \qquad (2.\ 22)$$
$$\big(s_{i-13/2} - 4s_{i-11/2} - 14s_{i-9/2} + 140s_{i-7/2} - 465s_{i-5/2}$$
$$+ 888s_{i-3/2} - 1092s_{i-1/2} + 888s_{i+1/2} - 465s_{i+3/2} + 140s_{i+5/2}$$
$$- 14s_{i+7/2} - 4s_{i+9/2} + s_{i+11/2}\big)$$

$$= \frac{1}{245248819200}h^{10}\Big(s^{(x)}_{i-13/2} + 531428s^{(x)}_{i-11/2} + 237231970s^{(x)}_{i-9/2}$$
$$+ 10708911188s^{(x)}_{i-7/2} + 121383780207s^{(x)}_{i-5/2} + 477020564424s^{(x)}_{i-3/2}$$
$$+ 743288515164s^{(x)}_{i-1/2} + 477020564424s^{(x)}_{i+1/2} + 121383780207s^{(x)}_{i+3/2}$$
$$+10708911188s^{(x)}_{i+5/2} + 237231970s^{(x)}_{i+7/2} + 531428s^{(x)}_{i+9/2}$$
$$+ s^{(x)}_{i+11/2}\Big) , \qquad (2.\ 23)$$

$$h^8 s^{(viii)}_{i-1/2} = \frac{1}{3}\big(-s_{i-11/2} + 13s_{i-9/2} - 69s_{i-7/2} + 204s_{i-5/2} - 378s_{i-3/2}$$
$$+ 462s_{i-1/2} - 378s_{i+1/2} + 204s_{i+3/2} - 69s_{i+5/2} + 13s_{i+7/2}$$
$$- s_{i+9/2}\big) + \frac{1}{5885971660800}h^{12}\Big(s^{(xii)}_{i-11/2} + 531427s^{(xii)}_{i-9/2}$$
$$+ 236700533s^{(xii)}_{i-7/2} + 10466896340s^{(xii)}_{i-5/2}$$
$$+ 108525964106s^{(xii)}_{i-3/2} + 268387374866s^{(xii)}_{i-1/2}$$
$$+ 108525964106s^{(xii)}_{i+1/2} + 10466896340s^{(xii)}_{i+3/2}$$
$$+ 236700533s^{(xii)}_{i+5/2} + 531427s^{(xii)}_{i+7/2} + s^{(xii)}_{i+9/2}\Big) ,$$
$$i = 6,\ 7, \ldots\ldots, n-5 , \qquad (2.\ 24)$$

$$\big(s_{i-13/2} + 68s_{i-11/2} - 350s_{i-9/2} + 308s_{i-7/2} + 1647s_{i-5/2}$$
$$- 5496s_{i-3/2} + 7644s_{i-1/2} - 5496s_{i+1/2} + 1647s_{i+3/2} + 308s_{i+5/2}$$
$$- 350s_{i+7/2} + 68s_{i+9/2} + s_{i+11/2}\big)$$
$$= \frac{1}{5109350400}h^{8}\Big(s^{(viii)}_{i-13/2} + 531428s^{(viii)}_{i-11/2} + 237231970s^{(viii)}_{i-9/2}$$
$$+ 10708911188s^{(viii)}_{i-7/2} + 121383780207s^{(viii)}_{i-5/2} + 477020564424s^{(viii)}_{i-3/2}$$
$$+ 743288515164s^{(viii)}_{i-1/2} + 477020564424s^{(viii)}_{i+1/2} + 121383780207s^{(viii)}_{i+3/2}$$
$$+10708911188s^{(viii)}_{i+5/2} + 237231970s^{(viii)}_{i+7/2} + 531428s^{(viii)}_{i+9/2}$$
$$+ s^{(viii)}_{i+11/2}\Big) , \qquad (2.\ 25)$$

$$h^6 s^{(vi)}_{i-1/2} = \frac{1}{240}\big(13s_{i-11/2} - 190s_{i-9/2} + 1305s_{i-7/2} - 4680s_{i-5/2}$$
$$+ 9690s_{i-3/2} - 12276s_{i-1/2} + 9690s_{i+1/2} - 4680s_{i+3/2}$$
$$+ 1305s_{i+5/2} - 190s_{i+7/2} + 13s_{i+9/2}\big)$$

$$- \frac{1}{470877732864000} h^{12} \Big(13 s^{(xii)}_{i-11/2} + 6908530 s^{(xii)}_{i-9/2}$$
$$+ 3065947097 s^{(xii)}_{i-7/2} + 131170685048 s^{(xii)}_{i-5/2}$$
$$+ 1223612246810 s^{(xii)}_{i-3/2} + 2695333801004 s^{(xii)}_{i-1/2}$$
$$+ 1223612246810 s^{(xii)}_{i+1/2} + 131170685048 s^{(xii)}_{i+3/2}$$
$$+ 3065947097 s^{(xii)}_{i+5/2} + 6908530 s^{(xii)}_{i+7/2} + 13 s^{(xii)}_{i+9/2} \Big) \ ,$$
$$i = 6,\ 7, \ldots\ldots, n-5\ , \qquad (2.\ 26)$$

$$\big(s_{i-13/2} + 716 s_{i-11/2} + 6226 s_{i-9/2} - 28900 s_{i-7/2} + 12975 s_{i-5/2}$$
$$+ 90648 s_{i-3/2} - 163332 s_{i-1/2} + 90648 s_{i+1/2} + 12975 s_{i+3/2}$$
$$- \ 28900 s_{i+5/2} + 6226 s_{i+7/2} + 716 s_{i+9/2} + s_{i+11/2} \big)$$
$$= \frac{1}{42577920} h^{6} \Big(s^{(vi)}_{i-13/2} + 531428 s^{(vi)}_{i-11/2} + 237231970 s^{(vi)}_{i-9/2}$$
$$+ 107089111188 s^{(vi)}_{i-7/2} + 121383780207 s^{(vi)}_{i-5/2} + 477020564424 s^{(vi)}_{i-3/2}$$
$$+ 743288515164 s^{(vi)}_{i-1/2} + 477020564424 s^{(vi)}_{i+1/2} + 121383780207 s^{(vi)}_{i+3/2}$$
$$+ 107089111188 s^{(vi)}_{i+5/2} + 237231970 s^{(vi)}_{i+7/2} + 531428 s^{(vi)}_{i+9/2}$$
$$+ s^{(vi)}_{i+11/2} \Big) \ , \qquad (2.\ 27)$$

$$h^4 s^{(iv)}_{i-1/2} = \frac{1}{15120} \big(-82 s_{i-11/2} + 1261 s_{i-9/2} - 9738 s_{i-7/2} + 52428 s_{i-5/2}$$
$$- \ 140196 s_{i-3/2} + 192654 s_{i-1/2} - 140196 s_{i+1/2}$$
$$+ 52428 s_{i+3/2} - 9738 s_{i+5/2} + 1261 s_{i+7/2} - 82 s_{i+9/2} \big)$$
$$+ \frac{1}{29665297170432000} h^{12} \Big(82 s^{(xii)}_{i-11/2} + 43576819 s^{(xii)}_{i-9/2}$$
$$+ 19305816986 s^{(xii)}_{i-7/2} + 812949948500 s^{(xii)}_{i-5/2}$$
$$+ 7222638646292 s^{(xii)}_{i-3/2} + 15216573195122 s^{(xii)}_{i-1/2}$$
$$+ 7222638646292 s^{(xii)}_{i+1/2} + 812949948500 s^{(xii)}_{i+3/2}$$
$$+ 19305816986 s^{(xii)}_{i+5/2} + 43576819 s^{(xii)}_{i+7/2} + 82 s^{(xii)}_{i+9/2} \Big) \ ,$$
$$i = 6,\ 7, \ldots\ldots, n-5\ , \qquad (2.\ 28)$$

$$\big(s_{i-13/2} + 6548 s_{i-11/2} + 305410 s_{i-9/2} + 1198148 s_{i-7/2}$$
$$- \ 3302673 s_{i-5/2} - 2622936 s_{i-3/2} + 8831004 s_{i-1/2}$$

$$
\begin{aligned}
& - 2622936 s_{i+1/2} - 3302673 s_{i+3/2} + 1198148 s_{i+5/2} \\
& + 305410 s_{i+7/2} + 6548 s_{i+9/2} + s_{i+11/2}\big) \\
= & \frac{1}{190080} h^4 \Big(s^{(iv)}_{i-13/2} + 531428 s^{(iv)}_{i-11/2} + 237231970 s^{(iv)}_{i-9/2} \\
& + 10708911188 s^{(iv)}_{i-7/2} + 121383780207 s^{(iv)}_{i-5/2} + 477020564424 s^{(iv)}_{i-3/2} \\
& + 743288515164 s^{(iv)}_{i-1/2} + 477020564424 s^{(iv)}_{i+1/2} + 121383780207 s^{(iv)}_{i+3/2} \\
& + 10708911188 s^{(iv)}_{i+5/2} + 237231970 s^{(iv)}_{i+7/2} + 531428 s^{(iv)}_{i+9/2} \\
& + s^{(iv)}_{i+11/2} \Big) , \qquad (2.\ 29)
\end{aligned}
$$

$$
\begin{aligned}
h^2 s''_{i-1/2} = & \frac{1}{25200} \big(8 s_{i-11/2} - 125 s_{i-9/2} + 1000 s_{i-7/2} - 6000 s_{i-5/2} \\
& + 42000 s_{i-3/2} - 73766 s_{i-1/2} + 42000 s_{i+1/2} \\
& - 6000 s_{i+3/2} + 1000 s_{i+5/2} - 125 s_{i+7/2} + 8 s_{i+9/2} \big) \\
& - \frac{1}{49442161950720000} h^{12} \Big(8 s^{(xii)}_{i-11/2} + 4251395 s^{(xii)}_{i-9/2} \\
& + 1882444472 s^{(xii)}_{i-7/2} + 78857458608 s^{(xii)}_{i-5/2} \\
& + 689483258240 s^{(xii)}_{i-3/2} + 1432258134554 s^{(xii)}_{i-1/2} \\
& + 689483258240 s^{(xii)}_{i+1/2} + 78857458608 s^{(xii)}_{i+3/2} \\
& + 1882444472 s^{(xii)}_{i+5/2} + 4251395 s^{(xii)}_{i+7/2} + 8 s^{(xii)}_{i+9/2} \Big) , \\
& i = 6,\ 7, \ldots\ldots, n-5 , \qquad (2.\ 30)
\end{aligned}
$$

$$
\begin{aligned}
& \big(s_{i-13/2} + 59036 s_{i-11/2} + 8998066 s_{i-9/2} + 160127660 s_{i-7/2} \\
& + 559437615 s_{i-5/2} - 108453192 s_{i-3/2} - 1240338372 s_{i-1/2} \\
& - 108453192 s_{i+1/2} + 559437615 s_{i+3/2} + 160127660 s_{i+5/2} \\
& + 8998066 s_{i+7/2} + 59036 s_{i+9/2} + s_{i+11/2} \big) \\
= & \frac{1}{528} h^2 \Big(s''_{i-13/2} + 531428 s''_{i-11/2} + 237231970 s''_{i-9/2} \\
& + 10708911188 s''_{i-7/2} + 121383780207 s''_{i-5/2} + 477020564424 s''_{i-3/2} \\
& + 743288515164 s''_{i-1/2} + 477020564424 s''_{i+1/2} + 121383780207 s''_{i+3/2} \\
& + 10708911188 s''_{i+5/2} + 237231970 s''_{i+7/2} + 531428 s''_{i+9/2} \\
& + s''_{i+11/2} \Big) \qquad (2.\ 31)
\end{aligned}
$$

and

$$\begin{aligned}
&\left(s_{i-13/2} - 12s_{i-11/2} + 66s_{i-9/2} - 220s_{i-7/2} + 495s_{i-5/2} - 792s_{i-3/2}\right.\\
&+ 924s_{i-1/2} - 792s_{i+1/2} + 495s_{i+3/2} - 220s_{i+5/2}\\
&\left.+\ 66s_{i+7/2} - 12s_{i+9/2} + s_{i+11/2}\right)\\
=\ &\frac{1}{1961990553600}h^{12}\left(s^{(xii)}_{i-13/2} + 531428s^{(xii)}_{i-11/2} + 237231970s^{(xii)}_{i-9/2}\right.\\
&+ 10708911188s^{(xii)}_{i-7/2} + 121383780207s^{(xii)}_{i-5/2} + 477020564424s^{(xii)}_{i-3/2}\\
&+ 743288515164s^{(xii)}_{i-1/2} + 477020564424s^{(xii)}_{i+1/2} + 121383780207s^{(xii)}_{i+3/2}\\
&+10708911188s^{(xii)}_{i+5/2} + 237231970s^{(xii)}_{i+7/2} + 531428s^{(xii)}_{i+9/2}\\
&\left.+\ s^{(xii)}_{i+11/2}\right)\ .\ i = 7,\ 8, \ldots\ldots, n-6 \qquad (2.32)
\end{aligned}$$

Following Siddiqi and Twizell [9], the new consistancy relations are defined, to determine the even-order derivatives of the splines.

In addition to the consistancy relations (2.24), (2.26), (2.28), (2.30) to define even-order derivaties $s^{(viii)}_{i-1/2}$, $s^{(vi)}_{i-1/2}$, $s^{(iv)}_{i-1/2}$, and $s^{(ii)}_{i-1/2}$, $i = 6,\ 7, \ldots\ldots, n-5$, the corresponding n n new consistancy relations are defined as under:

$$\begin{aligned}
h^8 s^{(viii)}_{i-1/2} =\ &\frac{1}{146152}\left(131669s_{i-11/2} - 1170538s_{i-9/2} + 4755889s_{i-7/2}\right.\\
&-11708024s_{i-5/2} + 19465978s_{i-3/2} - 22949948s_{i-1/2}\\
&+ 19465978s_{i+1/2} - 11708024s_{i+3/2}\\
&\left.+\ 4755889s_{i+5/2} - 1170538s_{i+7/2} + 131669s_{i+9/2}\right)\\
&+\frac{1}{35843605423718400}h^{10}\left(-131669s^{(x)}_{i-11/2} - 69971949470s^{(x)}_{i-9/2}\right.\\
&- 308939339868801s^{(x)}_{i-7/2} - 1259425317190504s^{(x)}_{i-5/2}\\
&- 10029740769032890s^{(x)}_{i-3/2} - 21599275265234932s^{(x)}_{i-1/2}\\
&- 10029740769032890s^{(x)}_{i+1/2} - 1259425317190504s^{(x)}_{i+3/2}\\
&\left.-\ 308939339868801s^{(x)}_{i+5/2} - 69971949470s^{(x)}_{i+7/2} - 131669s^{(x)}_{i+9/2}\right),\\
&i = 6,\ 7, \ldots\ldots, n-5\ , \qquad (2.33)
\end{aligned}$$

$$
\begin{aligned}
h^6 s^{(vi)}_{i-1/2} &= \frac{1}{1169216}\big(-130407s_{i-11/2} + 1011766s_{i-9/2} - 2360667s_{i-7/2} \\
&+ 449032s_{i-5/2} + 6521794s_{i-3/2} - 10983036s_{i-1/2} \\
&+ 6521794s_{i+1/2} + 449032s_{i+3/2} \\
&- 2360667s_{i+5/2} + 1011766s_{i+7/2} - 130407s_{i+9/2}\big) \\
&+ \frac{1}{286748843389747200}h^{10}\Big(130407s^{(x)}_{i-11/2} + 69301441058s^{(x)}_{i-9/2} \\
&+ 30676238680539s^{(x)}_{i-7/2} + 1281423420667928s^{(x)}_{i-5/2} \\
&+ 11099628319731134s^{(x)}_{i-3/2} + 22690797221056908s^{(x)}_{i-1/2} \\
&+ 11099628319731134s^{(x)}_{i+1/2} + 1281423420667928s^{(x)}_{i+3/2} \\
&+ 30676238680539s^{(x)}_{i+5/2} + 69301441058s^{(x)}_{i+7/2} + 130407s^{(x)}_{i+9/2}\Big) \\
& i = 6,\ 7, \ldots\ldots, n-5\ , \qquad (2.\ 34)
\end{aligned}
$$

$$
\begin{aligned}
h^4 s^{(iv)}_{i-1/2} &= \frac{1}{140305920}\big(1235997s_{i-11/2} - 8267714s_{i-9/2} - 502503s_{i-7/2} \\
&+ 246875368s_{i-5/2} - 881595446s_{i-3/2} + 1284508596s_{i-1/2} \\
&- 881595446s_{i+1/2} + 246875368s_{i+3/2} - 502503s_{i+5/2} \\
&-8267714s_{i+7/2} + 1235997s_{i+9/2}\big) \\
&+ \frac{1}{344098612067696640000}h^{10}\Big(-1235997s^{(x)}_{i-11/2} - 656840089990s^{(x)}_{i-9/2} \\
&- 291451685669913s^{(x)}_{i-7/2} - 12449551555628872s^{(x)}_{i-5/2} \\
&- 115290331839610570s^{(x)}_{i-3/2} - 233677147590742116s^{(x)}_{i-1/2} \\
&- 115290331839610570s^{(x)}_{i+1/2} - 12449551555628872s^{(x)}_{i+3/2} \\
&- 291451685669913s^{(x)}_{i+5/2} - 656840089990s^{(x)}_{i+7/2} - 1235997s^{(x)}_{i+9/2}\Big)\ , \\
& i = 6,\ 7, \ldots\ldots, n-5 \qquad (2.\ 35)
\end{aligned}
$$

and

$$
\begin{aligned}
h^2 s''_{i-1/2} &= \frac{1}{23571394560}\big(-11044753s_{i-11/2} + 68355754s_{i-9/2} + 101624707s_{i-7/2} \\
&- 3388908552s_{i-5/2} + 35394833166s_{i-3/2} - 64329720644s_{i-1/2} \\
&+ 35394833166s_{i+1/2} - 3388908552s_{i+3/2} + 101624707s_{i+5/2} \\
&+68355754s_{i+7/2} - 11044753s_{i+9/2}\big)
\end{aligned}
$$

$$+ \frac{1}{5780856682737303552000} h^{10} \Big(11044753 s_{i-11/2}^{(x)} + 5869466820542 s_{i-9/2}^{(x)}$$
$$+ 2607320271000701 s_{i-7/2}^{(x)} + 112518558035483112 s_{i-5/2}^{(x)}$$
$$+ 1072085463591560946 s_{i-3/2}^{(x)} + 2169470806548219572 s_{i-1/2}^{(x)}$$
$$+ 1072085463591560946 s_{i+1/2}^{(x)} + 112518558035483112 s_{i+3/2}^{(x)}$$
$$+ 2607320271000701 s_{i+5/2}^{(x)} + 5869466820542 s_{i+7/2}^{(x)} + 11044753 s_{i+9/2}^{(x)} \Big) .$$
$$i = 6,\ 7, \ldots\ldots, n-5 \qquad (2.36)$$

Since the system of equations (2.32) provides $n - 12$ equations in n unknowns ($s_{i-1/2}$. $i = 1, 2, \ \ldots \ , n$), twelve more equations are needed. These are defined in the next subsection in the form of end conditions, see[9]

2.2 End conditions

The following relations define the first six end conditions

$$\big(-924 s_0 + 1716 s_{1/2} - 1287 s_{3/2} + 715 s_{5/2} - 286 s_{7/2} + 78 s_{9/2}$$
$$- 13 s_{11/2} + s_{13/2}\big)$$
$$= -\frac{273}{2} h^2 s_0'' + \frac{637}{32} h^4 s_0^{(iv)} - \frac{4537}{1280} h^6 s_0^{(vi)} + \frac{105131}{122880} h^8 s_0^{(viii)}$$
$$- \frac{14765933}{44236800} h^{10} s_0^{(x)} + \frac{h^{12}}{19619905536000} \Big(266267950740 s_{1/2}^{(xii)}$$
$$+ 355636784217 s_{3/2}^{(xii)} + 110674869019 s_{5/2}^{(xii)} + 10471679218 s_{7/2}^{(xii)}$$
$$+ 236700542 s_{9/2}^{(xii)} + 531427 s_{11/2}^{(xii)} + s_{13/2}^{(xii)} \Big) , \qquad (2.37)$$

$$\big(660 s_0 - 1287 s_{1/2} + 1144 s_{3/2} - 858 s_{5/2} + 507 s_{7/2} - 221 s_{9/2}$$
$$+ 66 s_{11/2} - 12 s_{13/2} + s_{15/2}\big)$$
$$= \frac{171}{2} h^2 s_0'' - \frac{279}{32} h^4 s_0^{(iv)} + \frac{499}{1280} h^6 s_0^{(vi)} + \frac{99587}{286720} h^8 s_0^{(viii)}$$
$$- \frac{4497669}{11468800} h^{10} s_0^{(x)} + \frac{h^{12}}{19619905536000} \Big(355636784217 s_{1/2}^{(xii)}$$
$$+ 732579603976 s_{3/2}^{(xii)} + 476783332454 s_{5/2}^{(xii)} + 121383248779 s_{7/2}^{(xii)}$$
$$+ 10708911187 s_{9/2}^{(xii)} + 237231970 s_{11/2}^{(xii)}$$
$$+ 531428 s_{13/2}^{(xii)} + s_{15/2}^{(xii)} \Big) , \qquad (2.38)$$

$$\begin{aligned}
&\big(-330s_0 + 715s_{1/2} - 858s_{3/2} + 936s_{5/2} - 793s_{7/2} + 495s_{9/2} \\
&- 220s_{11/2} + 66s_{13/2} - 12s_{15/2} + s_{17/2}\big) \\
= &- \frac{123}{4}h^2 s_0'' - \frac{9}{64}h^4 s_0^{(iv)} + \frac{7319}{7680}h^6 s_0^{(vi)} - \frac{503369}{1720320}h^8 s_0^{(viii)} \\
&- \frac{18798307}{206438400}h^{10} s_0^{(x)} + \frac{h^{12}}{1961990553600}\Big(110674869019 s_{1/2}^{(xii)} \\
&+ 476783332454 s_{3/2}^{(xii)} + 743287983736 s_{5/2}^{(xii)} + 477020564423 s_{7/2}^{(xii)} \\
&+ 121383780207 s_{9/2}^{(xii)} + 10708911188 s_{11/2}^{(xii)} \\
&+ 237231970 s_{13/2}^{(xii)} + 531428 s_{15/2}^{(xii)} + s_{17/2}^{(xii)}\Big) \, , \qquad (2.39)
\end{aligned}$$

$$\begin{aligned}
&\big(110s_0 - 286s_{1/2} + 507s_{3/2} - 793s_{5/2} + 924s_{7/2} - 792s_{9/2} \\
&+ 495s_{11/2} - 220s_{13/2} + 66s_{15/2} - 12s_{17/2} + s_{19/2}\big) \\
= &\frac{17}{4}h^2 s_0'' + \frac{281}{192}h^4 s_0^{(iv)} - \frac{6943}{23040}h^6 s_0^{(vi)} - \frac{311959}{5160960}h^8 s_0^{(viii)} \\
&- \frac{9057103}{1857945600}h^{10} s_0^{(x)} + \frac{h^{12}}{1961990553600}\Big(10471679218 s_{1/2}^{(xii)} \\
&+ 121383248779 s_{3/2}^{(xii)} + 477020564423 s_{5/2}^{(xii)} + 743288515164 s_{7/2}^{(xii)} \\
&+ 477020564424 s_{9/2}^{(xii)} + 121383780207 s_{11/2}^{(xii)} \\
&+ 10708911188 s_{13/2}^{(xii)} + 237231970 s_{15/2}^{(xii)} + 531428 s_{17/2}^{(xii)} \\
&+ s_{19/2}^{(xii)}\Big) \, , \qquad (2.40)
\end{aligned}$$

$$\begin{aligned}
&\big(-22s_0 + 78s_{1/2} - 221s_{3/2} + 495s_{5/2} - 792s_{7/2} + 924s_{9/2} \\
&- 792s_{11/2} + 495s_{13/2} - 220s_{15/2} + 66s_{17/2} - 12s_{19/2} + s_{21/2}\big) \\
= &\frac{3}{4}h^2 s_0'' - \frac{23}{64}h^4 s_0^{(iv)} - \frac{239}{7680}h^6 s_0^{(vi)} - \frac{2183}{1720320}h^8 s_0^{(viii)} \\
&- \frac{19679}{619315200}h^{10} s_0^{(x)} + \frac{h^{12}}{1961990553600}\Big(236700542 s_{1/2}^{(xii)} \\
&+ 10708911187 s_{3/2}^{(xii)} + 121383780207 s_{5/2}^{(xii)} + 477020564424 s_{7/2}^{(xii)} \\
&+ 743288515164 s_{9/2}^{(xii)} + 477020564424 s_{11/2}^{(xii)} \\
&+ 121383780207 s_{13/2}^{(xii)} + 10708911188 s_{15/2}^{(xii)} + 237231970 s_{17/2}^{(xii)} \\
&+ 531428 s_{19/2}^{(xii)} + s_{21/2}^{(xii)}\Big) \qquad (2.41)
\end{aligned}$$

and

$$
\begin{aligned}
&\left(2s_0 - 13s_{1/2} + 66s_{3/2} - 220s_{5/2} + 495s_{7/2} - 792s_{9/2}\right.\\
&+ 924s_{11/2} - 792s_{13/2} + 495s_{15/2} - 220s_{17/2} + 66s_{19/2}\\
&\left. - \ 12s_{21/2} + s_{23/2}\right)\\
= \ & - \frac{1}{4}h^2 s_0'' - \frac{1}{192}h^4 s_0^{(iv)} - \frac{1}{23040}h^6 s_0^{(vi)} - \frac{1}{5160960}h^8 s_0^{(viii)}\\
&- \frac{1}{1857945600}h^{10} s_0^{(x)} + \frac{h^{12}}{1961990553600}\left(531427 s_{1/2}^{(xii)}\right.\\
&+ 237231970 s_{3/2}^{(xii)} + 10708911188 s_{5/2}^{(xii)} + 121383780207 s_{7/2}^{(xii)}\\
&+ 477020564424 s_{9/2}^{(xii)} + 743288515164 s_{11/2}^{(xii)}\\
&+ 477020564424 s_{13/2}^{(xii)} + 121383780207 s_{15/2}^{(xii)} + 10708911188 s_{17/2}^{(xii)}\\
&\left.+ \ 237231970 s_{19/2}^{(xii)} + 531428 s_{21/2}^{(xii)} + s_{23/2}^{(xii)}\right) \qquad (2.42)
\end{aligned}
$$

The remaining last six end conditions can be determined similarly.

3. Spline solution

For the spline solution, the following system of equations can be determined as, see [9]

$$
\left.
\begin{array}{ll}
(i) & \mathbf{MY} = \mathbf{C} + \mathbf{T} \ , \\
(ii) & \mathbf{MS} = \mathbf{C} \ , \\
(iii) & \mathbf{ME} = \mathbf{T} \ .
\end{array}
\right\} \qquad (3.1)
$$

where $\mathbf{Y} = \left(y_{i-1/2}\right)$, $\mathbf{T} = (t_i)$, $\mathbf{E} = \left(\hat{e}_{i-1/2}\right)$, $i = 1, 2 \ldots, n$.

$$
\begin{aligned}
\mathrm{M} &= \mathrm{M}_0 + \frac{1}{1961990553600}h^{12}\mathrm{BF} \ , && (3.2)\\
\mathbf{S} &= \left(s_{i-1/2}\right), \quad i = 1, 2, \ldots\ldots, n && (3.3)
\end{aligned}
$$

and

$$
\mathbf{C} = (\hat{c}_i), \quad i = 1, 2, \ldots\ldots, n \ . \qquad (3.4)
$$

Also, M_0 and B are thirteen-band symmetric matrices, with

$$
\mathrm{M}_0 = [\mathrm{M}_1 \ \ \mathrm{M}_2] \qquad (3.5)
$$

where

$$
M_1 = \begin{bmatrix}
1716 & -1287 & 715 & -286 & 78 & -13 & 1 \\
-1287 & 1144 & -858 & 507 & -221 & 66 & -12 \\
715 & -858 & 936 & -793 & 495 & -220 & 66 \\
-286 & 507 & -793 & 924 & -792 & 495 & -220 \\
78 & -221 & 495 & -792 & 924 & -792 & 495 \\
-13 & 66 & -220 & 495 & -792 & 924 & -792 \\
1 & -12 & 66 & -220 & 495 & -792 & 924 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & 1 & -12 & 66 & -220 & 495 & -792 \\
 & & 1 & -12 & 66 & -220 & 495 \\
 & & & 1 & -12 & 66 & -220 \\
 & & & & 1 & -12 & 66 \\
 & & & & & 1 & -12 \\
 & & & & & & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
M_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & & & & & & \\
-12 & 1 & & & & & \\
66 & -12 & 1 & & & & \\
-220 & 66 & -12 & 1 & & & \\
495 & -220 & 66 & -12 & 1 & & \\
-792 & 495 & -220 & 66 & -12 & 1 & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
924 & -792 & 495 & -220 & 66 & -12 & 1 \\
-792 & 924 & -792 & 495 & -220 & 66 & -13 \\
495 & -792 & 924 & -792 & 495 & -221 & 78 \\
-220 & 495 & -792 & 924 & -793 & 507 & -286 \\
66 & -220 & 495 & -793 & 936 & -858 & 715 \\
-12 & 66 & -221 & 507 & -858 & 1144 & -1287 \\
1 & -13 & 78 & -286 & 715 & -1287 & 1716
\end{bmatrix}
$$

while

$$B = [B_1 \;\; B_2 \;\; B_3 \;\; B_4 \;] \;, \tag{3.6}$$

where

$$B_1 = \begin{bmatrix} 266267950740 & 355636784217 & 110674869019 & 10471679218 \\ 355636784217 & 732579603976 & 476783332454 & 121383248779 \\ 110674869019 & 476783332454 & 743287983736 & 477020564423 \\ 10471679218 & 121383248779 & 477020564423 & 743288515164 \\ 236700542 & 10708911187 & 121383780207 & 477020564424 \\ 531427 & 237231970 & 10708911188 & 121383780207 \\ 1 & 531428 & 237231970 & 10708911188 \\ & \ddots & \ddots & \ddots \\ & 1 & 531428 & 237231970 \\ & & 1 & 531428 \\ & & & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
B_2 = \begin{bmatrix}
236700542 & 531427 & 1 & 0 \\
10708911187 & 237231970 & 531428 & 1 \\
121383780207 & 10708911188 & 237231970 & 531428 \\
477020564424 & 121383780207 & 10708911188 & 237231970 \\
743288515164 & 477020564424 & 121383780207 & 10708911188 \\
477020564424 & 743288515164 & 477020564424 & 121383780207 \\
121383780207 & 477020564424 & 743288515164 & 477020564424 \\
\ddots & \ddots & \ddots & \ddots \\
10708911188 & 121383780207 & 477020564424 & 743288515164 \\
237231970 & 10708911188 & 121383780207 & 477020564424 \\
531428 & 237231970 & 10708911188 & 121383780207 \\
1 & 531428 & 237231970 & 10708911188 \\
 & 1 & 531428 & 237231970 \\
 & & 1 & 531428 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
B_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & & \\
531428 & 1 & \\
237231970 & 531428 & 1 \\
10708911188 & 237231970 & 531428 \\
121383780207 & 10708911188 & 237231970 \\
\ddots & \ddots & \ddots \\
477020564424 & 121383780207 & 10708911188 \\
743288515164 & 477020564424 & 121383780207 \\
477020564424 & 743288515164 & 477020564424 \\
121383780207 & 477020564424 & 743288515164 \\
10708911188 & 121383780207 & 477020564423 \\
237231970 & 10708911187 & 121383248779 \\
531427 & 236700542 & 10471679218
\end{bmatrix}
$$

and

$$B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 531428 & 1 & \\ & & \\ \ddots & \ddots & \ddots \\ & & \\ 237231970 & 531428 & 1 \\ 10708911188 & 237231970 & 531427 \\ 121383780207 & 10708911187 & 236700542 \\ 477020564423 & 121383248779 & 10471679218 \\ 743287983736 & 476783332454 & 110674869019 \\ 476783332454 & 732579603976 & 355636784217 \\ 110674869019 & 355636784217 & 266267950740 \end{bmatrix}$$

The matrix F is defined as

$$F = \operatorname{diag}\left(\phi_{i-1/2}\right), \; i = 1, 2, \ldots, n \; , \tag{3.7}$$

and the vector $\mathbf{C} = (\hat{c}_i)$, $i = 1, 2, \ldots$, n can be defined as

$$\begin{aligned} \hat{c}_1 &= 924A_0 - \frac{273}{2}h^2A_2 + \frac{637}{32}h^4A_4 - \frac{4537}{1280}h^6A_6 + \frac{105131}{122880}h^8A_8 \\ &- \frac{14765933}{44236800}h^{10}A_{10} + \frac{h^{12}}{1961990553600}\big(266267950740\phi_{1/2} \\ &+ 355636784217\psi_{3/2} + 110674869019\psi_{5/2} + 10471679218\psi_{7/2} \\ &+ 236700542\psi_{9/2} + 531427\psi_{11/2} + \psi_{13/2}\big) \; , \qquad (3.8) \\ \hat{c}_2 &= -660A_0 + \frac{171}{2}h^2A_2 - \frac{279}{32}h^4A_4 + \frac{499}{1280}h^6A_6 + \frac{99587}{286720}h^8A_8 \\ &- \frac{4497669}{11468800}h^{10}A_{10} + \frac{h^{12}}{1961990553600}\big(355636784217\psi_{1/2} \\ &+ 732579603976\psi_{3/2} + 476783332454\psi_{5/2} + 121383248779\psi_{7/2} \\ &+ 10708911187\psi_{9/2} + 237231970\psi_{11/2} \\ &+ 531428\psi_{13/2} + \psi_{15/2}\big) \; , \qquad (3.9) \end{aligned}$$

$$\begin{aligned}
\hat{c}_3 &= 330A_0 - \frac{123}{4}h^2A_2 - \frac{9}{64}h^4A_4 + \frac{7319}{7680}h^6A_6 - \frac{503369}{1720320}h^8A_8 \\
&- \frac{18798307}{206438400}h^{10}A_{10} + \frac{h^{12}}{1961990553600}\Big(110674869019\psi_{1/2} \\
&+ 476783332454\psi_{3/2} + 743287983736\psi_{5/2} + 477020564423\psi_{7/2} \\
&+ 121383780207\psi_{9/2} + 10708911188\psi_{11/2} \\
&+ 237231970\psi_{13/2} + 531428\psi_{15/2} + \psi_{17/2}\Big), \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
\hat{c}_4 &= -110A_0 + \frac{17}{4}h^2A_2 + \frac{281}{192}h^4A_4 - \frac{6943}{23040}h^6A_6 - \frac{311959}{5160960}h^8A_8 \\
&- \frac{9057103}{1857945600}h^{10}A_{10} + \frac{h^{12}}{1961990553600}\Big(10471679218\psi_{1/2} \\
&+ 121383248779\psi_{3/2} + 477020564423\psi_{5/2} + 743288515164\psi_{7/2} \\
&+ 477020564424\psi_{9/2} + 121383780207\psi_{11/2} \\
&+ 10708911188\psi_{13/2} + 237231970\psi_{15/2} + 531428\psi_{17/2} \\
&+ \psi_{19/2}\Big), \qquad (3.11)
\end{aligned}$$

$$\begin{aligned}
\hat{c}_5 &= 22A_0 + \frac{3}{4}h^2A_2 - \frac{23}{64}h^4A_4 - \frac{239}{7680}h^6A_6 - \frac{2183}{1720320}h^8A_8 \\
&- \frac{19679}{619315200}h^{10}A_{10} + \frac{h^{12}}{1961990553600}\Big(236700542\psi_{1/2} \\
&+ 10708911187\psi_{3/2} + 121383780207\psi_{5/2} + 477020564424\psi_{7/2} \\
&+ 743288515164\psi_{9/2} + 477020564424\psi_{11/2} \\
&+ 121383780207\psi_{13/2} + 10708911188\psi_{15/2} + 237231970\psi_{17/2} \\
&+ 531428\psi_{19/2} + \psi_{21/2}\Big), \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
\hat{c}_6 &= -2A_0 - \frac{1}{4}h^2A_2 - \frac{1}{192}h^4A_4 - \frac{1}{23040}h^6A_6 - \frac{1}{5160960}h^8A_8 \\
&- \frac{1}{1857945600}h^{10}A_{10} + \frac{h^{12}}{1961990553600}\Big(531427\psi_{1/2} \\
&+ 237231970\psi_{3/2} + 10708911188\psi_{5/2} + 121383780207\psi_{7/2} \\
&+ 477020564424\psi_{9/2} + 743288515164\psi_{11/2} \\
&+ 477020564424\psi_{13/2} + 121383780207\psi_{15/2} + 10708911188\psi_{17/2} \\
&+ 237231970\psi_{19/2} + 531428\psi_{21/2} + \psi_{23/2}\Big) \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
\hat{c}_i &= \frac{1}{1961990553600}h^{12}\Big(\psi_{i-13/2} + 531428\psi_{i-11/2} + 237231970\psi_{i-9/2} \\
&+ 10708911188\psi_{i-7/2} + 121383780207\psi_{i-5/2}
\end{aligned}$$

$$+ 477020564424\omega_{i-3/2} + 743288515164\omega_{i-1/2}$$
$$+ 477020564424\omega_{i+1/2} + 121383780207\omega_{i+3/2}$$
$$+10708911188\omega_{i+5/2} + 237231970\omega_{i+7/2} + 531428\omega_{i+9/2}$$
$$+ \omega_{i+11/2}) \ .$$
$$i = 7,\ 8, \ldots \ldots , n-6 \ . \qquad (3.14)$$

$\hat{c}_{n-5}, \hat{c}_{n-3}, \ldots, \hat{c}_n$ are defined similar to $\hat{c}_6, \hat{c}_5, \cdots, \hat{c}_1$ respectively, except that the boundary values $B_0, B_2, \ldots, B_{10}$ will replace $A_0, A_2, \ldots, A_{10}$ respectively, at the other end.

After determining $s_{i-1/2}$, $i = 1, 2, \ldots, n$, s_0 and s_n can be determined using the differential equation (1.1).

Also, $s^{(xii)}_{i-1/2}$, $i = 1, 2, \ldots, n$, $s^{(xii)}_0$ and $s^{(xii)}_n$ can be determined using (1.1). The derivatives $s^{(x)}_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using (2.22) and (2.23). The derivatives $s^{(viii)}_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using (2.33) and (2.25). The derivatives $s^{(vi)}_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using (2.34) and (2.27). The derivatives $s^{(iv)}_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using (2.35) and (2.29) and $s''_{i-1/2}$, $i = 1, 2, \ldots, n$, can be determined using (2.36) and (2.31).

Now it is possible to determine the odd-order derivatives of the spline.

$s'_i, s'''_i, s^{(v)}_i, \ldots, s^{(xi)}_i$, $i = 1, 2, \ldots, n-1$ are determined using (2.21), (2.20), …, (2.16) respectively while s'_0, s'_n, s'''_0, s'''_n, …, $s^{(xi)}_0$, $s^{(xi)}_n$ through the following relations which were obtained during determining (2.16)—(2.21)

$$h(s'_i - s'_{i-1}) = h^2 s''_{i-1/2} + \frac{1}{24}h^4 s^{(iv)}_{i-1/2} + \frac{1}{1920}h^6 s^{(vi)}_{i-1/2} + \frac{1}{322560}h^8 s^{(viii)}_{i-1/2} + \frac{1}{92897280}h^{10} s^{(x)}_{i-1/2} + \frac{1}{40874803200}h^{12} s^{(xii)}_{i-1/2} \ . \qquad (3.15)$$

$$h(s'''_i - s'''_{i-1}) = h^2 s^{(iv)}_{i-1/2} + \frac{1}{24}h^4 s^{(vi)}_{i-1/2} + \frac{1}{1920}h^6 s^{(viii)}_{i-1/2} + \frac{1}{322560}h^8 s^{(x)}_{i-1/2}$$

$$+\frac{1}{92897280}h^{10}s_{i-1/2}^{(xii)}. \qquad (3.16)$$

$$h(s_i^{(v)}-s_{i-1}^{(v)}) = h^2 s_{i-1/2}^{(vi)}+\frac{1}{24}h^4 s_{i-1/2}^{(viii)}+\frac{1}{1920}h^6 s_{i-1/2}^{(x)}$$
$$+\frac{1}{322560}h^8 s_{i-1/2}^{(xii)}. \qquad (3.17)$$

$$h(s_i^{(vii)}-s_{i-1}^{(vii)}) = h^2 s_{i-1/2}^{(viii)}+\frac{1}{24}h^4 s_{i-1/2}^{(x)}+\frac{1}{1920}h^6 s_{i-1/2}^{(xii)}, \qquad (3.18)$$

$$h(s_i^{(ix)}-s_{i-1}^{(ix)}) = h^2 s_{i-1/2}^{(x)}+\frac{1}{24}h^4 s_{i-1/2}^{(xii)} \qquad (3.19)$$

and

$$h(s_i^{(xi)}-s_{i-1}^{(xi)}) = h^2 s_{i-1/2}^{(xii)}. \qquad (3.20)$$

4. Numerical results and conclusions

In this section, two problems are discussed to compare the maximum absolute errors with the analytical solutions, see[9]. Numerical results relating to the solution of twelfth order BVPs are rare in the liturature. The value of n used in Tables 1 and 2 is that which gives the smallest maximum error moduli for problems 4.1 and 4.2 . Some unexpected results were also obtained near the boundary of the given interval. These results were due to (2.23), (2.25), (2.27), (2.29) and (2.31) .

The absolute errors in the function values, were however, very small. The absolute errors in the function values and all derivatives were seen to be small at points remote from the boundaries as observed in [9].

Problem 4.1

Consider

$$\left.\begin{aligned} y^{(xii)} + xy &= -(120 + 23x + x^3)e^x\ , \qquad 0 \le x \le 1\ ,\\ y(0) &= 0 = y(1)\ ,\ y''(0) = 0\ ,\ y''(1) = -4e\ ,\\ y^{(iv)}(0) &= -8\ ,\ y^{(iv)}(1) = -16e\ ,\\ y^{(vi)}(0) &= -24\ ,\ y^{(vi)}(1) = -36e\ ,\\ y^{(viii)}(0) &= -48\ ,\ y^{(viii)}(1) = -64e\ ,\\ y^{(x)}(0) &= -80\ ,\ y^{(x)}(1) = -100e \end{aligned}\right\} \tag{4.1}$$

The analytical solution of the above differential system is

$$y(x) = x(1-x)e^x \tag{4.2}$$

The maximum errors (in absolute value) in $y_i^{(k)}$, $k = 0, 1, 2, \ldots, 11$, are shown in the Table 1 .

Problem 4.2

Consider

$$\left.\begin{aligned} y^{(xii)} + y &= 12[2x\sin(x) - 11\cos(x)]\ , \qquad -1 \le x \le 1\ ,\\ y(-1) &= 0 = y(1)\ ,\\ y''(-1) &= 4\sin(-1) + 2\cos(-1)\ ,\\ y''(1) &= -4\sin(1) + 2\cos(1)\ ,\\ y^{(iv)}(-1) &= -8\sin(-1) - 12\cos(-1)\ ,\\ y^{(iv)}(1) &= 8\sin(1) - 12\cos(1)\ ,\\ y^{(vi)}(-1) &= 12\sin(-1) + 30\cos(-1)\ ,\\ y^{(vi)}(1) &= -12\sin(1) + 30\cos(1)\ ,\\ y^{(viii)}(-1) &= -16\sin(-1) - 56\cos(-1)\ ,\\ y^{(viii)}(1) &= 16\sin(1) - 56\cos(1)\ ,\\ y^{(x)}(-1) &= 20\sin(-1) + 90\cos(-1)\ ,\\ y^{(x)}(1) &= -20\sin(1) + 90\cos(1)\ . \end{aligned}\right\} \tag{4.3}$$

The analytical solution of the above differential system is

$$y(x) = (x^2 - 1)\cos(x)\ . \tag{4.4}$$

The maximum errors (in absolute value) in $y_i^{(k)}$, $k = 0, 1, 2, \ldots, 11$, are shown in the Table 2 .

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Table 1: Maximum absolute errors for Problem 4.1 with $n = 22$.

$y_i^{(k)}$	$x \in [x_6, x_{n-6}]$	$x \notin [x_6, x_{n-6}]$
$k = 0$	0.4269×10^{-1}	0.7442×10^{28}
$k = 1$	0.1390	0.4645×10^{30}
$k = 2$	0.1947×10^{3}	0.2070×10^{32}
$k = 3$	0.1357×10^{4}	0.2412×10^{34}
$k = 4$	0.7795×10^{6}	0.2016×10^{36}
$k = 5$	0.2929×10^{7}	0.8863×10^{37}
$k = 6$	0.4679×10^{10}	0.1975×10^{39}
$k = 7$	0.1542×10^{12}	0.1966×10^{41}
$k = 8$	0.6821×10^{13}	$0..8637 \times 10^{42}$
$k = 9$	0.3001×10^{15}	0.1267×10^{44}
$k = 10$	0.5527×10^{3}	0.8791×10^{35}
$k = 11$	0.6169×10^{3}	0.1289×10^{37}

Table 2: Maximum absolute errors for Problem 4.2 with $n = 22$.

$y_i^{(k)}$	$x \in [x_6,\ x_{n-6}]$	$x \notin [x_6,\ x_{n-6}]$
$k = 0$	0.2822	0.1401×10^{29}
$k = 1$	0.4270	0.1211×10^{31}
$k = 2$	0.3248×10^{3}	0.1580×10^{32}
$k = 3$	0.1131×10^{4}	0.1841×10^{34}
$k = 4$	0.3249×10^{6}	0.1441×10^{35}
$k = 5$	0.6104×10^{6}	0.1874×10^{37}
$k = 6$	0.4876×10^{9}	0.2058×10^{38}
$k = 7$	0.8035×10^{10}	0.1025×10^{40}
$k = 8$	0.1777×10^{12}	0.7500×10^{40}
$k = 9$	0.3910×10^{13}	0.1650×10^{42}
$k = 10$	0.1542	0.1908×10^{33}
$k = 11$	0.1455	0.4199×10^{34}

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A NOTE ON LINEAR ASSOCIATIVE ALGEBRA OF RECTANGULAR MATRICES

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Abstract

Chawla [1] constructed n basic algebras over the vector space $S_{m,n}$ of all $m \times n$ matrices, where $m \geq n$, over a field F and thereby generated linear systems of matrix algebras of $m \times n$ matrices. Here in this note we are concerned with one of Chawla's basic algebra, namely, $A_{m,n} \equiv [S_{m,n}+, *]$ in which the multiplication * in $S_{m,n}$ is defined by

$$X_{mn} * Y_{mn} = X_{mn} I_{nm} Y_{mn}$$

where $X_{mn}, Y_{mn} \in S_{m,n}$ and I_{nm} is the rectangular matrix of n rows and m columns formed by the n unit vectors each of dimension m.

In this note, we prove that the algebra $A_{m,n}$, $m > n$, contains a subalgebra isomorphic to the total matrix algebra of all $n \times n$ matrices of F, which in fact, is itself a particular case of $A_{m,n}$ where $m = n$.

For the sake of definiteness and reference, we reproduce the algebra $A_{m,n} \equiv [S_{m,n}, +, *]$ in the form of

Theorem 1

The system $A_{m,n} \equiv [S_{m,n}, +, *]$ is a linear associative algebra.

Proof

It is evident that $S_{m,n}$ is closed under the multiplication *, since for any $X_{mn}, Y_{mn} \in S_{m,n}$, the matrix

$X_{mn} * Y_{mn} = X_{mn} I_{nm} Y_{nm}$ is an $m \times n$ matrix and belongs to $S_{m,n}$.

Further

$$\begin{aligned}(X_{mn} * Y_{mn}) * Z_{mn} &= (X_{mn} I_{nm} Y_{nm})\ I_{nm} Z_{mn} \\ &= (X_{mn} I_{nm})(Y_{mn} I_{nm} Z_{mn})\end{aligned}$$

by the rule of multiplication of conformal matrices. Thus

$$(X_{mn} * Y_{mn}) * Z_{mn} = X_{mn} * (Y_{mn} * Z_{mn})$$

Hence the multiplication * is associative.

For the left distributive law, consider

$$\begin{aligned}X_{mn} * (Y_{mn} + Z_{mn}) &= X_{mn} I_{nm} (Y_{mn} + Z_{mn}) \\ &= X_{mn} I_{nm} Y_{mn} + X_{mn} I_{nm} Z_{mn}\end{aligned}$$

by the rule of multiplication of conformal matrices.

Hence $X_{mn} * (Y_{mn} + Z_{mn} = X_{mn} * Y_{mn} + X_{mn} * Z_{mn}$, which proves that the left multiplication holds in $A_{m,n}$.

Similarly one may prove that the right distribution holds in $A_{m,n}$. This proves that $A_{m,n}$ is a ring.

For any scalar $\alpha \in F$, we have

$$\alpha \cdot (X_{mn} * Y_{mn}) = \alpha \cdot (X_{mn} I_{nm} Y_{mn})$$

$$\begin{aligned} &= (\alpha \cdot X_{mn})(I_{nm}Y_{mn}) \\ &= (X_{mn}I_{nm})(\alpha \cdot Y_{mn}) \end{aligned}$$

Hence

$$\begin{aligned} \alpha \cdot (X_{mn} * Y_{mn}) &= (\alpha \cdot X_{mn}) * Y_{mn} \\ &= X_{mn} * (\alpha \cdot Y_{mn}) \end{aligned}$$

This completes the proof of Theorem I.

Corollary

The total matrix algebra $A_{m,n}$ of $n \times n$ matrices is a particular case of the algebra $A_{m,n}$.

This follows by taking $m = n$ and then $I_{nm} = I_{nn}$ and hence the corollary.

Definition

Let $\bar{S}_{m,n}$ denote the subset of all elements of $S_{m,n}$, $m > n$, in each of which the last $m - n$ rows ar ezero-rows. Then any element $\bar{X}_{mn} \in \bar{S}_{m,n}$ can be written as

$$\bar{X}_{mn} = \begin{bmatrix} X_{nn} \\ 0 \end{bmatrix}$$

where X_{nn} is the $n \times n$ matrix formed by the first n rows in X_{mn} and 0 is the $(m - n) \times n$ matrix of zero-rows.

Theorem II

The system $\bar{A}_{m,n} \equiv [\bar{S}_{m,n}, +, *]$ is a subalgebra of the algebra $A_{m,n} \equiv [S_{m,n}, +, *]$ and it is isomorphic to the total matrix algebra $A_{n,}$ of all $n \times n$ matrices.

Proof

We first note that if $\bar{X}_{mn}, \bar{Y}_{mn} \in \bar{S}_{m,n}$, then each of the matrices $\bar{X}_{mn} - \bar{Y}_{mn}, \alpha\bar{X}_{mn} + \beta\bar{Y}_{mn}, \alpha, \beta \in F, \bar{X}_{mn} * \bar{Y}_{mn} = \bar{X}_{mn}$, has its last $m - n$ rows as zero-rows and thus each of them $\in \bar{S}_{m,n}$. This proves that $\bar{A}_{m,n}$ is a subalgebra of $A_{m,n}$.

To prove that $\bar{A}_{m,n}$ is isomorphic to the total matrix algebra $A_{n,n}$, consider the mapping $\rho : \bar{A}_{m,n} \to A_{n,n}$ defined by $\rho(\bar{X}_{mn}) = \rho \begin{bmatrix} X_{nn} \\ 0 \end{bmatrix}$

Then ρ is evidently one-one and onto mapping, since for each $\bar{X}_{mn}$, X_{nn} is uniquely determnined and conversely corresponding to each X_{nn} there is a unqiue $\bar{X}_{mn}$. Further

$$\begin{aligned}
\rho(\bar{X}_{mn} + \bar{Y}_{mn}) &= \rho \left[\begin{bmatrix} X_{nn} \\ 0 \end{bmatrix} + \begin{bmatrix} Y_{nn} \\ 0 \end{bmatrix} \right] \\
&= \rho \begin{bmatrix} X_{nn} + Y_{nn} \\ 0 \end{bmatrix} \\
&= X_{nn} + Y_{nn} \\
&= \rho(\bar{X}_{mn}) + \rho(\bar{Y}_{mn}), \quad \text{by definition of } \rho
\end{aligned}$$

Also

$$\begin{aligned}
\rho(\bar{X}_{mn} * \bar{Y}_{mn}) &= \rho \left[\begin{bmatrix} X_{nn} \\ 0 \end{bmatrix} I_{nm} \begin{bmatrix} Y_{nn} \\ 0 \end{bmatrix} \right] \\
&= \rho \left[\begin{bmatrix} X_{nn} \\ 0 \end{bmatrix} Y_{nn} \right] \\
&= \rho \begin{bmatrix} X_{nn} & Y_{nn} \\ & 0 \end{bmatrix} \\
&= X_{nn} Y_{nn} \\
&= \rho(\bar{X}_{mn})\rho(\bar{Y}_{mn}), \quad \text{by definition of } \rho
\end{aligned}$$

This completes the proof of the Theorem.

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ON s-CONTINUOUS, s-OPEN AND s-CLOSED FUNCTIONS

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Abstract

In this paper, we further explore the characterizations and properties of s-continuous and s-open functions. Moreover, we define s-closed functions and study its characterizations and properties.

Introduction

Recently, Cameron and Woods [3], and Abd.El.Monsef et al[2] have independently defined the notion of s-continuous functions (also called strongly semi- continuous [2]) and investigated several characterizations and properties of s-continuous and s-open functions. We define s-closed functions and explore the characterizations and properties of s-continuous, s-open and s-closed functions.

Preliminaries

A subset $\cup$ of X is said to be semi-open [11], if there exists an open set 0 in X such that $0 \subset \cup \subset$ cl 0. The complement of a semi-open set is called semi-closed. Scl ($\cup$) and sInt ($\cup$) denote the semi-closure and semi-interior of $\cup$. SC(X) (respect SO(X)) denotes the class of semi-closed (respect semi-open) subsets of X [4]. sBd A denotes the semi-boundary [1] of a set A and is defined by $sBdA = sclA \cap scl(X - A)$. For the properties of a semi-boundary of A, we refer to [1].

Definition 1

A function $f : X \to Y$ is said to be a s-continuous [3] (also called strongly semi-continuous [2]), if the inverse image of every semi-open set is open.

It is known [3] that an s-continuous function is irresolute, semi-continuous and continuous. For the characterization of s-continuous functions, we refer to [2].

Definition 2

A space X is said to be p-regular [10] (respect. semi-regular [6]) if for each semi-closed set F and $x \in X - F$, there exist disjoint open (respect. semi-open) sets U and V such that $x \in U$ and $F \subset V$.

Clearly every p-regular space is semi-regular as well as regular, but the converse is not true in general. In Example 3.2 [8], X is a regular space but it is not semi-regular and, consequently, it is not p-regular.

Example

Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then X is semi-regular but not p-regular.

Theorem 1

The image of a regular space under a clopen and s-continuous surjection is p-regular space.

Proof

Let $F \in SC(Y)$ and $y \in Y - F$. Let $x \in f^{-1}(y)$. Since f is s-continuous, therefore by theorem 2.2(iii)[2], $f^{-1}(F)$ is closed in X and $x \in X - f^{-1}(F)$. Since X is regular, therefore there exist open sets U and V in X such that $x \in U$, $f^{-1}(F) \subset V$ and $U \cap V = \phi$. Since f is closed, therefore by theorem 11.2 (i) [9], there exists an open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset V$. Therefore $U \cap f^{-1}(W) = \phi$ and hence $f(U) \cap W = \phi$. Since f is open, therefore $f(U)$ is open and $y \in f(U)$. This proves that Y is p-regular. This completes the proof.

Theorem 2

Let $f : X \to Y$ be s-continuous and semi-closed surjection with compact point inverses and X a regular space. Then Y is semi- regular.

Proof

Let $C \in SC(X)$ and $y \in Y - C$. Since f is s-continuous, therefore by theorem 2.2(iii)[2], $f^{-1}(C)$ is closed in X. Moreover, the compact sets $f^{-1}(Y)$ and $f^{-1}(C)$ are disjoint in a regular space. Thus there exist disjoint open sets F and G in X such that $f^{-1}(Y) \subset F$ and $f^{-1}(C) \subset G$. Since f is semi-closed, then by theorem 5[14], there exist semi-open sets V and W containing Y and C respectively such that $f^{-1}(V) \subset F$ and $f^{-1}(W) \subset G$. Since $F \cap G = \phi$, it follows that $f^{-1}(V \cap W) = \phi$ implies $V \cap W = \phi$. This proves that Y is semi-regular. This completes the proof.

Corollary

Let $f : X \to Y$ be s-continuous and closed surjection with compact point inverses. Then Y is p-regular if X is regular.

Definition 3

A function $f : X \to Y$ is said to be s-open [3] if the image of every semi-open set is open.

It is known [3] that every s-open function is open, semi-open and presemi-open. Now we give the characterizations of s-open functions as:

Theorem 3

For a function $f : X \to Y$, the following are equivalent:

(1) f is s-open

(2) f(slntA) $\subset$ Intf(A), for each $A \subset X$.

(3) $\text{slnt} f^{-1}(B) \subset f^{-1}$ (IntB), for each $B \subset Y$.

(4) $f^{-1}(clB) \subset \text{self}^{-1}(B)$, for each $B \subset Y$.

(5) $f^{-1}(BdB) \subset sBd(f^{-1}(B))$, for each $B \subset Y$.

Proof

(1) $\to$ (2). Obviously $f(slntA) \subset f(A)$. f is s-open gives f(slntA) is open in Y. But Intf(A) is the largest open set contained in f(A). Therefore f(slntA) $\subset$ Intf(A). This gives (2).

(2) $\rightarrow$ (3). For any $B \subset Y$, put $f^{-1}(B) = A \subset X$. Then by (2), f(slnt $f^{-1}(B)) \subset$ Intf($f^{-1}(B)) \subset$ IntB or f(slnt $f^{-1}(B)) \subset$ IntB or slnt $f^{-1}(B) \subset ff^{-1}$ (slnt (B)) $\subset f^{-1}$ (IntB) or slnt $f^{-1}(B) \subset f^{-1}$ (IntB). This gives (3).

(3) $\rightarrow$ (4). By (3), slnt $f^{-1}(B) \subset f^{-1}$ (IntB) implies $X - f^{-1}$ (IntB) $\subset X$ - slnt $f^{-1}(B)$ =scl $(X - f^{-1}(B)$ or $f^{-1}(Y) - f^{-1}$ (IntB) $\subset$ scl$(f^{-1}(Y) - f^{-1}(B))$ or $f^{-1}(Y - IntB) \subset$ scl $f^{-1}(Y - B)$ or $f^{-1}cl(Y - B) \subset$ scl $f^{-1}(clC) \subset$ self$^{-1}(C)$, where $C = Y - B$. This gives (4).

(4) $\rightarrow$ (5). For $B \subset Y, Bd\ B = clB \cap cl(Y - B)$ is a closed set in Y. Now $f^{-1}(BdB) = f^{-1}(clB) \cap f^{-1}cl(Y - B) = f^{-1}(clB) \cap f^{-1}cl(Y - B)$ or

$f^{-1}(BdB) = f^{-1}(clB) \cap f^{-1}cl(Y - B)$ gives $f^{-1}(BdB) \subset$ self$^{-1}(B) \cap scl(B) \cap scl(X - f^{-1}(B)) = sBdf^{-1}(B)$ or $f^{-1}(BdB) \subset sBdf^{-1}(B)$. This gives (5).

(5) $\rightarrow$ (1). Let U be an arbitrary semi-open set in X. Put $Y - f(U) = B$. Now we show that B is closed in Y. By (5), $U \cap f^{-1}(BdB) \subset U \cap sBdf^{-1}(B)$ or $f(U \cap f^{-1}(BdB)) \subset f(U \cap sBdf^{-1}(B))$. since $f(U) \cap BdB = f(U \cap f^{-1}BdB) \subset U \cap sBdf^{-1}(B)$ or $f(U \cap f^{-1}(BdB)) \subset f(U \cap sBdf^{-1}(B))$. Since $f(U) \cap BdB = f(U \cap f^{-1}BdB)$, therefore we have

$$f(U) \cap BdB \subset f(U \cap sBdf^{-1}(B)) \tag{1}$$

$B = Y - f(U)$ gives $f^{-1}(B) = f^{-1}(Y - f(U)) = f^{-1}(Y) - f^{-1}(U) \subseteq X - U$ or $f^{-1}(B) \subseteq X - U$ gives sclf$^{-1}(B) \subseteq scl(X - U) = X - slntU = X - U$ implies

$$sclf^{-1}(B) \cap U = \phi \tag{2}$$

Now

$$\begin{aligned} U \cap sBdf^{-1}(B) &= U \cap (sclf^{-1}(B) \cap scl(X - f^{-1}(B))) \\ &= U \cap sclf^{-1}(B)) \cap scl(X - f^{-1}(B)) = \phi(by(2)) \end{aligned}$$

Using $U \cap sBdf^{-1}(B) = \phi$, (1) becomes $f(U) \cap BdB = \phi$ or $BdB \subseteq Y - f(U) = B$ or $BdB \subseteq B$ gives B is closed. Hence $f(U)$ is open. This proves that f is s-open. This completes the proof.

The following theorem establishes the presemi-openness and s-openness of the composition of functions, the proof of which follows easily:

Theorem 4

For any functions: $f : X \to Y$ and $g : Y \to Z$, we have

(1) gof is s-open if f is s-open and g is open.

(2) gof is s-open if f is presemi, open and g is s-open.

(3) gof is open if f is semi-open and g is s-open.

(4) gof is presemi-open if f is s-open and g is semi-open.

Definition 4

A function $f : X \to Y$ is said to be s-closed if the image of every semi-closed set is closed.

Now, we characterize s-closed functions as:

Theorem 5

A function $f : X \to Y$ is s-closed iff clf (A) $\subset f(sclA)$, for each $A \subset X$.

Proof

Necessity. Obviously $f(A) \subset f(sclA)$. $f(sclA)$ is closed, since f is s-closed. But clf(A) is the smallest closed set containing $f(A)$. Therefore $clf(A) \subset f(scl(A))$.

Sufficiency. Let $A \in SC(X)$. We show that $f(A)$ is closed. By hypothesis, $clf(A) \subset f(sclA) = f(A)$ or $clf(A) \subset f(A)$. This prooves that $f(A)$ is closed. This completes the proof.

Theorem 6

A surjective function $f : X \to Y$ is s-closed if and only if for each subset B in Y and each semi-open set U in X containing $f^{-1}(B)$, there exists an open set V in Y containing B such that $f^{-1}(V) \subset U$.

Proof

Necessity. Let U be an arbitrary semi-open set in X containing $f^{-1}(B)$, where $B \subset Y$. Clearly $Y - f(X - U) = V$ (say) is open in Y. Since $f^{-1}(B) \subset U$ and

f is onto, then simple calculations give $B \subset V$. Moreover, we have $f^{-1}(V) \subset X - f^{-1}(f(X - U)) \subset U$ or $f^{-1}(V) \subset U$. This proves necessity.

Sufficiency. Let F be an arbitrary semi-closed set in X and $y \in Y - f(F)$. Then $f^{-1}(y) \subset f^{-1}(Y - f(F)) \subset X - f^{-1}f(F) \subset X - F$ or $f^{-1}(y) \subset X - F$. Since $X - F$ is semi-open, therefore, there exists an open set V_y containing y such that $f^{-1}(V_y) \subset X - F$. This implies $y \in V_y \subset Y - f(F)$. Thus $Y - f(F) = \cup\{V_y\ y \in Y - f(F)\}$ is open in Y or $f(F)$ is closed in Y. This proves that f is s-closed. This completes the proof.

Remark 1

If $f : X \to Y$ is s-continuous and closed (or irresolute [5] and s-closed) surjection, then using theorem 2.2(iii) [2], one can easily see that the classes $SC(X)$ and $C(X)$ (closed sets of X) coincide.

Remark 2

In general, an s-open function need not be s-closed as is evident from the following:

Example

Let $X = \{a, b, c\}, \tau_x = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $Y = \{a, b, c, d\}$, $\tau_y = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\} Y\}$. Let $i : X \to Y$ be an identity function. Then i is s-open but not s-closed.

Remark 3

However, for bijections, it is easily seen that the notations of s-open and s-closed coincide. Moreover, f is s-open iff f^{-1} is s-continuous. The proof follows from theorem 3(4) and theorem 2.2(iv) [2].

Definition 5

A space X is said to be s-closed [12] (respect semi-compact [7], almost compact [13]), if for every semi-open (respect semi - open, open) cover of X, there exists a finite subfamily such that union of their semi-closures (respect their union, union of their closures) cover X.

Note that every compact space is almost compact as well as semi-compact. Moreover every semi-compact space is s-closed. Finally, we state the following from theorem which follows from theorem 5 and remark 3.

Theorem 7

The inverse image of an almost compact space under s-open bijection is s-closed.

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FUZZY H-RELATIONS AND FUZZY H-ALGEBRAS OF BCI-ALGEBRAS

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Abstract

In this paper, we define and study fuzzy H-relations and fuzzy H-algebras of BCI-algebras.

Keywords

Fuzzy relation, fuzzy H-relation, strongest fuzzy H-relation, fuzzy H-algebra, left or right fuzzy H-relation, perfect fuzzy H-algebra, weakest fuzzy H-subset, normalized fuzzy H-algebra and fuzzy K-subset.

Introduction

The notion of fuzzy set was formulated by Zadeh [11] in 1965. Since then there have been wide-ranging applications of the theory of fuzzy sets. There have been also attempts to fuzzify the various mathematical structures like topological spaces [4] and groups [10] and also concepts like measure, probability and automata. The concept of a fuzzy relation on a set was defined by Zadeh. In 1991, X. Ougen [9] applied the concept of fuzzy set to BCK-algebras and discussed their properties. In 1993, B. Ahmad [1] and Y. B. Jun [6] applied it to BCI-algebras and studied several properties. Following [3], Jun and Meng [7] defined and studied fuzzy relations in a BCI-algebra. In [8] B. Ahmad and H. M. Khalid studied fuzzy relations and fuzzy subalgebras of BCI-algebras. In this paper, we define and study fuzzy H-relations and fuzzy H-algebras of BCI-algebras.

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(1) $(x^*y)^*(x^*z) \leq z^*y$

(2) $x^*(x^*y) \leq y$

(3) $x \leq x$

(4) $x \leq y$ and $y \leq x$ imply $x = y$

(5) $x \leq 0$ implies $x = 0$

where $x \leq y$ means $x^*y = 0$

(6) $x^*0 = x$[5]

First we review some definitions and results. A fuzzy set ([1], [6]) in a BCI-algebra X is a mapping μ from X into [0, 1]. For any fuzzy set μ in X and $t \in [0,1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of μ.

Let X be a BCI-algebra. A mapping $\mu : X \to [0,1]$ is called a fuzzy subalgebra [1] of X, if for all $x, y \in X$,

$$\mu(x^*y) \geq \mu(x) \wedge \mu(y) \tag{1}$$

where $\wedge$ denotes the minimum. It is easily seen that $\mu(0) \geq \mu(x)$, for all $x \in X$.

Next, we define fuzzy H-algebra as:

Definition 1

Let X be a BCI-algebra. A mapping $\mu : X \to [0,1]$ is called a fuzzy H-algebra of X, if for all $x, y \in X$,

$$\mu(x^*y) \geq \mu(x)\mu(y)$$

From (1) $\mu(x^*y) \geq \mu(x) \wedge \mu(y) \geq \mu(x)\mu(y)$ or $\mu(x^*y) \geq \mu(x)\mu(y)$ implies that every fuzzy subalgebra of X is a fuzzy H-algebra of X. For fuzzy subalgebras, we refer to [1]. Here we shall study some properties of a fuzzy H-algebra of X.

The following shows that fuzzy H-algebras do exist.

Example

Let $X = \{0, a, b, c\}$ in which * is defined as:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X, *, 0)$ is a BCI-algebra.

Define $\mu : X \to [0,1]$ as : $\mu(0) = 3/4, \mu(a) = 1/2$ $\mu(b) = \mu(c) = 1/4$. Then routine calculations show that μ is a fuzzy H-algebra of X.

Following [3], Jun and Meng [7] defined:

Definition 2[7]

A fuzzy relation on X is a fuzzy subset $\mu : X \times X \to [0,1]$.

Now we define:

Definition 3

Let μ be a fuzzy relation on X and σ a fuzzy subset of X. Then μ is called a fuzzy H-relation on σ, if for any $x, y \in X$,

$$\mu(x, y) \geq \sigma(x)\sigma(y)$$

Definition 4

Let μ be a fuzzy relation on X and σ a fuzzy subset of X. Then a fuzzy H-relation on σ, denoted μ_σ, given by

$$\mu_\sigma(x, y) = \sigma(x)\sigma(y)$$

is called the strongest fuzzy H-relation.

Definition 5

Let μ be a fuzzy relation on X. Then μ is called a fuzzy H-algebra of $X \times X$, if for all (x_1, y_1), $(x_2, y_2) \in X \times X$, we have

$$\mu((x_1, y_1)^*(x_2, y_2)) \geq \mu(x_1, y_1)\mu(x_2, y_2)$$

Definition 6

Let μ be a fuzzy relation on σ and σ a fuzzy subset of X. Then μ_σ is called a fuzzy H-algebra of $X \times X$, if for all (x_1, y_1), $(x_2, y_2) \in X \times X$,

$$\mu_\sigma((x_1, y_1)^*(x_2, y_2)) \geq \mu_\sigma(x_1, y_1)\mu_\sigma(x_2, y_2)$$

Theorem 1

Let σ be a fuzzy subset of a BCI-algebra X and μ_σ the strongest fuzzy H-relation on σ. Then μ_σ is a fuzzy H-algebra of $X \times X$, if σ is a fuzzy H-algebra of X.

Proof

Suppose σ is a fuzzy H-algebra of X. We prove that

$$\mu_\sigma((x_1, y_1)^*(x_2, y_2)) \geq \mu_\sigma(x_1, y_1)\mu_\sigma(x_2, y_1)$$

or

$$\mu_\sigma(x_1^* x_2, y_1^* y_2) \geq \mu_\sigma(x_1, y_1)\mu_\sigma(x_2, y_2)$$

Since σ is a fuzzy H-algebra of X, therefore for $x_1, x_2, y_1, y_2 \in X$, we have

$$\sigma(x_1 {}^* x_2) \geq \sigma(x_1)\sigma(x_2) \text{ and } \sigma(y_1^* y_2) \geq \sigma(y_1)\sigma(y_2)$$

which give

$$\sigma(x_1^* x_2)\sigma(y_1^* y_2) \geq (\sigma(x_1)\sigma(y_1))(\sigma(x_2)\sigma(y_2))$$

Hence

$$\mu_\sigma(x_1^* x_2,\ y_1^* y_2) \geq \mu_\sigma(x_1, y_1)\mu_\sigma(x_2, y_2)$$

This completes the proof.

Definition 7

If σ is a fuzzy subset of X and μ is a fuzzy H-relation on σ, then μ is called left (respt. right) fuzzy H-relation, if for all $x, y \in X$,

$$\mu(x,y) = \sigma(x)(\text{respt } \sigma(y))$$

The following is the partial converse of theorem 1:

Theorem 2

Let σ be a fuzzy subset of a BCI-algebra X and μ a fuzzy H-relation on σ. Then σ is a fuzzy H-algebra of X, if the left or right fuzzy H-relation μ is fuzzy H-algebra of $X \times X$.

Proof

We prove that for $x_1, x_2 \in X$,

$$\sigma(x_1^* x_2) \geq \sigma(x_1)\sigma(x_2)$$

Since μ is a fuzzy H-algebra of $X \times X$, we have

$$\mu(x_1^* x_2, y_1^* y_2) = \mu((x_1, y_1)^*(x_2, y_2))$$

$$\geq \mu(x_1, y_1)\mu(x_2, y_2)$$

or

$$\mu(x_1^* x_2, y_1^* y_2) \geq \mu(x_1, y_1)\mu(x_2, y_2)$$

Since μ is left fuzzy H-relation, we have

$$\sigma(x_1^* x_2) \geq \sigma(x_1)\sigma(x_2)$$

This proves that σ is a fuzzy H-algebra of X. This completes the proof.

Definition 8

Let σ be a fuzzy subset of a BCI-algebra X and μ a fuzzy H-algebra of $X \times X$ Then μ is called perfect fuzzy H-algebra, if for all $x, y \in X$,

$$\mu(x, y) \leq \mu(0, y)$$

Given a fuzzy relation μ on a BCI-algebra X and an arbitrary element $x \in X$, we can define a fuzzy K-subset [2] σ_x on X by

$$\sigma_x(y) = \mu(x, y) \tag{1}$$

for all $y \in X$.

Next we prove that the fuzzy K-subset σ_x is a fuzzy H-algebra of X if μ is a fuzzy H-algebra of $X \times X$:

Theorem 3

Let X be a BCI-algebra and μ a perfect fuzzy H-algebra of $X \times X$. Then for each $x \in X$, the fuzzy K-subset σ_x defined in (1) is a fuzzy H-algebra of X.

Proof

For $x, y_1, y_2 \in X$, we have

$$\begin{aligned} \sigma_x(y_1^* y_2) &= \mu(x, y_1 {}^* y_2) \\ &= \mu(x {}^* 0, y_1 {}^* y_2) \\ &= \mu((x, y_1) {}^*(0, y_2)) \\ &\geq \mu(x, y_1)\mu(0, y_2) \geq \mu(x, y_1)\mu(x, y_2) \\ &= x(y_1)x(y_2) \end{aligned}$$

Thus $\sigma_x(y_1 {}^* y_2) \geq x(y_1)x(y_2)$.

This proves that σ_x is a fuzzy H-algebra of X. This completes the proof.

Definition 9

Let σ be a fuzzy subset of a BCI-algebra X and μ a fuzzy relation on σ. Then the fuzzy subset σ_μ of X, defined by

$$\sigma_\mu(x) = \inf_{y \in X} \{\mu(x, y)\mu(y, x)\}$$

is called the weakest fuzzy H-subset of X.

Definition 10

A fuzzy H-algebra μ of $X \times X$ is called normalized, if $\mu(x,0) = \mu(0,x) = 1$, for all $x \in X$.

Finally, we prove the following:

Theorem 4

Let X be a BCI-algebra and μ a normalized fuzzy H-algebra of $X \times X$. Then the weakest fuzzy H-subset σ_μ of X is a fuzzy H-algebra of X.

Proof

Let $x, y \in X$. We prove that

$$\sigma_\mu(x * y) \geq \sigma_\mu(x)\sigma_\mu(y)$$

or

$$\inf_{u \in X}\{\mu(x * y, u)\mu(u, x * y)\} \geq \inf_{u \in X}\{\mu(x,u)\mu(u,x)\}\inf_{u \in X}\{\mu(y,u)\mu(u,y)\}$$

Since μ is a fuzzy H-algebra of X, we have

$$\begin{aligned} \mu(x * y, u) &= \mu(x * y, u * 0) \\ &= \mu((x,u) * (y,0)) \\ &\geq \mu(x,u)\mu(y,0) \\ &= \mu(x,u) \qquad \text{(since } \mu \text{ is normalized)} \end{aligned}$$

or $\mu(x * y, u) \geq \mu(x,u)$. Similarly, $\mu(u, x * y) \geq \mu(u,x)$. Consequently, $\mu(x * y, u)\mu(u, x * y) \geq \mu(x,u)\mu(u,x)$

or

$$\begin{aligned} \mu(x * y, u)\mu(u, x * y) &\geq \mu(x,u)\mu(u,x) \\ &\geq (\mu(x,u)\mu(u,x))(\mu(u,y)\mu(y,u)) \end{aligned}$$

$$\inf_{u \in X}\{\mu(x * y, u)\mu(u, x * y)] \geq \inf_{u \in X}\{\mu(x,u)\mu(u,x)\}\inf_{u \in X}\{\mu(u,y)\mu(y,u)\}.$$

or $\sigma_\mu(x * y) \geq \sigma_\mu(x)\sigma_\mu(y)$. This proves the theorem.

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PERFORMANCE OF USUAL VARIANCE ESTIMATOR IN THE PRESENCE OF NON-SAMPLING ERRORS

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Abstract

The bias of the usual estimator of the variance of the sample mean for two stage cluster sampling based on the model for interviewers and coders affects has been worked out. It is argued that for small scale studies, the bias may not be ignorable.

Key Words: True value; Survey conditions; Response error. Expected response;

1 Introduction

A commonly used method for obtaining data, to be used for statistical purposes, is a sample survey. It is, however, true that all surveys are subjected to some kind of error and some are quite misleading. The errors present in surveys can be divided into two main groups: sampling errors and non-sampling errors. Sampling errors decrease as the sample size increase. On the other hand, non-sampling errors are likely to increase with an increase in the sample size. In some surveys non-sampling errors are very large and interpretation of the results with out taking these errors into account may be dangerously misleading. A list of errors in surveys is given by Deming (1944).

Although statisticians had realized, very early in the development of the sam-

Hansen and Waksberg (1970) have pointed out this problem. Kish and Lansing (1954) measured the response errors in estimating the value of homes. They found that the mean square difference between the measurements tend to increase with the value of home. Hansen, Hurwitz and Bershad (1961) have calculated the response variances as well as the sampling variances of a number of variables measured in 1950 census in U.S.A.

The table presented by them shows that the ratio of the response variance to the sampling variance varies considerably for different variables. The minimum values, 0.1 and 0.2 are for groups of the age variable where as the highest values goes up to 4.2 which is for people earning less than $2500 from the sources other than the wage, salary and business. Fellegi (1964) used interpenetrating samples, proposed by Mahalonobis (1946), and re-enumeration technique jointly to measure the response error.

Two of the sources of non-sampling errors are interviewers and coders. Martin Collins and Graham Kalton (1980) pointed out that the coder's reliability is affected by their work load. Martin Collins and Gill Courtenay (1985) have suggested that field coding has some advantage over office coding. Crittenden and Hill (1971) studies the effect of open question on coding. Martin Collins (1974) found that coder's reliability is affected by the type of question. Mckenzie (1977) reports a study where the interviewer's load in each primary sampling unit is equally and randomly divided among four interviewers. Collins (1979) used nested designs to measure the interviewer's effect. Kish (1962) studied the influence of interviewers effects on the precision of survey estimates. Freeman and Butler (1976) suggested that men and older interviewers have greater variability than women and younger interviewers.

2. Model for Coder and Interviewer Effect

In practice, apart from the errors due to respondent, there are other factors as well which contribute to distort the inferences drawn from the results of a survey. Two common sources are interviewer and coder. In such a case, the value used finally in obtaining the estimates may be written as

$$Y_{hijqrs} = (X_{iq} + \alpha_{jq} + \epsilon_{ijq}) + \beta_{hq} + \eta_{hijqrs} \qquad (2.1)$$

$$(i = 1, 2, \cdots n;) \quad j = 1, 2, \cdots Q; \quad r = 1, 2, \cdots R; \quad s = 1, 2, \cdots S)$$

where Y_{hijqrs} denotes the value coded by the h-th coder on the r-th occasions for the q-th question in the i-th form, in a random sample of n units from N units, obtained by the j-th interviewer in s-th trial; where X_{iq} is the true value for the i-th unit for the q-th question; α_{jq} is the j-th interviewer's effect for the q-th question; β_{hq} is the h-th coder's effect for the q-th question; and ϵ_{ijq} and η_{hijqrs} are purely random components. It is assumed that coders and interviewers are randomly selected from infinite population of interviewers and coders respectively. Further it is assumed that interviewers and coders effects are independent of each other.

In practice, at analysis stage, each question is treated separately and the sources available do not allow to carry out interview more than once; even if the sources are available, the respondents may remember the answers they gave in the original survey and thus the responses may not be independent. Thus the model which is simple and practical, obtained, from 2.1, by dropping the subscript q and putting $s = 1$, may be given as

$$Y_{hijr} = (X_i + \alpha_j + \epsilon_{ij}) + \beta_h + \eta_{hijr} \tag{2.2}$$

3. Total Variance of the Estimator

The total variance of the estimator depends on sample design and interviewers and coders allocation. The sampling scheme considered here is two-stage sampling. It is assumed that one interviewer interviews the n respondents selected from the p-th cluster and these responses are coded by only one coder. Practically it is reasonable to assume that the interviewer work load is only one cluster i.e. each interviewer interviews the respondents in one cluster only but the assumption that each coder codes the responses obtained from one cluster only is made just for the sake of simplicity. Actually in practice each coder codes the responses obtained from more than one cluster but it will complicate the total variance. For two stage sampling the model in 2.2 becomes as

$$Y_{phijr} = (X_{pi} + \alpha_{pi} + \epsilon_{pij}) + \beta_{ph} + \eta_{phijr} \tag{3.1}$$

Where Y_{phijr} denotes the value coded by the h-th coder on the r-th occassions for the i-th response obtained by the j-th interviewer in a random sample of n units from N units in the p-th cluster; β_{ph} is the h-th coder's effect in the p-th cluster; and ϵ_{pij} and η_{phijr} are purely random components. However for the sake of simplicity we assume that $\alpha_{pj} = \alpha_j$ and $\beta_{ph} = \beta_h$ i.e. no interaction between interviewers and clusters and coders and clusters. It is also assumed that $\epsilon_{pij} = \epsilon_{ij}$ and $\eta_{phijr} = \eta_{hijr}$ but the subscript p has been retained because of the identification of i-th unit in the p-th cluster. Further for the time being, we shall assume that $r = i$. Thus the model in 3.1 can be written as

$$Y_{phij} = (X_{pi} + \alpha_j + \epsilon_{pij}) + \beta_h + \eta_{phij} \tag{3.2}$$

Now suppose that from a population of L clusters, l clusters are drawn. For the sake of simplicity, we shall assume simple random sampling at both stages. Further it is assumed that $N_p = N$ and $n_p = n$ i.e. the clusters are of equal size and equal number of units are drawn from each selected cluster. The sample mean for the p-th cluster and the overall sample mean can be given as,

$$\hat{\bar{Y}}_{p\cdots} = \frac{1}{n}\sum_{i=1}^{n} X_{pi} + \alpha_j + \frac{1}{n}\sum_{i=1}^{n} \epsilon_{pij} + \beta_h + \frac{1}{n}\sum_{i=1}^{n} \eta_{phij} \tag{3.3}$$

$$\hat{\bar{Y}}_{\ldots} = \frac{1}{ln}\sum_{p=1}^{l}\sum_{1=1}^{n} X_{pi} + \frac{1}{l}\sum_{j=1}^{l} \alpha_j + \frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n} \epsilon_{pij}$$

$$+\frac{1}{l}\sum_{h=1}^{l} \beta_h + \frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n} \eta_{phij} \tag{3.4}$$

Let E_L be the expectation over first stage of selection; E_p denotes the overall expectation over second stage of selection i.e. for the p-th cluster selected at the first stage, it includes the expectation over repetition of interviews and coding; over selection of interviewers and coders; and over all possible samples which can be drawn from the N units in the given p-th cluster; and E_S denotes the overall expectation. Similarly, $V_L V_P$ and V_S denote the variance over the first stage of selection, variane of second stage of selection and overall variance respectively. Thus using 3.4, we have

$$\text{Var}(\hat{\bar{Y}}_{\ldots}) = V_S\left(\frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n} X_{pi}\right) + V_S\left(\frac{1}{l}\sum_{j=1}^{l} \alpha_j + \frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n} \epsilon_{pij} + \right.$$

$$\left.\frac{1}{l}\sum_{h=1}^{l} \beta_h + \frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n} \eta_{phij}\right)$$

$$\text{Var}(\hat{\bar{Y}}_{\ldots}) = V_S\left(\frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n} X_{pi}\right) + E_L V_P\left(\frac{1}{l}\sum_{j=1}^{l} \alpha_j + \frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n} \epsilon_{pij} + \right.$$

$$\frac{1}{l}\sum_{h=1}^{l}\beta_h+\frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n}\eta_{phij}\Bigg)+E_PV_L\left(\frac{1}{l}\sum_{j=1}^{l}\alpha_j+\frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n}\epsilon_{pij}+\right.$$

$$\left.\frac{1}{l}\sum_{h=1}^{l}\beta_h+\frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n}\eta_{phij}\right)$$

$$\text{Var}(\hat{\bar{Y}}_{....})=V_S(\bar{X})+\frac{\sigma_\alpha^2}{l}+\frac{\sigma_\epsilon^2}{ln}+\frac{\sigma_\beta^2}{l}+\frac{\sigma_\eta^2}{ln}$$

where $\bar{X}=\frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n}X_{pi}$ is the overall sample mean and

$V_S(\bar{X})=\frac{1}{l\cdot L}\sum_{p=1}^{l}\left(\frac{N-n}{N-n}\right)S_{px}^2+\left(\frac{L-l}{L\cdot l}\right)S_{bx}^2$ is the variance of the sample mean under two stage cluster sampling. Thus

$$\text{Var}\ (\hat{\bar{Y}}_{....})=\frac{1}{l\cdot L}\sum_{p=1}^{l}\left(\frac{N-n}{N\cdot n}\right)S_{px}^2+\left(\frac{L-l}{L\cdot l}\right)S_{bx}^2+\frac{\sigma_\alpha^2}{l}+\frac{\sigma_\epsilon^2}{ln}+\frac{\sigma_\beta^2}{l}+\frac{\sigma_\eta^2}{ln}\qquad(3.5)$$

This is the total variance of the sample mean in two stage cluster sampling when (1) clusters are of equal size, (2) equal number of units are selected from the p-th selected cluster, (3) at both stages simple random sampling is used, (4) one interviewer and one coder is assigned to each cluster and (5) it is assumed that the model in 3.2 holds.

4. The Variance of the Sample Mean due to Sampling Errors Only And Its Unbiased Estimator

The variance of the sample mean, for the sampling scheme considered in section 3, in the absence of non-sampling errors is given as

$$\text{Var}(\hat{\bar{Y}})=\frac{1}{l\cdot L}\left(\frac{N-n}{N\cdot n}\right)\sum_{p=1}^{l}S_{pY}^2+\left(\frac{L-l}{L\cdot l}\right)S_{bY}^2\qquad(4.1)$$

where $S_{pY}^2=\frac{1}{N-1}\sum_{i=1}^{N}(Y_{pi}+\bar{Y}_p)^2$; $S_b^2=\frac{1}{L-1}\sum_{p=1}^{L}(\bar{Y}_p-\bar{Y})^2$; $\bar{Y}_p=\frac{1}{N}\sum_{p=1}^{L}Y_{pi}$ is the mean of the p-th cluster and $\bar{Y}=\frac{1}{NL}\sum_{i=1}^{N}\sum_{p=1}^{L}Y_{pi}$ is the population mean.

An unbiased estimator of the $\text{Var}(\bar{Y})$ given in 4.1, is

$$\hat{Var}(\hat{\bar{Y}}) = \frac{1}{l \cdot L}\left(\frac{N-n}{N \cdot n}\right)\sum_{p=1}^{l} s_p^2 + \left(\frac{L-l}{L \cdot l}\right) s_b^2 \tag{4.2}$$

where $s_p^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_{pi} - \bar{Y}_p)^2$ is the variance of the sample drawn from the p-th cluster and it is an unbiased estimator of the variance of the variance p-th cluster;

$s_b^2 = \frac{1}{l-1}\sum_{p=1}^{l}(\hat{\bar{Y}}_p - \hat{\bar{Y}})^2$ is the variance between cluster sample means and it is an unbiased estimator of the variance between cluster means;

$\hat{\bar{Y}}_p = \frac{1}{n}\sum_{i=1}^{n} Y_{pi}$ is the mean of the sample drawn from the p-th cluster and $\hat{\bar{Y}} = \frac{1}{nl}\sum_{i=1}^{n}\sum_{p=1}^{l} Y_{pi}$ is the overall sample mean.

5. The Expected Value of the $\hat{Var}$ $(\hat{\bar{Y}})$ in the Presence of Non-Sampling Errors

If the model in 3.2 holds, then the estimator given in 4.2 becomes as

$$\hat{Var}(\hat{\bar{Y}}) = \frac{1}{l \cdot L}\left(\frac{N-n}{N \cdot n}\right)\sum_{p=1}^{l} s_{pY}^2 + \left(\frac{L-l}{L \cdot l}\right) s_{bY}^2 \tag{5.1}$$

where

$$s_{pY}^2 = \frac{1}{n-1}\sum_{p=1}^{n}(Y_{phij} - \hat{\bar{Y}}_{p...})^2 \tag{5.2}$$

$$s_{bY}^2 = \frac{1}{l-1}\sum_{i=1}^{n}(\hat{\bar{Y}}_{p...} - \hat{\bar{Y}}_{....})^2 \tag{5.3}$$

By taking expectation on both sides of equation 5.1, we have,

$$E_S\{\hat{Var}(\hat{\bar{Y}}\} = \frac{1}{l \cdot L}\left(\frac{N-n}{N \cdot n}\right) E_L\left\{\sum_{p=1}^{l} E_p(s_{pY}^2)\right\} + \left(\frac{L-l}{L \cdot l}\right) E_S(s_{bY}^2)$$

$$E_S\{\hat{Var}(\hat{\bar{Y}}\} = \frac{1}{L^2}\left(\frac{N-n}{N \cdot n}\right)\sum_{p=1}^{l} E_p(s_{pY}^2) + \left(\frac{L-l}{L \cdot l}\right) E_S(s_{bY}^2) \tag{5.4}$$

Substituting from 3.2 and 3.3 into 5.2 we get

$$E_p(s^2_{pY}) = \frac{1}{n-1}E_p\left\{\sum_{i=1}^{n}(x_{pi} - \hat{\bar{X}}_p)^2 + \sum_{i=1}^{n}(\epsilon_{pij} - \frac{1}{n}\sum_{i=1}^{n}\epsilon_{pij})^2 + \right.$$

$$\left.\sum_{i=1}^{n}(\eta_{phij} - \frac{1}{n}\sum_{i=1}^{n}\eta_{phij})^2\right\}$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X}_p)^2 + \frac{1}{n-1}\left\{\sum_{i=1}^{n}E_p(\epsilon_{pij} - \frac{1}{n}\sum_{i=1}^{n}\epsilon_{pij})^2 + \right.$$

$$\left.\sum_{i=1}^{n}(\eta_{phij} - \frac{1}{n}\sum_{i=1}^{n}\eta_{phij})^2\right\}$$

But

$$\frac{1}{n-1}E_p\{\sum_{i=1}^{n}(\epsilon_{pij} - \frac{1}{n}\sum_{i=1}^{n}\epsilon_{pij})^2\} = \sigma^2_\epsilon$$

and

$$\frac{1}{n-1}E_p\{\sum_{i=1}^{n}(\eta_{pij} - \frac{1}{n}\sum_{i=1}^{n}\eta_{pij})^2\} = \sigma^2_\eta$$

Thus

$$E_p(s^2_{pY}) = S_{pX} + \sigma^2_\epsilon + \sigma^2_\eta$$

Now substituting from 3.3 and 3.4 into 5.3 we get

$$E_S(s^2_{bY}) = E\left[\frac{1}{l-1}\left\{\sum_{p=1}^{l}(\hat{\bar{X}}_{pi} - \hat{\bar{X}})^2 + \frac{1}{l-1}\sum_{p=1}^{l}\{(\sigma_j - \frac{1}{l}\sum_{i=1}^{l}\sigma_{pj})^2 + \right.\right.$$

$$(\frac{1}{n}\sum_{i=1}^{n}\epsilon_{pij} - \frac{1}{ln}\sum_{i=1}^{n}\sum_{p=1}^{l}\epsilon_{pij})^2 + (\beta_h - \frac{1}{l}\sum_{h=1}^{l}\beta_h)^2 +$$

$$\left.\left.\frac{1}{n}\sum_{i=1}^{n}\eta_{phij} - \frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n}\eta_{phij})^2\right\}\right] \qquad (5.6)$$

$$E_S\{\frac{1}{l-1}\{\sum_{p=1}^{l}(\hat{\bar{X}}_{pi}-\hat{\bar{X}})^2\} = \frac{1}{l-1}\left(\frac{N-n}{N\cdot n}\right)\sum_{p=1}^{l}S_{px}^2 + S_{bx}^2 \tag{5.7}$$

$$E_S\{\sum_{p=1}^{l}(\alpha_j - \frac{1}{l}\sum_{i=1}^{l}\alpha_{pj})^2\} = E_L E_P\{\sum_{p=1}^{l}(\alpha_j - \frac{1}{l}\sum_{i=1}^{l}\alpha_{pj})^2\}$$

$$= E_L\{\sum_{j=1}^{l}E_P(\alpha_j^2) - IE_P(\frac{1}{l}\sum_{j=1}^{l}\alpha_j)^2\} = (l-1)\sigma_\alpha^2 \tag{5.8}$$

Similarly

$$E_S(\sum_{p=1}^{l}(\beta_h - \frac{1}{l}\sum_{h=1}^{l}\beta_h)^2\} = (l-1)\sigma_\beta^2 \tag{5.9}$$

$$E_S\{\sum_{p=1}^{l}(\frac{1}{l}\sum_{i=1}^{n}\epsilon_{pi} - \frac{1}{ln}\sum_{i=1}^{n}\sum_{p=1}^{l}\epsilon_{pij})^2\} = E_L\{\sum_{p=1}^{l}E_P(\frac{1}{n}\sum_{i=1}^{n}\epsilon_{pij})^2 - E_P(\frac{1}{ln}\sum_{i=1}^{n}$$

$$\sum_{p=1}^{l}\epsilon_{pij})^2\} = E_L\left(\frac{I\sigma_\epsilon^2}{n} - \frac{\sigma_\epsilon^2}{n}\right) = (l-1)\frac{\sigma_\epsilon^2}{n} \tag{5.10}$$

Similarly

$$E_S\{\sum_{p=1}^{l}\frac{1}{n}\sum_{i=1}^{n}\eta_{phij} - \frac{1}{ln}\sum_{p=1}^{l}\sum_{i=1}^{n}\eta_{phij})^2\} = (l-1)\frac{\sigma_\eta^2}{n} \tag{5.11}$$

Therefore by substituting from 5.7 - 5.11. into 5.6, we get

$$E_S(s_{bY}^2) = \frac{1}{L}\left(\frac{N-n}{N\cdot n}\right)\sum_{p=1}^{l}S_{px}^2 + S_{bx}^2 + \sigma_\alpha^2 + \frac{\sigma_\epsilon^2}{n} + \sigma_\beta^2 + \frac{\sigma_\eta^2}{n} \tag{5.12}$$

And now substituting from 5.5 and 5.12 into 5.4, we get.

$$E_S\{\mathrm{Var}(\hat{\bar{Y}})\} = \frac{1}{L.l}\left(\frac{(N-n)}{N.n}\right)\sum_{p=1}^{l}S_{px}^2 + \left(\frac{L-l}{L.l}\right)S_{bx}^2 + \frac{1}{l}\left(\sigma_\alpha^2 + \frac{\sigma_\epsilon^2}{n} + \sigma_\beta^2 + \frac{\sigma_\eta^2}{n}\right)$$

$$\frac{1}{L}\left(\sigma_\alpha^2 + \frac{\sigma_\epsilon^2}{N} + \sigma_\beta^2 + \frac{\sigma_\eta^2}{N}\right)$$

Using 3.5 we get,

$$E_S\{\hat{\mathrm{Var}}(\hat{\bar{Y}}) = \{\hat{\mathrm{Var}}(\hat{\bar{Y}}_{....}) - \frac{1}{L}(\sigma_\alpha^2 + \frac{\sigma_\epsilon^2}{N} + \sigma_\beta^2 + \frac{\sigma_\eta^2}{N})$$

The above result shows that the bias in estimating the variance of the sample mean outlined in 3.5 by using the estimator, given in 5.1 which takes into account the sampling errors only, is a factor of L^{-1} and N^{-1} i.e. the text book estimator of the variance of the sample mean provides a good approximation to the variance of the sample mean when actually the sampling errors are present. However, if the results are required for the small area, for one cluster, say, the bias may not be ignorable and in such case it becomes important to estimate the components of the total variance which are due to both sampling and non-sampling errors if precise inferences are to be drawn. In case of a census the sampling variance will be zero and the total variance will be equal to $\frac{1}{L}(\sigma_\alpha^2 + \frac{\sigma_\epsilon^2}{N} + \sigma_\beta^2 + \frac{\sigma_\eta^2}{N})$. On a national level it will be trivial but often the results are also required for small areas. Further there may be other small scale studies based on census data; in such cases the variance may be quite important.

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